

The background of the cover is an abstract geometric pattern composed of numerous triangles of various sizes. The color palette is primarily green, ranging from dark forest green to bright lime green, with some yellow and cream-colored triangles interspersed, particularly in the upper right and lower right areas. The triangles are arranged in a way that creates a sense of depth and movement.

# **A Beautiful Journey** Through Olympiad Geometry

Stefan Lozanovski



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Version 1.0.0

[www.olympiadgeometry.com](http://www.olympiadgeometry.com)

Cover created by Damjan Lozanovski

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# Introduction

This book is aimed at anyone who wishes to prepare for the geometry part of the mathematics competitions and Olympiads around the world. No previous knowledge of geometry is needed. Even though I am a fan of non-linear storytelling, this book progresses in a linear way, so everything that you need to know at a certain point will have been already visited before. We will start our journey with the most basic topics and gradually progress towards the more advanced ones. The level ranges from junior competitions in your local area, through senior national Olympiads around the world, to the most prestigious International Mathematical Olympiad.

The word "Beautiful" in the book's title means that we will explore only synthetic approaches and proofs, which I find elegant and beautiful. We will not see any analytic approaches, such as Cartesian or barycentric coordinates, nor we will do complex number or trigonometry bashing.

## Structure

This book is structured in two parts. The first one provides an introduction to concepts and theorems. For the purpose of applying these concepts and theorems to geometry problems, a number of useful properties and examples with solutions are offered. At the end of each chapter, a selection of unsolved problems is provided as an exercise and a challenge for the reader to test their skills in relation to the chapter topics. The second part contains mixed problems, mostly from competitions and Olympiads from all around the world.

## Acknowledgments

I would like to thank my primary school math teacher Ms. Vesna Todorovikj for her dedication in training me and my friend Bojan Joveski for the national math competitions. She introduced me to problem solving and thinking logically, in general. I'll never forget the handwritten collection of geometry problems that she gave us, which made me start loving geometry.

I would also like to thank my high school math Olympiad mentor, Mr. Özgür Kırçak. He boosted my Olympiad spirit during the many Saturdays in "Olympiad Room" while eating burek, drinking tea and solving Olympiad problems. Under his guidance, I started preparing geometry worksheets and teaching the younger Olympiad students. Those worksheets are the foundation of this book.

Finally, I would like to thank all of my students for working through the geometry worksheets, shaping the Olympiad geometry curriculum together with me and giving honest feedback about the lessons and about me as a teacher. Their enthusiasm for geometry and thirst for more knowledge were a great inspiration for me to write this book.

## Support & Feedback

This book is part of my project for sharing knowledge with the whole world. If you are satisfied with the book contents, please support the project by donating at [olympiadgeometry.com](http://olympiadgeometry.com).

Tell me what you think about the book and help me make this Journey even more beautiful. Write a general comment about the book, suggest a topic you'd like to see covered in a future version or report a mistake at the same web site.

You can also follow us on Facebook ([facebook.com/olympiadgeometry](https://facebook.com/olympiadgeometry)) for the latest news and updates. Please leave you honest review there.

The Author

## Notations

Since the math notations slightly differ in various regions of the world, here is a quick summary of the ones we are going to use throughout our journey.

| Notation                       | Explanation                                                                                                                   |
|--------------------------------|-------------------------------------------------------------------------------------------------------------------------------|
| $\angle ABC$                   | angle $ABC$ ; or measurement of said angle                                                                                    |
| $\overline{AB}$                | length of the line segment $AB$                                                                                               |
| $\overrightarrow{AB}$          | vector $AB$                                                                                                                   |
| $\widehat{AB}$                 | arc $AB$                                                                                                                      |
| $(ABCD)$                       | circumcircle of the cyclic polygon $ABCD$                                                                                     |
| $\equiv$                       | coincide<br>Example: if $A - B - C$ are collinear, then $AB \equiv AC$ .                                                      |
| $\cap$                         | intersection                                                                                                                  |
| $\perp$                        | perpendicular                                                                                                                 |
| $\parallel$                    | parallel                                                                                                                      |
| $P_{\triangle ABC}$            | area of the triangle $ABC$                                                                                                    |
| $P_{ABCD}$                     | area of the polygon $ABCD$                                                                                                    |
| $d(P, AB)$                     | distance from the point $P$ to the line $AB$                                                                                  |
| $\angle(p, q)$                 | angle between the lines $p$ and $q$                                                                                           |
| $\alpha, \beta, \gamma, \dots$ | unless otherwise noted, the angles at the vertices $A, B, C, \dots$ in a polygon $ABC \dots$ ; or measurements of said angles |
| $a, b, c$                      | unless otherwise noted, the sides opposite the vertices $A, B, C$ in a triangle $ABC$ ; or lengths of said sides              |
| $\iff$                         | if and only if (shortened iff)<br>Example: $p \iff q$ means "if $p$ then $q$ AND if $q$ then $p$ ".                           |
| $\therefore$                   | therefore                                                                                                                     |
| $\because$                     | because                                                                                                                       |
| LHS \ RHS                      | The left-hand side \ the right-hand side of an equation                                                                       |
| WLOG                           | Without loss of generality                                                                                                    |
| ■                              | Q.E.D. (initialism of the Latin phrase "quod erat demonstrandum", meaning "which is what had to be proved".)                  |



# **Part I**

# **Lessons**



# Chapter 1

## Congruence of Triangles

Two triangles  $\triangle ABC$  and  $\triangle A_1B_1C_1$  are said to be congruent when their corresponding sides and corresponding angles are equal.

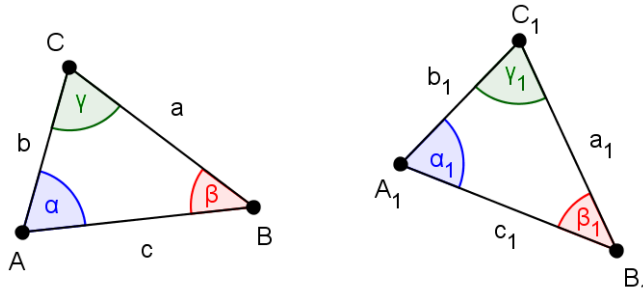


Figure 1.1: Congruent triangles.

$$\triangle ABC \cong \triangle A_1B_1C_1 \iff a = a_1, b = b_1, c = c_1, \alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1$$

However, in most of the problems, the equality of all these six pairs of elements will not be given, so we will need to use some criteria for congruence. With these criteria, we will prove the congruence of two triangles only by using the equality of three pairs of corresponding elements.

**Criterion SSS (side-side-side)** If three pairs of corresponding sides are equal, then the triangles are congruent.

**Criterion SAS (side-angle-side)** If two pairs of corresponding sides and the angles between them are equal, then the triangles are congruent.

**Criterion ASA (angle-side-angle)** If two pairs of corresponding angles and the sides formed by the common rays of these angles are equal, then the triangles are congruent.

These criteria are part of our axioms, so we will not prove them. However, in [Figure 1.2](#), you can see that we can construct exactly one triangle given the corresponding set of elements for each criteria. We can also see why there can not exist an ASS congruence criterion.

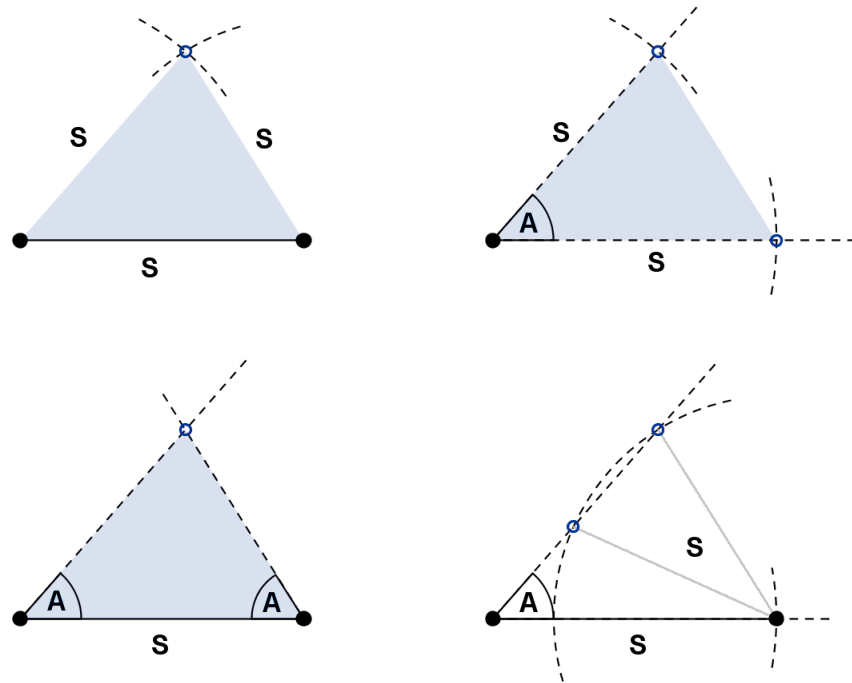


Figure 1.2: Criteria for congruence of triangles.

## Chapter 2

# Angles of a Transversal

When two lines  $p$  and  $q$  are intersected by a third line  $t$ , we get eight angles. The line  $t$  is called a *transversal*. The pairs of angles, depending on their position relative to the transversal and the two given lines are called:

**corresponding angles** if they lie on the same side of the transversal and one of them is in the interior of the lines  $p$  and  $q$ , while the other one is in the exterior (e.g.  $\alpha_1$  and  $\alpha_2$ );

**alternate angles** if they lie on different side of the transversal and both of them are either in the interior or in the exterior of the lines  $p$  and  $q$  (e.g.  $\beta_1$  and  $\beta_2$ ); or

**opposite<sup>1</sup> angles** if they lie on the same side of the transversal and both of them are either in the interior or in the exterior of the lines  $p$  and  $q$  (e.g.  $\gamma_1$  and  $\gamma_2$ ).

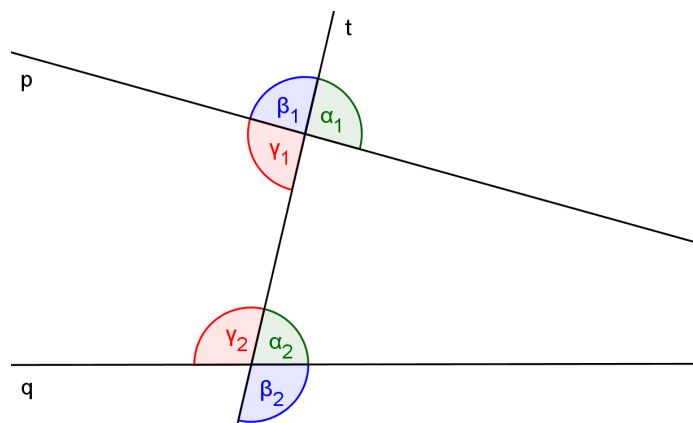


Figure 2.1: Angles of a transversal.

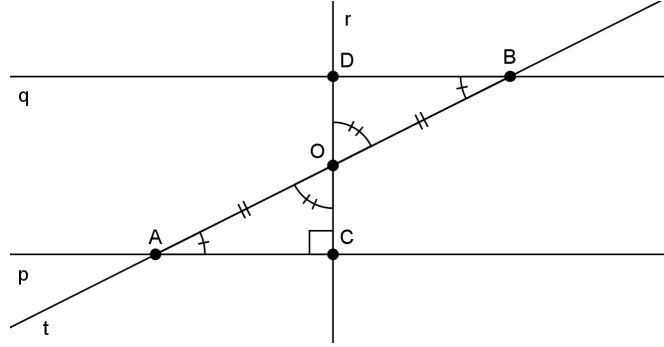
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<sup>1</sup>In some resources, the interior opposite angles are called *consecutive interior angles*, but there is no name for the exterior opposite angles, which have the same property. Since in some languages these angles are called opposite, in this book we'll call them that in English, too, even though I haven't seen this terminology used in other resources in English.

**Property 2.1.** If the lines  $p$  and  $q$  are parallel, then the corresponding angles are equal, the alternate angles are equal and the opposite angles are supplementary. The converse is also true.

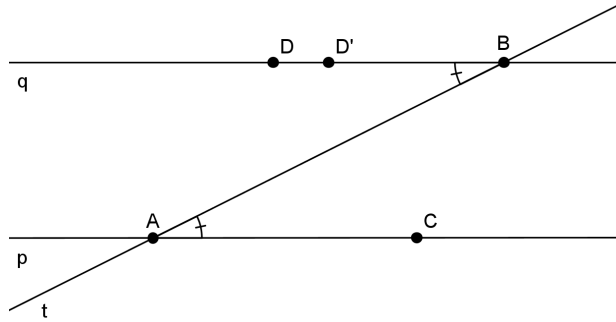
$$p \parallel q \iff \alpha_1 = \alpha_2, \beta_1 = \beta_2, \gamma_1 + \gamma_2 = 180^\circ$$

*Proof.* Let the transversal  $t$  intersect  $p$  and  $q$  at  $A$  and  $B$ , respectively and let  $O$  be the midpoint of the line segment  $AB$ , i.e.  $\overline{AO} = \overline{BO}$ . Let  $r$  be a line



through  $O$  that is perpendicular to  $p$ . Let  $r \cap p = C$  and  $r \cap q = D$ . Then  $\angle OCA = 90^\circ$ . Let's prove one of the directions, i.e. let  $\angle OAC = \angle OBD$ . The angles  $\angle AOC$  and  $\angle BOD$  are vertical angles and therefore equal. So, by the criterion ASA,  $\triangle AOC \cong \triangle BOD$ . Therefore, their corresponding elements are equal, i.e.  $\angle ODB = \angle OCA = 90^\circ$ . So,  $r \perp q$ . Therefore,  $p \parallel q$ .  $\square$

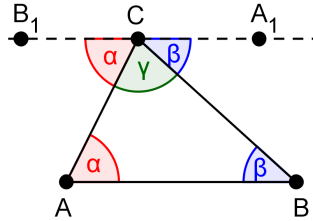
Now, let's prove the other direction. Let  $p \parallel q$ . Let  $t$  be a transversal, such



that  $t \cap p = A$  and  $t \cap q = B$ . Let  $C \in p$  and  $D \in q$ , such that  $C$  and  $D$  are on different sides of  $t$ . We want to prove that  $\angle BAC = \angle ABD$ . Let  $D'$  be a point such that  $\angle BAC = \angle ABD'$ . By the direction we just proved,  $AC \parallel BD'$ . Since  $B$  lies on both  $BD$  and  $BD'$  and  $BD' \parallel AC \parallel BD$ , then  $BD \equiv BD'$  and consequently,  $\angle ABD \equiv \angle ABD'$ . Therefore,  $\angle BAC = \angle ABD$ .

*Remark.* The other angles with vertices at  $A$  and  $B$  are either vertical to (and therefore equal) or form a linear pair (and therefore supplementary) with the angles  $\angle BAC$  and  $\angle ABD$ , so it is easy to prove the rest.  $\blacksquare$

**Example 2.1** (Sum of angles in a triangle). Prove that the sum of the interior angles in a triangle is 180 degrees.



*Proof.* Let  $ABC$  be a triangle. Let's draw a line  $B_1A_1$  which passes through  $C$  and is parallel to  $AB$ . Then, by [Property 2.1](#), we have:

$$\angle B_1CA = \angle CAB = \alpha \quad (\text{alternate interior angles; transversal } AC)$$

$$\angle A_1CB = \angle CBA = \beta \quad (\text{alternate interior angles; transversal } BC)$$

$$\angle ACB = \gamma$$

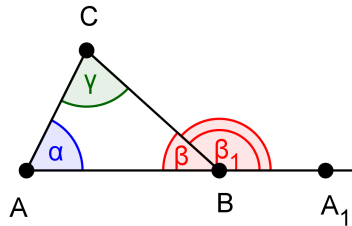
$$\therefore \angle B_1CA + \angle A_1CB + \angle ACB = \alpha + \beta + \gamma$$

$$\angle B_1CA_1 = \alpha + \beta + \gamma$$

$$180^\circ = \alpha + \beta + \gamma$$

■

**Example 2.2.** Prove that an exterior angle equals the sum of the two non-adjacent interior angles.



*Proof.* Let  $ABC$  be a triangle and let  $A_1$  be a point on the extension of  $AB$ .

$$\angle A_1BC + \angle ABC = 180^\circ \quad (\text{linear pair})$$

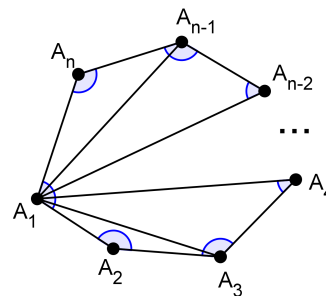
$$\angle ABC + \angle BCA + \angle CAB = 180^\circ \quad (\text{Sum of angles in a triangle})$$

$$\therefore \angle A_1BC = 180^\circ - \angle ABC = \angle BCA + \angle CAB$$

■

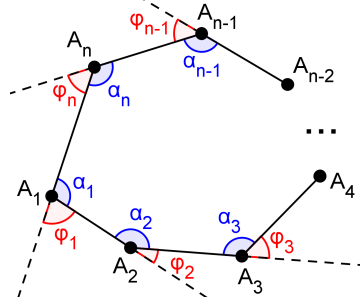
**Example 2.3.** Find the sum of the interior angles in an  $n$ -gon.

*Proof.* Let  $A_1A_2A_3 \dots A_n$  be a polygon with  $n$  sides. If we draw the diagonals from  $A_1$  to all the other  $(n-3)$  vertices, we get  $(n-2)$  distinct triangles. By [Example 2.1](#), the sum of all the interior angles in these triangles is  $(n-2) \cdot 180^\circ$ . Note that these angles actually form all the interior angles in the  $n$ -gon. So, the sum of the interior angles in an  $n$ -gon is  $(n-2) \cdot 180^\circ$ . ■



**Example 2.4.** Find the sum of the exterior angles in an  $n$ -gon.

*Proof.* Let  $A_1A_2A_3 \dots A_n$  be a polygon with  $n$  sides. Let  $\alpha_i$  and  $\varphi_i$  ( $i = 1, 2, \dots, n$ ) be the interior and exterior angles in the polygon, respectively.



Since each exterior and its corresponding interior angle form a linear pair, we have  $\alpha_i + \beta_i = 180^\circ$ ,  $i = 1, 2, \dots, n$ . If we sum these equations, we get

$$\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \varphi_i = n \cdot 180^\circ.$$

From [Example 2.3](#), we know that

$$\sum_{i=1}^n \alpha_i = (n - 2) \cdot 180^\circ.$$

In order to find the sum of the exterior angles, we need to subtract the two previous equations.

$$\sum_{i=1}^n \varphi_i = (n - (n - 2)) \cdot 180^\circ = 2 \cdot 180^\circ = 360^\circ.$$

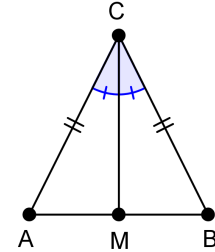
So, the sum of the exterior angles in any polygon does not depend on the number of sides  $n$  and is always  $360^\circ$ . ■

**Example 2.5** (Isosceles Triangle). In  $\triangle ABC$ , two of the sides are equal, i.e.  $\overline{CA} = \overline{CB}$ . Prove that  $\angle CAB = \angle CBA$ .

*Proof.* Let the angle bisector of  $\angle BCA$  intersect the side  $AB$  at  $M$ . Then,  $\angle ACM = \angle BCM$ . Combining with  $\overline{CA} = \overline{CB}$  and  $CM$ -common side, by SAS, we get that  $\triangle ACM \cong \triangle BCM$ . Therefore, their corresponding angles are equal, i.e.

$$\angle CAB \equiv \angle CAM = \angle CBM \equiv \angle CBA.$$

Additionally, as a consequence of the congruence, we can also get two other things:  $\overline{AM} = \overline{MB}$  and  $\angle AMC = \angle BMC$ , which means that  $CM \perp AB$ . Therefore, as a conclusion, the angle bisector, the median and the altitude from the vertex  $C$  in an isosceles triangle coincide with the side bisector of  $AB$ . ■

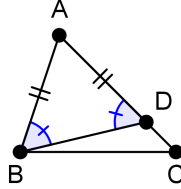


*Remark.* The converse is also true (if  $\angle CAB = \angle CBA$ , then  $\overline{CA} = \overline{CB}$ ). Can you prove it by yourself?

**Example 2.6** (Equilateral triangle). In  $\triangle ABC$ , all three sides are equal. Prove that all the angles are equal to  $60^\circ$ .

*Proof.* Combining [Example 2.1](#) and [Example 2.5](#), we directly get the desired result. ■

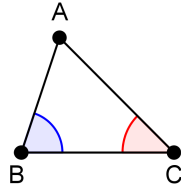
**Example 2.7.** In any triangle, a greater side subtends a greater angle.



*Proof.* In  $\triangle ABC$ , let  $\overline{AC} > \overline{AB}$ . Then we can choose a point  $D$  on the side  $AC$ , such that  $\overline{AD} = \overline{AB}$ . Since  $\triangle ABD$  is isosceles, we have  $\angle ABD = \angle ADB$ .

$$\angle ABC > \angle ABD = \angle ADB \stackrel{2.2}{=} \angle DBC + \angle DCB > \angle DCB \equiv \angle ACB \quad \blacksquare$$

**Example 2.8.** In any triangle, a greater angle is subtended by a greater side.

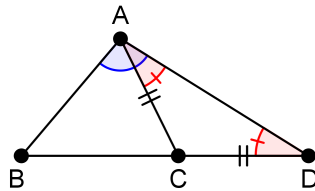


*Proof.* In  $\triangle ABC$ , let  $\angle ABC > \angle ACB$ . We want to prove that  $\overline{AC} > \overline{AB}$ . Let's assume the opposite, i.e.  $\overline{AC} \leq \overline{AB}$ .

- i) If  $\overline{AC} = \overline{AB}$ , then by [Example 2.5](#),  $\angle ABC = \angle ACB$ , which is not true.
- ii) If  $\overline{AC} < \overline{AB}$ , then by [Example 2.7](#),  $\angle ABC < \angle ACB$ , which is not true.

Therefore, our assumption is wrong, so  $\overline{AC} > \overline{AB}$ . ■

**Example 2.9** (Triangle Inequality). In any triangle, the sum of the lengths of any two sides is greater than the length of the third side.



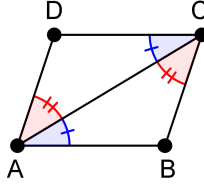
*Proof.* In  $\triangle ABC$ , let  $D$  be a point on the extension of the side  $BC$  beyond  $C$ , such that  $\overline{CD} = \overline{CA}$ . Then,  $\triangle CAD$  is isosceles, so  $\angle CAD = \angle CDA$ . Now, in  $\triangle BAD$  we have

$$\angle BAD = \angle BAC + \angle CAD > \angle CAD = \angle CDA \equiv \angle BDA,$$

which by [Example 2.8](#) means that  $\overline{BD} > \overline{AB}$ . Therefore,

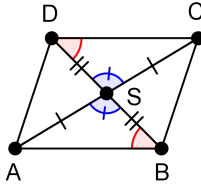
$$\overline{BC} + \overline{CA} = \overline{BC} + \overline{CD} = \overline{BD} > \overline{AB} \quad \blacksquare$$

**Example 2.10.** Let  $ABCD$  be a parallelogram. Prove that its opposite sides are equal.



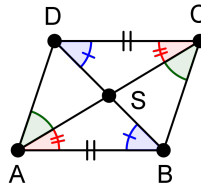
*Proof.* Let's draw the diagonal  $AC$ . Since  $AB \parallel CD$ , by [Property 2.1](#),  $\angle CAB = \angle ACD$ . Similarly, since  $BC \parallel AD$ ,  $\angle ACB = \angle CAD$ . Therefore, since  $AC$  is a common side for the triangles  $\triangle ABC$  and  $\triangle CDA$ , by the ASA criterion,  $\triangle ABC \cong \triangle CDA$ . Therefore, their corresponding elements, are equal, i.e.  $\overline{AB} = \overline{CD}$  and  $\overline{BC} = \overline{DA}$ . ■

**Example 2.11.** In the quadrilateral  $ABCD$ , the intersection point of the diagonals bisects them. Prove that  $ABCD$  is a parallelogram.



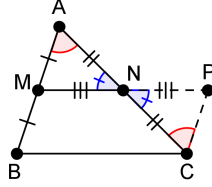
*Proof.* Let the intersection of the diagonals  $AC$  and  $BD$  be  $S$ . Then, from the condition, we have that  $\overline{AS} = \overline{SC}$  and  $\overline{BS} = \overline{SD}$ . Let's take a look at  $\triangle ABS$  and  $\triangle CDS$ . We have  $\overline{AS} = \overline{CS}$ ,  $\angle ASB = \angle CSD$  as vertical angles and  $\overline{BS} = \overline{DS}$ . So, by the SAS criterion,  $\triangle ABS \cong \triangle CDS$ . Therefore, the corresponding elements are equal, i.e.  $\angle ABS = \angle CDS$ . Since these angles are alternate angles of the transversal  $BD$  and the lines  $AB$  and  $CD$ , we have that  $AB \parallel CD$ . Similarly,  $\triangle BCS \cong \triangle DAS$  and  $\angle BCS = \angle DAS$ . Therefore,  $BC \parallel DA$ . ■

**Example 2.12.** In the quadrilateral  $ABCD$ ,  $\overline{AB} = \overline{CD}$  and  $AB \parallel CD$ . Prove that  $ABCD$  is a parallelogram.



*Proof.* Let the intersection of the diagonals  $AC$  and  $BD$  be  $S$ . Since  $AB \parallel CD$ , the alternate angles of the transversal  $BD$  are equal, i.e.  $\angle ABS = \angle CDS$ . Similarly,  $\angle BAS = \angle DCS$ . Combining with the fact that  $\overline{AB} = \overline{CD}$ , by the ASA criterion, we get that  $\triangle ABS \cong \triangle CDS$ . Therefore, as the corresponding elements are equal,  $\overline{AS} = \overline{CS}$  and  $\overline{BS} = \overline{DS}$ . Combining with the fact that  $\angle ASD = \angle CSB$  as vertical angles, by the SAS criterion we get that  $\triangle ASD \cong \triangle CSB$ . Therefore,  $\angle DAS = \angle BCS$ , so  $DA \parallel BC$ . ■

**Example 2.13** (Midsegment Theorem). In a triangle, the segment joining the midpoints of any two sides is parallel to the third side and half its length.



*Proof.* In  $\triangle ABC$ , let  $M$  and  $N$  be the midpoints of the sides  $AB$  and  $AC$ , respectively. Let  $P$  be a point on the ray  $MN$  beyond  $N$ , such that  $\overline{MN} = \overline{NP}$ . Since  $\angle MNA = \angle PNC$  as vertical angles, by SAS we have that  $\triangle AMN \cong \triangle CPN$ . Therefore,  $\overline{AM} = \overline{CP}$  and  $\angle MAN = \angle PCN$  which means that  $AM \parallel CP$ . Now, we have  $\overline{BM} = \overline{AM} = \overline{CP}$  and  $BM \parallel CP$ . By [Example 2.12](#), since the opposite sides in the quadrilateral  $MBCP$  are of equal length and parallel, it must be a parallelogram. Therefore,

$$\overline{MN} \equiv \overline{MP} \parallel \overline{BC}$$

and because of [Example 2.10](#),

$$\overline{MN} = \frac{1}{2}\overline{MP} = \frac{1}{2}\overline{BC}.$$

■

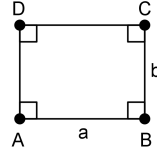
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**Related problems:** 1, 2 and 6.

## Chapter 3

# Area of Plane Figures

### Rectangle

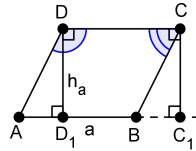


The area of a rectangle  $ABCD$  is defined as the product of the length  $a = \overline{AB} = \overline{CD}$  and the width  $b = \overline{BC} = \overline{AD}$  of the rectangle.

$$P_{ABCD} = a \cdot b$$

Using this fact, we will derive the formulae for the area of other plane figures.

