University of Northern Colorado Mathematics Contest 2023-2024

Solutions of Final Round

1. Replace each different letter with a different positive digit so that the following pair of equations are satisfied simultaneously. Which single digit was not used?

$$OF^A = BAT \text{ and } IN^A = BATH$$

Answer: 6

Solution:

The key to save time: the tens digit of a square number is odd if and only if the ones digit is 6.

Since $OF^A = BAT$ a 3-digit number, we have $A = 2$.

Ones digit of a square number is 0, 1, 4, 5, 6, or 9.

Obviously, T cannot be 0 or 5.

Since A, the tens digit of square OF^2, is even, ones digit of the square cannot be 6.

T is 1, 4, or 9.

Let us list the 3-digit squares ending in 21, 24, 29:

$$121, 324, 529, 729.$$

If T, the tens digit of square IN^2, is 1 or 9 (odd numbers), then ones digit H of the square is 6. However, 1216, 5296, and 7296 are not square numbers.

Then $BAT = 324$. $BATH = 3249$ is a square.

We have

$$18^2 = 324 \text{ and } 57^2 = 3249.$$

We have digits 0 and 6 not used.

The only positive digit not used is 6.

The answer is 6.

2. The White Knight has 45 identical wooden blocks that he will use to make a very jagged wall. He has decided that it will consist of nine stacks of blocks, each a different height. The first three and the last three stacks should all be very different heights: the difference between heights of the 1st and 2nd stack must be more than 5, as must the differences between the 2nd and 3rd stack, between the 7th and 8th stack, and between the 8th and 9th stack. How many different ways can the White Knight arrange his blocks into a very jagged wall, given his constraints?

Answer: 48

Solution:

Obviously, the 9 heights must be 1, 2, 3, 4, 5, 6, 7, 8, and 9 since $1 + 2 + \cdots + 9 = 45$.

We are finding the number of permutations

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \text{ of } (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

to satisfy $|a_1 - a_2| \geq 6$, $|a_2 - a_3| \geq 6$, $|a_7 - a_8| \geq 6$, and $|a_8 - a_9| \geq 6$.

We cannot have $a_1 > a_2 > a_3$. Otherwise, $a_3 = a_3 - a_2 + a_2 - a_1 + a_1 \geq 6 + 6 + 1 = 13$.

Similarly, we cannot have $a_1 < a_2 < a_3$, $a_7 > a_8 > a_9$, or $a_7 < a_8 < a_9$

Case 1: ($a_1 > a_2$ and $a_2 < a_3$) and ($a_7 > a_8$ and $a_8 < a_9$)

Then $\{a_2, a_8\} = \{1, 2\}$.

Without loss of generality, let $a_2 = 1$ and $a_8 = 2$.

With $a_8 = 2$, we must have $\{a_7, a_9\} = \{8, 9\}$.

Then we don't have two numbers for a_1 and a_3.

There is no permutation in this case.

Case 2: ($a_1 < a_2$ and $a_2 > a_3$) and ($a_7 < a_8$ and $a_8 > a_9$)

Similarly to case 1, there is no permutation in this case.

Case 3: ($a_1 > a_2$ and $a_2 < a_3$) and ($a_7 < a_8$ and $a_8 > a_9$)

Obviously, $a_2 = 1$ or 2.

If $a_2 = 2$, we must have $\{a_1, a_3\} = \{8, 9\}$.

Then there is no number for a_8.

We must have $a_2 = 1$.

By symmetry, we must have $a_8 = 9$.

Then $\{a_1, a_3\} = \{7, 8\}$, $\{a_7, a_9\} = \{2, 3\}$, and $\{a_4, a_5\, a_6\} = \{4, 5, 6\}$.

The number of permutations in this case is

$$2! \cdot 2! \cdot 3! = 2 \cdot 2 \cdot 6 = 24.$$

Case 4: ($a_1 < a_2$ and $a_2 > a_3$) and ($a_7 > a_8$ and $a_8 < a_9$)

Similarly to case 4, the number of permutations in this case is 24.

The total number of permutations is $2 \cdot 24 = 48$.

The answer is 48.

