S2 New Curriculum O-level mathematics

Term 2

Theme: Patterns and algebra

Chapter 6: Algebra 2

Quadratic equation

These are equations expressed in the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$

Example of quadratic equations

$2y^2 + 3y + 5 = 0$; a = 2, b = 3 and c = 5

$x^2 + 4x – 10 = 0$; a = 1, b = 4, c = -10

Roots of quadratic equation

The **roots** or **solutions** are the values of x that make the quadratic equation true—that is, they satisfy the equation. Each quadratic equation has two roots. Usually, denoted as α and β.

Forming quadratic equations

Suppose that the roots of a quadratic equation are α and β,

then $x – \alpha = 0$ and $x – \beta = 0$

When forming a quadratic equation

$(x – \alpha)(x – \beta) = 0$

$x^2 – (\alpha + \beta)x + \alpha\beta = 0$

This means that if the roots of a quadratic equation are given, its equation in terms of x is x^2-(sum of roots)x + product of roots = 0

Example 1

Form quadratic equation I terms of x with roots

(a) (2, 3)

Solution

Let x = 2 or x = 3

$x – 2 = 0$ $\quad$ $x – 3 = 0$

⇨ $(x – 2)(x – 3) = 0$

$x^2 – 5x – 6 = 0$

(b) (p, q)

Solution

Let x = p or x = q

$x – p = 0$ $\quad$ $x – q = 0$

⇨ $(x – p)(x – q) = 0$

$x^2 – (p + q)x – pq = 0$

(c) (-3, -2)

Solution

Let x = -3 or x = -2

$x + 3 = 0$ $\quad$ $x + 2 = 0$

⇨ $(x + 3)(x + 2) = 0$

$x^2 + 5x + 6 = 0$

Trail 1

Form quadratic equation with the following roots

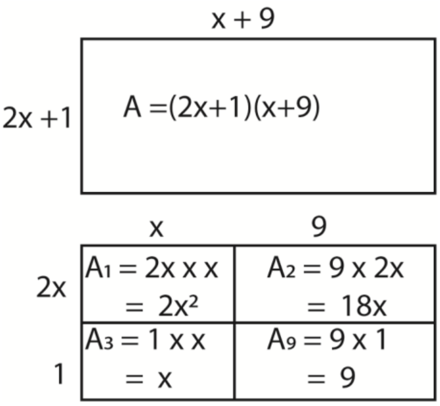
(a) (2, -2)
(b) (1, 3)
(c) (4, 5)

Equivalent quadratic equations

Equivalent quadratic expressions are different expressions of the equation that produce the same values for all inputs of x. i.e. they

- represent the same curve when graphed
- yield the same result for any value of x.
- can be transformed into one another using algebraic operations

Consider the area A of a rectangle of side (x + 3) by x.



$A = A_1 + A_2 + A_3 + A_4$

$= 2x^2 + 18x + x + 9$

$= 2x^2 + 19x + 9$

⇨ $(2x + 1)(x + 9)$ and $2x^2 + 19x + 9$ are equivalent quadratic equation

Trial 2

Form equivalent quadratic equations for rectangle with sides

(a) (x + 6) and (x + 2)
(b) (x – 2) and (x + 7)
(c) (x – 4) and (x – 8)
(d) (x + 1)and (x + 3)

Factorization of quadratic equations

Consider a quadratic equation

$ax^2 + bx + c = 0$

b = ac

To factorize the above equation, we find numbers α and β such that

$\alpha + \beta = b$ and $\alpha\beta = c$

Then, $ax^2 + bx + c = x^2 + \alpha x + \beta x + \alpha\beta$

$= x(x + \alpha) + \beta(x + \alpha)$

$= (x + \alpha)(x+\beta)$

Hence, the factors of $ax^2 + bx + c$ are $(x + \alpha)$ and $(x+\beta)$

Example 2

Factorise the following

(a) $x^2 + 13x + 22$
We find two numbers such that their sum is 13 and their product is (1x22) = 22. The numbers are 11 and 2

⇨ $x^2 + 13x + 22 = x^2 + 11x + 2x + 22$

$= x(x + 11) + 2(x + 11)$

$= (x + 2)(x + 11)$

$\therefore\ x^2 + 13x + 22= (x + 2)(x + 11)$

(b) $x^2 – 20x + 64$
We find two numbers such that their sum is -20 and their product is (1 x 64) = 64. The numbers are -16 and -4

⇨ $x^2 – 20x + 64 = x^2 – 16x – 4x + 64$

$= x(x – 16) -4 (x – 16)$

$= (x – 4)(x – 16)$

$\therefore\ x^2 – 20x + 64 =(x – 4)(x – 16)$

(c) $2x^2 + 5x - 3$

We find two numbers such that their sum is 5 and their product is (2 x -3)= -6. The numbers are 6 and -1

$2x^2 + 5x - 3 = 2x^2 + 6x - x - 3$

$= 2x(x + 3) - 1(x+3)$

$= (2x - 1)(x + 3)$

(d) $6x^2 - x - 12$

We find two numbers such that their sum is -1 and their product is (6 x -12) = -72.

