NMC 2014 A-LEVEL PAPER 1 QUESTIONS

SECTION A

Qn1. Let $2007! + 2008! + 2009! = (a!)(b^2)$, where a and b are positive integers. Without using a calculator, find the pair (a, b).

Qn2. The function f has the following properties: $f(1) = 1$ and $f(n) = n + f(n-1)$ for all integers $n \geq 2$. Compute $f(2014)$.

Qn3. Let S be a set of odd numbers $85, 99, 117$ and 121. Which of these numbers could be the discriminant D of the quadratic equation $ax^2 + bx + c = 0$, where $a > 0$, a, b, c are integers and $D = b^2 - 4ac$.

Qn4. In a mathematics class, students grades are based on tests only, all of which have equal weight. When Eva got a grade of 98 on a certain test, it raised her average by 1 point. But when she got a grade of 70 on the next test, her average then dropped by 2 points. Including these two tests, how many tests did she take?

Qn5. The sum of the $3-$digit numbers $35x$ and $4y7$ is divisible by 36. Find all possible ordered pairs (x, y).

Qn6. Jack, John and James are identical triplets. It is impossible to distinguish them by appearance. Jack and John always tell the truth, but James always lies i.e everything he says is false. You know that the triplets are between 20 and 30 years old, 20 and 30 included. One day you meet two of the triplets and ask them how old they are. A says "We are between 20 and 29 years old, 20 and 29 included". B makes the following statements: "We are between 21 and 30 years old, 21 and 30 included" and "One of us present is lying". How old are they?

Qn7. If $m^2 + n^2 = 12$ and $m^4 + n^4 = 136$, determine all possible integral values of m and n.

Qn8. The solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$. Find the relationship between p, m and n. Hence, find p if $nm = 4$ and $m = 3$.

Qn9. Let $a = -\sqrt{3} + \sqrt{5} + \sqrt{7}$, $b = \sqrt{3} - \sqrt{5} + \sqrt{7}$ and $c = \sqrt{3} + \sqrt{5} - \sqrt{7}$. Evaluate

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-a)(b-c)} + \frac{c^4}{(c-b)(c-a)}.$$

Qn10. Points A, B, C, and D lie in a plane, with A, B, and C in order in a straight line. The angles between some of these points, measured in degrees, satisfy $BDC = 78$, $DBC > DCB$, $ABD = 4x + y$ and $DCB = x + y$, for some numbers x and y. How many positive integer values can y take on, satisfying all these conditions?

SECTION B

Qn11. (a) What is the value of $\frac{2^{2014}+2^{2012}}{2^{2014}-2^{2012}}$.

(b) The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

Qn12. Let a, b and c be distinct positive integers that form a Geometric Progression and $b - a$ be a prime number between 0 and 10 such that

$$\log_4 a + \log_4 b + \log_4 c = 3.$$

Find the highest value of $a + b + c$.

Qn13. A bus, an hour after starting meets an accident which detains it for half an hour, after which it proceeds at $\frac{3}{4}$ of its former speed and arrives $3\frac{1}{2}$ hours late. Had the accident happened $90km$ further along the road, it would have arrived only 3 hours late. How long is the trip.

Qn14. Find all possible positive integers (a, b) that satisfy the equation

$$(a + 1) + (b + 1) + ab = 2014.$$

Qn15. One of the sides of a triangle PQR is divided into segments of 6 and 8 units by the point of tangency of the inscribed circle. If the radius of the circle is 4 units, find the length of the shortest side of the triangle.

NMC 2014 A-LEVEL PAPER 2 QUESTIONS

SECTION A

Qn1. Joey and his five brothers are aged $3, 5, 7, 9, 11$ and 13. One afternoon two of his brothers whose ages sum to 16 went to the beach, two brothers younger than 10 went to play baseball, and Joey and the 5-year-old stayed home. How old is Joey?

Qn2. The first pair of odd numbers is $(1, 3)$, the second pair is $(5, 7)$, the third pair is $(9, 11)$ and so on. What is the sum of the 252^{nd} pair of odd numbers?

Qn3. If $x = 2 + \sqrt{3}$, find the integer or fraction equal to $x^4 + \frac{1}{x^4}$.

Qn4. Scovia wrote down the numbers $26, 27, \cdots, 209$. How many digits did she write down?

Qn5. Let $f(x) + 1 = x^2 + 2x + 1$. List all values of x for which

$$f(f(f(f(f(x))))) = 16^{16}.$$

Qn6. How many 6 digit numbers end in 2014?

Qn7. In how many ways can five students A, B, C, D and E line up in one row if students B, C and D must stay together?

Qn8. Find the smallest possible integers a, b and c such that $abc = 7(a + b + c)$.

Qn9. Suppose a is chosen randomly from the set $\{1, 2, 3, 4, 5\}$ and b is chosen randomly from the set $\{6, 7, 8\}$, what is the probability that a^b is an even number?

Qn10. During the lunch break from $1:00$ *pm* to $2:00$ *pm*, Mr. Kakande eats, then marks scripts for his students, then goes to the restroom, then talks to a friend. Each activity after the first takes half as much time as the preceding activity. There are no intervening time intervals. At what time did Mr Kakande finish marking his students' scripts?

SECTION B

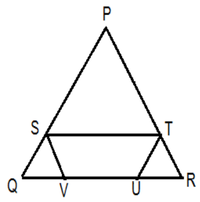
Qn11. A is a $2-$digit number and B is a $3-$digit number such that A increased by $B\%$ equals B reduced by $A\%$. Find all possible pairs (A, B).

Qn12. What is the number of triples (x, y, z) such that when any of these numbers is added to the product of the other two, the result is 2 ?

Qn13. A palindrome is a positive integer that is the same when read forwards or backwards. For example, 151 is a palindrome. What is the largest palindrome less than 200 that is the sum of three consecutive integers?

Qn14. When the tens digit of a three-digit number abc is deleted, a two-digit number ac is formed. How many numbers abc are there such that $abc = 9 \times ac + 4 \times c$? For example, $245 = 9 \times 25 + 4 \times 5$.

Qn15. In the diagram below, $\triangle PQR$ has $PQ = QR = RP = 30$. Points S and T are on PQ and PR, respectively, so that ST is parallel to QR. Points V and U are on QR so that TU is parallel to PQ and SV is parallel to PR. If $VS + ST + TU = 35$, what is the length of VU?



NMC 2013 A-LEVEL PAPER 1 QUESTIONS

SECTION A : 5 marks each

1. Function $f(x)$ obeys the rule $f(x+1) = f(x) - f(x-1)$. Given $f(2) = 5$ and $f(1) = 2$, find $f(2009)$.

2. Find the value of $\sqrt{7-\sqrt{48}} + \sqrt{5-\sqrt{24}} + \sqrt{3-\sqrt{8}}$.

3. Evaluate $1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \ldots - 2012! \cdot 2014 + 2013!$.

4. The solutions to the equation $x^2 + px + q = 0$ are cubes of the solutions of the equation $x^2 + mx + n = 0$. Write down an equation relating m, n and p.

5. An ice-cream van sells nine different flavours of ice-cream. A large group of children gathered by the van and each of them bought a double-scoop con with two different flavours of ice-cream. If none of the children chose the same two flavours, What is the largest possible number of children in the group?

6. Jonathan never lies on Monday, Tuesday, Wednesday and Thursday. David always tells the truth on Monday, Friday, Saturday and Sunday. On the rest of the days, they may tell the truth or they may lie. Both say they lied yesterday. What day is today?

7. Each square in the partially filled grid shown contains a number from 1 to 25, each number appearing exactly once. The numbers in each row and each column add up to 65. What is the value of $x + y$?

	7	25	18	
	21	19	y	
22	20			9
16			10	23
	x	6	24	

8. Given the system of equations.

$$\begin{aligned} x + y &= 2m^2 \\ y + z &= 6m \\ x + z &= 2 \end{aligned}$$

where m is a real number. For what values of m is $x \le y \le z$?

9. Jane and Sarah run at different but constant speeds. They ran two races on a track that measured $100m$ from start to finish. In the first race, when Jane crossed the finish line, Sarah was $5m$ behind. In the second race, Jane started $5m$ behind the original start line and they ran at the same speeds as in the first race. What was the outcome of the second race?

10. Find all functions $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $xy\left(3f(\frac{x}{y}) + 5f(\frac{y}{x})\right) = 3x^2 + 5y^2$. where $\mathbb{R}^+$ consists of positive real numbers and $\mathbb{R}$ consists of all real numbers.

SECTION B : 10 marks each

11. Twelve different balls are to be packed into 4 different boxes so that each box contains 3 balls. Find the number of ways of packing the balls in this manner.

12. Each of 52 cards has an integer written on it (repetitions allowed). The numbers on any three consecutive cards add up to 20. If the number on the first card is 2 and the number on the ninth card is 8. What is the number on the 52^{nd} card?

13. Let a, b and c be distinct positive real numbers that form a geometric sequence. Let $b - a$ be a square of a natural number n and

$$\log_6 a + \log_6 b + \log_6 c = 6$$

Find the value of $a + b + c$.

14. Let P, Q and R be real numbers such that

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = \frac{Pn + Q}{2^n} + R$$

holds for every positive integer n. What is the value of $P + Q + R$?

15. In the figure shown, $< ABC$ and $< BDA$ are both right angles. If $v + w = 35$ and $x + y = 37$. What is the value of $y + 8$?

NMC 2013 A-LEVEL PAPER 2 QUESTIONS

SECTION A : 5 marks each

1. The average of 16 **different** positive integers is 16. What is the second greatest possible value that any of these integers could have?

2. A security guard works for four days consecutivaly, then has the next day off, works four more days, has a day off , and so on. Today is a sunday and a day off. On how many days does the guard work before next having a day off on a Sunday?

3. If a is non zero and the three roots of $ax^3 + bx^2 + (a-4)x + 2 = 0$ are equal integers, with r the common value of these integers. Determine the value of r.

4. One Half the books on a teacher's bookshelf are mathematics books, a third of them are physics books, and $\frac{1}{15}$th of them are history books. The remainder of the books are romance novels. If 2 of the mathematics books, and 4 of the physics books are replaced by romance novels, then romance novels will comprise 15% of the books on the bookshelf. How many books, total, are there on the bookshelf?

5. If the equation $x^4 - ax^3 + 2x^2 - bx + 1 = 0$ has a real root. Show that $a^2 + b^2 \geq 8$.

6. Let x and y be real numbers. Find the maximum value of $x - y$ if $2(x^2 + y^2) = x + y$.

7. The real numbers x and y satisfy the equations $xy = sin(2x)$ and $\frac{x}{y} = tan(t)$ where $0 < t < \frac{\pi}{2}$.

   What is the value of $x^2 + y^2$?

8. Show that if $|x| < 1$, $|y| < 1$, then $|x - y| < 1 - xy$.

9. The five-digit number $a679b$ where a and b are digits, is divisible by 36. Find all possible such five-digit numbers.

10. On Monday, Ruth tells the truth for the whole day, otherwise he lies for the whole day. Today is Monday and Ruth has made exactly FOUR of the following statements. Which statement could he have not made today?
    A. I have a prime number of friends. B. I have as many male friends as female friends
    C. My name is Charles. D. I always speak the truth. E. Three of my friends are older than me. SA PAGE 50

SECTION B : 10 marks each

11. Immaculate is fascinated by triangle numbers $(1, 3, 6, 10, 15, 21, \text{etc})$ and recently coming across a clock, he found that he could rearrange the twelve numbers $1, 2, 3, \cdots, 12$ around the face so that each digit pair added up to a triangle number. He left 12 in its usual place. What number did he put where the 6 would usually be?

12. An "unfortunate" number is a positive integer which is equal to 13 times the sum of its digits. Find all "unfortunate" numbers less than 1000?

13. The letters S, M and C represent whole numbers. If $S \times M \times C = 240$, $S \times C + M = 46$, $S + M \times C = 64$. What is the value of $S + M + C$?

14. If the square is completed with the letters A, B, C, D and E so that no row, column or either of the two main diagonal lines contains the same letter more than once,which letter should replace the asterisk?

	*			A
		B		
D		C		
			E	

15. In the trapezium shown(not to scale), XY is parallel to two sides and passes through the point of intersection of the diagonals. What is the length XY?

NMC 2012 **A-LEVEL PAPER** 1 **QUESTIONS**

SECTION A

Qn1. Without using a calculator, evaluate,

$$S = \sin^2(10^0) + \sin^2(20^0) + \ldots + \sin^2(80^0).$$

Qn2. Given that $|m| - m + n = 10$ and $m + |n| + n = 12$, find the value of $m + n$.

Qn3. Determine the largest possible area of a rectangle with a fixed perimeter of 1000.

Qn4. In a community, two of the three people always lie while the other always tells the truth. A woman asks questions to three men from this community independently . She ask each man; which of you is tallest? John replies, not me. David replies, I am. Geoffrey replies, Not David. Who is tallest?

Qn5. Two particles move clockwise around a circle of circumference 300 cm. The faster particle moves at a constant speed of R cm per second and the slower particle moves at a constant speed of r cm per second. If the particles meet every after 50 seconds, find the difference between R and r.

Qn6. For how many integers x in the set $\{1, 2, 3, \ldots, 99, 100\}$ is $x^3 - x^2$ a square of an integer?

Qn7. Suppose that $f(n) = \log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{n-1} n$, find the value of $\sum_{k=2}^{500} f(2^k)$.

Qn8. A circle passes through two adjacent vertices of a square and is tangent to one side of the square. If the length of the side of the square is $2cm$, find the radius of the circle.

Qn9. A man caught some fish. The two heaviest fish had a combined weight which was 25% of the total weight of all the fish. The five lightest fish had a combined weight which was 45% of the total weight of all fish. He put the two heaviest fish in the freezer and ate the five lightest fish for lunch. His cat took all the remaining fish. How many fish did the cat eat?

Qn10. On Wednesday, four trucks drove in a line (one directly behind the other). None of the trucks bypassed any of the others, so the order of the trucks never changed. How many ways are there to rearrange the order of the trucks so that on the next day, no truck is directly behind a truck that it was directly behind the day before.

SECTION B

Qn11. The equation $x^2 - y^2 = 1,000,001$ has only two solutions where x and y are both in the solution set $S = \{1, 2, 3, 4, \ldots, \}$. Find the smallest value of x in the solution set.

Qn12. A circle is inscribed in a triangle whose sides are 10 cm, 10 cm and 12 cm. A second smaller circle is inscribed tangent to the first circle but also tangent to the two equal sides of the triangle. Find the radius of the second smaller circle.

Qn13. Bill starts walking along a 6km road from the beginning. Sometime later, his brother Phil rides his bike along the same road in the same direction starting at the same point as Bill. Phil can ride three times as fast as Bill can walk. When he passes Bill, Phil on the bike continues to the other end of the road and Bill turns around and heads back towards the starting point. When Phil get to the other end of the road, he turns his bike around and goes back along the same road, arriving at the same time as does Bill. How far from the end of the road did Phil pass Bill?

Qn14. In the Magic square below, the numbers in each row; the numbers in each column and the numbers on each diagonal have the same sum. Given the magic square shown with a, b, c, x, y, z all greater than zero, determine the product xyz in terms of a, b and c.

$\log a$	$\log b$	$\log x$
p	$\log y$	$\log c$
$\log z$	q	r

Qn15. The figure shows a graph of the function of x which models the height of a ramp. The function meets the following requirements:

(i) At $x = 0$, the value of the function is 0 and the slope of the graph of the function is also 0.

(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.

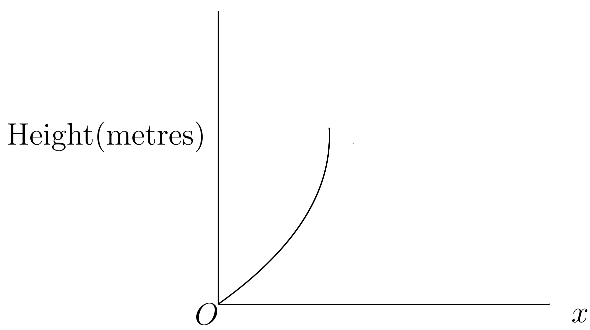
(iii) Between $x = 0$ and $x = 4$ the function is increasing.

(a) Let $f(x) = ax^2$ where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a non zero constant. Find the value of c so that g meets requirement (ii) above.

(c) Show that with the value of c obtained in (b) above, the function g does not satisfy requirement (iii) above.

(d) Let $h(x) = \frac{x^n}{k}$ where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.



NMC 2012 A-LEVEL PAPER 2 QUESTIONS

SECTION A : 5 marks each

Qn1 Suppose a and b are positive integers such that $(a+2b)(a-b)=10$. Find the possible value of $2a-b$.

Qn2 Without using a calculator, evaluate $\frac{1+2011^4+2012^4}{1+2011^2+2012^2}$

Qn3 A die with numbers 1, 2, 3, 4, 6, 8 on its six faces is rolled. After the roll, if an odd number appears on the top face, all odd numbers are doubled. If an even number appears on the top face, all even numbers are halved. What is the probability that a 2 will appear on the second roll?

Qn4 Let $p(x)$ be a polynomial of degree four such that $p(2)=p(-2)=p(-3)=-1$ and $p(1)=p(-1)=1$. What is the value of $p(0)$?

Qn5 If 1, x, y is a geometric sequence and x, y, 3 is an arithmetic sequence. Compute the highest value of $x+y$.

Qn6 A car uses 8.4litres of diesel for every 100km it is driven. A mechanic is able to modify the car's engine at a cost of \$400 so that it will only use 6.3litres of diesel per 100km. The owner determines the minimum distance that they would have to drive to recover the cost of the modifications. If diesel costs \$0.80 per litre, find this minimum distance, in km.

Qn7 In a triange ABC, we have that $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$ What is the value of angle C?

Qn8 A circle of radius 4 units is inscribed in an equilateral triangle. Given that the area inside the triangle and outside the circle is $\sqrt{3}x - \pi y$, find the value of $x+y$.

Qn9 If the roots of $x^2 - bx + c = 0$ are $\sin(\frac{\pi}{7})$ and $\cos(\frac{\pi}{7})$. What is the value of b^2?

Qn10 A written test has 30 multiple choice questions. A student will get 5 marks for each correct answer, 1 mark for each un answered question and 0 points for each wrong answer. Suppose that at the of the test, 4 students make the following statements:

Chris says "My test score is 147", Blair says "My test score is 143", Drew says "My test score is 141" and Alex says "My test score is 139"

Only one of the students could possibly be correct. Who is this?

SECTION B

Qn11 There are 6 litres of pure alcohol in container A and 6 litres of pure water in container B. An empty bottle is filled with alcohol from container A and then emptied into container B. After shaking, the bottled is filled with the mixture from container B and emptied into container A. The ratio of alcohol to water in container A is now 4 : 1. Assuming that there were no spills, what is the size of the bottle, in litres?

Qn12 A system of 5 equations in the unknowns x, y, z, u, v has a unique solution which is a permutation of numbers 1, 2, 3, 4, 5. Five participants in a Mathematics contest were asked to solve the system, but they were only able to provide the following partial answers before the end of the alloted time:

Jane: $z = 1, u = 2$; Bob: $y = 3, v = 2$; Cathy: $z = 5, u = 3$; David: $x = 4, v = 2$; Eva: $x = 4, y = 1$.

From these partial answers each of the participants had found the correct value of only one unknown. What is the value of $x + z + v$?

Qn13 Using the iterated formula that $a^{b^c} = a^{(b^c)}$; and given that x is a real number which satisfies the equation $2^{2^x} + 4^{2^x} = 42$, find the value of $\sqrt{2^{2^{2^x}}}$.

Qn14 Determine all integers x and y that satisfy $(x + 2)^4 - x^4 = y^3$.

Qn15 What is the value of x in the plane figure below?

NMC 2011 A-LEVEL PAPER 1 QUESTIONS

SECTION A : 5 marks each

1. Let a and b be positive numbers not equal to 1. What is the value of $(\log_a b^2)/(\log_{a^2} b)$?
2. The sum of the first five terms of an arithmetic sequence is 40 and the sum of the first ten terms of the sequence is 155. What is the sum of the first 15 terms of the sequence?
3. How many real solutions does the equation $\left(\frac{2x^2-5}{3}\right)^{x^2-3x} = 1$ have?
4. Given that $\frac{a}{b+c+d} = \frac{4}{3}$ and $\frac{a}{b+c} = \frac{3}{5}$, find $\frac{d}{a}$?
5. Suppose that x and y are two real numbers such that $x - y = 2$ and $x^2 + y^2 = 8$, find $x^3 - y^3$.
6. Suppose John is given 1 penny on the first day, 2 pennies on the second day, 4 pennies on the third day, 8 pennies on the fourth day and so on for 18 days while David is given $\$1,000$ per day for 18 days. If $\$1 = 10$ pennies, who will have the larger amount of money at the end of 18 days. Give your answer in dollars ($).
7. When Mum came home and found the cookie jar broken, she called all her four children for an explanation and the following discussion took place:
   Ann: I did not break the jar.
   Bob: I did not break the jar and Ann did not break the jar.
   Cal: I did not break the jar and Bob did not break the jar.
   Deb: Ann did not break the jar and Bob did not break the jar.
   Mum later found out that only one of the children lied. Which one of the children broke the cookie jar?
8. Let K be the set consisting of the first 11 positive integers. A subset of three distinct positive integers is said to be *acceptable* if it contains at most one odd element. If a subset of three different numbers is chosen at random from K, what is the probability that the subset will be *acceptable*?
9. Suppose that $a > 0$. If $x^{a/2} - 4 + 4x^{-a/2} = 0$, find the reciprocal of $\frac{1}{2} + x^a$.
10. Two ladders, one of which is twice as long as the other, rest on the floor and reach the same vertical height on the wall. The shorter ladder makes an angle of $60°$ with the floor. What angle does the longer ladder make with the floor?

SECTION B : 10 marks each

11. Suppose that f is a real-valued function whose domain is the set of positive integers and that f satisfies the following two properties

$$\begin{aligned} f(1) &= 23 \text{ and} \\ f(n+1) &= 8 + 3f(n) \text{ for all } n \geq 1. \end{aligned}$$

It follows that there are constants a, b and c such that $f(n) = a \cdot b^n - c$ for $n = 1, 2, \ldots$ Find the value of $a + b + c$.

12. (a) How many real numbers p are there so that the polynomial $x^{10} + px + 1 = 0$ has a real solution S and also has $\frac{1}{S}$ as a solution?

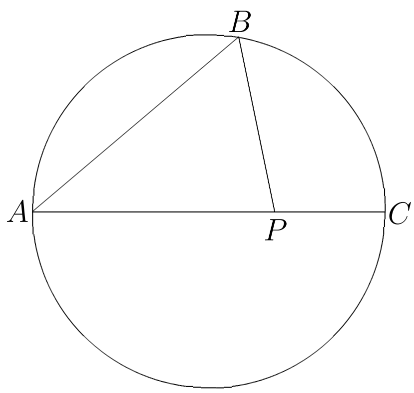
(b) Given that $T = 1997^2 - 1996^2 + 1955^2 - 1994^2 + \ldots + 3^2 - 2^2 + 1^2$, find the largest prime factor of T.

13. Given that $\cos 5x = A\cos^5 x + B\cos^4 x + C\cos^3 x + D\cos^2 x + \cos x + F$, find the sum of A, B, C, D, E and F.

14. In a magic square, each row, column, and diagonal has the same sum. All the odd numbered years from 1987 to 2003 inclusive are used to form the magic square without repeating any year. Given the locations of 1999 and 2003, what is the year that belongs in the box marked X?

	1999	
2003		
		X

15. In the diagram below, point P is chosen on diameter $\overline{AC}$ of a circle so that $\overline{PC} = 3$ cm. The area of the circle is 64π cm^2 and chord AB is 12 cm long. Compute the distance from P to B.



NMC 2011 A-LEVEL PAPER 2 QUESTIONS

NMC 2010 A-LEVEL PAPER 1 QUESTIONS

SECTION A

1. Evaluate the sum to infinity

$$\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\dots.$$

2. Two candles of the same height are lighted at the same time. The first is consumed in 4 hours and the second in 3 hours. Assuming that each candle burns at a constant rate. In how many hours after being lighted was the first candle twice the height of the second?

3. The sum of seven integers is -1. What is the maximum number of the seven integers that can be larger than 13?

4. The marked price of a coat was 40% less than the suggested retail price. Hasna purchased the coat for half the marked price at a promotion on the fiftieth anniversary sale. What percent less than the suggested retail price did Hasna pay?

5. In a group of cows and chicken, the number of legs was 14 more than twice the number of heads, find the number of cows in the group.

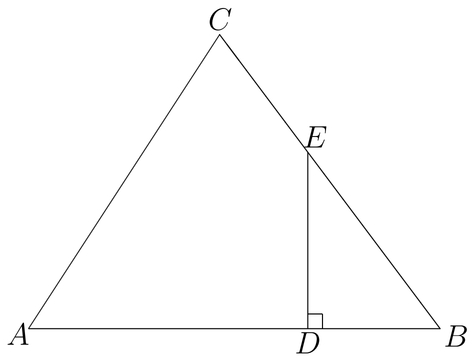
6. In solving a problem that reduces to a quadratic equation one student makes a mistake only in the constant term of the equation and obtains 8 and 2 for the roots. Another student makes a mistake only in the coefficient of the first degree term and finds -9 and -1 for the roots. write down the correct equation.

7. A total of 28 handshakes were exchanged at the conclusion of a party. Assuming that each participant was equally polite towards all the others. How many people attended the party?

8. If a worker receives a 20% decrease in wages. By what percentage should his current pay be increased in order to regain his original wage before the decrease?

9. Find the value of x if,
$$\sqrt{5x-1}+\sqrt{x-1}=2?$$

10. In the figure below, it's given that angle $C=90°$, $\overline{AD}=\overline{DB}$, $\overline{DE}\perp\overline{AB}$. $\overline{AB}=20cm$ and $\overline{AC}=12cm$. Find the area of the quadrilateral $ADEC$.



SECTION B : 10 marks each

11. (a) The sum of three numbers is 98, the ratio of the first to the second is $\frac{2}{3}$, and the ratio of the second to the third is $\frac{5}{8}$. Find the second number?

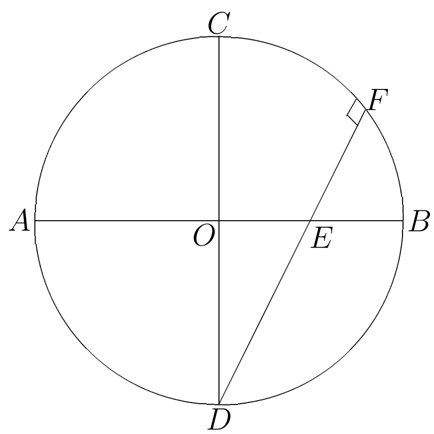
    (b) A farmer divides his herd of n cows among his four sons so that one son gets one half of the herd, a second son gets one fourth, a third gets one fifth and the fourth son gets 7 cows. Find the total number of cows and how many each son got?

12. A function f from integers to integers is defined as follows, $f(n) = n + 3$ if n is odd and $f(n) = \frac{n}{2}$ if n is even. Suppose k is odd and $f(f(f(k))) = 27$, what is the sum of the digits of k?

13. A wheel with a rubber tire has an outside diameter of 25 inches. Assuming that it's radius is reduced by a quarter of an inch. Find the percentage increase in the number of revolutions of the wheel in one mile.

14. Tom, Dick, and Harry started out on a 100 mile journey. Tom and Harry went by automobile at the rate of $25mph$ while Dick walked at the rate of $5mph$. After a certain distance, Harry got off and walked on at $5mph$ while Tom went back for Dick and got him to the destination at the same time that Harry arrived. Find the number of hours spend on the journey?

15. In the figure below, $\overline{AB}$ and $\overline{CD}$ are diameters of the circle with center O, $\overline{AB} \perp \overline{CD}$, and chord $\overline{DF}$ intersects $\overline{AB}$ at E. If $DE = 6cm$ and $EF = 2cm$ then find the area of the circle.

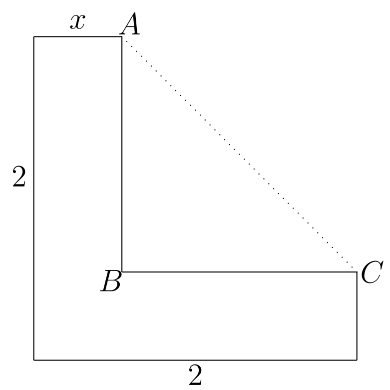


NMC 2010 A-LEVEL PAPER 2 QUESTIONS

1. If $x^2 + \frac{1}{x^2} = 3$, find the value of $\frac{(x+1)^2}{x^2}$.
2. Compute the exact value of the infinite series,

$$-\frac{4}{3} + \frac{1}{3^2} - \frac{4}{3^3} + \frac{1}{3^4} - \frac{4}{3^5} + \ldots$$

3. When 60 minutes elapse on a correct clock, 62 minutes register on clock F (fast) and only 56 minutes register on clock S (slow). If later in the day clock F reads $8:00PM$ and clock S reads $7:00PM$ what was the correct time when the two clocks were originally set?
4. If $x-2$ divides the polynomial $x^4 - 3ax^2 + (5a-2)x - 16$ without leaving a remainder, find the value of a.
5. Consider the function $f(x)$ which is such that $f(mn) = f(m-n)$ for all real numbers m and n, if $f(4) = 2$, find the value of $f(-3) + f(6)$.
6. If $x > 0$, and $\log_a b = 4$, find the value of $\frac{\log_a x}{\log_{ab} x}$?
7. How many distinct pairs of integers (x, y) satisfy the function $y = \frac{x+12}{2x-1}$?
8. Suppose that $a < b$ and that $3^2 + 4^2 + 5^2 + 12^2 = a^2 + b^2$ is satisfied by only one pair of positive numbers (a, b), What is the sum of the numbers a and b?
9. A six sided die has faces labeled 1 through 6. It is weighted so that a three is three times as likely to be rolled as a one, a three and a six are equally likely, and a one, a two, a four and a five are equally likely. What is the probability of rolling a three?
10. In the figure shown below, if the area of the letter "L" equals the area of the triangle ABC. What is the length x of the ends of the letter L?



SECTION B : 10 marks each

11. (a) If $g\left(\sqrt{\frac{1+x}{1-x}}\right) = 5x$, Determine the value of $g(2)$.

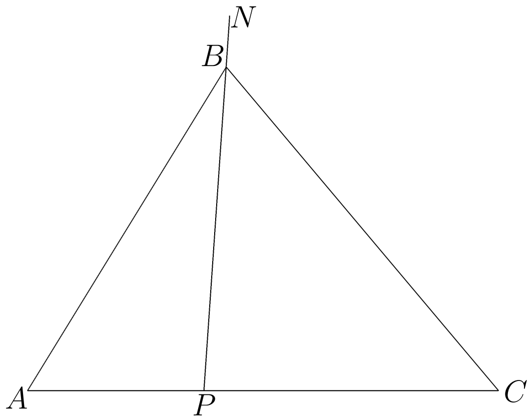
    (b) The number $\sqrt{19+\sqrt{336}}$ can be written in the form $\sqrt{a}+\sqrt{b}$ where a and b are whole numbers and $a > b$. What is the value of $a - b$?

12. A swimming team consists of eight members only, two of which are boys. The coach wants to take a delegation from the team to a special swimming camp. If the delegation must have either five or six members and must include at least one boy. How many ways are there to select the delegation?

13. Bill and Jill are hired to paint a line on a road. If Bill works by himself, he could paint the line in B hours. If Jill works by herself, she could paint the line in J hours. Bill starts painting the line from one end and one hour later Jill begins from the other end. They both work until the line is painted. Determine the number of hours that Bill works in terms of B and J.

14. According to a free fall model an object released from rest will fall $16t^2$ feet, t seconds after it's dropped. Suppose that two marbles are dropped one after the other from a cliff that is 64 feet high and that the second marble is released the instant the first marble has fallen exactly one foot. According to the free fall model, how far above the ground is the second marble at the instant the first strikes the ground.

15. The triangle ABC has sides $BC = 13, AC = 14$ and $AB = 15$ as shown below. Line N bisects angle B and crosses side AC at P. Find the distance from A to P.



NMC 2014 A-LEVEL PAPER 1 SOLUTIONS
SECTION A

1.

$$\begin{aligned} 2007! + 2008! + 2009! &= 2007! + 2008(2007!) + 2009(2008)(2007!). \\ &= 2007!(1 + 2008 + 2009(2008)). \\ &= 2007!(2009)(1 + 2008) \\ &= (2007!)(2009)^2. \end{aligned}$$

Comparing with $(a!)(b^2)$ we have $(a, b) = (2007, 2009)$

2.

$$\begin{gathered} f(n) = n + f(n-1) \\ f(2014) = 2014 + f(2013) \\ f(2013) = 2013 + f(2012) \\ f(2012) = 2012 + f(2011) \\ \vdots \\ f(2) = 2 + f(1) \\ \text{Therefore } f(2014) = 2014 + 2013 + \cdots 1 \end{gathered}$$

$$S_{2014} = \frac{n(n+1)}{2} = \frac{2014(2014+1)}{2} = \frac{2014(2015)}{2} = 2029105.$$

3. The discriminant of $ax^2 + bx + c = 0$ is $D = b^2 - 4ac$. Since all of the choices are odd numbers, and $4ac$ is even, then b^2 and hence b, must be odd. Let $b = 2n + 1$ where n is an integer. Then

$$b^2 - 4ac = (2n+1)^2 - 4ac = 4n^2 + 4n + 1 - 4ac = 4n^2 + 4n - 4ac + 1 = 4(n^2 + n - ac) + 1.$$

Therefore, the discriminant D must be one more than a multiple of 4. Of the choices, only 99 does not meet this requirement. The rest of the numbers could be a discriminant.

4. Suppose before scoring 98 and 70, the student's average grade on her first N tests was M. Then

$$\frac{NM + 98}{N + 1} = M + 1 \quad \text{and} \quad \frac{NM + 98 + 70}{N + 2} = M + 1 - 2$$

The first equation simplifies to $M + N = 97$ and the second to $2M - N = 170$. Thus solving these two equations simultaneously gives $M = 89$ and $N = 8$. Including the two tests, she took 10 tests.

5. We have $350+x+407+10y = 36j$ for some j. Since $757 = 36(21)+1$, we have $10y+x = 36k-1$ for some k. Since $0 \leq 10y + x \leq 99$, we have $10y + x = 35$ or 71. Now suppose $10y + x = 35$, then $x = 5$ and $y = 3$. Also, if $10y + x = 71$, then $x = 1$ and $y = 7$. Therefore the possible ordered pairs (x, y) are $(5, 3)$ and $(1, 7)$.

6. It follows from the fact that B says "One of us present is lying", that he must be telling the truth. Indeed, suppose that B were lying. Then his utterance "One of us present is lying" would be a false statement. Hence both present would be telling the truth, which would contradict the assumption that A is lying. Since we now know that what B is saying is true, we can deduce that A must be lying, hence that the triplets are not between 20 and 29 years old, and hence that they must be 30 years old.

7. **We have that $m^2 + n^2 = 12$ and $m^4 + n^4 = 136$.**

   From $(n^2 + m^2)^2 = m^4 + n^4 + 2n^2m^2$, we have

   $\frac{12^2}{2}$

8. Let x_1 and x_2 be the roots of $x^2 + mx + n = 0$. This implies x_1^3 and x_2^3 are the roots of $x^2 + px + q = 0$.

   Thus $x_1 + x_2 = -m$ and $x_1x_2 = n$; $\quad x_1^3 + x_2^3 = -p$ and $x_1^3x_2^3 = q$.

   From $(x_1 + x_2)^3 = (x_1^3 + x_2^3) + 3x_1x_2(x_1 + x_2) = -p + 3n(-m)$.

   Therefore $\quad m^3 = p + 3nm$.

   We thus have $p = m^3 - 3nm = 3^3 - 3(4) = 27 - 12 = 15$.

9. Let $a = -\sqrt{3} + \sqrt{5} + \sqrt{7}$, $b = \sqrt{3} - \sqrt{5} + \sqrt{7}$ and $c = \sqrt{3} + \sqrt{5} - \sqrt{7}$. Evaluate

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-a)(b-c)} + \frac{c^4}{(c-b)(c-a)}.$$

10.

$$\begin{aligned} \alpha + \beta &= 102. && (1) \\ 108\alpha &= 4x + y. && (2) \\ x + y &= \beta. && (3) \end{aligned}$$

Substituting Equations (2) and (3) into Equation (1) we get

$$\begin{aligned} 180 - 4x - y + x + y &= 102. \\ 78 - 3x &= 0. \\ x &= 26. \end{aligned}$$

Then $76y = \alpha > \beta = 26 + y$, so, $y < 25$. Therefore the values of y range from 1 to 24.

SECTION B

11. (a) Factoring 2^{2012} from each of the terms and simplifying gives

$$\frac{2^{2014}+2^{2012}}{2^{2014}-2^{2012}} = \frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)} = \frac{4+1}{4-1} = \frac{5}{3}.$$

(b) Let the three whole numbers be $a < b < c$. The set of sums of pairs of these numbers is $(a+b, a+c, b+c) = (12, 17, 19)$. Thus

$$2(a+b+c) = (a+b)+(a+c)+(b+c) = 12+17+19 = 48.$$

Dividing by 2 gives

$$a+b+c = 24.$$

If follows that $(a, b, c) = (24-19, 24-17, 24-12) = (5, 7, 12)$. Therefore the middle number is 7.

12. If a, b, c form a Geometric progression, then $\frac{b}{a} = \frac{c}{b} \Longrightarrow b^2 = ac$. From

$$\begin{aligned} \log_4 a + \log_4 b + \log_4 c &= 3. \\ \log_4(abc) &= 3 \\ abc &= 4^3. \end{aligned}$$

From $b^2 = ac$ and $abc = 4^3$, we have that $b^3 = 4^3 \Longrightarrow b = 4$. Now since $b-a$ is a prime number between 0 and 10, then

$$b - a = \{2, 3, 5, 7\}.$$

$$4 - a = \{2, 3, 5, 7\}.$$

The only integers satisfying this condition are $a = 2$ and $a = 1$.

However, if $a = 1$, then $abc = 4^3 \Longrightarrow c = 4^2 = 16$.

Thus $(a, b, c) = (1, 4, 6)$. This is a geometric progression whose sum is 21.

Now if $a = 2$, then $abc = 4^3 \Longrightarrow c = 8$.

Thus $(a, b, c) = (2, 4, 8)$. This is also a geometric progression whose sum is 14. Therefore the highest value of $a+b+c$ is 21.

13. Let X be the distance from the point of the accident to the end of the trip and R the former speed of the bus. Then the normal time for the trip in hours is $\frac{X}{R} + 1$.

Considering the time for each trip, we have

$$1 + \frac{1}{2} + \frac{X}{\frac{3}{4}R} = \frac{X}{R} + 1 + 3\frac{1}{2}$$

and

$$1 + \frac{90}{R} + \frac{1}{2} + \frac{X - 90}{\frac{3}{4}R} = \frac{X + R}{R} + 3$$

Simplifying these two equations gives

$$\frac{X}{R} = 9 \Longrightarrow X = 9R \tag{4}$$

and

$$-180 + 2X = 15R. \tag{5}$$

Solving equations (7) and (8) gives

$$\begin{aligned} -180 + 18R &= 15R. \\ -180 &= -3R. \\ R &= 60. \end{aligned}$$

Thus $X = 9R = 60 \times 9 = 540$. Therefore the trip is $540 + 60 = 600$km.

14.

$$\begin{aligned} (a+1) + (b+1) + ab &= 2014 \\ (a+1) + (b+1) + ab &= 2014. \\ a + b + ab + 2 &= 2014. \\ (a+1)(b+1) &= 2013. \\ \text{but } \ 2013 &= 3 \times 11 \times 61 \quad \text{prime factorisation.} \\ (a+1)(b+1) &= 3 \times 11 \times 61. \end{aligned}$$

The possibilities are

(a) $a + 1 = 1, b + 1 = 2013$ which means that $a = 0$. This is not possible.
(b) $a + 1 = 3$ and $b + 1 = 11 \times 61$. Hence $(a, b) = (2, 670)$.
(c) $a + 1 = 11$ and $b + 1 = 3 \times 61$. Hence $(a, b) = (10, 182)$.
(d) $a + 1 = 61$ and $b + 1 = 3 \times 11$. Hence $(a, b) = (60, 32)$.
(e) $a + 1 = 3 \times 11$ and $b + 1 = 61$. Hence $(a, b) = (32, 60)$.
(f) $a + 1 = 3 \times 61$ and $b + 1 = 11$. Hence $(a, b) = (182, 10)$.
(g) $a + 1 = 61 \times 11$ and $b + 1 = 3$. Hence $(a, b) = (670, 2)$.

Therefore the ordered pairs are $(a, b) = (2, 670), (10, 182), (60, 32), (32, 60), (182, 10), (670, 2)$.

15. Denote the sides of a triangle by a, b, c where $a = 8+6 = 14$, $b = 8+x$ and $c = x+6$. Therefore

$$\begin{aligned} 2s &= a+b+c \\ &= 14+8+x+x+6. \\ &= 2x+28. \\ s &= x+14. \end{aligned}$$

On the other hand, by Heron's Formula,

$$\begin{aligned} Area\ of\ \triangle ABC &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{(x+14)(x+14-14)(x+14-8-x)(x+14-x-6)}. \\ &= \sqrt{48x(x+14)}. \end{aligned}$$

From the fact that the area of a triangle is half the product of the perimeter and the radius of the inscribed circle, we have that

$$\begin{aligned} 4(x+14) &= \sqrt{48x(x+14)} \\ (x+14)^2 &= 3x(x+14). \\ x+14 &= 3x. \\ 2x &= 14. \\ x &= 7 \end{aligned}$$

Therefore the sides of the triangle are $c = 7+6 = 13$, $a = 6+8 = 14$ and $b = 8+7 = 15$.

Thus the shortest side is 13 units.

Alternatively
This could as well be done by construction method.

NMC 2014 A-LEVEL PAPER 2 SOLUTIONS

SECTION A

Qn1. The 5 year-old and the two brothers who went to play baseball account for three of the four brothers who are younger than 10. Because the only age pairs that sum to 16 are 3 and 13, 5 and 11 and 7 and 9, the brothers who went to the movies must be 3 and 13 years old. Hence the 7 year-old and 9 year-old brothers went to play baseball, and Joey is 11.

Qn2. The first number in the pair increase by 4 for each successive pair, thus the first number in the 252^{nd} pair is: $a+(n-1)d$ where $a=1, n=252, d=4$, that is $1+4(n-1)=1+4(252-1)=1+4(251)=1005$. The second number is two more than the first which is 1007. The sum is $1005+1007=2012$.

Qn3. Since $x=2+\sqrt{3}$, then $x+\frac{1}{x}=2+\sqrt{3}+\frac{1}{2+\sqrt{3}}=2+\sqrt{3}+\frac{1}{2+\sqrt{3}}\frac{2-\sqrt{3}}{2-\sqrt{3}}=2+\sqrt{3}+\frac{2-\sqrt{3}}{4-3}=4.$

Since $x+\frac{1}{x}=4$, then $x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=4^2-2=14.$

Therefore $x^4+\frac{1}{x^4}=\left(x^2+\frac{1}{x^2}\right)^2-2=14^2-2=194.$

Qn4. $f(x)+1=x^2+2x+1=(x+1)^2 \Longrightarrow f(x)=(x+1)^2-1$ and
$f(f(x))=(f(x)+1)^2-1=((x+1)^2-1+1)^2-1=(x+1)^4-1.$
$f(f(f(x)))=f((x+1)^4-1)=(f(x)+1)^4-1=((x+1)^2-1+1)^4-1=(x+1)^8-1.$
$f(f(f(f(f(x)))))=(x+1)^{32}-1=16^{16}$. This means that $(x+1)^{32}=16^{16} \Longrightarrow 32\log(x+1)=\frac{1}{2}\log 16.$
$x+1=4 \Longrightarrow x=3.$

Qn5. There are 74 two-digit numbers (from 26 to 99 inclusive) and 110 three-digit numbers (from 100 to 209 inclusive), so the number of digits is $74\times 2+110\times 3=148+330=478$.

Qn6. We are looking for numbers of the form $ab2014$, where a cannot be zero (it would not be a six digit number if a were zero). Thus $10\le ab\le 99$, so there are $99-10+1=90$ such numbers for example $(102014, 112014, \cdots, 992014)$.

Qn7. First arrange B, C and D next to each other. There are $3\times 2\times 1=6$ ways in which this can be done. Now regard B, C and D standing together as a single unit H. Now we arrange A, H and E, which can be done in $3\times 2\times 1=6$ ways. Thus there are a total of $6\times 6=36$ for the students to line up. **Alternatively** One can reason as follows: If B, C, D are next to one another, they can line up in $3\times 2\times 1=6$ different arrangements. With each arrange- ment, there are two with A and E both in front of them, two more with A and E both behind them, and two more with one in front and one behind. Thus the total number of arrangements is $6\times(2+2+2)=36$.

Qn8. One of the integers must be divisible by 7. Let this integer be a. For the smallest possible value, we should try $a = 7$. Then we have that $bc = 7 + b + c$, which yields

$$\begin{aligned} bc - b &= 7 + c. \\ b(c-1) &= 7 + c. \\ b(c-1) &= 8 + c - 1. \\ b(c-1) - (c-1) &= 8. \\ (b-1)(c-1) &= 8. \end{aligned}$$

Then $\{b, c\} = \{3, 5\}$ or $\{2, 9\}$. Thus $\{a, b, c\} = \{7, 3, 5\}$ yields the smallest sum.

Qn9. Since there are 5 choices for a and 3 choices for b, there are fifteen possible a^b numbars. If a is even, a^b is even and if a is odd, a^b is odd. So the choices of a and b which give an even value for a^b are those where a is even, or 6 of the choices (since there are two even choices for a and three ways of choosing b for each of these). (Notice that in fact the value of b does not affect whether a^b is even or odd, so the probability depends only on the choice of a.) Thus, the probability is $\frac{6}{15} = \frac{2}{5}$.

Qn10. Between 1 : 00pm and 2 : 00pm, there are exactly 60 minutes. Let t_1 be the amount of time Mr. Kakande uses for eating, then he uses $\frac{1}{2}t_1$ on marking scripts, $\frac{1}{4}t_1$ on going to the restroom and $\frac{1}{8}t_1$ on talking to a friend. Thus $t_1 + \frac{1}{2}t_1 + \frac{1}{4}t_1 + \frac{1}{8}t_1 = 60 \Longrightarrow t_1 = \frac{60\times 8}{15} = 4 \times 8 = 32$ minutes. Therefore, he finished marking scripts after $32 + \frac{1}{2} \times 32 = 32 + 16 = 48$ minutes. Mr. Kakande finished marking at 1 : 48 pm.

SECTION B

Qn11. We write the condition as an equation relating A and B as follows:

$$A \times \left(1 + \frac{B}{100}\right) = B \times \left(1 - \frac{A}{100}\right).$$

Now we multiply through by 100 and obtain

$$A(100 + B) = B(100A).$$

which on simplification yields

$$2AB + 100A - 100B = 0.$$

We can now solve for A to get

$$A = \frac{100B}{2B + 100} = \frac{50B}{B + 50} = 50 - \frac{2500}{B + 50}.$$

Since B is an integer, then $B + 50$ has to be a divisor of 2500, and there are only 15 divisors. These are $1, 2, 4, 5, 10, 20, 25, 50, 100, 125, 250, 500, 625, 1250, 2500$. Only three of these yield three-digit numbers B, namely $250, 500$ and 625 where B is respectively $250-50=200, 500-50=450$ and $625-50=575$.

Suppose $B=200$, then $A=50-\frac{2500}{250}=50-10=40$.

Suppose $B=450$, then $A=50-\frac{2500}{500}=50-5=45$.

Suppose $B=575$, then $A=50-\frac{2500}{625}=50-4=46$.

The corresponding pairs (A,B) are: $(40,200),(45,450),(46,575)$.

Alternatively,

Note that $A\%$ of B equals $B\%$ of A (both are equal to $\frac{AB}{100}$). Since A increased by this amount equals B decreased by this amount, it must equal half their difference:

$$\frac{AB}{100}=\frac{B-A}{2}.$$

Once again, we obtain that $AB+50A-50B=0$, and we can continue in the same way as before.

Qn12. We have

$$x+yz=2. \tag{6}$$
$$y+xz=2. \tag{7}$$
$$z+xy=2. \tag{8}$$

Subtracting Equations (10) from Equation (9), we obtain $x-y+yz-xz=0$.

Factoring yield $(1-z)(x-y)=0$, thus $z=1, x=y$.

From $z=1$, we have $(x,y,z)=(1,1,1)$.

From $x=y$, we obtain $x+xz=2$, $z+x^2=2$ from the Equations (9) and (11). Solving the system for x gives $(x-1)^2(x+2)=0$, thus $x=1$ and $x=-2$.

Therefore $(x,y,z)=(-2,-2,-2)$. Hence the number of triples is 2.

Qn13. Suppose that a palindrome p is the sum of the three consecutive integers $b-1, b, b+1$. In this case, $p=(b-1)+b+(b+1)=3b$, so p is a multiple of 3. The possible palindromes less than 200 are $11, 22, 33, 44, 55, 66, 77, 88, 99, 111, 121, 131, 141, 151, 161, 171, 181$ and 191. Note that b is a multiple of 3, so the possible palindromes are $33, 66, 99, 111, 141, 171$. The largest integer 171 can be written as $56+57+58$, so 171 is the largest palindrome less than 200 that is the sum of three consecutive integers

Qn14. The three-digit number abc is equal to $100a+10b+c$ in algebraic notation, and similarly ac is equal to $10a+c$. The given equation therefore becomes $100a+10b+c=9(10a+c)+4c$, which simplifies to $5(a+b)=6c$. Since a, b, c are integers between 0 and 9, and $a \neq 0$, the only possibility is $a+b=6$ and $c=5$. The six possible numbers then arise from taking $a=1,2,...,6$ and $b=6-a$. These numbers are $155, 245, 335, 425, 515, 605$.

Qn15. Since $\triangle PQR$ has $PQ = QR = RP$, then $\triangle PQR$ is an equilateral triangle and all its angles are each equal to $60°$. Since ST is parallel to QR, SV is parallel to PR and TU is parallel to PQ, then all angles in $\triangle PST$ and $\triangle SQV$ and $\triangle TUR$ are each equal to $60°$. Let $SQ = x$. Since $\triangle SQV$ is equilateral, then $QV = VS = SQ = x$. But $PQ = 30$, so $PS = 30-x$. Since $\triangle PST$ is equilateral, then $ST = TP = PS = 30-x$. But $PR = 30$ so $TR = 30-(30-x) = x$. Since $\triangle TUR$ is equilateral, then $TU = UR = TR = x$. But $VS+ST+TU = 35$ so $x+(30-x)+x = 35$ or $30+x = 35$ which implies $x = 5$. Therefore $VU = QR - QV - UR = 30 - x - x = 20$.

NMC 2013 A-LEVEL PAPER 1 SOLUTIONS
SECTION A

1. Given $f(2) = 5$ and $f(1) = 2$, $f(3) = f(2) - f(1) = 3$, $f(4) = f(3) - f(2) = -2$, $f(5) = -5$, $f(6) = -3$, $f(7) = 2(= f(1))$, $f(8) = 5(= f(2))$. The function $f(x)$ is periodic , with period 6. $2013 \bmod 6 = 3$ (Since $2013 = 6 \times 335 + 3$.) Therefore $f(2013) = f(3) = 3$.

2. Note that $7 - \sqrt{48} = (2 - \sqrt{3})^2$, $\quad 5 - \sqrt{24} = (\sqrt{3} - \sqrt{2})^2$, $\quad 3 - \sqrt{8} = (\sqrt{2} - 1)^2$.

$$\sqrt{7 - \sqrt{48}} + \sqrt{5 - \sqrt{24}} + \sqrt{3 - \sqrt{8}}. = \sqrt{(2 - \sqrt{3})^2} + \sqrt{(\sqrt{3} - \sqrt{2})^2} + \sqrt{(\sqrt{2} - 1)^2}$$

$$= 2 - \sqrt{3} + \sqrt{3} - \sqrt{2} + \sqrt{2} - 1 = 1$$

3. Note that $n! = (n + 1)! = n!(n + 2)$.
Using this observation, we get
$1!{\cdot}3 - 2!{\cdot}4 + 3!{\cdot}5 - 4!{\cdot}6 + \ldots - 2012!{\cdot}2014 + 2013! = 1! + 2! - (2! + 3!) - \cdots - 2012! - 2013! + 2013! = 1$

4. Let x_1, x_2 be solutions of $x^2 + mx + n = 0$, then x_1^3 and x_2^3 are solutions to $x^2 + px + q = 0$.
This implies $x_1 + x_2 = -m$ and $x_1x_2 = n \Longrightarrow x_1^3 + x_2^3 = -p$ and $x_1^3x_2^3 = q$.
From $(x_1 + x_2)^3 = x_1^3 + x_2^3 + 3x_1^2x_2 + 3x_1x_2^2 = (x_1 + x_2)^3 + 3x_1x_2(x_1 + x_2)$
$(-m)^3 = -p + 3n(-m)$. Therefore $m^3 = p + 3nm$.

5. For the first scoop, a child has a free choice that is 9 options. For the second scoop, there are only 8 choices left because the choice of say, stawberry and vanilla is identical to that of vanilla and strawberry. There are $\frac{9\times8}{2} = \frac{72}{2} = 36$ choices of flavours. Hence the largest possible number of children in the group is 36.

6. Jonathan never lies on Monday,Tuesday, Wednesday and Thursday. we are not sure whether he lies on Friday, Saturday or sunday. David never lies on Monday, friday, Saturday, Sunday. we are not sure whether he lies on Tuesday, Wednesday or Thursday.
Suppose today is saturday, then yesterday was Friday on which day we are not sure whether Jonathan lies or not and david never lies on Friday which is a contradiction.
By continuous elimination gives today as Friday.

7.

	7	25	18	
	21	19	y	
22	20	16-x	x-2	9
16	17-x	x-1	10	23
	x	6	24	

Using the second column, we have the unfilled box as $17 - x$ that is $65 - (7 + 21 + 20 + x)$ and using the fourth row, the unfilled box will have $x - 1$ that is $65 - (16 = 17 - x + 10 + 23)$. continuing in this fashion, we shall fill the table as shown above and considering the fourth column, we have $18 + y + x - 2 + 10 + 24 = 65$.
Hence $x + y = 65 - 40 = 15$.

8. For the given equations

$$x + y = 2m^2$$
$$y + z = 6m$$
$$x + z = 2$$

sum of the equations gives $x = y + z = m^2 + 3m + 1$. Therefore
$x = (x + y + z) - (y + z) = m^2 - 3m + 1$, $\quad y = (x + y + z) - (x + z) = m^2 + 3m - 1$ and
$z = (x + y + z) - (x + y) = -m^2 + 3m + 1$
The values of m is $x \leq y \leq z$ is the same as $m^2 - 3m + 1 \leq m^2 + 3m - 1 \leq -m^2 + 3m + 1$.
For $x \leq y$, we have $m^2 - 3m + 1 \leq m^2 + 3m - 1$ or $m \geq \frac{1}{3}$.
For $y \leq z$, we have $m^2 + 3m - 1 \leq -m^2 + 3m + 1$ or $m^2 \leq 1 \Longrightarrow -1 \leq m \leq 1$. Hence the values of m lie in the range $\frac{1}{3} \leq m \leq 1$

9. In the first race, Jane ran $100m$ in the same time Sarah ran $95m$. This means the ratio of their speeds is $100 : 95 = 20 : 19$. In the time Sarah runs $1m$, Jane runs $\frac{20}{19} = 1.053m$ or in the time Jane runs $1m$, Sarah runs $\frac{19}{20} = 0.95m$. n the second race, Jane must run $105m$ and Sarah must run $100m$. If Sarah finishes first, then Jane must not have completed 105 in the time that it takes Sarah to complete $100m$. Since Jane runs $1.053m$ for $1m$ that Sarah runs, then Jane will run $105m$ in the time Sarah runs $100m$. This implies Jane must finish first. When Jane runs $105m$, Sarah will run $105 \times \frac{9}{20} = 99\frac{3}{4}m$. At this time, Sarah was $100 - 99\frac{3}{4} = \frac{1}{4}m$ behind Jane when she crossed the finish line.