3. Let $R_n = 1^n + 2^n + 3^n + 4^n$. For how many integers *n* between 1 and 1000 (inclusive) is R_n a multiple of 5?

Answer: 750

Solution:

Lemma:

If $R_n = 0 \mod 5$ if and only if $n \neq 0 \mod 4$.

Proof of the lemma:

Note that

$$R_1 = 1 + 2 + 3 + 4 = 1 + 2 + (-2) + (-1) = 0 \mod 5,$$

$$R_2 = 1^2 + 2^2 + 3^2 + 4^2 = 1^2 + 2^2 + (-2)^2 + (-1)^2 = 0 \mod 5,$$

$$R_3 = 1^3 + 2^3 + 3^3 + 4^3 = 1^3 + 2^3 + (-2)^3 + (-1)^3 = 0 \mod 5.$$

So

$$R_i = 1^i + 2^i + 3^i + 4^i = 0 \mod 5 \text{ for } i = 1, 2, \text{ and } 3.$$

For all $a = 1$, 2, 3, and 4, a and 5 are relatively prime.

By Fermat Theorem, $a^4 = 1 \mod 5$.

If $n = 0 \mod 4$, let $n = 4k$ where k is a positive integer.

$$R_n = 1^n + 2^n + 3^n + 4^n = 1^{4k} + 2^{4k} + 3^{4k} + 4^{4k} = 1 + 1 + 1 + 1 = 4 \neq 0 \mod 5.$$

If $n \neq 0 \mod 4$, let $n = 4k + i$ where k is a positive integer and $i = 1$, 2, or 3.

$$R_n = 1^n + 2^n + 3^n + 4^n = 1^{4k+i} + 2^{4k+i} + 3^{4k+i} + 4^{4k+i} = 1^i + 2^i + 3^i + 4^i = R_i = 0 \mod 5.$$

The lemma is proved.

From 1 to 1000 inclusive, there are 250 numbers that are 0 mod 4 and 750 numbers that are not 0 mod 4.

The answer is 750.

4. The Gryphon was telling Alice and the Mock Turtle about the four trolls he met on vacation in the land of Knights and Knaves. “I knew that each troll was either a knight, who always told the truth, or a knave who always lied. They each proceeded to tell me how many of the four were knights, giving me a number from 1 to 4 inclusive. Strangely, as they spoke, their numbers never decreased. Anyway, I was able to deduce that they were all lying.” Alice remarked, “I guess it was lucky that they said the numbers they did!” The Gryphon replied: “I suppose, although there were other numbers they could have used and I still would have been able to deduce they were all knaves.”

   How many different responses could the group of trolls have given that would have still conclusively revealed their identities as four knaves?

Answer: 18

Solution:

There are $\binom{7}{4} = 35$ different possible responses (a_1, a_2, a_3, a_4) with $1 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 4$.

To understand why the answer is $\binom{7}{4} = 35$, see the appendix.

The key observation: if there are k knights, there must be exactly k k's in the 4 numbers of a response for $k = 1, 2, 3, 4$.

If there is one knight, the response must be $(1, b_1, b_2, b_3)$ with $2 \le b_2 \le b_3 \le b_4 \le 4$.

The number of possible responses is $\binom{5}{3} = 10$.

The 10 responses are

$$(1, 2, 2, 2),$$

$$(1, 2, 2, 3),$$

$$(1, 2, 2, 4),$$

$$(1, 2, 3, 3),$$

$$(1, 2, 3, 4),$$

$$(1, 2, 4, 4),$$

$$(1, 3, 3, 3),$$

$$(1, 3, 3, 4),$$

$$(1, 3, 4, 4),$$

$$(1, 4, 4, 4).$$

If there are 2 knights, the response must be $(1, 1, 2, 2)$, $(1, 2, 2, c)$ with $c = 3$ or 4, or $(2, 2, c_1, c_2)$ with $3 \le c_3 \le c_4 \le 4$.

The number of possible responses is $1 + 2 + \binom{3}{2} = 6$.