### Parallelogram



Let  $ABCD$  be a parallelogram. WLOG, let  $\angle ABC > 90^\circ$ . Let  $C_1$  and  $D_1$  be the feet of the perpendiculars from  $C$  and  $D$ , respectively, to the line  $AB$ . Since  $AD \parallel BC$ , by [Property 2.1](#),  $\gamma = 180^\circ - \delta$ .

$$\angle BCC_1 = 90^\circ - \gamma = \delta - 90^\circ = \angle ADD_1$$

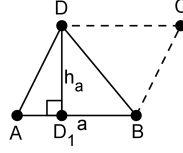
Additionally,  $\overline{CC_1} = d(AB, CD) = \overline{DD_1}$  and  $\angle CC_1B = 90^\circ = \angle DD_1A$ . Therefore, by the ASA criterion,  $\triangle BCC_1 \cong \triangle ADD_1$ . So  $P_{\triangle BCC_1} = P_{\triangle ADD_1}$ .

$$P_{ABCD} = P_{\triangle ADD_1} + P_{DD_1BC} = P_{\triangle BCC_1} + P_{DD_1BC} = P_{DD_1C_1C}$$

Since  $DD_1C_1C$  is a rectangle with length  $\overline{CD} = \overline{AB} = a$  and width  $\overline{CC_1} = h_a$ , we get

$$P_{ABCD} = a \cdot h_a$$

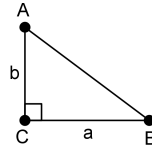
### Triangle



Let  $ABCD$  be a parallelogram. The diagonal  $BD$  divides the parallelogram in two triangles  $\triangle ABD$  and  $\triangle BCD$ . By [Example 2.10](#), the opposite sides of the parallelogram are equal, i.e.  $\overline{AB} = \overline{CD}$  and  $\overline{BC} = \overline{DA}$ . Therefore, since  $\angle BAD = 180^\circ - \angle ADC = \angle DCB$ , by the SAS criterion,  $\triangle BAD \cong \triangle DCB$ . Since congruent triangles have equal areas, then the area of each of the triangles is half the area of the parallelogram, i.e.

$$P_{\triangle ABD} = \frac{a \cdot h_a}{2}$$

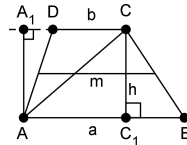
### Right Triangle



In right triangle, the altitude opposite of the side  $a$  is in fact the side  $b$ , so

$$P_{\triangle ABC} = \frac{a \cdot b}{2}$$

### Trapezoid



Let  $ABCD$  be a trapezoid, such that  $AB \parallel CD$ . Let  $A_1$  and  $C_1$  be the feet of the altitudes from  $A$  and  $C$  to the lines  $CD$  and  $AB$ , respectively.

$$P_{ABCD} = P_{\triangle ABC} + P_{\triangle CDA} = \frac{\overline{AB} \cdot \overline{CC_1}}{2} + \frac{\overline{CD} \cdot \overline{AA_1}}{2}$$

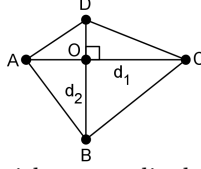
Let  $h = d(AB, CD)$ ,  $a = \overline{AB}$  and  $b = \overline{CD}$ . Then  $\overline{AA_1} = \overline{CC_1} = h$ . Therefore,

$$P_{ABCD} = \frac{a + b}{2} \cdot h$$

Since the midsegment in  $ABCD$ ,  $m$ , is the sum of the midsegments in  $\triangle ABC$  and  $\triangle CDA$ , the area of the trapezoid is sometimes expressed as

$$P_{ABCD} = m \cdot h$$

### Quadrilateral with perpendicular diagonals



Let  $ABCD$  be a quadrilateral with perpendicular diagonals. Let  $AC \cap BD = O$ . Then the triangles  $\triangle ABO$ ,  $\triangle BCO$ ,  $\triangle CDO$  and  $\triangle DAO$  are right triangles. Therefore,

$$\begin{aligned} P_{ABCD} &= P_{\triangle ABO} + P_{\triangle BCO} + P_{\triangle CDO} + P_{\triangle DAO} = \\ &= \frac{\overline{AO} \cdot \overline{BO}}{2} + \frac{\overline{BO} \cdot \overline{CO}}{2} + \frac{\overline{CO} \cdot \overline{DO}}{2} + \frac{\overline{DO} \cdot \overline{AO}}{2} = \\ &= \frac{(\overline{AO} + \overline{CO}) \cdot (\overline{BO} + \overline{DO})}{2} = \frac{\overline{AC} \cdot \overline{BD}}{2} \end{aligned}$$

Let the diagonals  $AC$  and  $BD$  be  $d_1$  and  $d_2$ , respectively. Then,

$$P_{ABCD} = \frac{d_1 \cdot d_2}{2}$$

## 3.1 Area of Triangles

We will now show some properties that are often used in geometry problems.

#### Property 3.1.

- (a) Two triangles that have base sides of equal length and a common altitude, have equal areas.
- (b) Two triangles that have a common base side and altitudes of equal length, have equal areas.

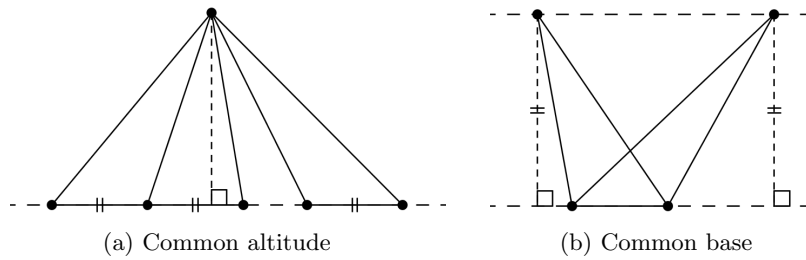
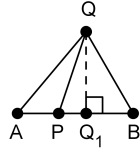


Figure 3.1: Triangles with equal area

*Proof.* Follows directly by the formula for area of triangle  $P_{\triangle ABC} = \frac{a \cdot h_a}{2}$ . ■

**Property 3.2.** Let  $A - P - B$  be collinear points in that order and let  $Q$  be a point that is not collinear with them. Then

$$\frac{P_{\triangle APQ}}{P_{\triangle BPQ}} = \frac{\overline{AP}}{\overline{PB}}.$$



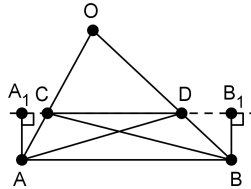
*Proof.* Let  $Q_1$  be the foot of the perpendicular from  $Q$  to  $AB$ . Then,

$$\frac{P_{\triangle APQ}}{P_{\triangle BPQ}} = \frac{\frac{\overline{AP} \cdot \overline{QQ_1}}{2}}{\frac{\overline{PB} \cdot \overline{QQ_1}}{2}} = \frac{\overline{AP}}{\overline{PB}}. \quad \blacksquare$$

We will use the proof of the following well-known theorem to present how these properties can be used.

**Example 3.1** (Thales' Proportionality Theorem). Let  $OAB$  be a triangle and let  $CD$  be a line that intersects its sides  $OA$  and  $OB$  at  $C$  and  $D$ , respectively. Prove that

$$AB \parallel CD \iff \frac{\overline{OC}}{\overline{CA}} = \frac{\overline{OD}}{\overline{DB}}$$



*Proof.* Let  $A_1$  and  $B_1$  be the feet of the perpendiculars from  $A$  and  $B$ , respectively, to the line  $CD$ . Then,

$$\begin{aligned} AB \parallel CD &\iff \overline{AA_1} = \overline{BB_1} \\ &\stackrel{\text{Property 3.1}}{\iff} P_{\triangle CDA} = P_{\triangle CDB} \\ &\iff \frac{P_{\triangle OCD}}{P_{\triangle CDA}} = \frac{P_{\triangle OCD}}{P_{\triangle CDB}} \\ &\stackrel{\text{Property 3.2}}{\iff} \frac{\overline{OC}}{\overline{CA}} = \frac{\overline{OD}}{\overline{DB}} \quad \blacksquare \end{aligned}$$

---

**Related problems:** 3, 4, 5, 7 and 8.

## Chapter 4

# Similarity of Triangles

Two triangles  $\triangle ABC$  and  $\triangle A_1B_1C_1$  are said to be similar when their corresponding angles are equal and their corresponding sides are proportional.

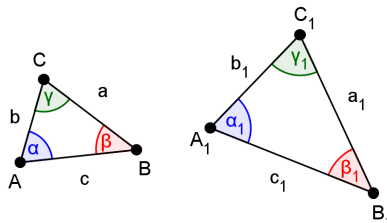


Figure 4.1: Similar triangles.

$$\triangle ABC \sim \triangle A_1B_1C_1 \iff \alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1, \frac{a_1}{a} = \frac{b_1}{b} = \frac{c_1}{c} = k$$

The positive real number  $k$  is called the *ratio of similarity*. If it is greater than 1, then  $\triangle A_1B_1C_1$  is proportionally greater than  $\triangle ABC$ . If it is less than 1, then  $\triangle A_1B_1C_1$  is proportionally smaller than  $\triangle ABC$ . If it is equal to 1, then  $\triangle ABC$  and  $\triangle A_1B_1C_1$  are congruent.

This ratio doesn't apply only for the lengths of the sides, but also for the lengths of other corresponding elements (for example, the length of an altitude, a median, etc). So, for the ratio of the areas of two similar triangles, we get:

$$\frac{P_1}{P} = \frac{\frac{a_1 \cdot h_{a_1}}{2}}{\frac{a \cdot h_a}{2}} = \frac{a_1}{a} \cdot \frac{h_{a_1}}{h_a} = k \cdot k = k^2 \quad \text{or} \quad k = \sqrt{\frac{P_1}{P}}.$$

There are also criteria for similarity of triangles.

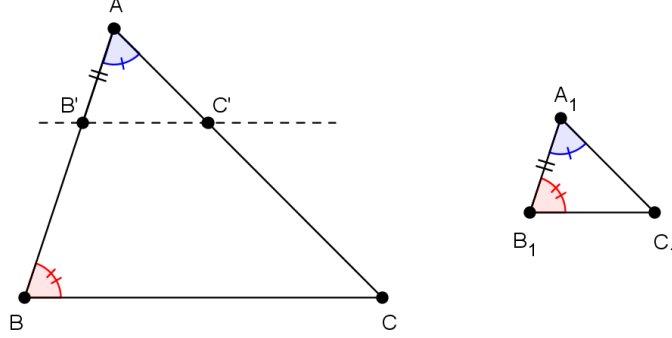
**Criterion AA (angle-angle)** If two pairs of corresponding angles are equal, then the triangles are similar.

**Criterion SSS (side-side-side)** If three pairs of corresponding sides are proportional, then the triangles are similar.

**Criterion SAS (side-angle-side)** If two pairs of corresponding sides are proportional and the angles between them are equal, then the triangles are similar.

We will now present the proofs of these criteria, for the sake of completeness. Although they use only the things that we learned until now, if you are a beginner, you may want to skip them (page 20) since the main point is to know how to use them. But if you are skeptical and don't believe that the criteria for similarity are really true, here are the proofs :)

*Proof (AA).* Let  $\triangle ABC$  and  $\triangle A_1B_1C_1$  be two triangles with  $\alpha = \alpha_1$  and  $\beta = \beta_1$ . By [Example 2.1](#),  $\gamma = \gamma_1$ , too. WLOG, let  $\overline{A_1B_1} < \overline{AB}$ . Then, we



can construct a point  $B' \in AB$ , such that  $\overline{AB'} = \overline{A_1B_1}$ . The parallel line to  $BC$  through  $B'$  intersects  $AC$  at  $C'$ . Then, by [Property 2.1](#),  $\angle AB'C' = \angle ABC$ . So, by the ASA criterion for congruent triangles, we have  $\triangle AB'C' \cong \triangle A_1B_1C_1$ .

Since  $BC \parallel B'C'$ , by [Thales' Proportionality Theorem](#), we have

$$\begin{aligned} \frac{\overline{AB'}}{\overline{B'B}} &= \frac{\overline{AC'}}{\overline{C'C}} \\ \frac{\overline{B'B}}{\overline{AB'}} &= \frac{\overline{C'C}}{\overline{AC'}} \\ \frac{\overline{B'B}}{\overline{AB'}} + 1 &= \frac{\overline{C'C}}{\overline{AC'}} + 1 \\ \frac{\overline{B'B} + \overline{AB'}}{\overline{AB'}} &= \frac{\overline{C'C} + \overline{AC'}}{\overline{AC'}} \\ \frac{\overline{AB}}{\overline{AB'}} &= \frac{\overline{AC}}{\overline{AC'}} \\ \frac{\overline{AB'}}{\overline{AB}} &= \frac{\overline{AC'}}{\overline{AC}} \end{aligned}$$

Now, by substituting the corresponding sides from the congruence we just proved, we get

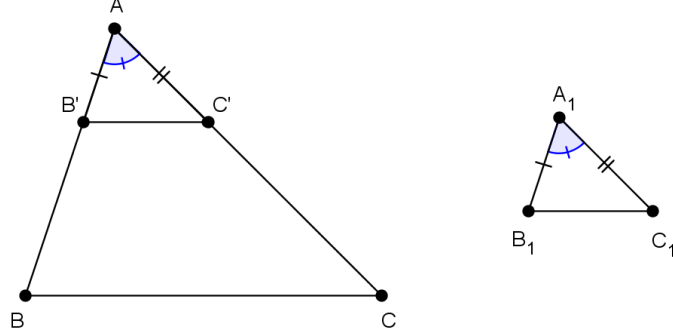
$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}}.$$

Similarly, by constructing a point  $A'' \in BA$  and then a line  $A''C''$  that is parallel to  $AC$ , we can get that

$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{B_1C_1}}{\overline{BC}}.$$

Therefore, all the three corresponding angles are equal and the three corresponding pairs of sides are proportional, so  $\triangle ABC \sim \triangle A_1B_1C_1$ . ■

*Proof (SAS).* Let  $\triangle ABC$  and  $\triangle A_1B_1C_1$  be two triangles with  $\alpha = \alpha_1$  and  $\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}} = k$ .



WLOG, let  $k < 1$ . Then, we can construct points  $B' \in AB$  and  $C' \in AC$ , such that  $\overline{AB'} = \overline{A_1B_1}$  and  $\overline{AC'} = \overline{A_1C_1}$ . By substituting the line segments with equal lengths, we get

$$k = \frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}}.$$

Similarly as in the previous proof, by algebraic transformations (taking the reciprocal value, subtracting 1 on both sides, and taking the reciprocal value once again), we get

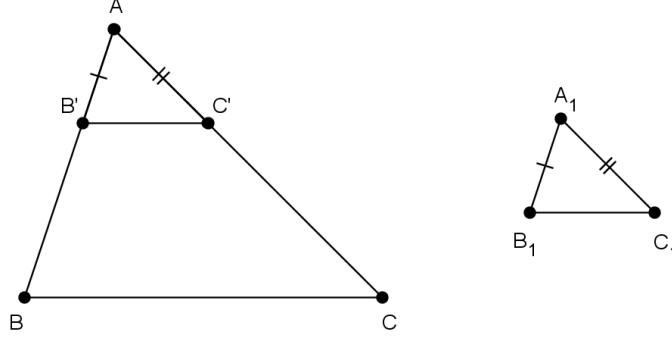
$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}},$$

which by [Thales' Proportionality Theorem](#) means that  $B'C' \parallel BC$ . Therefore, by [Property 2.1](#), we get that  $\angle AB'C' = \angle ABC$  and  $\angle AC'B' = \angle ACB$ .

By the SAS criterion for congruence, we get  $\triangle AB'C' \cong \triangle A_1B_1C_1$ . Therefore,  $\angle AB'C' = \angle A_1B_1C_1$  and  $\angle AC'B' = \angle A_1C_1B_1$ . By combining this with the previous result, we get that  $\beta = \beta_1$  and  $\gamma = \gamma_1$ .

In conclusion, all the angles in the triangles  $\triangle ABC$  and  $\triangle A_1B_1C_1$  are equal, so by the criterion AA that we previously proved, we get that  $\triangle ABC \sim \triangle A_1B_1C_1$ . ■

*Proof (SSS).* Let  $\triangle ABC$  and  $\triangle A_1B_1C_1$  be two triangles with  $\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}} = \frac{\overline{B_1C_1}}{\overline{BC}} = k$ .



WLOG, let  $k < 1$ . Then, we can construct points  $B' \in AB$  and  $C' \in AC$ , such that  $\overline{AB'} = \overline{A_1B_1}$  and  $\overline{AC'} = \overline{A_1C_1}$ . Therefore, we have

$$k = \frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}},$$

which, as in the previous proof, by algebraic transformations (taking the reciprocal value, subtracting 1 on both sides, and taking the reciprocal value once again), becomes

$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}}.$$

Therefore, by [Thales' Proportionality Theorem](#),  $B'C' \parallel BC$ , so by [Property 2.1](#),  $\angle AB'C' = \angle ABC$  and  $\angle AC'B' = \angle ACB$ . By the AA criterion that we earlier proved, we get that  $\triangle AB'C' \sim \triangle ABC$  and therefore

$$\frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}} = \frac{\overline{B'C'}}{\overline{BC}}.$$

By substituting the line segments with equal length that we constructed, we get

$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}} = \frac{\overline{B'C'}}{\overline{BC}}.$$

Combining this with the condition, we can conclude that

$$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{B_1C_1}}{\overline{BC}}, \text{ i.e. } \overline{B'C'} = \overline{B_1C_1}.$$

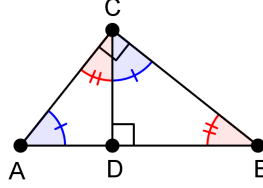
Now, by the SSS criterion for congruence, we get that  $\triangle AB'C' \cong \triangle A_1B_1C_1$  and therefore  $\angle B'AC' = \angle B_1A_1C_1$ . But  $\angle B'AC' \equiv \angle BAC$ , so  $\angle B_1A_1C_1 = \angle BAC$ . Combining this with the condition, by the SAS criterion for similarity that we earlier proved, we get that  $\triangle A_1B_1C_1 \sim \triangle ABC$ .  $\blacksquare$

**Example 4.1** (Euclid's laws). In a right triangle  $ABC$ , with the right angle at  $C$ , let  $D$  be the foot of the perpendicular from  $C$  to  $AB$ . Prove that:

$$\overline{CD}^2 = \overline{AD} \cdot \overline{DB}$$

$$\overline{AC}^2 = \overline{AD} \cdot \overline{AB}$$

$$\overline{BC}^2 = \overline{BD} \cdot \overline{BA}.$$



*Proof.* Let  $\angle CAB = \alpha$  and  $\angle CBA = \beta$ . Since  $\angle ACB = 90^\circ$  and we know that all the angles in a triangle add up to  $180^\circ$ , then  $\alpha + \beta = 90^\circ$ . Now looking at the triangles  $ACD$  and  $BCD$ , and remembering again the sum of angles in a triangle, we get that  $\angle ACD = 180^\circ - 90^\circ - \alpha = \beta$  and  $\angle BCD = 180^\circ - 90^\circ - \beta = \alpha$ .

$\triangle ADC \sim \triangle CDB$  (by the criterion AA)

$$\therefore \frac{\overline{AD}}{\overline{DC}} = \frac{\overline{CD}}{\overline{DB}}, \text{ i.e. } \overline{CD}^2 = \overline{AD} \cdot \overline{DB}$$

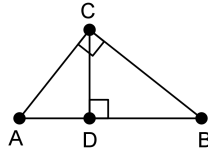
$\triangle ACD \sim \triangle ABC$  (by the criterion AA)

$$\therefore \frac{\overline{AC}}{\overline{AD}} = \frac{\overline{AB}}{\overline{AC}}, \text{ i.e. } \overline{AC}^2 = \overline{AD} \cdot \overline{AB}$$

$\triangle BCD \sim \triangle BAC$  (by the criterion AA)

$$\therefore \frac{\overline{BC}}{\overline{BD}} = \frac{\overline{BA}}{\overline{BC}}, \text{ i.e. } \overline{BC}^2 = \overline{BD} \cdot \overline{BA} \quad \blacksquare$$

**Example 4.2** (Pythagorean Theorem). Prove that the square of the hypotenuse in a right triangle is equal to the sum of the squares of the legs.



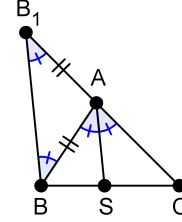
*Proof.* Let  $ABC$  be a right triangle with right angle at  $C$  and let  $CD$  be an altitude in that triangle. From [Example 4.1](#), we know that  $\overline{AC}^2 = \overline{AD} \cdot \overline{AB}$  and  $\overline{BC}^2 = \overline{BD} \cdot \overline{BA}$ . By adding these equations, we get

$$\overline{AC}^2 + \overline{BC}^2 = \overline{AB} \cdot (\overline{AD} + \overline{BD}) = \overline{AB}^2 \quad \blacksquare$$

**Example 4.3** (Angle Bisector Theorem). The angle bisector in a triangle divides the opposite side in segments proportional to the other two sides of the triangle.

*Proof.* Here is an idea how to prove this theorem. Let  $ABC$  be a triangle and let  $S$  be a point on  $BC$ , such that  $AS$  is an angle bisector in  $\triangle ABC$ . We need to prove that  $\frac{BS}{SC} = \frac{AB}{AC}$ . If we rearrange this equality, we

get that we need to prove that  $\frac{CS}{SB} = \frac{CA}{AB}$ . This resembles the Thales' Proportionality Theorem, with the exception that the points  $C$ ,  $A$  and  $B$  are not collinear. So if we take a point  $B_1$  on the extension of  $CA$ , such that  $AB_1 = AB$ , then we will only need to prove that  $SA$  is parallel to  $BB_1$ .



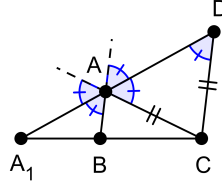
Let  $B_1 \in CA$ , such that  $AB_1 = AB$ . The triangle  $\triangle ABB_1$  is isosceles, so  $\angle ABB_1 = \angle AB_1B = \varphi$ . The angle  $\angle BAC$  is exterior angle of  $\triangle ABB_1$ , so  $\angle BAC = \angle ABB_1 + \angle AB_1B = 2\varphi$ . Since  $AS$  is an angle bisector,  $\angle BAS = \frac{1}{2}\angle BAC = \varphi$ . So,  $\angle BAS = \angle ABB_1$ , which means that  $SA \parallel BB_1$ . By the

Thales' Proportionality Theorem, we get that  $\frac{CS}{SB} = \frac{CA}{AB_1}$ . By substituting  $\overline{AB}$

for  $\overline{AB_1}$ , we get  $\frac{CS}{SB} = \frac{CA}{AB}$ . ■

**Example 4.4** (External Angle Bisector Theorem). Let the bisector of the exterior angle at vertex  $A$  in  $\triangle ABC$  intersect the line  $BC$  at  $A_1$ . Prove that

$$\frac{BA_1}{A_1C} = \frac{AB}{AC}.$$



*Proof.* WLOG, let  $\overline{AB} < \overline{AC}$ , i.e.  $\overline{A_1B} < \overline{A_1C}$ . Let  $D$  be a point on the line  $AA_1$ , such that  $AB \parallel CD$ . Then, by Property 2.1,  $\angle A_1AB = \angle A_1DC$ , so  $\triangle A_1AB \sim \triangle A_1DC$  and therefore

$$\frac{\overline{A_1B}}{\overline{A_1C}} = \frac{\overline{AB}}{\overline{DC}}. \quad (*)$$

Let  $\alpha'$  be the external angle at the vertex  $A$  in  $\triangle ABC$ . Then, as vertical angles,

$$\angle DAC = \frac{\alpha'}{2} = \angle A_1AB = \angle A_1DC \equiv \angle ADC,$$

so  $\triangle ADC$  is isosceles, i.e.  $\overline{AC} = \overline{DC}$ . By substituting in (\*), we get the desired ratio. ■

**Related problem:** 9.

# Chapter 5

## Circles

A circle is a set of point equidistant from one previously chosen point, called the center. The distance from the center to the circle is called the radius of the circle. We will usually notate a circle with center  $O$  and radius  $r$  as  $\omega(O, r)$ .

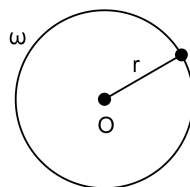
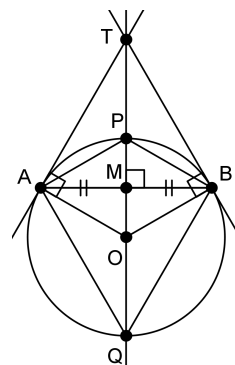


Figure 5.1: Circle  $\omega$  with center  $O$  and radius  $r$ .

### 5.1 Symmetry in a Circle

Let  $AB$  be a chord in a circle. If we connect the points  $A$  and  $B$  with the center  $O$ , we get an isosceles triangle  $ABO$ . If  $M$  is the midpoint of  $AB$ , then  $\triangle AMO \cong \triangle BMO$  (by SSS) and therefore  $\angle AMO = \angle BMO$ , i.e.  $OM \perp AB$ . Also,  $\angle AOM = \angle BOM$ , so if we denote by  $P$  and  $Q$  the intersections of  $OM$  with the circle, we get that  $\triangle OAP \cong \triangle OBP$  (by SAS) which yields  $\overline{AP} = \overline{BP}$  and consequently  $\widehat{AP} = \widehat{BP}$ . Similarly,  $\triangle AOQ \cong \triangle BOQ$  (by SAS) and  $\widehat{AQ} = \widehat{BQ}$ . Looking from a different perspective, this all means that the center of the circle  $O$  and the midpoints of the minor and major arc  $\widehat{AB}$ ,  $P$  and  $Q$ , all lie on the perpendicular bisector of the chord  $AB$ . Hence, the center of any circle can be found as the intersection of the perpendicular bisectors of any two chords.

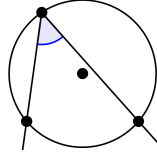


Moreover, let  $T$  be the intersection of the tangents at  $A$  and  $B$ . By the [Pythagorean Theorem](#),  $\overline{TA}^2 = \overline{TO}^2 - \overline{OA}^2 = \overline{TO}^2 - \overline{OB}^2 = \overline{TB}^2$ , i.e.  $\overline{TA} = \overline{TB}$ . So the tangent segments from a point to the circle are equal. Now,  $\triangle OAT \cong \triangle OBT$  (by SSS), so  $\angle TOA = \angle TOB$ , which combined with the previous findings, means that  $T$  also lies on the perpendicular bisector of the chord  $AB$ .

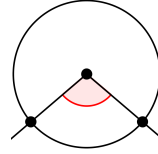
## 5.2 Angles in a Circle

An **inscribed angle** is an angle whose vertex lies on a circle and its rays intersect that circle.

A **central angle** is an angle whose vertex is the center of the circle and its rays intersect that circle.

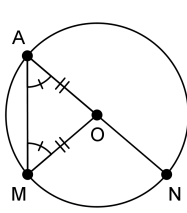


(a) Inscribed angle



(b) Central angle

Now, let's take a look at the relation between an inscribed angle and a central angle that subtend the same arc. Let  $\angle MAN$  and  $\angle MON$  be an inscribed and the central angle that subtend the arc  $\widehat{MN}$ , respectively. The center  $O$  can be in three positions relative to  $\angle MAN$ .



i)  $O$  lies on one of the rays of  $\angle MAN$ , WLOG let  $O$  lie on the ray  $AN$ .

$$\overline{OA} = r = \overline{OM}$$

$\therefore \triangle OAM$  is isosceles.

$$\therefore \angle OAM = \angle OMA$$

$$\therefore \angle MON = \angle OAM + \angle OMA = 2 \cdot \angle OAM \equiv 2 \cdot \angle MAN$$

ii)  $O$  is in the interior of  $\angle MAN$ .

Let  $A_1$  be the second intersection of  $AO$  with the circle.

$$\angle MOA_1 = 2 \cdot \angle MAA_1 \text{ (from case i)}$$

$$\angle A_1ON = 2 \cdot \angle A_1AN \text{ (from case i)}$$

$$\therefore \angle MON = \angle MOA_1 + \angle A_1ON = 2 \cdot \angle MAA_1 + 2 \cdot \angle A_1AN = 2 \cdot \angle MAN$$

iii)  $O$  is in the exterior of  $\angle MAN$ , WLOG  $O$  is closer to the ray  $AN$ .