10. $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $xy\left(3f(\frac{x}{y}) + 5f(\frac{y}{x})\right) = 3x^2 + 5y^2$

Divide by xy, the equation becomes $\left(3f(\frac{x}{y}) + 5f(\frac{y}{x})\right) = 3\frac{x}{y} + 5\frac{y}{x}$

put $y = 1$ and multiply the equation by 3 gives $9f(x) + 15f(\frac{1}{x}) = 9x + 15\frac{1}{x} \cdots\cdots\cdots$ (i)
putting $x = 1$ and $y = x$ and multiply by 5 gives $15f(\frac{1}{x}) + 25f(x) = 15\frac{1}{x} + 25x \cdots\cdots$ (ii)
solving equations (i) and (ii) simultaneously gives $15f(\frac{1}{x}) + 25f(x) = 15\frac{1}{x} + 25x$
$(25 - 9)f(x) = 25x - 9x \Longrightarrow 16f(x) = 16x$.
Hence $f(x) = x$ for all x.

SECTION B : 10 marks each

11. As the boxes are different, we can give a label to them and deal with them in order. The first box gets 3 of the balls- this can be done in

12. Let x be the number on the second card. Then, the number on the third card is $20 - 2 - x = 18 - x$. The number on the fourth card is $20 - x - (18 - x) = 2$.
The number on the fifth card is $20 - (18 - x) - 2 = x$ again. This implies the pattern repeats in the cycles of 3. The pattern is shown below $2, x, 18 - x, 2, x, 18 - x$. This follows that the number on the 52^{nd} is the same as the number on the third card. so, $8 = 18 - x \Longrightarrow x = 10$

13. If a, b, c form a geometric sequence then, $\frac{b}{a} = \frac{c}{b} \Longrightarrow b^2 = ac \cdots\cdots$ (i)
$\log_6 a + \log_6 b + \log_6 c = 6 \Longrightarrow \log_6(abc) = 6 \Longrightarrow abc = 6^6$
From (i), $b^3 = 6^6 = (6^2)^3$. Thus $b = 36$.

If $b-a$ is a sequence of a natural number then the possible choices are $\{11, 20, 27, 32, 35\}$.

14. let P, Q and R be real numbers
$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}=\frac{Pn+Q}{2^n}+R$$

If it is true for $n=n+1$, then

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\frac{n+1}{2^{n+1}}=\frac{P(n+1)+Q}{2^{n+1}}+R$$

$$\frac{Pn+Q}{2^n}+\frac{n+1}{2^{n+1}}=\frac{P(n+1)+Q}{2^{n+1}} \Longrightarrow 2(Pn+Q)+(n+1)=P(n+1)+Q$$

$$2Pn+2Q+n+1=nP+P+Q \Longrightarrow n(P+1)+(Q+1-P)=0$$

on equating coefficients, we get $P+1=0 \Longrightarrow P=-1$ and $(Q+1+1)=0 \Longrightarrow Q=-2$.
Therefore $P+Q+R=-1-2+R$.
If $n=1$ in the first equation , then we have that $R=2$.
Thus $P+Q+R=-1-2+2=-1$

15. Area of triangle ABC $=\frac{1}{2}vw=\frac{1}{2}xy \Longrightarrow xy=vw$. By Pythagoras theorem, $w^2+v^2=x^2$.
Consider $35^2=(v+w)^2=v^2+w^2+2vw=v^2+w^2+2xy=x^2+2yx=x^2+2yx+y^2-y^2$.
$\Longrightarrow 35^2=(x+y)^2-y^2=(37)^2-y^2 \Longrightarrow y=12$.
Thus, $y+8=12+8=20$

NMC 2013 A-LEVEL PAPER 2 SOLUTIONS

SECTION A

1. $xy = \sin(2x)$ and $\frac{x}{y} = \tan t \Longrightarrow x^2 + y^2 = y^2 \tan^2 t + y^2 = y^2(\tan^2 t + 1) = y^2 \sec^2 t$

   $= xy\frac{y}{x} \sec^2 t = \frac{\sin(2t)\cos t}{\tan t} = \frac{2\sin t\cos t}{\sin t\cos t} = 2.$

2. In order to make one number the largest possible, the other fifteen numbers should be the smallest possible numbers.
   Let y be the largest number possible, then

$$\frac{1+2+3+4+5+6+7+8+9+10+11+12+13+14+15}{16} = 16$$

   $120 + y = 256 \Longrightarrow y = 256 - 120 = 136.$

3. The number of work days until the guard's new Sunday off is a multiple of 7 and 5. The smallest multiple of 7 and 5 is 35, hence the number of work days before the guard's new Sunday off is 35 days of which there are 28 work days.

4. Denote Florence's age by ab. Suppose that $b \geq 6$, then $ab = a(b-6)$ which equation yields $a = 0$ contradicting the assumption that a is non zero. This implies that $b \leq 5$ and $ab = (a-1)(b+4)$ or $4a - b = 4$. The only non zero solution is $a = 2$ and $b = 4$. We therefore have that Florence's age is currently 24 and the product of the digits of the current age is 18.

5. Given the equation $x^4 - ax^3 + 2x^2 - bx + 1 = 0$. Let y be the root of the equation. Then

$$y^4 - ay^3 + 2y^2 - by + 1 = 0 \text{ and so } y^2 - ay + 2 - \frac{b}{y} + \frac{1}{y^2} = 0$$

$$\text{Completing the squares gives } (y - \frac{a}{2})^2 + (\frac{1}{y} - \frac{b}{2})^2 - \frac{a^2}{4} - \frac{b^2}{4} + 2 = 0$$

$$(y - \frac{a}{2})^2 + (\frac{1}{y} - \frac{b}{2})^2 = \frac{a^2}{4} - \frac{b^2}{4} + 2 \geq 0$$

$$\frac{a^2 + b^2 - 8}{4} \geq 0 \Longrightarrow a^2 + b^2 \geq 8.$$

6. Let x denote the girl-boy sets, then there are $(x+4)$ girl-girl sets. Therefore there are $35 - (x + x + 4) = 31 - 2x$ boy-boy sets, and so
   $x + 2(31 - 2x) = 38 \Longrightarrow x + 62 - 4x = 38 \Longrightarrow 3x = 24$ so that $x = 8$. Thus, there are 8 girl-boy sets and the boy-boy sets of twins are therefore $31 - 16 = 15$.

7. The positive odd integers are grouped into sets of 3 as follows: Set 1 : $\{1,3,5\}$; Set 2 : $\{7,9,11\}$; Set 3 : $\{13,15,17\}$.
   Recall that odd numbers are always denoted by $2n-1$ where $n \in \mathbb{N}$.
   By using positions to locate required values, Set 1 ends with the third odd number 5, Set 2 ends with the sixth odd number 11 and Set 3 ends with the ninth odd number 17. Therefore, the position of the last odd numbers is given by $\{3^{rd}, \{6^{th}, \{9^{th}, \cdots\}$ in the $\{1^{st}, \{2^{nd}, \{3^{rd}, \cdots\}$ positions respectively. Therefore the last odd number in the 2013^{th} set is the $3(2013)^{th}$ odd number or 6039^{th} odd number. Now since $2 \times 6039 - 1 = 12077$. The set 2013 required is $\{12073, 12075, 12077\}$.

8. Since $a679b$ is divisible by 36, it is also divisible by 4 and 9. If a number is divisible by 4, then the number formed by the last 2 digits must be a multiple of 4. Hence 4 divides $9b$ and so $b = 2$ or $b = 6$.
   Similarly, if a number is divisible by 9, the sum of the digits is a multiple of 9 and hence 9 divides $a + 6 + 7 + 9 + b = a + b + 22$.
   Now, when $b = 2$, it implies $a = 3$ because 9 divides $a + 24$ and when $b = 6$, it implies $a = 6$ because 9 divides $a + 28$. Therefore the possible numbers are 36792 and 86796.

9. For $n > 5$, we have $1 < \frac{n+3}{n-1} < 2$ and for $n < -3$, we have $0 < \frac{n+3}{n-1} < 1$ and so we need to consider nine integer values of n from -3 to $+5$ only.

   We find that $\frac{n+3}{n-1}$ is an integer except for $n = -2, n = 1$ and $n = 4$ giving six acceptable values of n.
   Alternatively,
   We can write $\frac{n+3}{n-1}$ as $\frac{n+3}{n-1} = \frac{n-1+4}{n-1} = 1 + \frac{4}{n-1}$. The number on the far left is an integer if and

   only if $\frac{4}{n-1}$ is an integer. Since the divisors of 4 are $1, 2$ and 4, the only possibilities are $n - 1 = \pm 1, \pm 2, \pm 3$ so that $n = 0, 2, -1, 3, -3, 5$.

10. Statement C is true and Statement D is false. There are two possibilities. Either Ruth is telling the truth today and does not does not make Statement D or she is lying and does not make Statement C. Suppose A, B and D are true, from statement E, Ruth has atleast 3 friends and from A, this number of friends is prime and from B the number of her friends is even. The only even prime number is 2 so these statements cannot all be true and hence Ruth cannot be telling the truth today. However, it is possible that all the 3 statements are false so Ruth is lying today and could not have made statement C.

SECTION B

11. Multiply $S + M \times C = 64$ by S to get $S^2 + S \times M \times C = 64S \Longrightarrow S^2 + 240 = 64S$ and solving this equation gives $S = 60$ or $S = 4$.

    Multiply $S \times C + M = 46$ by M to get $M^2 + S \times M \times C = 46S \Longrightarrow M^2 + 240 = 64M$ and solving this equation gives $M = 40$ or $M = 6$.

(i) If $M = 40$, this implies $S \times C \times 40 = 240 \Longrightarrow S \times C = 6$. Either $C = \frac{6}{40}$ or $C = \frac{6}{4}$ because the value of S is either 60 or 4.

(ii) If $M = 6$, this implies $S \times C \times 60 = 240 \Longrightarrow S \times C = 40$. Either $C = \frac{40}{60}$ or $C = \frac{40}{10} = 4$. Hence the only whole number solutions of the equations are $S = 4, M = 6$ and $C = 10$.

12. Let the vertices of the trapezium be A, B, C, D and $\overline{AC}$ meet $\overline{BD}$ at E as shown below. Triangle

ABE and CDE are similar since $< ABE =< CDE$ (alternate angles) and $< AEB =< CED$(vertically opposite angles). This implies $\frac{CE}{AE} = \frac{CD}{AB} = \frac{18}{3} = 3$.
Hence $AC = 4AE$ and by a similar argument $BD = 4BE$.
Triangles AXE and ADC are similar since $< XAE =< DAC$ and $< AXE =< ADC$ (corresponding angles), so that $\frac{XE}{DC} = \frac{AE}{DC} = \frac{1}{4}$. Hence $XE = 4.5cm$.
Applying similar argument to triangles BYE and BCD, we find that $YE = 4.5cm$ also. Hence the length $XY = 4.5 + 4.5 = 9cm$.

13. (i) Clearly, there is no single digit magic number.

(ii) Let ab be a two digit magic number, Then $10a + b = 13(a + b) \Longrightarrow 3a + 12b = 0$. This is an impossibility since both a and b are greater than zero. Hence there are no two digit magic numbers.

(iii) Let abc be a three digit magic number, Then $100a+10b+c = 13(a+b+c) \Longrightarrow 29a = b+4c$. The only possible value of a is 1 since otherwise $29a \geq 58$ and yet the maximum value of $b + 4c$ can only be 45. The equation $29a = b + 4c$ has exactly non negative integers $(b, c) = (1, 7), (5, 6)$ and $(9, 5)$.
Hence $117, 156, 195$ are the only magic numbers less than 1000.

14. The equation $ax^3 + bx^2 + (a - 4)x + 2 = 0 \Longrightarrow x^3 + \frac{b}{a}x^2 + \frac{a-4}{a}x + \frac{2}{a} = 0$.
Sum of roots, $3r = \frac{b}{a}$ and product of roots $r^3 = \frac{-2}{a} \cdots\cdots\cdots (i)$.
Since the equation has repeated roots, $x = r$ is also a solution to $f'(x) = 0$. Therefore $3ax^2 + 2bx + (a - 4) = 0$ so that the product of roots $r^2 = \frac{a-4}{3a} \cdots\cdots\cdots (ii)$. It follows from (i) and (ii) that $3r^2 = \frac{a-4}{a}$ and $-r = \frac{2}{a}$. Thus
$r(\frac{a-4}{3a}) = r^3 = -\frac{2}{a} \Longrightarrow r = \frac{6}{4-a}$, so that $3(\frac{6}{4-a})^2 = \frac{a-4}{a}$ which simplifies to the cubic equation $(a - 4)^3 - 108a = 0$ with roots $a = 2$ and $a = 16$.
This gives $r = 1$ or $r = -\frac{1}{2}$.

15.

	$*$			A
		B		
D	E_2	C	A_1	B_3
			E	

The square labelled A_1 is in the same row as C and D, the same column as E and the same diagonal as B, so it must be A. The square labelled E_2 is the next to be filled in as is in the same row as A, C and D and in the same diagonal as B. The square labelled B_3 now completes the fourth row. The bottom left-hand corner must now be A.This is because one of the three remaining squares in the diagonal which runs from bottom left to top right must be A, but A at the end of row 2 means that A cannot be in either of the last 2 squares of this diagonal. The square marked $*$ is in the same diagonal as A, B and C and in the same column as E so it must be D.

NMC 2012 A-LEVEL PAPER 1 SOLUTIONS

SECTION A

Qn1. $S = \cos^2 80° + \cos^2 70° + \cos^2 60° + \cos^2 50° + \sin^2 50° + \sin^2 60° + \sin^2 70° + \sin^2 80°$
$=(\cos^2 80° + \sin^2 80°) + (\cos^2 70° + \sin^2 70°) + (\cos^2 60° + \sin^2 60°) + (\cos^2 50° + \sin^2 50°)$
$= 1 + 1 + 1 + 1 = 4.$

Qn2. $|m| - m + n = 10$ and $m + |n| + n = 12$.

- Case I. When $m > 0, n > 0$, then $n = 10$
  and $m + 2n = 12 \Longrightarrow m + 20 = 12 \Longrightarrow m = -8$ which is not possible since m should be positive.

- Case II.When $m > 0, n < 0, |m| - m + n = 10$ and $m = 12$ implying $n = 10$. This is not possible since n should be less than zero.

- Case III. When $m < 0, n < 0$, then $-2m+n = 10$ and $m = 12$ which is not possible again.

- Case IV. When $m < 0, n > 0, -2m + n = 10 \Longrightarrow n = 10 + 2m$ and $m + 2n = 12$ and substituting for n, we get $m + 2(10 + 2m) = 12$
  $m + 20 + 4m = 12 \Longrightarrow m = \frac{-8}{5} \Longrightarrow n = 10 + 2 \times \frac{-8}{5} = \frac{34}{5}$ Thus $m + n = \frac{-8}{5} + \frac{36}{5} = \frac{28}{5}$.

Qn3. Let the length be x and width be $y \Longrightarrow A = xy$ and $P = 2(x + y) = 1000$
$\Longrightarrow x + y = 500 \Longrightarrow y = 500 - x$
$A = x(500 - x)$ and $\frac{dA}{dx} = 500 - 2x$.
For max, $500 - 2x = 0 \Longrightarrow 2x = 500$ so that $x = 250$.
$\frac{d^2A}{dx^2} = -2$ since $\frac{d^2A}{dx^2} < 0$. Therefore $x = 250$ and $y = 250$.
The largest possible area is $250 \times 250 = 62500$ square units.

Qn4. John: Not me, David: I am and Geoffrey: Not David
Suppose John tells the truth, then David tells a lie and Geoffrey tells the truth which is a contradiction since two of the three people always tells a lie.
Suppose David tells the truth, it implies John tells the truth too and Geoffrey tells a lie. A contradiction again, so then Geoffrey tells the truth implying that John and David tell a lie. Therefore John is the tallest.

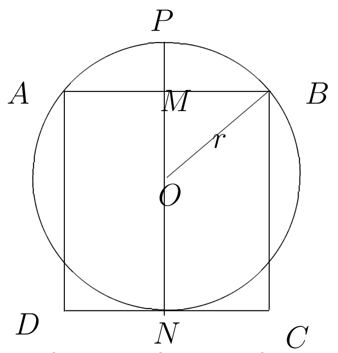
Qn5. Let k be the distance the slower particle travels in 50 seconds. Then $50r = K$ and $50R = k + 300$. Subtracting the two equations gives $50R - 50r = k + 300 - k$.
Thus $R - r = \frac{300}{6} = 6$ cm per second.

Qn6. Consider $x^3 - x^2 = x^2(x - 1)$, we need $x - 1$ to be a square of an integer and there are exactly ten perfect squares between $+1$ and 100. Therefore there are 10 integers x in the set $1, 2, 3, ..., 100$ for which $x^3 - x^2$ is a square of an integer.

Qn7. $f(n) = \log_2 3. \log_3 4. \log_4 5 ... \log_{n-1} n = \frac{\log_2 3}{\log_2 2} \cdot \frac{\log_2 4}{\log_2 3.} \cdot \frac{\log_2 5}{\log_2 4} \cdots \frac{\log_2 n-1}{\log_2 n-2} \cdot \frac{\log_2 n}{\log_2 n-1}.$

$f(n) = \log_2 n.$
$\sum_{k=1}^{500} f(2^k)$ where $f(2^k) = \log_2 2^k$.
$\sum_{k=1}^{500} k = 2 + 3 + 4 + 5 + 6 + ... + 498 + 499 + 500,$
$u_n = a + (n-1)d$ where $u_n = 500, a = 2, d = 1$
$500 = 2 + (n-1)$ which gives $n = 499$. Therefore $S_{499} = \frac{499}{2}(2 \times 2 + (499-1) \times 1) = 125249.$

Qn8. Let $ABCD$ be the vertices of a square. Let M be the midpoint of the segment AB. let r be the radius of the circle. Given $\overline{AB} = 2cm$, then $\overline{MB} = 1cm.\overline{MN} = 2cm$ and $\overline{PN} = 2rcm$. But $\overline{MP} = \overline{PN} - \overline{MN} = (2r-2)cm$. But $\overline{OP} = r$ so $\overline{OM} = \overline{OP} - \overline{MP} = r - (2r-2) = (2-r)cm$.



Using triangle OMB, $\overline{OM}^2 + \overline{MB}^2 = \overline{OB}^2 \Longrightarrow (2-r)^2 + 1^2 = r^2$
$4 - 4r + r^2 + 1 = r^2 \Longrightarrow 5 - 4r = 0$ so that $r = \frac{5}{4}$. The radius of the circle is $\frac{5}{4}cm$.

Qn9. Let W be the total weight of all fish the man caught. The fish taken by the cat had weight of 30% .

- If the cat took 4 fish or more, their average weight must have been at most $7.5\%W(\frac{30}{4} = 7.5)$ which is less than the average weight of the 5 lightest fish.
- If the cat took 1 or 2 fish, then their average weight must have been atleast $15\%W(\frac{30}{2})$ which is more than the average weight of the 2 heaviest fish.

Therefore the cat took 3 fish.

Qn10. Suppose we label the trucks $1, 2, 3, 4$ with the labels corresponding to the position of the trucks on Wednesday. There are 11 possible arrangements the next day. $1324, 1432, 2143, 2413, 2431, 3142, 3214$
Hence there are 11 ways.

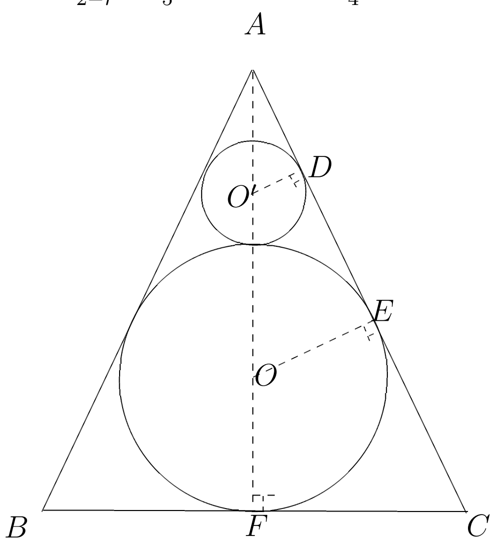
SECTION B

Qn11. $x^2 - y^2 = 1000001$ can be written as $x^2 - y^2 = 1000000 + 1 = 10^6 + 1$
$(x-y)(x+y) = 1(10^6 + 1)$ so that $x - y = 1 \Longrightarrow x = y + 1$.
$x + y = 1000001(10^6 + 1) \Longrightarrow y + 1 + y = 1000001 \Longrightarrow 2y = 1000000 \Longrightarrow y = 500000.$
Also $x^2 - y^2 = 10^6 + 1$, But $10^6 + 1 = (10^2 + 1)(10^4 - 10^2 + 1)$
$(x-y)(x+y) = (101)(9901).$
Thus $x - y = 101$ and $x + y = 9901$ and on solving simultaneously gives $x = 5001$ and $y = 4901$.
The smallest value of x is 5001.

Qn12. $\overline{CF} = \overline{CE} = 6cm$ since they are tangents to the circle. Thus $\overline{OF} = \overline{OE}$. Since $\overline{AC} = 10cm$ and $\overline{CE} = 6cm \Longrightarrow \overline{AE} = 4cm$.
Using right angled triangle AFC, $AF^2 + FC^2 = AC^2 \Longrightarrow AF^2 = 10^2 - 6^2 = 64$ so that $AF = 8$cm.
Triangles AEO and AFC are similar, so $\frac{AF}{AE} = \frac{FC}{OE}$ and on subtraction $\frac{8}{4} = \frac{6}{OE} \Longrightarrow OE = 3cm$.

Hence $\overline{OE} = \overline{GO} = \overline{OF} = 3cm$ which is the radius of the circle.
$\overline{AG} = \overline{AF} - \overline{GF} = 8 - 6 = 2cm$. Also $O'A = AG - O'G = 2 - r$ since $\overline{O'D}$ is parallel $\overline{OE}$, then triangles ADO' and AEO are similar and $\frac{O'D}{O'A} = \frac{OE}{OA}$.

But AEO is a right angled triangle and $\overline{AE} = 4cm, \overline{OE} = 3cm, \overline{AO} = 5cm$.
Thus by substitution $\frac{r}{2-r} = \frac{3}{5}$ so that $r = \frac{3}{4} = 0.75cm$.



Qn13. Let the speed of Bill be x. Then the speed of Phil is $3x \Longrightarrow$ the ratio of their speeds is $3:1$ respectively. The total time that Bill uses for the whole journey is $\frac{12}{3x} = \frac{4}{x}$.
Let t_1 be the time that Phil waited before he set off after Bill had left, then
$-t_1 + \frac{2(6-d)}{x} = \frac{4}{x}$ where $\frac{2(6-d)}{x}$ is the time taken for Bill to cover to and from the meeting point.
From the previous equation $-t_1 = \frac{4}{x} - \frac{12}{x} + \frac{2d}{x} \Longrightarrow -t_1 = \frac{-8}{x} + \frac{2d}{x}$
$t_1 = \frac{8-2d}{x}$.
Thus, Bill will have moved a distance of $8 - 2d$ when Bill starts and the remaining distance of $6 - (8 - 2d) = 2d - 2$.

Bill will cover $\frac{2d-2}{4}$ up to the meeting point and Phill will cover $\frac{3(2d-2)}{4}$, so the total distance

covered by Bill when Phil is also moving is $\frac{2(2d-2)}{4} + 8 - 2d = d - 1 + 8 - 2d = 7 - d$.
Using ratios, Distances $(7 - d) : 12$ and the speeds $1 : 3$.
Thus $\frac{7-d}{12} = \frac{1}{3} \Longrightarrow 3(7 - d) = 12 \Longrightarrow 9 = 3d$
$d = 3km$.

Qn14. let R_1, R_2, R_3 represent the three rows, and C_1, C_2, C_3 three columns, D_1 the diagonal from the top left to the top right and D_2 the diagonal from the top left to bottom right.

Sum of R_1= sum of D_1.
$\log a + \log b + \log x = \log z + \log y + \log x$.
This implies $\log ab = \log yz \Longrightarrow ab = yz$ or $z = \frac{ab}{y}$.
Also sum of C_1 = sum R_2.
$\log a + \log p + \log z = p + \log y + \log c$
$az = cy$ or $z = \frac{cy}{a}$.

since $z = \frac{ab}{y}$ and $z = \frac{cy}{a} \Longrightarrow \frac{ab}{y} = \frac{cy}{a}$ or $y^2 = \frac{a^2b}{c}$

$y = \sqrt{\frac{a^2b}{c}}$ or $y = \frac{a\sqrt{b}}{\sqrt{c}}$.

sum of C_3 = sum of D_2
$\log x + \log c + r = \log a + \log y + r$
$\log xc = \log ay \Longrightarrow x = \frac{ay}{c}$.

Thus $xyz = \frac{ay}{c}\frac{y}{1}\frac{cy}{a} = y^3$. Therefore $xyz = \frac{a^3b^{\frac{3}{2}}}{c^{\frac{3}{2}}}$.

Qn15. (a) $f(x) = ax^2 \Longrightarrow f(4) = 1 = a.4^2 \Longrightarrow a = \frac{1}{16}$. Since the slope of the function is 1,

$\Longrightarrow \frac{df}{dx} = 2a(4) = 1$.
$a = \frac{1}{8}$. Thus f cannot satisfy (ii) since $\frac{1}{16} \neq \frac{1}{8}$.

(b) $g(x) = cx^3 - \frac{x^2}{16} \Longrightarrow g(4) = 64c - 1 = 1$
$c = \frac{1}{32}$.

When $c = \frac{1}{32}$, $\frac{dg}{dx} = 3cx^2 - \frac{2x}{16}$.
Therefore $g'(4) = 3 \times \frac{1}{32} \times 16 - 2(\frac{1}{2}) = 1$.

(c) $g'(x) = \frac{3x^2}{32} - \frac{x}{8} = \frac{x(3x-4)}{32}$.

- when $x < 0$, $g'(x) > 0$, so $g(x)$ is increasing.
- When $0 < x < \frac{4}{3}$, $g'(x) < 0$. and so $g(x)$ is decreasing.
- When $\frac{4}{3} < 0$, $g'(x) > 0$ and so $g(x)$ is increasing.

This means $g(x)$ does not increase only when $0 < x < \frac{4}{3}$.
$g'(x) < 0$ for $0 < x < \frac{4}{3}$ so g does not satisfy condition (iii).

(d) $h(x) = \frac{x^n}{k} \Longrightarrow h(4) = \frac{4^n}{k} = 1 \Longrightarrow 4^n = k$.

$h'(x) = \frac{nx^{n-1}}{k}$. $\Longrightarrow h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1$. Hence $n = 4$ and $k = 4^4 = 256$.

$h(x) = \frac{x^4}{256} \Longrightarrow h(0) = 0.$

$h'(x) = \frac{4x^3}{256} \Longrightarrow h'(0) = 0$ and $h'(x) > 0$ for $0 < x < 4$.
Therefore $h(0) = 0, h'(0) = 0$ which meets requirement (i).
$h'(x) = \frac{4x^3}{256} = \frac{x^3}{64}$. This implies that $x > 0$ and $h'(x) > 0$. $\Longrightarrow h(x)$ is increasing and this meets the requirement (iii).

NMC 2012 A-LEVEL PAPER 2 SOLUTIONS

SECTION A: 5 MARKS EACH

Qn1. We are given that $(a+2b)(a-b)=10$. This implies that $(a+2b)$ and $(a-b)$ are factors of 10. We know that factors of 10 are $1,2,5$ and 10. Let us investigate these cases.

- Case I. When $a+2b=5$ then $a-b=2$ and solving the equations simultaneously gives $b=1$ and $a=3$. Thus $2a-b=2\times 3-1=5$.
- Case II. When $a+2b=10$,then $a-b=1$ and solving simultaneously gives $a=4$ and $b=3$. Thus $2a-b=5$.

Qn2. Let $x=2011$. Then,

$$
\begin{aligned}
\frac{1+2011^4+2012^4}{1+2011^2+2012^2} &= \frac{1+x^4+(x+1)^4}{1+x^2+(x+1)^2} \\
&= \frac{1+x^4+(x+1)^2(x+1)^2}{1+x^2+x^2+2x+1} \\
&= \frac{1+x^4+x^4+4x^3+6x^2+4x+1}{2(x^2+x+1)} \\
&= \frac{x^4+2x^3+3x^2+2x+1}{x^2+x+1} \\
&= \frac{(x^2+x+1)^2}{x^2+x+1} \\
&= x^2+x+1 \\
&= 2011^2+2011+1 \\
&= 4046133.
\end{aligned}
$$

Qn3. The sample space is $\{1,2,3,4,6,8\}$. The probability of an odd outcome is $\frac{2}{6}=\frac{1}{3}$ and the probability of an even outcome is $\frac{4}{6}=\frac{2}{3}$.

Case I. If an odd outcome appears first, the 'new' sample space is $\{2,2,6,4,6,8\}$. Therefore p (getting a; 2; on this roll) $=\frac{2}{6}=\frac{1}{3}$. Thus prob of getting a 2 on the second roll is $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$.

Case II. If the first roll is even, then, the 'new' sample space is $1,1,3,2,3,8$ and the prob of getting a 2 on the second die is $\frac{1}{6}$. The probability of a 2 on the roll is $\frac{2}{3}\times\frac{1}{6}=\frac{1}{9}$.

Therefore the probability of a 2 to appear on the top face is the sum of the two answers in Case I and Case II which is $\frac{1}{9}+\frac{1}{9}=\frac{2}{9}$.

Qn4. Since $p(2)=p(-2)=p(-3)=-1,\Longrightarrow 2,-2$ and -3 are roots of $p(x)=-1$. When $Q(x)=ax+b\Longrightarrow p(x)=(x-1)(x+2)(x+3)(ax+b)-1$. Using $p(1)=p(-1)=1$, we get $2=-12(a+b)\Longrightarrow -6a-6b=1$ ie putting $x=1$, and $2=6a-6b$ when $x=-1$. On solving simultaneously we get $a=\frac{1}{12}$ and $b=\frac{-1}{4}$. The polynomial is thus $p(x)=(x-1)(x+2)(x+3)(\frac{1}{12}x-\frac{1}{4})-1$ and putting $x=0$ gives $p(0)=2$.
Alt,
let $P(x)=ax^4+bx^3+cx^2+dx+e, p(2)=16a+8b+4c+2d+e=-1, p(-2)=16a-8b+4c-2d+e=-1, p(-3)=81a-27b+9c-3d+e=-1$,this means $p(1)=a+b+c+d+e=1$ and $p(-1)=a-b+c-d+e=1.p(2)-p(-2)=16b+4d=0$ and $p(1)-p(-1)=2b+2d=0\Longrightarrow b=-d=0$. also,$16a+4c+e=-1, 16a+4c+e=-1, 81a+9c+e=-1$ so that on solving we get $e=2$. Therefore $p(0)=0+0+0+0+0+2=2$.

Qn5. $1,x,y$ is a Geometric Progression. This implies that

$$\frac{x}{1}=\frac{y}{x}\Longrightarrow y=x^2. \tag{9}$$

$x,y,3$ is an Arithmetic Progression. This implies that

$$y-x=3-y\Longrightarrow 2y-3=x. \tag{10}$$

Solving equations (9) and (10) simultaneously gives $y=\frac{9}{4}$ or $y=1$ implying $x=\pm 1$ and $x=\pm\frac{3}{2}$. Therefore, the maximum value of $x+y=\frac{3}{2}+\frac{9}{4}=\frac{15}{4}$.

Qn6. The cost of Car's engine modification is equivalent to the cost of $\frac{400}{0.8}=500l$ of diesel, so the car would have to be driven a distance that would save $500l$ of diesel in order to make up the cost of the modification. Originally, the car consumes $8.4l$ per $100km$, and after it consumes $6.3l$ of diesel per $100km$, saving $8.4-6.3=2.1l$. Thus,to save $500l$, the car must have been driven $\frac{500}{2.1}\times 100=23809.52km$.

Qn7. $3\sin A+4\sin B=6$. Squaring gives

$$9\sin^2 A+24\sin A\sin B+16\cos^2 B \quad = \quad 36. \tag{11}$$

$4\sin B+3\cos A=1$. Squaring gives

$$16\sin^2 B+24\sin B\cos A+9\cos^2 A=1. \tag{12}$$

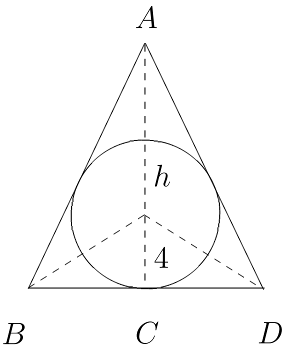
Solving equations (11) and (12) gives

$$9\sin^2 A+9\cos^2 A+24(\sin A\cos B+\sin B\cos A)+16\sin^2 B+16\cos^2 B=37.$$

$$\Longrightarrow 9=16+24\sin(A+B)=37\Longrightarrow \sin(A+B)=\frac{1}{2}\Longrightarrow A+B=30^\circ,150^\circ.$$

For $A+B=30^\circ$ it implies one of the angles must be less than 30°.By inspection $3\sin A+4\sin B<6$, so discard. Thus $A+B=150^\circ$ thus $<C=180^\circ-150^\circ=30^\circ$.

Qn8.



Let the height of $\triangle ABD$ be h. This implies that $\tan 30° = \frac{4}{CD} \Longrightarrow CD = 4\sqrt{3}$. Therefore,

$h = 4\sqrt{3}\tan 60 = 12$.

The area of triangle ABD is $\frac{1}{2} \times 12 \times 8\sqrt{3} = 48\sqrt{3}$ sq units.

Area A inside the triangle but outside the circle is $A = 48\sqrt{3} - 16\pi = \sqrt{3}x - \pi y. \Longrightarrow x = 48$ and $y = 16$. Hence $x + y = 48 + 16 = 64$.

Qn9. Let the roots be α and $\beta \Longrightarrow \alpha = \sin(\frac{\pi}{7})$ and $\beta = \cos(\frac{\pi}{7})$. Note that $\alpha^2 + \beta^2 = 1$, and using the given equation $x^2 - bx + c = 0$, we get $x^2 - bx + c = x^2 - (\alpha + \beta)x + \alpha\beta$ and we deduce that $b^2 = (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = 1 + 2c$.

Qn10. Drew is correct.

SECTION B: 10 MARKS EACH

Qn11. Let x be the size of the bottle in litres. Initially, $8l$ of pure alcohol from A are emptied into B and stirred implying that B contains a mixture of alcohol to total litres of mixture $= \frac{x}{6+x}$. xl of this mixture is emptied into A so that once again A has $6l$ of liquid. The amount of alcohol in mixture of alcohol and water in that bottle that was poured into A is $x \times \frac{x}{6+x} = \frac{x^2}{6+x}l$. This

implies that A ends up with $6 - x + \frac{x^2}{6+x}l$ of alcohol and $x - \frac{x^2}{6+x}l$ of water. But the ratio of

alcohol to water in A is $4:1 \Longrightarrow \frac{6-x+\frac{x^2}{6+x}}{x-\frac{x^2}{6+x}} = \frac{4}{1} \Longrightarrow \frac{36-x^2+x^2}{6x+x^2-x^2} = \frac{4}{1} \Longrightarrow x = \frac{3}{2}$. Therefore the size

of the bottle is $1.5l$.

Qn12. Assume $v = 2$, using that each contestant had found the correct value of exactly one unknown; then considering Bob and David $\Longrightarrow y \neq 3$ and $x \neq 4$ and therefore considering Eva $\Longrightarrow y = 1$. Since $y = 1$, then Jane's answer that $Z = 1$ is not correct, therefore $z \neq 1$ and $v \neq 2$. Using Bob and David's answers, $y = 3$ and $x = 4$. Therefore Cathy's answer $z = 5$ is correct. From the above, since $v \neq 2$, then $v = 1$, and $u = 2 \Longrightarrow v = 1, u = 2, y = 3, x = 4$ and $z = 5$. So, $x + z + v = 4 + 5 + 1 = 10$.

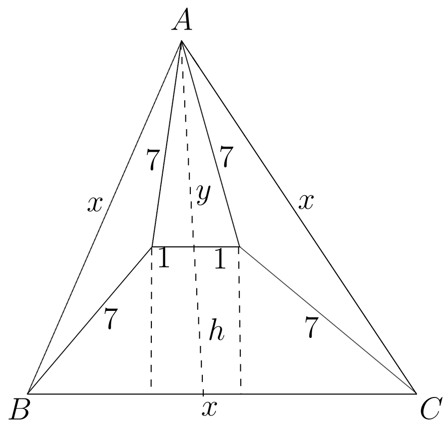
Qn13. let $y = 2^{2^x} \Longrightarrow y^2 = y.y = 2^{2^x} \times 2^{2^x} = 4^{2^x} \Longrightarrow 2^{2^x} + 4^{2^x} = 42 \Longrightarrow y + y^2 = 42$ which on solving gives $y = -7$ or $y = 6$. Since $y > 0$, we discard $y = -7$. Therefore $y = 6.\sqrt{2^{2^x}} = \sqrt{2^y} = \sqrt{2^6} = 2^3 = 8$.

Qn14. Using the difference of two squares,

$$\begin{aligned} & ((x+2)^2 - x^2)(x+2)^2 + x^2) = y^3 \\ \Longrightarrow \quad & ((x+2-x)(x+2+x))(x^2+4x+4+x^2) = y^3 \\ \Longrightarrow \quad & 8(x+1)((x+1)^2+1) = y^3. \end{aligned}$$

let $a = x+1 \Longrightarrow 8a(a^2+1) = y^3 \Longrightarrow a(a^2+1) = (\frac{y}{2})^3$. Let $b = \frac{y}{2} \Longrightarrow a(a^2+1) = b^3$. Since the R.H.S. is a perfect cube, so is a and a^2+1 for some integers q and $r \Longrightarrow a = q^3$ and $a^2+1 = r^3$ and substituting for $a = q^3$ in a^2+1, we get $(q^3)^2 + 1 = r^3 \Longrightarrow (q^2)^3 + 1 = r^3 + 0 \Longrightarrow q^2 = 0$ and $r^3 = 0 \Longrightarrow q = 0$ and $r = 1$. Substituting for $q = 0$ and $r = 1 \Longrightarrow a = 0$ and $x = -1 \Longrightarrow b = 0 \Longrightarrow y = 0$.

Qn15.



$y = \sqrt{7^2 - 1^2} = \sqrt{49 - 1} = \sqrt{48} = 4\sqrt{3}$.

$$h^2 + (\frac{x}{2} - 1)^2 = 7^2 \quad \Longrightarrow h^2 + (\frac{x}{2} - 1)^2 = 49 \tag{13}$$

Since ABC is an equilateral triangle,

$$\sin 60^\circ = \frac{h+y}{x} \quad \Longrightarrow h + y = \frac{\sqrt{3}}{2}x.$$

But $y = 4\sqrt{3}$. Therefore,

$$h + y = \frac{\sqrt{3}}{2}x - y = \frac{\sqrt{3}}{2}(x - 8) \tag{14}$$

Substituting for h in equation (14), gives $(\frac{\sqrt{3}}{2}(x-8))^2 + (\frac{x-2}{2})^2 = 49 \Longrightarrow 3(x-8)^2 + (x-2)^2 = 49 \times 4 = 196$. On expanding and simplifying we get $4x(x - 13) = 0 \Longrightarrow$. Either $4x = 0$, or $x - 13 = 0 \Longrightarrow x = 0$ or $x = 13$. Take $x = 13$.

NMC 2010 A-LEVEL PAPER 1 SOLUTIONS

SECTION A

1.

$$\begin{aligned}
\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\dots &= \left(\frac{1}{7}+\frac{1}{7^3}+\frac{1}{7^5}+\dots\right)+\left(\frac{2}{7^2}+\frac{2}{7^4}+\frac{2}{7^6}+\dots\right)\\
&= \frac{\frac{1}{7}}{1-\left(\frac{1}{7}\right)^2}+\frac{2\frac{2}{7^2}}{1-\left(\frac{1}{7}\right)^2}\\
&= \frac{7}{48}+\frac{2}{48}\\
&= \frac{9}{48}=\frac{3}{16}.
\end{aligned}$$

2. Let x be the height of the two candles. The burning rate of A is $\frac{x}{4}$ and the burning rate of B is $\frac{x}{3}$. At a time t when the height of A is twice the height of B, A will have lost $\frac{tx}{4}$ of its height and B will have lost $\frac{tx}{3}$ of its height. The remaining heights of candles A and B are $x-\frac{tx}{4}$ and $x-\frac{tx}{3}$ respectively. The remaining heights of the two candles satisfy the equation $x-\frac{x}{4}t=2\left(x-\frac{x}{3}t\right)$ or $1-\frac{1}{4}t=2\left(1-\frac{1}{3}t\right)$ from which t is found to be $2\frac{2}{5}$ hours.

3. It is not possible for all the seven integers to be greater than 13. If six of them where each 14, then the seventh could be $-(6\times 14)-1$ so that the sum is -1. Thus, the maximum number larger that 13 is 14.

4. Suppose that the suggested retail price was P. Then, the marked price was $0.6P$ and Hasna bought it at half the marked price which is $0.3P$. This means that Hasna bought it at 70% less than the suggested retail price.

5. Let the number of cows be a and the number of chicken be b. The number of legs are $4a+2b$ and the number of heads is $a+b$. So,we have $4a+2b=2(a+b)+14$ or $2a=14$ or $a=7$ implying that there were 7 cows in the group.

6. Let the quadratic equation be of the form $ax^2-bx+c=0$. Sum of the roots is $\frac{b}{a}$ and product of the roots is $\frac{c}{a}$. The first student makes a mistake only in the constant term implying that the sum of the root $\frac{b}{a}=8+2=10$ is correct. This implies that $b=10a$. The second students makes a mistake only in the linear coefficient implying that the product $\frac{c}{a}=-9\times -1=9$ is the correct one. This implies that $c=9a$. The equation is therefore $ax^2-10ax+9a=0$ *or* $x^2-10x+9=0$.

7. Let n be the total number of people at the party. If each person exchanged a handshake with a different person once, then we have $\binom{n}{2} = \frac{n(n-1)}{2}$ handshakes in all. Given that there were 28 handshakes, in all, we solve $\frac{n(n-1)}{2} = 28$ or $n^2 - n - 56 = 0$ implying that $n = 8$ or -7. We ignore -7 and take $n = 8$.

8. Let the original wage be w. After the decrease, his new wage is $w_1 = 80\%w$. If this wage bill is increased by $I\%$, the new wage bill will be $w_2 = 80\% \times (100 + I)\%$. Since the new bill is equal to the first bill, we solve for $80\% \times (100 + I)w\% = w$ implying that $I = 25$. Therefore, we have to increase the current wage by 25% to get the original wage.

9. The equation $\sqrt{5x-1} + \sqrt{x-1} = 2$ can be written as $\sqrt{5x-1} = 2 - \sqrt{x-1}$. Squaring gives

$$\begin{aligned} 5x - 1 &= 4 - 4\sqrt{x-1} + x - 1 \\ \Rightarrow\ 4x - 4 &= -4\sqrt{x-1} \\ \Rightarrow\ x - 1 &= -\sqrt{x-1} \\ \Rightarrow\ (x-1)^2 &= x - 1 \\ \Rightarrow\ (x-1)^2 - x - 1 &= 0 \\ \Rightarrow\ x^2 - 3x + 2 &= 0 \\ \Rightarrow\ x &= 1 \text{ or } x = 2. \end{aligned}$$

Of the two solutions, only $x = 1$ satisfies the equation $5x - 1 = 4 - 4\sqrt{x-1} + x - 1$. So $x = 1$ is the solution.

10. Since $\overline{AB} = 20cm$ and $\overline{AC} = 12cm$, then, $\overline{BC} = 16cm$. The height CA of $\triangle ACB$ is $16\tan B = 12$ which implies that $\tan B = \frac{12}{16}$. Likewise, the height DE of $\triangle EDB$ is $\overline{DE} = \overline{DB}\tan B = 10 \times \frac{12}{16}$. Therefore,

$$\begin{aligned} \text{Area of quadrilateral } ADEC &= \text{Area of } \triangle ACB - \text{Area of } \triangle EDB \\ &= \frac{1}{2}\overline{CB} \times \overline{AC} - \frac{1}{2}\overline{DB} \times \overline{DE} \\ &= \frac{1}{2} \times 16 \times 12 - \frac{1}{2} \times 10 \times 10 \times \frac{12}{16} \\ &= 58\frac{1}{2}. \end{aligned}$$

11. (a) Let the numbers be x, y and z. We are given that $x + y + z = 98$, $\frac{x}{y} = \frac{2}{3}$ and $\frac{y}{z} = \frac{5}{8}$. The second equation implies that $y = \frac{3}{2}x$ and the third equation implies that $z = \frac{8}{5}y = \frac{8}{5} \times \frac{3}{2} = \frac{12}{5}x$.

So, $x+y+z=98$ implies that $x+\frac{3}{2}x+\frac{12}{5}x=98$ which in turn imply that $\frac{49}{10}x=98$ from which we get $x=20$. The second number is therefore equal to $\frac{3}{2}x=30$.

(b) From the information given in the question, we have, $n=\frac{1}{2}n+\frac{1}{4}n+\frac{1}{5}n+7$ implying that $n=140$. Therefore, the first son gets $\frac{n}{2}=70$ cows, the second gets $\frac{n}{4}=35$ cows, the third gets $\frac{n}{5}=28$ cows and the fourth gets 7 cows.

12. Since k is odd, $f(k)=k+3$. Since $k+3$ is even,

$$f(f(k))=f(k+3)=\left(\frac{k+3}{2}\right).$$

If $\left(\frac{k+3}{2}\right)$ is odd, then,

$$f(f(f(k)))=f\left(\frac{k+3}{2}\right)=\frac{k+3}{2}+3=\frac{k+9}{2}=27$$

which, on solving gives $k=45$. This is not possible because

$$f(f(f(45)))=f(f(48))=f(24)=12.$$

Hence, $\left(\frac{k+3}{2}\right)$ is even and

$$f(f(f(k)))=f\left(\frac{k+3}{2}\right)=\frac{k+3}{4}=27$$

implying that $k=105$. Checking, we find that

$$f(f(f(105)))=f(f(108))=f(54)=27.$$

The sum of the digits of k is $1+0+5=6$.

13. Let D be the diameter of the wheel before it is reduced. The number of revolutions N_1 before the diameter is reduced are $N_1=\frac{1\,\text{mile}}{\pi D}$. The number of revolutions N_2 after the diameter is reduced by $\frac{1}{2}$ an inch is $N_2=\frac{1\,\text{mile}}{\pi\left(D-\frac{1}{2}\right)}$. The number of revolutions increase by

$$\begin{aligned} N_2-N_1 &= \frac{1}{\pi\left(D-\frac{1}{2}\right)}-\frac{1}{\pi D}=\frac{1}{2\pi D\left(D-\frac{1}{2}\right)}. \\ \frac{N_2-N_1}{N_1} &= \frac{1}{2\left(D-\frac{1}{2}\right)}=\frac{1}{2\times 24\frac{1}{2}}=\frac{1}{49}. \end{aligned}$$

The revolutions increase by $\frac{100}{49} = 2\%$.

14. Let t_1, t_2 and t_3 be the number of hours respectively that the car travels forwards, back to pick up, then forward to the destination. During this time, Harry, traveled forwards a distance given by $25t_1 + 5t_2 + 5t_3 = 100$ while Dick traveled forwards a distance of $5t_1 + 5t_2 + 25t_3 = 100$. On the other hand, Tom traveled a distance of $25t_1 + 25t_3$ forwards with an overlap for a distance of $25t_2$. Therefore, net forward distance traveled by tom is $25t_1 + 25t_3 - 25t_3 = 100$. We then solve the following equations

$$\begin{aligned} 25t_1 + 5t_2 + 5t_3 = 100 \quad &\Rightarrow \quad 5t_1 + t_2 + t_3 = 20 \qquad (15) \\ 5t_1 + 5t_2 + 25t_3 = 100 \quad &\Rightarrow \quad t_1 + t_2 + 5t_3 = 20 \qquad (16) \\ 25t_1 + 25t_3 - 25t_3 = 100 \quad &\Rightarrow \quad t_1 - t_2 + t_3 = 4 \qquad (17) \end{aligned}$$

Equation (16)+ (17) and Equation (15)- (16)yield

$$\begin{aligned} t_1 + 3t_3 &= 12 \\ t_1 - t_3 &= 0 \end{aligned}$$

whose solution is $t_1 = t_3 = 3$. Using Equation (15) generates $t_2 = 2$. Hence, $t_1 + t_2 + t_3 = 8$ hours.

15. Draw segment FC. Note that $\angle CFD$ is a right angle since arc CFD is a semi circle. Then, the right angled triangles DOE and DFC are similar. If we let $DO = r$. Then,

$$\frac{\overline{DO}}{\overline{DF}} = \frac{\overline{DE}}{\overline{DC}} \Rightarrow \frac{r}{8} = \frac{6}{2r} \Rightarrow 2r^2 = 48 \Rightarrow r^2 = 24.$$

Then, the area of the circle is $\pi r^2 = 24\pi cm^2$.

NMC 2010 A-LEVEL PAPER 2 SOLUTIONS

SECTION A

1. Let $y = x^2$. Then, $x^2 + \frac{1}{x^2} = 3 \;\Rightarrow\; y + \frac{1}{y} = 3 \;\Rightarrow y^2 - 3y + 1 = 0 \;\Rightarrow\; y = \frac{3 \pm \sqrt{5}}{2}$.

$$\begin{aligned}
\frac{(x+1)^2}{x^2} &= \left(1 + \frac{1}{x}\right)^2 \\
&= 1 + \frac{1}{x^2} + \frac{2}{x} \\
&= 1 + \frac{2}{3 \pm \sqrt{5}} \pm \frac{2\sqrt{2}}{\sqrt{3 \pm \sqrt{5}}} \\
&= \frac{3 \pm \sqrt{5} + 2 \pm 2\sqrt{2} \times \sqrt{3 \pm \sqrt{5}}}{3 \pm \sqrt{5}} \\
&= \frac{5 \pm \sqrt{5} \pm 2\sqrt{6 \pm 2\sqrt{5}}}{3 \pm \sqrt{5}}.
\end{aligned}$$

Note that in all there are four solutions.

2.

$$\begin{aligned}
-\frac{4}{3} + \frac{1}{3^2} - \frac{4}{3^3} + \frac{1}{3^4} - \frac{4}{3^5} + \ldots &= -4\left[\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \ldots\right] + \left[\frac{1}{3^2} + \frac{1}{3^4} + \ldots\right] \\
&= -4\left[\frac{\frac{1}{3}}{1 - \frac{1}{9}}\right] + \left[\frac{\frac{1}{3^2}}{1 - \frac{1}{3^2}}\right] = -\frac{3}{2} + \frac{1}{8} = -\frac{11}{8}.
\end{aligned}$$

3.

Fast Clock	$8:00PM$	$6:58PM$	$5:56PM$	$4:54PM$	$3:52PM$	$2:50PM$
Slow Clock	$7:00PM$	$6:04PM$	$5:08PM$	$4:12PM$	$3:16PM$	$2:20PM$
Fast Clock	$1:48PM$	$12:46PM$	$11:44AM$	$10:42AM$	$9:40AM$	
Slow Clock	$1:24PM$	$12:28PM$	$11:32AM$	$10:36AM$	$9:40AM$	

The correct time on both clocks was set at $9:40AM$.

4. $x-2$ divides $x^4-3ax^2+(5a-2)x-16$ implies that $x=2$ solves the equation $x^4-3ax^2+(5a-2)x-16=0$. That is, $16-12a+2(5a-2)-16=0 \Rightarrow a=-2$.

5. Since $f(4)=2$, we have $f(0)=f(2-2)=f(2\times 2)=2$ by taking $m=n=2$. Likewise,

$$\begin{aligned} f(-3) &= f(1\times -3)=f(1--3)=f(4)=2. \\ f(0) &= f(3\times 0)=f(3-0)=f(3)=2. \\ f(0) &= f(1\times 0)=f(1-0)=f(1)=2. \\ f(6) &= f(3\times 2)=f(3-2)=f(1)=2. \end{aligned}$$

Thus, $f(-3)+f(6)=2+2=4$.

6. Let us show first of all that $\log_{ab}x=\frac{\log_a x}{\log_a(ab)}$. Let $y=\log_{ab}x \Rightarrow (ab)^y=x$. Taking logs to base a gives $y\log_a(ab)=\log_a x$ from which we deduce that $y(\log_{ab}x)=\frac{\log_a x}{\log_a(ab)}$. Therefore,

$$\frac{\log_a x}{\log_{ab}x} = \frac{\log_a x}{\frac{\log_a x}{\log_a(ab)}}=\log_a(ab)=\log_a a+\log_a b=1+4=5.$$

7. Only pairs $(1,13),(13,1),(0,-12),(-12,0)$ and $(-2,-2)$ do satisfy the relation $y=x+12x-1$.

8. You can see that $3^2+4^2+5^2+12^2=5^2+13^2$. Therefore $a=5$ and $b=13$.

9. Let $P(k)$ be the probability of rolling side $k, k=1,2,\ldots,6$ and let $P(1)=p$. Since $\sum_{k=1}^{6}=1$ we have that $p+p+3p+p+p+3p=1 \Rightarrow p=\frac{1}{10}$. Therefore, $P(3)=3p=\frac{3}{10}$.

10.

$$\text{Area of triangle} = \frac{1}{2}\cdot b\cdot h=\frac{1}{2}(2-x)(2-x)=\frac{1}{2}(2-x)^2.$$

$$\text{Area of Letter } L = 2x+x(2-x).$$

Since the two areas are equal, we have

$$2x+(2-x)=\frac{1}{2}(2-x)^2 \Rightarrow 3x^2-12x+4=0 \Rightarrow x=\frac{6\pm 2\sqrt{6}}{3}.$$

SECTION B : 10 marks each

11. (a) When $2=\sqrt{\frac{1+x}{1-x}} \Leftrightarrow x=\frac{3}{5}$. Therefore, $g(2)=5\times\frac{3}{5}=3$.

(b) Let $\sqrt{a}+\sqrt{b}=\sqrt{19+\sqrt{336}}$. We want to find a and b.

$$(\sqrt{a}+\sqrt{b})^2 = (\sqrt{19+\sqrt{336}})^2$$

$$\begin{aligned}
&\Leftrightarrow \; a+b+2\sqrt{ab} = 19+\sqrt{336}\\
&\Leftrightarrow \; a+b+2\sqrt{ab} = 19+\sqrt{336}\\
&\Leftrightarrow \; a+b = 19 \text{ and } \sqrt{ab} = \sqrt{84}
\end{aligned}$$

By letting $b=\frac{84}{7}$, we get a quadratic equation $a+\frac{84}{a}=19$ whose solution is 12 or 7. Taking $a=12$ gives $b=7$ and taking $a=7$ gives $b=12$. The value of $a-b$ is either 5 or -5.

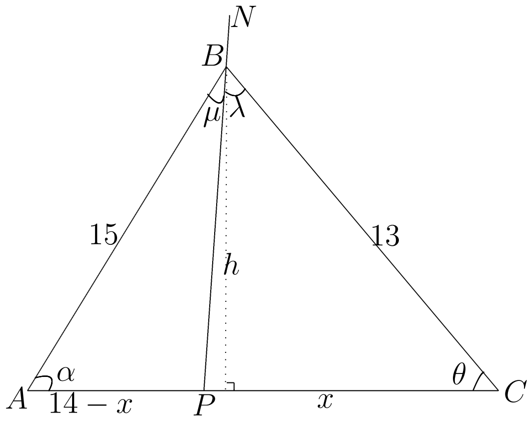
12. If the delegation consists of 5 members, then, there are $\binom{2}{1}\binom{6}{4}+\binom{2}{2}\binom{6}{4}=50$ ways to select them.

    If the delegation consists of 6 members, then, there are $\binom{2}{1}\binom{6}{5}+\binom{2}{2}\binom{6}{4}=27$ ways to select them. There are $50+27$ ways of selecting a delegation of either 5 or 6.

13. Let the distance of the line on the road be D. Jill's speed of work is $\frac{D}{J}$ per hour and Bill's speed is$\frac{D}{B}$. per hour. The combined rate of work for both $\frac{D}{B}+\frac{J}{B}$ per hour. Since Jill starts to paint one hour later, the section of the line painted by both Bill and Jill is $D-\frac{D}{B}$. Let t be the time it takes to paint this remaining portion. Then $\left(\frac{D}{B}+\frac{D}{J}\right)t = D-\frac{D}{B} \;\Rightarrow\; t=\frac{(B-1)J}{B+J}$. In total, Bill paints for $t+1=\frac{(B-1)J}{B+J}+1=\frac{B(J+1)}{B+J}$ hours.