The 6 responses are

$$(1, 1, 2, 2),$$

$$(1, 2, 2, 3),$$

$$(1, 2, 2, 4),$$

$$(2, 2, 3, 3),$$

$$(2, 2, 3, 4),$$

$$(2, 2, 4, 4).$$

If there are 3 knights, there are 3 possible responses:

$$(1, 3, 3, 3),$$

$$(2, 3, 3, 3),$$

$$(3, 3, 3, 4).$$

If there are 4 knights, the only one possible response is

$$(4, 4, 4, 4).$$

There are overlaps.

1. $(1, 2, 2, 3)$ and $(1, 2, 2, 4)$ are valid for that there is exactly one knight and for that there are exactly two knights.
2. $(1, 3, 3, 3)$ is valid for that there is exactly one knight and for that there are exactly three knights.

The total number of possible responses with at least one knight is

$$10 + 6 + 3 + 1 - 3 = 17.$$

The total numbers of responses with which it cannot be determined that there are 4 knaves is

$$35 - 17 = 18.$$

The answer is 18.

Appendix:

The number of combinations to choose m numbers from $\{1, 2, \cdots, n\}$ with replacements (we can choose a number any number of times) is equal to the number of combinations to choose m different numbers from $\{1, 2, \cdots, n + m - 1\}$.

The answer is $\binom{n+m-1}{m}$.

Proof:

Let $(a_1, a_2, \cdots, a_m)$ be a combination with

$$1 \le a_1 \le a_2 \le \cdots \le a_m \le n.$$

Let $(b_1, b_2, \cdots, b_m)$ be a combination with

$$1 \le b_1 < b_2 < \cdots < b_m \le n + m - 1.$$

There is one to one correspondence between $(a_1, a_2, \cdots, a_m)$ and $(b_1, b_2, \cdots, b_m)$:

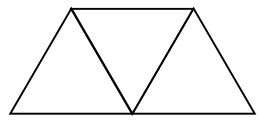
$$(a_1, a_2, \cdots, a_m) \overset{a_1 = b_1,\ a_2 = b_2 - 1,\ \cdots,\ a_m = b_m - (m-1)}{\Longleftrightarrow} (b_1, b_2, \cdots, b_m).$$

5. Hwei is designing a game similar to Tetris. In Tetris, each playing piece is made by joining four squares together along their sides. Hwei's new game is called tripentaris. In this game, each piece is made by joining five equilateral triangles together along their sides. Two pieces are considered the same in tripentaris if one can be transformed into the other by a sequence of translations, rotations, and/or reflections. How many possible pieces will there be in this game?

Answer: 4

Solution:

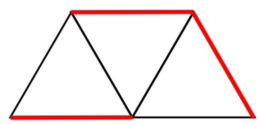
There is only one way to place three equilateral triangles together.



We will examine how many ways there are to place two more equilateral triangles.

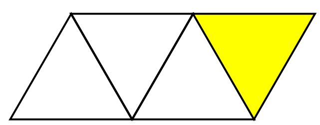
Let us place the 4th equilateral triangle.

There are three different edges to build 4th equilateral triangle:

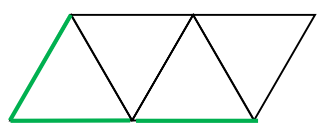


There are three cases to place the 4th triangle.

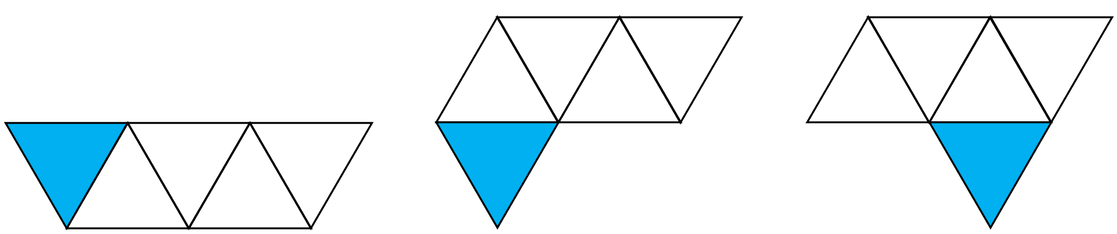
Case 1:



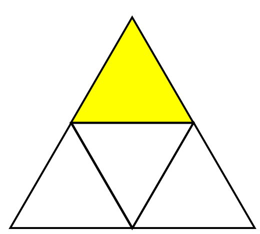
There are three different edges to build an equilateral triangle:



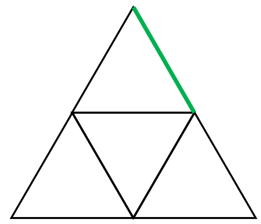
The 3 different arrangements are



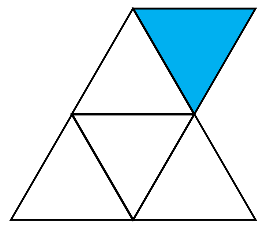
Case 2:



There is only one edge to build an equilateral triangle.

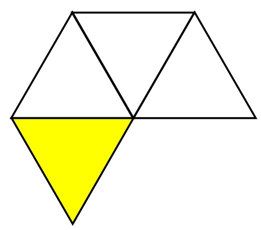


There is the arrangement:

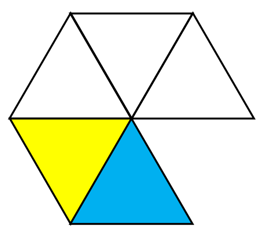
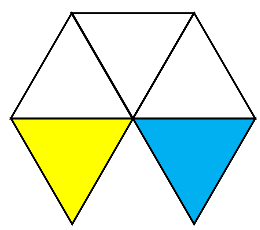


However, this is counted in the 3rd arrangement of case 1.

Case 3:



All six edges are different. There are 6 arrangements in this case. However, only the following two arrangements are different from those we have counted:

However, these two arrangements are the same.

In total, there are 4 different ways to place 5 equilateral triangles in one piece.

The answer is 4.

6. The quotient $\frac{2^{24}-1}{2^8-1}$ is an integer. If you write that integer in base 2, you get a sequence of 1's and 0's. How many 0's are in that sequence?

Answer: 14

Solution 1:

$$\frac{2^{24}-1}{2^8-1}=\frac{\left(2^8\right)^3-1}{2^8-1}=\frac{\left(2^8-1\right)\left(2^{16}+2^8+1\right)}{2^8-1}=2^{16}+2^8+1=10000000100000001_2$$

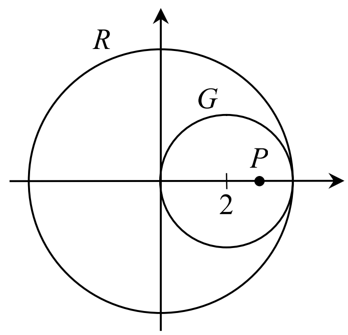
We see 14 zeros.

Solution 2:

$$\frac{2^{24}-1}{2^8-1}=\frac{11111111{,}11111111{,}11111111_2}{11111111_2}=10000000100000001_2$$

We see 14 zeros.

7. A spirograph is built by putting a circular gear G of radius 2 inside a circular ring R of radius 4, in such a way that the gear can rotate along the inside of the ring. There is a hole P in the gear, one unit away from the center of the gear, with a pen in it. In the starting position of the gear, the pen is collinear with the center of the gear and the center of the ring, as shown in the picture.