Let  $A_1$  be the second intersection of  $AO$  with the circle.

$$\angle MOA_1 = 2 \cdot \angle MAA_1 \text{ (from case i)}$$

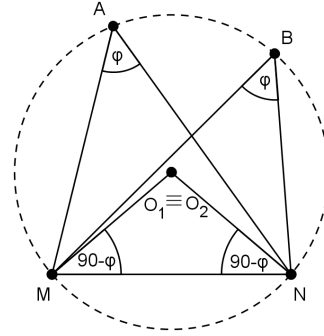
$$\angle NOA_1 = 2 \cdot \angle NAA_1 \text{ (from case i)}$$

$$\therefore \angle MON = \angle MOA_1 - \angle NOA_1 = 2 \cdot \angle MAA_1 - 2 \cdot \angle NAA_1 = 2 \cdot \angle MAN$$

Therefore, any inscribed angle is half the central angle that subtends the same arc. It also implies that all the inscribed angles that subtend the same arc are equal.

The converse is also true. The proof is "less attractive", but it will be presented for the sake of completeness :) We will prove that if two angles  $\angle MAN$  and  $\angle MBN$  are equal (and their vertices  $A$  and  $B$  lie on the same side of the line  $MN$ ), then their vertices,  $A$  and  $B$ , and the intersection points of their corresponding rays,  $M$  and  $N$ , are concyclic.

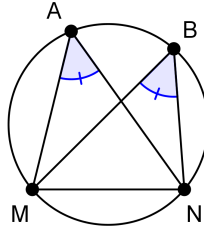
Let  $\omega_1(O_1, r_1)$  be the circumcircle of  $\triangle MAN$ . Let  $\varphi = \angle MAN = \angle MBN$ . Therefore,  $\angle MO_1N = 2 \cdot \angle MAN = 2\varphi$ . Since  $\triangle MO_1N$  is isosceles (because  $\overline{O_1M} = r_1 = \overline{O_1N}$ ),  $\angle O_1MN = \angle O_1NM = 90^\circ - \varphi$ . Similarly, if  $\omega_2(O_2, r_2)$  is the circumcircle of  $\triangle MBN$ , then  $\angle O_2MN = \angle O_2NM = 90^\circ - \varphi$ . Therefore, by the ASA criterion,  $\triangle MO_1N \cong \triangle MO_2N$ . Since  $A$  and  $B$ , and consequently  $O_1$  and  $O_2$  lie on the same side of  $MN$ , we get that  $O_1 \equiv O_2$ . Therefore,  $r_1 = \overline{O_1M} = \overline{O_2M} = r_2$ , so  $\omega_1 \equiv \omega_2$ , i.e. the points  $M, A, B$  and  $N$  lie on a single circle.



In conclusion, we get two important properties of the angles in a circle:

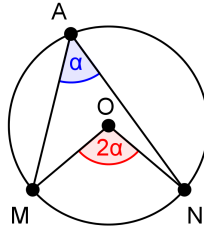
**Property 5.1.** Inscribed angles that subtend the same arc are equal. The converse is also true, i.e. if two angles are equal, then their vertices and the intersection points of their corresponding rays are concyclic.

$$M, A, B, N \in \omega \text{ (in that order)} \iff \angle MAN = \angle MBN \quad (5.1)$$



**Property 5.2.** The central angle is twice an inscribed angle that subtends the same arc.

$$M, A, N \in \omega(O, r) \implies \angle MON = 2 \cdot \angle MAN \quad (5.2)$$



Finally, let's investigate the angle between a tangent and a chord through the tangent point.

Let  $AB$  be a chord in  $\omega(O, r)$  and let  $TA$  be a tangent to  $\omega$  at  $A$ . Let  $\angle BAT = \alpha$ . Since  $TA$  is a tangent, then it must be perpendicular to  $OA$ , i.e.  $\angle OAT = 90^\circ$ .

$$\therefore \angle OAB = \angle OAT - \angle BAT = 90^\circ - \alpha$$

$$\overline{OA} = r = \overline{OB}$$

$\therefore \triangle OAB$  is isosceles

$$\therefore \angle OAB = \angle OBA = 90^\circ - \alpha$$

$$\angle AOB = 180^\circ - 2(90^\circ - \alpha) = 180^\circ - 180^\circ + 2\alpha = 2\alpha$$

Let  $\angle APB$  be any inscribed angle over the arc  $\widehat{AB}$ . Then,

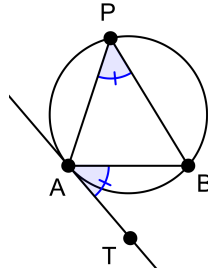
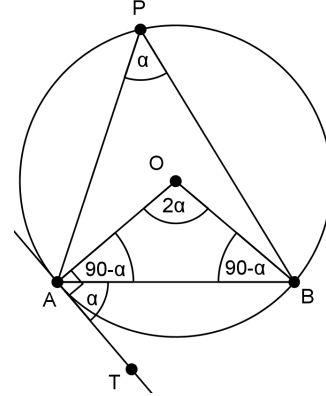
$$\angle APB \stackrel{(5.2)}{=} \frac{1}{2} \cdot \angle AOB = \frac{1}{2} \cdot 2\alpha = \alpha.$$

In conclusion, we get the following property:

**Property 5.3.** The angle between a tangent and a chord is equal to any inscribed angle that subtends that chord.

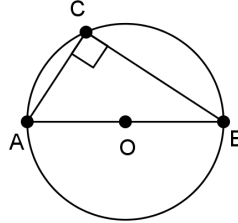
$$\angle TAB = \angle APB \tag{5.3}$$

The converse is also true, i.e. if an angle between a chord and a line through one of the endpoints of the chord is equal to an inscribed angle that subtends that chord, then that line must be tangent to the circle.



We will now see a few useful consequences of the relation between an inscribed and a central angle.

**Example 5.1** (Thales' Theorem). Every inscribed angle that subtends a diameter is a right angle.

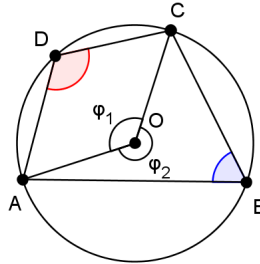


*Proof.* Let  $AB$  be a diameter in a circle with center  $O$ , and let  $C$  be another point on the circle.

$$\angle ACB \stackrel{(5.2)}{=} \frac{1}{2} \cdot \angle AOB = \frac{1}{2} \cdot 180^\circ = 90^\circ \quad \blacksquare$$

*Remark.* Moreover, we can see that inscribed angles that subtend an arc greater than half the circumference are obtuse and inscribed angles that subtend an arc smaller than half the circumference are acute.

**Example 5.2.** The opposite angles of a cyclic quadrilateral are supplementary.



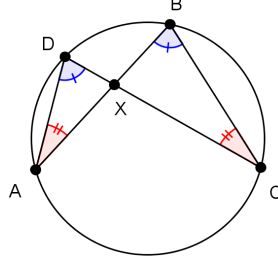
*Proof.* Let  $ABCD$  be a cyclic quadrilateral and let its circumcircle be centered at  $O$ . Let  $\varphi_1$  and  $\varphi_2$  be the central angles that subtend the arcs  $\widehat{ADC}$  and  $\widehat{ABC}$ , respectively.

$$\angle ABC \stackrel{(5.2)}{=} \frac{1}{2} \varphi_1 \text{ (over the arc } \widehat{ADC}\text{)}$$

$$\angle ADC \stackrel{(5.2)}{=} \frac{1}{2} \varphi_2 \text{ (over the arc } \widehat{ABC}\text{)}$$

$$\therefore \angle ABC + \angle ADC = \frac{1}{2} \cdot (\varphi_1 + \varphi_2) = \frac{1}{2} \cdot 360^\circ = 180^\circ \quad \blacksquare$$

**Example 5.3** (Intersecting Chords Theorem). Let  $AB$  and  $CD$  be two line segments that intersect at  $X$ . Then the quadrilateral  $ACBD$  is cyclic if and only if  $AX \cdot XB = CX \cdot XD$ .



*Proof.* Let's notice that

$$\angle AXD = \angle CXB. \quad (*)$$

Then,

$ACBD$  is cyclic

$$\stackrel{(5.1)}{\iff} \angle ADC = \angle ABC \quad \text{and} \quad \angle DAB = \angle DCB$$

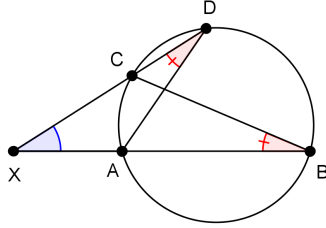
$$\iff \angle ADX = \angle XBC \quad \text{and} \quad \angle DAX = \angle XCB$$

$$\iff \triangle ADX \sim \triangle CBX$$

$$\stackrel{(*)}{\iff} \frac{AX}{XD} = \frac{CX}{XB}$$

$$\iff AX \cdot XB = CX \cdot XD \quad \blacksquare$$

**Example 5.4** (Intersecting Secants Theorem). Let  $AB$  and  $CD$  be two lines that intersect at  $X$ , such that  $X-A-B$  and  $X-C-D$ . Then the quadrilateral  $ABDC$  is cyclic if and only if  $XA \cdot XB = XC \cdot XD$ .



*Proof.* Let's notice that

$$\angle CXB \equiv \angle AXD. \quad (*)$$

Then,

$ABDC$  is cyclic

$$\stackrel{(5.1)}{\iff} \angle ABC = \angle ADC$$

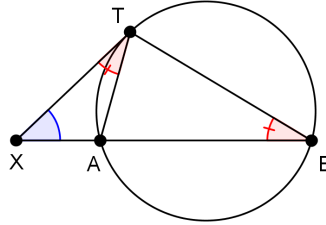
$$\iff \angle XBC = \angle XDA$$

$$\stackrel{(*)}{\iff} \triangle XBC \sim \triangle XDA$$

$$\stackrel{(*)}{\iff} \frac{XB}{XC} = \frac{XD}{XA}$$

$$\iff XA \cdot XB = XC \cdot XD \quad \blacksquare$$

**Example 5.5** (Secant-Tangent Theorem). Let  $ABT$  be a triangle and let  $X$  be a point on  $AB$ , such that  $X - A - B$ . Then  $XT$  is tangent to the circumcircle of  $\triangle ABT$  if and only if  $\overline{XT}^2 = \overline{XA} \cdot \overline{XB}$ .



*Proof.* Let's notice that

$$\angle TXA \equiv \angle BXT. \quad (*)$$

Then,

$XT$  is tangent to  $(ABT)$

$$\stackrel{(5.3)}{\iff} \angle ATX = \angle TBA$$

$$\iff \angle ATX = \angle TBX$$

$$\stackrel{(*)}{\iff} \triangle XTA \sim \triangle XBT$$

$$\stackrel{(*)}{\iff} \frac{\overline{XT}}{\overline{XA}} = \frac{\overline{XB}}{\overline{XT}}$$

$$\iff \overline{XT}^2 = \overline{XA} \cdot \overline{XB} \quad \blacksquare$$

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**Related problems:** 11, 12, 14, 15, 16, 17, 21, 23, 27 and 30.

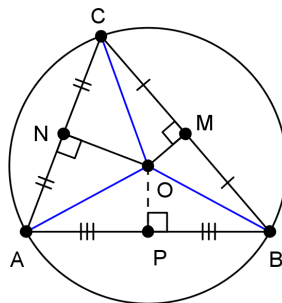
## Chapter 6

# A Few Important Centers in a Triangle

**Property 6.1** (Circumcenter). The three *side bisectors* of a triangle are concurrent. The point of concurrence is the center of a circle that passes through all three vertices of the triangle.

The point of concurrence is called the *circumcenter* of the triangle. The circle that is circumscribed around the triangle is called the *circumcircle* of the triangle.

*Proof.* Let  $M$ ,  $N$  and  $P$  be the midpoints of the sides  $BC$ ,  $CA$  and  $AB$ , respectively. Let  $O$  be the intersection of the side bisectors of  $BC$  and  $CA$ . Then  $OM \perp BC$  and  $ON \perp CA$ .



Let's take a look at the triangles  $\triangle OMB$  and  $\triangle OMC$ . They have a common side  $OM$ ,  $\angle OMB = 90^\circ = \angle OMC$  and  $\overline{MB} = \overline{MC}$ , so by SAS, they are congruent. Therefore, their corresponding sides are equal, i.e.  $\overline{OB} = \overline{OC}$ . Similarly,  $\triangle ONC \cong \triangle ONA$ , so  $\overline{OC} = \overline{OA}$ .

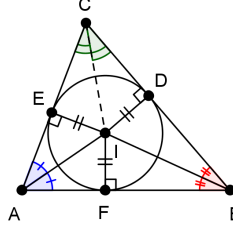
Therefore,  $\overline{OA} = \overline{OB}$ , so  $\triangle OAB$  is isosceles. Therefore, since  $P$  is the midpoint of  $AB$ , by [Example 2.5](#), we get that  $OP$  is the side bisector of  $AB$ .

Since  $\overline{OA} = \overline{OB} = \overline{OC}$ , then  $O$  is the center of a circle that passes through the vertices of  $\triangle ABC$ . ■

**Property 6.2** (Incenter). The three *angle bisectors* of a triangle are concurrent. The point of concurrence is the center of a circle that is tangent to all three sides of the triangle.

The point of concurrence is called the *incenter* of the triangle. The circle that is inscribed inside the triangle is called the *incircle* of the triangle.

*Proof.* Let  $I$  be the intersection of the angle bisectors of  $\angle CAB$  and  $\angle ABC$ . Let  $D$ ,  $E$  and  $F$  be the feet of the perpendiculars from  $I$  to the sides  $BC$ ,  $CA$  and  $AB$ , respectively.



Let's take a look at the triangles  $\triangle AIE$  and  $\triangle AIF$ . They are right triangles and  $\angle IAE = \frac{\alpha}{2} = \angle IAF$ , so they are similar. But they have a common corresponding side  $AI$ , so their ratio of similarity is 1, i.e. they are congruent. Therefore,  $\overline{IE} = \overline{IF}$ . Similarly,  $\triangle BIF \cong \triangle BID$ , so  $\overline{IF} = \overline{ID}$ .

Therefore,  $\overline{IE} = \overline{ID}$ . The triangles  $\triangle CIE$  and  $\triangle CID$  are right triangles, so by the [Pythagorean Theorem](#), we get

$$\overline{CE}^2 = \overline{IC}^2 - \overline{IE}^2 = \overline{IC}^2 - \overline{ID}^2 = \overline{CD}^2, \text{ i.e. } \overline{CE} = \overline{CD}.$$

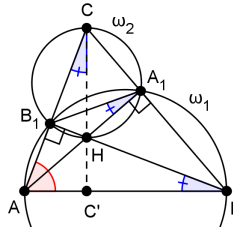
So, by SSS,  $\triangle CIE \cong \triangle CID$  and therefore  $\angle ICE = \angle ICD$ , i.e.  $CI$  is the angle bisector of  $\angle ECD \equiv \angle ACB$ .

Since  $\overline{ID} = \overline{IE} = \overline{IF}$ ,  $ID \perp BC$ ,  $IE \perp CA$  and  $IF \perp AB$ , then  $I$  is the center of a circle that is tangent to the sides of  $\triangle ABC$ . ■

**Property 6.3** (Orthocenter). The three *altitudes* of a triangle are concurrent.

The point of concurrence is called the *orthocenter* of the triangle.

*Proof.* Let the altitudes  $AA_1$  and  $BB_1$  intersect at  $H$ .



Since  $AA_1 \perp BC$  and  $BB_1 \perp AC$ , then  $\angle AA_1B = 90^\circ = \angle AB_1B$ . Therefore,  $ABA_1B_1$  is a cyclic quadrilateral. Let  $(ABA_1B_1)$  be  $\omega_1$ . Also,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so  $CB_1HA_1$  is a cyclic quadrilateral. Let  $(CB_1HA_1)$  be  $\omega_2$ . Let  $CH \cap AB = C'$ . Then,

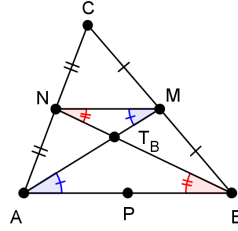
$$\angle ACC' \equiv \angle B_1CH \stackrel{\omega_2}{\equiv} \angle B_1A_1H \equiv \angle B_1A_1A \stackrel{\omega_1}{\equiv} \angle B_1BA \stackrel{\triangle ABB_1}{=} 90^\circ - \alpha$$

Finally, from [Sum of angles in a triangle](#)  $\triangle ACC'$ , we get that  $\angle AC'C = 90^\circ$ , i.e.  $CH \perp AB$ . ■

**Property 6.4** (Centroid). The three *medians* of a triangle are concurrent. The point of concurrence divides the medians in ratio 2 : 1.

The point of concurrence is called the *centroid* of the triangle.

*Proof.* Let  $M$ ,  $N$  and  $P$  be the midpoints of the sides  $BC$ ,  $CA$  and  $AB$ , respectively. Then,  $MN$  is a midsegment in  $\triangle ABC$ , so  $MN \parallel AB$  and  $\overline{AB} = 2 \cdot \overline{MN}$ .



Let the  $B$ -median intersect the  $A$ -median at a point  $T_B$ . Then, by [Property 2.1](#),  $\angle T_B AB = \angle T_B MN$  and  $\angle T_B BA = \angle T_B NM$ , so  $\triangle T_B AB \sim \triangle T_B MN$  and therefore

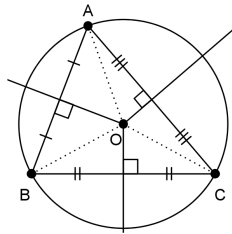
$$\frac{\overline{AT_B}}{\overline{T_B M}} = \frac{\overline{AB}}{\overline{MN}} = 2.$$

Similarly, if the  $C$ -median intersect the  $A$ -median at  $T_C$ , we can get

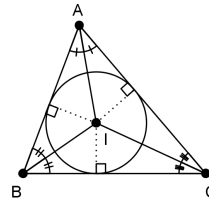
$$\frac{\overline{AT_C}}{\overline{T_C M}} = 2.$$

So  $T_B \equiv T_C \equiv T$ , i.e. the  $B$ -median and the  $C$ -median intersect the  $A$ -median at the same point  $T$ . Additionally,

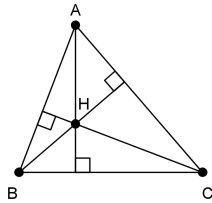
$$\frac{\overline{AT}}{\overline{TM}} = \frac{\overline{BT}}{\overline{TN}} = \frac{\overline{CT}}{\overline{TP}} = 2. \quad \blacksquare$$



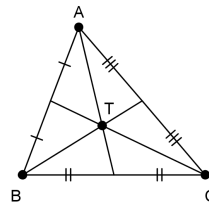
(a) Circumcenter



(b) Incenter



(c) Orthocenter

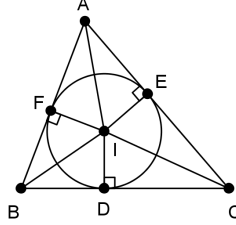


(d) Centroid

Figure 6.1: The four most important centers of a triangle  $ABC$ .

**Property 6.5.** Let  $s$  and  $r$  be the semiperimeter and the radius of the incircle, respectively, in a triangle  $\triangle ABC$ . Then,

$$P_{\triangle ABC} = r \cdot s.$$



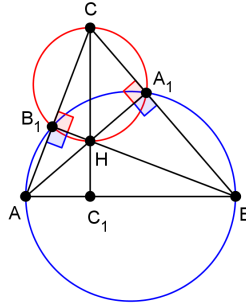
*Proof.* Let  $D$ ,  $E$  and  $F$  be the tangent points of the incircle with the sides  $BC$ ,  $CA$  and  $AB$ , respectively. Let  $I$  be the incenter of  $\triangle ABC$ . Then  $ID \perp BC$ ,  $IE \perp CA$  and  $IF \perp AB$ .

$$\begin{aligned} P_{\triangle ABC} &= P_{\triangle BCI} + P_{\triangle CAI} + P_{\triangle ABI} = \\ &= \frac{\overline{BC} \cdot \overline{ID}}{2} + \frac{\overline{CA} \cdot \overline{IE}}{2} + \frac{\overline{AB} \cdot \overline{IF}}{2} = \\ &= \frac{a \cdot r}{2} + \frac{b \cdot r}{2} + \frac{c \cdot r}{2} = \\ &= r \cdot \frac{a + b + c}{2} = \\ &= r \cdot s \end{aligned}$$

■

**Property 6.6.** Let  $AA_1$ ,  $BB_1$  and  $CC_1$  be the altitudes in a  $\triangle ABC$  and let  $H$  be its orthocenter. Then,

- $ABA_1B_1$ ,  $BCB_1C_1$  and  $CAC_1A_1$  are cyclic quadrilaterals
- $AB_1HC_1$ ,  $BC_1HA_1$  and  $CA_1HB_1$  are cyclic quadrilaterals



*Proof.* Since  $AA_1 \perp BC$  and  $BB_1 \perp AC$ , then  $\angle AA_1B = 90^\circ = \angle AB_1B$ . Therefore,  $ABA_1B_1$  is a cyclic quadrilateral. Similarly,  $BCB_1C_1$  and  $CAC_1A_1$  are cyclic quadrilaterals. □

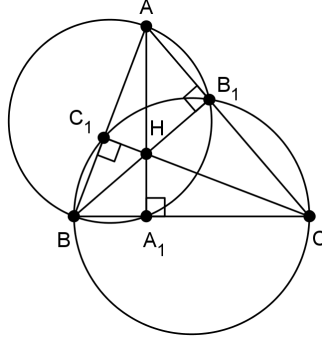
On the other hand,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so  $CA_1HB_1$  is a cyclic quadrilateral. Similarly,  $AB_1HC_1$  and  $BC_1HA_1$  are cyclic quadrilaterals. ■

**Property 6.7.** Let  $AA_1$ ,  $BB_1$  and  $CC_1$  be the altitudes in a  $\triangle ABC$  and let  $H$  be its orthocenter. Then,

$$\overline{AH} \cdot \overline{HA_1} = \overline{BH} \cdot \overline{HB_1} = \overline{CH} \cdot \overline{HC_1}.$$



*Proof.* From [Property 6.6](#), we know that  $ABA_1B_1$  is a cyclic quadrilateral. Since the altitudes  $AA_1$  and  $BB_1$  intersect at the orthocenter  $H$ , by the [Intersecting Chords Theorem](#), we get

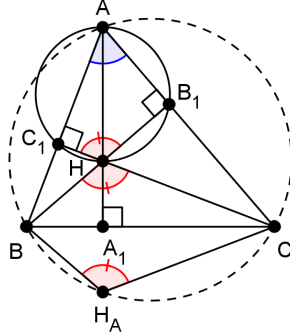
$$\overline{AH} \cdot \overline{HA_1} = \overline{BH} \cdot \overline{HB_1}.$$

Similarly, from the cyclic quadrilateral  $BCB_1C_1$ , we get

$$\overline{BH} \cdot \overline{HB_1} = \overline{CH} \cdot \overline{HC_1}.$$

■

**Property 6.8.** The reflections of the orthocenter with respect to the sides of a triangle lie on the circumcircle of the triangle.



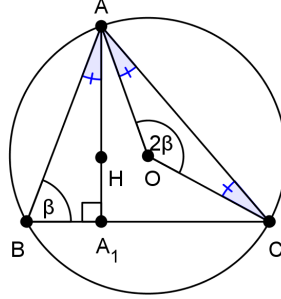
*Proof.* Let  $B_1$  and  $C_1$  be the feet of the altitudes from  $B$  and  $C$ , respectively, in  $\triangle ABC$ . Let  $H$  be its orthocenter. From [Property 6.6](#), we know that  $AB_1HC_1$  is a cyclic quadrilateral and therefore  $\angle B_1HC_1 = 180^\circ - \alpha$ . As vertical angles,  $\angle BHC = \angle B_1HC_1$ . Let  $H_A$  be the reflection of  $H$  with respect to the side  $BC$ . By symmetry,  $\angle BH_A C = \angle BHC$ . Therefore,  $\angle BH_A C = 180^\circ - \alpha$ . Finally,

$$\angle CAB + \angle BH_A C = \alpha + 180^\circ - \alpha = 180^\circ,$$

so  $H_A \in (ABC)$ . Similarly,  $H_B, H_C \in (ABC)$ .

■

**Property 6.9.** The orthocenter and the circumcenter in a triangle are isogonal conjugates<sup>1</sup>.



*Proof.* WLOG,  $\overline{AB} < \overline{AC}$ . Let  $O$  and  $H$  be the circumcenter and the orthocenter, respectively, in  $\triangle ABC$ . We need to prove that  $\angle HAB = \angle OAC$ .

Let  $A_1$  be the foot of the altitude from  $A$  to  $BC$ . Then, from  $\triangle ABA_1$ , we get

$$\angle HAB \equiv \angle A_1AB = 90^\circ - \angle ABA_1 = 90^\circ - \beta. \quad (1)$$

Since  $\angle ABC$  and  $\angle AOC$  are inscribed and central angle, respectively, over  $\widehat{AC}$  in  $(ABC)$ , we have

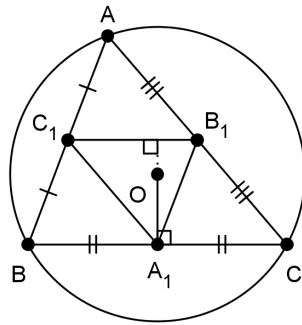
$$\angle AOC = 2 \cdot \angle ABC = 2\beta.$$

Since  $\overline{OA} = R = \overline{OC}$ , from sum of the angles in the isosceles  $\triangle AOC$ , we have

$$\angle OAC = \frac{180^\circ - \angle AOC}{2} = \frac{180^\circ - 2\beta}{2} = 90^\circ - \beta. \quad (2)$$

From (1) and (2), we get that  $\angle HAB = \angle OAC$ . Similarly,  $\angle HBC = \angle OBA$  and  $\angle HCA = \angle OCB$ , so  $H$  and  $O$  are isogonal conjugates in  $\triangle ABC$ . ■

**Property 6.10.** The circumcenter of a triangle is the orthocenter of its medial triangle<sup>2</sup>.

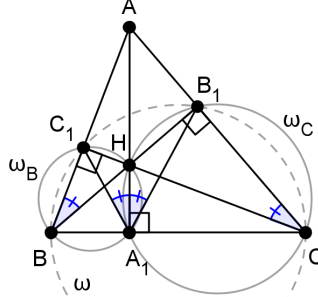


*Proof.* Let  $A_1$ ,  $B_1$  and  $C_1$  be the midpoints of the sides  $BC$ ,  $CA$  and  $AB$  in  $\triangle ABC$ , respectively. Let  $O$  be the circumcenter of  $\triangle ABC$ . Since  $A_1$  is the midpoint of the chord  $BC$  in  $(ABC)$ ,  $OA_1 \perp BC$ . Since  $B_1C_1$  is the midsegment in  $\triangle ABC$ ,  $B_1C_1 \parallel BC$ . Therefore,  $OA_1 \perp B_1C_1$ , i.e.  $A_1O$  is an altitude in  $\triangle A_1B_1C_1$ . Similarly,  $B_1O$  and  $C_1O$  are also altitudes in  $\triangle A_1B_1C_1$ , so  $O$  is the orthocenter of  $\triangle A_1B_1C_1$ . ■

<sup>1</sup>Two points  $P$  and  $P^*$  are called isogonal conjugates if  $XP^*$  is the reflection of  $XP$  across the angle bisector of the angle at the vertex  $X$  in a triangle  $\triangle ABC$ , where  $X$  is any of the vertices  $A$ ,  $B$  or  $C$ . In other words, the lines  $XP$  and  $XP^*$  make equal angles with the sides of the triangle that contain  $X$ , e.g.  $\angle PAB = \angle P^*AC$ .

<sup>2</sup>The medial triangle is the triangle with vertices the midpoints of a triangle.