14. The first marble falls to the ground after time $t=\sqrt{\frac{64}{16}}=2$ seconds. The second marble is dropped when the first one has fallen one foot. Time required to fall one foot is $\sqrt{\frac{1}{16}}=\frac{1}{4}$. So, by the time the first marble fall, the second will have traveled for $2-\frac{1}{4}=\frac{7}{4}$ seconds and it will have fallen a height h given by $h=16t^2=16\times\frac{49}{64}=49$. The distance left to hit the ground is therefore $64-49=15$ feet.

15.



Find h using triangles BMC and ABM.

$$h^2 = 13^2 - x^2 \tag{18}$$
$$h^2 = 15^2 - (14-x)^2 \tag{19}$$

Solving (18) and (19) gives $x = 5$ and $h = 12$.

$$\angle BCM = \sin^{-1}\left(\frac{12}{13}\right) = 67.3^0$$

$$\angle BAM = \sin^{-1}\left(\frac{12}{15}\right) = 53.1^0$$

$$\angle ABC = 180 - (67.3 + 53.1) = 59.6^0$$

$$\angle ABP = \angle PBC = \frac{1}{2}\angle ABC = 29.8^0.$$

$$\angle CPB = 180 - (29.8 + 67.3) = 82.9^0.$$

Distance $PM = \frac{12}{\tan 82.9} = 1.495$ and distance $AP = 14 - (5 + 1.495) = 7.505$.

NMC 2015 A-LEVEL PAPER 1 QUESTIONS

SECTION A

Qn1. Let a, b and c be real numbers such that $(a-6)^2 + (b-4)^2 + (c-5)^2 = 0$. Find the value of $a+b+c$.

Qn2. Let S_n and T_n be the respective sums of the first n terms of two arithmetic series. If $S_n : T_n = (7n+1) : (4n+27)$ for all n. Find the ratio of the eleventh term of the first series to the eleventh term of the second series.

Qn3. Find the number of terms in the simplified expansion of $\left((a+3b)^2(a-3b)^2\right)^9$.

Qn4. Find the sum of the maximum and the minimum values of $\sin^6 x + \cos^6 x$.

Qn5. Prove that if a, b and c are distinct real numbers such that $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0$, then

$\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2} = 0.$

Qn6. Find the number of pairs (x, y) of non-zero real numbers such that three of $x+y, x-y, xy$ and $\frac{x}{y}$ have the same value.

Qn7. Daniel has 3 red socks, 3 yellow socks and 3 blue socks. If he picks them without looking, what is the smallest number he must take in order to guarantee that he has four socks that form two pairs of socks of matching colours?

Qn8. Which numbers between 20 and 50 are exactly divisible by $3^{24} - 1$?

Qn9. A mathematical challenge book contains 213 problems from 12 years of competitions. At first there were 25 problems per year followed by 16 problems per year, and finally 20 per year. Find the number of years when there were 25 problems.

Qn10. Initially, 2014 is written on a blackboard. At any time, we may write a new integer on the blackboard, which is either 1 greater than the existing integer, or half of the existing even integer. Find the minimum number of integers on the blackboard when the integer 1 appears.

SECTION B

Qn11. Find all polynomials $P(x)$ that satisfy the equation $P(x^2) + 2x^2 + 10x = 2xP(x+1) + 3$.

Qn12. Suppose we perform the following four operations on a number h. Add 2, subtract 2, multiply by 2 and divide by 2, one after another, each exactly once, but not necessarily in that order of operation. With a clear explanation of the choice of sequence of these four operations on h, what is the largest number that we can end up with?

Qn13. An operation denoted by $*$ defines, for each pair of numbers (x, y), a number $x * y$ so that for all x, y and z, the identities $x * x = 0$ and $x * (y * z) = (x * y) + z$ hold. (Note that $+$ denotes ordinary addition of numbers). Find $2015 * 2000$.

Qn14. Suppose x, y and z are integers satisfying the following conditions.

(i) $x - y + z = 3$ $\quad$ (ii) $x > 10$ and $y \geq 8$ $\quad$ (iii) $y^2 + z^2 < 80$.

What is the largest value of x?

Qn15. (a) K, L, M and N are points on sides AB, BC, CD and DA, respectively, of the unit square $ABCD$ such that KM is parallel to BC and LN is parallel to AB. The perimeter of triangle KLB is equal to 1. What is the area of triangle MND?

(b) In triangle $ABC, AB = AC$. A line L is drawn through A parallel to BC. A circle of radius $1cm$ is drawn tangent to the line L, the extended line BC and to the circle inscribing triangle ABC. Determine the radius of the circle inscribing the triangle ABC.

NMC 2015 A-LEVEL PAPER 1 SOLUTIONS

SECTION A

Qn1. Since each of the squares is greater or equal to zero, the sum of these squares will be zero only if each of the squares is zero. Hence $(a-6)^2 = 0, (b-4)^2 = 0$ and $(c-5)^2 = 0$. This implies that $a = 6$, $b = 4$ and $c = 5$. Thus $a + b + c = 6 + 4 + 5 = 15$.

Qn2. Let a_1, a_2 denote the first terms and d_1, d_2 common difference in the Arithmetic Progressions with n^{th} sums S_n and T_n respectively. Then for all n,

$$\frac{S_n}{T_n} = \frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}.$$

The 11^{th} terms of these series are $u_{11} = a_1 + 10d_1$ and $v_{11} = a_2 + 10d_2$ respectively. The ratio

$$\frac{u_{11}}{v_{11}} = \frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$$

is an expression for $\frac{S_n}{T_n}$ for $n = 21$. Thus $\frac{u_{11}}{v_{11}} = \frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$.

Qn3. The binomial expansion of $(a+b)^n$ has $n+1$ terms. Hence we have $((a+3b)^2(a-3b)^2))^9 = ((a^2 - 9b^2)^2))^9 = (a^2 - 9b^2)^{18}$. Therefore the expansion has $18 + 1 = 19$ terms.

Qn4.

$$\begin{aligned}
(\sin^2 x + \cos^2 x)^3 &= \sin^6 x + 3\sin^4 x \cos^2 x + 3\sin^2 x \cos^4 x + \cos^6 x \\
(1)^3 &= \sin^6 x + 3\sin^4 x \cos^2 x + 3\sin^2 x \cos^4 x + \cos^6 x \\
\sin^6 x + \cos^6 x &= 1 - 3\sin^4 x \cos^2 x - 3\sin^2 x \cos^4 x \\
&= 1 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x) \\
&= 1 - 3\sin^2 x \cos^2 x \\
&= 1 - \frac{3}{4}(2\sin x \cos x)^2 = 1 - \frac{3}{4}\sin^2 2x.
\end{aligned}$$

The maximum value is 1 and occurs when $x = 0°$ and the minimum value is $\frac{1}{4}$ and occurs when $x = 45°$. The desired sum is $1 + \frac{1}{4} = \frac{5}{4}$.

Qn5. We have that

$$\begin{aligned} 0 &= \left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)\left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b}\right) \\ &= \frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2} + \frac{b+c}{(c-a)(a-b)} + \frac{c+a}{(a-b)(b-c)} + \frac{a+b}{(b-c)(c-a)} \\ &= \frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2} + \frac{b^2-c^2+c^2-a^2+a^2-b^2}{(a-b)(b-c)(c-a)} \\ &= \frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2} + \frac{0}{(a-b)(b-c)(c-a)}. \end{aligned}$$

Hence $\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2} = 0$ as required.

Qn6. Clearly $x + y \neq x - y$. We thus have $xy = \frac{x}{y} \implies xy^2 - x = 0$. So, $x(y^2 - 1) = 0$. Since x is non-zero, we must have that $y^2 - 1 = 0$ so that $y = \pm 1$. The only valid solution is $y = -1$. If $y = -1$, since $x - y = xy$, we have $1 + x = -x \implies x = \frac{-1}{2}$. Also from $x + y = xy$ we have $x - 1 = -x$ so that $x = \frac{1}{2}$. Hence the possible pairs are $(x, y) = (\frac{1}{2}, -1)$ and $(\frac{-1}{2}, -1)$.

Qn7. If Daniel picks 5 socks, he may have 3 red socks, 1 yellow and 1 blue or 1 red , 3 yellow and 1 blue socks or 1 red, 1 yellow and 3 blue socks. Each of these forms only one matching pair. If Daniel picks out 6 socks and leaves out 3 socks, he can leave out at least 2 socks of at most one colour. Hence of the six socks he takes, he can guarantee to have 2 matching pairs.

Qn8. $3^{24} - 1 = (3^{12} - 1)(3^{12} + 1) = (3^6 - 1)(3^6 + 1)(3^{12} + 1) = (3^3 - 1)(3^3 + 1)(3^6 + 1)(3^{12} + 1) = (26)(28)(3^6 + 1)(3^{12} + 1)$. Therefore $3^{24} - 1$ is a multiple of 26 and 28. Thus 26 and 28 are divisors of $3^{24} - 1$ that are between 20 and 50.

Qn9. Let x, y and z denote the respective numbers of years in which there were $25, 16$ and 20 problems. Then we have that $x + y + z = 12$ and $25x + 16y + 20z = 213$. Solving the two equations simultaneously and eliminating y gives $9x + 4z = 21$. Since both x and z are integer values, the equation $9x + 4z = 21$ has a solution only when $x = 1$ and $z = 3$. Therefore it is only one year that contained 25 problems.

Qn10. Since the objective is to downsize 2014 to 1, we should be halving as much as we can and increasing only when necessary. It follows that the most efficient sequence is $2014, 1007, 1008, 504, 252, 126, 63, 64, 32, 16, 8, 4, 2, 1$. Thus there are 14 numbers.

SECTION B

Qn11. $P(x)$ cannot be a constant polynomial since this would imply that the left hand side of the equation is a quadratic polynomial while the right hand side is linear.

Let $P(x)$ be of degree $n \geq 1$. Then the left hand side is of degree $2n$ and the right hand side degree $n+1$. We must have $2n = n+1$ which implies that $n = 1$. Hence $P(x)$ is of the form $ax + b$. Therefore

$$P(x^2) + 2x^2 + 10x = ax^2 + b + 2x^2 + 10x = 2x(a(x+1) + b) + 3 = 2ax^2 + 2(a+b)x + 3.$$

Comparing the constant terms, we have that $b = 3$.

Comparing the linear terms, we have $10 = 2a + 2b \Longrightarrow a = \frac{10-6}{2} = \frac{4}{2} = 2$.

Finally we check that the quadratic terms also agree. That is $a + 2 = 2a$ holds, which is the case. Hence the only polynomial that satisfies the functional equation is $P(x) = 2x + 3$.

Qn12. The following rules should be applied. $+$ and $-$ should not follow each other. Similarly, $\div$ and $\times$ should not follow each other as both would lead to the same number h we started with.

$+$ should preceed (come before) $\times$ since $(h+2)2 > 2h + 2$. Also $\div$ should preceed $+$ since $\frac{h}{2} + 2 > \frac{h+2}{2}$. This means that the order of the three signs is $\div, +, \times$.

It now remains to place $-$ either before $\div$ or after $\times$. Since $2h - 2 > (h-2)2 = 2h - 4$, it means $\times$ should preceed $-$. Since $\frac{h-2}{2} > \frac{h}{2} - 2$, then $-$ should preceed $\div$. Therefore the order of the signs is $-, \div, +, \times$ or $\div, +, \times, -$.

By comparing the numbers formed, $(\frac{h}{2} + 2)2 - 2 = h + 4 - 2 = h + 2$ and $\left(\frac{h-2}{2} + 2\right) 2 = h + 2$. The two numbers are the same, so $h + 2$ is the largest number we end up with.

Qn13. We have the following identities.

$$x * x = 0 \tag{1}$$
$$x * (y * z) = (x * y) + z \tag{2}$$

Put $x = y = z$ in equation (2). Then $x * (x * x) = (x * x) + x$.

From Equation 1, we have $x * 0 = 0 + x = x$ for all x.

Put $y = z$ in equation (2). Then $x * (y * y) = (x * y) + y$.

By equation (1), we have $x * 0 = (x * y) + y \Longrightarrow x * y = x - y$ for all x and y.

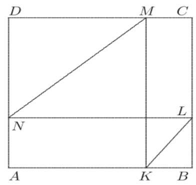
Therefore $2015 * 2000 = 2015 - 2000 = 15$.

Qn14. (i) $x - y + z = 3$ $\quad$ (ii) $x > 10$ and $y \geq 8$ $\quad$ (iii) $y^2 + z^2 < 80$.

Firstly from $y \geq 8$, we obtain that $y^2 \geq 64$. Using the inequality $y^2 + z^2 < 80$, we have that $z^2 < 80 - y^2 < 80 - 64 = 16$. That is $z^2 < 16$ and so $z \in \{-3, -2, -1, 0, 1, 2, 3\}$.

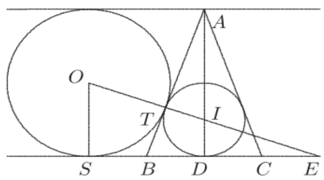
In addition, since $64 \leq y^2 < 80$ and y is an integer, then we must have that $y = 8$. But from equation (i), $10 < x = 3 + y - z = 3 + 8 - z \leq 11 - (-3)$ just by taking the lowest possible value of z, i.e $10 < x \leq 14$. Hence $x \leq 14$. The largest possible value of x is 14.

Qn15. (a) Let $AK = x$ and $AN = y$.



From $1-(1-x)-y = 1-KB-LB = KL = \sqrt{(1-x)^2+y^2}$, we have $x^2-2xy+y^2 = 1-2x+x^2+y^2$, which simplifies to $2x-2xy = 1 \Longrightarrow 2x(1-y) = 1$. Hence the area of triangle DMN is equal to $\frac{1}{2}DM \times DN = \frac{1}{2}x(1-y) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

(b) Let O be the centre of the larger circle and let this circle be tangent to BC at S. Let T be the point of common tangency of the two circles, and let I be the centre of the circle inscribing triangle ABC. Since $AB = AC$, I lies on the altitude AD and D is also the point of tangency of the incircle with BC.



Extend OI to meet BC at E. Let the inradius of ABC be r. Since $OS = 1$, then $AD = 2$. Triangles BAD and BET are congruent since $< ABD =< EBT$.
Now $< ADB = 90° =< ETB$ and $BD = BT$. Hence $ET = 2$ and $EI = 2-r$. Now triangles SOE and DIE are similar.
Hence $r = \frac{ID}{OS} = \frac{IE}{OE} = \frac{2-r}{3}$. It follows that $3r = 2-r \Longrightarrow r = \frac{1}{2}$.

NMC 2015 A-LEVEL PAPER 2 QUESTIONS

SECTION A

Qn1. Find the positive value of M for which $22222^2 \times 55555^2 = 10^2 \times M^2$?

Qn2. While Annet was reading a novel of less than 1000 pages, she noticed that the sum of the digits on the page she was reading was 19 and the sum of the digits on the next page was 2. What is the number of the page that Annet was reading?

Qn3. The sum of the two 5-digit numbers UMS14 and UMS15 is 123429. What is the value of U+M+S?

Qn4. Nineteen students, numbered 1 through 19 stood in a circle in a clockwise numerical order, all facing the centre. They began to count out loud in a clockwise order: The first called out the number 1, the second called out 2 and each student then called out the number 1 more than the number called to his right. What was the number of the student who called out 2015?

Qn5. Whenever brothers A, B and C are asked a question, TWO lie and ONE tells the truth. I asked who amongst them was the youngest. A said that he was not the oldest. B replied that C was the oldest while C said that A was the youngest. Who actually was the youngest among the three brothers?

Qn6. All but two of my math books are titled *Algebra*, all but two are titled *Trigonometry* and all but two are titled *Geometry*. I have atleast one math book titled *Algebra*. How many math books do I have?

Qn7. Let $F = \log\left(\frac{1+y}{1-y}\right)$. When each y in F is replaced by the expression $\frac{3y+y^3}{1+3y^2}$, a new function H

is formed. Find H in terms of F.

Qn8. Alex and Barbra both tried to solve the equation, $Ax^2 + Bx + C = 0$. However, each made exactly one error in copying the equation. The solutions of Alex's equation were -2 and $\frac{-3}{2}$ but his value of A was wrong. The solutions of Barbra's equation were 2 and 3, but her value of B was wrong. What are the solutions of the correct equation?

Qn9. Let $m + n = 1$ where $m = \log_{y^2} x$ and $n = \log_{x^2} y$, and $y > 0, y \neq 1, x \neq 1$. Find the value of x in terms of y.

Qn10. Johnson rides his bike 3 kilometres each day to school. The first kilometre is uphill and he rides at a constant speed of 4 km/hr. The second kilometre is downhill, where he moves at a constant speed that is an integer and is faster than his uphill speed and is less than 60 km/hr. For the final distance, he rides at a constant speed of 6 km/hr. If his average speed, in km/hr, for the entire trip to school is an integer, find the speed, in km/hr, that he moves with down the hill.

SECTION B

Qn11. The geometric sequence with n terms $t_1, t_2, \ldots, t_{n-1}, t_n$ has $t_1 \times t_n = 3$. Also, the product of all n terms equals 59049 (that is, $t_1 \times t_2 \times \ldots \times t_{n-1} \times t_n = 59049$). Determine the value of n.

Qn12. A telephone number has the form ABC-DEF-GHIJ, where each letter represents a different digit. The digits in each part of the number are in decreasing order; that is, $A > B > C, D > E > F$, and $G > H > I > J$. Furthermore, D, E, and F are consecutive even digits; G, H, I, and J are consecutive odd digits; and $A + B + C = 9$. What is the telephone number?

Qn13. When the mean, median, and mode of $10, 2, 5, 2, 4, 2, x$ are arranged in ascending order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

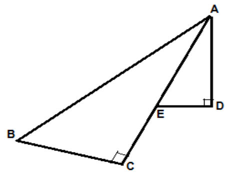
Qn14. The square below is a multiplicative magic square, that is, the product of the numbers in each row, column and diagonal is the same. If all the entries are positive integers, what is the sum of the possible values of g? [Hint: Express each unknown in terms of b].

50	b	c
d	e	f
g	h	2

Qn15. In the diagram below, $< ACB =< ADE = 90°$. If $\overline{AB} = 75$cm, $\overline{BC}$ =21cm, $\overline{AD} = 20$cm and $\overline{CE} = 47$cm. Determine the length of BD.

NMC 2015 A-LEVEL PAPER 2 SOLUTIONS

SECTION A



Qn1.

$$\begin{aligned} 22222^2 \times 55555^2 &= 10^2 \times M^2 \\ (2 \times 11111)^2 \times (5 \times 11111)^2 &= 10^2 \times M^2 \\ 2^2 \times 11111^2 \times 5^2 \times 11111^2 &= 10^2 \times M^2 \\ 10^2 \times 11111^4 &= 10^2 \times M^2 \\ M = 11111^2 &= 123454321. \end{aligned}$$

Qn2. The sum of digits of a 2-digit number cannot exceed $9 + 9 = 18$. Since the sum is 19, the page must have had at least 3 digits.
Since the digit sum of the next page is 2 and the book has less than 1000 pages, this page could be $110, 101$ or 200. Since $110 - 1 = 109$ and $1 + 0 + 9 = 10 \neq 19$, this page is discarded. Similarly page 101 is discarded and therefore this page must be 200 since $200 - 1 = 199$ and $1 + 9 + 9 = 19$ as required. The page that Annet was reading was $200 - 1 = 199$.

Qn3. Since the last two digits of UMS14 and UMS15 sum up to 29, we have that UMS+UMS=1234. Hence UMS=617, so, U=6, M=1, S=7. Thus, U+M+S= $6 + 1 + 7 = 14$.

Qn4. When 2015 is divided by 19, there is a quotient of 106 and a remainder of 1. So the student numbered 1 called out the number 2015.

Qn5. Suppose A lied, then he would be the oldest but in this case all the three would be liars hence, A told the truth and B and C lied. From $C's$ lie, we deduce that A was not the youngest, so, the youngest is either B or C. Since we know that B is lying, then C is not the oldest and therefore he/she must be the youngest.

Qn6. Suppose I have n books altogether, then $n - 2$ of them are algebra, $n - 2$ of them are trigonometry books and $n - 2$ of them are geometry books. Hence $n = 3(n - 2) \Longrightarrow n = 3$. Therefore I have 3 books altogether.

Qn7.

$$\frac{1 + \frac{3y+y^3}{1+3y^2}}{1 - \frac{3y+y^3}{1+3y^2}} = \frac{\frac{1+3y^2+3y+y^3}{1+3y^2}}{\frac{1+3y^2-3y-y^3}{1+3y^2}} = \frac{1 + 3y^2 + 3y + y^3}{1 + 3y^2 - 3y - y^3} = \frac{(1+y)^3}{(1-y)^3} = \left(\frac{1+y}{1-y}\right)^3.$$

$$H = \log\left(\frac{1+y}{1-y}\right)^3 = 3\log\left(\frac{1+y}{1-y}\right) = 3F.$$

Qn8. Alex's equation is $(x + 2)(2x + 3) = 0 \Longrightarrow 2x^2 + 3x + 4x + 6 = 0$ or $2x^2 + 7x + 6 = 0$.

Barbra's equation is $(x-2)(x-3)=0 \Longrightarrow x^2-5x+6=0$. From these two equations, $C=6, A=1$ and $B=7$.

The correct equation is $x^2+7x+6=0$. On solving the equation by factorisation method, we get $(x+6)(x+1)=0 \Longrightarrow x=-6$ or $x=-1$ as the solutions to the correct equation.

Qn9. $m=\log_{y^2} x \Longrightarrow x=y^{2m}$ and $n=\log_{x^2} y \Longrightarrow y=x^{2n}$. Thus $x=y^{2m}=(x^{2n})^{2m}=x^{4mn}$. Hence $1=4mn$ so that $n=\frac{1}{4m}$.

But $m+n=1$, so, $m+\frac{1}{4m}=1 \Longrightarrow 4m^2-4m+1=0$. Solving simultaneously gives $m=\frac{1}{2}$. Hence $n=\frac{1}{4\times 0.5}=\frac{1}{2}$. Therefore $x=y$

Qn10. First kilometre : Time $t_1=\frac{Distance}{speed}=\frac{1}{4}$hours.

Second kilometre: Time $t_2=\frac{Distance}{speed}=\frac{1}{x}$hours.

Third kilometre: Time $t_3=\frac{Distance}{speed}=\frac{1}{6}$hours.

$$Average\ speed=\frac{total\ distance}{total\ time}=\frac{3}{\frac{1}{4}+\frac{1}{x}+\frac{1}{6}}=\frac{3}{\frac{6x+24+4x}{24x}}=\frac{72x}{24+10x}$$

Required is to find an integer value x such that $4<x<60$ and satisfies the equation $\frac{72x}{24+10x}=k$ where k is an integer. This is possible if $24+10x$ is a factor of $72x$. This implies $24+10x$ is a multiple of 72. The possible multiples of 72 are $\{72,144,216,288,\cdots\}$.

Now, $24+10x\neq 72$ since $10x=58$ does not give an integer value of x.

Suppose $x=144$, then $24+10x=144 \Longrightarrow 10x=120$. Hence $x=12$ is the required integer value of x.

SECTION B

Qn11. Let the first term in the geometric progression be $t_1=a$ and r be the common ratio. Then the sequence is $a, ar, ar^2, \cdots ar^{n-1}$. Since $t_1t_n=3$ then $a.ar^{n-1}=3$ or $a^2r^{n-1}=3$.

$$\begin{aligned}
t_1t_2t_3\cdots t_n &= 59049, \quad \text{therefore}\\
(a)(ar)(ar^2)\cdots(ar^{n-1}) &= 59049\\
a^n rr^2r^3r^4\cdots r^{n-2}r^{n-1} &= 59049\\
a^n r^{1+2+3+4+\cdots+(n-1)} &= 59049\\
a^n r^{\frac{1}{2}(n-1)(n)} &= 59049.
\end{aligned}$$

From the equation $a^2r^{n-1}=3 \Longrightarrow (a^2r^{n-1})^n=3^n \Longrightarrow a^{2n}r^{(n-1)n}=3^n=(59049)^2$.

Thus $3^n=59049^2 \Longrightarrow n=\frac{2\times\log 59049}{\log 3}=20$. Hence the geometric progression has 20 terms.

Qn12. The last four digits $(GHIJ)$ are either 9753 or 7531, and the remaining odd digit (either 1 or 9) is A, B, or C. Since $A+B+C=9$, the odd digit among A, B, and C must be 1. Thus the sum of the two even digits in ABC is 8. The three digits in DEF are

864; 642; or 420, leaving the pairs 2 and 0, 8 and 0, or 8 and 6, respectively, as the two even digits in ABC. Of those, only the pair 8 and 0 has sum 8, so ABC is 810, and the required first digit is 8. The only such telephone number is $810 - 642 - 9753$.

Qn13. Suppose $x \leq 2$, then 2 would be both median and the mode, a contradiction, so $x > 2$. Suppose $2 < x < 4$, then the list is $2, 2, 2, x, 4, 5, 10$ of which the mode is 2, median is x and mean $\frac{25+x}{7}$. Thus the arithmetic progression is $2, x, \frac{25+x}{7}$. So,

$$\begin{aligned} x - 2 = \frac{25 - 6x}{7} \Longrightarrow 7x - 14 &= 25 - 6x \\ 13x = 39 \Longrightarrow x &= 3. \end{aligned}$$

Suppose $x \geq 4$, then the mode is 2, median is 4, and the mean is $\frac{25+x}{7}$ and the arithmetic progression $2, 4, \frac{25+x}{7}$.

$$4 - 2 = \frac{25 + x}{7} - 4 \Longrightarrow 6 = \frac{25 + x}{7} \Longrightarrow x = 42 - 25 = 17.$$

The sum of $x-$values is $3 + 17 = 20$.

Qn14. Since $100e = beh = ceg = def \Longrightarrow h = \frac{100}{b},\ \ g = \frac{100}{c}\ \ \ f = \frac{100}{d}$. By comparing rows 1 and 3, then we have $50 \times b \times c = 2 \times \frac{100}{b} \times \frac{100}{c} \Longrightarrow c = \sqrt{\frac{400}{b^2}} = \frac{20}{b} \Longrightarrow b \in \{1, 2, 4, 5, 10, 20\}$.

Thus the product of the entries in the first row is $50 \times b \times \frac{20}{b} = 1000$, so, $100e = 1000 \Longrightarrow e = 10$.
By comparing columns 1 and 3, then we have $50d \times \frac{100}{c} = 2 \times c \times \frac{100}{d}$ from which $d = \frac{c}{5} = \frac{4}{b} \Longrightarrow b \in \{1, 2, 4\}$. It follows from that $c = \frac{1000}{50b} \Longrightarrow c \in \{20, 10, 5.\}$
Now since $g = \frac{1000}{10c}$ then $g \in \{5, 10, 20\}$ and their sum is $5 + 10 + 20 = 35$.

Qn15. We use cosine law, on the triangle $\triangle ABD$ to determine the length of BD.

$$\begin{aligned} BD^2 &= AB^2 + AD^2 - 2(AB)(AD)\cos(< BAD) \\ \cos(< BAD) &= \cos(< BAC + < EAD) \\ &= \cos(< BAC)\cos(< EAD) - \sin(< BAC)\sin(< EAD) \\ &= \frac{AC}{AB} \times \frac{AD}{AE} - \frac{BC}{AB} \times \frac{ED}{AE} \end{aligned}$$

just because $\triangle ABC$ and $\triangle ADE$ are right angled. But $AB = 75$ and $BC = 21$.
By Pythagoras theorem, $AC = \sqrt{AB^2 - BC^2} = \sqrt{75^2 - 21^2} = 72$.
Now, $AC = 72, CE = 47 \Longrightarrow AE = AC - CE = 25$.
Since $AE = 25, AD = 20 \Longrightarrow ED = \sqrt{AE^2 - AD^2} = \sqrt{25^2 - 20^2} = \sqrt{225} = 15$.

$\cos(< BAD) = \frac{AC}{AB} \times \frac{AD}{AE} - \frac{BC}{AB} \times \frac{ED}{AE} = \frac{72\times20}{75\times25} - \frac{21\times15}{75\times25} = \frac{3}{5}$.

$BD = \sqrt{75^2 + 20^2 - (2 \times 75 \times 20 \times \frac{3}{5})} = \sqrt{4225} = 65$. Therefore $BD = 65$.

==

NMC 2016 A-LEVEL PAPER 1 QUESTIONS

SECTION A

Qn1. Suppose f is a function defined for all real numbers and

$$f(x) + 2f(3 - x) = x^2,$$

for all x. What is $f(1)$?

Qn2. The letters A, B, D and M represent the digits $1, 7, 8$ and 9, not necessarily in that order. Given that each of these digits is used exactly once, What is the largest possible sum of the three digit numbers BAD, DAM and MAD.

Qn3. Show that if $10a + b$ is a multiple of 7, then $a - 2b$ must also be a multiple of 7.

Qn4. In the sum below, each letter represents a different nonzero digit. If $R = 8$ and $F = 1$, determine the value of T and W.

$$TWO + TWO = FOUR$$

Qn5. Given that $\frac{a-b}{c-d} = 2$ and $\frac{a-c}{b-d} = 3$ for certain numbers a, b, c, d. Determine the value of $\frac{a-d}{b-c}$.

Qn6. Find the sum of all integers between 1 and 100 that leave remainder 2 upon division by 6.

Qn7. Ralph went to the store and bought 12 pairs of socks for a total of \$24. Some of the socks he bought cost \$1 a pair, some of the socks he bought cost \$3 a pair, and some of the socks he bought cost \$4 a pair. If he bought at least one pair of each type, how many pairs of \$1 socks did Ralph buy?

Qn8. Joshua writes the numbers $1, 2, 3, \ldots, 99, 100$. He marks out 1, skips the next number 2, marks out 3, and continues skipping and marking out the next number to the end of the list. Then he goes back to the start of his list, marks out the first remaining number 2, skips the next number 4, marks out 6, skips 8, marks out 10, and so on to the end. Joshua continues in this manner until only one number remains. What is that number?

Qn9. In the case of the Grand Hotel theft, inspector Derrick knows that the thief is either one or more of Cathy (room 23), Richards (room 27) and Miriam (room 36). He also knows that

• if Cathy or Richards is guilty of the theft, then Miriam is also guilty,

• Miriam and Richards are not both guilty,

• if Richards is not guilty, then so is Cathy.

What is the room number(s) of the thief or thieves?

Qn10. A boat travels around an equilateral triangle. The first $\frac{3}{4}$ of the distance requires 3.5 hours, while the last $\frac{3}{4}$ of the distance requires 4.5 hours. The middle third of the distance takes 10 minutes longer than the first third of the distance. How many hours did it take to travel around the triangle?

SECTION B

Qn11. Determine all pairs of real numbers a and x that satisfy the simultaneous equations
$5x^3 + ax^2 + 8 = 0$ and $5x^3 + 8x^2 + a = 0$.

Qn12. Let b be a number such that $0 < b \leq 1$ and $b^{2016} - 2b + 1 = 0$. Determine all possible values of $1 + b + b^2 + \cdots + b^{2016}$.

Qn13. Three distinct integers a, b and c satisfy the following three conditions.
•$abc = 17955$,
•a, b and c form an arithmetic sequence in that order, and
•$(3a + b), (3b + c), (3c + a)$ form a geometric sequence in that order. what is the value of $a + b + c$?

Qn14. (a) Amina read 15 books, one at a time. The first book took her 1 day to read, the second book took her 2 days to read, the third book took her 3 days to read, and so on, with each book taking her 1 more day to read than the previous book. Amina finished the first book on a Monday, and the second on a Wednesday. On what day of the week did she finish her 15th book?

(b) There are 300 white boxes and n red boxes in storage. Each box contains the same number of soccer balls. The total number of soccer balls in all of the boxes is $n^2 + 290n - 2490$. Determine n.

Qn15. ABC is a triangle with $< B = 90°, < A = 30°$. Points P, Q and R are on $\overline{AB}, \overline{BC}$ and $\overline{CA}$ respectively. PQR is an equilateral triangle, $\overline{BC} = 4cm$ and Q is the midpoint of $\overline{BC}$. Determine $\overline{PR}$.

NMC 2016 A-LEVEL PAPER 1 SOLUTIONS

SECTION A

Qn1. Substitute $x = 1$ and $x = 2$ into the equation to get

$$f(1) + 2f(2) = 1 \text{ and } f(2) + 2f(1) = 4.$$

solving these two equations simultaneously for $f(1)$ gives

$$\frac{1}{2} - \frac{1}{2}f(1) + 2f(1) = 4$$

$$\frac{3}{2}f(1) = \frac{7}{2}$$

$$f(1) = \frac{7}{3}.$$

Qn2. $BAD + DAM + MAD$ when expanded give us $(100B + 10A + D) + (100D + 10A + M) + (100M + 10A + D) = 102D + 101M + 100B + 30A$. Since we want the largest sum, we set $D = 9, M = 8, B = 7$ and $A = 1$ to get

$$(102 \times 9) + (101 \times 8) + (100 \times 7) + (30 \times 1) = 918 + 808 + 700 + 30 = 2456.$$

The largest possible sum of the three digit numbers is 2456.

Qn3. Since $10a + b$ is divisible by 7, then we write $10a + b = 7n$ where n is an integer. This means $b = 7n - 10a$. Substituting for b into $a - 2b$ we obtain $a - 2b = a - 2(7n - 10a) = a - 14n + 20a = 21a - 14n = 7(3a - 2n)$. Thus $a - 2b$ is a multiple of 7.

Qn4. $TWO + TWO = FOUR = 1OU8$. Clearly $2 \times O = x8$. If O is 4, then $x = 0$ or if O is 9, then $x = 1$. Suppose that $O = 9$, then $TW9 + TW9 = 19U8$. In this case, $2T = 19$. Since $2T$ cannot be odd, it implies that 1 has been carried forward and added to 18. So $2T + 1 = 19$ so that $T = 9$ which is a contradiction.

In this case we conclude that $O = 4$ and we have $TW4 + TW4 = 14U8$. Clearly $2T = 14$ hence $T = 7$. Further more $2W$ can take on values $2, 4, 6, 8$. However 4 and 8 are excluded since $R = 8$ and $O = 4$. Also 2 is excluded since $F = 1$. Therefore we conclude that $W = 3$.

Qn5.

$$a - b = 2c - 2d \Longrightarrow a + 2d = b + 2c \quad (1)$$
$$a - c = 3b - 3d \Longrightarrow a + 3d = 3b + c \quad (2)$$

$4\text{eqn}(3) - 3\text{eqn}(4)$ gives $a - d = -5b + 5c = -5(b - c)$.
Hence

$$\frac{a-d}{b-c} = -5.$$

Qn6. Let x be the integer between 1 and 100 such that $\frac{x}{6} = n \; rem \; 2$ where n is an integer. Then $x = 6n + 2;$ for $x = 0, 1, 2, 3, \cdots, 16$. Now

$$\sum_{n=0}^{16}(6n+2) = \sum_{n=0}^{16} 6n + \sum_{n=0}^{16} 2 = 6 \times \frac{16}{2} \times (16+1) + (2 \times 17) = 850.$$

Qn7. Let there be x pairs of \$1 socks, y pairs of \$3 socks and z pairs of \$4 socks. Then we have

$$x + y + z = 12 \tag{3}$$
$$x + 3y + 4z = 24 \tag{4}$$
$$x, y, z \geq 1 \tag{5}$$

Eqn(6)-Eqn(5) gives $2y + 3z = 12$.

It follows from the inequality 7 and the equation $2y + 3z = 12$ that y is a multiple of 3 and $2y$ is a multiple of 6. So since $0 < 2y < 12$, we must have that $2y = 6 \Longrightarrow y = 3$.

Since $y = 3$ then $z = \frac{12-6}{3} = 2$ and $x = 12 - y - z = 12 - 3 - 2 = 7$.
Thus Raphl bought 7 pairs of \$1 socks.

Qn8. Following the pattern of cancellation, we are crossing out;
Round 1: Every non-multiple of 2,
Round 2: Every non-multiple of 4,
Round 3: Every non-multiple of 8.
Following this pattern you are left with every multiple of 64 which is only 64 as the numbers are only 100.

Qn9. Suppose Richards is guilty of the theft then by the first statement Miriam is also guilty which contradicts the second statement. Thus Richards is not guilty. By the third statement, Cathy is also not guilty. Since at least one of the three persons is guilty, the only possibility is that Miriam is the thief.

Qn10. Let x, y, z denote the number of hours required to traverse each of the three sides. Then

$$x + y + \frac{z}{4} = \frac{7}{2}$$

$$\frac{x}{4} + y + z = \frac{9}{2}$$

$$y = x + \frac{1}{6}$$

Put the third equation into the first and second equations we get

$$\begin{aligned} x + x + \frac{1}{6} + \frac{z}{4} &= \frac{7}{2} \\ \frac{x}{4} + x + \frac{1}{6} + z &= \frac{9}{2} \end{aligned}$$

Hence we have

$$\begin{aligned} 2x + \frac{z}{4} &= \frac{20}{6} \\ \frac{5x}{4} + z &= \frac{26}{6} \end{aligned}$$

Solving these two equations simultaneously gives $x = \frac{4}{3}, z = \frac{8}{3}$ and $y = \frac{3}{2}$.
The total number of hours is

$$x + y + z = \frac{4}{3} + \frac{3}{2} + \frac{8}{3} = \frac{8 + 16 + 9}{6} = \frac{33}{6} = \frac{11}{2}\text{hours.}$$

SECTION B

Qn11. Subtract the two equations to get $ax^2 + 8 - 8x^2 - a = 0 \Longrightarrow (a - 8)(x^2 - 1) = 0$. Thus $a = 8$ or $x = 1$ or $x = -1$.

Suppose $a = 8$ then $5x^3 + 8x^2 + 8 = (x + 2)(5x^2 - 2x + 4) = 0$ and the second factor has no real roots since the discriminant $2^2 - 4(5)(4) = -76$ is negative. Hence $x = -2$ in this case.

If $x = -1$, we get $a = -5x^3 - 8x^2 = -5(-1)^3 - 8(-1)^2 = -3$.
If $x = 1$, we get $a = -5x^3 - 8x^2 = -5(1)^3 - 8(1)^2 = -13$.
Thus the three possible pairs are $(a, x) = (8, -2)$ or $(-3, -1)$ or $(-13, 1)$.

Alternatively

From $5x^3 + 8x^2 + a = 0$ we get $a = -5x^3 - 8x^2$. Thus

$$5x^3 + (-5x^3 - 8x^2)x^2 + 8 = -5x^5 - 8x^4 + 5x^3 + 8 = 0.$$

Factorising this polynomial equation
$-5x^5 - 8x^4 + 5x^3 + 8 = 5x^3(1 - x^2) + 8(1 - x^4) = 5x^3(1 - x^2) + 8(1 + x^2)(1 - x^2) = (1 - x^2)(5x^3 + 8x^2 + 8) = (1 - x)(1 + x)(x + 2)(5x^2 - 2x + 4) = 0$.

Again we find that $x = -1, +1, -2$ since the quadratic factor $5x^2 - 2x + 4$ has no real roots.

Qn12. Let b be a number such that $0 < b \leq 1$ and $b^{2016} - 2b + 1 = 0$. Then

• Assume $b = 1$, then

$$1 + b + b^2 + \cdots + b^{2016} = 1 + 1 + 1 + \cdots + 1 = 2017.$$

• Suppose $0 < b < 1$, then

$$\frac{b^{2016}-1}{b-1} = 1 + b + b^2 + \cdots + b^{2016}.$$

From $b^{2016} - 2b + 1 = 0$ we have

$$b^{2016} - 1 + 2 - 2b = 0 \Longrightarrow b^{2016} - 1 = -2(1-b).$$

Substituting for $b^{2016} - 1$ into $\frac{b^{2016}-1}{b-1} = 1 + b + b^2 + \cdots + b^{2016}$, we get

$$\frac{-2(1-b)}{b-1} = \frac{2(b-1)}{b-1} = 2 = 1 + b + b^2 + \cdots + b^{2016}.$$

Hence

$$1 + b + b^2 + \cdots + b^{2016} = 2.$$

Qn13. a, b, c form an arithmetic progression in that order which means $a = b - d$ and $c = b + d$. We note that $d \neq 0$ and is the common difference. Writing the terms of the GP in terms of b and d we have

$$\begin{aligned} 3a + b &= 3(b-d) + b = 4b - 3d, \\ 3b + c &= 3b + (b+d) = 4b + d, \\ 3c + a &= 3(b+d) + (b-d) = 4b + 2d. \end{aligned}$$

Since $3a + b, 3b + c$ and $3c + a$ form a GP then

$$\frac{3b+c}{3a+b} = \frac{3c+a}{3b+c}$$

$$\begin{aligned} (3b+c)^2 &= (3c+a)(3a+b) \\ 12bd &= -7d^2, \end{aligned}$$

$$d = \frac{-12}{7}b$$

This implies $a = b - d = b - \frac{-12}{7}b = \frac{19}{7}b$ and $c = b + d = b + \frac{-12}{7}b = \frac{-5}{7}b$. Since $abc = 17955$, we have $\frac{-12}{7}b \times b \times \frac{-5}{7}b = 17955 \Longrightarrow b^3 = -9261 \Longrightarrow b = -21$. Thus $a = \frac{19}{7}b = -57$ and $c = \frac{-5}{7}b = 15$. We conclude that $a + b + c = -57 + 15 + -21 = -63$.

Qn14. (a) The process took $1 + 2 + 3 + \cdots + 14 + 15 = 120$ days so that the last day was 119 days after the first day. Now since 119 is divisible by 7, both must have been on the same day of the week. Hence Amina finished her 15th book on Monday.

(b) The number of balls in each box is

$$\frac{n^2 + 290n - 2490}{n+300} = n - 10 + \frac{510}{n+300}$$

which is an integer. It follows that $\frac{510}{n+300}$ is an integer which is obviously positive and is less than $\frac{510}{300}$ i.e

it is less than 2. The only possibility is that $\frac{510}{n+300} = 1 \Longrightarrow n + 300 = 510$. Therefore $n = 210$. Hence there are 210 red boxes in the storage.

Qn15. Let $\overline{PB} = x, \overline{PQ} = y, \overline{BQ} = \overline{QC} = 2$. Since $< PQR = 60°$, it implies $< PQB+ < RQC = 120°$ so that $< ACB = 60°$. Further more, $< RQC+ < CRQ = 120°$ so that $< PQB =< CRQ$.

Let this angle be α and using the sine rule on $\triangle PBQ$, we have

$$\frac{PB}{\sin\alpha} = \frac{PQ}{\sin 90} \Longrightarrow \frac{x}{\sin\alpha} = y \Longrightarrow x = y\sin\alpha.$$

On $\triangle QCR$, we have

$$\frac{QC}{\sin\alpha} = \frac{RQ}{\sin 60} \Longrightarrow \frac{2}{\sin\alpha} \times \frac{y}{\frac{\sqrt{3}}{2}} \Longrightarrow y = \frac{\sqrt{3}}{\sin\alpha}.$$

Therefore

$$x = y\sin\alpha = \frac{\sqrt{3}}{\sin\alpha} \times \sin\alpha = \sqrt{3}.$$

We note by Pythagoras Theorem that

$$(\sqrt{3})^2 + 2^2 = (PQ)^2 \Longrightarrow PQ = \sqrt{3+4} = \sqrt{7} = PR.$$

Therefore $\overline{PR} = \sqrt{7}.$

NMC 2016 A-LEVEL PAPER 2 QUESTIONS

SECTION A

Qn1. Let a be a number such that $\log_a(10) + \log_a(10^2) + \log_a(10^3) + \ldots + \log_a(10^{10}) = 110$. What is the value of a?

Qn2. What is the smallest positive integer x for which $\frac{1}{32} = \frac{x}{10^y}$ for some positive integer y?

Qn3. A sequence is defined by $a_1 = 1$ and for $n \geq 1$ by;

$$a_{n+1} = \begin{cases} 0 & \text{if } a_n = 0 \text{ and } n \textit{ is } \text{odd} \\ 2 & \text{if } a_n = 0 \text{ and } n \textit{ is } \text{even} \\ 1 & \text{if } a_n = 1 \text{ and } n \textit{ is } \text{odd} \\ 0 & \text{if } a_n = 1 \text{ and } n \textit{ is } \text{even} \\ 1 & \text{if } a_n = 2 \end{cases}$$

How many of the numbers $a_1, a_2, a_3, \ldots, a_{2016}$ are equal to 2?

Qn4. How many members of the set $\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots, \frac{99}{7}\}$ are integer multiples of $\frac{2}{5}$?

Qn5. Find all ordered pairs (x, y) of real numbers satisfying the following equations:

$$x^{2016} + 2y^{2016} = x^{2015} + 2y^{2015} = x^{2014} + 2y^{2014}.$$

Qn6. Consider the equation $e^y + e^{-y} = 5$. The largest solution of the equation can be written in the form $y = \ln\left(\frac{A+\sqrt{B}}{C}\right)$. Find the value of $A + B + C$.

Qn7. Circles of radius 10 and 17 intersect at two points. The segment connecting these points of intersection has length 16. List all possible values for the distance between the centers of the circles?

Qn8. How many of the 2-digit numbers from 10 to 99 have the property that both digits are perfect squares? For example, 10 is the smallest such a number and 99 is the largest.

Qn9. Let $x = \sqrt{16 + \sqrt{16 + \sqrt{\cdots}}}$. What is the value of x?

Qn10. In a group of dogs and people, the number of legs was 28 more than twice the number of heads. How many dogs were there? [Assume none of the people or dogs is missing a leg.]

SECTION B

Qn11. On a particular street in Muyenga, there are exactly 14 houses, each numbered with an integer between 500 and 599, inclusive. The 14 house numbers form an arithmetic sequence in which 7 terms are even and 7 terms are odd. One of the houses is numbered 555 and none of the remaining 13 numbers has two equal digits. What is the smallest of the 14 house numbers?

Qn12. (a) Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$

(b) The function f, defined by

$$f(x) = \frac{ax + b}{cx + d}, \quad x \neq -\frac{d}{c}$$

where a, b, c, and d are nonzero real numbers, has the properties

$$f(19) = 19,\ f(97) = 97,\ \text{and}\ f(f(x)) = x,$$

for all values of x. Find the range of f.

Qn13. A strip of rubber is 80 cm long initially, and after every minute it is instantaneously and uniformly stretched by 40 cm. An ant moves at a rate of 42 cm per minute, starting at one end. Each time the strip is stretched, the ant is moved to a position on the modified strip proportional to its position on the strip before the stretching took place. How many minutes does it take the ant to cross the strip?

Qn14. Starting with an equilateral triangle, you inscribe a circle in the triangle, and then inscribe an equilateral triangle inside the circle. You then repeat this process four more times, each time inscribing a circle and then an equilateral triangle in the smallest triangle constructed up to that point, so that you end up drawing five triangles in addition to the one you started with. What is the ratio of of the area of the largest triangle to the area of the smallest triangles?

Qn15. Alistair, Conrad, Emma, and Salma compete in a three-sport race. They each swim 2 km, then bike 40 km, and finally run 10 km. Also, they each switch instantly from swimming to biking and from biking to running.

(a) Emma has completed $\frac{1}{13}$ of the total distance of the race. How many kilometers has she travelled?

(b) Conrad began the race at 8 : 00 a.m. and completed the swimming portion in 30 minutes. Conrad biked 12 times as fast as he swam, and ran 3 times as fast as he swam. At what time did he finish the race?

(c) Alistair and Salma also began the race at 8 : 00 a.m. Alistair finished the swimming portion in 36 minutes, and then biked at 28 km/h. Salma finished the swimming portion in 30 minutes, and then biked at 24 km/h. Alistair passed Salma during the bike portion. At what time did Alistair pass Salma?

NMC 2016 A-LEVEL PAPER 2 SOLUTIONS

SECTION A

Qn1.

$$\begin{aligned}
\log_a(10) + \log_a(10^2) + \log_a(10^3) + \cdots + \log_a(10^{10}) &= 110 \\
\log_a\left(10 \times 10^2 \times 10^3 \times \cdots \times 10^{10}\right) &= 110 \\
\log_a\left(10^{1+2+3+\cdots+10}\right) &= 110 \\
\log_a\left(10^{55}\right) &= 110 \\
55 \log_a 10 &= 110 \\
\log_a 10 &= 2
\end{aligned}$$

$$a^2 = 10 \quad \Rightarrow \quad a = \sqrt{10}$$

Qn2. Since $10^y \neq 0$, the equation $\frac{1}{32} = \frac{x}{10^y}$ is equivalent to $10^y = 32x$.

So the given question is equivalent to asking for the smallest positive integer x for which $32x$ equals a positive integer power of 10.

Now $32 = 2^5$ and so $32x = 2^5x$.

For $32x$ to equal a power of 10, each factor of 2 must be matched with a factor of 5.

Therefore, x must be divisible by 5^5 (that is, x must include at least 5 powers of 5), and so

$$x \geq 5^5 = 3125$$

But $32(5^5) = 2^5 5^5 = 10^5$, and so if $x = 5^5 = 3125$, then $32x$ is indeed a power of 10, namely 10^5. This tells us that the smallest positive integer x for which $\frac{1}{32} = \frac{x}{10^y}$ for some positive integer y is $x = 5^5 = 3125$.

Qn3. Starting with $a_1 = 1$, the sequence goes as

$$1, 1, 0, 0, 2, 1, 0, 0, 2, \ldots$$

and keeps repeating with period 4, that is $a_n = 2$ if and only if $n = 5, 9, \cdots$

1 greater than a positive multiple of 4 thus; a_{2016} will be the 503^{rd} 2 in $a_1, a_2, a_3, \ldots, a_{2016}$. So 503.

Qn4. Let $\frac{a}{7} = k \cdot \frac{2}{5}$ where k is an integer, then $5a = 14k \quad \Rightarrow \quad a$ must be a multiple of 14. The only multiples of 14 below 99 are

$$14, 28, 42, 46, 70, 84, 98$$

Thus there are 7 such numbers less than 100.

Qn5. Solving the three equations

$$\begin{aligned} x^{2016} + 2y^{2016} &= x^{2015} + 2y^{2015} \\ x^{2016} + 2y^{2016} &= x^{2014} + 2y^{2014} \\ x^{2015} + 2y^{2015} &= x^{2014} + 2y^{2014} \end{aligned}$$

Gives the solutions $\quad (x, y) = (0, 0),\ (0, 1),\ (1, 0),\ (1, 1)$

Qn6. Solving the equation

$$\begin{aligned} e^y + e^{-y} &= 5 \\ e^y + \frac{1}{e^y} &= 5 \\ (e^y)^2 + 1 &= 5e^y \\ (e^y)^2 - 5e^y + 1 &= 0 \end{aligned}$$

A quadratic equation in e^y, whose solution is given by

$$e^y = \frac{5 \pm \sqrt{25 - 4(1)(1)}}{2} = \frac{5 \pm \sqrt{21}}{2}$$

With a maximum value of

$$e^y = \frac{5 + \sqrt{21}}{2} \quad \Rightarrow \quad y = \ln\left(\frac{5 \pm \sqrt{21}}{2}\right)$$

The largest solution of the equation can be written in the form

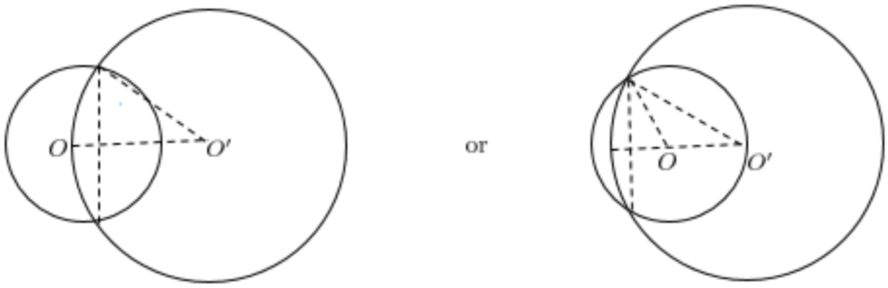
$$y = \ln\left(\frac{A + \sqrt{B}}{C}\right).$$

To have the value of $A + B + C = 5 + 21 + 2 = 28$.

Qn7. There are two options of drawing the two circles The line connecting the centers O and O' is the perpendicular bisection of this chord, intersecting it at point P.

If O and O' are centers of the circles, then

$$OO' = PO' \pm OP$$



and OP and PO' are bases of right triangles with height 8 and hypotenuse 10 and 17 respectively.

By Pythagoras theorem, OP and PO' are 6 and 15 respectively. Thus

$$OO' = PO' \pm OP = 15 \pm 6 = 9 \text{ units} \quad , \ 2 \text{ units}$$

Qn8. The only perfect square single digit numbers are $0, 1, 4, 9$

A 2-digit number can be formed from the set $\{0, 1, 4, 9\}$ only starting with $1, 4, 9$. That is three possible choices to start a 2-digit perfect square.

The next digit could take on any of the four digits since repetition is allowed. That is

$$3 \times 4 = 12$$

Qn9. The value of x can be found by solving the algebraic equation $x = \sqrt{16 + x}$. This gives $x^2 - x - 16 = 0$, and the positive root given by the quadratic formula is

$$x = \frac{1}{2} + \frac{\sqrt{65}}{2}$$

The other root is extraneous.

Remark: For this problem to have a real solution, it is essential for the associated sequence

$$a_{n+1} = \sqrt{16 + a_n}$$

to converge. To show this sequence does converge we use the fact that every nonempty set of real numbers that is bounded above has a least upper bound. That the sequence $\{a_n\}$ is increasing and bounded above (by 5) follows by an easy induction proof.

Qn10. If there were only people, there would be exactly twice the number of legs as heads. Each dog contributes two extra legs over and above this number. So 28 extra legs means 14 dogs.

SECTION B

Qn11. Since this arithmetic sequence of integers includes both even and odd integers, then the common difference in the sequence must be odd. (If it were even, then all of the terms in the sequence would be even or all would be odd.)

Suppose that this common difference is d. Note that 555 is one of the terms in the sequence.

If $d = 1$, then either or both of 554 and 556 would be in the sequence. Since no term other than 555 can contain repeated digits, then $d \neq 1$.

If $d = 3$, then either or both of 552 and 558 would be in the sequence. Since no term other than 555 can contain repeated digits, then $d \neq 3$.

If d = 5, then 550 cannot be in the sequence, so 555 would have to be the smallest term in the sequence. This would mean that 560 and 565 were both in the sequence.

Since no term other than 555 can contain repeated digits, then $d \neq 5$.

We can also rule out the possibility that $d \geq 9$: [Since there are 14 terms in the sequence and the common difference is d, then the largest term is 13d larger than the smallest term. Since all terms are between 500 and 599, inclusive, then the maximum possible difference between two terms is 99. This means that $13d \leq 99$. Since d is an odd integer, then cannot be 9 or greater.]

Since d is odd, d cannot be 1, 3 or 5, and $d \leq 7$, then it must be the case that $d = 7$. Can we actually construct a sequence with the desired properties? Starting with 555 and adding and subtracting 7s, we get the sequence

$$506,\ 513,\ 520,\ 527,\ 534,\ 541,\ 548,\ 555,\ 562,\ 569,\ 576,\ 583,\ 590,\ 597$$

This sequence has 14 terms in the desired range, including 7 even and 7 odd terms, and no term other than 555 has repeated digits. (Note that this sequence cannot be extended in either direction and stay in the desired range.) Therefore, the smallest of the 14 house numbers must be 506.

Qn12. (a) By setting $2^x = a$ and $3^x = b$, the equation becomes - *common parameters*

$$\begin{aligned} \frac{a^3+b^3}{a^2b+b^2a} &= \frac{7}{6} \\ \frac{a^2-ab+b^2}{ab} &= \frac{7}{6} \\ 6a^2-13ab+6b^2 &= 0 \\ (2a-3b)(3a-2b) &= 0 \end{aligned}$$

Therefore $2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$, which implies that $x = -1$ and $x = 1$

It is easy to check that both $x = -1$ and $x = 1$ satisfy the equation

(b) For all x, - *we start with a more general option*

$$\begin{aligned} \frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} &= x \\ \frac{(a^2+bc)x+b(a+d)}{c(a+d)x+bc+d^2} &= x \\ c(a+d)x^2+(d^2-a^2)x-b(a+d) &= 0, \end{aligned}$$

Wrong: equating coefficients of x,
Wrong: solving a quadratic in x, *no need for* x

But

$$\begin{aligned} c(a+d)x^2+(d^2-a^2)x-b(a+d) &= 0 \\ (a+d)\left[cx^2+(d-a)x-b\right] &= 0 \end{aligned}$$

which implies that $c(a+d)=0$. Since $c \neq 0$, we must have $a=-d$.