The numbers are -9 and 8

⇨ $6x^2 - x - 12 = 6x^2 - 9x + 8x - 12$

$= 3x(2x - 3) + 4(2x - 3)$

$= (3x + 4)(2x - 3)$

(e) $-35x^2 + 11x + 6$

We find two numbers such that their sum is 11 and their product is (-35 x 6) = -210.

The numbers are 21 and -10

$-35x^2 + 11x + 6 = -35x^2 - 10 - 21x + 6$

$= -5x(7x - 2) - 3(7x + 6)$

$= -(5x + 3)(7x - 2)$

$= (5x + 3)(2 - 7x)$

Trial 3

Factorise the following quadratic equations

(a) $x^2 + x - 30$
(b) $x^2 - 7x + 6$
(c) $x^2 + 19x - 20$
(d) $x^2 - 11x + 24$
(e) $x^2 + x - 72$
(f) $6x^2 - 4x - 2$
(g) $15 - 4x - 4x^2$
(h) $15x^2 + 7x - 2$
(i) $40x^2 + 38x - 15$

Solving quadratic equations

Quadratic equations may be solved by

(a) Factorization method
(b) Completing square method
(c) Graphical method

Solving quadratic equation by factorization method

Example 3

(a) $4x^2 + 7x + 3 = 0$

Solution	side work
$4x(x + 1) + 3(x + 1) = 0$	Product = 4 x 3 = 12
$(x + 1)(4x + 3) = 0$	Sum = 7
Either $x + 1 = 0$; $x = -1$	Factor = (4, 3)

Or $4x + 3 = 0$; $x = -\frac{3}{4}$

(b) $2x^2 + 5x + 3 = 0$

Solution	side work
$2x(x + 1) + 3(x + 1) = 0$	Product = 3 x 2 = 6
$(x + 1)(2x + 3) = 0$	Sum = 5
	Factors 3, 2

Either $x + 1 = 0$; $x = -1$

Or $2x + 3 = 0$; $x = -\frac{3}{2}$

(c) $x^2 + x - 20 = 0$

Solution	Side work
$x(x - 4) + 5(x - 4) = 0$	Product = -20
$(x - 4)(x + 5) = 0$	Sum = 1
Either $x - 4 = 0$; $x = 4$	Factors (5, - 4)

Or $x + 5 = 0$; $x = -5$

(d) $10x^2 + x - 3 = 0$

Solution	Side work
$5x(2x - 1) + 3(2x - 1) = 0$	product 10 x – 3 = -30
$(5x + 3)(2x - 1) = 0$	sum = 1
Either $5x + 3 = 0$; $x = -\frac{3}{5}$	Factors (6, -5)

Or $2x - 1 = 0$; $x = \frac{1}{2}$

Trial 2

Find the roots of the following quadratic equations using factorization methos

(i)	$x^2 + 3x + 2 = 0$	[-1, -2]
(ii)	$2x^2 – 5x + 3$	$\left[1, \frac{3}{2}\right]$
(iii)	$6x^2 – 7x + 3$	$\left[\frac{2}{3}, \frac{1}{2}\right]$
(iv)	$2x^2 – 5x – 3$	$\left[3, -\frac{1}{2}\right]$
(v)	$6x^2 –x – 1$	$\left[\frac{1}{2}, -\frac{1}{3}\right]$
(vi)	$x^2 – 3x + 2$	[1, 2]
(vii)	$x^2 - 11x + 24$	[8, 3]
(viii)	$x^2 + x – 72$	[-9, 8]
(ix)	$6x^2 – 4x – 2$	$\left[\frac{1}{3}, -\frac{3}{2}\right]$
(x)	$15 – 4x – 4x^2$	$\left[\frac{3}{2}, -\frac{5}{2}\right]$
(xi)	$40x^2 + 38x – 15$	$\left[\frac{3}{10}, -\frac{5}{4}\right]$

Solving quadratic equations by Completing squares approach

The idea is to create a perfect square on one side of the equation:

Given the equation $ax^2+ bx + c = 0$

- Dividing the equation by and transposing the constant term to the RHS
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
- Marking the LHS a perfect square, add a half the coefficient of x squared on both sides
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
- Factorise the terms on the LHS
$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$
$$= \frac{b^2-4ac}{4a^2}$$
Or
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$
- Taking square root on both sides of the equation
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$
$$= \pm\frac{\sqrt{b^2-4ac}}{2a}$$
- Solving
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$
$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This is the general quadratic equation formula for finding the square root of any quadratic equation. This formula is locally known as ***bull dozer formula***

Example 3

Solve the following equations by completing squares

(a) $2x^2 – x – 3 = 0$

Solution

$2x^2 – x – 3 = 0$

$x^2 - \frac{1}{2}x = \frac{3}{2}$

$x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = \frac{3}{2} + \left(-\frac{1}{4}\right)^2$

$\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$

$x - \frac{1}{4} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

Either $x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Or $x = -\frac{5}{4} + \frac{1}{4} = \frac{-4}{4} = -1$

Hence x = -1 and x = $\frac{3}{2}$

Or

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{1\pm\sqrt{(-1)^2-4(2\,x-3)}}{2x\ 2}$$

$$= \frac{1 \pm \sqrt{1+24}}{4}$$

$$= \frac{1 \pm 5}{4}$$

Either x = $\frac{-4}{4} = -1$

Or x = $\frac{6}{4} = \frac{3}{2}$

(b) $18x^2 + 7x - 1 = 0$

Solution

$18x^2 + 7x - 1 = 0$

$x^2 + \frac{7}{18}x = \frac{1}{18}$

$x^2 + \frac{7}{18}x + \left(\frac{7}{36}\right)^2 = \frac{1}{18} + \left(-\frac{7}{36}\right)^2$

$$\left(x + \frac{7}{36}\right)^2 = \frac{1}{18} + \frac{49}{1296} = \frac{121}{1296}$$

$$x + \frac{7}{36} = \sqrt{\frac{121}{1296}} = \pm\frac{11}{36}$$

Either $x = \frac{11}{36} - \frac{7}{36} = \frac{4}{36} = \frac{1}{9}$

Or $x = -\frac{11}{36} - \frac{7}{36} = -\frac{18}{36} = -\frac{1}{2}$

Hence x = $\frac{1}{9}$ or x =$-\frac{1}{2}$

Or

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(-7)^2-4(18\,x-1)}}{2x\ 18}$$

$$= \frac{1 \pm \sqrt{49+72}}{36}$$

$$= \frac{-7 \pm 11}{36}$$

Either x = $\frac{4}{36} = \frac{1}{9}$

Or x = $\frac{-18}{36} = \frac{-1}{2}$

(c) $3x^2 + 7x + 2 = 0$

Solution

$3x^2 + 7x + 2 = 0$

$$x^2 + \frac{7}{3}x = -\frac{2}{3}$$

$$x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{2}{3} + \left(\frac{7}{6}\right)^2$$

$$\left(x + \frac{7}{6}\right)^2 = -\frac{2}{3} + \frac{49}{36} = \frac{25}{36}$$

$$x + \frac{7}{6} = \sqrt{\frac{25}{36}} = \pm\frac{5}{6}$$

Either x =$\frac{5}{6} - \frac{7}{6} = -\frac{2}{6} = -\frac{1}{3}$

Or x =$-\frac{5}{6} - \frac{7}{6} = -\frac{12}{6} = -2$

Hence x = -2 and x =$-\frac{1}{3}$

Or

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(-7)^2-4(2\,x3)}}{2x\ 3}$$

$$= \frac{-7 \pm \sqrt{49-24}}{6}$$

$$= \frac{-7 \pm 5}{6}$$

Either x = $\frac{-12}{6} = -2$

Or x = $\frac{-2}{6} = -\frac{1}{3}$

Trial 3

Solve the following equations by using the method of completing squares and using quadratic formula

(a)	$7x^2 - 5x - 2 = 0$	$\left[1, -\frac{2}{7}\right]$
(b)	$3x^2 - 7x - 6 = 0$	$\left[3, -\frac{2}{3}\right]$
(c)	$6x^2 - 5x - 6 = 0$	$\left[\frac{3}{2}, -\frac{2}{3}\right]$
(d)	$x^2 - 7x + 10 = 0$	[5, 2]
(e)	$x^2 + 3x + 2 = 0$	[-1, -2]
(f)	$x^2 - 7x + 12$	(3, 4]