**Property 6.11.** The orthocenter of a triangle is the incenter of its orthic triangle<sup>3</sup>.



*Proof.* Let  $AA_1$ ,  $BB_1$  and  $CC_1$  be the altitudes in a  $\triangle ABC$ . Let  $H$  be the orthocenter of  $\triangle ABC$ . We want to prove that  $A_1H$  is the angle bisector of  $\angle C_1A_1B_1$ , i.e.  $\angle C_1A_1H = \angle HA_1B_1$ . From [Property 6.6](#), we know that  $BC_1HA_1$ ,  $CA_1HB_1$  and  $BCB_1C_1$  are cyclic quadrilaterals. Let's call them  $\omega_B$ ,  $\omega_C$  and  $\omega$ , respectively. Then,

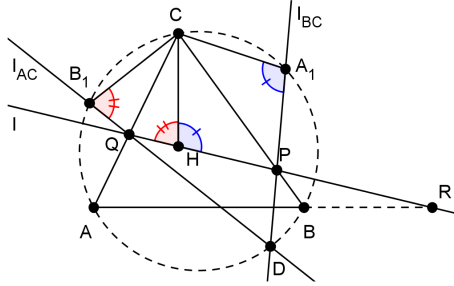
$$\angle C_1A_1H \stackrel{\omega_B}{=} \angle C_1BH \equiv \angle C_1BB_1 \stackrel{\omega}{=} \angle C_1CB_1 \equiv \angle HCB_1 \stackrel{\omega_C}{=} \angle HA_1B_1.$$

Similarly,  $B_1H$  and  $C_1H$  are angle bisectors of  $\angle A_1B_1C_1$  and  $\angle B_1C_1A_1$ , so  $H$  is the incenter of  $\triangle A_1B_1C_1$ . ■

**Property 6.12.** Let  $l$  be any line through the orthocenter of  $\triangle ABC$ . Prove that the reflections of the line  $l$  with respect to the lines  $AB$ ,  $BC$  and  $CA$  are concurrent at the circumcircle of  $\triangle ABC$ .

*Proof.* Let the line  $l$  intersect the lines  $BC$ ,  $CA$  and  $AB$  at  $P$ ,  $Q$  and  $R$ , respectively. We will examine the case when  $H$  is inside  $\triangle ABC$  (the other cases should be similar). Since one of these points will be on the extension of a side and two of these points will be on the sides of the triangle, WLOG, let  $R$  be on the extension of the side  $AB$ .

Let  $A_1$ ,  $B_1$  and  $C_1$  be the reflections of the orthocenter with respect to the sides  $BC$ ,  $CA$  and  $AB$ , respectively. From [Property 6.8](#), we know that  $A_1, B_1, C_1 \in (ABC)$ . Therefore,  $l_{BC}$ , the reflection of the line  $l$  with respect to the line  $BC$ , will contain  $A_1$  (and similarly for the other lines). Let  $D$  be the intersection of the lines  $l_{BC}$  and  $l_{AC}$ . We want to prove that  $D \in (ABC)$ .



$$\angle DA_1C + \angle CB_1D \equiv \angle PA_1C + \angle CB_1Q = \angle PHC + \angle CHQ = 180^\circ$$

$$\therefore D \in (A_1CB_1) \equiv (ABC)$$

Similarly, we can prove that the intersection of the lines  $l_{AB}$  and  $l_{BC}$  lies on  $(ABC)$ . ■

**Related problems:** 19 and 20.

<sup>3</sup>The orthic triangle is the triangle with vertices the feet of the altitudes of a triangle.

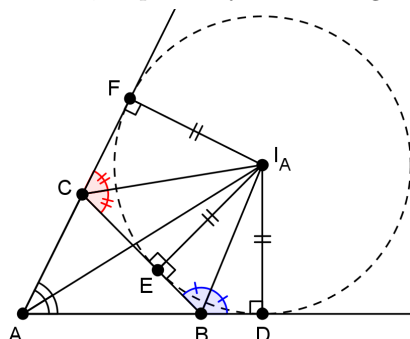
## Chapter 7

# Excircles

**Property 7.1** (Excenter). The external bisectors of two of the angles and the internal angle bisector of the third angle in a triangle are concurrent. The point of concurrence is the center of a circle that is externally tangent to one of the sides and the extensions of the other two sides of a triangle.

The point of concurrence is called an *excenter* of the triangle. The circle that is exscribed outside the triangle is called an *excircle* of the triangle. There are three excircles for each triangle.

*Proof.* Let  $I_A$  be the intersection of the external angle bisectors at  $B$  and  $C$ , in a triangle  $\triangle ABC$ . Let  $D$ ,  $E$ , and  $F$  be the feet of the perpendiculars from  $I_A$  to the lines  $AB$ ,  $BC$  and  $AC$ , respectively. The triangles  $\triangle BDI_A$  and  $\triangle BEI_A$



are similar because they have two equal angles. Moreover, they have a common corresponding side, so they are congruent. Therefore,  $\overline{I_AD} = \overline{I_AE}$ . Similarly,  $\overline{I_AF} = \overline{I_AE}$ . The triangles  $\triangle I_A DA$  and  $\triangle I_A FA$  are right triangles with two equal corresponding sides, so by the [Pythagorean Theorem](#), the third sides are also equal. By the criterion SSS, these triangles are congruent. Therefore, their corresponding angles  $\angle I_A AD$  and  $\angle I_A AF$  are equal, so  $AI_A$  is angle bisector of  $\angle BAC$ .

Since  $\overline{I_AD} = \overline{I_AE} = \overline{I_AF}$ ,  $I_AD \perp AB$ ,  $I_AE \perp BC$  and  $I_AF \perp CA$ , then  $I$  is the center of a circle that is tangent to the lines  $AB$ ,  $BC$  and  $CA$ . ■

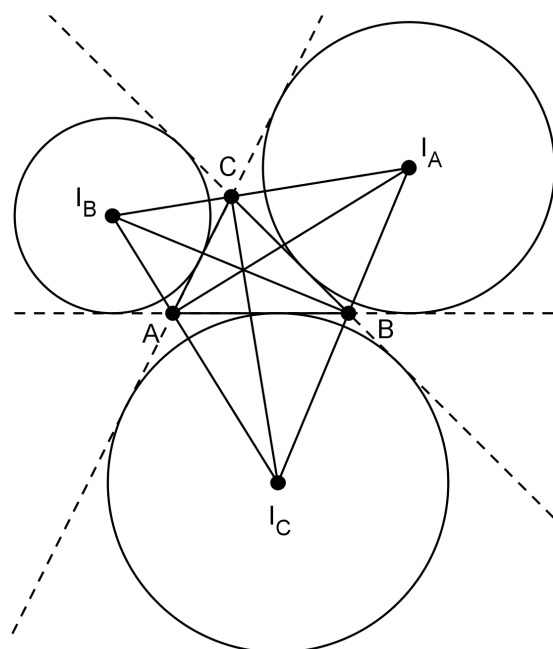


Figure 7.1: The three excircles of a triangle  $ABC$ .

**Example 7.1.** Let  $I$  be the incenter of  $\triangle ABC$ . Let  $A_1$  be the second intersection of the angle bisector of  $\angle BAC$  with the circumcircle of  $\triangle ABC$ . Prove that

$$\overline{A_1B} = \overline{A_1I} = \overline{A_1C}.$$

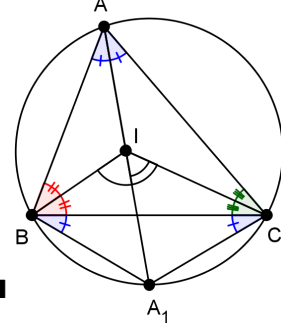
*Proof.*

$$\angle A_1BI = \angle A_1BC + \angle CBI = \angle A_1AC + \frac{\beta}{2} = \frac{\alpha}{2} + \frac{\beta}{2}$$

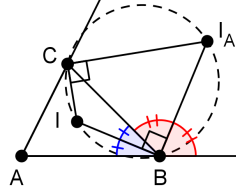
$$\angle A_1IB = \angle IAB + \angle IBA = \frac{\alpha}{2} + \frac{\beta}{2}$$

$\therefore \triangle A_1BI$  is isosceles, i.e.  $\overline{A_1B} = \overline{A_1I}$

Similarly,  $\overline{A_1C} = \overline{A_1I}$ . ■



**Example 7.2.** Let  $I$  and  $I_A$  be the incenter and  $A$ -excenter in  $\triangle ABC$ , respectively. Prove that the quadrilateral  $IBI_AC$  is cyclic.



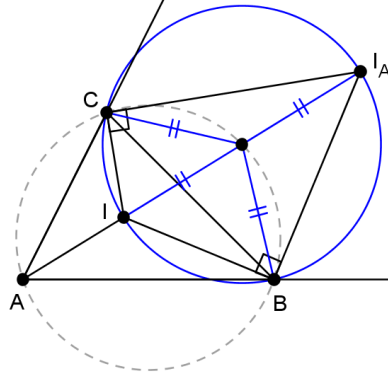
*Proof.* Since  $I$  and  $I_A$  are incenter and excenter,  $BI$  and  $BI_A$  are internal and external angle bisectors.

$$\therefore \angle IBI_A = \angle IBC + \angle CBI_A = \frac{\beta}{2} + \frac{180^\circ - \beta}{2} = 90^\circ$$

Similarly,  $\angle ICI_A = 90^\circ$ .

Therefore,  $\angle IBI_A + \angle ICI_A = 180^\circ$ , so  $IBI_AC$  is cyclic. ■

With these two examples, we actually proved that the circle with diameter  $II_A$  passes through  $B$  and  $C$  and its center is the intersection of  $AI$  with the circumcircle of  $\triangle ABC$ .

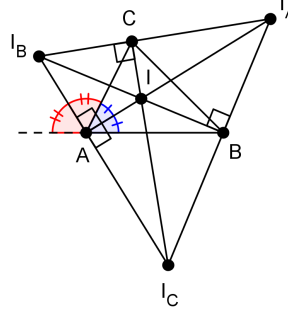


**Example 7.3.** Let  $I_A$ ,  $I_B$  and  $I_C$  be the excenters opposite of  $A$ ,  $B$ , and  $C$  in  $\triangle ABC$ , respectively. Prove that the incenter of  $\triangle ABC$  is the orthocenter of  $\triangle I_AI_BI_C$ .

*Proof.* Let  $I$  be the incenter of  $\triangle ABC$ . Since  $I$  and  $I_B$  are incenter and excenter,  $AI$  and  $AI_B$  are internal and external angle bisectors.

$$\therefore \angle IAI_B = \angle IAC + \angle CAI_B = \frac{\alpha}{2} + \frac{180^\circ - \alpha}{2} = 90^\circ$$

Similarly,  $\angle IAI_C = 90^\circ$ . Therefore, since  $\angle IAI_B + \angle IAI_C = 180^\circ$ ,  $A \in I_BI_C$  and  $I_AA \equiv IA \perp I_BI_C$ , so  $I_AA$  is an altitude in  $\triangle I_AI_BI_C$ . Similarly,  $I_BB$  and  $I_CC$  are altitudes, too, so  $I$  is the orthocenter of  $\triangle I_AI_BI_C$ . ■



**Example 7.4.** Let  $I$  and  $I_A$  be the incenter and the  $A$ -excenter in  $\triangle ABC$ . Prove that

$$\overline{AI} \cdot \overline{AI_A} = \overline{AB} \cdot \overline{AC}.$$

*Proof 1.* Let's look at the triangles  $\triangle AIC$  and  $\triangle ABI_A$ .

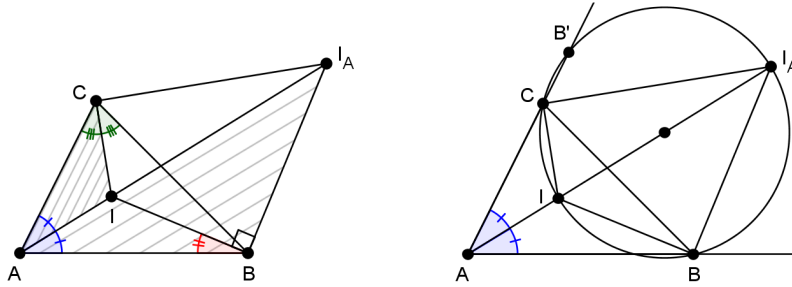
$$\angle CAI = \frac{\alpha}{2} = \angle I_AAB \quad (1)$$

$$\angle AIC = 180^\circ - (\angle IAC + \angle ICA) = 180^\circ - \left(\frac{\alpha + \gamma}{2}\right) = 90^\circ + \frac{\beta}{2}$$

$$\begin{aligned} \angle ABI_A &= \angle ABI + \angle IBI_A = \frac{\beta}{2} + 90^\circ \\ \therefore \angle AIC &= \angle ABI_A \end{aligned} \quad (2)$$

From (1) and (2), we can conclude that  $\triangle AIC \sim \triangle ABI_A$ .

$$\therefore \frac{\overline{AI}}{\overline{AC}} = \frac{\overline{AB}}{\overline{AI_A}}, \text{ i.e. } \overline{AI} \cdot \overline{AI_A} = \overline{AB} \cdot \overline{AC} \quad \blacksquare$$



*Proof 2.* Recall from [Example 7.2](#) that  $IBI_AC$  is a cyclic quadrilateral and that the center of this circle lies on  $AI$ . Notice that the line  $AC$  is a reflection of the line  $AB$  with respect to the angle bisector of  $\angle BAC$ ,  $AI$ . Let the second intersection of  $(IBI_AC)$  with  $AC$  be  $B'$ . By symmetry,  $\overline{AB} = \overline{AB'}$ . Now, by the [Intersecting Secants Theorem](#) for the point  $A$ , we have

$$\overline{AI} \cdot \overline{AI_A} = \overline{AB'} \cdot \overline{AC} = \overline{AB} \cdot \overline{AC} \quad \blacksquare$$

**Related problem:** 36.

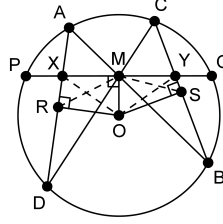
## Chapter 8

# A Few Useful Lemmas

### 8.1 Butterfly Theorem

**Example 8.1** (Butterfly Theorem). Let  $M$  be the midpoint of a chord  $PQ$  of a circle  $\omega$ , through which two other chords  $AB$  and  $CD$  are drawn. Let  $AD \cap PQ = X$  and  $BC \cap PQ = Y$ . Prove that  $M$  is also the midpoint of  $XY$ .

*Proof.* Let  $O$  be the center of  $\omega$ . Since  $M$  is the midpoint of  $PQ$ ,  $OM \perp PQ$ . Therefore, in order to show that  $\overline{XM} = \overline{MY}$ , we need to prove that  $\angle MOX = \angle MOY$ .



Let  $R$  and  $S$  be the feet of the perpendiculars from  $O$  to  $AD$  and  $BC$ , respectively. Therefore,  $\overline{AR} = \overline{RD}$  and  $\overline{BS} = \overline{SC}$ .

$$\angle DAM \equiv \angle DAB \stackrel{\omega}{=} \angle DCB \equiv \angle MCB \quad \text{and} \quad \angle AMD = \angle CMB,$$

$$\therefore \triangle AMD \sim \triangle CMB$$

$$\therefore \frac{\overline{AD}}{\overline{AM}} = \frac{\overline{CB}}{\overline{CM}}$$

$$\therefore \frac{\overline{AR}}{\overline{AM}} = \frac{\overline{CS}}{\overline{CM}} \quad (\because \frac{\overline{AD}}{\overline{AR}} = 2 = \frac{\overline{CB}}{\overline{CS}})$$

$$\therefore \triangle AMR \sim \triangle CMS \quad (\because \angle RAM \equiv \angle DAB \stackrel{\omega}{=} \angle DCB \equiv \angle MCS)$$

$$\therefore \angle MRA = \angle MSC \quad (*)$$

Since  $OM \perp PQ$  and  $OR \perp AD$ ,

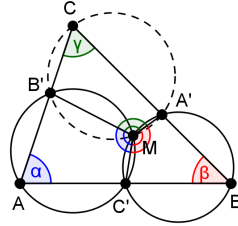
$$\angle ORX + \angle OMX \equiv \angle ORA + \angle OMP = 90^\circ + 90^\circ = 180^\circ.$$

Therefore,  $OMXR$  is cyclic. Similarly,  $OMYS$  is cyclic. Therefore,

$$\angle MOX = \angle MRX \equiv \angle MRA \stackrel{(*)}{=} \angle MSC \equiv \angle MSY = \angle MOY \quad \blacksquare$$

## 8.2 Miquel's Theorem

**Example 8.2.** Let  $ABC$  be a triangle, with arbitrary points  $A'$ ,  $B'$  and  $C'$  on sides  $BC$ ,  $CA$  and  $AB$ , respectively (or their extensions). The circumcircles of  $\triangle AB'C'$ ,  $\triangle A'BC'$  and  $\triangle A'B'C$  intersect in a single point, called the Miquel point.



*Proof.* Let's assume that the points  $A'$ ,  $B'$  and  $C'$  are on the sides (not on the extensions). In the other cases, the proof follows a similar structure.

Let  $M = (AB'C') \cap (A'BC')$ . We will prove that  $M$  lies on  $(A'B'C)$ , too. Since  $AB'MC'$  and  $BC'MA'$  are cyclic, we have

$$\angle B'MC' = 180^\circ - \alpha \quad \text{and} \quad \angle C'MA' = 180^\circ - \beta.$$

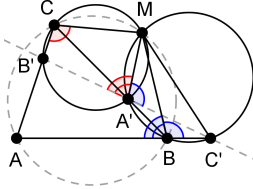
$$\therefore \angle A'MB' = 360^\circ - (\angle B'MC' + \angle C'MA') = \alpha + \beta$$

$$\therefore \angle A'CB' + \angle A'MB' = \gamma + \alpha + \beta = 180^\circ$$

$$\therefore M \in (A'B'C)$$

■

**Example 8.3.** Let  $ABC$  be a triangle, with arbitrary points  $A'$ ,  $B'$  and  $C'$  on sides  $BC$ ,  $CA$  and  $AB$ , respectively (or their extensions). The Miquel point lies on the circumcircle of  $\triangle ABC$  if and only if the points  $A'$ ,  $B'$  and  $C'$  are collinear.



*Proof.* We will see the configuration when one of the points is on the extension of the sides, WLOG let it be  $C'$ . The other case, where all three points are on the extensions of the sides follows a similar structure.

Let  $M$  be the Miquel point of  $\triangle ABC$ . Then  $MA'BC'$  and  $MCB'A'$  are cyclic quadrilaterals.

$$\angle MA'C' = \angle MBC' = 180^\circ - \angle MBA$$

$$\angle MA'B' = 180^\circ - \angle MCB' \equiv 180^\circ - \angle MCA$$

$$\therefore \angle MA'C' + \angle MA'B' = 360^\circ - (\angle MBA + \angle MCA)$$

The points  $C' - A' - B'$  are collinear iff the left-hand side is  $180^\circ$ . The right-hand side is  $180^\circ$  iff  $ABMC$  is a cyclic quadrilateral, i.e.  $M \in (ABC)$ . ■

**Related problems:** 25 and 55.

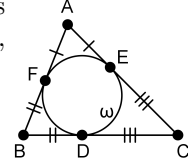
### 8.3 Tangent Segments

From [section 5.1](#), we know that the tangent segments from a point to the circle are of equal length. We will now present some useful properties based on this fact.

**Example 8.4.** Let  $\omega$  be the incircle in  $\triangle ABC$ . Let  $D$  be the tangent point of  $\omega$  with the side  $BC$ . Prove that  $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$ .

*Proof.* Let  $E$  and  $F$  be the tangent points of  $\omega$  with the sides  $CA$  and  $AB$ , respectively. Then, as tangent segments from  $A$ ,  $B$  and  $C$  to  $\omega$ , we get

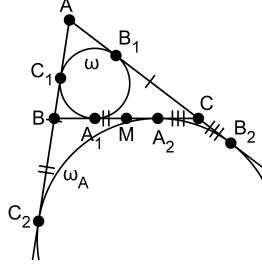
$$\overline{AF} = \overline{AE}, \quad \overline{BF} = \overline{BD} \quad \text{and} \quad \overline{CD} = \overline{CE}.$$



$$\therefore \overline{AB} + \overline{CD} = \overline{AF} + \overline{FB} + \overline{CD} = \overline{AE} + \overline{EC} + \overline{BD} = \overline{AC} + \overline{BD} \quad \blacksquare$$

**Example 8.5.** Let  $\omega$  and  $\omega_A$  be the incircle and the  $A$ -excircle in  $\triangle ABC$ . Let  $A_1$ ,  $B_1$  and  $C_1$  be the tangent points of  $\omega$  with the sides  $BC$ ,  $CA$  and  $AB$ , respectively. Let  $A_2$ ,  $B_2$  and  $C_2$  be the tangent points of  $\omega_A$  with the lines  $BC$ ,  $CA$  and  $AB$ . Prove that:

- $\overline{AB} + \overline{BA_2} = \overline{AC} + \overline{CA_2}$ ;
- $\overline{BA_2} = \overline{CA_1}$ , i.e.  $\overline{A_1M} = \overline{MA_2}$ , where  $M$  is the midpoint of  $BC$ ;



*Proof.* As tangent segments from the points  $A$ ,  $B$  and  $C$  to  $\omega_A$ , we get

$$\overline{AB_2} = \overline{AC_2}, \quad \overline{BA_2} = \overline{BC_2} \quad \text{and} \quad \overline{CA_2} = \overline{CB_2}.$$

$$\therefore \overline{AB} + \overline{BA_2} = \overline{AB} + \overline{BC_2} = \overline{AC_2} = \overline{AB_2} = \overline{AC} + \overline{CB_2} = \overline{AC} + \overline{CA_2} \quad \square$$

Since the sum of both sides equals the whole perimeter of  $\triangle ABC$ , then each side is equal to its semiperimeter  $s$ .

$$\therefore \overline{BA_2} = s - \overline{AB}$$

From [Example 8.4](#), we have

$$\overline{AC} + \overline{BA_1} = \overline{AB} + \overline{CA_1}.$$

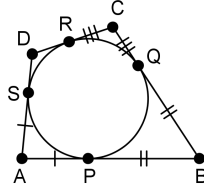
Again, the sum of both sides equals the whole perimeter of  $\triangle ABC$ , so each side is equal to its semiperimeter  $s$ .

$$\therefore \overline{CA_1} = s - \overline{AB}$$

Thus, we can conclude that  $\overline{BA_2} = \overline{CA_1}$ . Since  $\overline{BM} = \overline{CM}$ , then we also have  $\overline{A_1M} = \overline{MA_2}$ .  $\blacksquare$

**Example 8.6** (Tangential quadrilateral). Let  $ABCD$  be a quadrilateral such that there exists an incircle that is tangent to its sides. Prove that the sums of the opposite sides are equal, i.e.

$$\overline{AB} + \overline{CD} = \overline{BC} + \overline{AD}.$$



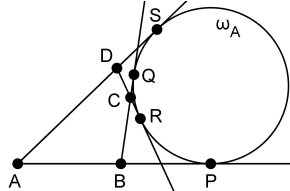
*Proof.* Let  $P$ ,  $Q$ ,  $R$  and  $S$  be the tangent points of the incircle with the sides  $AB$ ,  $BC$ ,  $CD$  and  $DA$ , respectively. Then, as tangent segments,

$$\overline{AP} = \overline{AS}, \quad \overline{BP} = \overline{BQ}, \quad \overline{CQ} = \overline{CR} \quad \text{and} \quad \overline{DR} = \overline{DS}.$$

$$\therefore \overline{AB} + \overline{CD} = \overline{AP} + \overline{PB} + \overline{CR} + \overline{RD} = \overline{AS} + \overline{BQ} + \overline{CQ} + \overline{DS} = \overline{BC} + \overline{AD} \quad \blacksquare$$

**Example 8.7** (Ex-tangential quadrilateral). Let  $ABCD$  be a quadrilateral such that there exists an excircle  $\omega_A$  that is tangent to the rays  $AB$  (beyond  $B$ ) and  $AD$  (beyond  $D$ ) and is also tangent to the lines  $BC$  and  $CD$ . Prove that

$$\overline{AB} + \overline{BC} = \overline{AD} + \overline{DC}.$$

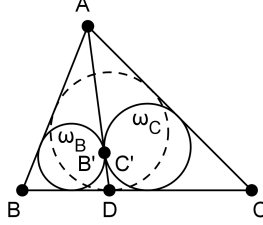


*Proof.* Let  $P$ ,  $Q$ ,  $R$  and  $S$  be the tangent points of the excircle with the lines  $AB$ ,  $BC$ ,  $CD$  and  $DA$ , respectively. Then, as tangent segments,

$$\overline{AP} = \overline{AS}, \quad \overline{BP} = \overline{BQ}, \quad \overline{CQ} = \overline{CR} \quad \text{and} \quad \overline{DR} = \overline{DS}.$$

$$\begin{aligned} \therefore \overline{AB} + \overline{BC} &= \overline{AP} - \overline{BP} + \overline{BQ} - \overline{CQ} = \\ &= \overline{AP} - \overline{CQ} = \\ &= \overline{AS} - \overline{CR} = \\ &= \overline{AS} - \overline{DS} + \overline{DR} - \overline{CR} = \\ &= \overline{AD} + \overline{DC} \end{aligned} \quad \blacksquare$$

**Example 8.8.** Let  $ABC$  be a triangle, and let  $D$  be the point where the incircle touches the side  $BC$ . Let  $\omega_B$  and  $\omega_C$  be the incircles of  $\triangle ABD$  and  $\triangle ACD$ , respectively. Prove that  $\omega_B$  and  $\omega_C$  are tangent to each other.



*Proof.* Let  $B'$  and  $C'$  be the tangent points of  $\omega_B$  and  $\omega_C$ , respectively, to the side  $AD$ . We need to prove that  $B' \equiv C'$ .

Using [Example 8.4](#) on  $\triangle ABD$  and  $\triangle ACD$ , we get

$$\overline{AB} + \overline{DB'} = \overline{BD} + \overline{AB'} \quad \text{and} \quad \overline{CD} + \overline{AC'} = \overline{AC} + \overline{DC'}$$

By adding these two equations side by side, we get

$$\overline{AB} + \overline{CD} + \overline{AC'} + \overline{DB'} = \overline{AC} + \overline{BD} + \overline{AB'} + \overline{DC'}$$

From, [Example 8.4](#), we know that  $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$ , so

$$\overline{AC'} + \overline{DB'} = \overline{AB'} + \overline{DC'}.$$

By adding  $\overline{AB'} + \overline{AC'}$  on both sides, we get:

$$2 \cdot \overline{AC'} + \overline{AD} = 2 \cdot \overline{AB'} + \overline{AD}$$

$$\therefore \overline{AC'} = \overline{AB'}, \text{ i.e. } B' \equiv C'$$

■

**Related problem:** 43.

## 8.4 Simson Line Theorem

**Example 8.9** (Simson Line Theorem). Let  $P$  be a point on the circumcircle  $\omega$  of a triangle  $ABC$ . If  $D$ ,  $E$  and  $F$  are the feet of the perpendiculars from  $P$  to the lines  $AB$ ,  $BC$  and  $CA$ , prove that the points  $D$ ,  $E$  and  $F$  are collinear.

*Proof.* WLOG let  $P$  be on the arc  $\widehat{BC}$  that doesn't contain  $A$ .

$$\angle PDB = 90^\circ = \angle PEB$$

$\therefore PEDB$  is cyclic

$$\angle CEP + \angle CFP = 180^\circ$$

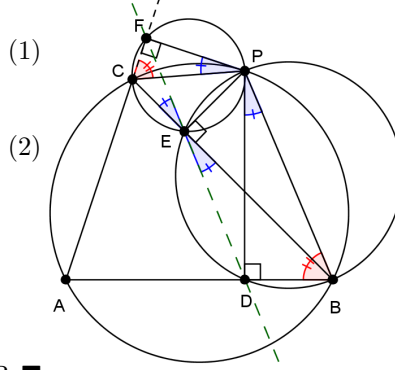
$\therefore CEPF$  is cyclic

We will prove that  $\angle CEF = \angle BED$ .

$$\angle CEF \stackrel{(2)}{=} \angle CPF \stackrel{\triangle CFP}{=} 90^\circ - \angle FCP$$

$$\angle BED \stackrel{(1)}{=} \angle BPD \stackrel{\triangle BDP}{=} 90^\circ - \angle DBP$$

$$\angle FCP = 180^\circ - \angle ACP \stackrel{\omega}{=} \angle ABP \equiv \angle DBP \blacksquare$$



**Example 8.10.** Let  $P$  be a point on the circumcircle  $\omega$  of  $\triangle ABC$  and let  $H$  be its orthocenter. Prove that the reflections of  $P$  with respect to the sides of  $\triangle ABC$  are collinear with  $H$ .