The conditions $f(19)=19$ and $f(97)=97$ lead to the equations

$$19^2c = 2\cdot 19a+b \text{ and } 97^2c = 2\cdot 97a+b$$

Hence $(97^2-19^2)c = 2(97-19)a$
It follows that $a=58c$, which in turn leads to $b=-1843c$. Therefore

$$f(x)=\frac{58x-1843}{x-58}=58+\frac{1521}{x-58}$$

which never has the value 58.

Thus the range of f is $\mathbb{R}-\{58\}$

Qn13. After one minute, it will be $\frac{42}{80}$ of the length on the rubber strip.

After stretching, it will be at $\frac{63}{120}$ because he stretch is by $40cm$ from $80cm$ which imply an increase by $\frac{1}{2}$ of the length. So the proportional stretch is $\frac{1}{2}\times 42=21$, so $42+21=63$.

After the second minute, it will be at $\frac{105}{120}$, that is 42×63 along the rubber.

After the second stretching, it will be at $\frac{140}{160}$. Since the stretching is from 120 to 160 which is $\frac{1}{3}$ of 120, the proportional stretch is $\frac{1}{3}\times 105=35$, added to 105 to get 140.

The remaining $\frac{20}{160}$ is covered in $\frac{20}{42}$ minutes, that is a total time is $2\frac{20}{40}$ minutes.

Qn14. Each time you draw a new triangle, it has its vertices at the midpoints of the three sides of the old triangle. The new triangle is similar to the old triangle, and indeed this process divides the old triangle into four smaller triangles which are congruent and which each have area $\frac{1}{4}$ of the old triangle. The answer is therefore $4^5=1024$.

Qn15. (a) When Emma has completed $\frac{1}{13}$ of the total distance, she has travelled $\frac{1}{13}\times 52=4$ km

(b) Since Conrad completed the 2 km swim in 30 minutes (which is half an hour), then his speed was $2\div\frac{1}{2}=4$ km/h Since Conrad biked 12 times as fast as he swam, then he biked at $12\times 4=48$ km/h.
Since Conrad biked 40 km, then the bike portion took him $\frac{40}{48}=\frac{5}{6}$ *hours*.
Since 1 hour equals 60 minutes, then the bike portion took him $\frac{5}{6}\times 60=50$ minutes.
Since Conrad ran 3 times as fast as he swam, then he ran at $3\times 4=12$ km/h.
Since Conrad ran 10 km, then the running portion took him $\frac{10}{12}=\frac{5}{6}$ hours.
This is again 50 minutes.

Therefore, the race took him $30+50+50=130$ minutes, or 2 hours and 10 minutes. Since Conrad began the race at $8:00$ a.m., then he completed the race at $10:10$ a.m.

(c) Suppose that Alistair passed Salma after t minutes of the race. Since Alistair swam for 36 minutes, then he had biked for $t-36$ minutes (or $\frac{t-36}{60}$ hours)when they passed.

Since Salma swam for 30 minutes, then she had biked for $t-30$ minutes (or $\frac{t-30}{60}$ hours) when they passed.

When Alistair and Salma passed, they had travelled the same total distance. At this time, Alistair had swum 2 km. Since he bikes at 28 km/h, he had biked $28\times\frac{t-36}{60}$ km.

Similarly, Salma had swum 2 km. Since she bikes at 24 km/h, she had biked $24\times\frac{t-30}{60}$ km.

Since their total distances are the same, then

$$\begin{aligned}
2 + 28 \times \frac{t-36}{60} &= 2 + 24 \times \frac{t-30}{60} \\
28 \times \frac{t-36}{60} &= 24 \times \frac{t-30}{60} \\
28(t-36) &= 24(t-30) \\
7(t-36) &= 6(t-30) \\
7t - 252 &= 6t - 180 \\
t &= 72
\end{aligned}$$

Therefore, Alistair passed Salma 72 minutes into the race. Since the race began at 8 : 00 a.m., then he passed her at 9 : 12 a.m.

Alternatively, we could notice that Salma started the bike portion of the race 6 minutes before Alistair. Since 6 minutes is $\frac{1}{10}$ of an hour, she biked a distance of $\frac{1}{10} \times 24 = 2.4$ km before Alistair started riding.

In other words, she has a 2.4 km head start on Alistair. Since Alistair bikes 4 km/h faster than Salma, it will take $\frac{2.4}{4} = 0.6$ hours for him to make up this difference.

Since 0.6 hours equals $0.6 \times 60 = 36$ minutes and he swam for 36 minutes before starting the bicycle portion, he will catch up a total of $36 + 36 = 72$ minutes into the race.

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 1.

SATURDAY 18TH MARCH, 2017 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*

(b) *There are* **two** *sections and each section carries* 50 *marks. Attempt all questions in exactly three hours fifteen minutes* ($9:00am$ *to* $12:15pm$)*. Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*

(c) *Answer scripts of only qualified students should be returned to the contest coordinator, latest* **Monday 20th March, 2017**.

(d) **ALL** *participants who qualify for Paper* 2 *are cordially invited to the certificate and prize giving ceremony on Saturday* 29*th July,* 2017 *at NOON at Makerere University, Kampala.*

(e) *National Mathematics Contest* 2017 *Paper* 2 *will be done on* 17*th June,* 2017 *at various centres.*

SECTION A : 5 marks each

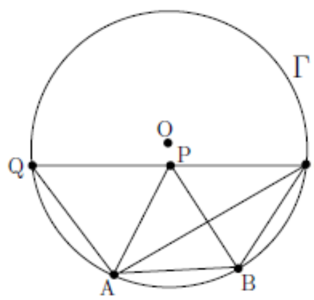
1. Determine the number a such that $\log_a(10) + \log_a\left(10^2\right) + \cdots + \log_a\left(10^{10}\right) = 110$.

2. Let (a, b, c) be a Pythagorean triple, i.e., a triplet of positive integers with $a^2 + b^2 = c^2$. Prove that $\left(\frac{c}{a} + \frac{c}{b}\right)^2 > 8$.

3. Solve the following system of equations. Carefully justify your answer.

$$\frac{4x^2}{1+4x^2} = y, \quad \frac{4y^2}{1+4y^2} = z, \quad \frac{4z^2}{1+4z^2} = x$$

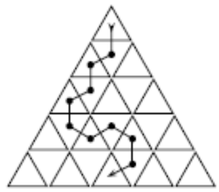
4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let another circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let CP meet Γ again at Q. Prove that $PQ = r$.



5. Find all real solutions to the equation $4x^2 - 40[x] + 51 = 0$. Here, if x is a real number, then $[x]$ denotes the greatest integer that is less than or equal to x say $[3.7] = 3, [-2.1] = -3, [5] = 5$.

6. Evaluate the sum $\displaystyle\sum_{n=1}^{1994}(-1)^n \frac{n^2+n+1}{n!}$.

7. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



8. Twenty five boys and twenty five girls sit around a table. Prove that, it is always possible to find a person both of whose neighbours are girls.

9. Let $ABCD$ be a convex quadrilateral with

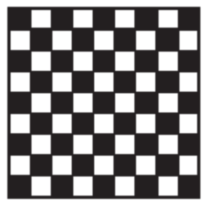
$$\angle CBD = 2\angle ADB, \angle ABD = 2\angle CDB, \text{and} \quad AB = CB.$$

Prove that $AD = CD$.

10. If α, β, γ are roots of $x^3 - x - 1 = 0$, compute $\dfrac{1+\alpha}{1-\alpha} + \dfrac{1+\beta}{1-\beta} + \dfrac{1+\gamma}{1-\gamma}$.

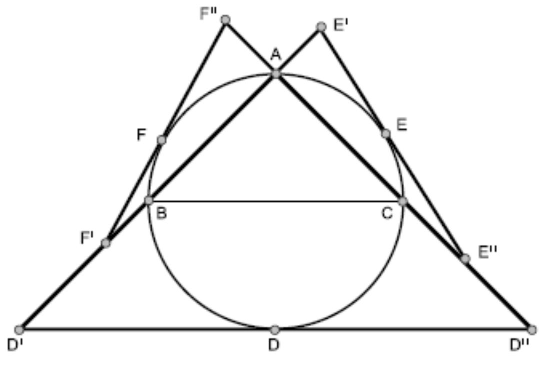
SECTION B : 10 marks each

11. How many ways can 8 mutually non-attacking rooks be placed on the 9×9 chessboard (shown here)



so that all 8 rooks are on squares of the same colour? [Two rooks are said to be attacking each other if they are placed in the same row or column of the board.]

12. The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc AB and arc AC are supplementary. To each of the three arcs, we draw a tangent such that its point of tangency is the midpoint of that portion of the tangent intercepted by the extended lines AB and AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where the tangent at D intersects the extended lines AB and AC. Similarly for E on arc AC and F on arc AB. Prove that triangle DEF is equilateral.



13. Let $\mathbb{N}$ be a set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that $xf(y) + yf(x) = (x+y)f\left(x^2+y^2\right)$ for all x and y in $\mathbb{N}$.

14. Find all real positive solutions (if any) to

$$\begin{aligned} x^3 + y^3 + z^3 &= x + y + z \\ x^2 + y^2 + z^2 &= xyz \end{aligned}$$

15. (a) Show that at any party there are two people who have the same number of friends at the party (assume that all friendships are mutual).

(b) Seven darts are thrown onto a circular dartboard of radius 10 units. Show that there will always be two darts which are at most 10 units apart?

SECTION A : 5 marks each

1. By the rules for logs, the expression given is

$$\begin{aligned}\log_a\left(10\cdot 10^2\cdot 10^3\cdots 10^{10}\right) &= \log_a\left(10^{1+2+3+\cdots+10}\right)\\ &= \log_a\left(10^{\frac{10\cdot 11}{2}}\right)\\ &= \log_a\left(10^{55}\right)\\ &= 55\log_a(10)\\ &= \frac{55}{\log_{10}(a)}\end{aligned}$$

Therefore, from

$$\frac{55}{\log_{10}(a)} = 110 \quad\Rightarrow\quad \log_{10}(a) = \frac{1}{2}$$

so that

$$a = \sqrt{10}$$

2. *Alternative Solution 1*

Let (a, b, c) be a Pythagorean triple. View a, b as lengths of the legs of a right angled triangle with hypotenuse of length c; let θ be the angle determined by the sides with lengths a and c. Then

$$\begin{aligned}\left(\frac{c}{a}+\frac{c}{b}\right)^2 &= \left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)^2\\ &= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{(\sin\theta\cos\theta)^2}\\ &= 4\left(\frac{1+\sin 2\theta}{\sin^2 2\theta}\right)\\ &= \frac{4}{\sin^2 2\theta}+\frac{4}{\sin 2\theta}\end{aligned}$$

Note that because $0 < \theta < 90°$, we have $0 < \sin^2\theta \leq 1$, with equality only if $\theta = 45°$. But then $a = b$ and we obtain $\sqrt{2} = c/a$, contradicting a, c both being integers. Thus, $0 < \sin 2\theta < 1$ which gives $\left(\frac{c}{a}+\frac{c}{b}\right)^2 > 8$.

Alternative Solution 2

Defining θ as in Solution 1 above, we have $\frac{c}{a}+\frac{c}{b} = \sec\theta+\csc\theta$. By the AM-GM inequality, we have $\frac{(\sec\theta+\csc\theta)}{2} \geq \sqrt{\sec\theta\csc\theta}$. So

$$\frac{c}{a}+\frac{c}{b} \geq \frac{2}{\sqrt{\sec\theta\csc\theta}} = \frac{2\sqrt{2}}{\sqrt{\sin 2\theta}} \geq 2\sqrt{2}$$

Since a, b, c are integers, we have $\left(\frac{c}{a}+\frac{c}{b}\right) > 2\sqrt{2}$ which gives $\left(\frac{c}{a}+\frac{c}{b}\right)^2 > 8$.

Alternative Solution 3

By simplifying and using the AM-GM inequality,

$$\left(\frac{c}{a}+\frac{c}{b}\right)^2 = c^2\left(\frac{a+b}{ab}\right)^2 = \frac{(a62+b^2)(a+b)^2}{a^2b^2} \geq \frac{2\sqrt{a^2b^2}\left(2\sqrt{ab}\right)^2}{a^2b^2} = 8$$

with equality only if $a = b$. By using the same argument as in Solution 1, a cannot equal b and the inequality is strict.

Alternative Solution 4

$$\begin{aligned}\left(\frac{c}{a}+\frac{c}{b}\right)^2 &= \frac{c^2}{a^2}+\frac{c^2}{b^2}+\frac{2c^2}{ab}\\ &= 1+\frac{b^2}{a^2}+\frac{a^2}{b^2}+1+\frac{2\left(a^2+b^2\right)}{ab}\\ &= 2+\left(\frac{a}{b}-\frac{b}{a}\right)^2+2+\frac{2}{ab}\left((a-b)^2+2ab\right)\\ &= 4+\left(\frac{a}{b}-\frac{b}{a}\right)^2+\frac{2(a-b)^2}{ab}+4\\ &\geq 8\end{aligned}$$

with equality only if $a = b$, which (as argued previously) cannot occur.

3. *Alternative Solution 1*

For any $t, 0 \leq 4t^2 < 1 + 4t^2$, so $0 \leq \dfrac{4t^2}{1+4t^2} < 1$. Thus x, y and z must be non-negative and less than 1.

Observe that if one of xy or z is 0, then $x = y = z = 0$.

If two of the variables are equal, say $x = y$, then the first equation becomes

$$\frac{4x^2}{1+4x^2} = x$$

This has the solution $x = 0$, which gives $x = y = z = 0$ and $x = \frac{1}{2}$ which gives x $= y = z = \frac{1}{2}$.

Finally, assume that x, y and z are non-zero and distinct. Without loss of generality we may assume that either $0 < x < y < z < 1$ or $0 < x < z < y < 1$. The two proofs are similar, so we do only the first case.

We will need the fact that $f(t) = \dfrac{4t^2}{1+4t^2}$ is increasing on the interval $(0, 1)$.

To prove this, if $0 < s < t < 1$ then

$$\begin{aligned} f(t) - f(s) &= \frac{4t^2}{1+4t^2} - \frac{4s^2}{1+4s^2} \\ &= \frac{4t^2 - 4s^2}{(1+4x^2)(1+4s^2)} \\ &> 0 \end{aligned}$$

So $0 < x < y < z \;\Rightarrow\; f(x) = y < f(y) = z < f(z) = x$, a contradiction. Hence $x = y = z = 0$ and $x = y = z = \frac{1}{2}$ are the only real solutions.

Alternative Solution 2

Notice that x, y and z are non-negative. Adding the three equations gives

$$x + y + z = \frac{4x^2}{1+4x^2} + \frac{4y^2}{1+4y^2} + \frac{4z^2}{1+4z^2}$$

This can be rearranged to give

$$\frac{x(2x-1)^2}{1+4x^2} + \frac{y(2y-1)^2}{1+4y^2} + \frac{z(2z-1)^2}{1+4z^2} = 0$$

Since each term is non-negative, each term must be 0, and hence each variable is either 0 or $\frac{1}{2}$. The original equations then show that $x = y = z = 0$ and $x = y = z = \frac{1}{2}$ are the only two solutions.

Alternative Solution 3

Notice that x, y, and z are non-negative. Multiply both sides of the inequality

$$\frac{y}{1+4y^2} \geq 0$$

by $(2y-1)^2$, and rearrange to obtain

$$y - \frac{4y^2}{1+4y^2} \geq 0$$

and hence that $y \geq z$. Similarly, $z \geq x$, and $x \geq y$. Hence, $x = y = z$ and, as in Solution 1, the two solutions follow.

Alternative Solution 4

As for solution 1, note that $x = y = z = 0$ is a solution and any other solution will have each of x, y and z positive.

The arithmetic-geometric mean inequality (or direct computation) shows that

$$\frac{1 + 4x^2}{2} \geq \sqrt{1 \cdot 4x^2} = 2x$$

and hence $x \geq \dfrac{4x^2}{1 + 4x^2} = y$, with equality if and only if $1 = 4x^2$, that is, $x = \frac{1}{2}$. Similarly, $y \geq z$ with equality if and only if $y = \frac{1}{2}$ and $z \geq x$ with equality if and only if $z = \frac{1}{2}$. Adding $x \geq y, y \geq z$ and $z \geq x$ gives $x + y + x \geq x + y + z$. Thus equality must occur in each inequality, so $x = y = z = \dfrac{1}{2}$.

4. *Alternative Solution 1*

Let the center of Γ be O, the radius r. Since $BP = BC$, let

$$\theta = \measuredangle BPC = \measuredangle BCP.$$

Quadrilateral $QABC$ is cyclic, so

$$\measuredangle BAQ = 180^\circ - \theta$$

and hence

$$\measuredangle PAQ = 120^\circ - \theta.$$

Also

$$\measuredangle APQ = 180^\circ - \measuredangle APB - \measuredangle BPC = 120^\circ - \theta$$

so $PQ = AQ$ and

$$\measuredangle AQP = 2\theta - 60^\circ.$$

Again because quadrilateral $QABC$ is cyclic,

$$\measuredangle ABC = 180^\circ - \measuredangle AQC = 240^\circ - 2\theta.$$

Triangles OAB and OCB are congruent, since $OA = OB = OC = r$ and $AB = BC$. Thus

$$\measuredangle ABO = \measuredangle CBO = \frac{1}{2}\measuredangle ABC = 120^\circ - \theta.$$

We have now shown that in triangles AQP and AOB,

$$\measuredangle PAQ = \measuredangle BAO = \measuredangle APQ = \measuredangle ABO.$$

Also $AP = AB$, so $\triangle AQP \cong \triangle AOB$. Hence $QP = OB = r$.

Alternative Solution 2

Let the center of Γ be O, the radius r. Since A, P and C lie on a circle centered at B,

$$60^\circ = \measuredangle ABP = 2\measuredangle ACP, so \measuredangle ACP = \measuredangle ACQ = 30^\circ.$$

Since Q, A, and C lie on Γ,

$$\measuredangle QOA = 2\measuredangle QCA = 60^\circ.$$

So $QA = r$ since if a chord of a circle subtends an angle of 60° at the center, its length is the radius of the circle.

Now $BP = BC$, so

$$\measuredangle BPC = \measuredangle BCP = \measuredangle ACB + 30^\circ.$$

Thus

$$\measuredangle APQ = 180^\circ - \measuredangle APB - \measuredangle BPC = 90^\circ - \measuredangle ACB.$$

Since Q, A, B and C lie on Γ and $AB = BC$,

$$\measuredangle AQP = \measuredangle AQC = \measuredangle AQB + \measuredangle BQC = 2\measuredangle ACB.$$

Finally,

$$\measuredangle QAP = 180 - \measuredangle AQP - \measuredangle APQ = 90 - \measuredangle ACB.$$

So

$$\measuredangle PAQ = \measuredangle APQ$$

hence

$$PQ = AQ = r.$$

5. Rearranging the equation we get $4x^2 + 51 = 40[x]$. It is known that $x \geq [x] > x - 1$, so

$$\begin{aligned} 4x^2 + 51 = 40[x] &> 40(x-1) \\ 4x^2 - 40x + 91 &> 0 \\ (2x-13)(2x-7) &> 0 \end{aligned}$$

Hence $x < 13/2$ or $x < 7/2$. Also

$$\begin{aligned} 4x^2 + 51 = 40[x] &\leq 40x \\ 4x^2 - 40x + 51 &\leq 0 \\ (2x-17)(2x-3) &\leq 0 \end{aligned}$$

Hence $3/2 \leq x \leq 17/2$. Combining these inequalities give $3/2 \leq x < 7/2$ or $13/2 < x \leq 17/2$.

CASE 1 $3/2 \leq x < 7/2$

For this case, the possible values for $[x]$ are $1, 2$ and 3.

If $[x] = 1$ then $4x^2 + 51 = 40 \cdot 1$ so $4x^2 = -11$, which has no real solutions.

If $[x] = 2$ then $4x^2 + 51 = 40 \cdot 2$ so $4x^2 = 29$ and $x = \frac{\sqrt{29}}{2}$. Notice that $\dfrac{\sqrt{16}}{2} < \dfrac{\sqrt{29}}{2} < \dfrac{\sqrt{36}}{2}$ so $2 < x < 3$ and $[x] = 2$.

If $[x] = 3$ then $4x^2 + 51 = 40 \cdot 3$ and $x = \frac{\sqrt{69}}{2}$. But $\dfrac{\sqrt{69}}{2} > \dfrac{\sqrt{64}}{2} = 4$. So, this solution is rejected.

CASE 2 $13/2 < x \leq 17/2$

For this case, the possible values for $[x]$ are $6, 7$ and 8.

If $[x] = 6$ then $4x^2 + 51 = 40 \cdot 6$ so $x = \frac{\sqrt{189}}{2}$. Notice that $\dfrac{\sqrt{144}}{2} < \dfrac{\sqrt{189}}{2} < \dfrac{\sqrt{196}}{2}$ so $6 < x < 7$ and $[x] = 6$.

If $[x] = 7$ then $4x^2 + 51 = 40 \cdot 7$ so $x = \frac{\sqrt{229}}{2}$. Notice that $\dfrac{\sqrt{196}}{2} < \dfrac{\sqrt{229}}{2} < \dfrac{\sqrt{256}}{2}$ so $7 < x < 8$ and $[x] = 7$.

If $[x] = 8$ then $4x^2 + 51 = 40 \cdot 8$ so $x = \frac{\sqrt{269}}{2}$. Notice that $\dfrac{\sqrt{256}}{2} < \dfrac{\sqrt{269}}{2} < \dfrac{\sqrt{324}}{2}$ so $8 < x < 9$ and $[x] = 8$.

The solutions are

$$x = \frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}$$

6.

$$
\begin{aligned}
\sum_{n=1}^{1994}(-1)^n\frac{n^2+n+1}{n!} &= \sum_{n=1}^{1994}(-1)^n\frac{n^2}{n!}+\sum_{n=1}^{1994}(-1)^n\frac{n+1}{n!}\\
&= \sum_{n=1}^{1994}(-1)^n\frac{n}{(n-1)!}+\sum_{n=1}^{1994}(-1)^n\frac{n+1}{n!}\\
&= \sum_{n=0}^{1993}(-1)^{n+1}\frac{n+1}{n!}+\sum_{n=1}^{1994}(-1)^n\frac{n+1}{n!}\\
&= -1+\sum_{n=1}^{1993}(-1)^{n+1}\frac{n+1}{n!}+\sum_{n=1}^{1994}(-1)^n\frac{n+1}{n!}\\
&= -1+\frac{1995}{1994!}
\end{aligned}
$$

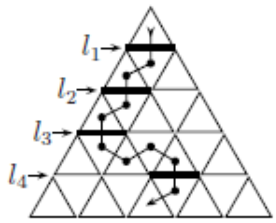
By telescoping sums.

7. We shall show that $f(n) = (n1)!$.

Label the horizontal line segments in the triangle $l_1, l_2, \ldots$ as in the diagram below. Since the path goes from the top triangle to a triangle in the bottom row and never travels up, the path must cross each of $l_1, l_2, \ldots, l_{n-1}$ exactly once. The diagonal lines in the triangle divide l_k into k unit line segments and the path must cross exactly one of these k segments for each k. (In the diagram below, these line segments have been highlighted.) The path is completely determined by the set of $n-1$ line segments which are crossed. So as the path moves from the kth row to the $(k+1)$st row, there are k possible line segments where the path could cross l_k. Since there are

$$1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!$$

ways that the path could cross the $n-1$ horizontal lines, and each one corresponds to a unique path, we get $f(n) = (n-1)!$. Therefore $f(2005) = (2004)!$.



8. For the sake of contradiction, we assume that there is a seating arrangement such that there is no one sitting in between girls. We call a block, any group of girls (boys) sitting next to each other and sandwiched by boys(girls) from both sides.

   By assumption, each girl block has at most two girls and there is at least 2 boys in the gap between two consecutive girl blocks.

   Hence there are at least $\frac{25}{2} = 13$ girl blocks and at least 2×13 boys sitting in between the 13 gaps between girl blocks. But we only have 25 boys, a contradiction. Therefore, our assumption was wrong and it is always possible to find someone sitting between two girls.

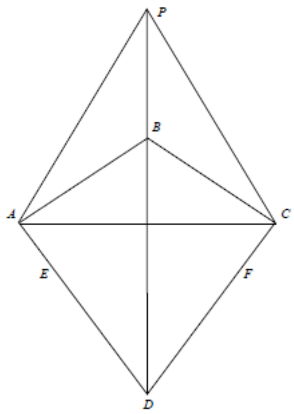
   Advanced learners can also apply the Pigeonhole principle.

9. *Alternative Solution 1*

Extend DB to a point P on the circle through A and C centered at B. Then

$$\angle CPD = \frac{1}{2}\angle CBD = \angle ADB \text{ and } \angle APD = \frac{1}{2}\angle ABD = \angle CDB,$$

so $APCD$ is a parallelogram. Now PD bisects AC so BD is an angle bisector of isosceles triangle ABC.



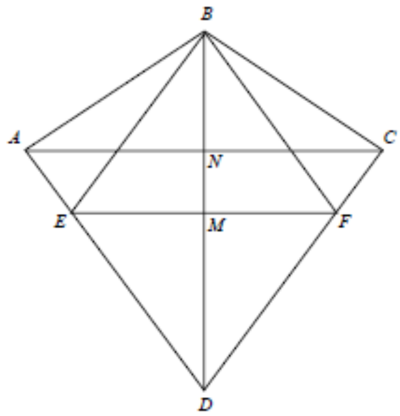
We have

$$\angle ADB = \frac{1}{2}\angle CBD = \frac{1}{2}\angle ABD = \angle CDB$$

so DB is the angle bisector of $\angle ADC$. As DB bisects the base of triangle ADC, this triangle must be isosceles and $AD = CD$.

Alternative Solution 2

Let the bisector of $\angle ABD$ meet AD at E. Let the bisector of $\angle CBD$ meet CD at F. Then $\angle FBD = \angle BDE$ and $\angle EBD = \angle BDF$, which imply $BE \parallel FD$ and $BF \parallel ED$.



Thus $BEDF$ is a parallelogram whence

$$BD \text{ intersects } EF \text{ at its midpoint } M. \tag{1.1}$$

On the other hand since BE is an angle bisector, we have $\frac{AB}{BD} = \frac{AE}{ED}$. Similarly we have $\frac{CB}{BD} = \frac{CF}{FD}$. By assumption $AB = CB$ so $\frac{AE}{ED} = \frac{CF}{FD}$ which implies $EF \parallel AC$. Thus $\triangle DEF$ and $\triangle DAC$ are similar, which implies by (1.11) that BD intersects AC at its midpoint N.

Since $\triangle ABC$ is isosceles, this implies $AC \perp BD$. Thus $\triangle NAD$ and $\triangle NCD$ are right triangles with equal legs and hence are congruent. Thus $AD = CD$.

10. If $f(x) = x^3 - x - 1 = (x-\alpha)(x-\beta)(x-\gamma)$ has roots α, β, γ standard results about roots of polynomials give

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \text{ and} \\ \alpha\beta\gamma &= 1. \end{aligned}$$

Then
$$S = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = \frac{N}{(1-\alpha)(1-\beta)(1-\gamma)}$$
where the numerator simplifies to

$$\begin{aligned} N &= 3 - (\alpha+\beta+\gamma) - (\alpha\beta + \alpha\gamma + \beta\gamma) + 3\alpha\beta\gamma \\ &= 3 - (0) - (-1) + 3(1) \\ &= 7. \end{aligned}$$

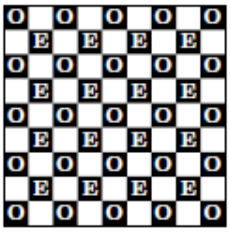
The denominator is $f(1) = -1$ so the required sum is -7.

SECTION B : 10 marks each

11. *Alternative Solution 1*

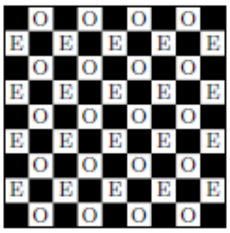
We will first count the number of ways of placing 8 mutually non-attacking rooks on black squares and then count the number of ways of placing them on white squares. Suppose that the rows of the board have been numbered 1 to 9 from top to bottom.

First notice that a rook placed on a black square in an odd numbered row cannot attack a rook on a black square in an even numbered row. This effectively partitions the black squares into a 5×5 board and a 4×4 board (squares labelled O and E respectively, in the diagram at right) and rooks can be placed independently on these two boards. There are 5! ways to place 5 non-attacking rooks on the squares labelled O and 4! ways to place 4 non-attacking rooks on the squares labelled E.



This gives 5!4! ways to place 9 mutually non-attacking rooks on black squares and removing any one of these 9 rooks gives one of the desired configurations. Thus there are $9 \cdot 5!4!$ ways to place 8 mutually non-attacking rooks on black squares.

Using very similar reasoning we can partition the white squares as shown in the diagram at right. The white squares are partitioned into two 5×4 boards such that no rook on a square marked O can attack a rook on a square mark E. At most 4 non-attacking rooks can be placed on a 5×4 board and they can be placed in $5 \cdot 4 \cdot 3 \cdot 2 = 5!$ ways. Thus there are $(5!)2$ ways to place 8 mutually non-attacking rooks on white squares. In total there are



$$9 \cdot 5!4! + (5!)2 = (9 + 5)5!4! = 14 \cdot 5!4! = 40320$$

ways to place 8 mutually non-attacking rooks on squares of the same colour.

Alternative Solution 2

Consider rooks on black squares first. We have 8 rooks and 9 rows, so exactly one row will be without rooks. There are two cases: either the empty row has 5 black squares or it has 4 black squares. By permutation these rows can be made either last or second last. In each case well count the possible number of ways of placing the rooks on the board as we proceed row by row.

In the first case we have 5 choices for the empty row, then we can place a rook on any of the black squares in row 1 (5 possibilities) and any of the black squares in row 2 (4 possibilities). When we attempt to place a rook in row 3, we must avoid the column containing the rook that was placed in row 1, so we have 4 possibilities. Using similar reasoning, we can place the rook on any of 3 possible black squares in row 4, etc. The total number of possibilities for the first case is

$$5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 = (5!)2.$$

In the second case, we have 4 choices for the empty row (but assume its the second last row). We now place rooks as before and using similar logic, we get that the total number of possibilities for the second case is

$$4 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 4(5!4!).$$

Now, do the same for the white squares. If a row with 4 white squares is empty (5 ways to choose it), then the total number of possibilities is (5!)2. Its impossible to have a row with 5 white squares empty, so the total number of ways to place rooks is

$$(5!)2 + 4(5!4!) + (5!)2 = (5 + 4 + 5)5!4! = 14(5!4!).$$

12. *Alternative Solution 1*

A prime indicates where a tangent meets AB and a double prime where it meets AC. It is given that

$$DD' = DD'', EE' = EE'' \text{ and } FF' = FF''.$$

It is required to show that arc EF is a third of the circumference as is arc DBF.

AF is the median to the hypotenuse of right triangle $AF'F''$, so that $FF' = FA$ and therefore

$$\text{arc}\, AF = 2\angle F''FA = 2(\angle FF'A + \angle FAF') = 4\angle FAF' = 4\angle FAB = 2\,\text{arc}\, BF$$

whence $\text{arc}\, FA = (2/3)\,\text{arc}\, BFA$. Similarly, $\text{arc}\, AE = (2/3)\,\text{arc}\, AEC$. Therefore, $\text{arc}\, FE$ is $2/3$ of the semicircle, or $1/3$ of the circumference as desired.

As for $\text{arc}\, DBF$,

$$\text{arc}\, BD = 2\angle BAD = \angle BAD + \angle BD'D = \angle ADD'' = (1/2)\,\text{arc}\, ACD.$$

But, $\text{arc}\, BF = (1/2)\,\text{arc}\, AF$, so $\text{arc}\, DBF = (1/2)\,\text{arc}\, FAED$. That is, $\text{arc}\, DBF$ is $1/3$ the circumference and the proof is complete.

Alternative Solution 2

Since $AE'E''$ is a right triangle, $AE = EE' = EE''$ so that $\angle CAE = \angle CE''E$. Also $AD = D'D = DD''$, so that

$$\angle CDD'' = \angle CAD = \angle CD''D.$$

As $EADC$ is a concyclic quadrilateral,

$$\begin{aligned}
180^\circ &= \angle EAD + \angle ECD \\
&= \angle DAC + \angle CAE + \angle ECA + \angle ACD \\
&= \angle DAC + \angle CAE + \angle CEE'' + \angle CE''E + \angle CDD'' + \angle CD''D \\
&= \angle DAC + \angle CAE + \angle CAE + \angle CAE + \angle CAD + \angle CAD \\
&= 3(\angle DAC + \angle DAE) \\
&= 3(\angle DAE)
\end{aligned}$$

Hence $\angle DFE = \angle DAE = 60^\circ$. Similarly, $\angle DEF = 60^\circ$. It follows that triangle DEF is equilateral.

13. *Alternative Solution 1*

We claim that f is a constant function. Suppose, for a contradiction, that there exist x and y with $f(x) < f(y)$, choose x, y such that $f(y) - f(x) > 0$ is minimal. Then

$$f(x) = \frac{xf(x) + yf(x)}{x+y} < \frac{xf(y) + yf(x)}{x+y} < \frac{xf(y) + yf(y)}{x+y} = f(y)$$

so for $f(x) < f(x^2+y^2) < f(y)$ and $0 < f(x^2+y^2) < f(y) - f(x)$, contradicting the choice of x and y. Thus, f is a constant function. Since $f(0)$ is in $\mathbb{N}$, the constant must be from $\mathbb{N}$.

Also, for any c in $\mathbb{N}$, $xc + yc = (x+y)c$ for all x and y, so $f(x) = c, c \in \mathbb{N}$ are the solutions to the equation.

Alternative Solution 2

We claim f is a constant function. Define $g(x) = f(x) - f(0)$. Then

$$g(0) = 0, g(x) \geq f(0)$$

and

$$xg(y) + yg(x) = (x+y)g\left(x^2+y^2\right)$$

for all $x, y \in \mathbb{N}$.

Letting $y = 0$ shows $g\left(x^2\right) = 0$ (in particular, $g(1) = g(4) = 0$), and letting $x = y = 1$ shows $g(2) = 0$. Also, if x, y and z in $\mathbb{N}$ satisfy $x^2 + y^2 = z^2$, then

$$g(y) \quad = \quad -\frac{y}{x}g(x) \tag{1.2}$$

Letting $x = 4$ and $y = 3$, (1.2) shows that $g(3) = 0$.

For any even number $x = 2n > 4$, let $y = n^2 - 1$. Then $y > x$ and $x^2 + y^2 = \left(n^2+1\right)^2$. For any odd number $x = 2n+1 > 3$, let $y = 2(n+1)^n$. Then $y > x$ and $x^2 + y^2 = \left((n+1)^2 + n^2\right)^2$. Thus for every $x > 4$ there is $y > x$ such that (1.2) is satisfied.

Suppose for a contradiction, that there is $x > 4$ with $g(x) > 0$. Then we can construct a sequence $x = x_0 < x_1 < x_2 < \ldots$ where $g\left(x_{i+1}\right) = -\dfrac{x_{i+1}}{x_i}g\left(x_i\right)$. It follows that $|g\left(x_{i+1}\right)| > |g\left(x_i\right)|$ and the signs of $g(x_i)$ alternate. Since $g(x)$ is always an integer, $|g\left(x_{i+1}\right)| \geq |g\left(x_i\right)| + 1$. Thus for some sufficiently large value of i, $g(x_i) < -f(0)$, a contradiction.

As for Proof 1, we now conclude that the functions that satisfy the given functional equation are $f(x) = c, c \in N$.

Alternative Solution 3

Suppose that W is the set of nonnegative integers and that $f : W \to W$ satisfies:

$$xf(y) + yf(x) \quad = \quad (x+y)f\left(x^2+y^2\right)$$

We show that f is a constant function.

14. *Alternative Solution 1*

Let $f(x, y, z) = (x^3 - x) + (y^3 - y) + (z^3 - z)$. The first equation above is equivalent to $f(x, y, z) = 0$. If $x, y, z \geq 1$, then $f(x, y, z) \geq 0$ with equality only if $x = y = z = 1$.

But if $x = y = z = 1$, then the second equation is not satisfied. So in any solution to the system of equations, at least one of the variables is less than 1. Without loss of generality, suppose that $x < 1$. Then

$$x^2 + y^2 + z^2 > y^2 + z^2 \geq 2yz > yz > xyz.$$

Therefore the system has no real positive solutions.

Alternative Solution 2

We will show that the system has no real positive solution. Assume otherwise.

The second equation can be written $x^2 - (yz)x + (y^2 + z^2)$. Since this quadratic in x has a real solution by hypothesis, its discrimant is nonnegative. Hence

$$y^2z^2 - 4y^2 - 4z^2 \geq 0.$$

Dividing through by $4y^2z^2$ yields

$$\frac{1}{4} \geq \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{1}{y^2}.$$

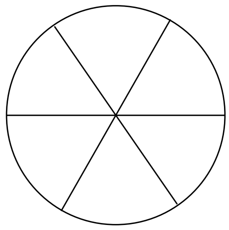
Hence $y^2 \geq 4$ and so $y \geq 2$, y being positive. A similar argument yields $x, y, z \geq 2$. But the first equation can be written as

$$x(x^2 - 1) + y(y^2 - 1) + z(z^2 - 1) = 0,$$

contradicting $x, y, z \geq 2$. Hence, a real positive solution cannot exist.

15. (a) Let n be the number of people at the party. Not counting themselves, a person can have $0, 1, \cdots, n-2$ or $n-1$ friends. If all n people had a different number of friends, then one person would have 0 friends and another would have $n-1$. This is not possible since the person with $n-1$ friends is friends with everyone at the party, including the person with 0 friends.

(b) To prove that the final statement is always true, we first divide the circle into six equal sectors as shown:



Allowing each sector to be a pigeonhole and each dart to be a pigeon, we have seven pigeons to go into six pigeonholes. By pigeonhole principle, there is at least one sector containing a minimum of two darts. Since the greatest distance between two points lying in a sector is 10 units, the statement is proven to be true in any case.

In fact, it is also possible to prove the scenario with only six darts. In such a case, the circle is this time divided into five sectors and all else follows. However, take note that this is not always true anymore with only five darts or less.

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 2.

SATURDAY 17TH JUNE, 2017 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.

(b) There are **two** *sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*

(c) Answer scripts of only qualified students should be returned to the contest coordinator, latest **Monday 19th June, 2017**.

(d) **ALL** *participants who qualified and sat for Paper 2 are cordially invited to the certificate and/or prize giving ceremony on Saturday 29th July, 2017 at NOON at Makerere University, Kampala.*

SECTION A : 5 marks each

Qn. 1. Two positive numbers a and b satisfy $2 + \log_2 a = 3 + \log_3 b = \log_6(a + b)$. Compute the value of

$$\frac{1}{a} + \frac{1}{b}.$$

Qn. 2. Given the sequence $\{x_n\}$ defined by $x_{n+1} = \dfrac{1 + x_n\sqrt{3}}{\sqrt{3} - x_n}$ with $x_1 = 1$, compute the value of $x_{2016} - x_{618}$.

Qn. 3. Let $f(x) = \dfrac{9^x}{9^x + 3}$. Evaluate the sum $f\left(\dfrac{1}{1996}\right) + f\left(\dfrac{2}{1996}\right) + f\left(\dfrac{3}{1996}\right) + \cdots + f\left(\dfrac{1995}{1996}\right)$.

Qn. 4. Prove that $\dfrac{1}{1999} < \dfrac{1}{2} \cdot \dfrac{3}{4} \cdot \dfrac{5}{6} \cdots \dfrac{1997}{1998} < \dfrac{1}{44}$

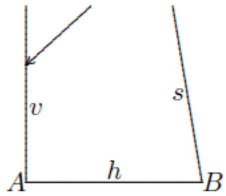
Qn. 5. You are trying to guess an integer between 1 and 1000, inclusive. Each time you make a guess, which must be an integer, you are told whether your number is too high, too low, or correct. What is the smallest number of guesses required to guarantee that you will have guessed the number?

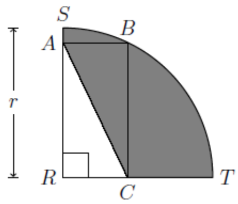
Qn. 6. In triangle ABC, $BC = \sqrt{2}, CA = 6$, and angle $ACB = 135$ degrees. If C' is the midpoint of AB, what is the length of CC'?

Qn. 7. Evaluate $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{1}{3 \cdot 4 \cdot 5} + \cdots + \dfrac{1}{98 \cdot 99 \cdot 100}$ as a simple fraction.

Qn. 8. Alan and Bill walk at 4 miles per hour, and Chad drives 40 miles per hour. They set out from the same point in the same direction with Alan walking, and Bill riding with Chad. After an hour, Bill gets out of the car and starts walking in the same direction they were going, while Chad turns the car around and drives back to pick up Alan. When the car gets back to Alan, Alan gets into the car, the car turns around and drives until it meets Bill. How many miles has Bill traveled when the car meets him? You may assume that no time is required to turn around or to change passengers.

Qn. 9. In the diagram below, line v is perpendicular to line h, and the angle at B is 75 degrees. A light ray has angle of reflection equal to angle of incidence. Our convention is to consider the angle which is ≤ 90 degrees. A ray has initial angle of incidence with v of 50 degrees. It follows a path in which it next hits wall h, then s, and then v again. What will be the number of degrees of its angle of incidence with wall v this (second) time?





Qn. 10. In the figure, arc SBT is one quarter of a circle with center R and radius r. The length plus the width of rectangle $ABCR$ is 8, and the perimeter of the shaded region is $10 + 3\pi$. Find the value of r.

SECTION B : 10 marks each

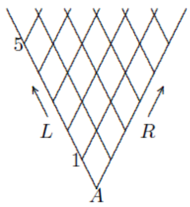
Qn. 11. Find all real numbers x such that $x = \left(x - \frac{1}{x}\right)^{1/2} + \left(1 - \frac{1}{x}\right)^{1/2}$.

Qn. 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = 1$ and for any $x, y \in \mathbb{R}$ such that

$$f(xy + 1) = f(x)f(y) - f(y) - x + 2.$$

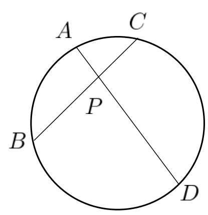
Determine $f(x)$.

Qn. 13. The bottom part of a network of roads is shown to the right. It extends up to level 99 in a similar fashion. 2^{100} people leave point A (level 0). At every intersection, half go in direction L and half in direction R. How many people will end up at position 1 at level 99?



Qn. 14. Let P be any point in the interior of a circle. Let two lines through P intersect the circle, one line, in points A and D, and another line, in points B and C as shown in the diagram. Prove that

$$AP \cdot PD = BP \cdot PC$$



Qn. 15. Without using a calculator:

(a) What integer equals $\sqrt{2000 \cdot 2008 \cdot 2009 \cdot 2017 + 1296}$?

(b) If

$$1.0000042376^2 = 1.00000xyz521795725376,$$

what is the value of $x + y + z$?

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 2.

SATURDAY 17TH JUNE, 2017 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.

(b) There are **two** *sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*

(c) Answer scripts of only qualified students should be returned to the contest coordinator, latest **Monday 19th June, 2017**.

(d) **ALL** *participants who qualified and sat for Paper 2 are cordially invited to the certificate and/or prize giving ceremony on Saturday 29th July, 2017 at NOON at Makerere University, Kampala.*

SECTION A : 5 marks each

Qn. 1. Two positive numbers a and b satisfy $2 + \log_2 a = 3 + \log_3 b = \log_6(a + b)$. Compute the value of $\frac{1}{a} + \frac{1}{b}$.

Suppose

$$k = 2 + \log_2 a = 3 + \log_3 b = \log_6(a + b)$$

then we have $a = 2^{k-2}, b = 3^{k-3}, a + b = 6^k$, and therefore,

$$\frac{1}{a} + \frac{1}{b} = \frac{a + b}{ab} = \frac{6^k}{2^{k-2}3^{k-3}} = 2^2 \times 3^3 = 108$$

Qn. 2. Given the sequence $\{x_n\}$ defined by $x_{n+1} = \frac{1 + x_n\sqrt{3}}{\sqrt{3} - x_n}$ with $x_1 = 1$, compute the value of $x_{2016} - x_{618}$.

Note that $x_2 = 2 + \sqrt{3}, x_3 = -2 - \sqrt{3}, x_4 = -1, x_5 = -2 + \sqrt{3}, x_6 = 2 - \sqrt{3}, x_7 = 1 = x_1$, so that we have $x_{n+6} = x_n$ for all positive integers n; i.e., the sequence is periodic. Since

$$2016 - 618 = 1398 = 233 \times 6,$$

we have $x_{2016} - x_{618} = 0$.

Qn. 3. Let $f(x)=\dfrac{9^x}{9^x+3}$. Evaluate the sum

$$f\left(\frac{1}{1996}\right)+f\left(\frac{2}{1996}\right)+f\left(\frac{3}{1996}\right)+\cdots+f\left(\frac{1995}{1996}\right)$$

Note that

$$f(1-x)=\frac{9^{1-x}}{9^{1-x}+3}=\frac{9}{9+3\times 9^x}=\frac{3}{9^x+3}$$

From which we get

$$f(x)+f(1-x)=\frac{9^x}{9^x+3}+\frac{3}{9^x+3}=1$$

Therefore

$$\begin{aligned}\sum_{k=1}^{1995} f\left(\frac{k}{1996}\right) &= \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right)+f\left(\frac{1996-k}{1996}\right)\right]+f\left(\frac{998}{1996}\right)\\ &= \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right)+f\left(1-\frac{k}{1996}\right)\right]+f\left(\frac{1}{2}\right)\\ &= 997+\frac{3}{3+3}\\ &= 997\frac{1}{2}\end{aligned}$$

Qn. 4. Prove that

$$\frac{1}{1999}<\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998}<\frac{1}{44}$$

Let $P=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998}$. But $\frac{1}{2}>\frac{1}{3}$ since $2<3$, $\frac{3}{4}>\frac{3}{5}$ since $4<5,\ldots,\ldots,\frac{1997}{1998}>\frac{1997}{1999}$ because $1998<1999$.

So

$$P>\frac{1}{3}\cdot\frac{3}{5}\cdots\frac{1997}{1999}=\frac{1}{1999} \tag{1.1}$$

Also $\frac{1}{2}<\frac{2}{3}$ because $1\cdot 3<2\cdot 2$, and $\frac{3}{4}<\frac{4}{5}$ because $3\cdot 5<4\cdot 4,\ldots\ \frac{1997}{1998}<\frac{1998}{1999}$ because $1997\cdot 1999=1998^2-1<1998^2$.

So

$$\begin{aligned}P<\frac{2}{3}\cdot 4\cdot 5\cdots\frac{1998}{1999} &= \left[\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{1998}{1997}\right]\frac{1}{1999}\\ &= \frac{1}{P}\frac{1}{1999}\end{aligned}$$

Hence

$$P^2<\frac{1}{1999}<\frac{1}{1936}=\frac{1}{44^2}\quad\text{and}\quad P<\frac{1}{44} \tag{1.2}$$

Then (1.1) and (1.2) give

$$\frac{1}{1999}<P<\frac{1}{44}$$

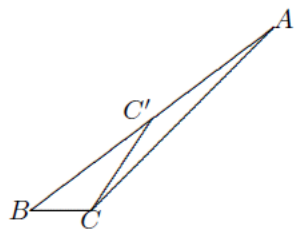
Qn. 5. You are trying to guess an integer between 1 and 1000, inclusive. Each time you make a guess, which must be an integer, you are told whether your number is too high, too low, or correct. What is the smallest number of guesses required to guarantee that you will have guessed the number?

They must be 10 guesses. This is because $2^{10} = 1024 > 1000$: The best that you can guarantee after 1 guess is to limit it to 500 numbers, then $250, 125, 62, 31, 15, 7, 3,$ and 1. After 9 guesses, you know what the answer is, but you still must make the tenth guess.

Qn. 6. In triangle ABC, $BC = \sqrt{2}, CA = 6$, and angle $ACB = 135$ degrees. If C' is the midpoint of AB, what is the length of CC'?

Place B at $(0, 0)$, C at $\left(\sqrt{2}, 0\right)$ and A at $\left(4\sqrt{2}, 3\sqrt{2}\right)$. Then C' is at $\left(2\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$ and so the desired length is

$$\sqrt{\left(\sqrt{2}\right)^2 + \left(\frac{3}{2}\sqrt{2}\right)^2} = \sqrt{2}\sqrt{1 + \frac{9}{4}} = \frac{\sqrt{26}}{2}$$



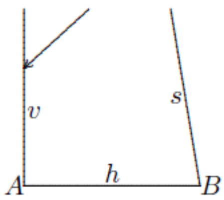
Qn. 7. Evaluate $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{98\cdot 99\cdot 100}$ as a simple fraction.

Note that
$$\frac{1}{(n+1)n(n-1)}=\frac{1}{2n(n-1)}-\frac{1}{2(n+1)n}$$
Thus the desired sum is $\frac{1}{4}-\frac{1}{2(100)(99)}$, as all other intermediate terms cancel out. This equals
$$\frac{50\cdot 99-1}{200\cdot 99}.$$

Qn. 8. Alan and Bill walk at 4 miles per hour, and Chad drives 40 miles per hour. They set out from the same point in the same direction with Alan walking, and Bill riding with Chad. After an hour, Bill gets out of the car and starts walking in the same direction they were going, while Chad turns the car around and drives back to pick up Alan. When the car gets back to Alan, Alan gets into the car, the car turns around and drives until it meets Bill. How many miles has Bill traveled when the car meets him? You may assume that no time is required to turn around or to change passengers.

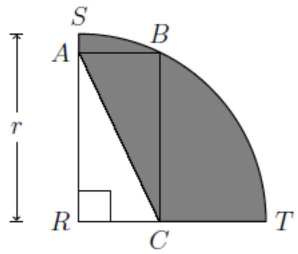
Let t denote the number of hours during which Chad was driving back to pick up Alan. Then $4(1+t)=40-40t$, hence $t=\frac{9}{11}$. Although you can compute it by solving equations, you can also reason it out that Bill will have to drive forward the second time for exactly one hour in order to catch up with Bill. The reason for this is that if we forget about the time period when they were both walking, we see that when they eventually meet, they both will have been in the car the same amount of time. So, during the two hours when one was walking and one riding each will have traveled 44 miles, and during the other 9/11 hour, each will have traveled 36/11 miles. Thus $47\frac{3}{11}$ and $\frac{520}{11}$

Qn. 9. In the diagram below, line v is perpendicular to line h, and the angle at B is 75 degrees. A light ray has angle of reflection equal to angle of incidence. Our convention is to consider the angle which is ≤ 90 degrees. A ray has initial angle of incidence with v of 50 degrees. It follows a path in which it next hits wall h, then s, and then v again. What will be the number of degrees of its angle of incidence with wall v this (second) time?



If it hits h at C, the angle there is 40 degrees. Then it hits s at point D with angle $180 - 75 - 40 = 65$. If it next hits v at point E, then $ABDE$ is a quadrilateral with three of its angles $90, 75$, and $(180 - 65)$. Since a quadrilateral has 360 degrees, this makes the desired angle $360 - 280 = 80$.

Qn. 10. In the figure below, arc SBT is one quarter of a circle with center R and radius r. The length plus the width of rectangle $ABCR$ is 8, and the perimeter of the shaded region is $10 + 3\pi$. Find the value of r.



Set $x = RC$ and $y = AR$. It must then be $x + y = 8$ (by hypothesis) and $x^2 + y^2 = r^2$ (since the rectangle $ABCR$ has one vertex in the circle). The perimeter of the shaded area is then

$$AS+SBT+TC+AC = (r-y)+r\frac{\pi}{2}+(r-x)+\sqrt{x^2+y^2} = 3r+r\frac{\pi}{2}-(x+y) = 3r+r\frac{\pi}{2}-8.$$

All we need to do is solve for r in the equation

$$18 + 3\pi = \left(3 + \frac{\pi}{2}\right) r.$$

SECTION B : 10 marks each

Qn. 11. Find all real numbers x such that

$$x = \left(x - \frac{1}{x}\right)^{\frac{1}{2}} + \left(1 - \frac{1}{x}\right)^{\frac{1}{2}}$$

Since $\left(x - \frac{1}{x}\right)^{\frac{1}{2}} \geq 0$ and $\left(1 - \frac{1}{x}\right)^{\frac{1}{2}} \geq 0$, then $0 \leq \left(x - \frac{1}{x}\right)^{\frac{1}{2}} + \left(1 - \frac{1}{x}\right)^{\frac{1}{2}} = x$. Note that $x \neq 0$. Else, $\frac{1}{x}$ would not be defined, so $x > 0$. Squaring both sides gives,

$$\begin{aligned} x^2 &= \left(x - \frac{1}{x}\right)^{\frac{1}{2}} + \left(1 - \frac{1}{x}\right)^{\frac{1}{2}} + 2\sqrt{\left(x - \frac{1}{x}\right) + \left(1 - \frac{1}{x}\right)} \\ x^2 &= x + 1 - \frac{2}{x} + 2\sqrt{x - 1 - \frac{1}{x} + \frac{1}{x^2}}. \end{aligned}$$

Multiplying both sides by x and rearranging, we get

$$\begin{aligned} x^3 - x^2 - x + 2 &= 2\sqrt{x^3 - x^2 - x + 1} \\ \left(x^3 - x^2 - x + 1\right) - 2\sqrt{x^3 - x^2 - x + 1} + 1 &= 0 \\ \left(\sqrt{x^3 - x^2 - x + 1} - 1\right)^2 &= 0 \\ \sqrt{x^3 - x^2 - x + 1} &= 1 \\ x^3 - x^2 - x + 1 &= 1 \\ x(x^2 - x - 1) &= 0 \\ x^2 - x - 1 &= 0 \quad \text{since } x \neq 0 \end{aligned}$$

Thus $x = \dfrac{1 \pm \sqrt{5}}{2}$. We must check to see whether theses are indeed solutions.

Let $\alpha = \dfrac{1 + \sqrt{5}}{2}$, $\beta = \dfrac{1 - \sqrt{5}}{2}$. Note that $\alpha + \beta = 1$, $\alpha\beta = -1$ and $\alpha > 0 > \beta$.

Since $\beta < 0, \beta$ is not a solution.

Now, if $x = \alpha$, then

$$\begin{aligned} \left(\alpha - \frac{1}{\alpha}\right)^{1/2} + \left(1 - \frac{1}{\alpha}\right)^{1/2} &= (\alpha + \beta)^{1/2} + (1 + \beta)^{1/2} \quad \text{since } \alpha\beta = -1 \\ &= 1^{1/2} + (\beta^2)^{1/2} \quad \text{since } \alpha + \beta = 1, \text{ and } \beta^2 = \beta + 1 \\ &= 1 - \beta \quad \text{since } \beta < 0 \\ &= \alpha \quad \text{since } \alpha + \beta = 1 \end{aligned}$$

So $x = \alpha = \dfrac{1 + \sqrt{5}}{2}$ is a unique solution to the equation.

Qn. 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = 1$ and for any $x, y \in \mathbb{R}$ such that

$$f(xy+1) = f(x)f(y) - f(y) - x + 2.$$

Determine $f(x)$.

Since for $x, y \in \mathbb{R}$,

$$f(xy+1) = f(x)f(y) - f(y) - x + 2 \tag{1.3}$$

We have

$$f(xy+1) = f(y)f(x) - f(x) - y + 2 \tag{1.4}$$

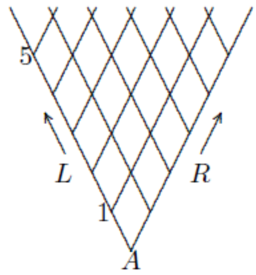
From (1.3) and (1.4)

$$\begin{aligned} f(x)f(y) - f(y) - x + 2 &= f(y)f(x) - f(x) - y + 2 \\ f(x) + y &= f(y) + x \end{aligned}$$

For $y = 0$, we obtain

$$f(x) = x + 1$$

Qn. 13. The bottom part of a network of roads is shown below. It extends up to level 99 in a similar fashion. 2^{100} people leave point A (level 0). At every intersection, half go in direction L and half in direction R. How many people will end up at position 1 at level 99?