(g) $p^2 - 5p + 6 = 0$ [2, 3]

(h) $2x^2 + 3x - 2 = 0$ $\left[-2,\ \frac{1}{2}\right]$

(i) $2x^2 + 9x + 7 = 0$ $\left[-1,\ \frac{7}{2}\right]$

(j) $-2x^2 + x + 6 = 0$ $\left[2,\ \frac{3}{2}\right]$

Example 4

(a) Express $x^2 + 6x + 5$ in the form $(x + a)^2 + b$. Hence solve the equation $x^2 +6x + 5 = 0$

$x^2 + 6x + 5$

$= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 5$

$= \left(x + \frac{6}{2}\right)^2 - \frac{36}{2} + 5$

$= (x+3)^2 - 4$

⇨ a = 3 and b = -4

Alternatively

$x^2 + 6x + 5 = (x + a)2 + b$

$=x^2 + 2ac + a^2 + b$

Equating coefficients

2a = 6 => a = 2

a2 + b = 5

$(3)^2 + b = 5$

b = -4

Hence, $(x+3)^2 - 4 = 0$

x + 3 = ±2

For x+ 3 = 2

x = -1

For x + 3 = -2

x = -5

(b) Express $2x^2 + x - 10$ in the form $a(x + b)^2 + c$. Hence solve equation $2x^2 + x - 10 = 0$

Solution

$2x^2 + x - 10 = a(x + b)^2 + c$

$= a(x^2 + 2bx + b^2) +c$

$= ax^2 +2abx + ab^2 + c$

Equating coefficients

a = 2

$2ab = 1 \Rightarrow b = \frac{1}{4}$

$ab^2 + c = -10$

$2 \text{ x } \left(\frac{1}{4}\right)^2 + c = -10$

$c = -\frac{81}{8}$

Hence

$2(x + \frac{1}{4})^2 - \frac{81}{8} = 0$

$(x + \frac{1}{4})^2 = \frac{81}{16}$

$x + \frac{1}{4} = \pm\frac{9}{3}$

$x = 2 \text{ or } \frac{5}{5}$

Trial 3

1. Express $x^2 + x -12$ in the form $(x + a)^2 + b$.
   Hence solve $x^2 + x - 12 = 0$ [3 -4]
2. Express $x^2 +10x + 9$ in the form $(x + a)^2 + b$.
   Hence solve $x^2 + 10x +9 = 0$ [-9 -1]
3. Express $3x^2 - 7x + 2$ in the form $a(x + b)^2 + c$.
   Hence solve $3x^2 - 7x + 2 = 0$
4. Express $9x^2 + 18x - 7$ in the form $a(x + b)^2 + c$.
   Hence solve $9x^2 + 18x - 7 = 0$ $[\frac{1}{3}\ -\frac{7}{3}]$
5. Express $6x^2 - x - 12$ in the form $a(x + b)^2 + c$.
   Hence solve $6x^2 - x - 12 = 0$ $[\frac{2}{3}\ -\frac{4}{3}]$

Solving quadratic equations using Graphical approach

Here, the roots of the equations are established by plotting suitable graphs. It may done by:-

(i) Single graph plot
(ii) Two graph plot

Single graph method

Given the function y = f(x), tabulate the selected values of x closer to zero within a given range which must be substituted in the function given to obtain the corresponding values of y. These values are plotted on the same x and y axes and joined by a curve. The roots of the function f(x) = 0 are the values of x where the function y = f(x) crosses the x-axis.

Two graphs method

Here the function y = f(x) is split into two and the equations plotted separately on the same axes. The points of intersection of the two graphs are noted and the corresponding values of x are read off on the x – axis.