*Proof.* From [Example 8.9](#), we know that the feet of perpendiculars from  $P$  to the sides of  $\triangle ABC$  lie on the  $P$ -Simson line of  $\triangle ABC$ . Then, by [Thales' Proportionality Theorem](#), the reflections of  $P$  with respect to the sides of  $\triangle ABC$  will also be collinear. We just need to prove that  $H$  lies on that line. Since the distance from a point to the foot of the perpendicular to a line is half the distance from the point to its reflection with respect to the line, we need to prove that the  $P$ -Simson line bisects the line segment  $PH$ .

WLOG let  $P$  be on the arc  $\widehat{BC}$  that doesn't contain  $A$ . Let  $D$  and  $E$  be the feet of the perpendiculars from  $P$  to  $AB$  and  $BC$ , respectively. Let  $H_C$  be the reflection of  $H$  with respect to the side  $AB$ . By [Property 6.8](#), we know that  $H_C \in \omega$ .

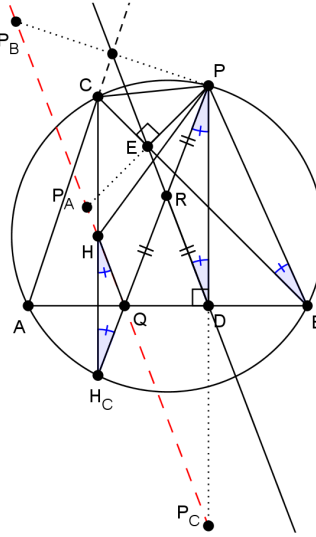
$$DP \parallel H_C H \quad (\because DP \perp AB \wedge H_C H \perp AB) \quad (1)$$

$$\text{Let } PH_C \cap AB = Q. \text{ Then, } \angle EDP \stackrel{(PEDB)}{=} \angle EBP \equiv \angle CBP \stackrel{\omega}{=} \angle CH_C P \equiv \angle HH_C Q = \angle H_C H Q. \quad (2)$$

$$\text{Because of (1), we get that } ED \parallel HQ. \quad (3)$$

$$\text{Let } ED \cap PH_C = R.$$

Then,  $\angle RPD \stackrel{(1)}{=} \angle RH_C C \equiv \angle PH_C C \stackrel{(2)}{=} \angle EDP \equiv \angle RDP$ . Since  $\triangle PDQ$  is right triangle, we can also get  $\angle RQD = \angle RDQ$ . Therefore,  $\overline{RP} = \overline{RD} = \overline{RQ}$ , i.e.  $ED$  bisects  $PQ$ . Combining with (3), we get that the  $P$ -Simson line  $ED$  bisects the line segment  $PH$ .  $\blacksquare$



## 8.5 Euler Line

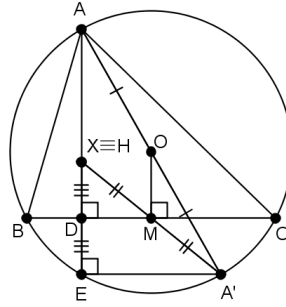
**Example 8.11.** Let  $M$  be the midpoint of the side  $BC$  in  $\triangle ABC$ . Let  $A'$  be the antipode of  $A$  on the circumcircle, i.e. the point on the circumcircle such that  $AA'$  is a diameter. Finally, let  $H$  be the orthocenter of  $\triangle ABC$ . Prove that the points  $H - M - A'$  are collinear.

*Proof.* Let  $D$  be the feet of the altitude from  $A$  to  $BC$ . Let  $A'M \cap AD = X$ . We want to prove that  $X \equiv H$ .

The lines  $OM$  and  $AX$  are parallel because they are both perpendicular to  $BC$ . Since  $O$  is the midpoint of  $AA'$ ,  $OM$  is midsegment in  $\triangle AXA'$ . Therefore,  $\overline{XM} = \overline{MA'}$ .

Let  $AD$  intersect the circumcircle of  $\triangle ABC$  again at  $E$ . Then,  $\angle AEA' = 90^\circ$  as an inscribed angle over the diameter, which means that  $AE \perp EA'$ . Since  $AE \equiv AD \perp BC$ , we have  $EA' \parallel BC$ , i.e.  $EA' \parallel DM$ .

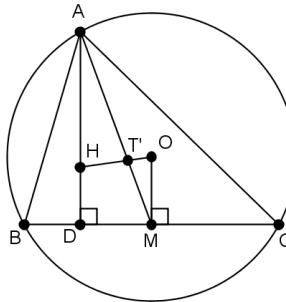
Since  $\overline{XM} = \overline{MA'}$  and  $DM \parallel EA'$ ,  $DM$  is midsegment in  $\triangle XEA'$ , so  $\overline{XD} = \overline{DE}$ . We know (from [Property 6.8](#)) that the reflections of the orthocenter lie on the circumcircle, so  $X \equiv H$ . ■



**Example 8.12.** Let  $M$  be the midpoint of the side  $BC$  in  $\triangle ABC$ . Let  $H$  and  $O$  be the orthocenter and circumcenter of  $\triangle ABC$ , respectively. Prove that  $\overline{AH} = 2 \cdot \overline{OM}$ .

*Proof.* Using the same notations as in [Example 8.11](#) and continuing from there, we got that  $OM$  is midsegment in  $\triangle AHA'$ . So  $\overline{AH} = 2 \cdot \overline{OM}$ . ■

**Example 8.13 (Euler Line).** Let  $H$ ,  $T$  and  $O$  be the orthocenter, centroid and circumcenter in  $\triangle ABC$ , respectively. Prove that the points  $H - T - O$  are collinear and  $\overline{HT} = 2 \cdot \overline{TO}$ .



*Proof.* Let  $M$  be the midpoint of  $BC$ . Let  $T' = AM \cap HO$ . We will prove that  $T' \equiv T$ . The lines  $AH$  and  $OM$  are parallel because they are both perpendicular to  $BC$ . Therefore,  $\triangle AHT' \sim \triangle MOT'$  and

$$\therefore \frac{\overline{AH}}{\overline{MO}} = \frac{\overline{AT'}}{\overline{MT'}} = \frac{\overline{HT'}}{\overline{OT'}}$$

Combining with  $\overline{AH} = 2 \cdot \overline{OM}$  (from [Example 8.12](#)), we get that the ratio of similarity is 2.

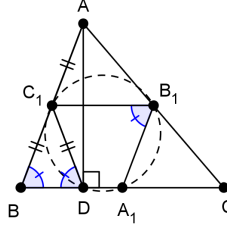
$$\therefore \frac{\overline{AT'}}{\overline{T'M}} = 2 : 1,$$

which means that  $T'$  is the centroid of the triangle, i.e.  $T' \equiv T$ . So the points  $H - T - O$  are collinear. This line is known as the *Euler line* of  $\triangle ABC$ .

From the same similarity, we also get that  $\overline{HT} = 2 \cdot \overline{TO}$ . ■

## 8.6 Nine Point Circle

**Example 8.14.** Let  $A_1$ ,  $B_1$  and  $C_1$  be the midpoints of the sides  $BC$ ,  $CA$  and  $AB$  in  $\triangle ABC$ , respectively. Let  $D$  be the foot of the altitude from  $A$  to  $BC$ . Prove that  $D$  lies on the circumcircle of  $\triangle A_1B_1C_1$ .

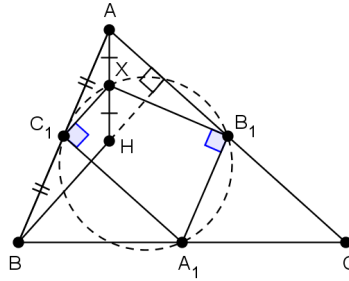


*Proof.*  $\triangle ABD$  is a right triangle and  $C_1$  is the midpoint of the hypotenuse, so  $\overline{C_1D} = \overline{C_1B}$ . Therefore,  $\angle C_1DB = \angle C_1BD = \beta$ .  $B_1C_1$  is a midsegment in  $\triangle ABC$ , so  $B_1C_1 \parallel BC$ . Similarly,  $A_1B_1 \parallel AB$ . Therefore,  $\angle C_1B_1A_1 = \beta$ .

$$\angle C_1B_1A_1 + \angle C_1DA_1 = \beta + (180^\circ - \beta) = 180^\circ$$

$$\therefore D \in (A_1B_1C_1) \quad \blacksquare$$

**Example 8.15.** Let  $A_1$ ,  $B_1$  and  $C_1$  be the midpoints of the sides  $BC$ ,  $CA$  and  $AB$  in  $\triangle ABC$ , respectively. Let  $H$  be the orthocenter in  $\triangle ABC$ . Let  $X$  be the midpoint of  $AH$ . Prove that  $X$  lies on the circumcircle of  $\triangle A_1B_1C_1$ .



*Proof.*

$$C_1X \parallel BH \quad (\because C_1X \text{ is midsegment in } \triangle ABH)$$

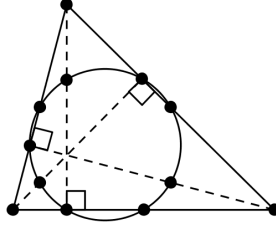
$$BH \perp CA$$

$$CA \parallel C_1A_1 \quad (\because C_1A_1 \text{ is midsegment in } \triangle ABC)$$

$$\therefore C_1X \perp C_1A_1, \text{ i.e. } \angle XC_1A_1 = 90^\circ$$

Similarly,  $\angle XB_1A_1 = 90^\circ$ . Therefore,  $\angle XC_1A_1 + \angle XB_1A_1 = 180^\circ$ , so  $X$  lies on the circumcircle of  $A_1B_1C_1$ .  $\blacksquare$

With these two examples, we proved that the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments from each vertex to the orthocenter (totally nine points) all lie on one circle. This circle is called the *nine point circle* of the triangle.

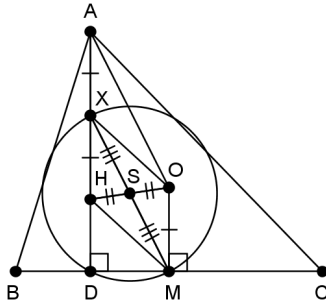


Now, let's try to find the center and the radius of this circle.

Let  $M$  be the midpoint of the side  $BC$  in  $\triangle ABC$ . Let  $H$  and  $O$  be the orthocenter and circumcenter of  $\triangle ABC$ , respectively. Let  $X$  be the midpoint of  $AH$  and let  $D$  be the foot of the altitude from  $A$ . As we know from [Example 8.12](#),  $\overline{AH} = 2 \cdot \overline{OM}$ , so

$$\overline{XH} = \frac{1}{2} \cdot \overline{AH} = \overline{OM}.$$

Also,  $XH \parallel OM$  because they are both perpendicular to  $BC$ . So  $XHMO$  is a parallelogram, which means that the intersection point of its diagonals, let it be  $S$ , is their midpoint, i.e.  $\overline{HS} = \overline{SO}$  and  $\overline{XS} = \overline{SM}$ .

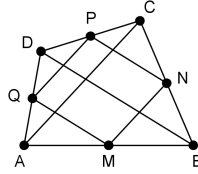


Since  $D$ ,  $M$  and  $X$  lie on the nine point circle of  $\triangle ABC$  and since  $\angle XDM = 90^\circ$ , the center of the nine point circle must be on the midpoint of  $XM$ , i.e. the point  $S$  and  $SX$  is a radius in that circle. Also, since  $SX$  is midsegment in  $\triangle HOA$ ,  $\overline{SX} = \frac{1}{2} \cdot \overline{OA} = \frac{1}{2} \cdot R$ . In conclusion,

**Property 8.1.** The center of the nine point circle lies on the Euler line, more specifically it is the midpoint of  $OH$ . The radius of the nine point circle is one half of the radius of the circumcircle of  $\triangle ABC$ .

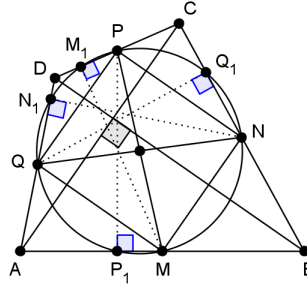
## 8.7 Eight Point Circle

**Example 8.16.** Let  $ABCD$  be a convex quadrilateral. Let  $M, N, P$  and  $Q$  be the midpoints of the sides  $AB, BC, CD$  and  $DA$ , respectively. Prove that  $MNPQ$  is a parallelogram.



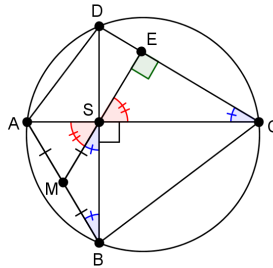
*Proof.*  $MN$  is midsegment in  $\triangle ABC$ . Therefore,  $MN \parallel AC$ . Similarly,  $PQ$  is midsegment in  $\triangle DAC$ , so  $PQ \parallel AC$ . Therefore,  $MN \parallel PQ$ . Similarly,  $MQ \parallel NP$ . Therefore,  $MNPQ$  is a parallelogram. ■

**Example 8.17** (Eight Point Circle). Let  $ABCD$  be a convex quadrilateral with perpendicular diagonals. Let  $M, N, P$  and  $Q$  be the midpoints of the sides  $AB, BC, CD$  and  $DA$ , respectively. Let  $M_1, N_1, P_1$  and  $Q_1$  be the feet of the perpendiculars from  $M, N, P$  and  $Q$ , respectively, to the opposite sides in the quadrilateral. Prove that the points  $M, N, P, Q, M_1, N_1, P_1$  and  $Q_1$  all lie on a single circle.



*Proof.* Combining the proof of [Example 8.16](#) with  $AC \perp BD$ , we get that  $MNPQ$  is a rectangle, i.e. a quadrilateral inscribed in a circle where the diagonals  $MP$  and  $NQ$  are diameters. From the definition of  $M_1$ ,  $MM_1 \perp CD$ , i.e.  $\angle MM_1P = 90^\circ$ , so  $M_1 \in (MNPQ)$ . Similarly,  $N_1, P_1$  and  $Q_1$  also lie on  $(MNPQ)$ . ■

**Example 8.18** (Brahmagupta Theorem). Let  $ABCD$  be a cyclic quadrilateral with perpendicular diagonals that intersect at  $S$ . Let  $M$  be the midpoint of the side  $AB$ . Prove that  $MS \perp CD$ .



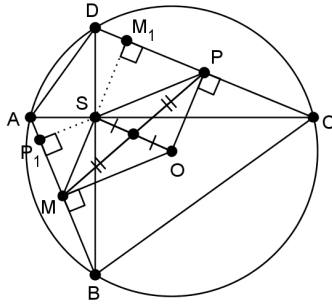
*Proof.* Let  $MS \cap CD = E$ .  $M$  is the midpoint of the hypotenuse in the right  $\triangle ABS$ , so  $\overline{MB} = \overline{MS}$ , i.e.  $\angle MBS = \angle MSB$ .

$$\angle SCE \equiv \angle ACD = \angle ABD \equiv \angle MBS = \angle MSB$$

$$\angle ESC + \angle SCE = \angle MSA + \angle MSB = \angle ASB = 90^\circ$$

$$\angle SEC = 180^\circ - (\angle ESC + \angle SCE) = 90^\circ, \text{ i.e. } MS \perp CD \quad \blacksquare$$

**Example 8.19.** Let  $ABCD$  be a cyclic quadrilateral with perpendicular diagonals that intersect at  $S$ . Let  $O$  be the center of  $(ABCD)$ . Prove that the eight point circle of  $ABCD$  is centered at the midpoint of  $OS$ .



*Proof.* Let  $M, N, P$  and  $Q$  be the midpoints of the sides  $AB, BC, CD$  and  $DA$ , respectively. Let  $M_1, N_1, P_1$  and  $Q_1$  be the feet of the perpendiculars from  $M, N, P$  and  $Q$ , respectively, to the opposite sides in the quadrilateral. From [Example 8.17](#), we know that  $MP$  is a diameter of the eight point circle, so its center is the midpoint of  $MP$ . We need to prove that the midpoint of  $MP$  coincides with the midpoint of  $OS$ . We will prove that  $MOPS$  is a parallelogram.

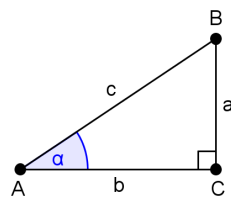
Since  $M$  is a midpoint of the chord  $AB$  and  $O$  is the center of  $(ABCD)$ , we get that  $OM \perp AB$ . From [Example 8.18](#), we know that the lines  $MM_1, NN_1, PP_1$  and  $QQ_1$  pass through  $S$ , so  $PS \equiv PP_1 \perp AB$ . Therefore,  $OM \parallel PS$ . Similarly,  $OP \parallel MS$ . Therefore,  $MOPS$  is a parallelogram.  $\blacksquare$

**Related problems:** 32 and 35.

## Chapter 9

# Basic Trigonometry

### Trigonometric Functions in Right Triangle



Let  $ABC$  be a right triangle ( $\gamma = 90^\circ$ ). Then, we define the sine and cosine functions as follows:

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

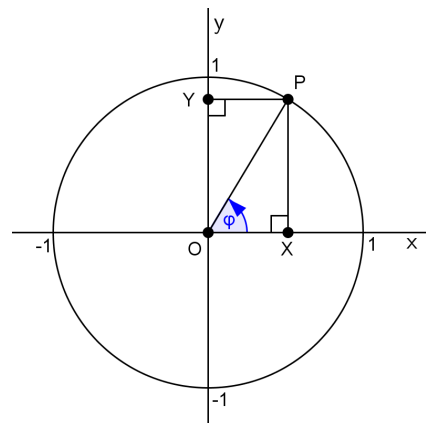
Although they may sound scary, they are nothing more than ratios of sides :)

### The Unit Circle

We defined the basic trigonometric functions for angles  $0^\circ < \varphi < 90^\circ$ . Let's try to extend them for all values of  $\varphi$ .

Let's take a look at the unit circle. That is a circle which is centered at the origin  $O(0, 0)$  of the coordinate plane and has a radius of length 1. Let's represent any angle  $\varphi$  with a point  $P$  on the unit circle, such that the angle starting from the positive  $x$ -axis and going in the counter-clockwise direction to the line  $OP$  is equal to  $\varphi$ .

Let  $0^\circ < \varphi < 90^\circ$ . Let  $P$  be a point on the unit circle that represents the angle  $\varphi$ . Let  $X$  and  $Y$  be the feet of the perpendicular from  $P$  to the  $x$ - and  $y$ -axis, respectively. Then the triangle  $\triangle OPX$  is a right triangle with hypotenuse  $\overline{OP} = 1$ , so by the definitions above, we get that  $\cos \varphi = \overline{OX}$  and  $\sin \varphi = \overline{PX} = \overline{OY}$ . That's right, the cosine and sine values are in fact the  $x$ - and  $y$ -component of the point  $P$  in the coordinate system.



So why not extend this definition for all possible values of  $\varphi$ ? Those are, in fact, the actual definitions for the cosine and sine functions. The cosine is the  $x$ -component and the sine is the  $y$ -component of the point  $P$  representing the angle  $\varphi$ . For example,  $\cos(120^\circ) = -\frac{1}{2}$  and  $\sin(90^\circ) = 1$ . As we can see, the ranges of both the cosine and sine functions are  $[-1, 1]$ .

Using the unit circle and the definition above, very simply, using congruence of triangles, or the Pythagorean Theorem, we can prove various properties, like:

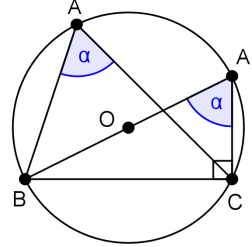
$$\begin{aligned}\sin(180^\circ - \alpha) &= \sin \alpha & \cos(180^\circ - \alpha) &= -\cos \alpha \\ \sin(90^\circ + \alpha) &= \cos \alpha & \cos(90^\circ + \alpha) &= -\sin \alpha \\ \cos^2 \alpha + \sin^2 \alpha &= 1\end{aligned}$$

**Property 9.1** (Law of Sines). In a triangle  $\triangle ABC$  with circumradius  $R$ ,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R.$$

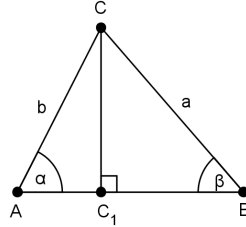
*Proof.* Let  $\omega$  be the circumcircle of  $\triangle ABC$  and let  $O$  be its center. Let  $A_1$  be the second intersection of  $BO$  and  $\omega$ . Then  $\angle BA_1C = \angle BAC = \alpha$ . On the other hand,  $\angle A_1CB = 90^\circ$  as an inscribed angle over the diameter, so  $\triangle A_1BC$  is a right triangle. By definition,

$$\sin \alpha = \frac{\overline{BC}}{\overline{A_1B}} = \frac{a}{2R}, \text{ i.e. } \frac{a}{\sin \alpha} = 2R \quad \blacksquare$$



**Property 9.2** (Law of Cosines). In a triangle  $\triangle ABC$ , for any side

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$



*Proof.* Let  $C_1$  be the feet of the altitude from  $C$  to  $AB$ . Let's investigate the case when  $C_1$  is between  $A$  and  $B$ . From the two right triangles  $\triangle ACC_1$  and  $\triangle BCC_1$  we get  $\overline{AC_1} = b \cos \alpha$  and  $\overline{BC_1} = a \cos \beta$ . Since  $\overline{AB} = \overline{AC_1} + \overline{BC_1}$ , we get

$$c = a \cos \beta + b \cos \alpha.$$

We get exactly the same result even when  $C_1$  is not between  $A$  and  $B$  because of the property  $\cos(180^\circ - \alpha) = -\cos \alpha$ . Multiplying the last equation by  $c$  on both sides, we get

$$c^2 = ac \cos \beta + bc \cos \alpha.$$

Similarly, we can get

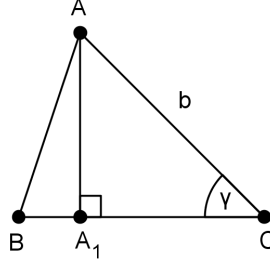
$$ab \cos \gamma + ac \cos \beta = a^2$$

$$ab \cos \gamma + bc \cos \alpha = b^2$$

By adding the last three equations side by side, we get the desired result.  $\blacksquare$

**Property 9.3.** The area of a triangle  $\triangle ABC$  can be expressed as

$$P_{\triangle ABC} = \frac{1}{2}ab \sin \gamma.$$



*Proof.* Let  $A_1$  be the feet of the altitude from  $A$  to  $BC$ . Then  $\triangle CAA_1$  is a right triangle, so we have

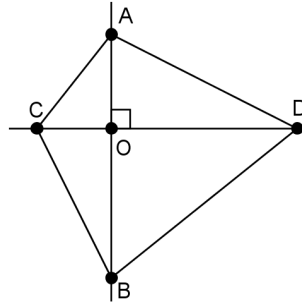
$$\sin \gamma = \frac{\overline{AA_1}}{\overline{AC}}, \text{ i.e. } \overline{AA_1} = b \sin \gamma.$$

Then, the area of  $\triangle ABC$  is

$$P_{\triangle ABC} = \frac{1}{2} \cdot \overline{BC} \cdot \overline{AA_1} = \frac{1}{2}ab \sin \gamma. \quad \blacksquare$$

**Property 9.4.** Let  $AB$  and  $CD$  be two intersecting lines. Then,

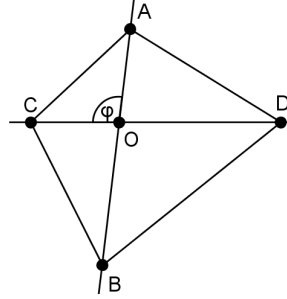
$$AB \perp CD \iff \overline{CA}^2 - \overline{CB}^2 = \overline{DA}^2 - \overline{DB}^2.$$



*Proof.* Let  $AB \cap CD = O$ . We will firstly prove the first direction, so let  $AB \perp CD$ . Then, the triangles  $\triangle ACO$ ,  $\triangle BCO$ ,  $\triangle ADO$  and  $\triangle BDO$  are right triangles, so by the Pythagorean Theorem, we get

$$\overline{CA}^2 - \overline{CB}^2 = \overline{OA}^2 - \overline{OB}^2 = \overline{DA}^2 - \overline{DB}^2 \quad \square$$

Now, let's prove the other direction. Let  $\overline{CA}^2 - \overline{CB}^2 = \overline{DA}^2 - \overline{DB}^2$ . We will discuss the case where  $O$  is between  $A$  and  $B$  and between  $C$  and  $D$ . Let  $\angle AOC = \varphi$ . By using the [Law of Cosines](#) for the triangles  $\triangle ACO$ ,  $\triangle BCO$ ,



$\triangle ADO$  and  $\triangle BDO$ , and using the property  $\cos(180^\circ - \varphi) = -\cos \varphi$ , we get:

$$\begin{aligned}\overline{CA}^2 - \overline{CB}^2 &= \overline{OA}^2 + \overline{OC}^2 - 2\overline{OA}\overline{OC}\cos \varphi - \overline{OB}^2 - \overline{OC}^2 - 2\overline{OB}\overline{OC}\cos \varphi = \\ &= \overline{OA}^2 - \overline{OB}^2 - 2\overline{AB}\overline{OC}\cos \varphi\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{DA}^2 - \overline{DB}^2 &= \overline{OA}^2 + \overline{OD}^2 + 2\overline{OA}\overline{OD}\cos \varphi - \overline{OB}^2 - \overline{OD}^2 + 2\overline{OB}\overline{OD}\cos \varphi = \\ &= \overline{OA}^2 - \overline{OB}^2 + 2\overline{AB}\overline{OD}\cos \varphi\end{aligned}$$

Substituting in the condition, we get

$$\begin{aligned}\overline{OA}^2 - \overline{OB}^2 - 2\overline{AB}\overline{OC}\cos \varphi &= \overline{OA}^2 - \overline{OB}^2 + 2\overline{AB}\overline{OD}\cos \varphi \\ 0 &= 2\overline{AB}\overline{CD}\cos \varphi\end{aligned}$$

Since  $\overline{AB}$  and  $\overline{CD}$  are positive lengths, then  $\cos \varphi = 0$ . Therefore,  $\varphi = 90^\circ$ , i.e.  $AB \perp CD$ .

In the cases where  $O$  is not between  $A$  and  $B$  or  $C$  and  $D$ , the proof follows exactly the same steps. There might be a different operation when dealing with the line segments (addition or subtraction) or different angle ( $\varphi$  or  $180^\circ - \varphi$ ) depending on the configuration, but the result will always be the same because of the property  $\cos(180^\circ - \varphi) = -\cos \varphi$ . ■

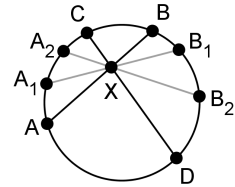
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**Related problems:** 10 and 50.

# Chapter 10

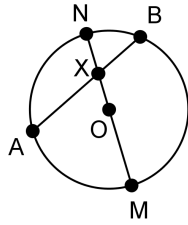
## Power of a Point

Let  $AB$  and  $CD$  be two intersecting chords in a circle and let their intersection be  $X$ . By the [Intersecting Chords Theorem](#),  $\overline{AX} \cdot \overline{XB} = \overline{CX} \cdot \overline{XD}$ . This means that for a fixed point  $X$  in the fixed circle  $\omega(O, r)$ , the product  $\overline{AX} \cdot \overline{XB}$  will be constant and will not depend on the choice of the chord  $A_iB_i$  which passes through  $X$ , i.e.



$$\overline{AX} \cdot \overline{XB} = \overline{A_1X} \cdot \overline{XB_1} = \overline{A_2X} \cdot \overline{XB_2} = \text{const.}$$

So, this product must depend on the position of  $X$  (relative to  $\omega$ ) and on  $\omega$  itself.



Well, let's try to express this product as a function of the known elements, i.e. the radius of the circle and the distance from the center of the circle to the point  $X$ . Let's draw a diameter through  $X$  (in order to include the center in all of this) and let  $M$  and  $N$  be its endpoints. Then, as previously proved,

$$\begin{aligned} \overline{AX} \cdot \overline{XB} &= \overline{MX} \cdot \overline{XN} = (\overline{MO} + \overline{OX})(\overline{ON} - \overline{OX}) = \\ &= (r + \overline{OX})(r - \overline{OX}) = r^2 - \overline{OX}^2. \end{aligned}$$

This is, in fact, the absolute value (since  $X$  is inside the circle) of what we will call the power of  $X$  with respect to  $\omega$ .