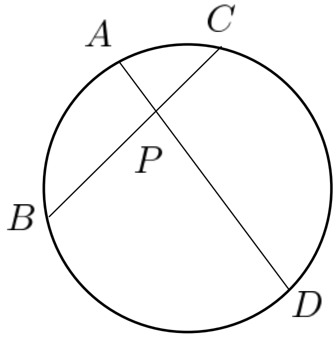
This is like Pascal's triangle. At level n, the fraction of the people in horizontal position i is $\frac{\binom{n}{i}}{2^i}$, and so the number of people in position 1 at level 99 is

$$\frac{\binom{99}{1}2^{100}}{2^{99}} = 198$$

Some students might use a more lengthy way of drawings and listings.

Qn. 14. Let P be any point in the interior of a circle. Let two lines through P intersect the circle, one line, in points A and D, and another line, in points B and C as shown in the diagram. Prove that

$$AP \cdot PD = BP \cdot PC$$



Follows from similarities of triangles ABP and CDP, a consequence of the equality of their angles:

- $\angle BAD = \angle BCD$ being angles subtended by same chord BD.
- $\angle ABC = \angle ADC$ being angles subtended by same chord AC.

$$\angle APB = \angle CPD \quad - \quad \text{clear}$$

So from similarity of triangles, ABP and CDP:

$$\frac{AP}{CP} = \frac{BP}{DP}\ , \quad \text{hence} \quad AP \cdot PD = BP \cdot PC$$

Qn. 15. Without using a calculator:

(a) What integer equals $\sqrt{2000 \cdot 2008 \cdot 2009 \cdot 2017 + 1296}$?

Let $A = 2000$ and $B = A^2 + 17A + 36$. The expression inside the square root is

$$A(A+17)(A+8)(A+9)+1296 = (A^2+17A)(A^2+17A+72)+1296 = (B-36)(B+36)+1296 = B^2.$$

Thus the answer is $B = 20002 + 34000 + 36 = 4034036$.

(b) If

$$1.0000042376^2 = 1.00000xyz521795725376,$$

what is the value of $x + y + z$?

We know that $(1 + t)^2 = 1 + 2t + t^2$. Here $t = .0000042376$. The t^2 is much too small to affect the first three nonzero digits of $2t$. We obtain $(x, y, z) = (8, 4, 7)$. Therefore $x + y + z = 19$

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 1.

SATURDAY 17TH MARCH, 2018 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*

(b) *There are* **two** *sections and each section carries* 50 *marks. Attempt all questions in exactly three hours fifteen minutes* ($9:00am$ *to* $12:15pm$)*. Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*

(c) *Answer scripts of only qualified students should be returned to the contest coordinator, latest* **Monday 19th March, 2018**.

(d) **ALL** *participants who qualify for Paper* 2 *are cordially invited to the certificate and prize giving ceremony on Saturday* 28*th July,* 2018 *at NOON at Makerere University, Kampala.*

(e) *National Mathematics Contest* 2018 *Paper* 2 *will be done on* 23*rd June,* 2018 *at various centres.*

SECTION A : 5 marks each

1. Let $\dfrac{113}{50} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}}$, where a, b, c, d and e are integers. Find c and e.

2. Suppose real numbers x, y, z satisfy the equation $\dfrac{x}{y+z} + \dfrac{y}{z+x} + \dfrac{z}{x+y} = 1$.
Compute the value of
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}$$

3. Let ABC be a triangle and D be a point such that A and D are on opposite sides of BC. Given that $\angle ACD = 75°, AC = 2, BD = \sqrt{6}$, and AD is an angle bisector of both $\triangle ABC$ and $\triangle BCD$, find the area of quadrilateral $ABDC$.

4. Liesl has a bucket. Henri drills a hole in the bottom of the bucket. Before the hole was drilled, Tap A could fill the bucket in 16 minutes, tap B could fill the bucket in 12 minutes, and tap C could fill the bucket in 8 minutes. A full bucket will completely drain out through the hole in 6 minutes. Liesl starts with the empty bucket with the hole in the bottom and turns on all three taps at the same time. How many minutes will it take until the instant when the bucket is completely full?

5. If $f(x) = a + bx$, what are the real values of a and b such that
$$\begin{aligned} f(f(f(1))) &= 29 \\ f(f(f(0))) &= 2 \end{aligned}$$

6. Add all the digits in 2008^{2008}. Add all the digits in the resulting number. Keep going until the result has only one digit. What is this digit?

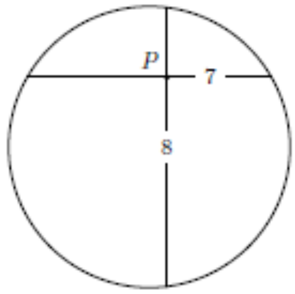
7. Let $x = \log_{10}(81)$ and $y = \log_{10}(25)$. If one expresses $\log_{10}(6)$ as $ax + by + c$ where a, b, and c are rational numbers, then what is $a + b + c$?

8. Three identical circles of radius 3 lie externally tangent to each other. A fourth, larger circle is drawn around the other three circles so that the smaller three circles are internally tangent to the larger circle. Compute the radius of the larger circle.

9. Ed writes the first 2018 positive integers down in order: $1, 2, 3, \ldots, 2018$. Then for each power of 2 that appears, he crosses out that number as well as the number 1 greater than that power of 2. After he is done, how many numbers are not crossed out?

10. If f is a function such that $f(x-1) = x^2 - 3x$, then $f(x+1) =$?

SECTION B : 10 marks each



11. Two perpendicular chords of a circle intersect at point P. One chord is 7 units long, divided by P into segments of length 3 and 4, while the other chord is divided into segments of length 2 and 6. What is the diameter of the circle?

12. (a) Let $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$. Determine $f\left(\dfrac{3x+x^3}{1+3x^2}\right)$ in terms of $f(x)$.

    (b) If $x^2 + xy + y^2 = 84$ and $x - \sqrt{xy} + y = 6$, what is xy?

    (c) How many real solutions does the following equation have?

$$x^2 + \sqrt{x-2} = 4 + \sqrt{4-2x}.$$

13. Cindy has a collection of identical rectangular prisms. She stacks them, end to end, to form 1 longer rectangular prism. Suppose that joining 11 of them will form a rectangular prism with 3 times the surface area of an individual rectangular prism. How many will she need to join end to end to form a rectangular prism with 9 times the surface area?

14. If $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}$ and $\dfrac{1}{x+1} + \dfrac{1}{y+1} = \dfrac{3}{8}$, compute $\dfrac{1}{x-1} + \dfrac{1}{y-1}$ as a fraction of integers.

15. (a) Let $a_1, a_2, a_3, a_4, a_5, \ldots$ be a geometric progression with positive ratio such that $a_1 > 1$ and

$$(a_{1357})^3 = a_{34}.$$

    Find the smallest integer n such that $a_n < 1$.

    (b) If $a > 0$ and $b > 0$ are real numbers satisfying

$$\frac{1}{a} + \frac{1}{b} \leq 2\sqrt{2} \quad \text{and} \quad (a-b)^2 = 4(ab)^3,$$

    what is the value of $\log_a b$?

SECTION A : 5 marks each

1. By expansion

$$\begin{aligned}
\frac{113}{50} &= 2+\frac{13}{50}\\
&= 2+\frac{1}{\frac{50}{13}}\\
&= 2+\cfrac{1}{3+\cfrac{11}{13}}\\
&= 2+\cfrac{1}{3+\cfrac{1}{\frac{13}{11}}}\\
&= 2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{2}{11}}}\\
&= 2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{\frac{11}{2}}}}\\
&= 2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2}}}}
\end{aligned}$$

Therefore $(a,b,c,d,e)=(2,3,1,5,2)$. Therefore $(c,e)=(1,2)$

2. We first note that $x+y+z \neq 0$. Otherwise,

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=-3$$

Thus, we have

$$\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)(x+y+z)=(x+y+z).$$

Simplifying it, we have

$$\frac{x^2}{y+z}+x+\frac{y^2}{z+x}+y+\frac{z^2}{x+y}+z=(x+y+z),$$

which yields

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}=0$$

3. Since AD is an angle bisector of both ABC and BCD, $\angle BAD = \angle CAD$ and $\angle BDA = \angle CDA$. Then by angle-side-angle congruence, $\triangle ABD \cong \triangle ACD$, and $CD = BD = \sqrt{6}$. Since $\angle ACD = 75°$, we can use the area formula

$$[ACD] = \frac{1}{2}AC \cdot CD \cdot \sin 75° = \frac{1}{2} \cdot 2 \cdot \sqrt{6} \cdot \frac{\sqrt{2}+\sqrt{6}}{4} = \frac{\sqrt{3}+3}{2}$$

Because $[ABD] = [ACD]$ we have $[ABCD] = 2[ACD] = 3 + \sqrt{3}$.

Solution 2: We begin as in the original solution by noticing $\triangle ABD \cong \triangle ACD$. This implies that $ABCD$ is a kite, which means that $AD \perp BC$. Let E be the intersection of AD and BC. Note that then $\triangle CAE$ is a $30-60-90$ right triangle, while $\triangle CDE$ is a $45-45-90$ right triangle. This gives us $AD = 1 + \sqrt{3}$ and $BC = 2\sqrt{3}$, so the area of kite $ABCD$ is $\frac{1}{2}AD \cdot BC = 3 + \sqrt{3}$

4. Since taps A, B and C can fill the bucket in 16 minutes, 12 minutes and 8 minutes, respectively, then in 1 minute taps A, B and C fill $\frac{1}{16}$, $\frac{1}{12}$ and $\frac{1}{8}$ of the bucket, respectively.

   Similarly, in 1 minute, $\frac{1}{6}$ of the bucket drains out through the hole. Therefore, in 1 minute with the three taps on and the hole open, the fraction of the bucket that fills is

$$\frac{1}{16} + \frac{1}{12} + \frac{1}{8} - \frac{1}{6} = \frac{3+4+6-8}{48} = \frac{5}{48}.$$

   Thus, to fill the bucket under these conditions will take $\frac{48}{5}$ minutes.

5. We have that

$$\begin{aligned} f(f(f(1))) &= f(f(a+b)) = f(a+ab+b^2)) = a+ab+ab^2+b^3 = 29 \\ f(f(f(0))) &= f(f(a)) = f(a+ab)) = a+ab+ab^2 = 2 \end{aligned}$$

Therefore $2+b^3 = 29 \;\Leftrightarrow\; b = 3$ and $a = \dfrac{2}{13}$.

6. You can recognise the pattern

n	2008^n	Sum of digits
0	1	1
1	2008	1
2	4,032,064	1
3	8,096,384,512	1
4	16,257,540,100,096	1

Or, we know that, the sum of digits

$$\begin{aligned} 2008^{2008} &= (2008 \mod 9)^{2008} \\ &= 1^{2008} \\ &= 1 \end{aligned}$$

7. Solution: Since $81 = 3^4$, we get $\log_{10}(3) = \frac{x}{4}$. Also, $25 = 5^2$, so we obtain $\log_{10}(5) = \frac{y}{2}$.

   Note that $\log_{10}(5) + \log_{10}(2) = 1$, so $\log_{10}(2) = 1 - \frac{y}{2}$. Thus,

$$\log_{10}(6) = \log_{10}(2) + \log_{10}(3) = 1 - \frac{y}{2} + \frac{x}{4}.$$

   Therefore $a = \frac{1}{4}$, $b = -\frac{1}{2}$ and $c = 1$, such that $a + b + c = \frac{1}{4} - \frac{1}{2} = 1 = \frac{3}{4}$

8. Three identical circles of radius 3 lie externally tangent to each other. A fourth, larger circle is drawn around the other three circles so that the smaller three circles are internally tangent to the larger circle. Compute the radius of the larger circle.

   Solution 1 *Let* A, B, C *be the centers of the three smaller circles, and let* O *be the center of the larger circle. Note that the radius of the larger circle is simply* $3 + OA$. *Because the three smaller circles are tangent to each other, we have* $AB = AC = BC = 3 \cdot 2 = 6$, *so* $\triangle ABC$ *is an equilateral triangle. Now if we let* D *be the midpoint of* AB, *we can see that* ADO *is a* $30 - 60 - 90$ *triangle. This gives us the ratio* $\frac{\sqrt{3}}{2} = \frac{3}{OA}$, *which we solve to get* $OA = 2\sqrt{3}$. *Therefore, the radius of the larger circle is* $3 + 2\sqrt{3}$.

9. Ed writes the first 2018 positive integers down in order: $1, 2, 3, \ldots, 2018$. Then for each power of 2 that appears, he crosses out that number as well as the number 1 greater than that power of 2. After he is done, how many numbers are not crossed out?

   There are 11 powers of 2 less than 2018:

$$2^0, 2^1, \ldots, 2^{10} = 1024.$$

   For each power of 2, we cross out two numbers. However, we end up crossing out 2 twice, because it is crossed out by $2^0 + 1$ and by 2^1. Therefore, the total number of integers we cross out is $11 \cdot 2 - 1 = 21$. It follows that $2018 - 21 = 1997$ numbers are left.

10. We know $f(x-1) = x^2 - 3x$. Let $y = x - 1 \quad \Leftrightarrow \quad x = y + 1$. Then

$$\begin{aligned} f(y) &= f(x-1) = x^2 - 3x \\ &= (y+1)^2 - 3(y+1) \\ &= y^2 + 2y + 1 - 3y - 3 \\ f(y) &= y^2 - y - 2 \end{aligned}$$

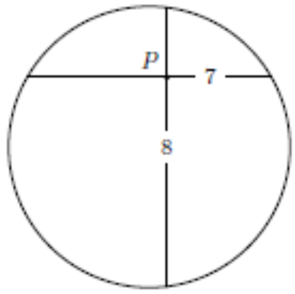
or $f(x) = x^2 - x - 2 \quad \Leftrightarrow$

$$\begin{aligned} f(x+1) &= (x+1)^2 - (x+1) - 2 \\ &= x^2 + x - 2 \end{aligned}$$

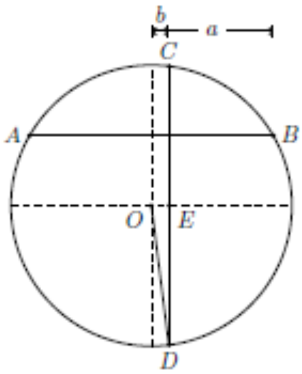
SECTION B : 10 marks each

For each question in SECTION B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

11. Two perpendicular chords of a circle intersect at point P. One chord is 7 units long, divided by P into segments of length 3 and 4, while the other chord is divided into segments of length 2 and 6. What is the diameter of the circle?



Solution 2 *Label points A, B, C, D, E and the center of the circle, O, in the diagram below (right), draw diameters parallel to both chords, and assign distances a, b as depicted. Since $\overline{CD}$ is the longer*



chord of length 8, the distance DE must equal 4. Since $\overline{AB}$ is the shorter chord of length 7, it must be that $a = 3$ and $a + b = 3.5$. This gives $b = 0.5$. By the Pythagorean Theorem, it must be that $r = OD = \dfrac{\sqrt{65}}{2}$.

12. (a)

$$\begin{aligned} f\left(\frac{3x+x^3}{1+3x^2}\right) &= \ln\left(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}\right) \\ &= \ln\left(\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}\right) \\ &= \ln\left(\frac{(1+x)^3}{(1-x)^3}\right) \\ &= 3f(x) \end{aligned}$$

(b) Rewrite the second equation as $x+y=6+\sqrt{xy}$. Squaring then gives

$$x^2+y^2+2xy=36+xy+12\sqrt{xy}.$$

Subtracting this equation from $x^2+xy+y^2=84$ and simplifying gives $48=12\sqrt{xy}$, thus $xy=16$.

(c) The square roots are defined only when $x-2\geq 0$ and $4-2x\geq 0$. The two inequalities are satisfied only when $x=2$. Since $x=2$ is a solution, the equation has exactly one real solution.

13. Let n be the number of boxes stacked together, and let m be the resulting multiplier on the surface area (i.e. the resulting box has m times the surface area of an individual box). Letting x, y be the lengths of the sides of the box and z be the height of the box, we may write

$$\begin{aligned} 2xy + 2x(nz) + 2y(nz) &= m(2xy + 2xz + 2yz) \\ (n-m)(x+y)z &= (m-1)xy \\ \frac{n-m}{m-1} &= \frac{xy}{(x+y)z} \end{aligned}$$

Note that the right side is a constant because x, y, z are fixed. Furthermore, we are given $n = 11$ and $m = 3$, so plugging this in gives us $\frac{11-3}{3-1} = 4 = \frac{xy}{(x+y)z}$. Therefore, we must find n such that $\frac{n-9}{9-1} = 4$, which solves to $n = 41$.

Solution 2: Note that after stacking 10 additional boxes, you gain 2 additional boxes worth of surface area. This ratio is constant, so if we need n additional boxes to get 8 additional boxes worth of surface area, we have the equation $\frac{n}{8} = \frac{10}{2}$. Solving yields $n = 40$, which means we need a total of 41 boxes.

14. If $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}$ and $\dfrac{1}{x+1}+\dfrac{1}{y+1}=\dfrac{3}{8}$, compute $\dfrac{1}{x-1}+\dfrac{1}{y-1}$.

Solution 3 *With two equations and two unknowns, we can solve for the expressions $a=xy$ and $b=x+y$. The first equation can be written as*

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=\frac{b}{a}=\frac{1}{2}$$

which implies that $a=2b$. Similarly, from the second equation we have

$$\frac{1}{x+1}+\frac{1}{y+1}=\frac{(x+y)+2}{xy+(x+y)+1}=\frac{b+2}{a+b+1}=\frac{3}{8}$$

which implies that $8(b+2)=3(a+b+1)$, or $3a-5b=13$. Solving the system of equations gives us $a=26$ and $b=13$. Therefore, we have

$$\frac{1}{x-1}+\frac{1}{y-1}=\frac{(x+y)-2}{xy-(x+y)+1}=\frac{b-2}{a-b+1}=\frac{11}{14}$$

15. (a) Let $a_1, a_2, a_3, a_4, a_5, \ldots$ be a geometric progression with positive ratio such that $a_1 > 1$ and $(a_{1357})^3 = a_{34}$. Find the smallest integer n such that $a_n < 1$.

Let r be the ratio between the terms. The n-th term of the sequence can therefore be written as $a_n = a_1 r^{n-1}$. This allows us to write

$$(a_{1357})^3 = a_{34} \;\Rightarrow\; \left(a_1 r^{1356}\right)^3 = a_1 r^{33} \;\Rightarrow\; a_1^2 r^{4035} = 1.$$

Since $a_1 > 1$, we must have $0 < r < 1$, so the geometric progression is decreasing. However, the above equation also gives

$$a_1^2 r^{4035} = \left(a_1 r^{2017}\right)\left(a_1 r^{2018}\right) = a_{2018} a_{2019} = 1.$$

Because the sequence is decreasing, we must have $a_{2018} > 1 > a_{2019}$, which gives us the answer 2019.

(b) From $\dfrac{1}{a} + \dfrac{1}{b} \le 2\sqrt{2}$, we have $0 < a + b \le 2\sqrt{2}ab$, then $0 < (a+b)^2 \le 8(ab)^2$. So we get

$$4(ab)^3 = (a-b)^2 = (a+b)^2 - 4ab \le 8(ab)^2 - 4ab$$

Now let us focus on $4(ab)^3 \le 8(ab)^2 - 4ab$. Since $ab > 0$, we may divide both sides of previous inequality by $4ab$ and rearrange the inequality, giving $(ab)^2 - 2ab + 1 \le 0$ and then $(ab-1)^2 \le 0$, which can only be true if $ab = 1$. So $b = a^{-1}$, i.e., $\log_a b = -1$.

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 2.

SATURDAY 23RD JUNE, 2018 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*

(b) *There are two sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm).*

(c) *Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.*

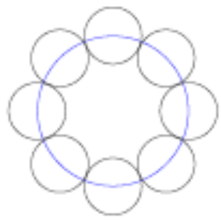
(d) ***Indicate your names, class, school and district on all your answer sheets.***

(e) *Answer scripts of only qualified students should be returned to the contest coordinator, latest Monday 25th June, 2018.*

(f) *ALL participants who qualified for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 28th July, 2018 at 8:00am at Makerere University, Kampala.*

SECTION A : 5 marks each

Qn. 1. Eight circles of radius 1cm have centers on a larger common circle and adjacent circles are tangent. Find the area of the common circle.



Qn. 2. Determine all the real solutions of the following equation:

$$2^{(2^x)} - 3 \cdot 2^{\left(2^{x-1}+1\right)} + 8 = 0.$$

Qn. 3. Compute the integer k, $k > 2$, for which

$$\log_{10}[(k-2)!] + \log_{10}[(k-1)!] + 2 = 2\log_{10}(k!).$$

Qn. 4. Let N be a 3-digit number with three distinct non-zero digits. We say that N is *mediocre* if it has the property that when all six 3-digit permutations of N are written down, the average is N. For example, $N = 481$ is mediocre, since it is the average of $418, 481, 148, 184, 814, 841$. Determine the largest mediocre number.

Qn. 5. Prove that for all real numbers x, y,

$$\big(x^2+1)(y^2+1\big) + 4\,(x-1)\,(y-1) \geq 0.$$

Determine when equality holds.

Qn. 6. If

$$x = \left(1 + \frac{1}{n}\right)^{n} \quad \text{and} \quad y = \left(1 + \frac{1}{n}\right)^{n+1},$$

show that $y^x = x^y$.

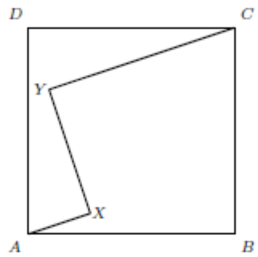
Qn. 7. Find all number triples (x, y, z) such that when any of these numbers is added to the product of the other two, the result is 2.

Qn. 8. Suppose that n people each know exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything that A knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything?

Qn. 9. Simplify

$$\left(\frac{1 \cdot 2 \cdot 4 + 2 \cdot 4 \cdot 8 + \ldots + n \cdot 2n \cdot 4n}{1 \cdot 3 \cdot 9 + 2 \cdot 6 \cdot 18 + \cdots + n \cdot 3n \cdot 9n}\right)^{1/3}.$$

Qn. 10. In the sketch below $ABCD$ is a square, AX has length 1cm, XY has length 2cm, and YC has length 3cm. Furthermore, $\angle AXY = 90°$ and $\angle XYC = 90°$. Determine the area of the square.



SECTION B : 10 marks each

Qn. 11. E and F are points on the sides CA and AB, respectively, of an equilateral triangle ABC such that EF is parallel to BC. G is the intersection point of medians in triangle AEF and M a point on the segment BE. Prove that $\angle MGC = 60°$ if and only if M is the midpoint of BE.

Qn. 12. Determine all solutions of the following system of equations

$$\begin{aligned} y &= 4x^3 + 12x^2 + 12x + 3 \\ x &= 4y^3 + 12y^2 + 12y + 3 \end{aligned}$$

Qn. 13. In a test, Nabatanzi solved four-fifth of the problems and Amoding solved 35 problems. Half of the problems were solved by both of them. The number of problems solved by neither was a positive one-digit number. What was this number?

Qn. 14. A dog standing at the centre of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the centre of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

Qn. 15. The non-zero numbers a, b, c, x, y and z are such that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. Determine the value of

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)}.$$

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 1.

SATURDAY 23RD JUNE, 2018 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*

(b) *There are two sections and each section carries* 50 *marks. Attempt all questions in exactly three hours fifteen minutes (*$9:00am$ *to* $12:15pm$*).*

(c) *Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.*

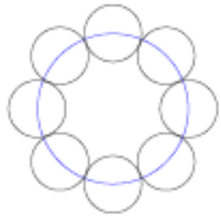
(d) ***Indicate your names, class, school and district on all your answer sheets****.*

(e) *Answer scripts of only qualified students should be returned to the contest coordinator, latest Monday 25th June, 2018.*

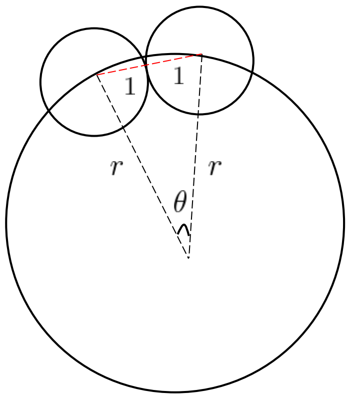
(f) *ALL participants who qualified for Paper* 2 *are cordially invited to the certificate and prize giving ceremony on Saturday* 28*th July,* 2018 *at 8:00am at Makerere University, Kampala.*

SECTION A : 5 marks each

Qn. 1. Eight circles of radius 1cm have centers on a larger common circle and adjacent circles are tangent. Find the area of the common circle.



Solution: For r, the radius of the big circle, we have



$$\theta = \frac{360}{8} = \frac{\pi}{4}$$

applying the cosine rule

$$2^2 = r^2 + r^2 - 2r^2\cos\left(\frac{\pi}{4}\right) \quad \Rightarrow \quad r^2 = \frac{4}{2 - 2\cos\left(\frac{\pi}{4}\right)} = \frac{4}{2-\sqrt{2}}$$

Therefore, the area of the big circle is given by

$$A = \pi r^2 = \frac{4\pi}{2-\sqrt{2}} = \frac{4\pi}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = 4\pi\left(\frac{2+\sqrt{2}}{2}\right) = 2\pi\left(2+\sqrt{2}\right)$$

Qn. 2. Determine all the real solutions of the following equation:

$$2^{(2^x)} - 3 \cdot 2^{\left(2^{x-1}+1\right)} + 8 = 0.$$

Solution: Let $y = 2^{x-1}$. Then $2^x = 2y$. Then the left-hand side of the equation becomes

$$2^{2y} - 3 \cdot 2^{y+1} + 8 = 0.$$

Equivalently,

$$2^{2y} - 6 \cdot 2^y + 8 = 0.$$

This factors as

$$(2^y - 4)(2^y - 2) = 0.$$

Therefore, $2^y = 2$ or 4. This yields the solutions $y = 1, 2$. Therefore, $2^{x-1} = 1, 2$, which yields solutions $x - 1 = 0, 1$. Hence, $x = 1, 2$.

We now verify these are indeed solutions. If $x = 1$, then

$$2^{2^1} - 3 \cdot 2^{2^0+1} + 8 = 2^2 - 3 \cdot 2^2 + 8 = 4 - 12 + = 0.$$

Hence, $x = 1$ is a solution. If $x = 2$, then

$$2^{2^2} - 3 \cdot 2^{2^1+1} + 8 = 2^4 - 3 \cdot 2^3 + 8 = 16 - 24 + 8 = 0.$$

Hence, the solutions are indeed $x = 1, 2$.

Qn. 3. Compute the integer k, $k > 2$, for which

$$\log_{10}[(k-2)!] + \log_{10}[(k-1)!] + 2 = 2\log_{10}(k!).$$

Solution:

$$\begin{aligned}
\log_{10}[(k-2)!] + \log_{10}[(k-1)!] + 2 &= 2\log_{10}(k!) \\
\log_{10}\left[\frac{k!}{(k-1)(k)}\right] + \log_{10}\left[\frac{k!}{k}\right] + 2 &= 2\log_{10}(k!) \\
\log_{10}[k!] - \log_{10}[(k-1)(k)] + \log_{10}[k!] - \log_{10}[k] + 2 &= 2\log_{10}(k!) \\
\log_{10}[(k-1)(k)] + \log_{10}[k] &= 2 \\
\log_{10}\left[(k-1)(k^2)\right] &= 2 \\
(k-1)(k^2) &= 100 \\
k^3 - k^2 &= 100
\end{aligned}$$

Let

$$\begin{aligned}
f(k) &= k^3 - k^2 - 100 \\
f(0) &= -100 \\
f(1) &= -100 \\
f(2) &= -96 \\
f(3) &= -82 \\
f(4) &= -52 \\
f(5) &= 0
\end{aligned}$$

Therefore, one of the factors is $k = 5$. We have that

$$k^3 - k^2 - 100 = (k-5)\left(k^2 + 4k + 20\right).$$

Since for $k^2 + 4k + 20 = 0$, $b^2 < 4ac$, it violates the condition for $k > 2$. Therefore $k = 5$.

Qn. 4. Let N be a 3-digit number with three distinct non-zero digits. We say that N is *mediocre* if it has the property that when all six 3-digit permutations of N are written down, the average is N. For example, $N = 481$ is mediocre, since it is the average of $418, 481, 148, 184, 814, 841$. Determine the largest mediocre number.

Solution: Suppose abc is a mediocre number. The 6 permutations are $\{abc, acb, bac, bca, cab, cba\}$. The sum of these numbers is $222(a + b + c)$ and the average is $37(a + b + c))$. Since abc is mediocre we have $100a + 10b + c = 37(a + b + c)$.

We can rearrange this equation to get $63a = 27b + 36c$. Notice that $63 = 27 + 36$ so we must have a strictly between b and c. Thus, $a \neq 9$.

Notice if that $b = a + 1$ then we have $36a = 27 + 36c$ which has no integer solutions, and if $c = a + 1$ then we have $27a = 27b + 36$ which has no integer solutions. Since a is strictly between b and c if $a = 8$ then b or c would have to be 9, but this cannot happen. Thus, $a \neq 8$.

If $a = 7$ then either $b = 9$ $c = 9$. This would reduce to $441 = 243 + 36c$ or $441 = 27b + 324$

Neither of these has integer solutions, so $a \neq 7$.

If $a = 6$ then we have $378 = 27b + 36c$ which reduces to $42 = 3b + 4c$. This gives $b = 14 - \dfrac{4}{3}c$. To get integer digit values we must have $c = 3, 6$, or 9 which gives corresponding b values of $10, 6$, and 2 respectively. We can omit the first two, since 10 is not a digit and 666 is not mediocre. Thus, the largest mediocre number is 629.

Qn. 5. Prove that for all real numbers x, y,

$$\left(x^2+1)(y^2+1\right)+4\,(x-1)\,(y-1)\geq 0.$$

Determine when equality holds.

Solution: Expanding and refactoring the left hand side of the given inequality gives

$$(xy+1)^2+(x+y-2)^2.$$

This is clearly non-negative, so the inequality holds. This holds with equality when each of the terms is 0. So $xy=-1$ and $x+y=2$. Solving this system yields

$$(x,y)=\left(1-\sqrt{2},\ 1+\sqrt{2}\,\right),\ \left(1+\sqrt{2},\ 1-\sqrt{2}\,\right).$$

Qn. 6. If

$$x = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad y = \left(1 + \frac{1}{n}\right)^{n+1},$$

show that $y^x = x^y$.

Solution:

$$\begin{aligned}
y^x &= \left(\left(1 + \frac{1}{n}\right)^{n+1}\right)^{\left(1+\frac{1}{n}\right)^n} = \left(1 + \frac{1}{n}\right)^{(n+1)\left(1+\frac{1}{n}\right)^n}; \\
x^y &= \left(\left(1 + \frac{1}{n}\right)^{n}\right)^{\left(1+\frac{1}{n}\right)^{n+1}} \\
&= \left(1 + \frac{1}{n}\right)^{n\left(1+\frac{1}{n}\right)^{n+1}} \\
&= \left(1 + \frac{1}{n}\right)^{n\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)} \\
&= \left(1 + \frac{1}{n}\right)^{(n+1)\left(1+\frac{1}{n}\right)^{n}}
\end{aligned}$$

Qn. 7. Find all number triples (x, y, z) such that when any of these numbers is added to the product of the other two, the result is 2.

Solution: The system to be solved is:

$$x + yz = 2; \quad y + zx = 2; \quad z + xy = 2.$$

Subtracting the second from the first and the third from the second yields, respectively,

$$(x - 1)(1 - z) = 0, \quad (y - z)(1 - x) = 0.$$

Each of the four cases

$$\begin{aligned} x - y &= 0 = y - z, \quad x - y = 0 = 1 - x, \\ 1 - z &= 0 = y - z \quad 1 - z = 0 = 1 - x, \end{aligned}$$

implies

$$x = y = z = 1$$

or

$$x = y = z = -2$$

Qn. 8. Suppose that n people each know exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything that A knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything?

Solution: Let us denote the n people by $p_1, p_2, \ldots, p_n$, and a call from p_i to p_j by $p_i \to p_j$.

Let $f(n)$ denote the minimum number of (one-way) calls which can leave everybody fully informed. The particular sequence

$$p_1 \to p_n, p_2 \to p_n, \ldots, p_{n\cdot 1} \to p_n, p_n \to p_{n\cdot 1}, p_n \to p_{n\cdot 2}, \ldots, p_n \to p_1$$

contains $2n-2$ calls and leaves everybody informed, showing that $f(n) \leq 2n-2$. Now suppose we have a given sequence of calls which leaves everybody fully informed.

Consider the "crucial" call at the end of which the receiver (call him p) is fully informed, but no one else is. Clearly each of the $n-1$ people (other than p) must have placed at least one call no later than this crucial call (how else could p be fully informed?) and each of these $n-1$ people (being not fully informed at the end of the crucial call) must receive at least one call after the crucial one. Hence the given sequence contains at least $2(n-1)$ calls. Hence

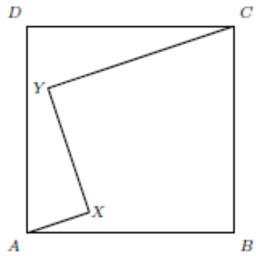
$$f(n) = 2n-2.$$

Qn. 9. Simplify

$$\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+\ldots+n\cdot 2n\cdot 4n}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+\cdots+n\cdot 3n\cdot 9n}\right)^{1/3}$$

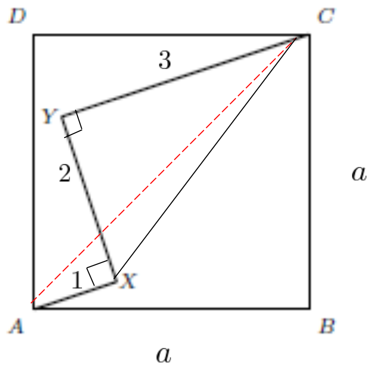
Solution: Since $k\cdot 2k\cdot 4k=8k^3$ and $k\cdot 3k\cdot 9k=27k^3$ for $1\le k\le n$,

$$\begin{aligned}\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+\ldots+n\cdot 2n\cdot 4n}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+\ldots+n\cdot 3n\cdot 9n}\right)^{1/3} &= \left(\frac{8\left[1^3+2^3+\ldots+n^3\right]}{27\left[1^3+2^3+\ldots+n^3\right]}\right)^{1/3}\\ &= \left(\frac{8}{27}\right)^{1/3}\\ &= \frac{2}{3}.\end{aligned}$$



Qn. 10. In the sketch below $ABCD$ is a square, AX has length 1cm, XY has length 2cm, and YC has length 3cm. Furthermore, $\angle AXY = 90^\circ$ and $\angle XYC = 90^\circ$. Determine the area of the square.

Solution: 10 From the figure



$$\begin{aligned} AX &= 1 \\ XY &= 2 \\ YC &= 3 \\ XC &= \sqrt{2^2+3^2} = \sqrt{13} \\ \angle YXC &= \sin^{-1}\frac{3}{\sqrt{13}} \\ \angle AXC &= 90^\circ + \sin^{-1}\frac{3}{\sqrt{13}} \end{aligned}$$

By Cosine rule,

$$\begin{aligned} AC &= \sqrt{1^2 + \left(\sqrt{13}\right)^2 - 2(1)\left(\sqrt{13}\right)\cdot\cos\left(90^\circ + \sin^{-1}\frac{3}{\sqrt{13}}\right)} \\ &= \sqrt{1^2 + \left(\sqrt{13}\right)^2 + 2(1)\left(\sqrt{13}\right)\cdot\sin\left(\sin^{-1}\frac{3}{\sqrt{13}}\right)} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Using $\triangle ABC$ and the Pyhtogras theorem,

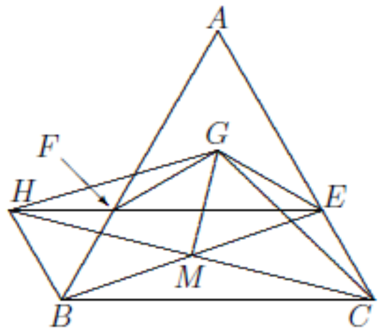
$$\begin{aligned} a^2 + a^2 &= AC^2 \\ 2a^2 &= AC^2 \quad \Rightarrow \\ a &= \frac{AC}{\sqrt{2}} = \frac{2\sqrt{5}}{\sqrt{2}} = \sqrt{2}\sqrt{5} = \sqrt{10} \end{aligned}$$

The area of the square is then given by $A = a \times a = a^2 = \left(\sqrt{10}\right)^2 = 10$ square units.

SECTION B : 10 marks each

Qn. 11. E and F are points on the sides CA and AB, respectively, of an equilateral triangle ABC such that EF is parallel to BC. G is the intersection point of medians in triangle AEF and M a point on the segment BE. Prove that $\angle MGC = 60°$ if and only if M is the midpoint of BE.

Solution: Let H be on the extension of EF such that $BCEH$ is a parallelogram. Then $\angle BHE = \angle BCE = 60°$ and $\angle BFH = \angle AFE = 60°$, so FBH is also an equilateral triangle, and in particular $HF = FB = EC$. Also $HFG = 180° - \angle GFE = 180° - 30° = 150°$, and similarly $\angle CEG = 150°$, thus $\angle HFG = \angle CEG$.



Since $\angle GFE = 30° = \angle GEF$, we know $FG = GE$. Hence triangles HFG and CEG are congruent (by SAS) so that $GH = GC$ and $\angle HGF = \angle CGE$. It follows that $\angle HGC = \angle HGF + \angle FGC = \angle CGE + \angle FGC = \angle EGF = 180° - \angle GFE - \angle GEF = 180° - 30° - 30° = 120°$.

Now M is the midpoint of the diagonal BE of the parallelogram if and only if it is the midpoint of the diagonal HC. Since triangle HGC is isosceles, this is equivalent to GM being the bisector of $\angle HGC$. In other words, $\angle MGC = 60°$.

Qn. 12. Determine all solutions of the following system of equations

$$\begin{aligned} y &= 4x^3 + 12x^2 + 12x + 3 \\ x &= 4y^3 + 12y^2 + 12y + 3 \end{aligned}$$

Solution: We can rewrite the two equations as:

$$\begin{aligned} y + 1 &= 4(x + 1)^3 \\ x + 1 &= 4(y + 1)^3 \end{aligned}$$

Substituting the second equation into the first, we get

$$(y + 1) = 256(y + 1)^9$$

By inspection we see that $y = -1$ satisfies the equation. Assume $y \neq -1$, then the equation becomes:

$$\frac{1}{2^8} = (y + 1)^8$$

The only real solutions to this are $y + 1 = \pm\frac{1}{2}$.

This gives $y = -\frac{1}{2}, -1, -\frac{3}{2}$ as possible solutions.

Substituting into the above equations, we get the following three solutions:

$$\left(-\frac{1}{2}, -\frac{1}{2}\right), \ (-1, -1), \ \left(-\frac{3}{2}, -\frac{3}{2}\right).$$

Qn. 13. In a test, Nabatanzi solved four-fifth of the problems and Amoding solved 35 problems. Half of the problems were solved by both of them. The number of problems solved by neither was a positive one-digit number. What was this number?

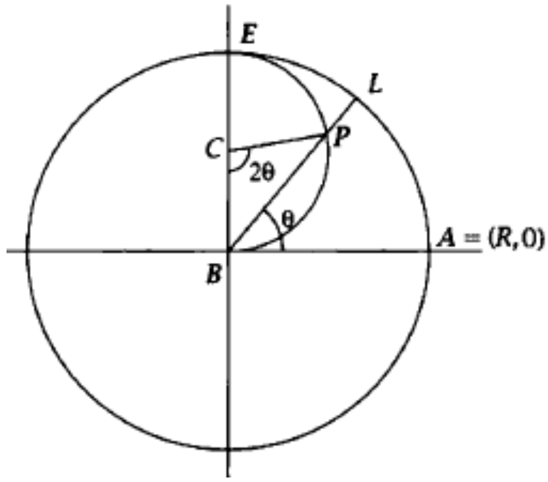
Solution: The fraction of problems solved only by Nabatanzi was $\frac{4}{5} - \frac{1}{2} = \frac{3}{10}$ so that the total number of problems is a multiple of 10. The fraction of problems solved by Amoding was at most $1 - \frac{3}{10} = \frac{7}{10}$. Thus the total number of problems was at least 50.

If it were 50, then 10 problems were solved by Amoding alone, and as Nabatanzi solved $\frac{4}{5} \times 50 = 40$ problems, the number of problems solved by neither is 0. The total number of problems could only be as large as 70 since 35 problems would be solved by both. In this case, the number of problems solved by neither was $\frac{1}{5} \times 70 = 14$.

It follows that the total number of problems must be 60, of which 30 were solved by both, 5 by Amoding alone, $\frac{3}{10} \times 60 = 18$ problems by Nabatanzi alone, and $60 - 30 - 5 - 18 = 7$ by neither of them.

Qn. 14. A dog standing at the centre of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the centre of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

Solution a: Let the centre of the circular arena be $B(0,0)$ and the radius R. Assume that the rabbit runs counterclockwise starting from $A(R,0)$. We shall show that if the dog runs along a circular arc with centre $C\left(0,\dfrac{R}{2}\right)$ and radius $\dfrac{R}{2}$, such that at time t its speed is the same as that of the rabbit, then the dog and the rabbit are on the same radial line. This will show that the circle is a possible path for the dog. The uniqueness statement given in the problem implies that this is the only path.



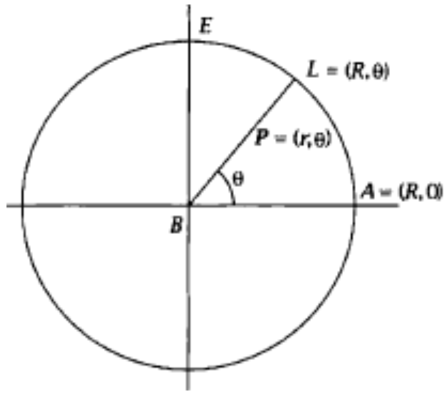
Since the dog and the rabbit are on the same radial line, they meet at the required point $E(0,R)$. Let L be the position of the rabbit at any time t and P be the point where BL meets the circle with centre C and radius $\dfrac{R}{2}$. It is sufficient to show that the arcs AL and BP are equal in length, since this implies that P is the position of the dog at time t.

Let $ABL=\theta$. Then angle $BCP=2\theta$ since angle $BPE=\dfrac{\pi}{2}$. Hence, arc $AL=R\theta$ and arc $BP=\dfrac{R}{2}(2\theta)=R\theta$. This completes the proof.

Solution b: Let the rabbit begin at $A(R,0)$ and the dog at $B(0,0)$. At any moment, suppose the dog is at P and the rabbit is at L. We are given that at any time t,

(a) $B, P,$ and L are collinear, and

(b) the speed of P is the same as of L.



Let the position of the dog be given in polar coordinates as $P(r,\theta)$.
Then condition (a) implies:

$$\begin{aligned}\text{speed of the dog} &= \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\\ \text{speed of the rabbit} &= \left(R\frac{d\theta}{dt}\right)\end{aligned}$$

Condition (b) implies

$$\begin{aligned}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 &= R^2\left(\frac{d\theta}{dt}\right)^2, \quad \text{or} \\ \left(\frac{dr}{dt}\right)^2 &= (R^2 - r^2)\left(\frac{d\theta}{dt}\right)^2\end{aligned}$$

Hence,

$$\frac{dr}{\sqrt{R^2 - r^2}} = d\theta.$$

Integrating each side gives $\arcsin\frac{r}{R} = \theta + C$. Hence, $r = R\sin(\theta + c)$. Initially, $r = 0$ and $\theta = 0$ so $c = 0$. The path traced by the dog, is therefore, $r = R\sin\theta$ which is a circle of radius $\frac{R}{2}$ and centre $\left(0, \frac{R}{2}\right)$. This completes the proof.

Qn. 15. The non-zero numbers a, b, c, x, y and z are such that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. Determine the value of

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)}$$

Solution: Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = r$. Then

$$\begin{aligned} \frac{xyz}{abc} &= r^3 \\ x &= ra \\ y &= rb \quad \text{and} \\ z &= rc \end{aligned}$$

It follows that $x+y = r(a+b),\ y+z = r(b+c)$ and $z+x = r(c+a)$, so that

$$\frac{x+y}{a+b} = \frac{y+z}{b+c} = \frac{z+x}{c+a} = r.$$

The given expression is equal to

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)} = r^3\left(\frac{1}{r}\right)^3 = 1$$

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 1.

SATURDAY 6TH APRIL, 2019 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.

(b) There are **two** *sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*

(c) Indicate your names, birth date, gender, class, school and district ***on all*** *your answer sheets.*

(d) Answer scripts of only registered students should be returned to the contest coordinator, latest **Monday 8th April, 2019**.

(e) **ALL** *participants who qualify for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 27th July, 2019 at NOON at Makerere University, Kampala.*

(f) National Mathematics Contest 2019 Paper 2 will be done on 29th June, 2019 at various centres.

SECTION A : 5 marks each

1. Find the root of the equation $2+\log\sqrt{1+x}+3\log\sqrt{1-x}=\log\sqrt{1-x^2}$.

2. Let ABC be a triangle. Let P be the point on line BC such that B is between P and C, and $BP = BA$. Similarly, let Q be the point on line BC such that C is between Q and B, and $CQ = CA$. If R is the second intersection of the circumcircles of $\triangle ACP$ and $\triangle ABQ$, prove that $\triangle PQR$ is isosceles.

3. A deck contains 52 cards of 4 different suits. Vanya is told the number of cards in each suit. He picks a card from the deck, guesses its suit, and sets it aside; he repeats until the deck is exhausted. Show that if Vanya always guesses a suit having no fewer remaining cards than any other suit, he will guess correctly at least 13 times.

4. Find, with proof, all real number solutions to the following:
$$\left(a^2+1\right)\left(b^2+1\right)=(ab+1)(a+b).$$

5. Solve the equation
$$\sqrt{-3x^2+18x+37}+\sqrt{-5x^2+30x-41}=\sqrt{x^2-6x+109}$$

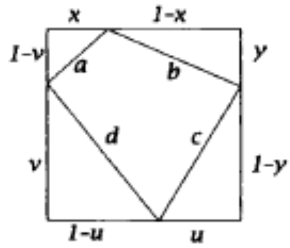
6. For positive real numbers a, b, c such that $abc \le 1$, prove that
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \;\ge\; a+b+c.$$

7. Determine the value of the expression

$$1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + 10 + 11 - 12 + \cdots + 94 + 95 - 96 + 97 + 98 - 99.$$

8. A quadrilateral has one vertex on each side of a square of side-length 1. Show that the lengths a, b, c and d of the sides of the quadrilateral satisfy the inequalities

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4.$$



9. Without using a calculator, evaluate

$$\sqrt[3]{2\sqrt{13}+5} - \sqrt[3]{2\sqrt{13}-5}.$$

10. Let a, b and c be real numbers such that $a + b + c = 5$ and $\dfrac{1}{b+c} + \dfrac{1}{c+a} + \dfrac{1}{a+b} = 6$. Evaluate the value of $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$.

SECTION B : 10 marks each

11. Determine x, if

$$\begin{aligned} 4^{\frac{x}{y}+\frac{y}{x}} &= 32 \\ \log_3(x-y) + \log_3(x+y) &= 1 \end{aligned}$$

12. Let $ABCD$ be a parallelogram with $AC/BD = k$. The bisectors of the angles formed by AC and BD intersect the sides of $ABCD$ at $K,\ L,\ M,\ N$. Prove that the ratio of the areas of $KLMN$ and $ABCD$ is $\dfrac{2k}{(k+1)^2}$.

13. (a) Find the number of subsets of $\{1, 2, \cdots, 10\}$ that contain its own size. For example, the set $(1, 3, 6)$ has 3 elements and contains 3.

    (b) Every vertex of the unit squares on an $n \times m$ grid is coloured either blue, green, or red, such that all the vertices on the boundary of the board are coloured red. We say that a unit square on the board is properly coloured if exactly one pair of adjacent vertices of the square are the same colour. Show that the number of properly coloured squares is even.

14. Let $x, y, z \geq 0$ be such that $x + y + z = 3$. Prove that

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \quad \geq \quad \frac{1}{9} + \frac{2}{27} \cdot (xy + yz + zx)$$

When does equality hold?

15. Let a positive integer k be given. Prove that there are infinitely many pairs of integers (a, b) with $|a| > 1$ and $|b| > 1$ such that $a^2 + b^2 + k$ is divisible by $ab + a + b$.

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 1 SOLUTIONS.

SATURDAY 6TH APRIL, 2019 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*

(b) *There are* **two** *sections and each section carries* 50 *marks. Attempt all questions in exactly three hours fifteen minutes (*$9:00am$ *to* $12:15pm$*). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*

(c) *Indicate your names,gender, class, school and district* ***on all*** *your answer sheets.*

(d) *Answer scripts of only qualified students should be returned to the contest coordinator, latest* **Monday 8th April, 2019**.

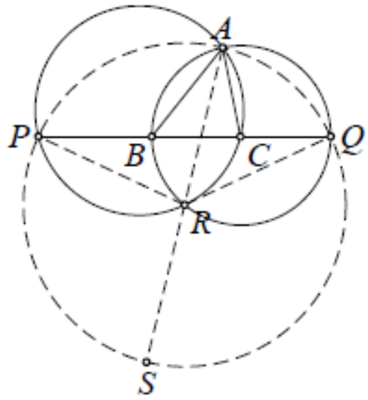
(e) **ALL** *participants who qualify for Paper* 2 *are cordially invited to the certificate and prize giving ceremony on Saturday* 27*th July,* 2019 *at NOON at Makerere University, Kampala.*

(f) *National Mathematics Contest* 2019 *Paper* 2 *will be done on* 29*th June,* 2019 *at various centres.*

SECTION A : 5 marks each

1. The equation is defined for $1+x>0, 1-x>0$ and $1-x^2>0$. The third one follows from the first two, so we can exclude it. What remains is $x \in (-1,1)$

$$\begin{aligned}
2+\log\sqrt{1+x}+3\log\sqrt{1-x} &= \log\sqrt{1-x^2} \\
2+\log\sqrt{1+x}+3\log\sqrt{1-x} &= \log\sqrt{1-x}+\log\sqrt{1+x} \\
2+2\log\sqrt{1-x} &= 0 \\
\log\sqrt{1-x} &= -1 \\
\sqrt{1-x} &= 10^{-1} = \frac{1}{10} \\
1-x &= \frac{1}{100} \\
x &= \frac{99}{100}
\end{aligned}$$

2. 

Let S be the intersection of AR with the circumcircle of $\triangle PAQ$. Then:

$$\begin{aligned}\angle PSA &= \angle AQP = \frac{1}{2}\angle ACP = \frac{1}{2}\angle ARP \Rightarrow \\ \angle PSA &= \angle SPR \Rightarrow \\ RP &= RS\end{aligned}$$

Similarly $RQ = RS$. Hence $RP = RQ$.

3. Let M denote the maximum number of cards remaining in any single suit. As Vanya proceeds, M will only decrease if the current card is in a suit with M cards remaining, and no other suit has M cards remaining. In this case, however, Vanya will correctly guess that suit. Therefore, Vanya will guess correctly every time M decreases. Since $M \geq \frac{52}{4} = 13$ initially and it is 0 by the end, Vanya will be correct at least 13 times.

4. We have

$$\begin{aligned}& 2(a^2+1)(b^2+1) - 2(ab+1)(a+b) \\ = \; & 2(a^2b^2 + a^2 + b^2 + 1 - a^2b - ab^2 - a - b) \\ = \; & (a^2b^2 - 2a^2b + a^2) + (a^2b^2 - 2ab^2 + b^2) + (a^2 - 2a + 1) + (b^2 - 2b + 1) \\ = \; & a^2(b-1)^2 + b^2(a-1)^2 + (a-1)^2 + (b-1)^2 \\ = \; & (b^2+1)(a-1)^2 + (a^2+1)(b-1)^2\end{aligned}$$

which is positive unless $a = 1$ and $b = 1$, which forms a solution. Therefore, the only solution is $a = b = 1$.

Remark: Alternatively, note that by Cauchy-Schwarz inequality, we have

$$(a^2+1)(b^2+1) \geq (ab+1)^2 \quad \text{and} \quad (a^2+1)(1+b^2) \geq (a+b)^2$$

and multiplying together yields $[(a^2+1)(b^2+1)]^2 \geq [(ab+1)(a+b)]^2$, so we can find all the solutions to the equation by consider the equality cases in the above inequalities.

5. After setting $t = (x-3)^2$, the equation reduces to

$$\sqrt{64 - 3t} + \sqrt{4 - 5t} = \sqrt{t + 100}.$$

Then $t = 0$ is one solution, and the left-hand side is decreasing in t while the right-hand side is increasing in t. Therefore, $t = 0$ (and hence $x = 3$) is the unique solution.

6. Using the weighted AM-GM, we have

$$\frac{2a}{b}+\frac{b}{c} \geq 3\cdot\sqrt[3]{\frac{a^2}{bc}} \geq 3a.$$

Similarly,

$$\begin{aligned}\frac{2b}{c}+\frac{c}{a} &\geq 3\cdot\sqrt[3]{\frac{b^2}{ca}} \geq 3b\\ \frac{2c}{a}+\frac{a}{b} &\geq 3\cdot\sqrt[3]{\frac{c^2}{ab}} \geq 3c\end{aligned}$$

Adding all the three inequalities,

$$\begin{aligned}\frac{2a}{b}+\frac{b}{c}+\frac{2b}{c}+\frac{c}{a}+\frac{2c}{a}+\frac{a}{b} &\geq 3a+3b+3c\\ 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) &\geq 3(a+b+c)\\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a} &\geq a+b+c\end{aligned}$$

giving the result.

Alternatively,

Replacing a, b, c by ta, tb, tc with $t=\dfrac{1}{\sqrt[3]{abc}}$ preserves the LHS while increases the RHS and makes at $at\cdot bt\cdot ct = abct^3 = 1$. Hence we may assume without loss of generosity that $abc = 1$.
Then there exists positive real numbers x, y, z such that

$$a=\frac{y}{x},\ b=\frac{z}{y},\ c=\frac{x}{z}$$

The rearrangement of the inequality gives

$$x^3+y^3+z^3 \geq x^2y+y^2z+z^2x$$

Thus

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{x^3+y^3+z^3}{xyz}\geq\frac{x^2y+y^2z+z^2x}{xyz}=a+b+c$$

as desired

7. Removing the initial 0, the remaining 99 terms can be written in groups of the form

$$(3k-2)+(3k-1)-3k \text{ for each } k \text{ from 1 to 33.}$$

The expression $(3k-2)+(3k-1)-3k$ simplifies to $3k-3$. Therefore, the given sum equals

$$0+3+6+\cdots+93+96$$

Since k ran from 1 to 33, then this sum includes 33 terms and so equals

$$\frac{33}{2}(0+96)=33(48)=33(50)-33(2)=1650-66=1584$$

8. First note that

$$a^2+b^2+c^2+d^2=x^2+(1-x)^2+y^2+(1-y)^2+u^2+(1-u)^2+v^2+(1-v)^2.$$

Now consider

$$x^2+(1-x)^2=2\left\{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\right\}.$$

Since $0\le x\le 1$, it follows easily that $\frac{1}{2}\le x^2+(1-x)^2\le 1$, and similarly with x replaced by y,u,v. Adding the four inequalities yields the desired result.

9. Let $A=\sqrt[3]{2\sqrt{13}+5}$ and $B=\sqrt[3]{2\sqrt{13}-5}$. Then we wish to solve for $A-B$. Notice that

$$\begin{aligned} A^3-B^3 &= \left(2\sqrt{13}+5\right)-\left(2\sqrt{13}-5\right)=10 \\ AB &= \sqrt[3]{\left(2\sqrt{13}+5\right)\left(2\sqrt{13}-5\right)}=3 \end{aligned}$$

such that

$$\begin{aligned} (A-B)^3 &= A^3-3A^2B+3AB^2-B^3 \\ &= A^3-B^3-3AB(A-B) \\ &= 10-3(3)(A-B) \\ &= 10-9(A-B) \\ (A-B)^3+9(A-B)-10 &= 0 \\ \left[(A-B)-1\right]\left[(A-B)^2+(A-B)+10\right] &= 0 \end{aligned}$$

since $(A-B)$ is real, $A-B=1$.