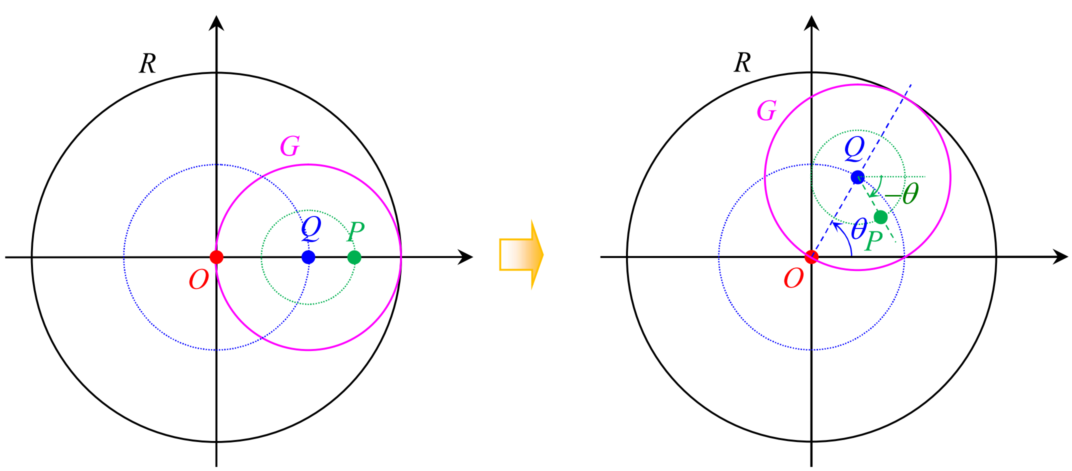


In the spirograph, the pen draws a curve C as the gear turns along the edge of the ring. The curve is complete when the pen comes back to its starting point. When drawing the complete curve, how many times does it happen that the pen is collinear with the center of the ring and the center of the gear (counting the starting and ending points together as one time)?

Answer: 4

Solution:

Let O be the center of the larger circle and Q be the center of the smaller circle.



The black circle is the ring and the pink circle is the gear.

Observation 1: Q is rotating around O and P is rotating around Q.

Let the blue circle be the locus of Q around O and the green circle be the locus of P around Q.

Let Q rotate around O counterclockwise. Then P rotates around Q clockwise.

Observation 2: let Q make one revolution around O. The number of revolutions of the gear is the ratio of the circumference (radius) of the blue circle to the circumference (radius) of the pink circle.

The radius of the blue circle is $4-2=2$, and the radius of the pink circle is 2.

The ratio is 1.

That is, the gear makes one revolution if Q makes one revolution.

Observation 3: for O, Q, and P to be in a line, the difference of two rotation angles must be a multiple of $180°$.

Let θ be the rotation angle of Q around O.

If Q rotates around O by an angle θ, then P rotates around Q by an angle $-\theta$.

We need $\theta-\left(-\theta\right)=2\theta=180°k$ where k is a positive integer.

θ changes from $0°$ to $360°$. Then 2θ changes from $0°$ to $2\cdot 360°=720°$.

The number of times that O, Q, and P are in a line is

$$\left\lceil \frac{720°}{180°} \right\rceil = 4$$

where $\lceil x \rceil$ represents the least integer greater than or equal to x.

The answer is 4.

8. Tweedledum and Tweedledee like to play the game "Taking Away Fibonacci Checkers." The way they play is as follows: they start with a pile of checkers and take turns removing checkers from the pile. On each player's turn, they remove some checkers from the pile. The number of checkers they remove on their turn must be one of the numbers from the Fibonacci Sequence (1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$), where each number is the sum of the preceding two numbers. The winner of the game is whoever takes the last checker from the pile. Tweedledum always goes first. Assuming that neither player makes a mistake when playing, what is the largest number $N<35$ such that Tweedledee should be able to win the game when they start the game with N checkers?

Answer: 30

Solution:

We define *the winning number* if the player in the turn wins when he has this number of checkers and practices the winning strategy.

All Fibonacci numbers are winning numbers.

We define *the losing number* if the player in the turn loses when he has this number of checkers and his opponent practices the winning strategy.

1 is a winning number.

2 is a winning number.

3 is a winning number.

4 is a losing number because the player in the turn has to take one checker and leave 3, take 2 checkers and leave 2, or take 3 checkers and leaves 1, a winning number to his opponent.

5 is a winning number.

6 is a winning number because the player in the turn can take 2 checkers and leave 4, a losing number to his opponent.

7 is a winning number because the player in the turn can take 3 checkers and leave 4, a losing number to his opponent.

8 is a winning number.

9 is a winning number because the player in the turn can take 5 checkers and leave 4, a losing number to his opponent.

10 is a losing number because the player in the turn has to take one checker and leaves 9, take 2 checkers and leaves 8, take 3 checkers and leave 7, take 5 checkers and leave 5, or take 8 checkers and leave 2, a winning number to his opponent.

11 is a winning number because the player in the turn can take one checker and leave 10, a losing number to his opponent.

12 is a winning number because the player in the turn can take 2 checkers and leave 10, a losing number to his opponent.