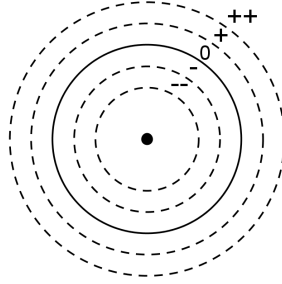
We will define the *power of the point  $X$*  with respect to the circle  $\omega(O, r)$  as a real number which reflects the relative distance of the point  $X$  to the circle  $\omega$ :

$$\text{pow}(X, \omega) = \overline{OX}^2 - r^2.$$

Consequently, we can conclude the following property:

**Property 10.1.** Points that are on equal distances from the center have equal powers with respect to the circle.

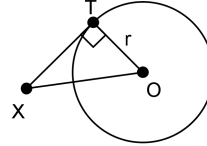
By the definition, it also means that the points inside the circle (for which  $0 \leq \overline{OX} < r$ ) will have negative power, the points on the circle (for which  $\overline{OX} = r$ ) will have zero power and the points outside the circle (for which  $\overline{OX} > r$ ) will have positive power with respect to the circle.



**Property 10.2.** If a point  $X$  is outside the circle  $\omega(O, r)$ , then the power of the point equals the square of the length of the tangent segment from  $X$  to the tangent point  $T$ .

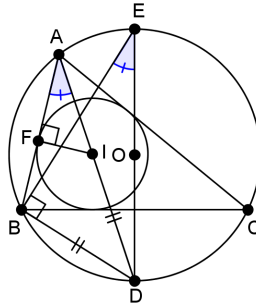
*Proof.* By the definition of power of a point and using the Pythagorean Theorem:

$$\text{pow}(X, \omega) = \overline{OX}^2 - r^2 = \overline{OX}^2 - \overline{OT}^2 = \overline{XT}^2 \quad \blacksquare$$



**Example 10.1** (Euler's Theorem in Geometry). Let  $O$  and  $I$  be the circumcenter and incenter of  $\triangle ABC$ , respectively. Let  $R$  and  $r$  be the circumradius and inradius, respectively. Prove that

$$\overline{OI}^2 = R(R - 2r)$$



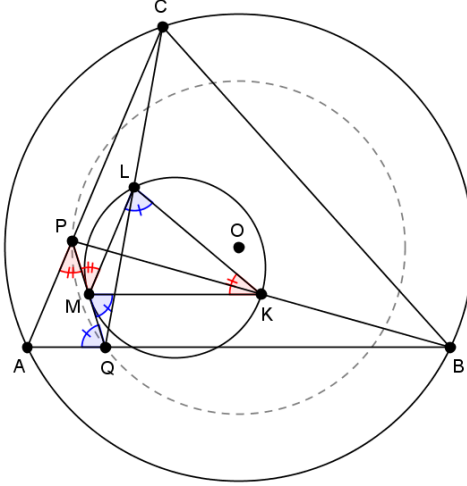
*Proof.* Let  $D$  be the second intersection of  $AI$  with the circumcircle  $\omega$  of  $\triangle ABC$ . We need to prove that  $2Rr = R^2 - \overline{OI}^2 = |\text{pow}(I, \omega)| = \overline{AI} \cdot \overline{ID}$ . By [Example 7.1](#), we know that  $\overline{DI} = \overline{DB}$ . So, we need to find similar triangles that contain the sides  $AI$ ,  $BD$ ,  $r$  and  $2R$ . Let  $E$  be the diametrically opposite point of  $D$  on  $\omega$ ; Thus  $\overline{ED} = 2R$  and  $\angle EBD = 90^\circ$ . Let  $F$  be tangent point of the incircle with the side  $AB$ ; Thus  $\overline{IF} = r$  and  $\angle IFA = 90^\circ$ . Since  $\angle FAI \equiv \angle BAD = \angle BED$ , the right triangles  $\triangle AIF$  and  $\triangle EDB$  are similar. Therefore,

$$\frac{\overline{AI}}{\overline{IF}} = \frac{\overline{ED}}{\overline{DB}} \quad (*)$$

$$\overline{AI} \cdot \overline{ID} = \overline{AI} \cdot \overline{DB} \stackrel{(*)}{=} \overline{ED} \cdot \overline{IF} = 2Rr \quad \blacksquare$$

*Remark.* From this theorem, we can derive the *Euler inequality*. Since  $\overline{OI}^2$  is non-negative and  $R$  is always positive, we can conclude that  $R - 2r$  is non-negative, i.e.  $R \geq 2r$ . Equality holds iff  $O \equiv I$ , i.e. when  $\triangle ABC$  is equilateral.

**Example 10.2** (IMO 2009/2). Let  $ABC$  be a triangle with circumcenter  $O$ . The points  $P$  and  $Q$  are interior points of the sides  $CA$  and  $AB$ , respectively. Let  $K$ ,  $L$  and  $M$  be the midpoints of the segments  $BP$ ,  $CQ$  and  $PQ$ , respectively, and let  $\Gamma$  be the circle passing through  $K$ ,  $L$  and  $M$ . Suppose that the line  $PQ$  is tangent to the circle  $\Gamma$ . Prove that  $\overline{OP} = \overline{OQ}$ .



*Proof.* We need to prove that  $P$  and  $Q$  are on the same distance from the circumcenter  $O$ , so by [Property 10.1](#), we need to prove that their power with respect to the circumcircle  $(ABC)$  are equal, i.e. we need to prove that

$$\overline{AP} \cdot \overline{PC} = \overline{AQ} \cdot \overline{QB}.$$

Since  $MK$  and  $ML$  are midsegments in  $\triangle QBP$  and  $\triangle PCQ$ , we have

$$MK \parallel QB \quad \text{and} \quad ML \parallel PC \tag{1}$$

$$\overline{MK} = \frac{1}{2} \cdot \overline{QB} \quad \text{and} \quad \overline{ML} = \frac{1}{2} \cdot \overline{PC} \tag{2}$$

Since  $\Gamma$  is tangent to  $PQ$  at  $M$ , and using (1), we get

$$\angle KLM = \angle KMQ = \angle MQA \equiv \angle PQA$$

$$\angle LKM = \angle LMP = \angle MPA \equiv \angle QPA$$

Therefore,  $\triangle APQ \sim \triangle MKL$ .

$$\therefore \frac{\overline{AP}}{\overline{AQ}} = \frac{\overline{MK}}{\overline{ML}} \stackrel{(2)}{=} \frac{\overline{QB}}{\overline{PC}}, \text{ i.e. } \overline{AP} \cdot \overline{PC} = \overline{AQ} \cdot \overline{QB} \quad \blacksquare$$

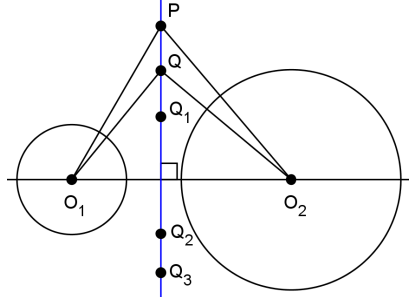
Before continuing, let's mention that sometimes it's useful to consider a point as a degenerate circle, i.e. a circle with radius zero. Then, the power of a point  $P$  with respect to the degenerate circle  $\omega(A, 0)$  is

$$pow(P, \omega) = \overline{AP}^2 - r^2 = \overline{AP}^2$$

If it is unclear to you what this means, wait just a bit; We will use this in [Example 10.3](#).

## 10.1 Radical axis

We learned about the power of a point with respect to a circle. Now, let's find the locus of the points that have equal power with respect to two given circles  $\omega_1(O_1, r_1)$  and  $\omega_2(O_2, r_2)$ . Let  $P$  be a point that satisfies this condition. Then,



by the definition of power of a point,

$$\overline{PO_1}^2 - r_1^2 = \overline{PO_2}^2 - r_2^2.$$

Let  $Q$  be another point that satisfies the condition. Similarly, we have

$$\overline{QO_1}^2 - r_1^2 = \overline{QO_2}^2 - r_2^2.$$

From the two equations above, we get:

$$\overline{PO_1}^2 - \overline{PO_2}^2 = r_1^2 - r_2^2 = \overline{QO_1}^2 - \overline{QO_2}^2,$$

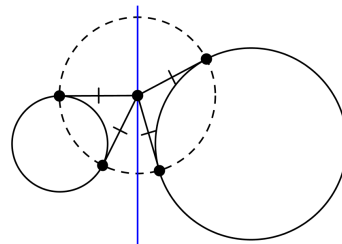
which, by [Property 9.4](#) means that  $PQ \perp O_1O_2$ . But this will also be true for all other points  $Q_1, Q_2, \dots$  that have same power with respect to both circles, i.e.  $PQ_1 \perp O_1O_2, PQ_2 \perp O_1O_2, \dots$ , which means that the set of all such points is a straight line perpendicular to  $O_1O_2$ .

**The radical axis** of two circles is a line that is the locus of all points that have equal powers with respect to both circles.

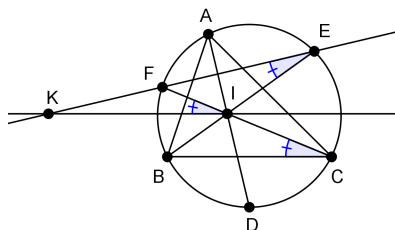
**Property 10.3.** The radical axis of two circles is perpendicular to the line connecting the centers of the circles.

As a consequence, the radical axis of two intersecting circles will be the line that passes through their intersection points, because those points have zero power with respect to both circles. The radical axis of two tangent circles will be their common tangent through their tangent point, because it is perpendicular to the line connecting the centers and because the tangent point has zero power with respect to both circles.

Let's recall that if a point is outside the circle, the power of the point with respect to the circle equals the square of the length of the tangent segment from the point to the circle. Hence, the tangent segments from such point  $Q_i$  to both circles are of equal length, which means that each point on the radical axis is a center of a circle that intersects both given circles orthogonally.



**Example 10.3** (BMO 2015). Let  $\triangle ABC$  be a scalene triangle with incentre  $I$  and circumcircle  $\omega$ . Lines  $AI$ ,  $BI$  and  $CI$  intersect  $\omega$  for the second time at points  $D$ ,  $E$  and  $F$ , respectively. The parallel lines from  $I$  to the sides  $BC$ ,  $AC$  and  $AB$  intersect  $EF$ ,  $DF$  and  $DE$  at points  $K$ ,  $L$  and  $M$ , respectively. Prove that the points  $K$ ,  $L$  and  $M$  are collinear.



*Proof.* Because of the parallel lines  $KI$  and  $BC$ ,

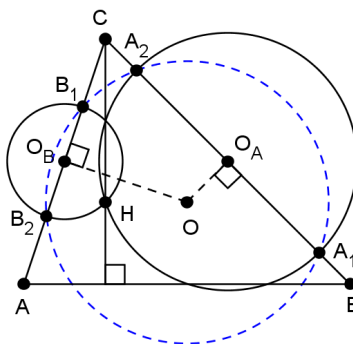
$$\angle KIF = \angle BCI \equiv \angle BCF \stackrel{\omega}{=} \angle BEF \equiv \angle IEK$$

In addition to this,  $\angle IKE \equiv \angle IKF$  is a common angle for the triangles  $\triangle KIF$  and  $\triangle KEI$ , so the triangles are similar and therefore

$$\frac{\overline{KI}}{\overline{KF}} = \frac{\overline{KE}}{\overline{KI}}, \text{ i.e. } \overline{KI}^2 = \overline{KF} \cdot \overline{KE}$$

The left hand side is the power of the point  $K$  with respect to the degenerate circle  $I$  and the right hand side is the power of the point  $K$  with respect to the circle  $(EFD)$ . Therefore,  $K$  lies on the radical axis  $r$  of  $I$  and  $(EFD)$ . Similarly,  $L$  and  $M$  also lie on  $r$ , so they are collinear. ■

**Example 10.4** (IMO 2008/1). Let  $H$  be the orthocenter of an acute-angled triangle  $ABC$ . The circle  $\Gamma_A$  centered at the midpoint of  $BC$  and passing through  $H$  intersects the sideline  $BC$  at points  $A_1$  and  $A_2$ . Similarly, define the points  $B_1$ ,  $B_2$ ,  $C_1$  and  $C_2$ . Prove that the six points  $A_1$ ,  $A_2$ ,  $B_1$ ,  $B_2$ ,  $C_1$  and  $C_2$  are concyclic.



*Proof.* A precise drawing may give us a hint about this problem. If we draw the figure correctly, we will see that the second intersection of the circles  $\Gamma_A$  and  $\Gamma_B$  lies on the altitude  $CH$ . So, we firstly need to prove that this is indeed true, i.e. we need to prove that  $CH$  is the radical axis of  $\Gamma_A$  and  $\Gamma_B$  and then use this fact to solve the problem.

Let  $O_A$  and  $O_B$  be the centers of  $\Gamma_A$  and  $\Gamma_B$  (i.e. the midpoints of  $BC$  and  $CA$ ), respectively. Since  $O_A O_B$  is a midsegment in  $\triangle ABC$ ,  $O_A O_B \parallel AB$ . Since  $CH$  is altitude in  $\triangle ABC$ ,  $AB \perp CH$ . Therefore,  $O_A O_B \perp CH$ . By [Property 10.3](#) and recalling that  $H \in \Gamma_A$  and  $H \in \Gamma_B$ , we can conclude that  $CH$  is the radical axis of  $\Gamma_A$  and  $\Gamma_B$ .

Now, using the fact that any point on the radical axis has equal power with respect to both circles, we get that

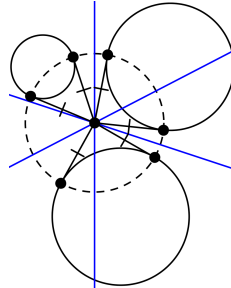
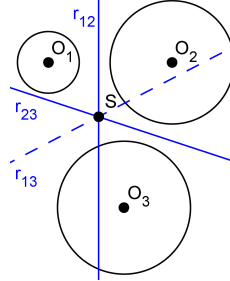
$$\overline{CA_1} \cdot \overline{CA_2} = \overline{CB_1} \cdot \overline{CB_2}.$$

Therefore,  $A_1 A_2 B_1 B_2$  is a cyclic quadrilateral. The center of  $(A_1 A_2 B_1 B_2)$  can be found as the intersection of the side bisectors of  $A_1 A_2$  and  $B_1 B_2$ . But, since the midpoint of  $BC$  coincides with the midpoint of  $A_1 A_2$  and the midpoint of  $CA$  coincides with the midpoint of  $B_1 B_2$ , then the side bisectors of  $A_1 A_2$  and  $B_1 B_2$  intersect at the circumcenter  $O$  of  $\triangle ABC$ . Similarly,  $(B_1 B_2 C_1 C_2)$  is a circle centered at  $O$ . Therefore, the six points  $A_1, A_2, B_1, B_2, C_1$  and  $C_2$  are concyclic. ■

## 10.2 Radical center

Let's find the locus of the points that have equal power with respect to three circles  $\omega_1, \omega_2$  and  $\omega_3$ , whose centers are not collinear. By definition, the set of points that satisfy  $\text{pow}(X, \omega_1) = \text{pow}(X, \omega_2)$  is the radical axis of  $\omega_1$  and  $\omega_2$  and the set of points that satisfy  $\text{pow}(X, \omega_2) = \text{pow}(X, \omega_3)$  is the radical axis of  $\omega_2$  and  $\omega_3$ . These axes are not parallel (because the centers of the circles are not collinear), so let their intersection be  $S$ . By transitivity, we get that for this point  $S$ , the following is true  $\text{pow}(S, \omega_1) = \text{pow}(S, \omega_3)$ , which means that the radical axis of  $\omega_1$  and  $\omega_3$  also passes through  $S$ . So,  $S$  is the only point that has equal power to all three circles and it is called the *radical center* of the three circles.

Note that if the radical center lies outside of all three circles, then the tangent segments from it to all three circles will be of equal length. So, the radical center is the center of the unique circle (called the *radical circle*) that intersects the three given circles orthogonally.



## Geometric construction of radical axis

Now, let's see how we can geometrically construct the radical axis of two non-concentric circles.

i)  $\omega_1 \cap \omega_2 = \{A, B\}$

The points  $A$  and  $B$  lie on both circles, so they both have zero power to both circles. Since we know that the radical axis is a line, we can construct it by drawing the line through the points  $A$  and  $B$ .

ii)  $\omega_1 \cap \omega_2 = \{T\}$

As discussed in the previous case, the point  $T$  lies on the radical axis. Since we proved that the radical axis is a line perpendicular to the line joining the centers, we can construct it easily as the common tangent of the circles through  $T$ .

iii)  $\omega_1 \cap \omega_2 = \emptyset$

Let's draw another circle  $\omega_3$  that intersects both  $\omega_1$  and  $\omega_2$ . Let's construct the radical axes of  $\omega_1$  and  $\omega_3$ ,  $r_{1,3}$ , and  $\omega_2$  and  $\omega_3$ ,  $r_{2,3}$ . The intersection of  $r_{1,3}$  and  $r_{2,3}$ ,  $S$ , is the radical center of the three circles, so it must lie on the radical axis  $r_{1,2}$  that we are trying to construct. Now, we can continue in two different ways: we can either do the same thing with another circle that intersect  $\omega_1$  and  $\omega_2$ , thus finding another point that lies on  $r_{1,2}$ ; or we can use the fact the radical axis  $r_{1,2}$  is perpendicular to the line joining the centers,  $O_1O_2$ .

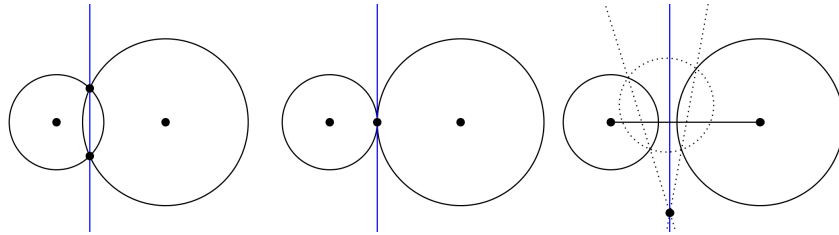


Figure 10.1: Radical axis of two circles

# Chapter 11

## Collinearity



Three points are *collinear* if they lie on a single line. We will now present a few approaches that will help us prove that three points are collinear when solving geometry problems.

### 11.1 Manual Approach

There are three most common angle chasing ways to prove that three points  $A$ ,  $B$  and  $C$  are collinear.

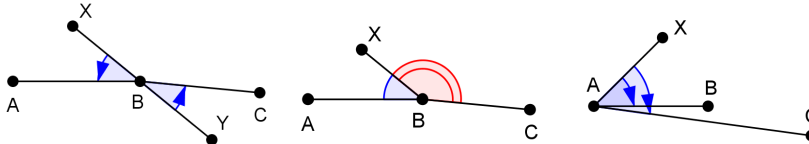


Figure 11.1: Three collinearity configurations

In the first configuration, we will need two extra points that are already collinear with our "middle" point  $B$ . Let those points be  $X$  and  $Y$ . If  $\angle XBA = \angle YBC$ , then the points  $A$ ,  $B$  and  $C$  are collinear.

In the second configuration, we will need one extra point  $X$  that doesn't lie on the supposed line  $A - B - C$ . If  $\angle ABX + \angle XBC = 180^\circ$ , then the points  $A$ ,  $B$  and  $C$  are collinear.

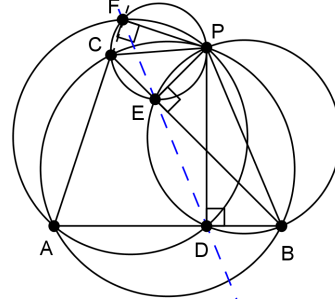
In the third configuration, we will also need one extra point  $X$  that doesn't lie on the supposed line  $A - B - C$ . If  $\angle XAB = \angle XAC$ , then the points  $A$ ,  $B$  and  $C$  are collinear.

In the proof of the following theorem, that we already saw as [Example 8.9](#), we will demonstrate all three approaches.

**Example 11.1** (Simson Line Theorem). Let  $P$  be a point on the circumcircle  $\omega$  of a triangle  $ABC$ . If  $D$ ,  $E$  and  $F$  are the feet of the perpendiculars from  $P$  to the lines  $AB$ ,  $BC$  and  $CA$ , prove that the points  $D$ ,  $E$  and  $F$  are collinear.

*Proof.* WLOG let  $P$  be on the arc  $\widehat{BC}$  that doesn't contain  $A$ .

$$\begin{aligned} \angle PDB &= 90^\circ = \angle PEB \\ \therefore PEDB &\text{ is cyclic} & (1) \\ \angle CEP + \angle CFP &= 180^\circ \\ \therefore CEPF &\text{ is cyclic} & (2) \\ \angle ADP + \angle AFP &= 180^\circ \\ \therefore ADPF &\text{ is cyclic} & (3) \end{aligned}$$



We will now finish the proof in three different ways, demonstrating all of the approaches mentioned before.

*I way:* We will prove that  $\angle CEF = \angle BED$ .

$$\angle CEF \stackrel{(2)}{=} \angle CPF \stackrel{\triangle CFP}{=} 90^\circ - \angle FCP$$

$$\angle BED \stackrel{(1)}{=} \angle BPD \stackrel{\triangle BDP}{=} 90^\circ - \angle DBP$$

$$\angle FCP = 180^\circ - \angle ACP \stackrel{\omega}{=} \angle ABP \equiv \angle DBP$$

*II way:* We will prove that  $\angle FEP + \angle PED = 180^\circ$ .

$$\angle FEP \stackrel{(2)}{=} \angle FCP = 180^\circ - \angle PCA$$

$$\angle PED \stackrel{(1)}{=} 180 - \angle PBD \equiv 180^\circ - \angle PBA$$

$$\angle FEP + \angle PED = 360^\circ - (\angle PCA + \angle PBA) \stackrel{\omega}{=} 360^\circ - 180^\circ = 180^\circ$$

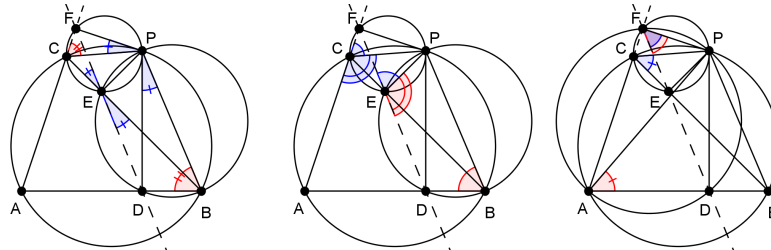
*III way:* We will prove that  $\angle PFE = \angle PFD$ .

$$\angle PFE \stackrel{(2)}{=} \angle PCE \equiv \angle PCB$$

$$\angle PFD \stackrel{(3)}{=} \angle PAD \equiv \angle PAB$$

$$\angle PCB \stackrel{\omega}{=} \angle PAB$$

■



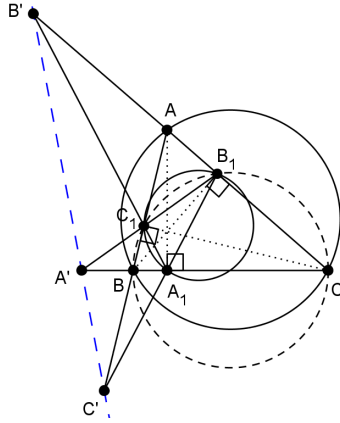
## 11.2 Radical Axis

In some problems, we can prove that the three points lie on a radical axis of some two circles.

**Example 11.2** (Orthic axis). Let  $AA_1$ ,  $BB_1$  and  $CC_1$  be the altitudes in  $\triangle ABC$ . Let  $A'$  be the intersection of the lines  $BC$  and  $B_1C_1$  and similarly define the points  $B'$  and  $C'$ . Prove that  $A'$ ,  $B'$  and  $C'$  lie on a line.

*Proof.* Since  $\angle BB_1C = 90^\circ = \angle BC_1C$ , the quadrilateral  $BCB_1C_1$  is cyclic. Therefore, by the intersecting secant theorem, we have

$$\overline{A'B} \cdot \overline{A'C} = \overline{A'C_1} \cdot \overline{A'B_1}.$$



The left-hand side is in fact the power of the point  $A'$  to the circle  $(ABC)$  and the right-hand side is the power of the point  $A'$  to the circle  $(A_1B_1C_1)$ . Since it has same power with respect to both circles, then it must lie on the radical axis of those circle. Similarly,  $B'$  and  $C'$  also lie on the radical axis of  $(ABC)$  and  $(A_1B_1C_1)$ , so  $A'$ ,  $B'$  and  $C'$  are collinear. ■

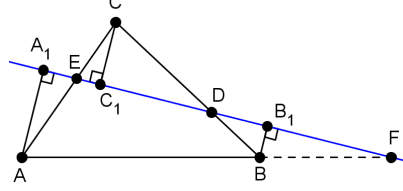
## 11.3 Menelaus' Theorem

**Example 11.3** (Menelaus' Theorem). Let  $ABC$  be a triangle. Let  $D$ ,  $E$  and  $F$  be points on the lines  $BC$ ,  $CA$  and  $AB$ , respectively, such that odd number of them (one or three) are on the extensions of the sides. The points  $D$ ,  $E$  and  $F$  are collinear if and only if

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



*Proof.* Let  $D$ ,  $E$  and  $F$  be collinear and let the line defined by them be  $p$ . Let  $A_1$ ,  $B_1$  and  $C_1$  be the feet of the perpendiculars from  $A$ ,  $B$  and  $C$ , respectively, to the line  $p$ .



$$\triangle AA_1F \sim \triangle BB_1F \quad (\because \angle AA_1F = 90^\circ = \angle BB_1F, \angle AFA_1 \equiv \angle BFB_1)$$

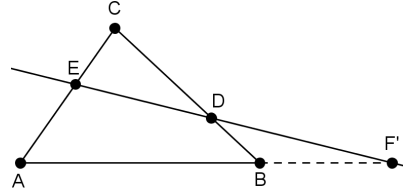
$$\therefore \frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AA_1}}{\overline{BB_1}}$$

$$\text{Similarly, } \frac{\overline{BD}}{\overline{DC}} = \frac{\overline{BB_1}}{\overline{CC_1}} \text{ and } \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CC_1}}{\overline{AA_1}}.$$

$$\therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AA_1}}{\overline{BB_1}} \cdot \frac{\overline{BB_1}}{\overline{CC_1}} \cdot \frac{\overline{CC_1}}{\overline{AA_1}} = 1 \quad \square$$

Now, let's prove the other direction. Let

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



We will discuss the case when two of the points are on the sides of the triangle and one on the extension of the other side. The other case, when all three points are on the extensions of the sides is analogous. WLOG, let  $D$  and  $E$  be on the sides  $BC$  and  $CA$ , respectively and  $F$  be on the extension of the side  $AB$ . We should prove that the points  $D$ ,  $E$  and  $F$  are collinear. Let the line  $DE$  intersect the line  $AB$  at  $F'$  (note that  $F'$  cannot lie between  $A$  and  $B$ ). Because the points  $D$ ,  $E$  and  $F'$  are collinear, we can use the direction of the Menelaus' Theorem that we just proved. So,

$$\frac{\overline{AF'}}{\overline{F'B}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Combining with the given condition, we get

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AF'}}{\overline{F'B}}.$$

Since  $\overline{AF} - \overline{FB} = \overline{AB} = \overline{AF'} - \overline{F'B}$ , by subtracting 1 from both sides in the above equation, we get

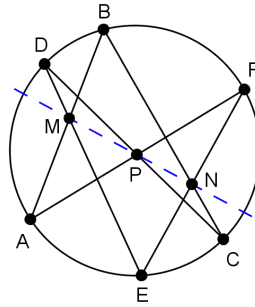
$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AB}}{\overline{F'B}}.$$

We conclude that  $\overline{FB} = \overline{F'B}$ . Because both  $F$  and  $F'$  are on the extension of the side  $AB$ , we get that  $F \equiv F'$ , i.e. the points  $D$ ,  $E$  and  $F$  are collinear. ■

We will see how this theorem can be used in both directions, while proving the next theorem.