Example 5

Solve the following equation by using graphical approach

(a) $2x^2 + 3x – 3 = 0$

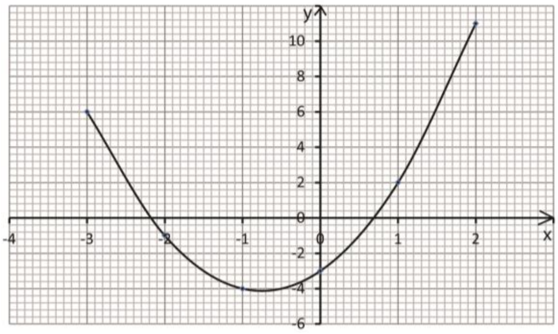
Solution

Using a single graphs

Let $y = 2x^2 + 3x – 3$ taking $-3 \leq x \leq 2$

Table of values

X	-3	-2	-1	0	1	2
y	6	-1	-4	-3	2	11



From the graph, the roots of the equation are x = -2.2 and x = 0.7

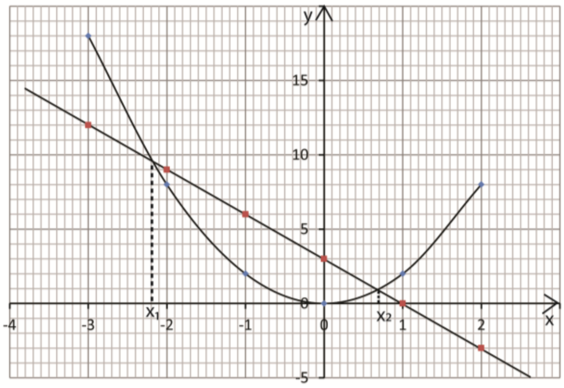
Using two graphs

By splitting the equation $2x^2 + 3x – 3 = 0$ into two we have $2x^2 = -3x + 3$

Let $y_1 = 2x^2$ and $y_2 = 3 - 3x$

Table of results

X	-3	-2	-1	0	1	2
y_1	18	8	2	0	2	8
y_1	12	9	6	3	0	-3



From the graph, the roots of the equation are x = -2.2 and x = 0.7

(b) $x^2 + x – 6 = 0$

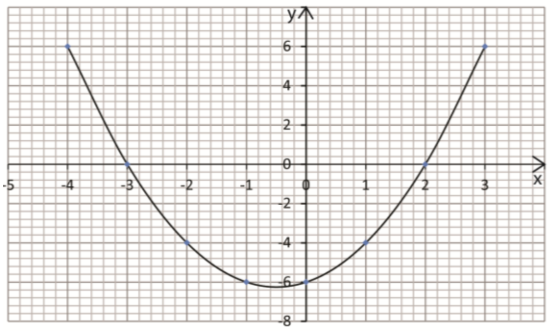
Solution

Using a single graphs

Let $y = x^2 + x – 6 = 0$, taking $-4 \leq x \leq 3$

Table of values

X	-4	-3	-2	-1	0	1	2	3
y	6	0	-4	-6	-6	-4	0	6



From the graph, the roots of the equation are x = -3 and x = 2

Using two graphs

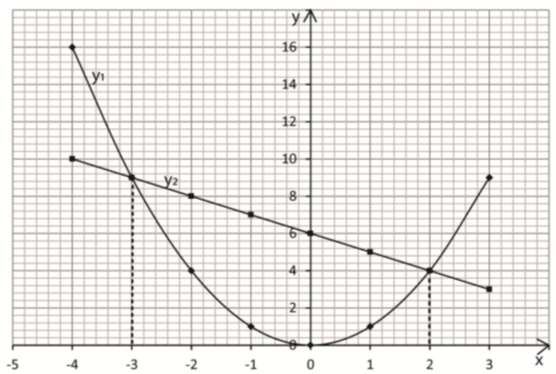
By splitting the equation $x^2 + x - 6 = 0$

into two we have $x^2 = 6 - x$

Let $y_1 = x^2$ and $y_2 = 6 - x$

Table of results

X	-4	-3	-2	-1	0	1	2	3
y_1	16	9	4	1	0	1	4	9
y_1	10	9	8	7	6	5	4	3



From the graph, the roots of the equation are x = -3 and x = 2

Trial 4

1. Use graphical method to solve the following quadratic equations.

   (a) $x^2 + 3x + 2 = 0$ [-1, -2]
   (b) $2x^2 – 5x + 3$ $\left[1, \frac{3}{2}\right]$
   (c) $6x^2 – 7x + 3$ $\left[\frac{2}{3}, \frac{1}{2}\right]$
   (d) $2x^2 – 5x – 3$ $\left[3, -\frac{1}{2}\right]$
   (e) $6x^2 – x – 1$ $\left[\frac{1}{2}, -\frac{1}{3}\right]$
   (f) $x^2 – 3x + 2$ [1, 2]
2. A rectangle of length (4x – 1) cm and breadth 2x cm has an area of 10cm2. Find
   (a) The value of x [1.25]
   (b) It length and breadth [4cm, 2.5cm]
   (c) Perimeter [13cm]
3. The sum of two numbers is 24. The difference in their square is 144. Find the numbers. [15, 9]
4. When thirty times a number is increased by 32, the result is twice the square of the number. Find the number [16]

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Thanks

Dr. Bbosa Science