13 is a winning number.

14 is a losing number because the player in the turn cannot take 4 checkers and leave 10, or take 10 checkers and leave 4, a losing number to his opponent.

$15-1=14 \quad \to \quad 15$ is a winning number.

$16-2=14 \quad \to \quad 16$ is a winning number.

$17-3=14 \quad \to \quad 17$ is a winning number.

$18-8=10 \quad \to \quad 18$ is a winning number.

$19-5=14 \quad \to \quad 19$ is a winning number.

Note that $20-6=14$, $20-10=10$, and $20-16=4$. None of 6, 10, and 14 is a Fibonacci number.

So 20 is a losing number.

21 is a winning number.

$22-2=20 \quad \to \quad 22$ is a winning number.

$23-3=20 \quad \to \quad 23$ is a winning number.

$24-8=10 \quad \to \quad 24$ is a winning number.

$25-5=20 \quad \to \quad 25$ is a winning number.

Note that $26-6=20$, $26-12=14$, $26-16=10$, and $26-22=4$. None of 6, 12, 16, and 22 is a Fibonacci number.

So 26 is a losing number.

$27 - 1 = 26 \quad \rightarrow \quad 27$ is a winning number .

$28 - 2 = 26 \quad \rightarrow \quad 28$ is a winning number .

$29 - 3 = 26 \quad \rightarrow \quad 29$ is a winning number .

Note that $30 - 4 = 26$, $30 - 10 = 20$, $30 - 16 = 14$, $30 - 20 = 10$, and $30 - 26 = 4$. None of 4, 10, 16, 20, and 26 is a Fibonacci number.

So 30 is a losing number.

$31 - 1 = 30 \quad \rightarrow \quad 31$ is a winning number .

$32 - 2 = 30 \quad \rightarrow \quad 32$ is a winning number .

$33 - 3 = 30 \quad \rightarrow \quad 33$ is a winning number .

34 is a winning number.

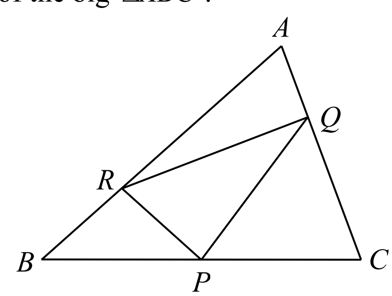
We have the losing numbers up to 34:

$$4,\ 10,\ 14,\ 20,\ 24,\ 30,\ \cdots$$

The largest number below 35 is 30.

The answer is 30.

9. On triangle ABC (which could be scalene), P is the midpoint of BC, $CQ = 2QA$, $AR = 2RB$. If the areas of the four triangles formed inside ΔABC are four consecutive integers, what is the area of the big ΔABC?

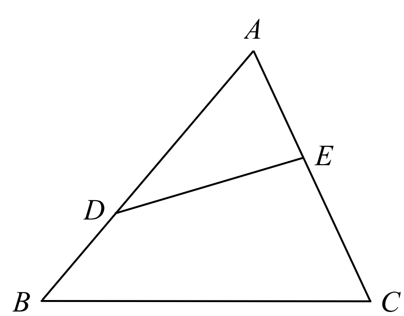


Answer: 18

Solution:

We use $[XYZ]$ to represent the area of ΔXYZ.

Lemma:

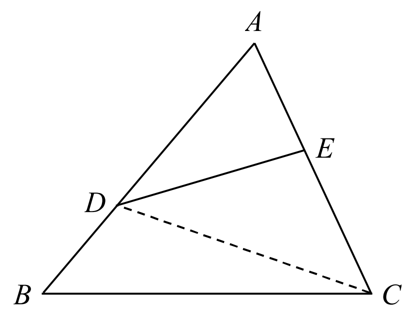


D and E are on sides of AB and AC, respectively, of ΔABC. Then

$$\frac{[ADE]}{[ABC]} = \frac{AD \cdot AE}{AB \cdot AC}.$$

Proof 1 of the Lemma:

Draw CD.



Look at ΔADC. DE is a Cevian. We have $\frac{[ADE]}{[ADC]} == \frac{AE}{AC}$.