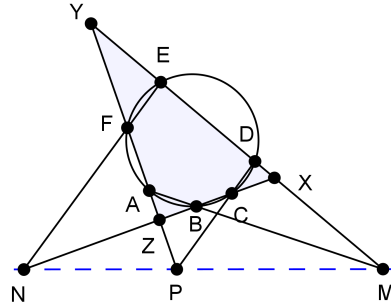
## 11.4 Pascal's Theorem

**Example 11.4** (Pascal's theorem). Let  $A, B, C, D, E$  and  $F$  be points on a circle (not necessarily in cyclic order). Let  $M = AB \cap DE$ ,  $N = BC \cap EF$  and  $P = CD \cap FA$ . Then  $M, N$  and  $P$  are collinear.



*Proof.* Let  $X = BC \cap DE$ ,  $Y = DE \cap FA$  and  $Z = FA \cap BC$ .

If we use [Menelaus' Theorem](#) on  $\triangle XYZ$  and the collinear points  $A - B - M$ ,



we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YA}}{\overline{AZ}} \cdot \frac{\overline{ZB}}{\overline{BX}} = 1.$$

If we use [Menelaus' Theorem](#) on  $\triangle XYZ$  and the collinear points  $P - C - D$ , we get:

$$\frac{\overline{XD}}{\overline{DY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZC}}{\overline{CX}} = 1.$$

If we use [Menelaus' Theorem](#) on  $\triangle XYZ$  and the collinear points  $F - N - E$ , we get:

$$\frac{\overline{XE}}{\overline{EY}} \cdot \frac{\overline{YF}}{\overline{FZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} = 1.$$

By multiplying these 3 equations and reordering the members, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} \cdot \frac{(\overline{YA} \cdot \overline{YF}) \cdot (\overline{ZB} \cdot \overline{ZC}) \cdot (\overline{XD} \cdot \overline{XE})}{(\overline{AZ} \cdot \overline{FZ}) \cdot (\overline{BX} \cdot \overline{CX}) \cdot (\overline{DY} \cdot \overline{EY})} = 1.$$

From the [Intersecting Secants Theorem](#) for the point  $X$ , we get:

$$\overline{XD} \cdot \overline{XE} = \overline{XC} \cdot \overline{XB}.$$

From the [Intersecting Secants Theorem](#) for the point  $Y$ , we get:

$$\overline{YF} \cdot \overline{YA} = \overline{YE} \cdot \overline{YD}.$$

From the [Intersecting Secants Theorem](#) for the point  $Z$ , we get:

$$\overline{ZB} \cdot \overline{ZC} = \overline{ZA} \cdot \overline{ZF}.$$

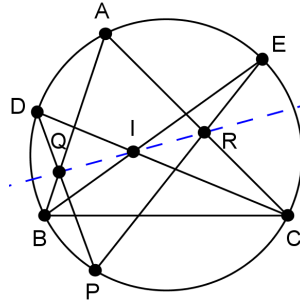
From the previous four equations, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} = 1,$$

which by [Menelaus' Theorem](#) means that the points  $M$ ,  $N$  and  $P$  are collinear. ■

Here is an example to show how the Pascal's Theorem can be used in a problem.

**Example 11.5.** Let  $D$  and  $E$  be the midpoints of the minor arcs  $\widehat{AB}$  and  $\widehat{AC}$  on the circumcircle of  $\triangle ABC$ , respectively. Let  $P$  be on the minor arc  $\widehat{BC}$ ,  $Q = PD \cap AB$  and  $R = PE \cap AC$ . Prove that the line  $QR$  passes through the incenter  $I$  of  $\triangle ABC$ .

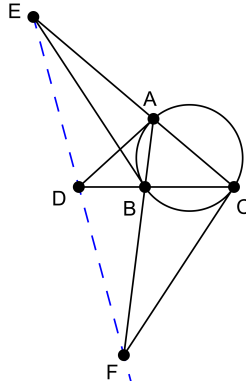


*Proof.* Since  $D$  is the midpoint of the arc  $\widehat{AB}$ ,  $CD$  is the angle bisector of  $\angle BCA$ . Similarly,  $BE$  is the angle bisector of  $\angle ABC$ . Therefore,  $CD \cap BE = I$ . Now, we apply Pascal's Theorem to the points  $C, D, P, E, B$  and  $A$  and we get that the points  $CD \cap EB = I$ ,  $DP \cap BA = Q$  and  $PE \cap AC = R$  are collinear. ■

We remark that there are limiting cases of Pascal's Theorem. For example, we may move  $A$  to approach  $B$ . In the limit,  $A$  and  $B$  will coincide and the line  $AB$  will become the tangent line at  $B$ . Here is an example to show how this works.

**Example 11.6.** Let  $\omega$  be the circumcircle of  $\triangle ABC$ . Let the tangent lines to  $\omega$  at  $A, B$  and  $C$  intersect the lines  $BC, CA$  and  $AB$  at points  $D, E$  and  $F$ , respectively. Prove that the points  $D, E$  and  $F$  are collinear.

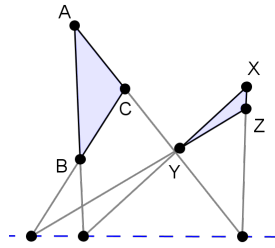
*Proof.* Let's apply the Pascal's Theorem to the points  $A, A, B, B, C$  and  $C$ . We get that the points  $AA \cap BC = D$ ,  $AB \cap CC = F$  and  $BB \cap CA = E$  are collinear. ■



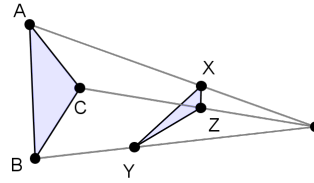
## 11.5 Desargues' Theorem

Two triangles  $\triangle ABC$  and  $\triangle XYZ$  are *perspective from a line* if the points  $AB \cap XY$ ,  $BC \cap YZ$  and  $CA \cap ZX$  are collinear.

Two triangles  $\triangle ABC$  and  $\triangle XYZ$  are *perspective from a point* if the lines  $AX$ ,  $BY$  and  $CZ$  are concurrent.

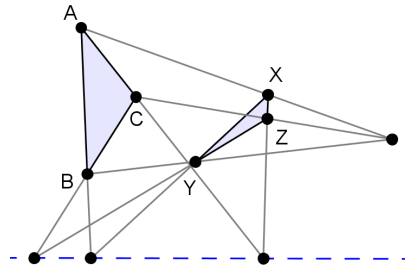


(a) Perspective from a line.



(b) Perspective from a point.

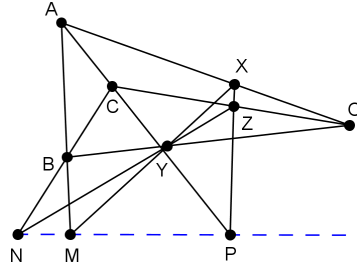
**Example 11.7** (Desargues' Theorem). Two triangles are perspective from a line if and only if they are perspective from a point.



*Proof.* Let  $\triangle ABC$  and  $\triangle XYZ$  be perspective from a point, i.e.  $AX$ ,  $BY$  and  $CZ$  are concurrent, and let the point of concurrence be  $O$ . Let  $AB \cap XY = M$ ,  $BC \cap YZ = N$  and  $CA \cap ZX = P$ .

We firstly apply the Menelaus' Theorem to  $\triangle OAB$  and the points  $M - Y - X$ :

$$\frac{\overline{OX}}{\overline{XA}} \cdot \frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BY}}{\overline{YO}} = 1.$$



Then, we apply the Menelaus' Theorem to  $\triangle OBC$  and the points  $N - Z - Y$ :

$$\frac{\overline{OY}}{\overline{YB}} \cdot \frac{\overline{BN}}{\overline{NC}} \cdot \frac{\overline{CZ}}{\overline{ZO}} = 1.$$

Finally, we apply the Menelaus' Theorem to  $\triangle OCA$  and the points  $P - X - Z$ :

$$\frac{\overline{OZ}}{\overline{ZC}} \cdot \frac{\overline{CP}}{\overline{PA}} \cdot \frac{\overline{AX}}{\overline{XO}} = 1.$$

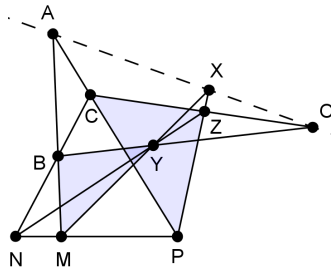
By multiplying these three equations, we get:

$$\frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BN}}{\overline{NC}} \cdot \frac{\overline{CP}}{\overline{PA}} = 1,$$

which by the Menelaus' Theorem for  $\triangle ABC$ , means that the points  $M$ ,  $N$  and  $P$  are collinear, i.e. the triangles  $\triangle ABC$  and  $\triangle XYZ$  are perspective from a line.  $\square$

Now, let's prove the other direction.

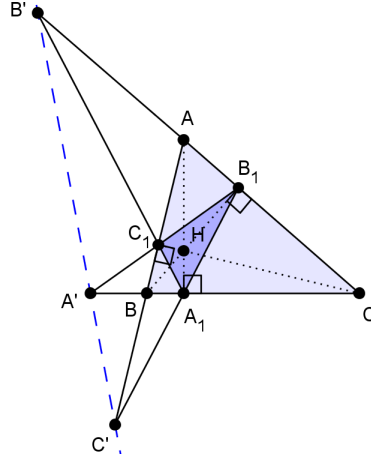
Let  $\triangle ABC$  and  $\triangle XYZ$  be perspective from a line, i.e. the points  $M = AB \cap XY$ ,  $N = BC \cap YZ$  and  $P = CA \cap ZX$  are collinear. Let  $O = BY \cap CZ$ . We should prove that  $AX$  also passes through  $O$ .



Let's take a look at  $\triangle PCZ$  and  $\triangle MBY$ . The lines  $PM$ ,  $CB$  and  $ZY$  are concurrent at  $N$ , so the triangles are perspective from a point. By the direction of the Desargues' Theorem that we just proved, it follows that the triangles must be perspective from a line, i.e. the points  $PC \cap MB = A$ ,  $CZ \cap BY = O$  and  $ZP \cap YM = X$  are collinear. With this, we proved that  $AX$  passes through  $O$ , so  $\triangle ABC$  and  $\triangle XYZ$  are perspective from a point.  $\blacksquare$

Let's see it in action. We will give an alternate proof to [Example 11.2](#):

**Example 11.8** (Orthic axis). Let  $AA_1$ ,  $BB_1$  and  $CC_1$  be the altitudes in  $\triangle ABC$ . Let  $A'$  be the intersection of the lines  $BC$  and  $B_1C_1$  and similarly define the points  $B'$  and  $C'$ . Prove that  $A'$ ,  $B'$  and  $C'$  lie on a line.



*Proof.* Since  $AA_1$ ,  $BB_1$  and  $CC_1$  are concurrent at the orthocenter of  $\triangle ABC$ , the triangles  $\triangle ABC$  and  $\triangle A_1B_1C_1$  are perspective from a point. Then, by the Desargues' Theorem, they are also perspective from a line, i.e. the points  $AB \cap A_1B_1 = C'$ ,  $BC \cap B_1C_1 = A'$  and  $CA \cap C_1A_1 = B'$  are collinear. ■

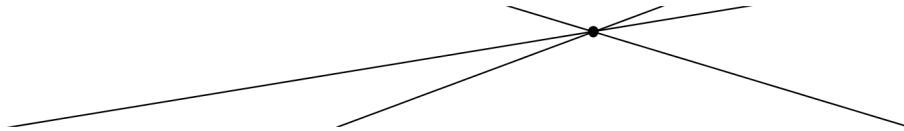
*Remark.* This proof can be used for more generalized problem, where  $AA_1$ ,  $BB_1$  and  $CC_1$  are any cevians in  $\triangle ABC$  that are concurrent.

We will end this chapter here, but we must mention that collinearity plays an important role in the chapter Homothety, so we will continue this theme later in our journey. —

**Related problems:** ([Menelaus' Theorem](#)) 37, 38 and 51.

# Chapter 12

## Concurrence

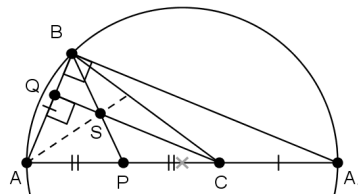


Three lines are *concurrent* if they pass through a common point. We will firstly present a few approaches to proving concurrence by using things we have already visited during our journey. Then, we will learn a new theorem related to concurrence.

### 12.1 Manual Approach

The most basic way to prove that three lines are concurrent is to take the intersection of two of them and then somehow prove that the third line passes through this intersection.

**Example 12.1.** Let  $C$  be a point on the diameter  $AA_1$  in a circle  $\omega$ . Let  $B$  be a point on  $\omega$ , such that  $\overline{AB} = \overline{CA_1}$ . Prove that in  $\triangle ABC$ , the internal angle bisector at the vertex  $A$ , the median from the vertex  $B$  and the altitude from the vertex  $C$  are concurrent.



*Proof.* We will take the intersection of the median and the altitude and we will prove that the angle bisector passes through this point. Let  $P$  be the midpoint of  $AC$  and let  $Q$  be the foot of the altitude from the vertex  $C$  in  $\triangle ABC$ . Let  $S = BP \cap CQ$ . We need to prove that  $AS$  bisects the angle  $\angle CAB$ .

$$CQ \perp AB \quad (\because CQ \text{ is altitude in } \triangle ABC)$$

$$A_1B \perp AB \quad (\because AA_1 \text{ is diameter})$$

$$\begin{aligned} &\therefore CQ \parallel A_1B, \text{ i.e. } CS \parallel A_1B \\ &\therefore \frac{\overline{PC}}{\overline{CA_1}} = \frac{\overline{PS}}{\overline{SB}} \text{ (by Thales' Proportionality Theorem)} \end{aligned}$$

Substituting  $\overline{PC} = \overline{AP}$  and  $\overline{CA_1} = \overline{AB}$ , we get

$$\frac{\overline{AP}}{\overline{AB}} = \frac{\overline{PS}}{\overline{SB}},$$

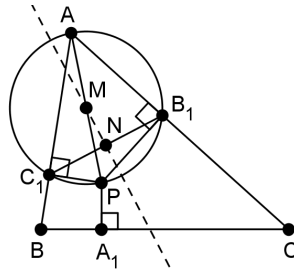
which by the [Angle Bisector Theorem](#) means that  $AS$  is the angle bisector of  $\angle PAB$ . Since  $\angle PAB \equiv \angle CAB$ , we get that  $AS$  bisects  $\angle CAB$ . ■

*Remark.* This approach can, in fact, be used not only for proving concurrent lines, but also for proving that any three objects (lines or circles or any combination of those) pass through a common point. For example, if we need to prove that three circles pass through a point, we will take the intersection of two of the circles and then prove that this intersection lies on the third circle. Otherwise, if we need to prove that two lines intersect on a circle, we can either take the intersection of the lines and prove that this intersection lies on the circle, or we can take the intersection of one of the lines and the circle and prove that this intersection lies on the other line.

## 12.2 Special Lines

Another way to prove that three lines are concurrent is by proving that they are "special lines" (such as side bisectors, angle bisectors, altitudes, ...) in a triangle in the figure. This is because we already know that these special lines concur at one of the important centers that we mentioned in [chapter 6](#).

**Example 12.2.** Let  $P$  be an arbitrary point inside the triangle  $ABC$ . Let  $A_1$ ,  $B_1$  and  $C_1$  be the feet of the perpendiculars from  $P$  to  $BC$ ,  $CA$  and  $AB$ , respectively. Prove that the lines that pass through the midpoints of  $PA$  and  $B_1C_1$ ,  $PB$  and  $C_1A_1$ , and  $PC$  and  $A_1B_1$  are concurrent.



*Proof.* We will prove that these lines are in fact side bisectors in  $\triangle A_1B_1C_1$ , so they will concur at the circumcircle of  $\triangle A_1B_1C_1$ . Let  $M$  and  $N$  be the midpoints of  $PA$  and  $B_1C_1$ , respectively.

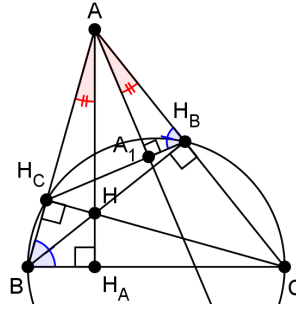
$$\angle PC_1A + \angle PB_1A = 180^\circ \quad (\because PC_1 \perp AB, PB_1 \perp AC)$$

Therefore,  $AC_1PB_1$  is cyclic. Since  $\angle AC_1P = 90^\circ$ ,  $AP$  is a diameter in  $(AC_1PB_1)$ , so  $M$  is its center. Since  $N$  is the midpoint of the chord  $B_1C_1$  and  $M$  is the center,  $MN$  is the side bisector of  $B_1C_1$ . Similarly, the other two lines are also side bisectors in  $\triangle A_1B_1C_1$ , so they are concurrent. ■

## 12.3 Special Point

If the lines in question are not "special lines", there is another way that the important centers can help us—by proving somehow that the lines pass through a "special point", i.e. an important center in a triangle in the figure.

**Example 12.3** (MMO 2015). Let  $AH_A$ ,  $BH_B$  and  $CH_C$  be altitudes in  $\triangle ABC$ . Let  $p_A$ ,  $p_B$  and  $p_C$  be the perpendicular lines from vertices  $A$ ,  $B$  and  $C$  to  $H_BH_C$ ,  $H_CH_A$  and  $H_AH_B$ , respectively. Prove that  $p_A$ ,  $p_B$  and  $p_C$  are concurrent.



*Proof.* We will prove that the lines pass through the circumcenter of  $\triangle ABC$ . Let  $A_1 = p_A \cap H_BH_C$ .

$$BCH_BH_C \text{ is cyclic } (\because \angle BH_BC = 90^\circ = \angle BH_CC)$$

$$\therefore \angle AH_BA_1 \equiv \angle AH_BH_C = \angle H_CBC \equiv \angle ABC = \beta$$

$$\angle CAA_1 \equiv \angle H_BAA_1 = 90^\circ - \angle AH_BA_1 = 90^\circ - \beta \quad (\because AA_1 \perp H_BH_C)$$

$$\angle BAH_A = 90^\circ - \angle ABH_A = 90^\circ - \beta$$

In conclusion,  $\angle CAA_1 = \angle BAH_A$ , so  $AH_A$  and  $AA_1 \equiv p_A$  are symmetric with respect to the angle bisector of  $\angle BAC$ . Since the orthocenter lies on the altitude  $AH_A$ , its isogonal conjugate, the circumcenter (Property 6.9), must lie on  $p_A$ . Similarly, the circumcenter of  $\triangle ABC$  lies on  $p_B$  and  $p_C$ , so the lines  $p_A$ ,  $p_B$  and  $p_C$  are concurrent. ■

## 12.4 Radical Center

Recall section 10.2, where we saw that the pairwise radical axes of three circles concur at the radical center. This is another approach of proving concurrence in geometry problems.

**Example 12.4** (IMO 1995/1). Let  $A$ ,  $B$ ,  $C$  and  $D$  be four distinct points on a line, in that order. The circles with diameters  $AC$  and  $BD$  intersect at  $X$  and  $Y$ . The line  $XY$  meets  $BC$  at  $Z$ . Let  $P$  be a point on the line  $XY$  other than  $Z$ . The line  $CP$  intersects the circle with diameter  $AC$  at  $C$  and  $M$ , and the line  $BP$  intersects the circle with diameter  $BD$  at  $B$  and  $N$ . Prove that the lines  $AM$ ,  $DN$  and  $XY$  are concurrent.

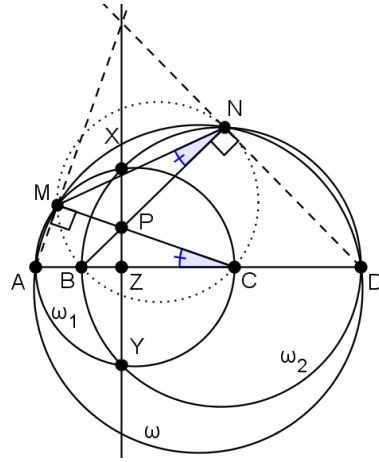
*Proof.* We will prove that these three lines are radical axis of three circles. Let the circle with diameter  $AC$  be  $\omega_1$  and the circle with diameter  $BD$  be  $\omega_2$ .

$$\overline{PM} \cdot \overline{PC} \stackrel{\omega_1}{=} \overline{PX} \cdot \overline{PY} \stackrel{\omega_2}{=} \overline{PB} \cdot \overline{PN}$$

$\therefore BCNM$  is cyclic (\*)

Since  $AC$  and  $BD$  are diameters of  $\omega_1$  and  $\omega_2$ , then  $\angle AMC = 90^\circ = \angle BND$ .

$$\begin{aligned} \angle MND &= \angle MNB + \angle BND \stackrel{(*)}{=} \\ &= \angle MCB + 90^\circ \equiv \\ &\equiv \angle MCA + 90^\circ \stackrel{\triangle AMC}{=} \\ &= 90^\circ - \angle MAC + 90^\circ \equiv \\ &\equiv 180^\circ - \angle MAD \\ \therefore MADN &\text{ is cyclic} \end{aligned}$$

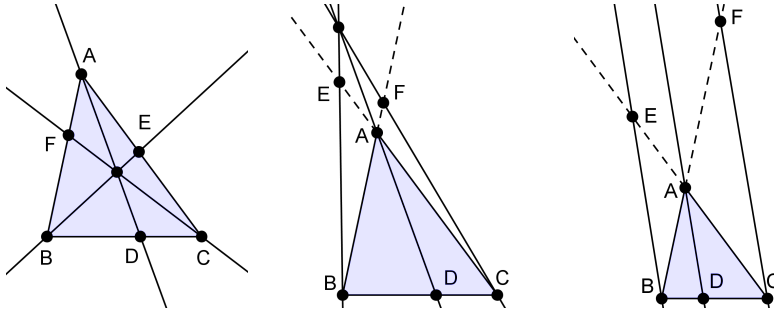


Now, we have three circles:  $(MAYCX)$ ,  $(MADN)$  and  $(NXBYD)$ . Their pairwise radical axes are  $MA$ ,  $DN$  and  $XY$ , so they are concurrent at the radical center of these three circles. ■

## 12.5 Ceva's Theorem

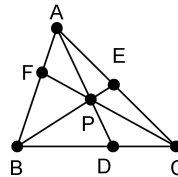
**Example 12.5** (Ceva's Theorem). Let  $ABC$  be a triangle. Let  $D$ ,  $E$  and  $F$  be points on the lines  $BC$ ,  $CA$  and  $AB$ , respectively, such that even number of them (zero or two) are on the extensions of the sides. The lines  $AD$ ,  $BE$  and  $CF$  are concurrent or parallel if and only if

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



*Proof.* Let the lines  $AD$ ,  $BE$  and  $CF$  be concurrent at  $P$ . Assume that the point  $P$  is inside the triangle  $ABC$ . (When  $P$  is outside, the proof is similar)

$$\begin{aligned} \frac{P_{\triangle CAF}}{P_{\triangle CFB}} &= \frac{\overline{AF}}{\overline{FB}} \\ \frac{P_{\triangle PAF}}{P_{\triangle PFB}} &= \frac{\overline{AF}}{\overline{FB}} \end{aligned}$$



$$\therefore \frac{P_{\triangle CAF} - P_{\triangle PAF}}{P_{\triangle CFB} - P_{\triangle PFB}} = \frac{\overline{AF}}{\overline{FB}}$$

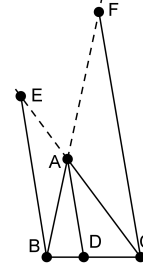
$$\text{i.e. } \frac{P_{\triangle CAP}}{P_{\triangle BCP}} = \frac{\overline{AF}}{\overline{FB}}.$$

$$\text{Similarly, } \frac{P_{\triangle ABP}}{P_{\triangle CAP}} = \frac{\overline{BD}}{\overline{DC}} \text{ and } \frac{P_{\triangle BCP}}{P_{\triangle ABP}} = \frac{\overline{CE}}{\overline{EA}}.$$

$$\therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1. \quad \square$$

Let the lines  $AD$ ,  $BE$  and  $CF$  be parallel. Exactly one of the points must be on the side of the triangle, WLOG let that point be  $D$  (the other two points are on the extensions of the sides). By [Thales' Proportionality Theorem](#), we get

$$\begin{aligned} \frac{\overline{AF}}{\overline{FB}} &= \frac{\overline{DC}}{\overline{CB}} \quad (\because DA \parallel CF) \\ \frac{\overline{CE}}{\overline{EA}} &= \frac{\overline{CB}}{\overline{BD}} \quad (\because DA \parallel BE) \\ \therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} &= \frac{\overline{DC}}{\overline{CB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CB}}{\overline{BD}} = 1. \quad \square \end{aligned}$$

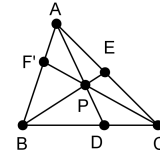


Now, let's prove the other direction. Let

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$$

and let even number of the points be on the extensions of the sides. Let the intersection of the lines  $AD$  and  $BE$  be  $P$ . In the case when there is no intersection, i.e. when  $AD \parallel BE$ , it can be easily proven that  $AD$ ,  $BE$  and  $CF$  are parallel.

Let  $CP$  intersect  $AB$  at  $F'$ . Similarly as in the proof of Menelaus' Theorem, we are using the direction of Ceva's Theorem that we just proved (for  $AD$ ,  $BE$  and  $CF'$  which do concur) and we get:



$$\frac{\overline{AF'}}{\overline{F'B}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Combining with the condition, we get:

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AF'}}{\overline{F'B}}.$$

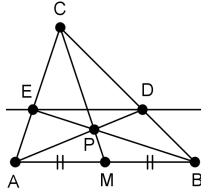
Keeping in mind that  $\overline{AF} + \overline{FB} = \overline{AB} = \overline{AF'} + \overline{F'B}$ , by adding 1 to both sides, we get:

$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AB}}{\overline{F'B}},$$

which means  $\overline{FB} = \overline{F'B}$ , i.e.  $F \equiv F'$ . We should note that in the last part, we assumed that  $F'$  is between  $A$  and  $B$ . That is a safe assumption because there are either zero or two points on the extensions of the sides; If there are zero, then  $D$  and  $E$  are on the sides  $BC$  and  $CA$ , so  $F'$  must also lie on the side  $AB$ ; If there are two points on the extensions, then WLOG let them be  $D$  and  $E$  and  $F'$  will again lie on the side  $AB$ . ■

In the next few examples, we will show how we can use Ceva's Theorem in both directions.

**Example 12.6.** In  $\triangle ABC$ , let  $M$  be the midpoint of the side  $BC$ . Let  $P$  be an arbitrary point on the segment  $CM$  ( $P \neq C$ ,  $P \neq M$ ). Let  $AP \cap BC = D$  and  $BP \cap AC = E$ . Prove that  $ED \parallel AB$ .



*Proof.* The lines  $AD$ ,  $BE$  and  $CM$  are concurrent, so we can use [Ceva's Theorem](#):

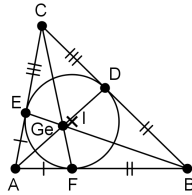
$$\frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Since  $M$  is the midpoint of  $AB$ ,  $\overline{AM} = \overline{MB}$ , so by canceling and then rearranging, we get:

$$\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CD}}{\overline{DB}},$$

which by [Thales' Proportionality Theorem](#) means that  $ED \parallel AB$ . ■

**Example 12.7** (Gergonne Point). Let  $D$ ,  $E$  and  $F$  be the tangent points of the incircle of  $\triangle ABC$  with the sides  $BC$ ,  $CA$  and  $AB$ , respectively. Prove that  $AD$ ,  $BE$  and  $CF$  are concurrent.



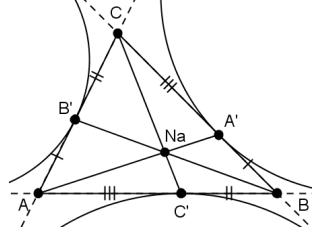
*Proof.*  $\overline{AF} = \overline{AE} = x$  as tangent segments from the point  $A$  to the incircle. Similarly,  $\overline{BF} = \overline{BD} = y$  and  $\overline{CD} = \overline{CE} = z$ .

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x} = 1,$$

so by [Ceva's Theorem](#),  $AD$ ,  $BE$  and  $CF$  are concurrent. ■

*Remark.* This point of concurrence is known as the *Gergonne Point* of the triangle  $ABC$ .