10. Note that

$$\begin{aligned} \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &= \frac{5-(b+c)}{b+c}+\frac{5-(c+a)}{c+a}+\frac{5-(a+b)}{a+b} \\ &= \frac{5}{b+c}+\frac{5}{c+a}+\frac{5}{a+b}-3 \\ &= 5(6)-3 \\ &= 27 \end{aligned}$$

SECTION B : 10 marks each

11. Checking if the system is defined for two variables is a hard task, so we shall find the eventual solutions to the system and check directly if the system is defined for them. We shall only write

$$\begin{aligned} x+y &> 0 \\ x-y &> 0 \end{aligned}$$

for now.

$$\begin{aligned} \frac{x}{y}+\frac{y}{x} &= \log_4 32 \\ \log_3(x^2-y^2) &= 1 \end{aligned}$$

$$\begin{aligned} \frac{x}{y}+\frac{y}{x} &= \frac{1}{2}\log_2 32 \\ x^2-y^2 &= 3 \end{aligned}$$

$$\begin{aligned} \frac{x^2+y^2}{xy} &= \frac{5}{2} \\ x^2-y^2 &= 3 \end{aligned}$$

$$\begin{aligned} x^2+y^2 &= \frac{5}{2}xy \\ x^2-y^2 &= 3 \end{aligned}$$

$$\begin{aligned} x^2-2xy+y^2 &= \frac{1}{2}xy \\ (x-y)(x+y) &= 3 \end{aligned}$$

$$\begin{aligned} (x-y)^2 &= \frac{xy}{2} \\ (x-y)(x+y) &= 3 \end{aligned}$$

We divide the first with the square of the second:

$$\frac{1}{(x+y)^2}=\frac{xy}{18} \quad \text{or} \quad (x+y)^2=\frac{18}{xy}.$$

We get the system

$$\begin{aligned} x^2+2xy+y^2 &= \frac{18}{xy} \\ x^2-2xy+y^2 &= \frac{xy}{2} \end{aligned}$$

we now subtract them

$$\begin{aligned} 4xy &= \frac{18}{xy} - \frac{xy}{2} \\ \frac{9}{2}xy &= \frac{18}{xy} \\ (xy)^2 &= 4 \\ xy &= \pm 2 \end{aligned}$$

But it can't be negative, because $2(x-y)^2 = xy$, therefore $xy = 2$. Substituting back into the system, we get

$$\begin{aligned} (x-y)^2 &= x^2 - 2xy + y^2 = x^2 + y^2 - 4 = 1 \\ x^2 - y^2 &= 3 \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= 5 \\ x^2 - y^2 &= 3 \end{aligned}$$

we subtract them: $2x^2 = 8$, so $x_1 = 2$ and $x_2 = -2$. $y_1 = \frac{2}{x_1} = 1$ and $y_2 = \frac{2}{x_2} = -1$. We now have to check if they are solutions. $x_1 + y_1 = 3 > 0$, $x_1 - y_1 = 1 > 0$, so $(2, 1)$ is a solution to the system. $x_2 + y_2 = -3 < 0$, so $(-2, -1)$ is not a solution. The final solution is $x = 2$.

12. Suppose K, L, M, N are on AB, BC, CD, DA, respectively. Let $P = AC \cap BD$. Since $ABCD$ is a parallelogram, $AP = PC$ and $BP = PD$. For any polygon $X_1X_2\cdots X_n$, we denote its area by $[X_1X_2\cdots X_n]$.

By the angle bisector theorem, $AK/KB = AP/PB = AC/BD = k$. Similarly, $AN/ND = k$. Hence, $AK = AB \models AN/AD = \frac{k}{k+1}$. Consequently, KN is parallel to BD. Furthermore,

$$[AKN] = \frac{k^2}{(k+1)^2}[ABD] = \frac{k^2}{2(k+1)^2}[ABCD].$$

Similarly,

$$\begin{aligned} [BKL] &= \frac{1}{2(k+1)^2}[ABCD], \\ [CLM] &= \frac{k^2}{2(k+1)^2}[ABCD], \\ [DMN] &= \frac{1}{2(k+1)^2}[ABCD]. \end{aligned}$$

Hence,

$$\begin{aligned} [KLMN] &= [ABCD] - [AKN] - [BKL] - [CLM] - [DMN] \\ &= \left[1 - \frac{k^2}{(k+1)^2} - \frac{1}{(k+1)^2}\right][ABCD] \\ &= \frac{2k}{(k+1)^2}[ABCD], \end{aligned}$$

as desired

13. (a) For each $k \in \{1, 2, \cdots, 10\}$, we want to find how many elements there are of size k. Note that $k \neq 0$ since $0 \notin \{1, 2, \cdots, 10\}$. Every subset of size k must contain k. The other $k-1$ elements can be anything. Therefore, there are $\binom{9}{k-1}$ subsets of size k that contains k. Therefore, the number of subsets that contain its own size is

$$\sum_{k=1}^{10} \binom{9}{k-1} = \sum_{k=0}^{9} \binom{9}{k} = 2^9 = 512$$

(b) We will call an edge of a square good if both its endpoints are of the same colour, and bad otherwise. Any square can have either no good edges (call such squares type A), one good edge (type B), two good edges (type C), or four good edges (type D). Note that it cannot have three good edges.

Let the number of squares of type $A,\ B,\ C,\ D$ be $a,\ b,\ c,\ d$ respectively. For each square, count the number of good edges, and add the numbers. We must get an even number, since each edge not on the boundary is counted twice, and all $2(m+n)$ edges on the boundary are good. On the other hand, this number is $b + 2c + 4d$. Hence b is even and the result follows.

Comments: We are counting in two different ways the number of (square, good edge) pairs. This kind of argument is very important in combinatorics problems!

14. Since $y \geq 0$, the AM-GM inequality implies

$$\frac{x^3}{y^3+8} + \frac{y+2}{27} + \frac{y^2-2y+4}{27} \quad \geq \quad 3 \cdot \sqrt[3]{\frac{x^3}{27^2}} = \frac{x}{3}$$

Similarly,

$$\begin{aligned} \frac{y^3}{z^3+8} + \frac{z+2}{27} + \frac{z^2-2z+4}{27} &\geq \frac{y}{3} \quad \text{and} \\ \frac{z^3}{x^3+8} + \frac{x+2}{27} + \frac{x^2-2x+4}{27} &\geq \frac{z}{3} \end{aligned}$$

Adding all the three inequalities, we have

$$\begin{aligned} \frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} &\geq \frac{x+y+z}{3} + \frac{y+z+x}{27} - \frac{6}{9} - \frac{y^2+z^2+x^2}{27} \\ &= \frac{4}{9} - \frac{(x+y+z)^2 - 2xy - 2yz - 2zx}{27} \\ &= \frac{1}{9} + \frac{2}{27} \cdot (xy + yz + zx) \end{aligned}$$

For equality to hold, we must have $\frac{y+2}{27} = \frac{y^2-2y+4}{27} \quad \Rightarrow \quad y^2 - 3y + 2 = 0$, so y equals 1 or 2. The same holds for z and x. Since $x + y + z = 3$, the only possibility is $x = y = z = 1$, and it is easy to check that equality does indeed hold in this case.

15. By inspection, we see that if $(a, b) = (0, 1)$, then $a^2 + b^2 + k$ is divisible by $ab + a + b$.

$$\frac{a^2+b^2+k}{ab+a+b} \quad = \quad k+1 \tag{1.1}$$

Let us rearrange this as a quadratic in a:

$$a^2 - (k+1)(b+1)a + (b-1)(b-k) \quad = \quad 0.$$

As a quadratic in a, the sum of the roots is $(k+1)(b+1)$. Hence, if (a,b) is a solution to (1.1), then so is $((k+1)(b+1)-a,b)$, and hence by symmetry of (1.1), so is $(b,(k+1)(b+1)-a)$.

Define a sequence (a_n) as follows: $a_1=0, a_2=1$, and

$$a_n = (k+1)\left(a_{n-1}+1\right) - a_{n-2}$$

for all $n \geq 2$. Then by the above reasoning, $(a,b)=(a_n,a_{n+1})$ is a solution of (1.1) for all $n \geq 0$. Furthermore, the sequence (a_n) is increasing, giving an infinite number of positive integer solutions.

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 2.

SATURDAY 29TH JUNE, 2019 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*

(b) *There are two sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm).*

(c) *Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.*

(d) ***Indicate your names, birth date, gender, class, school and district on all your answer sheets.***

(e) *Answer scripts of only qualified students should be returned to the contest coordinator, latest Monday 1st July, 2019.*

(f) *ALL participants who have sat for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 27th July, 2019 at 8:00am at Makerere University, Kampala.*

SECTION A : 5 marks each

Qn. 1. Suppose x, y and z are three real numbers such that $3x$, $4y$, $5z$ is a geometric sequence and $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ is an arithmetic sequence. What is the value of $\frac{x}{z} + \frac{z}{x}$?

Qn. 2. An equilateral triangle is inscribed in a circle of area $1m^2$. Then the second circle is inscribed in the triangle. Find the radius of the second circle.

Qn. 3. Given $x^2 + xy + y^2 = 84$ and $x - \sqrt{xy} + y = 6$, determine the value of xy.

Qn. 4. Four cards are laid out in front of you. You know for sure that on one side of each card is a single number, and on the other side of each card is a single geometric shape. The same number or the same geometric shape might be found on more than one of these four cards.

You see (on the top side) respectively a 2, a 5, a triangle, and a square. Your friend says: "Every card with a square has a 4 on the other side." Your task is to determine whether your friend is correct by choosing some of the cards to be flipped over. The chosen cards will only be flipped after you have made your choice(s). What is the fewest number of cards you can choose if you want to know for sure whether or not your friend is correct?

Qn. 5. List all real values of x which satisfy

$$\frac{6}{\sqrt{x-8}-9} + \frac{1}{\sqrt{x-8}-4} + \frac{7}{\sqrt{x-8}+4} + \frac{12}{\sqrt{x-8}+9} = 0$$

Qn. 6. Let $F_0 = 0$ and $F_1 = 1$. For all $n \geq 2$, define F_n to be the remainder of $F_{n-1} + F_{n-2}$ divided by 3. The sequence starts as follows: $0, 1, 1, 2, 0, 2, \ldots$ What is

$$F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}?$$

Qn. 7. Let ABC be a triangle with $\angle BAC \neq 90^0$. Let O be the circumcenter of the triangle ABC and let Γ be the circumcircle of the triangle BOC: Suppose that Γ intersects the line segment AB at P different from B; and the line segment AC at Q different from C: Let ON be a diameter of the circle Γ: Prove that the quadrilateral $APNQ$ is a parallelogram.

Qn. 8. How many integer triples (x, y, z) satisfy the following equation $x^2 + y^2 + z^2 = 2xyz$.

Qn. 9. How many different ways can one order the integers 1 through 5 so that no three consecutive integers in the ordering are in increasing order?

Qn. 10. Let $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ and let $f_n(x) = \underbrace{f(f(f(\cdots(f}_{n}(x))\cdots)))$. In other words, $f_1(x) = f(x)$ and then we recursively define $f_{n+1}(x)$ as $f(f_n(x))$. What is $f_{99}(1)$?

SECTION B : 10 marks each

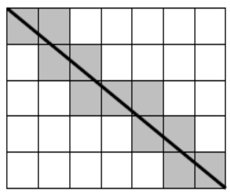
Qn. 11. Given a sequence $\{a_n\}$ of numbers satisfying $a_0 = 1$, $a_{n+1} = \dfrac{7a_n + \sqrt{45a_n^2 - 36}}{2}$, $n \in \mathbb{N}$. Show that

(a) for each $n \in \mathbb{N}$, a_n is a positive integer.

(b) for each $n \in \mathbb{N}$, $a_n \cdot a_{n+1} - 1$ is a perfect square.

Qn. 12. Let ABC be an acute triangle with $\angle BAC = 30^0$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2 respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2 respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P.
Prove that $\angle BPC = 90^0$.

Qn. 13. A diagonal of this 5×7 rectangle passes through 11 squares. (We say that a line passes through a square if the square and the line have at least two common points.) How many square will the diagonal of a 2016×2018 rectangle pass through?



Qn. 14. (a) Suppose f is a function such that for every real number x, $f(x) + f(1-x) = 10$ and $f(1+x) = 4 + f(x)$. What is the value of $f(100) + f(-100)$?

(b) Find all positive real numbers x and y that satisfy the following system of equations:

$$\begin{aligned} x^y &= y^{x-y} \\ x^x &= y^{12y} \end{aligned}$$

Qn. 15. You and your partner went to a dinner party in which there were four other couples. After the dinner was over, you asked everyone except yourself: "How many people did you shake hands with tonight?" To your surprise, no two people gave the same number, so that someone did not shake any hands, someone else shook only one person's hand, a third person only shook two people's hands, and so on. Assume nobody shook hands with their own partner or with themselves. How many people did your partner shake hands with that evening at the party?

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER 2 SOLUTIONS.

SATURDAY 29TH JUNE, 2019 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.

(b) There are two sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm).

(c) Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.

(d) ***Indicate your names, birth date, gender, class, school and district on all your answer sheets.***

(e) Answer scripts of only qualified students should be returned to the contest coordinator, latest Monday 1st July, 2019.

(f) ALL participants who have sat for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 27th July, 2019 at 8:00am at Makerere University, Kampala.

SECTION A : 5 marks each

Qn. 1. Suppose x, y and z are three real numbers such that $3x$, $4y$, $5z$ is a geometric sequence and $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ is an arithmetic sequence. What is the value of $\frac{x}{z} + \frac{z}{x}$?

Solution : *Since $3x, 4y, 5z$ is a geometric sequence,* $\frac{4y}{3x} = \frac{5z}{4y}$, *so that*

$$16y^2 = 15xz \tag{1.1}$$

Also, since $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ is an arithmetic sequence,

$$\frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{y}$$

or

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz}.$$

Which imply that

$$x + z = \frac{2xz}{y} \tag{1.2}$$

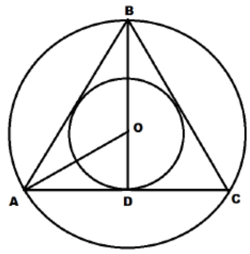
Therefore,

$$\begin{aligned}
\frac{x}{z}+\frac{z}{x} &= \frac{x^2+z^2}{xz} \\
&= \frac{(x+z)^2}{xz}-2 \\
&= \frac{(2xz/y)^2}{xz}-2 \qquad \textit{using Eqn (1.2)} \\
&= \frac{4xz}{y^2}-2 \\
&= \frac{64}{14}-2 \qquad \textit{using Eqn (1.1)} \\
&= \frac{34}{15}
\end{aligned}$$

■

Qn. 2. An equilateral triangle is inscribed in a circle of area $1m^2$. Then the second circle is inscribed in the triangle. Find the radius of the second circle.

Solution : *In an equilateral triangle the centres of inscribed and circumscribed circles are located at the same point O. AO and BO are bisectors and AD is the height. Therefore AOD is a right triangle and $\angle OAD$ is 30^0. Thus,*



$$|OD| = \frac{|AO|}{2}$$

. *Since the area of the circumscribed circle is* $1m^2$*,*

$$\pi|AO|^2 = 1$$

and

$$|AO| = \frac{1}{\sqrt{\pi}}.$$

Therefore, $|OD| = \frac{1}{2\sqrt{\pi}}$. ***Answer: The radius of the second cicle is*** $\frac{1}{2\sqrt{\pi}}m$.

■

Qn. 3. Given $x^2 + xy + y^2 = 84$ and $x - \sqrt{xy} + y = 6$, determine the value of xy.

Solution : *Rewrite the second equation as* $x + y = 6 + \sqrt{xy}$. *Squaring both then gives* $x^2 + y^2 + 2xy = 36 + xy + 12\sqrt{xy}$. *Subtracting this equation from* $x^2 + xy + y^2 = 84$ *and simplifying gives* $48 = 12\sqrt{xy}$, *thus* $xy = 16$ ■

Qn. 4. Four cards are laid out in front of you. You know for sure that on one side of each card is a single number, and on the other side of each card is a single geometric shape. The same number or the same geometric shape might be found on more than one of these four cards.

You see (on the top side) respectively a 2, a 5, a triangle, and a square. Your friend says: "Every card with a square has a 4 on the other side." Your task is to determine whether your friend is correct by choosing some of the cards to be flipped over. The chosen cards will only be flipped after you have made your choice(s). What is the fewest number of cards you can choose if you want to know for sure whether or not your friend is correct?

Solution : *Clearly you must flip over the square, to see if there is a* 4 *on the other side. You must also flip over the* 2 *and the* 5*: if either has a square on the other side, it contradicts your friend's statement. There is no need to flip over the triangle: the other side will have some number, and it doesn't matter whether it's a* 4 *or not. Therefore, the answer is a* 3. ■

Qn. 5. List all real values of x which satisfy

$$\frac{6}{\sqrt{x-8}-9}+\frac{1}{\sqrt{x-8}-4}+\frac{7}{\sqrt{x-8}+4}+\frac{12}{\sqrt{x-8}+9}=0$$

Solution : *Let* $y=\sqrt{x-8}$. *The equation becomes* $\dfrac{18y-54}{y^2-81}+\dfrac{8y-24}{y^2-16}=0$, *or* $(y-3)\left(\dfrac{18}{y^2-81}+\dfrac{8}{y^2-16}\right)=0$. *Thus* $y=3$ *or* $26y^2-(8\cdot 81+16\cdot 18)=0$. *The second equation yields* $y^2=36$, *so* $y=6$. *Finally* $x=3^2+8$ *or* 6^2+8. *Therefore the values are* 17 *and* 44 ■

Qn. 6. Let $F_0 = 0$ and $F_1 = 1$. For all $n \geq 2$, define F_n to be the remainder of $F_{n-1} + F_{n-2}$ divided by 3. The sequence starts as follows:

$$0, 1, 1, 2, 0, 2, \ldots$$

What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$?

Solution : *The first few terms of the sequence are* $0,\ 1,\ 1,\ 2,\ 0,\ 2,\ 2,\ 1,\ 0,\ 1,\ 1,$ *so the sequence is periodic with a period* 8. *Thus* $F_{2017} = F_1$ *since* $2017 = 8.252 + 1$, $F_{2018} = F_2$, *etc. The sum is* $1 + 1 + 2 + 0 + 2 + 2 + 1 + 0 = 9$. ■

Qn. 7. Let ABC be a triangle with $\angle BAC \neq 90^0$. Let O be the circumcenter of the triangle ABC and let Γ be the circumcircle of the triangle BOC: Suppose that Γ intersects the line segment AB at P different from B; and the line segment AC at Q different from C: Let ON be a diameter of the circle Γ: Prove that the quadrilateral $APNQ$ is a parallelogram.

Solution : *From the assumption that the circle Γ intersects both of the line segments AB and AC, it follows that the 4 points N, C, Q, O are located on Γ in the order of N, C, Q, O or in the order of N, C, O, Q.*
The following argument for the proof of the assertion of the problem is valid in either case. Since $\angle NQC$ and $\angle NOC$ are subtended by the same arc $\widehat{NC}$ of Γ at the points Q and O, respectively, on Γ , we have $\angle NQC = \angle NOC$: We also have $\angle BOC = 2\angle BAC$, since $\angle BOC$ and $\angle BAC$ are subtended by the same arc $\widehat{BC}$ of the circum-circle of the triangle ABC at the center O of the circle and at the point A on the circle, respectively. From $OB = OC$ and the fact that ON is a diameter of Γ, it follows that the triangles OBN and OCN are congruent, and therefore we obtain $2\angle NOC = \angle BOC$:
Consequently, we have $\angle NQC = \frac{1}{2}\angle BOC = \angle BAC$; which shows that the 2 lines AP, QN are parallel. In the same manner, we can show that the 2 lines AQ, PN are also parallel. Thus, the quadrilateral $APNQ$ is a parallelogram. ■

Qn. 8. How many integer triples (x, y, z) satisfy the following equation?

$$x^2 + y^2 + z^2 = 2xyz$$

Solution : *Since the left-hand side is even, either all of x, y, and z, or exactly one of x, y and z are even. If only one term is even, then the right side is divisible by 4 and the left side leaves a remainder of 2 after dividing by 4. So this can't be. Therefore, assume that $x = 2x_1$, $y = 2y_1$, $z = 2z_1$ for integers x, y, and z. Plugging these equations in, we obtain $4x_1^2 + 4y_1^2 + 4z_1^2 = 16x_1y_1z_1$, or $x_1^2 + y_1^2 + z_1^2 = 4x_1y_1z_1$. This is nearly the same as our original equation! By an argument identical to the one above, all of x_1, y_1 and z_1 must again be even.*
The pattern keeps repeating, and so we conclude that x, y, and z are all evenly divisible by an arbitrary power of 2. Therefore they are all zero. ■

Qn. 9. How many different ways can one order the integers 1 through 5 so that no three consecutive integers in the ordering are in increasing order?

Solution : *Let A_1 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ such that $a_1 < a_2 < a_3$; let A_2 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of $1, 2, 3, 4, 5$, such that $a_2 < a_3 < a_4$; and let A_3 be a set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ such that $a_3 < a_4 < a_5$. We need to count how many permutations of $1, 2, 3, 4, 5$ are not in the union of the sets A_1, A_2, and A_3.*

First, there are $5! = 120$ permutations of $1, 2, 3, 4, 5$.

Next by inclusion-exclusion

$$| A_1 \cup A_2 \cup A_3 | = | A_1 | + | A_2 | + | A_3 | - | A_1 \cap A_2 | - | A_1 \cap A_3 | - | A_2 \cap A_3 | + | A_1 \cap A_2 \cap A_3 | .$$

We have $|A_1| = |A_2| = |A_3| = \binom{5}{3} \cdot 2 = 20$ (for example, to compute $|A_1|$ note that there are $\binom{5}{3}$ ways to choose a_1, a_2, and a_3 and two ways to pick a_4 out of the remaining two numbers).

Also $A_1 \cap A_2$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_1 < a_2 < a_3 < a_4$, there are 5 such permutations (there are 5 choices for a_5). Similarly, $A_2 \cap A_3$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_2 < a_3 < a_4 < a_5$, there are 5 such permutations.

Finally, $A_1 \cap A_2 \cap A_3$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_1 < a_2 < a_3 < a_4 < a_5$, that is the set $\{1, 2, 3, 4, 5\}$.
Thus, $| A_1 \cup A_2 \cup A_3 | = 20 + 20 + 20 - 5 - 1 - 5 + 1 = 50$.

There are also other alternative methods. ■

Qn. 10. Let $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ and let $f_n(x) = \underbrace{f(f(f(\cdots(f}_{n}(x))\cdots)))$. In other words, $f_1(x) = f(x)$ and then we recursively define $f_{n+1}(x)$ as $f(f_n(x))$. What is $f_{99}(1)$?

Solution : *By recursion formula,*

$$
\begin{aligned}
f_1(x) = f(x) &= \frac{x}{\sqrt{1+x^2}} \\
f_2(x) = f(f(x)) &= \frac{x}{\sqrt{1+2x^2}} \\
\vdots &= \vdots \\
f_{99}(x) &= \frac{x}{\sqrt{1+99x^2}}
\end{aligned}
$$

Thus $f_{99}(1) = \dfrac{1}{10}$. ■

SECTION B : 10 marks each

Qn. 11. Given a sequence $\{a_n\}$ of numbers satisfying $a_0 = 1$, $a_{n+1} = \dfrac{7a_n + \sqrt{45a_n^2 - 36}}{2}$, $n \in \mathbb{N}$. Show that

(a) for each $n \in \mathbb{N}$, a_n is a positive integer.

Solution : *By assumption; $a_1 = 5$ and $\{a_n\}$ is strictly increasing with*

$$2a_{n+1} - 7a_n = \sqrt{45a_n^2 - 36}$$

Square both sides;

$$\begin{aligned}
(2a_{n+1} - 7a_n)^2 &= 45a_n^2 - 36 \\
4a_{n+1}^2 - 28a_na_{n+1} + 49a_n^2 &= 45a_n^2 - 36 \\
4a_{n+1}^2 - 28a_na_{n+1} + 4a_n^2 + 36 &= 0 \\
a_{n+1}^2 - 7a_na_{n+1} + a_n^2 + 9 &= 0 \qquad (1.3) \\
a_n^2 - 7a_{n-1}a_n + a_{n-1}^2 + 9 &= 0 \qquad (1.4)
\end{aligned}$$

From (1.3)-(1.4) we have

$$\begin{aligned}
a_{n+1}^2 - a_n^2 - 7a_na_{n+1} + 7a_na_{n+1} + a_n^2 - a_{n-1}^2 &= 0 \\
a_{n+1}^2 - 7a_na_{n+1} + 7a_na_{n+1} - a_{n-1}^2 &= 0 \\
a_{n+1}^2 - a_{n-1}^2 &= 7a_na_{n+1} - 7a_na_{n+1} \\
(a_{n+1} + a_{n-1})(a_{n+1} - a_{n-1}) &= 7a_n(a_{n+1} - a_{n+1}) \\
(a_{n+1} + a_{n-1}) &= 7a_n
\end{aligned}$$

such that

$$a_{n+1} = 7a_n - a_{n-1} \qquad (1.5)$$

From $a_0 = 1, a_2 = 5$ and Eqn (1.5), then a_n is always a positive integer. ∎

(b) for each $n \in \mathbb{N}$, $a_n \cdot a_{n+1} - 1$ is a perfect square.

Solution : *From Eqn (1.3),*

$$(a_{n+1} + a_n)^2 = 9(a_{n+1} \cdot a_n - 1) \quad \Rightarrow \quad a_{n+1} \cdot a_n - 1 = \left(\frac{a_{n+1} + a_n}{3}\right)^2$$

Therefore, $a_n \cdot a_{n+1} - 1$ is a perfect square. ∎

Qn. 12. Let ABC be an acute triangle with $\angle BAC = 30^0$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2 respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2 respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P.
Prove that $\angle BPC = 90^0$.

Solution : *Since $\angle B_1BB_2 = 90^0$, the circle having B_1B_2 as its diameter goes through the points $B : B_1 : B_2$: From $B_1A : B_1C = B_2A : B_2C = BA : BC$, it follows that this circle is the Apolonius circle with the ratio of the distances from the points A and C being $BA : BC$. Since the point P lies on this circle, we have*

$$PA : PC = BA : BC = \sin C : \sin A$$

; from which it follows that $PA \sin A = PC \sin C$: Similarly, we have $PA \sin A = PB \sin B$, and therefore,

$$PA \sin A = PB \sin B = PC \sin C$$

: Let us denote by $D : E : F$ the foot of the perpendicular line drawn from P to the line segment BC, CA and AB, respectively. Since the points E, F lie on a circle having PA as its diameter, we have by the law of sines $EF = PA \sin A$: Similarly, we have $FD = PB \sin B$ and $DE = PC \sin C$.
Consequently, we conclude that DEF is an equilateral triangle. Furthermore, we have $\angle CPE = \angle CDE$, since the quadrilateral $CDPE$ is cyclic. Similarly, we have $\angle FPB = \angle FDB$. Putting these together, we get;

$$\begin{aligned}
\angle BPC &= 360^0 - (\angle CPE + \angle FPB + \angle EPF) \\
&= 360^0 - \{(\angle CDE + \angle FDB) + (180^0 - \angle FAE)\} \\
&= 360^0 - (120^0 + 150^0) \\
&= 90^0.
\end{aligned}$$

which proves the assertion of the problem.

Alternate Solution:
Let O be the midpoint of the line segment B_1B_2. Then the points B and P lie on the circle with center at O and going through the point B_1. From

$$\angle OBC = \angle OBB_1 - \angle CBB_1 = \angle OB_1B - \angle B_1BA = \angle BAC$$

it follows that the triangles OCD and OBA are similar, and therefore we have that

$$OC \cdot OA = OB^2 = OP^2$$

Thus we conclude that the triangles OCP and OPA are similar, and therefore, we have $\angle OPC = \angle PAC$: Using this fact, we obtain

$$\begin{aligned}
\angle PBC - \angle PBA &= (\angle B_1BC + \angle PBB_1) - (\angle ABB_1 - \angle PBB_1) \\
&= 2\angle PBB_1 = \angle POB_1 = \angle PCA - \angle OPC \\
&= \angle PCA - \angle PAC
\end{aligned}$$

from which we conclude that $\angle PAC + \angle PBC = \angle PBA + \angle PCA$ *Similarly, we get* $\angle PAB + \angle PCB = \angle PBA + \angle PCA$. *Putting these facts together and taking into account the fact that*

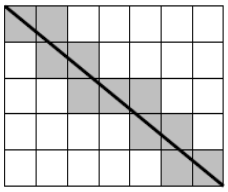
$$(\angle PAC + \angle PBC) + (\angle PAB + \angle PCB) + (\angle PBA + \angle PCA) = 180^0$$

we conclude that $\angle PBA + \angle PCA = 60^0$ *and finally that*

$$\angle BPC = (\angle PBA + \angle PAB) + (\angle PCA + \angle PAC) = \angle BAC + (\angle PBA + \angle PCA) = 90^0$$

proving the assertion of the problem. ■

Qn. 13. A diagonal of this 5×7 rectangle passes through 11 squares. (We say that a line passes through a square if the square and the line have at least two common points.) How many



square will the diagonal of a 2016×2018 rectangle pass through?

Solution : *First, we show that if we consider a rectangle with integer sides k and l where k and l are relatively prime positive integers with $k < l$, then any of its diagonals passes through $k + l - 1$ squares. We can assume that the bottom left corner of the rectangle is the origin of the coordinate system $(0, 0)$ and that the the remaining three vertices of the rectangle are $(l, 0)$, (l, k) and $(0, k)$. Also, let us consider the diagonal with endpoints $(0, k)$ and $(l, 0)$. Since the absolute value of the slope of the diagonal is less than 1, for each $t = 0, 1, \ldots, l - 1$ the diagonal passes either through one or two of the squares which are in the strip between the vertical lines $x = t$ and $x = t + 1$. It passes through one square if the diagonal does not intersect any of the horizontal lines $y = 1, y = 2, \ldots y = k - 1$ and it passes through two squares when the diagonal does intersect one of these lines.*

Thus, the number of squares the diagonal passes through equals the number of vertical strips (which is k) + the number of intersections of the diagonal and the horizontal lines $y = 1$, $y = 2, \ldots$, $y = k - 1$ (which is $k - 1$). So, indeed when k and l are relatively prime a diagonal passes through $k + l - 1$ squares.

Next, consider a 2016×2018 rectangle. We can assume that it has vertices $(0, 0)$, $(2018, 0)$, $(2018, 2016)$, $(0, 2016)$ and that the diagonal has endpoints $(0, 2016)$ and $(2018, 0)$. The center of the rectangle is $(1009, 1008)$. Since 1009 and 1008 are relatively prime, the diagonal passes through $1008 + 1009 - 1 = 2016$ squares of the rectangle with vertices $(0, 1008)$, $(1009, 1008)$, $(1009, 2016)$, $(0, 2016)$; it passes through $1008 + 1009 - 1 = 2016$ squares of the rectangle with vertices $(1009, 0)$, $(2018, 0)$, $(2018, 1008)$, $(1009, 1008)$; and it passes through no other squares. Therefore, passes through 4032 rectangles. ■

Qn. 14. (a) Suppose f is a function such that for every real number x, $f(x) + f(1 - x) = 10$ and $f(1 + x) = 4 + f(x)$. What is the value of $f(100) + f(-100)$?

Solution : *Replace x by $-x$ in the second property, and get $f(1 - x) = 4 + f(-x)$. Substitute this in the first property and get $f(x) + 4 + f(-x) = 10$ for all x.*
$\Rightarrow f(100) + f(-100) = 6$ ■

(b) Find all positive real numbers x and y that satisfy the following system of equations:

$$\begin{aligned} x^y &= y^{x-y} \\ x^x &= y^{12y} \end{aligned}$$

Solution :

$$ylnx = (x - y)lny \qquad\qquad lnx = \frac{(x - y)}{y}lny$$

$$xlnx = 12ylny \qquad\qquad \frac{x(x - y)}{y}lny = 12ylny$$

$$\Leftrightarrow$$

$$lnx = \frac{x - y}{y}lny$$

$$\left(\frac{x(x - y)}{y} - 12y\right) lny = 0$$

From the last equation either $lny = 0$ or $\dfrac{x^2 - xy - 12y^2}{y} = 0$

If $lny = 0$, then $lnx = 0$. Therefore, $x = 1$, $y = 1$ is a solution.

If $\dfrac{x^2 - xy - 12y^2}{y} = 0$, then $\dfrac{x^2}{y^2} - \dfrac{x}{y} - 12 = 0$ and $\left(\dfrac{x}{y} - 4\right)\left(\dfrac{x}{y} + 3\right) = 0$

Since x and y are positive numbers, $x = 4y$ and

$ln(4y) = 3ln(y)$. Therefore, $4y = y^3$, $y = 2$, and $x = 8$.

Thus, the system has two solutions

x=1	*and*	*x=8*
y=1		*y=2*

■

Qn. 15. You and your partner went to a dinner party in which there were four other couples. After the dinner was over, you asked everyone except yourself: "How many people did you shake hands with tonight?" To your surprise, no two people gave the same number, so that someone did not shake any hands, someone else shook only one person's hand, a third person only shook two people's hands, and so on. Assume nobody shook hands with their own partner or with themselves. How many people did your partner shake hands with that evening at the party?

Solution : *The maximum possible number of people whose hands anyone shook is 8, and the only way* 9 *people can give different answers if all integers from* 0 *through* 8 *are included. Let* A *be the one who shook the hands of* 8 *people. Since everyone shook hands with* A*, except for* A*'s partner and* A *him- or herself, then* A*'s partner has to be the one who shook* 0 *people's hands*
Similar reasoning shows that the partner of the one who shook 7 *people's hands, shook only* 1 *person's hand (specifically,* A*'s hand). The partner of the one who shook* 6 *(resp.* 5*) people's hands, shook* 2 *(resp.* 3*) people's hands.*
All these people are distinct from yourself, and also from your partner. This leaves 4 *for the answer given by your partner.*
Incidentally, with this information, it is possible to deduce that you also shook hands with 4 *people- the ones who shook* $8, 7, 6$ *and* 5 *people's hands.* ■

NATIONAL MATHEMATICS CONTEST

A-LEVEL PAPER.

SATURDAY 28TH MAY, 2022 9:00 AM - 12:15 PM

INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

(a) This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.

(b) There are **two** *sections and each carries* 50 *marks. Attempt all questions in exactly three hours fifteen minutes (*$9:00am$ *to* $12:15pm$*).*

(c) Indicate your names, birth date, gender, class, school and district ***on all*** *your answer sheets.*

(d) Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.

(e) Answer scripts of only registered students should be returned to the contest coordinator, latest Monday 30th May, 2022.

(f) ALL participants who score a pass mark are cordially invited to the certificate and prize giving ceremony on Saturday 23*rd July,* 2022 *at 8:00am at Makerere University, Kampala.*

SECTION A : 5 marks each

1. What is the value of the following product?

$$2^{2022}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{32}\right)\cdots\cos\left(\frac{\pi}{2^{2022}}\right)$$

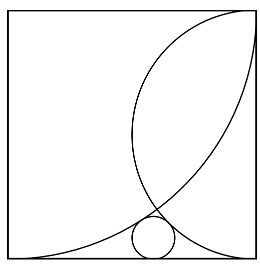
2. Find the function $f(x)$ such that $f(x)+f\left(\dfrac{1}{1-x}\right)=x$

3. Let $a_1=1, a_2=1+\dfrac{1}{1}, a_3=1+\dfrac{1}{1+\dfrac{1}{1}}, a_4=1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1}}}$ and continued in the same manner.

   Find a_{10}.

4. Suppose $x\neq y$ and the first terms of the two arithmetic sequences are x, a_1, a_2, a_3, y, a_4 and b_1, x, b_2, b_3, y, b_4. What is the value of $\dfrac{b_4-b_3}{a_2-a_1}$?

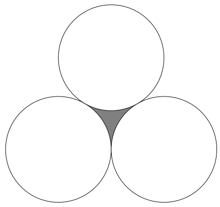
5. In a square with side length 1, a quarter circle is drawn with center at the top left corner and passing through the top right and bottom left corners, and a half circle is drawn with the right side of the square as its diameter. A smaller circle is drawn tangent to the quarter circle, the half circle, and the bottom edge (as shown). Determine its radius.



6. Given $x+\dfrac{1}{x}=-1$. Determine $x^{88}+\dfrac{1}{x^{88}}$.

7. Assume that real numbers a and b satisfy $ab + \sqrt{ab+1} + \sqrt{a^2+b} \cdot \sqrt{b^2+a} = 0$. Find, with proof, the value of $a\sqrt{b^2+a} + b\sqrt{a^2+b}$.

8. Given three circles of radius r, tangent to each other as shown in the following diagram, what is the area for the shaded region?



9. Darth Vader urgently needed a new Death Star battle station. He sent requests to four planets asking how much time they would need to build it. The Mandalorians answered that they can build it in one year, the Sorganians in one and a half year, the Nevarroins in two years, and the Klatooinians in three years. To expedise the work Darth Vader decided to hire all of them to work together. The Rebels need to know when the Death Star is operational. Can you help the Rebels to find the number of days needed if all four planets work together? We assume that one year=365 days.

10. The numbers 1447, 1005 and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

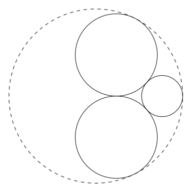
SECTION B : 10 marks each

11. Find all the real values that satisfy
$$\frac{27^x + 343^x}{63^x + 147^x} = \frac{37}{21}$$

12. (a) Find the product $a \cdot b$, given that
$$a = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{\cdots}}}}$$
$$b = \sqrt{9 - \sqrt{9 - \sqrt{9 - \sqrt{\cdots}}}}$$

(b) Find all sequences $a_1, a_2, a_3, \dots$ of real numbers such that for all positive integers $m, n \geq 1$, we have
$$\begin{aligned} a_{m+n} &= a_m + a_n - mn \\ a_{mn} &= m^2 a_n + n^2 a_m + 2a_m a_n. \end{aligned}$$

13. Two circles of radius 1 and one circle of radius 1/2 are drawn on a plane so that each of them is touching the other two at one point as shown. What is the radius of the largest circle (dashed) tangent to all three of these circles?



14. (a) Let a, b, c, and d be integers satisfying $a \log_{10} 2 + b \log_{10} 3 + c \log_{10} 5 + d \log_{10} 7 = 2022$. What is $a + b + c + d$?

(b) Consider the sequence $\{a_n\}$ given by $a_{n+1} = \dfrac{2(n+2)}{(n+1)} a_n$, with $a_1 = 2$. Compute the value of
$$\frac{a_{2022}}{a_1 + a_2 + \cdots + a_{2021}}.$$

15. Given a triangle A, B, C, X is on side AB, Y is on side AC, and P and Q are on side BC such that $AX = AY, BX = BP$ and $CY = CQ$. Let XP and YQ intersect at T. Prove that AT passes through the midpoint of PQ.

1. **Solution :** *The key is to repeatedly use the equality:*

$$\cos x = \frac{\sin 2x}{2\sin x}$$

The product then equals:

$$\begin{aligned}
& 2^{2022}\frac{\sin\left(\frac{\pi}{2}\right)}{2\sin\left(\frac{\pi}{4}\right)}\cdot\frac{\sin\left(\frac{\pi}{4}\right)}{2\sin\left(\frac{\pi}{8}\right)}\cdots\frac{\sin\left(\frac{\pi}{2^{2021}}\right)}{2\sin\left(\frac{\pi}{2^{2022}}\right)} \\
= \ & \frac{2^{2022}}{2^{2021}}\frac{\sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2^{2022}}\right)} \\
= \ & 2\frac{1}{\sin\left(\frac{\pi}{2^{2022}}\right)} \\
= \ & 2\csc\frac{\pi}{2^{2022}}
\end{aligned}$$

■

2. **Solution :** *Replace x with* $\dfrac{1}{1-x}$

$$f(x) + f\left(\frac{1}{1-x}\right) = x \qquad (1.1)$$

$$f\left(\frac{}{1-x}\right) + f\left(\frac{1}{1-\frac{1}{1-x}}\right) = \frac{1}{1-x}$$

But $\dfrac{1}{1-\frac{1}{1-x}} = \dfrac{1}{\frac{(1-x)-1}{1-x}} = \dfrac{1}{\frac{-x}{1-x}} = \dfrac{1-x}{-x} = \dfrac{-1}{x}$

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} \qquad (1.2)$$

Replace x with $\dfrac{x-1}{x}$

$$\begin{aligned}
f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-\frac{x-1}{x}}\right) &= \frac{x-1}{x} \\
f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-\frac{x-1}{x}}\right) &= \frac{x-1}{x} \\
f\left(\frac{x-1}{x}\right) + f\left(\frac{x}{1}\right) &= \frac{x-1}{x} \\
f\left(\frac{x-1}{x}\right) + f(x) &= \frac{x-1}{x} \qquad (1.3)
\end{aligned}$$

(1.1)+(1.3)−(1.2) we have,

$$
\begin{aligned}
2f(x) &= x + \left(\frac{x-1}{x}\right) - \frac{1}{(1-x)} \\
2f(x) &= x + \frac{x-1}{x} + \frac{1}{x+1} \\
2f(x) &= \frac{x(x)(x-1)+(x-1)(x-1)+x}{x(x-1)} \\
2f(x) &= \frac{x^2(x-1)+(x-1)(x-1)+x}{x(x-1)} \\
2f(x) &= \frac{x^3-x^2+x^2-2x+1+x}{x(x-1)} \\
2f(x) &= \frac{x^3-x+1}{x(x-1)} \\
f(x) &= \frac{x^3-x+1}{2x(x-1)}
\end{aligned}
$$

■

3. **Solution :** $\dfrac{89}{55}$.
We note that $a_1 = \dfrac{1}{1}, a_2 = \dfrac{2}{1}, a_3 = \dfrac{3}{2}, a_4 = \dfrac{5}{3}$ *and, in general, if* $a_n = \dfrac{p}{q}$ *then* $a_{n+1} = \dfrac{p+q}{p}$ *. We can prove this via the identity* $a_{n+1} = 1 + \dfrac{1}{a_n}$.

In any event, we observe that the numerator and denominator are Fibbonacci numbers and just carry out this pattern to a_{10}*:*

$$a_5 = \frac{8}{5}, a_6 = \frac{13}{8}, a_7 = \frac{21}{13}, a_8 = \frac{34}{21}, a_9 = \frac{55}{34}, a_{10} = \frac{89}{55}$$

■

4. **Solution :** *Let* d_1 *be the difference of the first arithmetic sequence, and let* d_2 *be the difference of the second arithmetic sequence. Considering the first sequence we get* $y - x = 4d_1$*, and considering the second sequence we obtain* $y - x = 3d_2$*. In that case,* $\dfrac{b_4 - b_3}{a_2 - a_1} = \dfrac{2d_2}{d_1} = \dfrac{8}{3}$. ■

5. **Solution :** *Assume the diagram is oriented so that the bottom left has coordinates* $(0,0)$ *and the top right* $(1,1)$*. Write* (a,x) *for the coordinates of the smallest circle; then the radii of the three circles are (from smallest to largest)* $x, \frac{1}{2}$ *, and* 1.
The line from $(0,1)$ *to* (a,x) *has length* $1+x$*, as it comprises two radii. Therefore we have* $a^2 + (1-x)^2 = (1+x)^2$*, and hence* $a^2 = 4x$*. Similarly, considering a line from* $\left(1, \dfrac{1}{2}\right)$ *to* (a,x) *we conclude that* $(1-a)^2 + \left(\dfrac{1}{2} - x\right)^2 = \left(\dfrac{1}{2} + x\right)^2$*, which reduces to* $(1-a)^2 = 2x$.
So $4x = 2(1-a)^2 = a^2$*, so that* $a^2 - 4a + 2 = 0$*. This quadratic equation has the solutions* $a = 2 \pm \sqrt{2}$*, of which only* $2 - \sqrt{2}$ *makes sense geometrically. Therefore*

$$x = \frac{3}{2} - \sqrt{2}$$

■

6. **Solution :**

Method I

Multiplying by x we obtain,

$$\begin{aligned} x^2+1 &= x \\ x^2+x+1 &= 0 \\ x &= \frac{1\pm\sqrt{1-4(1)(1)}}{2} \\ x &= \frac{1\pm\sqrt{3}i}{2} \end{aligned}$$

Since $x^2+x+1=0$

$$\begin{aligned} a^3-b^3 &= (a-b)(a^2+ab+b^2) \\ x^3-1^3 &= (x-1)(x^2+x+1) \\ x^3-1^3 &= (x-1)(0) \\ x^3-1 &= 0 \\ \Rightarrow\quad x^3 &= 1 \end{aligned}$$

Therefore

$$\begin{aligned} x^{88} &= x\cdot x^{87} \\ 87 &= x^{3\cdot 29} \quad = (x^3)^{29} \quad 1^{29} \quad = 1 \\ \Rightarrow\quad x^{88} &= x\cdot x^{87} \quad = x\cdot x^{3(29)} \quad = x\cdot (x^3)^{29} \\ &\Leftrightarrow \\ x^{88}+\frac{1}{x^{88}} &= x\cdot(x^3)^{29}+\frac{1}{x\cdot(x^3)^{29}} \\ &= x\cdot(1)^{29}+\frac{1}{x\cdot(1)^{29}} \\ &= x+\frac{1}{x} \quad = -1 \end{aligned}$$

Alternatively;

Method II

One an apply the Demorve's theorem where;

$$\begin{aligned} x^n &= r^n\big(\cos n\theta + i\sin n\theta\big) \\ x &= \frac{1}{2}\pm\frac{\sqrt{3}i}{2} \\ \left(\frac{1}{2}\pm\frac{\sqrt{3}i}{2}\right)^{88} &= r^{88}\big(\cos 88\theta \pm \sin 88\theta\big) \\ r &= \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \quad = 1 \end{aligned}$$

$$\begin{aligned}
\theta &= tan^{-1}\left(\frac{y}{x}\right) \quad = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) \quad = \tan^{-1}\left(\sqrt{3}\right) \quad = \frac{\pi}{3} \\
\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^{88} &= (1)^{88}\left(\cos\left(88\cdot\left(\frac{\pi}{3}\right)\right)+i\sin\left(88\cdot\left(\frac{\pi}{3}\right)\right)\right) \\
x^{88}+\frac{1}{x^{88}} &= \left(\cos\left(88\cdot\left(\frac{\pi}{3}\right)\right)+i\sin\left(88\cdot\left(\frac{\pi}{3}\right)\right)\right)+\frac{1}{\left(\cos\left(88\cdot\left(\frac{\pi}{3}\right)\right)+i\sin\left(88\cdot\left(\frac{\pi}{3}\right)\right)\right)}
\end{aligned}$$

and

$$\begin{aligned}
\theta &= tan^{-1}\left(\frac{y}{x}\right) \quad = \tan^{-1}\left(\frac{-\sqrt{3}2}{\frac{1}{2}}\right) \quad = \tan^{-1}\left(-\sqrt{3}\right) \quad = -\frac{\pi}{3} \\
\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^{88} &= (1)^{88}\left(\cos\left(88\cdot\left(\frac{\pi}{3}\right)\right)-i\sin\left(88\cdot\left(\frac{\pi}{3}\right)\right)\right) \\
&= \left(\cos\left(88\cdot\left(\frac{\pi}{3}\right)\right)-i\sin\left(88\cdot\left(\frac{\pi}{3}\right)\right)\right) \\
x^{88}+\frac{1}{x^{88}} &= \left(\cos\left(88\cdot\left(\frac{\pi}{3}\right)\right)-i\sin\left(88\cdot\left(\frac{\pi}{3}\right)\right)\right)+\frac{1}{\left(\cos\left(88\cdot\left(\frac{\pi}{3}\right)\right)-i\sin\left(88\cdot\left(\frac{\pi}{3}\right)\right)\right)}
\end{aligned}$$

Then one can use the above $x's$ to obtain $\quad x^{88}+\dfrac{1}{x^{88}}$ ■

7. **Solution :** *Let us rewrite the given equation as follows.*

$$ab+\sqrt{a^2+b}\sqrt{b^2+a}=-\sqrt{ab+1}.$$

Squaring this gives us

$$\begin{aligned}
a^2b^2+2ab\sqrt{a^2+b}\sqrt{b^2+a}+(a^2+b)(b^2+a) &= ab+1 \\
(a^2b^2+a^3)+2ab\sqrt{a^2+b}\sqrt{b^2+a}+(a^2b^2+b^3) &= 1 \\
\left(a\sqrt{b^2+a}+b\sqrt{a^2+b}\right)^2 &= 1 \\
a\sqrt{b^2+a}+b\sqrt{a^2+b} &= \pm 1.
\end{aligned}$$

Next, we show that $a\sqrt{b^2+a}+b\sqrt{a^2+b}>0$. Note that

$$ab=-\sqrt{ab+1}-\sqrt{a^2+b}\cdot\sqrt{b^2+a}<0,$$

so a and b have opposite signs. Without loss of generality, we may assume $a>0>b$. Then rewrite

$$a\sqrt{b^2+a}+b\sqrt{a^2+b}=a\left(\sqrt{b^2+a}+b\right)-b\left(a-\sqrt{a^2+b}\right)$$

and, since $\sqrt{b^2+a}+b$ and $a-\sqrt{a^2+b}$ are both positive, the expression above is positive. Therefore,

$$a\sqrt{b^2+a}+b\sqrt{a^2+b}=1,$$

and the proof is finished. ■

8. **Solution :** *If we connect the centers of each of the circles, we obtain an equilateral triangle, each side of which has a length of* $2r$. *This triangle has a height of* $r\sqrt{3}$, *and thus the area of this triangle is* $\frac{(r\sqrt{3})(2r)}{2} = \sqrt{3}r^2$. *To find the shaded area, we take the area of the triangle, and we subtract three copies of the area of the 3 sectors of the circles with central angles*

$$\frac{\pi}{3} \to A = \frac{1}{2}r^2\left(\frac{\pi}{3}\right) = \frac{\pi r^2}{6}.$$

Thus the shaded area is

$$\sqrt{3}r^2 - 3\left(\frac{\pi r^2}{6}\right) = \sqrt{3}r^2 - \left(\frac{\pi r^2}{2}\right) = \left(\sqrt{3} - \frac{\pi}{2}\right)r^2$$

■

9. **Solution :** *Denote by* x *the total amount of work. In one day the Mandalorians complete* $\frac{x}{365}$, *the Sorganians* $\frac{2}{3}\cdot\frac{x}{365}$, *the Nevarroins* $\frac{1}{2}\cdot\frac{x}{365}$, *and the Klatooinians* $\frac{1}{3}\cdot\frac{x}{365}$. *Thus, together in one day they complete*

$$\frac{x}{365} + \frac{2}{3}\cdot\frac{x}{365} + \frac{1}{2}\cdot\frac{x}{365} + \frac{1}{3}\cdot\frac{x}{365} = \frac{x}{365}\cdot\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}\right)$$
$$= \frac{5}{2}\cdot\frac{x}{365} = \frac{x}{146}.$$

Therefore, working together they will build the battle station in 146 *days.* ■

10. **Solution :** ***Solution 1***

Suppose that the two identical digits are both 1. *Since the thousands digit must be* 1, *only one of the other three digits can be* 1. *This means the possible forms for the number are* $11xy$, $1x1y$, $1xy1$

Because the number must have exactly two identical digits, $x \neq y$, $x \neq 1$, *and* $y \neq 1$. *Hence, there are* $3 \cdot 9 \cdot 8 = 216$ *numbers of this form.*

Now suppose that the two identical digits are not 1. *Reasoning similarly to before, we have the following possibilities:* $1xxy$, $1xyx$, $1yxx$.

Again, $x \neq y$, $x \neq 1$, *and* $y \neq 1$. *There are* $3 \cdot 9 \cdot 8 = 216$ *numbers of this form.*

Thus the answer is $216 + 216 = \boxed{432}$.

Solution 2

Consider a sequence of 4 *digits instead of a* 4*-digit number. Only looking at the sequences which have one digit repeated twice, we notice that the probability that the sequence starts with 1 is* $\frac{1}{10}$. *This means we can find all possible sequences with one digit repeated twice, and then divide by* 10.

If we let the three distinct digits of the sequence be a, b, *and* c, *with* a *repeated twice, we* *can make a table with all possible sequences:*

$$\begin{array}{lll} aabc & abac & abca \\ baac & baca & \\ bcaa & & \end{array}$$

There are 6 *possible sequences.*

Next, we can see how many ways we can pick a, b, and c. This is $10(9)(8) = 720$, *because there are* 10 *digits, from which we need to choose* 3 *with regard to order. This means there are* $720(6) = 4320$ *sequences of length* 4 *with one digit repeated. We divide by 10 to get* $\boxed{432}$ *as our answer.*

Solution 3 (Complementary Counting)

We'll use complementary counting. We will split up into 3 *cases: (1) no number is repeated, (2)* 2 *numbers are repeated, and* 2 *other numbers are repeated, (3)* 3 *numbers are repeated, or (4)* 4 *numbers are repeated.*

Case 1: There are 9 *choices for the hundreds digit (it cannot be* 1*),* 8 *choices for the tens digit (it cannot be* 1 *or what was chosen for the hundreds digit), and* 7 *for the units digit. This is a total of* $9 \cdot 8 \cdot 7 = 504$ *numbers.*

Case 2: One of the three numbers must be 1*, and the other two numbers must be the same number, but cannot be* 1 *(That will be dealt with in case 4). There are* 3 *choices to put the* 1*, and there are* 9 *choices (any number except for* 1*) to pick the other number that is repeated, so a total of* $3 \cdot 9 = 27$ *numbers.*

Case 3: We will split it into 2 *subcases: one where* 1 *is repeated* 3 *times, and one where another number is repeated* 3 *times. When* 1 *is repeated* 3 *times, then one of the digits is not* 1*. There are* 9 *choices for that number, and* 3 *choices for its location,so a total of* $9 \cdot 3 = 27$ *numbers. When a number other than* 1 *is repeated* 3 *times, then there are* 9 *choices for the number, and you don't have any choices on where to put that number. So in Case 3 there are* $27 + 9 = 36$ *numbers*

Case 4: There is only 1 *number:* 1111.