Look at ΔACB. CD is a Cevian. We have $\frac{[ADC]}{[ABC]} = \frac{AD}{AB}$.

Multiplying two equations, we obtain

$$\frac{[ADE]}{[ABC]} = \frac{AD \cdot AE}{AB \cdot AC}.$$

Proof 2 of the Lemma:

$$\frac{[ADE]}{[ABC]} = \frac{\frac{1}{2} AD \cdot AE \cdot \sin \angle A}{\frac{1}{2} AB \cdot AC \cdot \sin \angle A} = \frac{AD \cdot AE}{AB \cdot AC}.$$

By the lemma,

$$\frac{[AQR]}{[ABC]} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}, \quad \frac{[BRP]}{[ABC]} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}, \text{ and } \frac{[CPQ]}{[ABC]} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.$$

Then

$$\frac{[PQR]}{[ABC]} = 1 - \left(\frac{2}{9} + \frac{1}{6} + \frac{1}{3} \right) = \frac{5}{18}.$$

We have

$$[BRP] = \frac{3}{18}[ABC], \ [AQR] = \frac{4}{18}[ABC], \ [PQR] = \frac{5}{18}[ABC], \text{ and } [CPQ] = \frac{6}{18}[ABC].$$

For all to be integers, we need $[ABC]$ to be a multiple of 18.

The smallest value of $[ABC]$ is 18.

And the four triangles have areas of consecutive integers:

$$[BRP]=3,\ [AQR]=4,\ [PQR]=5, \text{ and } [CPQ]=6.$$

The answer is 18.

10. If you write the sum $\sum_{n=1}^{50}\frac{n}{n^4+n^2+1}$ as a fraction in lowest terms, what is the numerator?

Answer: 1275

Solution:

The general term is

$$\frac{n}{n^4+n^2+1}=\frac{n}{n^4+2n^2+1-n^2}=\frac{n}{\left(n^2+1\right)^2-n^2}=\frac{n}{\left(n^2-n+1\right)\left(n^2+n+1\right)}$$

$$=\frac{1}{2}\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)=\frac{1}{2}\left(\frac{1}{n^2-n+1}-\frac{1}{(n+1)^2-(n+1)+1}\right)$$

So

$$\frac{1}{1^4+1^2+1}+\frac{2}{2^4+2^2+1}+\frac{3}{3^4+3^2+1}+\cdots+\frac{50}{50^4+50^2+1}$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1^2-1+1}-\frac{1}{2^2-2+1}+\frac{1}{2^2-2+1}-\frac{1}{3^2-3+1}+\cdots+\frac{1}{50^2-50+1}-\frac{1}{51^2-51+1}\right)$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1^2-1+1}-\frac{1}{51^2-51+1}\right)=\frac{1}{2}\cdot\left(1-\frac{1}{2551}\right)=\frac{1275}{2551}$$

The numerator is 1275.

11. Essay question: Explain your solution to question 2 about the White Knight's wall.

Solution:

It is explained in the solution.

12. Essay question: Prove that your answer to question 8 about "Taking Away Fibonacci Checkers" is correct.

Solution:

It is proved in the solution.

This is a cool nim-problem where there is no pattern (at least I did not find it yet): describing the losing numbers with a simple formula or a simple sentence.

I wrote a code to calculate all losing numbers.

I found the first 100 losing numbers:

2 10
3 14
4 20
5 24
6 30
7 36
8 40
9 46
10 50
11 56
12 60
13 66
14 72
15 76
16 82
17 86
18 92
19 96
20 102
21 108
22 112
23 118
24 122
25 128
26 132
27 138
28 150
29 160
30 169
31 176
32 186
33 192
34 196
35 202
36 206
37 212
38 218
39 222
40 228

41 232
42 238
43 242
44 248
45 254
46 260
47 264
48 270
49 274
50 280
51 284
52 290
53 296
54 300
55 306
56 310
57 316
58 322
59 326
60 332
61 338
62 342
63 348
64 352
65 358
66 364
67 368
68 374
69 378
70 384
71 388
72 394
73 400
74 406
75 410
76 416
77 420
78 426
79 430

