Global Talent Lab

EAMO - 2023

eastern africa universities mathematics programme

The First East African Mathematical Olympiad

Date: April 20, 2023 **Time limit: 3.5 hours**

PROBLEMS and SOLUTIONS

1. Two cylindrical candles have the same length but are made of different material. The burning speed is therefore different. The first one burn in 4 hours, the second one in 5 hours. Candles are ignited at the same time. At what time one should ignite them so the length of one of candles is twice the length of the second one at $21:00$ ($09:00$ PM) ?

   Solution: We can assume that both candles have the length 1 unit when they are ignited. Suppose that the candle which burn faster has the length a at $21:00$. Then the length of the second candle is $2a$.

   The length of the first candle decreased by $1-a$. Since the speed of burning is $\frac{1}{4}$ unit per hour, the time passed must be equal $\dfrac{1-a}{\frac{1}{4}} = 4(1-a)$ hours.

   In the similar way we have that for the second candle the time passed equals $\dfrac{1-2a}{\frac{1}{5}} = 5(1-a)$ hours.

   Hence, $4(1-a) = 5(1-2a)$. Solving this equation gives $a = \frac{1}{6}$. The candles have then burned in $4(1-\dfrac{1}{6}) = \dfrac{10}{3}$ hours, i.e. 3 hours and 20 minutes. Then must then be ignited at $17:40$.

2. A palindrom-number is a positive integer which is the same when reading from right to left or from left to right, for example the number 4704074. How many five-digits palindrom-numbers are divisible by 37? (The first dight in a five digits number is not 0.)

   Solution: A five-digits palindrom-number A can be expressed as $A = 10000a + 1000b + 100c + 10b + a = 10001a + 1010b + 100c = 37(270a + 27b + 3c) + 11(a + b - c)$. The number A is then divisible by 37 if and only if $a + b - c$ is divisible by 37.

   Since $0 \leq a, b, c \leq 9$ and $a \neq 0$, then $-8 \leq a + b - c \leq 18$. Hence the number $a + b - c$ is divisible by 37 if and only if $a + b - c = 0$. Thus $c = a + b$ and $a + b \leq 9$.

   For $a = 1$ there are nine choices for b. For $a = 2$ there are eight choices for b, and so on. For $a = 9$ there is only one choice for b, namely $b = 0$.

   It follows that in total there are $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$ five-digits palindrom-numbers.

3. Let Γ be a circle with the center at point O and let AB be a cord in Γ not passing through O. Let now C be a point different from A and B, lying on the circumcircle of $\triangle ABO$. Let finally D be the point of intersection (different from A) between the line AC and Γ. Show that the $\triangle BCD$ is isosceles.

   Solution: One must consider two cases: case (1): the points C and O are on the same side of the line AB (figure 1 below), and case (2): the points C and O are on different sides of the line AB (figure 2a and 2b below).

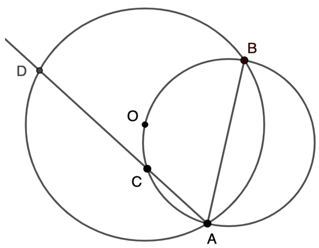

fig 1

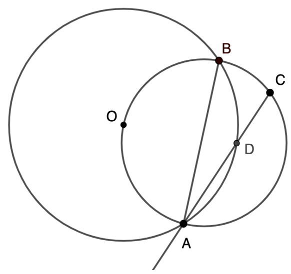

fig 2a

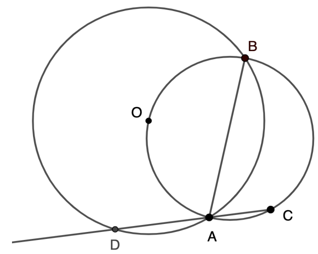

fig 2b

   In each case we use the following facts:

   (a) in a triangle an external angle equals the sum of not adjacent internal angles,

   (b) inscribed angle is half the side of the corresponding mid-point angle,

   (c) two angles inscribed on the same arc are equal, and

   (d) the sum of opposite angles in an inscribed quadrilateral is $180°$.

   (1) : We have $\angle AOB = by\,(c) = \angle ACB = by\,(a) = \angle CDB + \angle DBC$. Since $\angle CDB = by\,(b) = \frac{1}{2}\angle AOB$ then also $\angle DBC = \frac{1}{2}\angle AOB$.

   From this we conclude that $\angle CDB = \angle CBD$, which implies $|DC| = |BC|$.

   (2a) : In this case we find that $\angle BDA = by\,(b) = \pi - \frac{1}{2}\angle AOB$ but also $\angle BDA = by\,(a) = \angle DBC + \angle BCD$. Since $\angle BCD = by\,(d) = \pi - \angle AOB$, then $\angle DBC = \frac{1}{2}\angle AOB$.

   We also have $\angle CDB = \pi - \angle BDA = \frac{1}{2}\angle AOB$. The obtained equality $\angle DBC = \frac{1}{2}\angle AOB = \angle CDB$ implies $|DC| = |BC|$.

   (2b) : Here we have $\angle CDB = \angle ADB = (b) = \frac{1}{2}\angle AOB$ and $\angle BCD = \angle BCA = by\,(d) = \pi - \angle AOB$.

   From this we conclude that $\angle CBD = \pi - \angle BDA - \angle BCD = \pi - \frac{1}{2}\angle AOB - \pi + \angle AOB = \frac{1}{2}\angle AOB$.

   Hence $\angle CDB = \angle CBD$ and thus $|DC| = |BC|$.

4. An Airbus A320 airplane has 56 rows with 6 sits in each row. What is the largest number of passengers this plane can take if no two rows have the same sits occupied?

   Solution: In each row the passangers can be distributed in $2^6 = 64$ ways (empty row included). Of those 64 different distributions one can form 32 full rows (the distributions can be "paired" in such a way the they complement each other. If one row for example has occupied sits 1, 2, 3 and 6 we pair it with the row where only sits 4 and 5 are occupied). That could give us $32 \cdot 6 = 192$ passengers. However we only have 56 rows so we can remove the rows with least number of sits occupied. i.e. one empty row, six rows with one passenger and one row with two passengers. This gives us $192 - 8 = 184$ passengers.

5. Solve the system of equations for $x, y, z, t \in \mathbb{R}_{\geq 0}$ (non-negative real numbers):

$$\begin{cases} x = \dfrac{\sqrt[3]{yzt}}{y+z+t} \\ y = \dfrac{\sqrt[3]{xzt}}{x+z+t} \\ z = \dfrac{\sqrt[3]{xyt}}{x+y+t} \\ t = \dfrac{\sqrt[3]{xyz}}{x+y+z} \end{cases}.$$

Solution: Note first that none of x, y, z, t can be equal 0: if any variable is 0 then the remaining three are also equal 0 and we get 0 in the denominators.

The $AM - GM$ inequality implies that $\dfrac{y+z+t}{3} \geq \sqrt[3]{yzt}$, and therfore $x = \dfrac{\sqrt[3]{yzt}}{y+z+t} \leq \dfrac{1}{3}$. In similar way we have the inequalities $y \leq \dfrac{1}{3}$, $z \leq \dfrac{1}{3}$ and $t \leq \dfrac{1}{3}$

After we multiply the equations sidewise we get

$$xyzt = \frac{xyzt}{(y+z+t)(x+z+t)(x+y+t)(x+y+z)},$$

i.e. $(y+z+t)(x+z+t)(x+y+t)(x+y+z) = 1$.

Since each bracket is at most equal 1 then they all must be 1, hence $x = y = z = t = \dfrac{1}{3}$.