There are a total of 1000 4*-digit numbers that begin with* 1 *(from* 1000 *to* 1999*), so by complementary counting you get* $1000 - (504 + 27 + 36 + 1) = \boxed{432}$ *numbers.* ■

11. **Solution :**

$$\begin{aligned}
\frac{(3^3)^x + (7^3)^x}{(3^2\cdot 7)^x + (3\cdot 7^2)^x} &= \frac{37}{21}\\
\frac{(3^3)^x + (7^3)^x}{(3\cdot 7)^x\,[3^x + 7^x]} &= \frac{37}{21}\\
a^3 + b^3 &= (a+b)(a^2 - ab + b^2)\\
\frac{(3^x)^3 + (7^x)^3}{(3\cdot 7)^x\,[3^x + 7^x]} &= \frac{(3^x + 7^x)\left((3^x)^2 - (3^x)(7^x) + (7^x)^2\right)}{(3\cdot 7)^x\,[3^x + 7^x]}\\
&= \frac{\cancel{(3^x + 7^x)}\left((3^x)^2 - (3^x)(7^x) + (7^x)^2\right)}{(3\cdot 7)^x\,\cancel{[3^x + 7^x]}}\\
&= \frac{3^{2x} - 3^x 7^x + 7^{2x}}{(3\cdot 7)^x}\\
\frac{3^{2x} - 3^x 7^x + 7^{2x}}{(3\cdot 7)^x} &= \frac{37}{21}\\
\frac{3^x\cdot 3^x}{3^x\cdot 7^x} - \frac{3^x 7^x}{3^x\cdot 7^x} + \frac{7^x\cdot 7^x}{3^x\cdot 7^x} &= \frac{37}{21}\\
\frac{3^x}{7^x} - 1 + \frac{7^x}{3^x} &= \frac{37}{21}\\
\left(\frac{3}{7}\right)^x - 1\left(\frac{7}{3}\right)^x &= \frac{37}{21}
\end{aligned}$$

$$\textit{Let } u = \left(\frac{3}{7}\right)^x \quad \Rightarrow \quad \left(\frac{3}{7}\right)^x = \frac{1}{\left(\frac{3}{7}\right)^x} = \frac{1}{u}$$

$$\begin{aligned}
u - 1 + \frac{1}{u} &= \frac{37}{21}\\
u^2 - u + 1 &= \frac{37}{21}u\\
21u^2 - 21u + 21 &= 37u\\
21u^2 - 58u + 21 &= 0
\end{aligned}$$

Solving the quadratic equation

$$\begin{aligned}
21u^2 - 58u + 21 &= 0\\
21u^2 - 9u - 49u + 21 &= 0\\
3u(7u-3) - 7(7u-3) &= 0\\
(7u-3)(3u-7) &= 0\\
u = \frac{3}{7} \quad &or \quad u = \frac{7}{3}\\
\left(\frac{3}{7}\right)^x = \left(\frac{3}{7}\right) \quad &or \quad \left(\frac{3}{7}\right)^x = \left(\frac{7}{3}\right)\\
\left(\frac{3}{7}\right)^x = \left(\frac{3}{7}\right)^1 \quad &or \quad \left(\frac{3}{7}\right)^x = \left(\frac{3}{7}\right)^{-1}\\
x = 1 \quad &or \quad x = -1
\end{aligned}$$

■

12. (a) **Solution :** *Notice that* $a^2 = 6 + a$; *therefore,* $a = 3$ *(we cannot have* $a = -2$*). Similarly, note that* $b^2 = 9 - b$; *thus,* $b = \frac{(\sqrt{37}-1)}{2}$ *(we keep only the positive value). From these, it is simple to compute their product as* $\frac{3}{2}(\sqrt{37}-1)$. ■

(b) **Solution :** *Multiplying the second equation by 2 and adding* m^2n^2 *to both sides, we find*

$$2a_{mn} + (mn)^2 = (2a_m + m^2)(2a_n + n^2).$$

Therefore, define $b_n = 2a_n + n^2$, *so the second equation can be written as* $b_{mn} = b_m b_n$. *Multiplying the first equation by 2 and adding* $(m+n)^2$, *we find*

$$2a_{m+n} + (m+n)^2 = (2a_m + m^2) + (2a_n + n^2).$$

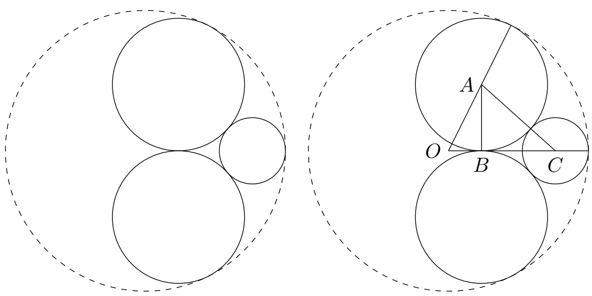
Therefore, $b_{m+n} = b_m + b_n$. *Therefore, it suffices to find sequences* $b_1, b_2, b_3, \ldots$ *that satisfy both* $b_{mn} = b_m b_n$ *and* $b_{m+n} = b_m + b_n$.

We claim that if $b_{m+n} = b_m + b_n$ *for all positive integers* m, n, *then* $b_n = nb_1$. *Certainly this is true for* $n = 1$. *If* $b_k = kb_1$, *then* $b_{k+1} = b_k + b_1 = kb_1 + b_1 = (k+1)b_1$, *so by induction, it follows that* $b_n = nb_1$ *for all positive integers* n. *Additionally, we observe that any sequence of this form will satisfy* $b_{m+n} = b_m + b_n$.

Substituting $m = n = 1$ *into* $b_{mn} = b_m b_n$, *we find* $b_1 = b_1^2$, *so* $b_1 = 0$ *or* $b_1 = 1$. *Combining this with the previous paragraph, the only possible sequences are* $b_n = 0$ *or* $b_n = n$. *Indeed, both sequences satisfy both* $b_{m+n} = b_m + b_n$ *and* $b_{mn} = b_m b_n$.

Therefore, since $a_n = \frac{b_n - n^2}{2}$, *it follows that the two possible sequences are* $a_n = -\frac{n^2}{2}$ *or* $a_n = \frac{n - n^2}{2}$. ■

13. **Solution :** *In the diagram, label points* O *(the center of the largest circle),* A *(the center of one circle of radius* 1*),* B *(the intersection of the two circles of radius* 1*), and* C *(the center of the circle of radius* $1/2$*).*



The points O *and* A *are collinear with the point of tangency of the circle of radius* 1 *with the largest circle, and the radius can then be expressed as* $OA + 1$.

The points O, B *and* C *are collinear with the point of tangency of the circle of radius* $1/2$ *with the largest circle. The radius can also be expressed as* $OB + BC + 1/2$.

Note that $AC = 1 + 1/2 = 3/2$; *by the Pythagorean Theorem in triangle* $\triangle ABC$, *we have* $BC = \sqrt{5}/2$.

This gives two equations with two variables:

$$\begin{aligned} OA + 1 &= OB + \frac{1+\sqrt{5}}{2} && \text{the two radii have the same length} \\ OA^2 &= OB^2 + 1 && \text{Pythagorean Theorem on triangle } \triangle OAB \end{aligned}$$

Solving for OA, for instance, gives us that $r = OA + 1 = 1 + \dfrac{\sqrt{5}}{2}$. ■

14. (a) **Solution :** *Raising both sides to the tenth power, we get*

$$2^a 3^b 5^c 7^d = 10^{2022},$$

so $a = c = 2022$ and $b = d = 0$. So the answer is 4044. ■

(b) **Solution :** *From the definition, we have*

$$\begin{aligned} a_n &= \frac{2(n+1)}{n} a_{n-1} \\ &= \frac{2(n+1)}{n} \cdot \frac{2n}{n-1} a_{n-2} \\ &= \cdots \\ \\ &= \frac{2(n+1)}{n} \cdot \frac{2n}{n-1} \cdots \frac{2 \cdot 3}{2} a_1 \\ &= 2^{n-1}(n+1) \end{aligned}$$

The sum of the first n terms of $\{a_n\}$, denoted by S_n, is given by:

$$S_n = 2 + 2 \times 3 + 2^2 \times 4 + \cdots + 2^{n-1}(n+1).$$

Then

$$2S_n = 2^2 + 2^2 \times 3 + 2^3 \times 4 + \cdots + 2^n(n+1),$$

so the difference of the above two equations gives

$$\begin{aligned} S_n &= 2^n(n+1) - (2 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1}) \\ &= 2^n(n+1) - 2^n \\ &= 2^n n. \end{aligned}$$

Therefore,

$$\frac{a_{2022}}{a_1 + a_2 + \cdots + a_{2021}} = \frac{2^{2021} \times 2023}{2^{2021} \times 2021} = \frac{2023}{2021}.$$

■

15. **Solution :**

Let AT intersect BC at M. Let R, S be on sides AB, AC, respectively such that $MR||XP$ and $MS||YQ$. Then since $BX = BP, BR = BM$. Therefore, $XR = PT$. Similarly, $YS = QT$.

We need to show that $PT = QT$. It suffices to show that $XR = YS$.

Since $RM||XT$ and $SM||YT$, $\dfrac{AR}{AX} = \dfrac{AM}{AT} = \dfrac{AS}{AY}$. Since $AX = AY, AR = AS$. Therefore, $RX = SY$, as desired.

■

SECTION A : 5 marks each

1. Let a and b be positive integers, prove that the equation $x^2 + (a+b)^2 x + 4ab = 1$ has rational solutions if and only if $a = b$.

Solution : ($\Leftarrow$) *if $a = b$, the equation becomes*

$$x^2 + 4a^2 x + 4a^2 - 1 = 0$$

and its solutions are:

$$x_1 = -1$$

and

$$x_2 = 1 - 4a^2,$$

which are clearly rational

($\Rightarrow$) *Assume that the equation*

$$x^2 + 5x + p = 0,$$

with

$$s = (a+b)^2$$

and

$$p = 4ab - 1,$$

has rational roots.

Then the discriminant $\Delta = s^2 - 4p$ must be a square, hence

$$s^2 - 4p = n^2$$

for some n a positive integer ($n \in \mathbb{N}$).

Since $4p > 0$ it follows that $n < s$, but s and n have the same parity.

Therefore, $n \leq s - 2$.

We obtain

$$s^2 - 4p \leq (s-2)^2 \quad \Leftrightarrow \quad s \leq p + 1,$$

which is equivalent to

$$(a+b)^2 \leq 4ab.$$

This results in

$$(a-b)^2 = 0,$$

hence

$$a = b$$

■

2. Find all pairs of integers m and n that satisfy $m^2 = 57 + n^2$.

Solution :

$$m^2 - n^2 = 57.$$

That is,

$$(m-n)(m+n) = 3 \cdot 19.$$

Case 1: $m - n = 1$ *and* $m + n = 57$

We obtain $(m, n) = (29, 28)$. *In fact,* $29^2 - 28^2 = 57$.

Case 2: $m - n = 3$ *and* $m + n = 19$

We obtain $(m, n) = (11, 8)$. *In fact,* $11^2 - 8^2 = 57$.

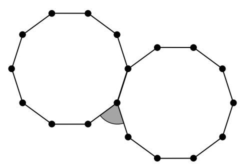
There are two pairs: $(m, n) = (29, 28)$ *and* $(11, 8)$.

■

3. Given an $n \times n \times n$ grid of unit cubes, a cube is good if it is a sub-cube of the grid and has side length at least two. If a good cube contains another good cube and their faces do not intersect, the first good cube is said to *properly* contain the second. What is the size of the largest possible set of good cubes such that no cube in the set properly contains another cube in the set?

Solution : *Let S be the set of good cubes with side length 2 or 3. For any $s \in S$, we define C_s to be the set of good cubes that have s as their centre. Then we can see that for any s and any pair of distinct cubes in C_s, that one cube properly contains the other. Also, for any good cube c_1 in the grid, there exists a cube $c_2 \in S$ such that c_2 is the centre of c_1. Thus, the size of largest possible set of good cubes with no cube properly containing each other is at most $|S|$. Finally, we observe that no cube in S can properly contain another cube in S. Thus $|S| = (n-1)^3 + (n-2)^3$ is the size of the largest possible set.* ■

4. This floor plan of the portal to the Addams castle shows twin chambers and a wedge shaped pit (shaded) where a trap door will be placed. Find the degree measure of the acute angle at the tip of the wedge. The chambers are congruent regular ten-sided polygons.



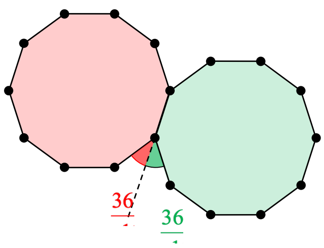
Solution 1 : *One exterior angle of a regular n-sided polygon measures*

$$\frac{360°}{n}$$

One exterior angle of a regular decagon (10-sided polygon) measures

$$\frac{360°}{10} = 36°.$$

The marked angle contains two exterior angles of regular decagons as shown.



The marked angle in the original diagram measures

$$2 \cdot 36^\circ = 72^\circ.$$

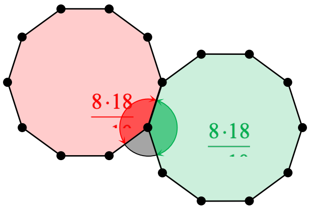
■

Solution 2 : *One interior angle of a regular n-sided polygon measures*

$$\frac{(n-2)\cdot 180^\circ}{n}$$

One interior angle of a regular decagon measures

$$\frac{8 \cdot 180^\circ}{10} = 144^\circ$$



The marked angle in the original diagram measures

$$360^\circ - 2 \cdot 144^\circ = 72^\circ$$

■

5. What is the sum of the digits of the integer equal to $3 \times 10^{500} - 2022 \times 10^{497} - 2022$?

Solution :

$$\begin{aligned} 3 \times 10^{500} - 2022 \times 10^{497} - 2022 &= 3000 \times 10^{497} - 2022 \times 10^{497} - 2022 \\ &= (3000 - 2022) \times 10^{497} - 2022 \\ &= 978 \times 10^{497} - 2022 \\ &= (978 \times 10^{497} - 1) - 2021 \end{aligned}$$

The integer $978 \times 10^{497} - 1$ is the 500-digit integer with leading digits 977 followed by 497 $9'$s. Thus, the digits of the given integer, from left to right, are 9, 7, and 7 followed by $497 - 4 = 493$ $9'$s and the final four digits are $9 - 2 = 7$, $9 - 0 = 9$, $9 - 2 = 7$, and $9 - 1 = 8$. Therefore, the sum of the digits of the integer is

$$9 + 7 + 7 + (493 \times 9) + 7 + 9 + 7 + 8 = 4491.$$

■

6. Suppose $f(x) = x^2 + 12x + 30$.

Find all real numbers x for which $f(f(f(f(f(x))))) = 0$.

Solution :

$$f(x) = (x+6)^2 - 6.$$

So

$$\begin{aligned} f(f(x)) &= f((x+6)^2-6) \\ &= (x+6)^4 - 6, \\ &\dots \\ f(f(f(f(f(x))))) &= (x+6)^{32} - 6. \end{aligned}$$

The equation becomes

$$x + 6^{32} = 6.$$

The 32 *roots are*

$$x = -6 + \sqrt[32]{6}e^{\frac{2k\pi}{32}i}$$

for

$$k = 0, 1, 2, \dots, 31.$$

We have only two real roots when $k = 0$ and $k = 16$ (the regular 32-gon has only two vertices on the x-axis):

$$x = -6 + \sqrt[32]{6}$$

and

$$x = -6 - \sqrt[32]{6}.$$

■

7. Let $a_0, a_1, a_3, \dots$ be a be a sequence of integers (positive, negative, or zero) such that for all nonnegative integers n and k,

$$a_{n+k}^2 - (2k+1)a_n a_{n+k} + (k^2+k)a_n^2 = k^2 - k.$$

Find all possible sequence a_n.

Solution 1 : *We can factor the equation as*

$$\left[a_{n+k} - ka_n\right]\left[a_{n+k} - (k+1)a_n\right] = k(k-1). \quad (1.1)$$

Substituting in $k = 1$, we find that either $a_{n+1} = 2a_n$ or $a_{n+1} = a_n$ for all integers n. If p is a prime such that $p \mid a_r$, then by repeatedly applying $a_{n+1} = 2a_n$ or $a_{n+1} = a_n$, we find that p divides a_n for all integers $n \geq r$. Substituting $k = 2$ and $n = r$ into (1.1) we notice that the LHS is divisible by p^2, and the RHS is equal to 2; hence $p^2 \mid 2$, which is a contradiction. Thus a_n is never divisible by any prime for any n, so $a_n = \pm 1$.

In particular, $a_{n+1} = a_n$ for all integers n, and a_n is constant-either 1 or -1. Substituting either of these sequences into (1.1) shows that both are valid solutions. ■

Solution 2 : *After multiplying by 4 and completing the square twice, we find*

$$2a_{n+k} - (2k+1)a_n)^2 - (2k-1)^2 = a_n^2 - 1.$$

As the closest square to $(2k-1)^2$ is $(2k-2)^2$ (for k positive), we see that if $x \neq 2k-1$ is an integer, then $\mid x^2 - (2k-1)^2 \mid \geq 4k-3$. Thus if $\mid a_n^2 - 1 \mid \neq 0$, then $\mid a_n^2 - 1 \mid \geq 4k-3$ for all positive integers k. But this contradicts the fact that $\mid a_n^2 - 1 \mid$ is finite. Therefore, $a_n^2 = 1$, and $\mid 2a_{n+k} - (2k+1)a_n \mid = 2k-1$. Now if $a_{n+k} = -a_n$, then $\mid 2a_{n+k} - (2k+1)a_n \mid = 2k+3$, which is not equal to $2k-1$, hence $a_{n+k} \neq -a_n$. But as $a_n^2 = 1$ for all n, we see that either $a_n = 1$ for all n, or $a_n = -1$ for all n. It is easy to check that both of these sequences work. ■

Solution 3 : *Factoring the equation as in (1.1) with $k=1$, we find that either $a_{n+1} = a_n$ or $a_{n+1} = 2a_n$ for all nonnegative integers n. It follows that, for all nonnegative integers n, either*

(i) $a_{n+2} = a_n$, *or*

(ii) $a_{n+2} = 2a_n$, *or*

(iii) $a_{n+2} = 4a_n$.

Now factor the equation with $k=2$, and we get

$$(a_{n+2} - 2a_n)(a_{n+2} - 3a_n) = 2.$$

Fix $n \geq 0$. Consider the cases (7i), (7ii), and (7iii). In case (7ii), the above gives $0 = 2$, a contradiction. In case (7i), the above gives $(-a_n)(-2a_n) = 2$, so, $a_n = 1$. In case (7iii), $(2a_n)(a_n) = 2$. In any case, $a_n^2 = 1$, so that $a_n = \{-1, 1\}$ for all n.
Because a_{n+1} has to be either a_n or $2a_n$, and $a_n \in \{-1, 1\}$ for all n, a_n must be constant. Thus the only sequences that satisfy the desired property are $a_n = 1$ for all $n \geq 0$, and $a_n = -1$ for all $n \geq 0$. ■

8. Find the smallest number of nightshade seedlings Uncle Fester could have if when he arranges them with 8 in each row, he has three left over and when he puts 9 in each row, he has 4 left over.

Solution 1 : *We are searching the number n:*

$$n = 3 = -5 \mod 8,$$
$$n = 4 = -5 \mod 9.$$

The smallest positive value of n is

$$lcm(8,9) - 5 = 72 - 5 = 67$$

■

Solution 2 :

We are searching the number n:

$$n = 3 \mod 8,$$
$$n = 4 \mod 9.$$

If you don't see the same "negative" remainders in solution 8, we may do the following.

Let $n = 8k + 3$ for some nonnegative integer k. We have

$$8k + 3 = 4 \mod 9.$$

Do mod calculations:

$$8k = 1 \mod 9 \to -k = 1 \mod 9 \to k = -1 = 8 \mod 9.$$

The smallest nonnegative value of k is 8.

The smallest positive value of n is $n = 8 \cdot 8 + 3 = 67$.

The answer is 67.

■

Solution 3 :

If you don't know the "mod" sign, we may think this way.

Let Uncle Fester borrow 5 nightshade seedlings.

When he arranges the original nightshade seedlings +5 borrowed with 8 in each row, he has none left over and when he puts 9 in each row, he has none left over.

Then the least number of nightshade seedlings is the least common multiple of 8 and 9. It is $8 \cdot 9 = 72$.

Now ask Uncle Fester to return 5 nightshade seedlings.

The answer is $72 - 5 = 67$.

■

Solution 4 :

We list the numbers that yield remainder 3 upon division by 8 until we hit a number that yields remainder 4 upon division by 9:

$$3, 11, 19, 27, 35, 43, 51, 59, 67, \ldots.$$

We see that 67 yields remainder 4 upon division by 9.

The answer is 67.

■

9. Let ABC be a triangle with $|AB| < |AC|$, where $|\cdot|$ denotes length. Suppose D, E, F are points on side BC such that D is the foot of the perpendicular on BC from A, AE is the angle bisector of $\angle BAC$, and F is the midpoint of BC. Further suppose that $\angle BAD = \angle DAE = \angle EAF = \angle FAC$. Determine all possible values of $\angle ABC$.

Solution :

The only possible value is $\angle ABC = 90^\circ$.

Let $\theta = \angle BAD = \angle DAE = \angle EAF = \angle FAC$.

Let O be the circumcentre of ΔABC. Then note that

$\angle BAD = 90 - \angle ABD = 90 - \angle ABC = 90 - \frac{1}{2}\angle AOC = \angle OAC$. Therefore, $\theta = \angle OAC = \angle FAC$.

Therefore, A, F, O are collinear.

Since $FB = FC$ and $OB = OC, F, O$ lie on the perpendicular bisector of BC. If F, O are distinct points, then A is also on the perpendicular bisector of BC. But $|AB| < |AC|$, thus this is impossible. Therefore, $F = O$. Thus, the midpoint of BC is the circumcentre of ΔABC. We conclude that $\angle BAC = 90°$.

■

10. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying for all $x, y \in \mathbb{R}^+$ the equality

$$f(xf(y)) = f(xy) + x$$

Solution : *Let*

$$P(x,y) = f(xf(y)) = (f(xy) + x)$$

$$\begin{aligned} P(f(x), y) &= f(f(x)f(y)) = (f(f(x)y) + f(x)) = f(f(x)f(y)) + y + f(x) \\ P(f(y), x) &= f(f(x)f(y)) = f(f(x)f(y)) + x + f(y) = P(f(x), y) \end{aligned}$$

So

$$f(x) - x = f(y) - y = c$$

$$f(x) = x + c$$

Putting it into original equation we get $c = 1$. Therefore $f(x) = x + 1$ is appropriate.

■

SECTION B : 10 marks each

11. The notation $[2, 3, 5, 13]$ is an abbreviation for the fraction

$$2 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{13}}}$$

Solve for the positive integer a that satisfies this equation:

$$[a+1, a, a-1] - [a-1, a, a+1] = [1, 1, 665, 2]$$

Solution : *The equation is*

$$a + 1 + \cfrac{1}{a + \cfrac{1}{a-1}} - a + 1 - \cfrac{1}{a + \cfrac{1}{a+1}} = 1 + \cfrac{1}{1 + \cfrac{1}{665 + \cfrac{1}{2}}}$$

Let us simplify:

$$\begin{aligned} \frac{a-1}{a^2-a+1} - \frac{a+1}{a^2+a+1} &= -\frac{2}{1333}, \\ \frac{a^3-1-(a^3+1)}{(a^2-a+1)(a^2+a+1)} &= -\frac{2}{1333}, \\ \frac{-2}{(a^2-a+1)(a^2+a+1)} &= -\frac{2}{1333}, \\ (a^2-a+1)(a^2+a+1) &= 1333 \\ &= 31 \cdot 43. \end{aligned}$$

Note that

$$6^2 - 6 + 1 = 31$$

and

$$6^2 + 6 + 1 = 43$$

We obtain

$$a = 6$$

■

12. The integers a and b have a property that the expression

$$\frac{2n^3 + 3n^2 + an + b}{n^2 + 1}$$

is an integer for every integer n. What is the value of the expression above when $n = 4$?

Solution :

$$\begin{aligned}\frac{2n^3 + 3n^2 + an + b}{n^2 + 1} &= \frac{2n^3 + 2n + 3n^2 + 3 + (a-2)n + (b-3)}{n^2+1} \\ &= \frac{2n(n^2+1) + 3(n^2+1) + (a-2)n + (b-3)}{n^2+1} \\ &= 2n + 3 + \frac{(a-2)n + (b-3)}{n^2+1}\end{aligned}$$

The quantity $2n + 3$ is an integer if n is an integer, so the given expression in n is an integer exactly when $\frac{(a-2)n + (b-3)}{n^2+1}$ is an integer.
Suppose that $a - 2 \neq 0$ and consider the function $f(x) = x^2 + 1$, $g(x) = (a-2)x + b - 3$, and $h(x) = f(x) - g(x) = x^2 - (a-2)x + 4 - b$.
The graph of $h(x)$ is a parabola with a positive coefficient of x^2, which means that there are finitely many integers n with the property that $h(n) \leq 0$ (there could be no such integers). Since $h(n) = f(n) - g(n)$, we have shown that there are only finitely many integers n for which $g(n) < f(n)$ is false.
The graph of $g(x)$ is a line that is neither vertical nor horizontal since $a - 2 \neq 0$, so there must be infinitely many integers for which $g(n) > 0$.
Since infinitely many integers satisfy $0 < g(n)$ and only finitely many integers fail $g(n) < f(n)$, there must exist an integer n with the property that $0 < g(n) < f(n)$.
This means there exists an integer n for which

$$0 < \frac{g(n)}{f(n)} < 1$$

but

$$\frac{g(n)}{f(n)} = \frac{(a-2)n + b - 3}{n^2+1}$$

is not an integer. Therefore, we have shown that if $a - 2 \neq 2$, then there is an integer n for which the expression in the problem is not an integer, so we conclude that $a - 2 = 0$ or $a = 2$. We now have that the expression in the question is equivalent to

$$2n + 3 + \frac{b-3}{n^2+1}$$

which is an integer exactly when $\dfrac{b-3}{n^2+1}$ *ia an integer.*
Similarly, if $b-3$ *is negative, then there are integers* n *for which* $\dfrac{b-3}{n^2+1}$ *is not an integer.*
This shows that if $b-3 \neq 0$*, then there are integers* n *for which the expression in the question is not an integer.*
We can conclude that $b-3=0$¡ *or* $b=3$*.*
Therefore, we have that $a=2$ *and* $b=3$*, so the expression given in the problem is equivalent to*

$$2n+3+\frac{(2-2)n+(3-3)}{n^2+1}=2n+3$$

Thus, the answer to the question is $2(4)+3=11$*.* ■

13. Morticia gives a number between 1 and 65 to Lurch, and then Pugsley tries to guess what it is by asking Lurch questions. Pugsley asks first "Is your number greater than 20?" and Lurch answers. Then Pugsley asks "Is your number even or odd?" and Lurch answers. "Hmm," says Pugsley, "Is your number a perfect square?" When Lurch answers, Pugsley says "I think I almost have it. I will know if you tell me whether one of the digits in your number is a 4." Lurch answers and Pugsley says "I have it!" Just then, Morticia looks up and says "Oh Pugsley-I forgot to tell you! Lurch is having a backwards day today. Every answer he gives is wrong." Pugsley furrows his brow. After a minute he says, "Thanks. That is ok. I can still tell what his number is!" What does Pugsley think is Lurch's number?

Solution : *We analyze in two cases.*

Case 1: the answer is "no" to "is your number greater than 20*?"*

The numbers are from 2 *to* 20 *inclusive.*

We divide the numbers into two groups: even and odd

Even:	$2, 4, 6, 8, 10, 12, 14, 16, 18, 20$
Odd:	$3, 5, 7, 9, 11, 13, 15, 17, 19$

There is a digit 4 *only in some even numbers. Hence the answer to "is your number even or odd?" is "even".*

In the even numbers there are 2 *square numbers:* 4 *and* 16*. One number contains a digit* 4 *and the other doesn't.*

In the even numbers there are 8 *non-square numbers:* $2, 6, 8, 10, 12, 14, 18, 20$*. There are* 7 *numbers not containing a digit* 4*.*

Since Pugsley can know the number by knowing whether of one of digits of the number is a 4*, the answer to "is your number a perfect square?" is "yes".*

If the answer is "yes" to "whether one of the digits is a 4*, Lurch's number is* 4*.*

If the answer is "no" to "whether one of the digits is a 4*, Lurch's number is* 16*.*

Now we examine which situation with the opposite answers still allows Pugsley to determine Lurch's number.

We use the opposite answers for all.

The numbers are greater than $20 : 21$ *to* 63 *inclusive.*

The numbers are odd: $21, 23, 25, 27, 29, \ldots, 47, 49, 51, 53, \ldots, 63$*.*

The numbers are non-squares: $21, 23, 27, 29, \ldots, 47, 51, 53, \ldots, 63$*.*

If the answer is "no" to "whether one of the digits is a 4*, the number can be one of* $21, 23, 27, 29, \ldots, 39, 51, 53, \ldots, 63$*. Pugsley cannot determine the number.*

If the answer is "yes" to "whether one of the digits is a 4*, the number can be one of* $41, 43, 45, 47$*. Pugsley cannot determine the number either.*

Therefore, there is no solution in this case.

Case 2: the answer is "yes" to "is your number greater than 20*?"*

The numbers are from 21 *to* 63 *inclusive.*

We divide the numbers into two groups: even and odd

$$\begin{aligned} \textit{Even:} &\quad 22, 24, 26, 28, \ldots, 62 \\ \textit{Odd:} &\quad 21, 23, 25, 27, \ldots, 63 \end{aligned}$$

Subcase a: to "is your number even or odd?", the answer is "even"

Since there is only one square number: 36*, the answer to "is your number a perfect square?" is "no".*

Then Pugsley cannot determine Lurch's number because there are two or more numbers containing a digit 4 *and there are two or more numbers not containing a digit* 4 *in the nonsquares.*

Subcase b: to "is your number even or odd?", the answer is "odd"

There are two squares 25 *and* 49*. One contains a digit* 4*, and the other doesn't. Therefore, the answer to "is your number a perfect square?" is "yes".*

If the answer is "yes" to "whether one of the digits is a 4*?", the number is* 49*.*

If the answer is "no" to "whether one of the digits is a 4*?", the number is* 25*.*

Now we examine which situation with the opposite answers still allows Pugsley to determine Lurch's number.

We use the opposite answers for all.

The numbers are less than or equal to $20 : 2$ *to* 20 *inclusive.*

The numbers are even: $2, 4, 6, 8, 10, 12, 14, 16, 18$*.*

The numbers are non-squares: $2, 6, 8, 10, 12, 14, 18$*.*

If the answer is "no" to "whether one of the digits is a 4*?", the number cannot be determined.*

If the answer is "yes" to "whether one of the digits is a 4*?", the number is* 14*. Pugsley can still determine Lurch's number.*

Therefore, Lurch's number is 14*.*

Let us make a summary:

Lurch's number is 14*.*

To "is your number greater than 20*?", the liar's answer is "yes" (the true answer is "no").*

To "is your number even or odd?", the liar's answer is "odd" (the true answer is "even").

To "is your number a perfect square?", the liar's answer is "yes" (the true answer is "no").
To "whether one of the digits is a 4?", the liar's answer is "no" (the true answer is "yes").
The answer to the problem is 14.

■

14. Find $x \in \left(0, \frac{3}{4}\right)$ so that

$$\log_x(1-x) + \log_2 \frac{1-x}{x} = \frac{1}{(\log_2 x)^2}.$$

Solution :

Let

$$a = \log_2 x$$

and

$$b = \log_2(1-x).$$

Then

$$\frac{b}{a} + b - a = \frac{1}{a^2}$$

hence

$$(a+1)(a^2 - a - ab + 1) = 0$$

If $a^2 - a - ab + 1 = 0$, *then* $a \neq 0$ *and*

$$\begin{aligned} b &= \frac{a^2 - a + 1}{a} \\ &= a + \frac{1}{a} - 1, \end{aligned}$$

yielding

$$a + \frac{1}{a} = b + 1$$

But $x \in \left(0, \frac{3}{4}\right)$ *gives* $1 - x \in \left(\frac{1}{4}, 1\right)$,
Hence

$$b = \log_2(1-x) \in (-2, 0)$$

Therefore, $a + \frac{1}{a} \in (-1, 1) \to$ *impossible,*
Since $\left|t + \frac{1}{t}\right| \geq 2$, *for all* $t \neq 0$
This leaves $a + 1 = 0$, *hence*

$$\begin{aligned} \log_2 x &= -1 \\ \Rightarrow \quad x &= 2^{-1} \\ &= \frac{1}{2} \end{aligned}$$

■

QUESTIONS AND SOLUTIONS

SECTION A : 5 marks each

1. Find the value of $F(13.333)$ for the function F defined by $F(x) = 2x$ for $x < 1$ and $F(x) = F(x-1)$ for $x \geq 1$.

Solution

$$\begin{aligned} F(13.333) &= F(12.333) \\ &= \dots \\ &= F(1.333) \\ &= F(0.333) \\ &= 2 \cdot 0.333 \\ &= 0.666 \end{aligned}$$

2. Let x, y, z be positive real numbers. Use the **AM-GM** (arithmetic mean - geometric mean) inequality to prove:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{x+z}.$$

Solution

From the **AM-GM** (arithmetic mean - geometric mean) inequality, we have:

$$(x+y)^2 \geq 4xy,$$

and from this inequality one gets:

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}. \tag{1.1}$$

Similarly,

$$\frac{1}{y} + \frac{1}{z} \geq \frac{4}{y+z}, \quad \text{and} \quad \frac{1}{x} + \frac{1}{z} \geq \frac{4}{x+z}. \tag{1.2}$$

Now, summing up all the inequalities in (1.1) and (1.2), together with some simplification, yields:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{x+z},$$

as required.

3. Prove that for any integer n, the number

$$2023^{5^{n+1}} + 2023^{5^n} + 1$$

is not a prime number.

Solution

Let $x = 2023^{5^m}$. Then

$$2023^{5^{n+1}} + 2023^{5^n} + 1 = x^5 + x + 1 = (x^2 + x + 1)(x^3 - x^2 + 1),$$

and since both factors are clearly greater than 1, the conclusion that $2023^{5^{n+1}} + 2023^{5^n} + 1$, is not a prime number follows.

Alternatively, Using the fact that $2023 \equiv 1(\mod 3)$, we have that $2023^{5^n} \equiv 1^{5^n} \equiv 1(\mod 3)$, similarly, $2023^{5^{n+1}} \equiv 1^{5^{n+1}} \equiv 1(\mod 3)$,
So, $2023^{5^{n+1}} + 2023^{5^n} + 1 \equiv 1 + 1 + 1 \equiv 3 \equiv 0(\mod 3)$
Therefore $2023^{5^{n+1}} + 2023^{5^n} + 1$ is divisible by 3 for any n, Hence, it is not a prime.

4. Triangle ABC has side length equal to a, b, c. Find a necessary and sufficient condition on the angles A, B, C of the triangle ABC such that $a^2,\ b^2,\ c^2$ can be side lengths of another triangle.

Solution

The positive real numbers a^2, b^2, c^2 can be the side length of a triangle if and only if:

$$a^2 + b^2 > c^2, \quad b^2 + c^2 > a^2, \quad \text{and} \quad a^2 + c^2 > b^2. \tag{1.3}$$

By the Cosine Rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \text{and} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

we get that (1.3) is equivalent to:

$$\cos A > 0, \quad \cos B > 0, \quad \text{and} \quad \cos C > 0.$$

Thus, the necessary and sufficient condition is that triangle ABC is acute, that is, all the angles of triangle ABC are less than 90^o.

5. Find the product of the roots of the equation:

$$y^2 - 31y + 220 = 2^y(31 - 2y - 2^y).$$

Solution

The equation $y^2 - 31y + 220 = 2^y(31 - 2y - 2^y)$ is equivalent to the equation:

$$(y + 2^y)^2 - 31(y + 2^y) + 220 = 0. \tag{1.4}$$

Consequently, the roots of (1.4) satisfy:

$$y + 2^y = 11 \quad \text{or} \quad y + 2^y = 20. \tag{1.5}$$

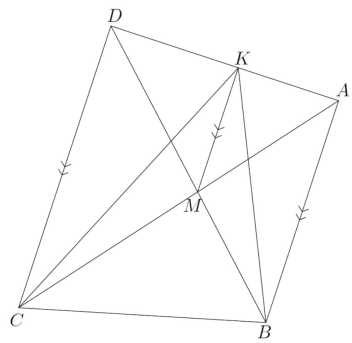
Since the function $f(y) = y + 2^y$, for $y \in \mathbb{R}$ is increasing, then each of the equations in (1.5) has at most one root. One can easily verify that $y_1 = 3$ satisfies the first equation in (1.5) and $y_2 = 4$ satisfies the second equation in (1.5). Thus, the product of roots is given by:

$$y_1 y_2 = 3 \times 4 = 12.$$

6. Let $ABCD$ be a quadrilateral such that $\angle BAD = \angle ADC = 90^o$. Diagonals AC and BD meet at M. Let K be a point on side AD such that $\angle ABK = \angle DCK$. Prove that KM bisects $\angle BKC$.

Solution

Since $\angle BAD = \angle ADC = 90^o$ and $\angle ABK = \angle DCK$, triangles CDK and BAK are similar, so we have $CD/AB = DK/KA$. Since CD and AB are parallel, triangles CDM and ABM are also similar, we also have $CD/AB = DM/MB$. Hence, $DK/KA = DM/MB$. Therefore, CD, AB, and KM are all parallel. Finally, $\angle MKC = \angle DCK = \angle ABK = \angle MKB$, as required.



7. An integer has 3 digits in base 10. When it is interpreted in base 6 and multiplied by 4, the result is 2 more than 3 times the same number when it is interpreted in base 7. What is the largest integer, in base 10, that has this property?

Solution

Let the three-digit number be rst. Since $4 \times rst_6 = 3 \times rst_7 + 2$, we have:

$$\begin{aligned} 4(36r + 6s + t) &= 3(49r + 7s + t) + 2 \\ 144r + 24s + 4t &= 147r + 21s + 3t + 2 \\ \Rightarrow \quad 3s + t &= 3r + 2. \end{aligned}$$

Hence, 3 divides $t - 2$. That is, $t - 2 = 3k$ for $k = 0, 1, 2$. Since $t < 6$, we have that $t - 2 = 0$ and $t - 2 = 3$ or $t = 2$ and $t = 5$. If $t = 2$, then $3t = 3s$ or $r = s$. If $t = 5$, then $3s + 3 = 3r$ or $r = s + 1$. Since all digits are at most 5, the largest required number, in base 10, is $rst = 552$.

8. If a, b, c are real numbers with $a - b = 4$, find the maximum value of $ac + bc - c^2 - ab$.

Solution

$$\begin{aligned} ac + bc - c^2 - ab &= (a-c)(c-b) \\ &= \left[\sqrt{(a-c)(c-b)}\right]^2 \\ &\leq \left[\frac{(a-c)+(c-b)}{2}\right]^2 \\ &= \left[\frac{a-b}{2}\right]^2 \end{aligned}$$

$$= \left[\frac{4}{2}\right]^2$$
$$= 4$$

The maximum is attained when $a = 4, b = 0$, and $c = 2$.

9. Malcolm writes a positive integer on a piece of paper. Malcolm doubles this integer and subtracts 1, writes this second integer on the same piece of paper. Malcolm then doubles the second integer and adds 1, writes this third integer on the paper. If all of the numbers Malcolm writes down are prime, determine all possible values for the first integer.

Solution

Let n be the first integer. Then $2n - 1$ is the second and $4n - 1$ is the third. We can write n as $3k, 3k + 1$, or $3k + 2$ for some non-negative integer k.

When $n = 3k$, we must have $k = 1$, since n is a prime. In this instance the three number are $3, 5, 11$, which are all prime.

When $n = 3k + 1$, we have $4n - 1 = 12k + 3 = 3(4k + 3)$. Since this is a multiple of 3, it must be 3, and we have $k = 0$. This means $n = 1$, and the first number written down is not prime.

When $n = 3k + 2$, we have $2n - 1 = 6k + 3 = 3(2k + 1)$. Since this is a multiple of 3, it must be 3, and we have $k = 0$. In this instance, the three numbers are $2, 3, 7$, which are all prime.

Thus, the first integer is either 2 or 3.

10. Consider a function $f : (0, \infty) \to \mathbb{R}$ and a positive real number $a > 0$ such that $f(a) = 1$. Prove that if

$$f(x)f(y) + f\left(\frac{a}{x}\right) f\left(\frac{a}{y}\right) = 2f(xy)$$

for all $x, y \in (0, \infty)$, then f is a constant function.

Solution

Setting $x = y = 1$ yields:

$$f^2(1) + f^2(a) = 2f(1), \quad \text{or} \quad (f(1) - 1)^2 = 0, \quad \text{so} \quad f(1) = 1.$$

Substituting $y = 1$, gives:

$$f(x)f(1) + f\left(\frac{a}{x}\right) f(a) = 2f(x), \quad \text{or} \quad f(x) = f\left(\frac{a}{x}\right), \quad x > 0.$$

Take now, $y = \frac{a}{x}$ and observe that:

$$f(x)f\left(\frac{a}{x}\right) + f\left(\frac{a}{x}\right) f(x) = 2f(a), \quad \text{or} \quad f(x)f\left(\frac{a}{x}\right) = 1.$$

Therefore, $f^2(x) = 1$, $x > 0$.

Now, set $x = y = \sqrt{t}$, which gives:

$$f^2(\sqrt{t}) + f^2\left(\frac{a}{\sqrt{t}}\right) = 2f(t), \tag{1.6}$$

and because the left-hand side of (1.6) is positive, it follows that f is positive and $f(x) = 1$ for all x. Then, f is a constant function, as required.

Section B: 10 marks each

11. If a cock costs \$ 5, a hen costs \$ 3, and three chicks together cost \$ 1, how many cocks, hens, and chicks, totalling 100, can be bought for \$ 100?

Solution

Let x be the number of cocks, y be the number of hens, and z be the number of chicks bought. This problem can be then written as:

$$5x + 3y + \frac{1}{3}z = 100 \quad \text{and} \quad x + y + z = 100.$$

Eliminating one of the unknowns, we are left with a linear Diophantine equation in the other two variables. In particular, since $z = 100 - x - y$, we have $5x + 3y + \frac{1}{3}(100 - x - y) = 100$, or

$$7x + 4y = 100. \tag{1.7}$$

The equation (1.7) is a linear Diophantine equation with the general solution:

$$x = 4t, \quad y = 25 - 7t, \quad \text{so that} \quad z = 75 + 3t,$$

where t is an arbitrary integer. For this problem, t must be chosen to satisfy simultaneously the inequalities:

$$4t > 0 \quad 25 - 7t > 0, \quad \text{and} \quad 75 + 3t > 0. \tag{1.8}$$

The last two inequalities in (1.8) are equivalent to: $-25 < t < 3\frac{4}{7}$. From the first inequality in (1.8), we have that $t > 0$, then the possible integer values of t are: $t = 1, 2, 3$; leading to the solutions:

$$\begin{aligned} t = 1: &\quad x = 4,\ y = 18,\ z = 78; \\ t = 2: &\quad x = 8,\ y = 11,\ z = 81; \\ t = 3: &\quad x = 12,\ y = 4,\ z = 84. \end{aligned}$$

12. If the numbers a_n are defined inductively by $a_1 = 11$, $a_2 = 21$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$, prove that $a_n = 5(2^n) + 1$ holds for all $n \geq 1$.

Solution

We prove the result by induction on n. For $n = 1$ and $n = 2$, we have:

$$a_1 = 11 = 5(2^1) + 1 = 10 + 1 = 11 \quad \text{and} \quad a_2 = 21 = 5(2^2) + 1 = 20 + 1 = 21.$$

Hence the formula holds in both cases, and this provides the basis for this induction.

For the induction step, we choose an integer $k \geq 3$ and assume that the formula is true for $n = 1, 2, \ldots, k-1$. Then, in particular:

$$a_{k-1} = 5(2^{k-1}) + 1 \quad \text{and} \quad a_{k-2} = 5(2^{k-2}) + 1.$$

By the way these numbers are defined, we have:

$$\begin{aligned} a_k &= 3a_{k-1} - 2a_{k-2} = 3\left(5(2^{k-1}) + 1\right) - 2\left(5(2^{k-2}) + 1\right) \\ &= 15(2^{k-1}) + 3 - 5(2^{k-1}) - 2 = 10(2^{k-1}) + 1 = 5(2^k) + 1. \end{aligned}$$

Thus, $a_k = 5(2^k) + 1$ and since the formula is true for $n = k$ whenever it is true for $n = 1, 2, \ldots, k-1$; we conclude by strong induction that $a_n = 5(2^n) + 1$ holds for all $n \geq 1$.

Alternatively, From the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$, we get the the characteristic polynomial $x^2 - 3x + 2$, whose roots are $x = 1$ and $x = 2$.

Using the general theory of recurrence relations, we have the general (Binet's formula):

$$a_n = a \cdot (1)^n + b \cdot (2)^2 \tag{1.9}$$

for some contestants a and b that are determined using the initial conditions:

For $n = 1$, we see that $a_1 = 11$ and (1.9) becomes:

$$11 = a + 2b \tag{1.10}$$

For $n = 2$, we see that $a_2 = 21$ and (1.9) becomes:

$$21 = a + 4b \tag{1.11}$$

Solving (1.10) and (1.11) simultaneously gives:

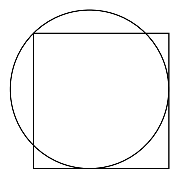
$(1.11) - (1.10) \Rightarrow 2b = 10 \Rightarrow b = 5$

put in $(1.9) \Rightarrow 11 = a + 2 \cdot 5 \Rightarrow a = 11 - 10 = 1$

So substituting back in (1.9), we get:

$$a_n = 5(2)^n + 1 \quad \text{for all} \quad n \geq 1$$

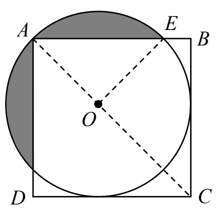
13. Pugsley tries to fit a round circle into a square hole, but part of the circle sticks out. What is the total area of the two excess regions that Pugsley will carve from the circle? The circle has radius 10, and Pugsley has it placed so that it passes through one corner of the square and is tangent to two sides of the square.



Solution

Let $ABCD$ be the square with A on the circle.

Let O be the center of the circle. Let AB intersect the circle at E.



Draw the diagonal AC of the square. Obviously, O is on AC.

With $\angle EAO = 45°$ and $OA = OE$, AOE is an isosceles right triangle.

Then we have $\angle AOE = 90°$.

The area of the shaded circular segment AE is

$$\frac{1}{4} \cdot 10^2\pi - \frac{1}{2} \cdot 10^2 = 25\pi - 50.$$

The total area of the shaded regions is

$$2 \cdot (25\pi - 50) = 50\pi - 100$$

14. Prove that if a, b, c are nonzero real numbers with $a + b + c = 0$, then

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{a^2+c^2}{a+c} = \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}.$$

Solution

Since $a + b + c = 0$, we obtain:

$$a+b=-c, \quad b+c=-a, \quad \text{and} \quad a+c=-b,$$

or by squaring and rearranging we get:

$$a^2+b^2=c^2-2ab, \quad b^2+c^2=a^2-2bc, \quad \text{and} \quad a^2+c^2=b^2-2ac.$$

Thus, the given equation in question is equivalent to:

$$\frac{c^2-2ab}{-c} + \frac{a^2-2ab}{-a} + \frac{b^2-2ac}{-b} = \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab},$$

and consequently equivalent to:

$$-(a+b+c) + 2\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b}\right) = \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}.$$

This equation is then equivalent to:

$$2(a^2b^2 + b^2c^2 + a^2c^2) = a^4 + b^4 + c^4.$$

On the other hand, from $a + b + c = 0$, we get that $(a + b + c)^2 = 0$, or

$$a^2 + b^2 + c^2 = -2(ab + bc + ac).$$

Squaring again yields:

$$a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + a^2c^2) = 4(a^2b^2 + b^2c^2 + a^2c^2) + 8abc(a + b + c),$$

or

$$a^4 + b^4 + c^4 = 2(a^2b^2 + b^2c^2 + a^2c^2),$$

as required.

15. Let $m \geq 100$ be an integer. Flavia writes the numbers $m, m + 1, m + 2, \ldots, 2m$ each on different cards. She then shuffles these $m + 1$ cards and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution

It is enough to find three cards numbered x, y, and z such that $x+y$, $x+z$, and $y+z$ are perfect squares. This is because two of the numbers x, y, or z will be in the same pile. Solving the system of equations:

$$x + y = (2t-1)^2, \quad x + z = (2t)^2, \quad y + z = (2t+1)^2,$$

yields

$$x = 2t^2 - 4t, \quad y = 2t^2 + 1, \quad z = 2t^2 + 4t.$$

Using these values of x, y, and z it suffices to show that for each integer $m \geq 100$, there exists a positive integer t such that $m \leq 2t^2 - 4t$ and $2t^2 + 4t \leq 2m$. That is,

$$t^2 + 2t \leq m \leq 2t^2 - 4t.$$

To this end, let t be the greatest integer such that $t^2 + 2t \leq m$. Note that $t \geq 9$ because $m \geq 100$. Since t is the greatest integer satisfying $t^2 + 2t \leq n$, it follows that

$$(t+1)^2 + 2(t+1) \geq m + 1 \quad \text{if and only if} \quad m \leq t^2 + 4t + 4.$$

To complete the proof, it is enough to show that $t^2 + 4t + 4 \leq 2t^2 - 4t$. This is equivalent to $(t-4)^2 \geq 20$. But this is true since $k \geq 9$.

Some important remarks:

- The above proof shows that the conclusion of the problem is true for all $m \geq 99$.
- The conclusion of the problem is false for $m = 98$. Indeed, a counterexample occurs if the first pile contains the even numbers from 98 to 126, the odd numbers from 129 to 161, and the even numbers from 162 to 196, while the second pile contains the rest of the numbers.

NATIONAL MATHEMATICS CONTEST
A-LEVEL PAPER 1 QUESTIONS AND SOLUTIONS.

SECTION A: 5 **marks each**

1. Find the real-valued function $f(x)$ such that $f\left(x-\frac{1}{x}\right)=x^3-\frac{1}{x^3}$.

Solution:

Using the fact that

$$\left(x-\frac{1}{x}\right)^3=x^3-3x^2\cdot\frac{1}{x}+3x\cdot\frac{1}{x^2}-\frac{1}{x^3}=x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right),$$

we get that

$$x^3-\frac{1}{x^3}=\left(x-\frac{1}{x}\right)^3+3\left(x-\frac{1}{x}\right).$$

Thus,

$$f\left(x-\frac{1}{x}\right)=x^3-\frac{1}{x^3}=\left(x-\frac{1}{x}\right)^3+3\left(x-\frac{1}{x}\right),$$

which implies that $\boxed{f(x)=x^3+3x}$ is the required function.

2. Show that the infinite tower, $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}=2$.

[**Hint**: use the property of the **Lambert** W function that $W\left(\psi e^{\psi}\right)=\psi$ for any function ψ]

Solution:

Let $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}=x$. So, $\sqrt{2}^x=x$. Taking the natural logarithm on both sides, we get $\ln\sqrt{2}^x=\ln x$, which is equivalent to $x\ln\sqrt{2}=\ln x$. This is equivalent to $e^{\ln x}\ln\sqrt{2}=\ln x$. Thus, $\ln\sqrt{2}=\ln xe^{-\ln x}$. Multiplying through both sides by -1, we get that

$$-\ln\sqrt{2}=-\ln xe^{-\ln x}.$$

But $-\ln\sqrt{2}=-\ln 2^{\frac{1}{2}}=-\frac{1}{2}\ln 2=\frac{1}{2}\ln 2^{-1}=\frac{1}{2}\ln\frac{1}{2}=\ln\frac{1}{2}e^{\ln\frac{1}{2}}$. Therefore, we get that

$$\ln\frac{1}{2}e^{\ln\frac{1}{2}}=-\ln xe^{-\ln x}=\ln x^{-1}e^{\ln x^{-1}}=\ln\frac{1}{x}e^{\ln\frac{1}{x}}.$$

Now, applying the Lambert W function to both sides of the above equation, we get that

$$\ln\frac{1}{2}=W\left(\ln\frac{1}{2}e^{\ln\frac{1}{2}}\right)=W\left(\ln\frac{1}{x}e^{\ln\frac{1}{x}}\right)=\ln\frac{1}{x}.$$

This gives that $\ln\frac{1}{2}=\ln\frac{1}{x}$, or $e^{\ln\frac{1}{2}}=e^{\ln\frac{1}{x}}$, or $\frac{1}{2}=\frac{1}{x}$, which implies that $\boxed{x=2}$, as required.

3. Given that $f\left(\dfrac{x+2}{x-2}\right) = \dfrac{x^2+4x+4}{8x}$, find the function $f(x)$.

Solution:

First, we observe that

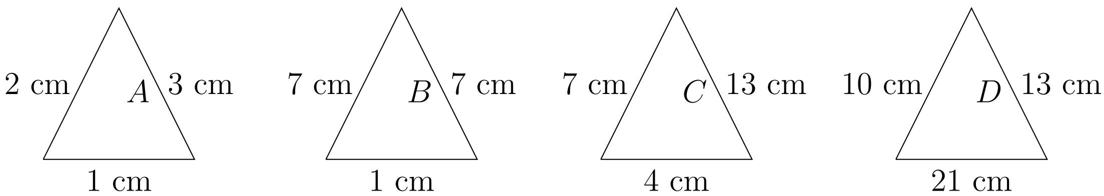
$$f(0) = f\left(\frac{-2+2}{-2-2}\right) = \frac{(-2)^2+4(-2)+4}{8(-2)} = \frac{4-8+4}{-16} = \frac{0}{-16} = 0.$$

Now, let $y = \frac{x+2}{x-2}$. This implies that $y(x-2) = x+2$, or $yx - x = 2y + 2$, or $x(y-1) = 2(y+1)$. This implies that $x = 2\left(\frac{y+1}{y-1}\right)$. Thus, we get

$$\begin{aligned}
f(y) &= \frac{\left(2\left(\dfrac{y+1}{y-1}\right)\right)^2 + 4\left(2\left(\dfrac{y+1}{y-1}\right)\right) + 4}{8\left(2\left(\dfrac{y+1}{y-1}\right)\right)} \\
&= \frac{\left(\dfrac{y+1}{y-1}\right)^2 + 2\left(\dfrac{y+1}{y-1}\right) + 1}{4\left(\dfrac{y+1}{y-1}\right)} \\
&= \frac{(y+1)^2 + 2(y+1)(y-1) + (y-1)^2}{4(y+1)(y-1)} \\
&= \frac{y^2+2y+1+2y^2-2+y^2-2y+1}{4(y^2-1)} \\
\therefore f(y) &= \frac{4y^2}{4(y^2-1)} = \frac{y^2}{y^2-1}.
\end{aligned}$$

Finally, we also observe that $f(0) = \dfrac{0^2}{0^2-1} = \dfrac{0}{-1} = 0$. Therefore, $\boxed{f(x) = \dfrac{x^2}{x^2-1}}$.

4. Which of the following triangles A, B, C, D does not exist?



Solution:

We use the fact that if a, b, c are the sides of a triangle, then following conditions $a+b \geq c$, $a+c \geq b$, and $b+c \geq a$, must be satisfied.

- For triangle A, we see that $2+3 = 5 \geq 1$, $1+2 = 3 \geq 3$, and $1+3 = 4 \geq 2$. Thus, triangle A exists.
- For triangle B, we see that $1+7 = 8 \geq 7$, $7+7 = 14 \geq 1$, and $7+1 = 8 \geq 7$. Thus, triangle B also exists.

- For triangle C, we see that $7+13=20 \geq 4$, $4+13=17 \geq 7$, but $7+4=11<13$. Thus, **triangle C does not exist**.
- For triangle D, we observe that $10+21=31 \geq 13$, $21+13=34 \geq 10$, and $10+13=23 \geq 13$. Thus, triangle D also exists.

5. An unreliable typist can guarantee that when they try to type a word with different letters, every letter of the word will appear exactly once in what they type, and each letter will occur at most one letter late (though it may occur more than one letter early). Thus, when trying to type **MATHS**, the typist may type **MATHS, MTAHS** or **TMASH**, but not **ATMSH**. Determine, with proof, the number of possible spellings of **OLYMPIADS** that might be typed.

Solution:

The number of possible spellings is equivalent to the number of cycles formed from the number of letters. Let the number of different letters in a word to be typed be n.

- For $n=2$, e.g. $(\bullet\bullet)$ two arrangements $(\bullet\bullet)$, $(\bullet)(\bullet)$ are possible. That is, the number of cycles $=2=2^1=2^{2-1}$.
- For $n=3$, e.g. $(\bullet\bullet\bullet)$ four arrangements $(\bullet\bullet\bullet)$, $(\bullet)(\bullet\bullet)$, $(\bullet\bullet)(\bullet)$, $(\bullet)(\bullet)(\bullet)$ are possible. That is, the number of cycles $=4=2^2=2^{3-1}$.
- For $n=4$, e.g. $(\bullet\bullet\bullet\bullet)$ eight possible arrangements $(\bullet\bullet\bullet\bullet)$, $(\bullet)(\bullet\bullet\bullet)$, $(\bullet\bullet\bullet)(\bullet)$, $(\bullet\bullet)(\bullet\bullet)$, $(\bullet\bullet)(\bullet)(\bullet)$, $(\bullet)(\bullet\bullet)(\bullet)$, $(\bullet)(\bullet)(\bullet\bullet)$, $(\bullet)(\bullet)(\bullet)(\bullet)$ are possible. That is, the number of cycles $=8=2^3=2^{4-1}$.
- In general, for a word with n different letters, the number of possible cycles is 2^{n-1}.
- For **OLYMPIADS**, $n=9$. Thus, the number of possible spellings that can be typed is $=2^{9-1}=2^8=256$ spellings.

6. Find all positive integers n such that the integer $n\times 2^n+1$ is a perfect square.

[**Hint**: An integer x is a perfect square if there is an integer y such that $x=y^2$]

Solution:

First, we try to check out the solutions for the first few values of n:

n	$n\times 2^n+1$	Is square?
1	$1\times 2^1+1=3$	No
2	$2\times 2^2+1=9=3^2$	**Yes**
3	$3\times 2^3+1=25=5^2$	**Yes**
4	$4\times 2^4+1=65$	No
5	$5\times 2^5+1=161$	No

Next, we prove that indeed the values $n=2$ and $n=3$ are the only solutions. Let $n\cdot 2^n+1=m^2$. That is, $n\cdot 2^n=m^2-1=(m-1)(m+1)$. Thus, either $m+1=a\cdot 2^{n-1}$ or $m-1=a\cdot 2^{n-1}$.