80 436

81 442

82 446

83 452

84 456

85 462

86 468

87 472

88 478

89 484

90 488

91 494

92 498

93 504

94 510

95 514

96 520

97 524

98 530

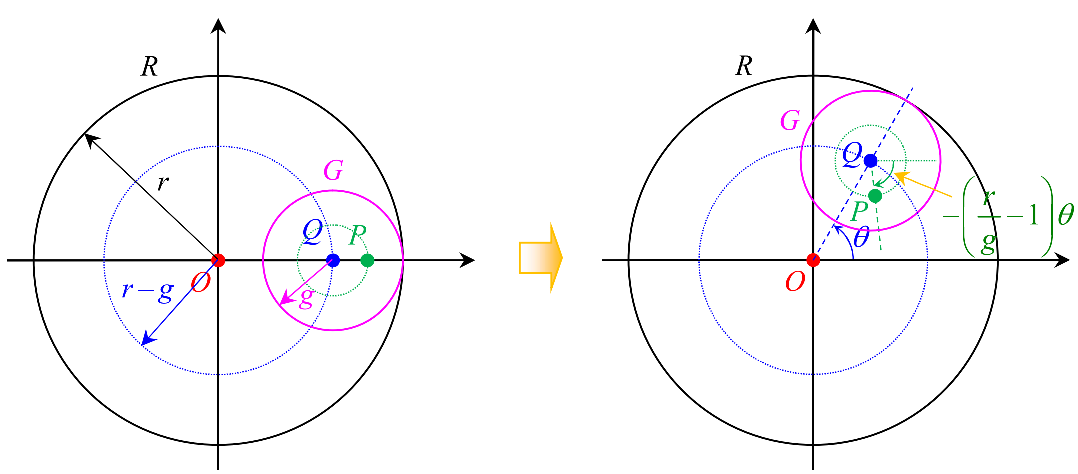
99 534

100 540

13. Essay question: Consider the general situation for Question 7, of a circular gear of radius g with a pen hole distance p from the center traveling around the inside of a ring of radius r. What can we say about the symmetries of the path traced out by the pen? Make the most general conjectures you can about this situation and prove them if possible.

Solution:

Let O be the center of the larger circle and Q be the center of the smaller circle.



The black circle is the ring and the pink circle is the gear.

Observation 1: Q is rotating around O and P is rotating around Q.

Let the blue circle be the locus of Q around O and the green circle be the locus of P around Q.

Let Q rotate around O counterclockwise. Then P rotates around Q clockwise.

Observation 2: let Q make one revolution around O. The number of revolutions of the gear is the ratio of the circumference (radius) of the blue circle to the circumference (radius) of the pink circle.

The radius of the blue circle is $r-g$, and the radius of the pink circle is g. The ratio is $\frac{r-g}{g}=\frac{r}{g}-1$.

That is, the gear makes one revolution if Q makes $\frac{r}{g}-1$ revolutions.

Observation 3: for O, Q, and P to be in a line, the difference of two rotation angles must be a multiple of $180°$.

Let θ be the rotation angle of Q around O.

If Q rotates around O by an angle θ, then P rotates around Q by an angle $-\left(\frac{r}{g}-1\right)\theta$.

We need $\theta-\left[-\left(\frac{r}{g}-1\right)\theta\right]=\frac{r}{g}\theta=180°k$ where k is a positive integer.

θ changes from $0°$ to $360°$. Then $\frac{r}{g}\theta$ changes from $0°$ to $360°\frac{r}{g}$.

The number of times that O, Q, and P are in a line is

$$\left\lceil\frac{360° r/g}{180°}\right\rceil=\left\lceil\frac{2r}{g}\right\rceil.$$

where $\lceil x\rceil$ represents the least integer greater than or equal to x.

The distance p doesn't matter if $p>0$.

We have the following conclusions:

1. If Q makes one revolution, O, Q, and P are in a line $\left\lceil\frac{2r}{g}\right\rceil$ times.
2. If $\frac{r}{g}$ is rational, we may assume r and g are relatively prime integers. Then after g revolutions by Q around O, the pen comes back to the original position to repeat the path.
3. If $\frac{r}{g}$ is irrational, the pen never comes back to the original position.