Case 1: If $m-1=a\cdot 2^{n-1}$, then $n\cdot 2^n=(m+1)a\cdot 2^{n-1}$, which implies that $2n=a(m+1)$. That is, $2n\geq m+1\geq m-1\geq 2^{n-1}$ or $2n\geq 2^{n-1}$ or $n\geq 2^{n-2}$, which is only true if $n\leq 4$.

Case 2: If $m+1=a\cdot 2^{n-1}$, then $n\cdot 2^n=(m-1)a\cdot 2^{n-1}$, which implies that $2n=a(m-1)$. That is, $2n\geq m-1=(m+1)-2\geq 2^{n-1}-2$, or $2n\geq 2^{n-1}-2$, or $n\geq 2^{n-2}-1$, which only holds when $n\leq 4$.

Thus, in both cases the solutions must satisfy the inequality $1\leq n\leq 4$. Thus, $\boxed{n=2,3}$ are the only possible values of n such that $n\times 2^n+1$ is a perfect square.

7. Let x, y, z be real numbers. Prove that if

$$\frac{xy+1}{x+1} = \frac{yz+1}{y+1} = \frac{zx+1}{z+1} \tag{1.1}$$

then $x = y = z$. [**Hint**: Use the **AM-GM Inequality**]

Solution:

From the first equality in (1.1), we have

$$\frac{xy+1}{x+1} = \frac{yz+1}{y+1} \quad \Rightarrow \quad (xy+1)(y+1) = (yz+1)(x+1),$$

so that

$$xy^2 + xy + y = xyz + yz + x. \tag{1.2}$$

Similarly, we also have that

$$yz^2 + yz + x = xyz + zx + y, \tag{1.3}$$
$$zx^2 + zx + x = xyz + xy + z. \tag{1.4}$$

Now, adding (1.2), (1.3) and (1.4) gives: $xy^2 + yz^2 + zx^2 = 3xyz$, or

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 3. \tag{1.5}$$

Applying the **AM-GM Inequality** on the left-hand side of (1.5) gives:

$$\frac{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}{3} \geq \left(\frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}\right)^{\frac{1}{3}} = 1 \quad \Rightarrow \quad \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3.$$

From this, equality is achieved only when

$$\frac{x}{y} = \frac{y}{z} = \frac{z}{x} = 1 \quad \Rightarrow \quad \boxed{x = y = z}.$$

8. Let g be a function with the following properties:

(i) $g(1) = 1$; and

(ii) $g(5n) = nf(n)$ for any positive integer n.

What is the value of $g(5^{100})$?

Solution:

Using the property (ii), we get that

$$\begin{aligned} g(5^{100}) &= g(5 \cdot 5^{99}) = 5^{99} \cdot g(5^{99}) = 5^{99} \cdot g(5 \cdot 5^{98}) = 5^{99} \cdot 5^{98} \cdot g(5^{98}) = 5^{99+98} \cdot g(5^{98}) \\ &= \cdots = 5^{99+98+\cdots+2+1+0} \cdot g(5^0) = 5^{99+98+\cdots+2+1+0} \cdot g(1) = 5^{99+98+\cdots+2+1+0} \cdot 1 \\ \therefore g(5^{100}) &= 5^{99+98+\cdots+2+1+0} = 5^{\frac{99 \cdot 100}{2}} = \boxed{5^{4950}}. \end{aligned}$$

9. Chess is played on an 8×8 square board. How may ways are there to put 3 identical **rooks** on the chessboard so that no two **rooks** are in the same row or column?

Solution:

Place the rooks in order so that they can not capture, leading to

$$\binom{8}{1}\binom{8}{1} = 64, \quad \binom{7}{1}\binom{7}{1} = 49, \quad \text{and} \quad \binom{6}{1}\binom{6}{1} = 36 \quad \text{possibilities.}$$

Then, divide their product $64 \cdot 49 \cdot 36$ by $3! = 6$ to account for the reorderings. Therefore, the answer is $\dfrac{64 \cdot 49 \cdot 36}{6} = \boxed{18816}$ possible ways.

10. Let ABC be an acute-angled triangle. Point P is such that $AP = AB$ and $PB \| AC$. Point Q is such that $AQ = AC$ and $CQ \| AB$. Line segments CP and BQ meet at point X. Prove that the circumcenter of triangle ABC lies on the circumcircle of triangle PXQ.

Solution:

Let D be the vertex of parallelogram $ABDC$. Then $APDC$ and $AQDB$ are isosceles trapezoids. Therefore the perpendicular bisectors to line segments PD and QD coincide with the perpendicular bisectors to AC and AB respectively, the circumcenter O of triangle ABC is also the circumcenter of DPQ and $\angle POQ = 2\angle BAC$. Also since, $\angle XPD = \angle ADP$, $\angle XQD = \angle ADQ$ we obtain that $\angle PXQ = 2\angle BAC$. Thus, points O, P, Q, X are concyclic.

SECTION B: 10 marks each

11. Find all values of y that satisfy the equation $(y+1)^2 + (y+2)^3 + (y+3)^4 = 2$.

Solution:

Let $x = y + 2$, so the equation in question becomes $(x-1)^2 + x^3 + (x+1)^4 = 2$. But $(x-1)^2 = x^2 - 2x + 1$ and $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$ (by Pascal's triangle). Thus, we get that

$$\begin{aligned} x^2 - 2x + 1 + x^3 + x^4 + 4x^3 + 6x^2 + 4x + 1 &= 2 \\ \Rightarrow x^4 + 5x^3 + 7x^2 + 2x + 2 &= 2 \\ \Rightarrow x^4 + 5x^3 + 7x^2 + 2x &= 0 \\ \Rightarrow x(x^3 + 5x^2 + 7x + 2) &= 0. \end{aligned}$$

Either: $x = 0$, which means that $y + 2 = 0$. So, $\boxed{y = -2}$ is a solution.

Or: $x^3 + 5x^2 + 7x + 2 = 0$. Let $f(x) = x^3 + 5x^3 + 7x + 2$, by the **Rational Root Theorem**, possible roots of $f(x)$ are all the divisors of 2, that is, $x = \pm 1, \pm 2$. Depending on the way $f(x)$ is defined, it is clear that $f(1) \neq 0$ and $f(2) \neq 0$. So, we try

$$f(-1) = (-1)^3 + 5(-1)^2 + 7(-1) + 2 = -1 + 5 - 7 + 2 = -1 \neq 0$$

and

$$f(-2) = (-2)^3 + 5(-2)^2 + 7(-2) + 2 = -8 + 20 - 14 + 2 = 0.$$

Thus, $x = -2$ is a root, or $(x+2)$ is a factor. To find the other factors, we do a **Long Divison**:

$$\begin{array}{r|l} & x^2 + 3x + 1 \\ \hline x + 2 & x^3 + 5x^2 + 7x + 2 \\ & -x^3 - 2x^2 \\ \hline & 3x^2 + 7x \\ & -3x^2 - 6x \\ \hline & x + 2 \\ & -x - 2 \\ \hline & 0 \end{array}$$

Thus, $x^3 + 5x^2 + 7x + 2 = (x+2)(x^2 + 3x + 1)$. **Either**: $x + 2 = 0$, this means that $y + 2 + 2 = 0$ or $y + 4 = 0$. Therefore $\boxed{y = -4}$ is a solution. **Or**: $x^2 + 3x + 1 = 0$. Here, we use the **Quadratic Formula** to find the remaining solutions. That is, $x = \dfrac{-3 \pm \sqrt{3^2 - 4}}{2} = \dfrac{-3 \pm \sqrt{5}}{2}$. This means that $y + 2 = \dfrac{-3 \pm \sqrt{5}}{2}$, or $y = \dfrac{7 \pm \sqrt{5}}{2}$. So, $\boxed{y = \dfrac{7 + \sqrt{5}}{2}}$ and $\boxed{y = \dfrac{7 - \sqrt{5}}{2}}$ are the remaining solutions. All solutions are

$$\boxed{y \in \left\{ -4, \ -2, \ \frac{7 - \sqrt{5}}{2}, \ \frac{7 + \sqrt{5}}{2} \right\}}.$$

12. Find all pairs of nonzero integers (x, y) such that the integer $x^2 + y^2$ is a common divisor of the integers $x^5 + y$ and $y^5 + x$.

Solution:

We are given that $x^2 + y^2 \mid x^5 + y$ and $x^2 + y^2 \mid y^5 + x$. This is equivalent to saying that $x^2 + y^2 \mid x(x^5 + y)$ and $x^2 + y^2 \mid y(y^5 + x)$, and hence $x^2 + y^2 \mid x(x^5 + y) + y(y^5 + x)$. That is,

$$x^2 + y^2 \mid x^6 + y^6 + 2xy. \tag{1.6}$$

From the well-known identity for the sum of cubes, $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$, we also have that $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)((x^2)^2 - x^2y^2 + (y^2)^2)$. This implies that

$$x^2 + y^2 \mid x^6 + y^6. \tag{1.7}$$

From (1.6) and (1.7), we get that $x^2+y^2 \mid 2xy$. This implies that $x^2+y^2 \le 2|xy|$, or $x^2-2|xy|+y^2 \le 0$, or $(|x| - |y|)^2 \le 0$, that is, $|x| = |y|$. Therefore, we have two cases:

- **Case 1**: If $x = y$, then $2x^2 \mid x^5 + x$, which implies that $2x^2 \mid x(x^4 + 1)$, or $2x \mid x^4 + 1$. This means that $x \mid 1$. That is, either $x = 1$ or $x = -1$.
- **Case 2**: If $x = -y$, then $2x^2 \mid x^5 - x$, which implies that $2x^2 \mid x(x^4 - 1)$, or $2x \mid x^4 - 1$. This means that $x \mid 1$, that is, either $x = 1$ or $x = -1$.

Hence, the required solutions are $\boxed{(x, y) \in \{(1, 1), (-1, -1), (1, -1), (-1, 1)\}}$.

13. Let a and b be real numbers with $a \neq 0$. Let $P(x) = ax^4 - 4ax^3 + (5a + b)x^2 - 4bx + b$. Show that all roots of $P(x)$ are real and positive if and only if $a = b$.

Solution:

($\Leftarrow$) Suppose that $a = b$, then with the aid of **Pascal's Triangle**, we have that

$$P(x) = ax^4 - 4ax^3 + 6ax^2 - 4ax + a = a(x^4 - 4x^3 + 6x^2 - 4x + 1) = a(x - 1)^4.$$

Thus, $P(x) = 0$ implies that $a(x - 1)^4 = 0$, which implies that, since $a \neq 0$, then $\boxed{x = 1}$ is the only root, which is real and positive.

($\Rightarrow$) Let r_1, r_2, r_3, r_4 be the real and positive roots of the polynomial $P(x)$. Then $P(x) = 0$ implies that $ax^4 - 4ax^3 + (5a + b)x^2 - 4bx + b = 0$. Since, $a \neq 0$, we may divide through by a to get that:

$$x^4 - 4x^3 + \left(5 + \frac{b}{a}\right)x^2 - 4\left(\frac{b}{a}\right)x + \frac{b}{a} = 0.$$

Then, the roots r_1, r_2, r_3, r_4, must satisfy the following equations:

$$r_1 + r_2 + r_3 + r_4 = 4, \tag{1.8}$$

$$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = 5 + \frac{b}{a}, \tag{1.9}$$

$$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = 4\left(\frac{b}{a}\right), \tag{1.10}$$

$$r_1r_2r_3r_4 = \frac{b}{a}. \tag{1.11}$$

Dividing (1.10) by (1.11) gives:

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 4. \tag{1.12}$$

Then, (1.8) and (1.12) imply that

$$r_1 + r_2 + r_3 + r_4 = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 4.$$

Now, applying the **AM-GM Inequality**, independenty, on (1.8) and (1.12) gives:

$$r_1r_2r_3r_4 \leq \left(\frac{r_1 + r_2 + r_3 + r_4}{4}\right)^4 = 1 \quad \Rightarrow \frac{b}{a} \leq 1 \quad \Rightarrow a \geq b. \tag{1.13}$$

$$\frac{1}{r_1r_2r_3r_4} \leq \left(\frac{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}}{4}\right)^4 = 1 \quad \Rightarrow \frac{a}{b} \leq 1 \quad \Rightarrow a \leq b. \tag{1.14}$$

Thus, (1.13) and (1.14) imply that $\boxed{a = b}$, as required.

14. Mrs. Olupot gave an exam in a Mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Olupot noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Olupot entered?

Solution:

Let $v,\ w, x\ y,\ z$ be the scores entered in the spread sheet in that order. Then the sequence of averages:

$$v,\ \frac{v+w}{2},\ \frac{v+w+x}{3},\ \frac{v+w+x+y}{4},\ \frac{v+w+x+y+z}{5},$$

are all integers. From this sequence, we observe that:

(i) v is an integer,

(ii) $v + w$ is divisible by 2,

(iii) $v + w + x$ is divisible by 3,

(iv) $v + w + x + y$ is divisible by 4, and

(v) $v + w + x + y + z = 71 + 76 + 80 + 82 + 91 = 400$, which is divisible by 5.

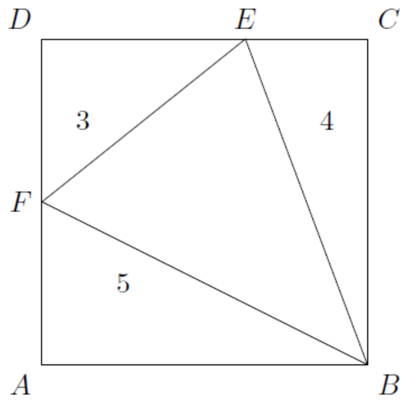
Since $v + w + x + y$ is divisible by 4 and 400 is divisible by 4, then z must also be divisible by 4. Therefore, the possible values of z are: $z = 76$ or $z = 80$ (the only scores divisible by 4). So, two cases arise.

- **Case 1**: For $z = 76$, we have $v + w + x + y = 400 - 76 = 324$. Since $v + w + x$ is divisible by 3 and 324 is divisible by 3, it follows that y must also be divisible by 3. But none of the scores, $71,\ 80,\ 82,\ 91$ is divisible by 3. Thus, $z \neq 76$.

- **Case 2**: For $z = 80$, we have $v + w + x + y = 400 - 80 = 320$. Since $v + w + x$ is divisible by 3 and 320 leaves remainder 2 when divided by 3, it follows that y must also leave remainder 2 when divided by 3. Since $71 \equiv 2 \pmod 3$, $76 \equiv 1 \pmod 3$, $91 \equiv 1 \pmod 3$, we get that $y = 71$. Thus, $v+w+x = 320-71 = 249$. Since $v+w$ is divisible by 2 (or even) and 249 is odd, it follows that x must be odd. The only possibility remaining for x is $x = 91$. Thus, there are two possible cases: $(v, w, x, y, z) = (76, 82, 91, 71, 80)$ or $(v, w, x, y, z) = (82, 76, 91, 71, 80)$. In both cases $z = 80$.

Therefore, the last score that was entered in the spreadsheet is $\boxed{z = 80}$.

15. In the square $ABCD$ shown in the figure below, the areas of ΔEDF, ΔBCE and ΔFAB are 3 cm^2, 4 cm^2 and 5 cm^2, respectively. Find the area of ΔEFB.

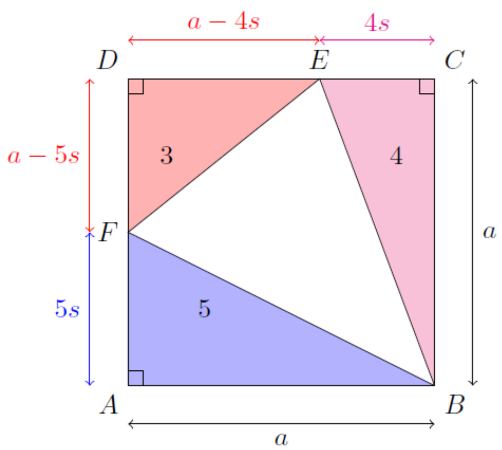


Solution:

Let a be the length of the side of the square. Considering the ratios of areas of the triangles ΔBCE and ΔFAB:

$$\frac{\text{Area of } \Delta BCE}{\text{Area of } \Delta FAB} = \frac{\frac{1}{2} \cdot a \cdot \overline{CE}}{\frac{1}{2} \cdot a \cdot \overline{AF}} = \frac{4}{5} \quad \Rightarrow \frac{\overline{CE}}{\overline{AF}} = \frac{4}{5} \quad \Rightarrow \overline{CE} : \overline{AF} = 4 : 5.$$

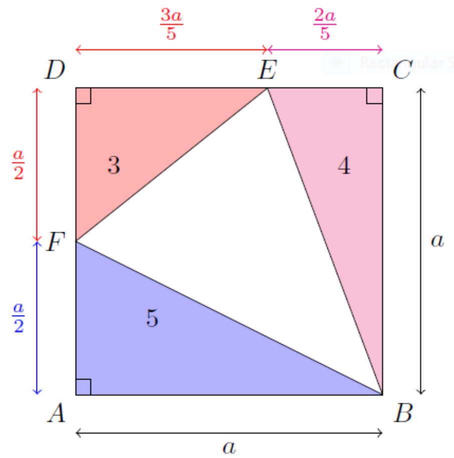
Thus, we may write $\overline{CE} = 4s$ and $\overline{AF} = 5s$ for some $s > 0$. From this, we get that $\overline{DE} = a - 4s$ and $\overline{DF} = a - 5s$. Now, consider the square below.



Next, considering the ratios of the areas of triangles ΔDEF and ΔBCE, we get that:

$$\frac{\text{Area of } \Delta DEF}{\text{Area of } \Delta BCE} = \frac{\frac{1}{2}\cdot(a-4s)\cdot(a-5s)}{\frac{1}{2}\cdot a\cdot 4s} = \frac{3}{4} \quad \Rightarrow \frac{(a-4s)(a-5s)}{4as} = \frac{3}{4}.$$

This implies that $(a-4s)(a-5s) = 3as$ or $a^2 - 12as + 20s^2 = 0$ or $(a-2s)(a-10s) = 0$. Thus, either $s = \frac{a}{2}$ or $s = \frac{a}{10}$. The possibility that $a = \frac{a}{2}$ can be ruled out by observing that $5s = \overline{AF} < \overline{AD} = a$, that is, $s < \frac{a}{5}$. Using $s = \frac{a}{10}$, we can express the side lenths in terms of a only as follows: $\overline{AF} = 5s = \frac{a}{2}$ and $\overline{CE} = 4s = \frac{2a}{5}$. The new square is given in the figure below:



Considering the area of ΔFAB, we get that:

$$5 = \text{Area of } \Delta FAB = \frac{1}{2}\cdot a\cdot\frac{a}{2} \quad \Rightarrow a^2 = 20 \quad \Rightarrow \boxed{a = 2\sqrt{5}\text{ cm}}.$$

Thus, we get that:

$$\begin{aligned}\text{Area of } \Delta EFB &= \text{Area of } \square ABCD - (\text{Area of } \Delta EDF + \text{Area of } \Delta BCE + \text{Area of } \Delta FAB)\\ &= a^2 - (3+4+5) = 20 - 12 = 8\text{ cm}^2.\end{aligned}$$

Therefore, $\boxed{\text{Area of } \Delta EFB = 8\text{ cm}^2}$.

NATIONAL MATHEMATICS CONTEST
A-LEVEL PAPER 2 SOLUTIONS.

SECTION A: 5 marks each

1. Solve the equation $(x^2+3x-4)^3+(2x^2-5x+3)^3=(3x^2-2x-1)^3$.

Solution 1 : *If*

$$x^2+3x-4=a, 2x^2-5x+3=b$$

then

$$3x^2-2x-1=a+b$$

So we have

$$a^3+b^3=(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

or

$$3ab(a+b)=0$$

So we need to solve

$$x^2+3x-4=0 \Rightarrow x=1,-4$$
$$2x^2-5x+3=0 \Rightarrow x=1,\frac{3}{2}$$
$$3x^2-2x-1=0 \Rightarrow x=1,-\frac{1}{3}$$

And the solution set $x \in \left\{-4,-\frac{1}{3},1,\frac{3}{2}\right\}$ ■

Solution 2 :

$$\left(x^2+3x-4\right)^3+\left(2x^2-5x+3\right)^3=\left(3x^2-2x-1\right)^3$$
$$(x-1)^3(x+4)^3+(x-1)^3(2x-3)^3-(x-1)^3(3x+1)^3=0$$
$$(x-1)^3\left((x+4)^3+(2x-3)^3-(3x+1)^3\right)=0$$
$$(x-1)^3\left((x+4)^3-(x+4)\left(19x^2-13x+7\right)\right)=0$$
$$(x-1)^3(x+4)\left(21x-18x^2+9\right)=0$$
$$x=\left\{-4,-\frac{1}{3},1,\frac{3}{2}\right\}$$

■

Solution 3 :

$$\left(x^2+3x-4\right)^3+\left(2x^2-5x+3\right)^3=\left(3x^2-2x-1\right)^3$$
$$\Leftrightarrow -18x^6+3x^5+192x^4-378x^3+222x^2+15x-36=0$$
$$\Leftrightarrow -3(x-1)^3(x+4)(2x-3)(3x+1)=0$$
$$\Leftrightarrow x \in \left\{-4,-\frac{1}{3},1,\frac{3}{2}\right\}$$

■

Solution 4 : $\Rightarrow 9x^6 - 51x^5 + 201x^4 - 350x^3 + 219x^2 + 9x - 37 = 27x^6 - 54x^5 + 9x^4 + 28x^3 - 3x^2 - 6x - 1 \Rightarrow -18x^6 + 3x^5 + 192x^4 - 378x^3 + 222x^2 + 15x - 36 = 0$ $\Rightarrow -3(x-1)^3(x+4)(2x-3)(3x+1) = 0$ *after IVT and RRT So* $x \in \left\{-4, -\frac{1}{3}, 1, \frac{3}{2}\right\}$.
■

2. Prove that for all positive real numbers a, b and c, $\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$.

Solution 1 : *Note that*

$$a^4 + b^4 + c^4 = \frac{(a^4 + b^4)}{2} + \frac{(b^4 + c^4)}{2} + \frac{(c^4 + a^4)}{2}.$$

Applying the arithmetic-geometric mean inequality to each term, we see that the right side is greater than or equal to

$$a^2b^2 + b^2c^2 + c^2a^2.$$

We can rewrite this as

$$\frac{a^2(b^2 + c^2)}{2} + \frac{b^2(c^2 + a^2)}{2} + \frac{c^2(a^2 + b^2)}{2}.$$

Applying the arithmetic mean-geometric mean inequality again we obtain

$$a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab.$$

Dividing both sides by abc (which is positive) the result follows. ■

Solution 2 : *Notice the inequality is homogeneous. That is, if* a, b, c *are replaced by* $ka, kb, kc, k > 0$ *we get the original inequality. Thus we can assume, without loss of generality, that* $abc = 1$.
Then

$$\begin{aligned} \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} &= abc\left(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}\right) \\ &= a^4 + b^4 + c^4. \end{aligned}$$

So we need prove that $a^4 + b^4 + c^4 \geq a + b + c$.

By the Power Mean Inequality,

$$\frac{a^4 + b^4 + c^4}{3} \geq \left(\frac{a+b+c}{3}\right)^4,$$

so

$$a^4 + b^4 + c^4 \geq (a+b+c) \cdot \frac{(a+b+c)^3}{27}.$$

By the arithmetic mean-geometric mean inequality,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} = 1,$$

so

$$a + b + c \geq 3.$$

Hence,

$$a^4 + b^4 + c^4 \geq (a+b+c) \cdot \frac{(a+b+c)^3}{27} \geq (a+b+c)\frac{3^3}{27} = a + b + c.$$

■

Solution 3 : *Rather than using the Power-Mean inequality to prove $a^4 + b^4 + c^4 \geq a + b + c$ in Proof 2, the Cauchy-Schwartz-Bunjakovsky inequality can be used twice.*

$$(a^4 + b^4 + c^4)(1^2 + 1^2 + 1^2) \geq (a^2 + b^2 + c^2)^2$$

$$(a^2 + b^2 + c^2)(1^2 + 1^2 + 1^2) \geq (a + b + c)^2$$

So

$$\frac{a^4 + b^4 + c^4}{3} \geq \frac{(a^2 + b^2 + c^2)^2}{9} \geq \frac{(a + b + c)^4}{81}$$

Continue as in Solution 2. ■

3. There are 20 students in a high school class, and each student has exactly three close friends in the class. Five of the students have bought tickets to an upcoming concert. If any student sees that at least two of their close friends have bought tickets, then they will buy a ticket too. Is it possible that the entire class buys tickets to the concert? (Assume that friendship is mutual; if student A is close friends with student B, then B is close friends with A).

Solution 1 : *It is impossible for the whole class to buy tickets to the concert. If two students A and B are close friends, and A has bought a ticket to the concert while B has not, then A is enticing B. We call this pair (A, B) an enticement.*

In order for a student to change their mind and buy a ticket, they first be enticed by at least 2 of their 3 close friends. That means they can only entice at most 1 other friend. Therefore, the total number of enticements among the students decreases by 1 whenever a student changes their mind to buy a ticket.

Initially, the maximum number of enticements is 15 (each of the initial 5 students with tickets has 3 friends to entice). Assume, for the sake of contradiction, that the entire class ends up buying tickets. After the first 14 people buy tickets, the number of enticements is at most $15 - 14 = 1$. This is not enough to convince the last person to buy a ticket, since they need 2 enticements.

Therefore, it is impossible that the entire class buys tickets. ■

Solution 2 : *We shall use the term* friendship *to denote an unordered pair of students who are close friends. Since each of the 20 students is part of exactly 3 friendships, there are exactly 30 friendships in the class. (We could also represent friendships as edges in an undirected graph whose vertices are the 20 students.)*

We say that a friendship is used *if one of the students in that friendship buys a ticket after the original five buyers, and the other student already has a ticket at that time. Each time a ticket is purchased after the original five purchases, at least two friendships are used. Observe that no friendship gets used twice.*

If all 20 students buy tickets, then three friendships are used when the last student buys a ticket. This would imply that the number of used friendships is at least $14 \times 2 + 3 = 31$, which is more than the number of friendships. This contradiction proves that it is not possible that the entire class buys tickets. ■

4. The circumradius R of a triangle with side lengths a, b, c satisfies $R = \dfrac{a\sqrt{bc}}{b + c}$. Find the angles of the triangle.

Solution 1 : *since area* $= \dfrac{abc}{4R} = \dfrac{\sqrt{bc}(b+c)}{4} = \dfrac{1}{2}bc\sin A \leq \dfrac{bc}{2}$ *we obtain* $\dfrac{b+c}{2} \leq \sqrt{bc}$ *It follows* $\dfrac{\sqrt{bc}(b+c)}{4} \leq \dfrac{bc}{2}$ *hence* $b = c$ *and* $\sin A = 1 \Longrightarrow \angle A = 90^\circ$ *so* $\angle B = \angle C = 45^\circ$
■

Solution 2 : *From sine rule* $\dfrac{a}{2\sin A} = R$, *substituting we get* $b + c = 2\sin A\sqrt{bc}$ *But from* $AM - GM$ *we have* $b+c \geq 2\sqrt{bc}$, *hence* $\sin A = 1$ *or* $A = 90^o$ *and moreover* $b = c$, *giving* $B = C = 45^o$ ■

5. Let $a > 0$ and x_1, x_2, x_3 be real numbers with $x_1 + x_2 + x_3 = 0$. Prove that;

$$\log_2(1 + a^{x_1}) + \log_2(1 + a^{x_2}) + \log_2(1 + a^{x_3}) \geq 3.$$

Solution : *Use the property of the log function to get that*

$$LHS = log_2\left(a^{x_1+x_2+x_3} + a^{x_1+x_2} + a^{x_3+x_2} + a^{x_1+x_3} + a^{x_1} + a^{x_2} + a^{x_3} + 1\right).$$

Use that $x_1 + x_2 + x_3 = 0$, *this simplifies to*

$$LHS = 2 + \left(a_{x_1} + \frac{1}{a^{x_1}}\right) + \left(a_{x_2} + \frac{1}{a^{x_2}}\right) + \left(a_{x_3} + \frac{1}{a^{x_3}}\right).$$

Using AM-GM we get that this expression is greater than or equal to

$$2 + 2 + 2 + 2 = 8.$$

So the $LHS \geq 8$. *Take log on both sides to get that*

$$log_2(LHS) \geq log_2(8) = 3.$$

■

6. Find all non-negative integers x, y such that $x^3y + x + y = xy + 2xy^2$.

Solution 1 : *Case 1:* $xy = 0$. *In this case the other is also visibly zero, so we're done.*

Case 2: $xy \neq 0$. *Observe that we get* $x + y = xy(1 + 2y - x)$. *In particular,* $xy \mid x + y \Rightarrow xy \leq x + y \Rightarrow (x-1)(y-1) \leq 1$. *Now if* $(x-1)(y-1) = 0$, *we have* $x = 1$ *or* $y = 1$, *and in both cases one can check that the other is* 1 *as well. Otherwise,* $x - 1 = y - 1 = 1$(*It can't be* -1 *because that would mean* $xy = 0$). *In other words,* $x = y = 2$. *Since all the solutions* $(0,0), (1,1), (2,2)$ *work, we're done with the problem and these are the solutions.* ■

Solution 2 : *The only solutions are* $(0,0), (1,1), (2,2)$, *which obviously work.*

Taking the equation modulo xy, *we see that* xy *must divide* $x + y$. *Since* x, y *are non-negative, this also implies that* $xy \leq x + y$, *so* $(x-1)(y-1) \leq 1$.

If x, y *were both greater than 1, we would have* $x - 1$ *and* $y - 1$ *are positive integers multiplying to a number at most 1, meaning that* $x = y = 2$.

If one of x, y *were 0, then clearly the other one is zero also.*

If $x = 1$, *then* $y \mid y + 1 \Rightarrow y = 1$. *If* $y = 1$, *then* $x \mid x + 1 \Rightarrow x = 1$.

We have exhausted all cases, so we are done . ■

Solution 3 : *If* $xy \neq 0 \Rightarrow x \mid y$ *and* $y \mid x$ *so,* $x = y \Rightarrow x^4 + 2x = x^2 + 2x^3 \Rightarrow x^2(x^2-1) = 2x(x^2-1)$ *thus:* • $x^2 - 1 = 0 \Rightarrow x = 1$ *Hence,* $(x, y) = (1, 1)$ • $x^2 - 1 \neq 0 \Rightarrow x^2 = 2x \Rightarrow x = 2$ *Hence,* $(x, y) = (2, 2)$ *If* $xy = 0$ *obviously* $(x, y) = (0, 0)$ *Therefore,*

$$(x, y) = (0, 0); (1, 1); (2, 2)$$

■

7. Call a triple (a, b, c) of positive numbers mysterious if

$$\sqrt{a^2 + \frac{1}{a^2c^2} + 2ab} + \sqrt{b^2 + \frac{1}{b^2a^2} + 2bc} + \sqrt{c^2 + \frac{1}{c^2b^2} + 2ca} = 2(a+b+c).$$

Prove that if the triple (a, b, c) is mysterious, then so is the triple (c, b, a).

Solution : *If* $abc < 1$ *we have*

$$\begin{aligned} &\sqrt{a^2 + \frac{1}{a^2c^2} + 2ab} + \sqrt{b^2 + \frac{1}{b^2a^2} + 2bc} + \sqrt{c^2 + \frac{1}{c^2b^2} + 2ca} \\ &> \sqrt{a^2+b^2+2ab} + \sqrt{b^2+c^2+2bc} + \sqrt{c^2+a^2+2ca} \\ &= (a+b) + (b+c) + (c+a) = 2(a+b+c). \end{aligned}$$

Analogously

$$\sqrt{a^2 + \frac{1}{a^2c^2} + 2ab} + \sqrt{b^2 + \frac{1}{b^2a^2} + 2bc} + \sqrt{c^2 + \frac{1}{c^2b^2} + 2ca} = 2(a+b+c)$$

for $abc = 1$ *and*

$$\sqrt{a^2 + \frac{1}{a^2c^2} + 2ab} + \sqrt{b^2 + \frac{1}{b^2a^2} + 2bc} + \sqrt{c^2 + \frac{1}{c^2b^2} + 2ca} < 2(a+b+c)$$

for $abc > 1$.

So (a, b, c) *is mysterious iff* $abc = 1$. *In particular if* (a, b, c) *is mysterious, so is* (c, b, a).
■

8. Determine the number of real solutions a to the equation

$$\left\lfloor \frac{1}{2}a \right\rfloor + \left\lfloor \frac{1}{3}a \right\rfloor + \left\lfloor \frac{1}{5}a \right\rfloor = a$$

Here, if x is a real number, then $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.

Solution : *Let* $a = 30k + r$, *where* k *is an integer,* r *is a real number between* 0 *and* 29 *inclusive and* $lcm(2, 3, 5) = 30$.

Then $\left\lfloor \frac{1}{2}a \right\rfloor = \left\lfloor \frac{1}{2}(30k + r) \right\rfloor = 15k + \left\lfloor \frac{r}{2} \right\rfloor$.

Similarly $\left\lfloor \frac{1}{3}a \right\rfloor = 10k + \left\lfloor \frac{r}{3} \right\rfloor$ *and* $\left\lfloor \frac{1}{5}a \right\rfloor = 6k + \left\lfloor \frac{r}{5} \right\rfloor$.

Now, $\left\lfloor \frac{1}{2}a \right\rfloor + \left\lfloor \frac{1}{3}a \right\rfloor + \left\lfloor \frac{1}{5}a \right\rfloor = a$,

so $\left(15k + \left\lfloor \frac{r}{2} \right\rfloor\right) + \left(10k + \left\lfloor \frac{r}{3} \right\rfloor\right) + \left(6k + \left\lfloor \frac{r}{5} \right\rfloor\right) = 30k + r$

and hence $k = r - \left\lfloor \frac{r}{2} \right\rfloor - \left\lfloor \frac{r}{3} \right\rfloor - \left\lfloor \frac{r}{5} \right\rfloor$.

Clearly, r *has to be an integer, or* $r - \left\lfloor \frac{r}{2} \right\rfloor - \left\lfloor \frac{r}{3} \right\rfloor - \left\lfloor \frac{r}{5} \right\rfloor$ *will not be an integer, and therefore, cannot equal* k.

On the other hand, if r *is an integer, then* $r - \left\lfloor \frac{r}{2} \right\rfloor - \left\lfloor \frac{r}{3} \right\rfloor - \left\lfloor \frac{r}{5} \right\rfloor$ *will also be an integer, giving exactly one solution for* k.

For each $r (0 \leq r \leq 29), a = 30k + r$ *will have a different remainder* $\bmod 30$*, so no two different values of* r *give the same result for* a.

Since there are 30 *possible values for* $r (0, 1, 2, \ldots, 29)$*, there are then* 30 *solutions for* a.
■

9. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each ordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

   What is the maximum number of elements that S can contain?

Solution 1 : *It can be checked that all the sums of pairs for the set* $\{1, 2, 3, 5, 8\}$ *are different.*

Suppose, for a contradiction, that S *is a subset of* $\{1, \ldots, 9\}$ *containing* 6 *elements such that all the sums of pairs are different. Now the smallest possible sum for two numbers from* S *is* $1 + 2 = 3$ *and the largest possible sum is* $8 + 9 = 17$*. That gives* 15 *possible sums:* $3, \ldots, 17$.

Also there are $\binom{6}{2} = 15$ *pairs from* S*. Thus, each of* $3, \ldots, 17$ *is the sum of exactly one pair. The only pair from* $\{1, \ldots, 9\}$ *that adds to* 3 *is* $\{1, 2\}$ *and to* 17 *is* $\{8, 9\}$*. Thus* $1, 2, 8, 9$ *are in* S*. But then* $1 + 9 = 2 + 8$*, giving a contradiction. It follows that the maximum number of elements that* S *can contain is* 5. ■

Solution 2 : *It can be checked that all the sums of pairs for the set* $\{1, 2, 3, 5, 8\}$ *are different.*

Suppose, for a contradiction, that S *is a subset of* $\{1, \ldots 9\}$ *such that all the sums of pairs are different and that* $a_1 < a_2 < \ldots < a_6$ *are the members of* S.

Since $a_1 + a_6 \neq a_2 + a_5$*, it follows that* $a_6 - a_5 \neq a_2 - a_1$.
Similarly $a_6 - a_5 \neq a_4 - a_3$ *and* $a_4 - a_3 \neq a_2 - a_1$*. These three differences must be distinct positive integers, so,*

$$(a_6 - a_5) + (a_4 - a_3) + (a_2 - a_1) \geq 1 + 2 + 3 = 6.$$

Similarly $a_3 - a_2 \neq a_5 - a_4$*, so*

$$(a_3 - a_2) + (a_5 - a_4) \geq 1 + 2 = 3.$$

Adding the above 2 inequalities yields

$$a_6 - a_5 + a_5 - a_4 + a_4 - a_3 + a_3 - a_2 + a_2 - a_1 \geq 6 + 3 = 9,$$

and hence $a_6 - a_1 \geq 9$*. This is impossible since the numbers in* S *are between* 1 *and* 9.
■

10. Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$ and $AB = CB$. Prove that $AD = CD$.

Solution 1 : *Extend DB to a point P on the circle through A and C centered at B. Then*

$$\angle CPD = \frac{1}{2}\angle CBD = \angle ADB$$

and

$$\angle APD = \frac{1}{2}\angle ABD = \angle CDB,$$

so $APCD$ is a parallelogram. Now PD bisects AC so BD is an angle bisector of isosceles triangle ABC. We have

$$\angle ADB = \frac{1}{2}\angle CBD = \frac{1}{2}\angle ABD = \angle CDB$$

so DB is the angle bisector of $\angle ADC$. As DB bisects the base of triangle ADC, this triangle must be isosceles and $AD = CD$. ■

Solution 2 : *Let the bisector of $\angle ABD$ meet AD at E. Let the bisector of $\angle CBD$ meet CD at F. Then*

$$\angle FBD = \angle BDE$$

and

$$\angle EBD = \angle BDF,$$

which imply $BE \mid\mid FD$ and $BF \mid\mid ED$. Thus $BEDF$ is a parallelogram whence

$$BD \ \text{ intersects } \ EF \ \text{ at its midpoint } \ M. \tag{1.1}$$

On the other hand since BE is an angle bisector, we have $\dfrac{AB}{BD} = \dfrac{AE}{ED}$. Similarly we have

$$\frac{CB}{BD} = \frac{CF}{FD}.$$

By assumption $AB = CB$ so

$$\frac{AE}{ED} = \frac{CF}{FD}$$

which implies $EF \mid\mid AC$. Thus $\triangle DEF$ and $\triangle DAC$ are similar, which implies by (1.1) that BD intersects AC at its midpoint N. Since $\triangle ABC$ is isosceles, this implies $AC \perp BD$. Thus $\triangle NAD$ and $\triangle NCD$ are right triangles with equal legs and hence are congruent. Thus $AD = CD$.

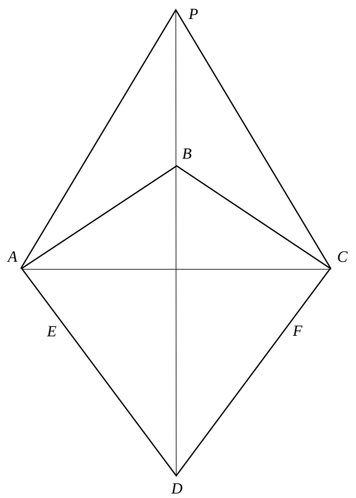


Diagram for Solution 1

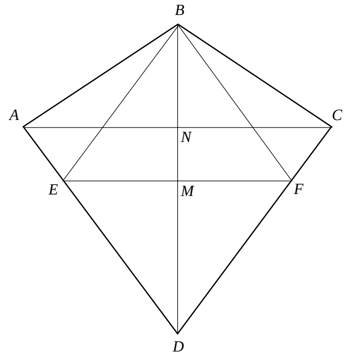


Diagram for Solution 2

■

<u>SECTION B: 10 marks each</u>

11. Find all real numbers x such that $x = \left(x - \dfrac{1}{x}\right)^{1/2} + \left(1 - \dfrac{1}{x}\right)^{1/2}$.

Solution : *Since*

$$\left(x - \frac{1}{x}\right)^{1/2} \geq 0$$

and

$$\left(1 - \frac{1}{x}\right)^{1/2} \geq 0,$$

then

$$0 \leq \left(x - \frac{1}{x}\right)^{1/2} + \left(1 - \frac{1}{x}\right)^{1/2} = x.$$

Note that $x \neq 0$*. Else,* $\dfrac{1}{x}$ *would not be defined so* $x > 0$*.*

Squaring both sides gives,

$$\begin{aligned} x^2 &= \left(x - \frac{1}{x}\right) + \left(1 - \frac{1}{x}\right) + 2\sqrt{\left(x - \frac{1}{x}\right)\left(1 - \frac{1}{x}\right)} \\ x^2 &= x + 1 - \frac{2}{x} + 2\sqrt{x - 1 - \frac{1}{x} + \frac{1}{x^2}} \end{aligned}$$

Multiplying both sides by x and rearranging, we get

$$\begin{aligned} x^3 - x^2 - x + 2 &= 2\sqrt{x^3 - x^2 - x + 1} \\ (x^3 - x^2 - x + 1) - 2\sqrt{x^3 - x^2 - x + 1} + 1 &= 0 \\ (\sqrt{x^3 - x^2 - x + 1} - 1)^2 &= 0 \\ \sqrt{x^3 - x^2 - x + 1} &= 1 \\ x^3 - x^2 - x + 1 &= 1 \\ x(x^2 - x - 1) &= 0 \\ x^2 - x - 1 &= 0 \ \textit{since} \ \ x \neq 0: \end{aligned}$$

Thus $x = \dfrac{1 \pm \sqrt{5}}{2}$. *We must check to see if these are indeed solutions.*

Let $\alpha = \dfrac{1+\sqrt{5}}{2}, \beta = \dfrac{1-\sqrt{5}}{2}$. *Note that* $\alpha + \beta = 1, \alpha\beta = -1$ *and* $\alpha > 0 > \beta$.

Since $\beta < 0, \beta$ *is not a solution.*

Now, if $x = \alpha$ *, then*

$$\begin{aligned} \left(\alpha - \frac{1}{\alpha}\right)^{1/2} + \left(1 - \frac{1}{\alpha}\right)^{1/2} &= (\alpha+\beta)^{1/2} + (1+\beta)^{1/2} && (\textit{since} \quad \alpha\beta - -1) \\ &= 1^{1/2} + (\beta^2)^{1/2} && (\textit{since} \ \ \alpha + \beta = 1 \ \ \textit{and} \ \ \beta^2 = \beta + 1) \\ &= 1 - \beta && (\textit{since} \ \ \beta < 0) \\ &= \alpha && (\textit{since} \ \ \alpha + \beta = 1). \end{aligned}$$

So $x = \alpha$ *is the unique solution to the equation.* ■

12. (a) Find all prime numbers p and q such that $p^3 - 3^q = 10$.

Solution 1 : *If* $p \neq 13$, *we have*

$$p^3 \in \{1, 5, 8, 12\} \pmod{13}$$

and

$$3^q \in \{1, 3, 9\} \pmod{13}$$

$$\Rightarrow 10 = p^3 - 3^q \in \{0, 2, 3, 4, 5, 7, 9, 11, 12\} \pmod{13}, \ \textit{a contradiction.}$$

Therefore, $p = 13 \implies 3^q = 13^3 - 10 \implies q = 7$. *Thus,* $(p, q) = (13, 7)$ *is the only solution.* ■

Solution 2 : *Rearrange to get* $3^q + 10 = p^3$. *Notice that the residues* 3^x *give* mod 13 *are* $1, 3, 9$ *which means the possible* mod 13 *values for* p^3 *are* $11, 0, 6$ *while the cubic residues mod 13 possible are* $0, 1, 5, 7, 8, 9, 12$, *which means* $p^3 \equiv 0 \mod 13$, *which means* $p = 13$. *Substituting this back, we have* $q = 7$ ■

(b) Let a and b be real numbers such that

$$\frac{a}{a^2-5} = \frac{b}{5-b^2} = \frac{ab}{a^2b^2-5}$$

where $a+b \neq 0$. Determine the value of a^4+b^4.

Solution 1 : *By cross multiplication, we have*

$$5a - ab^2 = a^2b - 5b \implies 5(a+b) = ab(a+b) \implies ab = 5$$

Hence,

$$\frac{a}{a^2-5} = \frac{b}{5-b^2} = \frac{1}{4}$$

Again

$$\frac{a}{a^2-5} = \frac{1}{4} \implies a^2 - 4a - 5 = 0 \implies a \in \{-1, 5\}$$

Similarly, $b \in \{1, -5\}$. *Therefore,* $(a,b) = (1,5)$ *or* $(-1,-5)$ *or* $(5,1)$ *or* $(-5,-1)$ *as* $ab = 5$. *Which implies*

$$a^4 + b^4 = 626$$

■

Solution 2 : *Replacing* b *with* $-b$ *we got* $\frac{a}{a^2-5} = \frac{b}{b^2-5} = -\frac{ab}{a^2b^2-5}$ *ans easy to see, that* $ab = -5$

$$a+b = \frac{a^2-b^2}{a-b} = \frac{a^2-5}{a} = 4$$
$$a^2+b^2 = (a+b)^2 - 2ab = 26$$
$$a^4+b^4 = (a^2+b^2) - 2a^2b^2 = 626$$

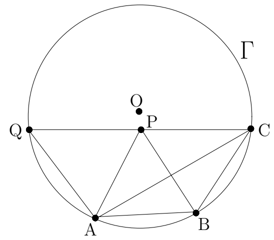
■

Solution 3 : *Let* $\frac{1}{c}$ *the common value. We have three equations :*

$$\begin{aligned} a^2 - ac - 5 &= 0 \\ b^2 + bc - 5 &= 0 \\ (ab)^2 - (ab)c - 5 &= 0 \end{aligned}$$

First and third imply that a *and* ab *both are roots of* $X^2 - cX - 5 = 0$ *and so : : Either* $b = 1$ *(same root), either* $a(ab) = -5$ *and so* $b = -\frac{5}{a^2}$
Subtracting first and second, we get $(a+b)(a-b-c) = 0$ *and so, since* $a+b \neq 0$ *:* $b = a - c = \frac{5}{a}$ *(from first)*
And so two cases :
1) $b = 1$ *and* $b = \frac{5}{a}$ *and so* $(a,b) = (5,1)$ *which indeed fits*
2) $b = -\frac{5}{a^2}$ *and* $b = \frac{5}{a}$ *and so* $(a,b) = (-1,-5)$, *which indeed fits*
And in both cases : $a^4 + b^4 = 626$ ■

13. (a) Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let CP meet Γ again at Q. Prove that $PQ = r$.



Solution 1 :

Let the center of Γ be O, the radius r. Since $BP = BC$, let

$$\theta = \measuredangle BPC = \measuredangle BCP.$$

Quadrilateral $QABC$ is cyclic, so $\measuredangle BAQ = 180^\circ - \theta$ and hence

$$\measuredangle PAQ = 120^\circ - \theta.$$

Also $\measuredangle APQ = 180^\circ - \measuredangle APB - \measuredangle BPC = 120^\circ - \theta$, so

$$PQ = AQ and \measuredangle AQP = 2\theta - 60^\circ.$$

Again because quadrilateral $QABC$ is cyclic,

$$\measuredangle ABC = 180^\circ - \measuredangle AQC = 240^\circ - 2\theta.$$

Triangles OAB and OCB are congruent, since $OA = OB = OC = r$ and $AB = BC$.
Thus

$$\measuredangle ABO = \measuredangle CBO = \frac{1}{2}\measuredangle ABC = 120^\circ - \theta.$$

We have now shown that in triangles AQP and

$$AOB, \measuredangle PAQ = \measuredangle BAO = \measuredangle APQ = \measuredangle ABO.$$

Also $AP = AB$, so $\triangle AQP \cong \triangle AOB$. Hence $QP = OB = r$.

■

Solution 2 :

Let the center of Γ be O, the radius r. Since A, P and C lie on a circle centered at $B, 60^\circ = \measuredangle ABP = 2\measuredangle ACP$, so

$$\measuredangle ACP = \measuredangle ACQ = 30^\circ.$$

Since Q, A, and C lie on

$$\Gamma, \measuredangle QOA = 2\measuredangle QCA = 60^\circ$$

So $QA = r$ since if a chord of a circle subtends an angle of 60° at the center, its length is the radius of the circle.
Now $BP = BC$, so

$$\measuredangle BPC = \measuredangle BCP = \measuredangle ACB + 30^\circ.$$

Thus

$$\measuredangle APQ = 180^\circ - \measuredangle APB - \measuredangle BPC = 90^\circ - \measuredangle ACB$$

Since Q, A, B and C lie on Γ and

$$AB = BC, \measuredangle AQP = \measuredangle AQC = \measuredangle AQB + \measuredangle BQC = 2\measuredangle ACB$$

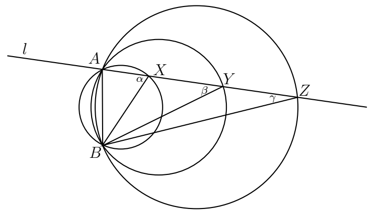
Finally,

$$\measuredangle QAP = 180 - \measuredangle AQP - \measuredangle APQ = 90 - \measuredangle ACB$$

So $\measuredangle PAQ = \measuredangle APQ$ hence $PQ = AQ = r$.

■

(b) Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where AX intersects the other two circles (labelled so that Y is between X and Z).



Solution 1 : *Let l be a line through A different from AB and join B to A, X, Y and Z as in the above diagram. No matter how l is chosen, the angles AXB, AYB and AZB always subtend the chord AB. For this reason the angles in the triangles BXY and BXZ are the same for all such l. Thus the ratio $XY : YZ$ remains constant by similar triangles.*

Note that this is true no matter how X, Y and Z lie in relation to A. Suppose X, Y and Z all lie on the same side of A (as in the diagram) and that $\measuredangle AXB = \alpha, \measuredangle AYB = \beta$ and $\measuredangle AZB = \gamma$. Then

$$\measuredangle BXY = 180^\circ - \alpha, \measuredangle BYX = \beta, \measuredangle BYZ = 180^\circ - \beta$$

and $\measuredangle BZY = \gamma$. Now suppose l is chosen so that X is now on the opposite side of A from Y and Z. Now since X is on the other side of the chord $AB, \measuredangle AXB = 180^\circ - \alpha$, but it is still the case that $\measuredangle BXY = 180^\circ - \alpha$ and all other angles in the two pertinent triangles remain unchanged. If l is chosen so that X is identical with A, then l is tangent to the first circle and it is still the case that $\measuredangle BXY = 180^\circ - \alpha$. All other cases can be checked in a similar manner.

■

Solution 2 : *Let m be the perpendicular bisector of AB and let O_1, O_2, O_3 be the centres of the three circles. Since AB is a chord common to all three circles, O_1, O_2, O_3 all lie on m.*

Let l be a line through A different from AB and suppose that X, Y, Z all lie on the same side of AB, as in the above diagram. Let perpendiculars from O_1, O_2, O_3 meet l at P, Q, R, respectively. Since a line through the centre of a circle bisects any chord, $AX = 2AP, AY = 2AQ$ and $AZ = 2AR$. Now

$$XY = AY - AX = 2(AQ - AP) = 2PQ$$

and, similarly, $YZ = 2QR$ Therefore $XY : YZ = PQ : QR$. But $O_1P || O_2Q || O_3R$, so

$$PQ : QR = O_1O_2 : O_2O_3$$

Since the centres of the circles are fixed, the ratio $XY : YZ = O_1O_2 : O_2O_3$ does not depend on the choice of l.

If X, Y, Z do not all lie on the same side of AB, we can obtain the same result with a similar proof. For instance, if X and Y are opposite sides of AB, then we will have $XY = AY + AX$, but since in this case $PQ = AQ + AP$, it is still the case that $XY = 2PQ$ and result still follows, etc. ■

14. William is thinking of an integer between 1 and 50, inclusive. Victor can choose a positive integer m and ask William: "does m divide your number?", to which William must answer truthfully. Victor continues asking these questions until he determines William's number. What is the minimum number of questions that Victor needs to guarantee this?

Solution : *The minimum number is 15 questions. First, we show that 14 or fewer questions is not enough to guarantee success. Suppose Victor asks at most 14 questions, and William responds with "no" to each question unless $m = 1$. Note that these responses are consistent with the secret number being 1. But since there are 15 primes less than 50, some prime p was never chosen as m. That means the responses are also consistent with the secret number being p. Therefore, Victor cannot determine the number for sure because 1 and p are both possible options.*

Now we show that Victor can always determine the number with 15 questions. Let N be William's secret number. First, Victor asks 4 questions, with $m = 2, 3, 5, 7$. We then case on William's responses.

Case 1. William answers "no" to all four questions.

N can only be divisible by primes that are 11 or larger. This means N cannot have multiple prime factors (otherwise $N \geq 112 > 50$), so either $N = 1$ or N is one of the 11 remaining primes less than 50. Victor can then ask 11 questions with $m = 11, 13, 17, ..., 47$, one for each of the remaining primes, to determine the value of N.

Case 2. William answers "yes" to $m = 2$, and "no" to $m = 3, 5, 7$.

There are only 11 possible values of N that match these answers ($2, 4, 8, 16, 22, 26, 32, 34, 38, 44$, and 46). Victor can use his remaining 11 questions on each of these possibilities.

Case 3. William answers "yes" to $m = 3$, and "no" to $m = 2, 5, 7$.

There are 5 possible values of N ($3, 9, 27, 33$, and 39). Similar to Case 1&2, Victor can ask about these 5 numbers to determine the value of N.

Case 4. William answers "yes" to multiple questions, or one "yes" to $m = 5$ or $m = 7$.

Let k be the product of all m's that received a "yes" response. Since N is divisible by each of these m's, N must be divisible by k. Since $k \geq 5$, there are at most 10 multiples of k between 1 and 50. Victor can ask about each of these multiples of k with his remaining questions.

■

15. (a) Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \geq 4xyz - 4$.

Solution : *Note that*

$$x^2 \geq 4x - 4, y^2 \geq 4y - 4, \quad \textit{and} \quad z^2 \geq 4z - 4,$$

and therefore

$$x^2 + xy^2 + xyz^2 \geq (4x - 4) + x(4y - 4) + xy(4z - 4) = 4xyz - 4.$$

■

(b) The product of the positive real numbers x, y, z is 1. Show that if $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x + y + z$, then $\frac{1}{x^k} + \frac{1}{y^k} + \frac{1}{z^k} \geq x^k + y^k + z^k$ for all positive integers k.

Solution 1 :

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x + y + z \Leftrightarrow xy + xz + yz - x - y - z \geq 0 \Leftrightarrow (1-x)(1-y)(1-z) \geq 0.$

If $1 - x \geq 0,\ 1 - y \geq 0,\ 1 - z \geq 0$ *then* $\{x, y, z\} \subset (0, 1]$. *Hence,* $\{x^k, y^k, z^k\} \subset (0, 1]$. *Hence,* $(1 - x^k)(1 - y^k)(1 - z^k) \geq 0$. *That is,*

$$\frac{1}{x^k} + \frac{1}{y^k} + \frac{1}{z^k} \geq x^k + y^k + z^k.$$

If $1 - x \leq 0,\ 1 - y \leq 0$ *and* $1 - z \geq 0$ *then* $1 - x^k \leq 0,\ 1 - y^k \leq 0,\ 1 - z^k \geq 0$. *That is,*

$$(1 - x^k)(1 - y^k)(1 - z^k) \geq 0$$

and

$$\frac{1}{x^k} + \frac{1}{y^k} + \frac{1}{z^k} \geq x^k + y^k + z^k.$$

■

Solution 2 : *Consider the function :* $f(t) = (t - x)(t - y)(t - z)$. *Then* $f(1) = (1 - x)(1 - y)(1 - z) \geq 0$. *Since* x, y, z *are positive, we have* $s(1 - x^{2k}) = s(1 - x^k) = s(1 - x)$ *(where* $s(n)$ *denotes the sign of* n*), so*

$$(1 - x^{2k})(1 - y^{2k})(1 - z^{2k}) \geq 0$$

Dividing both sides by $(xyz)^k$ *we'll get the desired result from AM-GM inequality.* ■