**Example 12.8** (Nagel Point). Let  $A'$ ,  $B'$  and  $C'$  be the tangent points of the  $A$ -excircle,  $B$ -excircle and  $C$ -excircle with the sides  $BC$ ,  $CA$  and  $AB$  in the  $\triangle ABC$ , respectively. Prove that  $AA'$ ,  $BB'$  and  $CC'$  are concurrent.



*Proof.* From [Example 8.5](#), we know that  $\overline{AB} + \overline{BA'} = \overline{AC} + \overline{CA'}$ . We see that LHS and RHS add up to the perimeter of  $\triangle ABC$ , so each of them is equal to the semiperimeter  $s$ . Therefore,  $\overline{BA'} = s - c$  and  $\overline{CA'} = s - b$ . Similarly,  $\overline{CB'} = s - a$ ,  $\overline{AB'} = s - c$ ,  $\overline{AC'} = s - b$  and  $\overline{BC'} = s - a$ .

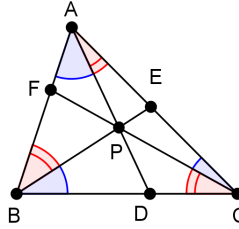
$$\frac{\overline{AC'}}{\overline{C'B}} \cdot \frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} = \frac{s-b}{s-a} \cdot \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} = 1,$$

so by [Ceva's Theorem](#),  $AA'$ ,  $BB'$  and  $CC'$  are concurrent. ■

*Remark.* This point of concurrence is known as the *Nagel Point* of the triangle  $ABC$ .

**Example 12.9** (Trigonometric Ceva's Theorem). Given a triangle  $ABC$  and points  $D$ ,  $E$  and  $F$  that lie on the lines  $BC$ ,  $CA$  and  $AB$ , respectively; the lines  $AD$ ,  $BE$  and  $CF$  are concurrent or parallel if and only if

$$\frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} = 1.$$



*Proof.* By using the [Law of Sines](#) in  $\triangle ABD$ , we get

$$\frac{\overline{BD}}{\sin \angle BAD} = \frac{\overline{AB}}{\sin \angle BDA}, \text{ i.e. } \sin \angle BAD = \frac{\overline{BD} \cdot \sin \angle BDA}{\overline{AB}}.$$

Similarly, for  $\triangle ACD$ , we get

$$\sin \angle CAD = \frac{\overline{CD} \cdot \sin \angle CDA}{\overline{AC}}.$$

Since  $D \in BC$ , then the angles  $\angle BDA$  and  $\angle CDA$  are always equal or supplementary. Therefore,  $\sin \angle BDA = \sin \angle CDA$ . By dividing the previous equations, we get

$$\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{\overline{BD}}{\overline{CD}} \cdot \frac{\overline{AC}}{\overline{AB}}.$$

Analogously, we can get similar equations for the cevians  $BE$  and  $CF$ . By multiplying these three equations, we get

$$\frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} = \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}},$$

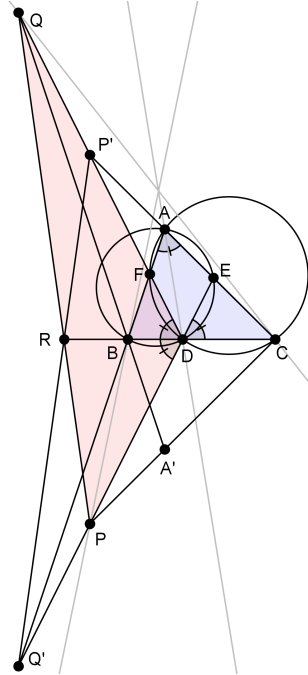
so by [Ceva's Theorem](#) we are done.  $\blacksquare$

## 12.6 Desargues' Theorem

Here is an example that shows how we can use [Desargues' Theorem](#) when we need to prove concurrence.

**Example 12.10** (RMM 2016). Let  $ABC$  be a triangle and let  $D$  be a point on the segment  $BC$ ,  $D \neq B$  and  $D \neq C$ . The circle  $(ABD)$  meets the segment  $AC$  again at an interior point  $E$ . The circle  $(ACD)$  meets the segment  $AB$  again at an interior point  $F$ . Let  $A'$  be the reflection of  $A$  in the line  $BC$ . The lines  $A'C$  and  $DE$  meet at  $P$ , and the lines  $A'B$  and  $DF$  meet at  $Q$ . Prove that the lines  $AD$ ,  $BP$  and  $CQ$  are concurrent (or all parallel).

*Proof.* Let  $\sigma$  denote reflection in the line  $BC$ . Since  $\angle BDF = \angle BAC = \angle CDE$  (because of the cyclic quadrilaterals  $ABDE$  and  $ACDF$ ), the lines  $DE$  and  $DF$  are images of one another under  $\sigma$ , so the lines  $AC$  and  $DF$  meet at  $P' \equiv \sigma(P)$ , and the lines  $AB$  and  $DE$  meet at  $Q' \equiv \sigma(Q)$ . Consequently, the lines  $PQ$  and  $P'Q' \equiv \sigma(PQ)$  meet at some point  $R$  on the line  $BC$ . Since the points  $Q' = AB \cap DP$ ,  $R = BC \cap PQ$  and  $P' = CA \cap QD$  are collinear, the triangles  $\triangle ABC$  and  $\triangle DPQ$  are perspective from a line. Therefore, by [Desargues' Theorem](#), they are also perspective from a point, i.e. the lines  $AD$ ,  $BP$  and  $CQ$  are concurrent.  $\blacksquare$

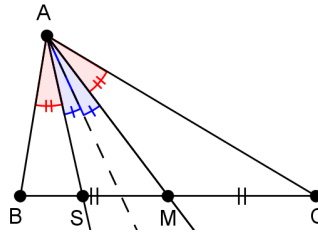


**Related problems:** 44, 47 and 54. [Ceva's Theorem](#): 39.

## Chapter 13

# Symmedian

**Symmedian** is the reflection of the median across the corresponding angle bisector.



We will now see and prove a few properties of the symmedians.

**Property 13.1.** The symmedian  $AS$  divides the opposite side in the ratio of the square of the sides, i.e.

$$\frac{\overline{BS}}{\overline{CS}} = \left( \frac{\overline{AB}}{\overline{AC}} \right)^2.$$

*Proof.* Since the symmedian is the reflection of the median with respect to the angle bisector, we have  $\angle BAS = \angle CAM$  and  $\angle BAM = \angle CAS$ .

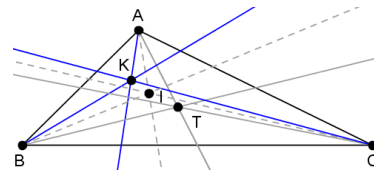
$$\frac{\overline{BS}}{\overline{MC}} = \frac{P_{\triangle BAS}}{P_{\triangle MAC}} = \frac{\overline{BA} \cdot \overline{AS}}{\overline{AM} \cdot \overline{AC}}$$

$$\frac{\overline{BM}}{\overline{SC}} = \frac{P_{\triangle BMA}}{P_{\triangle CSA}} = \frac{\overline{BA} \cdot \overline{AM}}{\overline{AS} \cdot \overline{AC}}$$

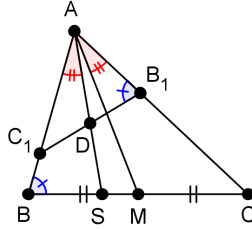
By multiplying these equalities, we are done. ■

**Property 13.2** (Lemoine Point). The three symmedians in a triangle are concurrent.

*Proof.* Using [Property 13.1](#), by [Ceva's Theorem](#), it immediately follows that the three symmedians in a triangle are concurrent. This point of concurrence is called the *Lemoine Point* of the triangle. ■



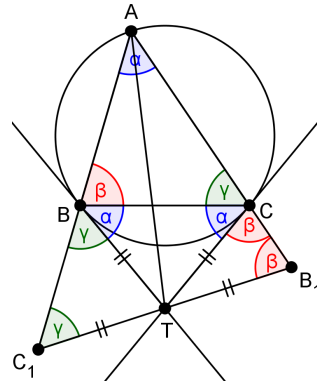
**Property 13.3.** A symmedian drawn from a vertex of a triangle bisects the antiparallels to the opposite side (with respect to the adjacent sides).



*Proof.* Let  $AS$  and  $AM$  be the symmedian and the median from the vertex  $A$  in  $\triangle ABC$ , respectively. Let  $B_1 \in AC$  and  $C_1 \in AB$ , such that  $B_1C_1$  is antiparallel to  $BC$  with respect to the lines  $AB$  and  $AC$ , i.e.  $\angle AB_1C_1 = \angle ABC$ . Therefore,  $\triangle ABC \sim \triangle AB_1C_1$ . Let  $AS \cap B_1C_1 = D$ . By the definition of symmedian,  $\angle BAS = \angle CAM$ , which means that the similarity "maps"  $AM$  in  $\triangle ABC$  to  $AS \equiv AD$  in  $\triangle AB_1C_1$ . Therefore,  $AD$  is median in  $\triangle AB_1C_1$ , i.e. the symmedian  $AS$  bisects  $B_1C_1$  which is antiparallel to the opposite side  $BC$ . ■

**Property 13.4.** Given a triangle  $ABC$  and its circumcircle, let the intersection of the tangents at the points  $B$  and  $C$  intersect at  $T$ . Then,  $AT$  is a symmedian in  $\triangle ABC$ .

*Proof.* Since the angle between a tangent and a chord is equal to any inscribed angle that subtends the same chord,  $\angle CBT = \angle CAB = \alpha$  and  $\angle BCT = \angle BAC = \alpha$ , so  $\triangle BCT$  is isosceles and therefore  $\overline{TB} = \overline{TC}$ . Let  $B_1 \in AC$  and  $C_1 \in AB$ , such that  $B_1C_1$  is an antiparallel line to  $BC$  (with respect to the lines  $AB$  and  $AC$ ) that passes through  $T$ . Then,  $\angle AB_1C_1 = \angle ABC = \beta$ . Now,  $\angle TCB_1 = 180^\circ - (\angle ACB + \angle BCT) = 180 - (\gamma + \alpha) = \beta$  and  $\angle TB_1C \equiv \angle C_1B_1A = \beta$ , so  $\triangle TCB_1$  is isosceles, i.e.  $\overline{TC} = \overline{TB_1}$ . Similarly,  $\overline{TB} = \overline{TC_1}$ . In conclusion,  $\overline{TC_1} = \overline{TB} = \overline{TC} = \overline{TB_1}$ , so  $T$  is the midpoint of  $B_1C_1$ . By [Property 13.3](#), it follows that  $AT$  is the symmedian from the vertex  $A$  in  $\triangle ABC$ . ■



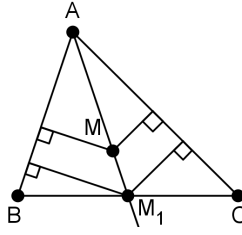
*Remark.* The previous example shows that the Lemoine point of a triangle is the Gergonne point of the tangential triangle.

**Property 13.5.** The  $A$ -symmedian is the locus of the points  $P$  such that

$$\frac{d(P, AB)}{d(P, AC)} = \frac{\overline{AB}}{\overline{AC}}.$$

*Proof.* We will firstly prove that the median is the locus of the points  $M$  such that

$$\frac{d(M, AB)}{d(M, AC)} = \frac{\overline{AC}}{\overline{AB}}.$$

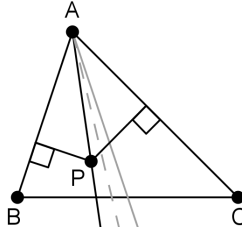


Let  $M$  be a point in the interior of  $\angle BAC$ . Let  $AM$  meet  $BC$  at  $M_1$ . By similarity of triangles, we get that

$$\frac{d(M, AB)}{d(M_1, AB)} = \frac{\overline{AM}}{\overline{AM_1}} = \frac{d(M, AC)}{d(M_1, AC)}.$$

By rearranging, we get

$$\begin{aligned} \frac{d(M_1, AB)}{d(M_1, AC)} &= \frac{d(M, AB)}{d(M, AC)} = \frac{\overline{AC}}{\overline{AB}} \\ \Leftrightarrow d(M_1, AB) \cdot \overline{AB} &= d(M_1, AC) \cdot \overline{AC} \\ \Leftrightarrow P_{\triangle ABM_1} &= P_{\triangle ACM_1} \\ \Leftrightarrow \overline{BM_1} &= \overline{M_1C} \quad \square \end{aligned}$$



Since the symmedian is the reflection of the median with respect to the angle bisector, by symmetry we have that it is the locus of the points  $P$  such that

$$\frac{d(P, AB)}{d(P, AC)} = \frac{d(M, AC)}{d(M, AB)} = \frac{\overline{AB}}{\overline{AC}} \quad \blacksquare$$

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**Related problems:** 26, 29, 31 and 41.

# Chapter 14

## Homothety

### Definition and properties

A homothety with center  $O$  and ratio  $k$  is a function that sends every point on the plane  $P$  to a point  $P'$  such that

$$\overrightarrow{OP'} = k \cdot \overrightarrow{OP}.$$

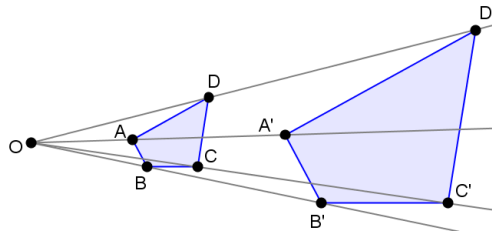


Figure 14.1: A homothety with center  $O$  and ratio  $k = 2.5$

From this definition, we can directly conclude the following properties:

**Property 14.1.** The image point, the original point and the center of the homothety are collinear.

**Property 14.2.** A homothety always sends a figure to a similar figure, such that the corresponding sides are parallel.

If  $k > 0$ , then the image and the original will be on the same side of the center; If  $k < 0$ , the image and the original will be on different sides of the center, i.e. the center will be between them. If  $|k| > 1$ , then the homothety is a magnification; If  $|k| < 1$ , then it is a reduction.

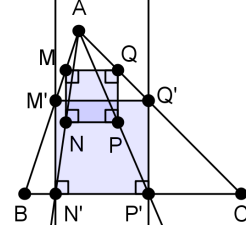
We will use the notation  $\mathcal{X}_{O,k} : P \rightarrow P'$  to denote that  $P'$  is the image of  $P$  under the homothety centered at  $O$  with ratio  $k$ .

### Getting started

As an exercise, let's try to construct a square that is "inscribed" in a  $\triangle ABC$ , such that one vertex lies on the side  $AB$ , one on the side  $AC$  and two adjacent

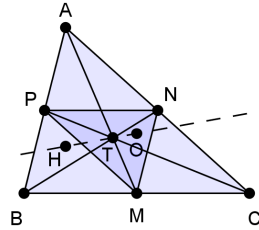
vertices of the square lie on the side  $BC$ . Firstly try by yourself and then see the solution presented below.

Construct any square  $MNPQ$  such that  $M \in AB$  and  $Q \in AC$  and  $MQ \parallel NP \parallel BC$ . We will now define a homothety centered at  $A$  that will send  $MNPQ$  to the desired square. Let  $AN \cap BC = N'$ . We define the ratio of the homothety  $k = \frac{AN'}{AN}$ , so that  $\mathcal{X} : N \rightarrow N'$ . Since  $NP \parallel BC$  and  $N' \in BC$ , the image of  $P$  will be a point  $P'$  on  $BC$ . Also the center of the homothety ( $A$ ), the original ( $P$ ) and the image ( $P'$ ) must be collinear, so  $P' = AP \cap BC$ .



Now, let's find  $M'$ .  $M'N'$  should be parallel to  $MN$ , but also  $A - M - M'$  should be collinear, so  $M'$  is the intersection of the perpendicular to  $BC$  through  $N'$  and the line  $AM \equiv AB$ . We can find  $Q'$  similarly to  $M'$ . The resulting quadrilateral  $M'N'P'Q'$  is similar to its original  $MNPQ$ , so it must be a square. It is also "inscribed" in  $\triangle ABC$  per the given conditions, so we are done.

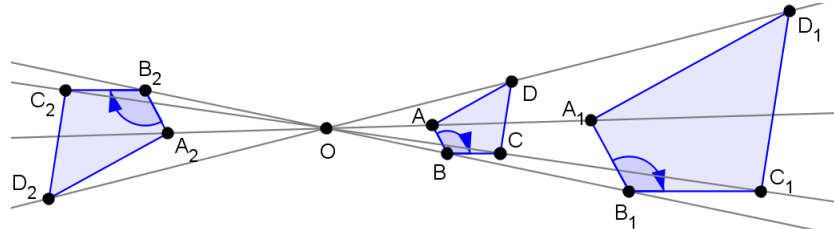
As another exercise, we will give an alternate proof of [Example 8.13](#), where we proved that in any triangle  $ABC$ , the orthocenter  $H$ , the centroid  $T$  and the circumcenter  $O$  are collinear and that  $\overline{HT} = 2 \cdot \overline{TO}$ .



Let  $M$ ,  $N$  and  $P$  be the midpoints of the sides  $BC$ ,  $CA$  and  $AB$ , respectively. Remember that the medians  $AM$ ,  $BN$  and  $CP$  intersect at the centroid  $T$  and moreover, it divides them in ratio 2:1. Therefore,  $\mathcal{X}_{T, -1/2} : \triangle ABC \rightarrow \triangle MNP$ . Also,  $\mathcal{X} : H_{ABC} \rightarrow H_{MNP}$ , so the points  $H_{ABC} - T_{ABC} - H_{MNP}$  are collinear. From [Property 6.10](#), we know that  $O_{ABC} \equiv H_{MNP}$ , so  $H_{ABC} - T_{ABC} - O_{ABC}$  are collinear. Because  $|k| = \frac{1}{2}$ , we can also conclude that  $\overline{HT} = 2 \cdot \overline{TO}$ .

## 14.1 Homothetic center of circles

Homothetic centers may be external ( $k > 0$ ) or internal ( $k < 0$ ). If the center is internal, the two geometric figures are scaled,  $180^\circ$ -rotated and translated images of one another. Otherwise, if the center is external, the two figures are scaled and translated similar to one another. Sometimes, the external and internal homothetic centers (centers of similitude) are called *exsimilicenter* and *insimilicenter*, respectively.



Circles are geometrically similar to one another and "rotation invariant". Hence, a pair of circles has both types of homothetic centers, internal and external (unless the centers coincide or the radii are equal; we will discuss these

special cases later). These two homothetic centers lie on the line joining the centers of the two given circles.

How can we find those homothetic centers? Let's draw two parallel diameters  $A_1B_1$  and  $A_2B_2$ , one for each circle. These make the same angle with the line connecting the centers. The lines  $A_1A_2$ , and  $B_1B_2$ , intersect each other and the line connecting the centers at the external homothetic center. Conversely, the lines  $A_1B_2$  and  $B_1A_2$  intersect each other and the line connecting the centers at the internal homothetic center. As a limiting case of this construction, a line tangent to both circles passes through one of the homothetic centers, as it forms right angles with both the corresponding diameters, which are thus parallel. The common external tangents pass through the external homothetic center, while the common internal tangents pass through the internal homothetic center. If the circles have the same radius (but different centers), they have no external homothetic center. If the circles have the same center, they have only one homothetic center and that is the common center of the circles.

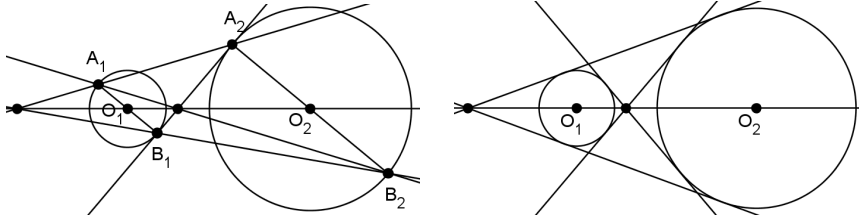
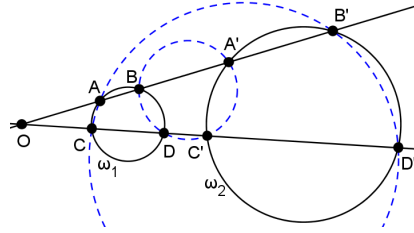


Figure 14.2: Internal and external homothetic center of two circles.

A line through a homothetic center that intersects the circles, will intersect each circle at two places. Of these four points, any two are said to be *homologous* if radii drawn to them make the same angle with the line connecting the centers (eg.  $A$  and  $A'$ ). Out of these four, any two that lie on different circles and are not homologous are said to be *antihomologous* (eg.  $A$  and  $B'$ ).



We will now prove that any two pairs of antihomologous points (defined by lines through the same homothetic center) are concyclic. Let  $O$  be a homothetic center of  $\omega_1$  and  $\omega_2$ . Let a line through  $O$  intersect  $\omega_1$  at  $A$  and  $B$  and  $\omega_2$  at  $A'$  and  $B'$  (such that  $A$  and  $A'$  are closer to  $O$  than  $B$  and  $B'$ , respectively). Then,

$$\overline{OA} \cdot \overline{OB'} = \overline{OA} \cdot (k \cdot \overline{OB}) = k \cdot \overline{OA} \cdot \overline{OB}.$$

If we similarly define points  $C, D, C'$  and  $D'$  for a different line through  $O$ , we have

$$\overline{OC} \cdot \overline{OD'} = \overline{OC} \cdot (k \cdot \overline{OD}) = k \cdot \overline{OC} \cdot \overline{OD}.$$

From the intersecting secants theorem for  $\omega_1$ , we have  $\overline{OA} \cdot \overline{OB} = \overline{OC} \cdot \overline{OD}$ , so combining the previous equations, we get  $\overline{OA} \cdot \overline{OB'} = \overline{OC} \cdot \overline{OD'}$  which means that the points  $A, B', C$  and  $D'$  (which are two pairs of antihomologous points)

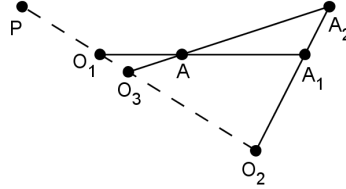
are concyclic. Just as a trivia for our more curious readers, we will mention that because of this concyclicity, the intersection of the lines  $AC$  and  $B'D'$  lies on the radical axis of the circles  $\omega_1$  and  $\omega_2$ . Can you see why?

## 14.2 Composition of homotheties

Composition of two homotheties,  $\mathcal{X}_1$  with center  $O_1$  and ratio  $k_1$ , and  $\mathcal{X}_2$  with center  $O_2$  and ratio  $k_2$  is a homothety  $\mathcal{X}_3$  (unless  $k_1 \cdot k_2 = 1$ ).

**Property 14.3.** Let  $\mathcal{X}_3$  be the composition of homotheties  $\mathcal{X}_2 \circ \mathcal{X}_1$ . Then, the center of  $\mathcal{X}_3$  lies on the line  $O_1O_2$  and the ratio of  $\mathcal{X}_3$  is  $k_1 \cdot k_2$ .

*Proof.* Let  $A$  be a point. Let  $A_1$  and  $A_2$  be points such that  $\mathcal{X}_1 : A \rightarrow A_1$  and  $\mathcal{X}_2 : A_1 \rightarrow A_2$ . In other words,  $\mathcal{X}_3 : A \rightarrow A_2$  because  $\mathcal{X}_3(A) = \mathcal{X}_2 \circ \mathcal{X}_1(A) = \mathcal{X}_2(\mathcal{X}_1(A)) = \mathcal{X}_2(A_1) = A_2$ .



Let's prove that the center of  $\mathcal{X}_3$ ,  $O_3$ , lies on  $O_1O_2$ . Let  $\mathcal{X}_2 : O_1 \rightarrow P$ . The point  $P$  doesn't have a special meaning, but it will help us prove our claim. We will also use the fact that the center of homothety is fixed under a homothety, i.e.  $\mathcal{X}_1 : O_1 \rightarrow O_1$ .

$$\mathcal{X}_3(O_1) = \mathcal{X}_2(\mathcal{X}_1(O_1)) = \mathcal{X}_2(O_1) = P$$

From this equation, we have  $\mathcal{X}_3 : O_1 \rightarrow P$  and  $\mathcal{X}_2 : O_1 \rightarrow P$ . Using [Property 14.1](#), which says that the center of homothety, the original, and the image are collinear, we have  $O_3 - O_1 - P$  and  $O_2 - O_1 - P$ , i.e. all four points are collinear, so the center of  $\mathcal{X}_3$ , which is  $O_3$ , lies on  $O_1O_2$ .

We will now find the ratio of  $\mathcal{X}_3$ . Let's get back to the points  $A$ ,  $A_1$  and  $A_2$  that we defined earlier. By definition of  $A_1$ , we get  $\overline{O_1A_1} = k_1 \cdot \overline{O_1A}$ . By definition of  $A_2$ , we get  $\overline{O_2A_2} = k_2 \cdot \overline{O_2A_1}$ . Because  $\mathcal{X}_3 : A \rightarrow A_2$ , we have  $\overline{O_3A_2} = k_3 \cdot \overline{O_3A}$ . Keep in mind the fact that we just proved, that  $O_3$  is collinear with  $O_1$  and  $O_2$ . Because of this collinearity, we can apply Menelaus' Theorem to  $\triangle AA_1A_2$  and the points  $O_2 - O_3 - O_1$

$$\frac{\overline{AO_1}}{\overline{O_1A_1}} \cdot \frac{\overline{A_1O_2}}{\overline{O_2A_2}} \cdot \frac{\overline{A_2O_3}}{\overline{O_3A}} = 1$$

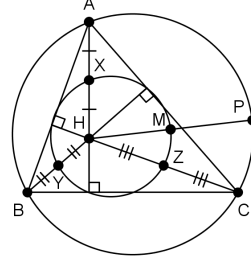
$$\frac{1}{k_1} \cdot \frac{1}{k_2} \cdot \frac{k_3}{1} = 1$$

Finally,  $k_3 = k_1 \cdot k_2$ . ■

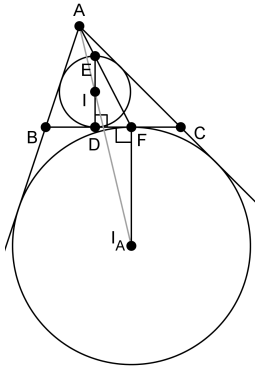
### 14.3 Useful Lemmas

**Example 14.1.** Prove that the nine point circle bisects any line segment from the orthocenter to the circumcircle.

*Proof.* Let  $ABC$  be a triangle with orthocenter  $H$ . Let  $X$ ,  $Y$  and  $Z$  be the midpoints of  $AH$ ,  $BH$  and  $CH$ , respectively. Then, there is a homothety centered at  $H$ , with ratio 2, that sends  $\triangle XYZ$  to  $\triangle ABC$ . Since the circumcircle of  $\triangle XYZ$  is the nine point circle of  $\triangle ABC$ , this homothety sends the nine point circle of  $\triangle ABC$  to its circumcircle. Let  $P$  be any point on the circumcircle of  $\triangle ABC$ . Let the nine point circle intersect  $HP$  at  $M$ . Then,  $\mathcal{H}_{H,2} : M \rightarrow P$ . Therefore,  $\overline{HP} = 2 \cdot \overline{HM}$ , i.e.  $\overline{HM} = \overline{MP}$ . ■



**Example 14.2** (Diameter of the incircle). Let the incircle of  $\triangle ABC$  touch the side  $BC$  at  $D$  and let  $DE$  be a diameter of the incircle. Let  $AE \cap BC = F$ . Prove that  $\overline{BD} = \overline{FC}$ .

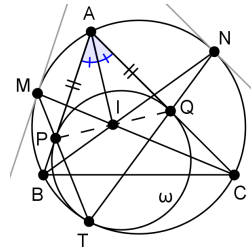


*Proof.* Consider the homothety with center  $A$  that sends the incircle to the  $A$ -excircle. The diameter  $DE$  of the incircle must be mapped to the diameter of the excircle that is perpendicular to  $BC$ . It follows that  $E$  must get mapped to the point of tangency between the excircle and  $BC$ . Since the image of  $E$  must lie on the line  $AE$ , it must be  $F$ . That is, the excircle is tangent to  $BC$  at  $F$ . In [Example 8.5](#), we already proved that the tangent points of the incircle and the excircle to the side  $BC$  are equidistant from the midpoint, so  $\overline{BD} = \overline{FC}$ . ■

*Remark.* Notice that, as a consequence to this collinearity, the line joining the incenter and the midpoint of  $BC$  is parallel to the line  $AF$ .

**Example 14.3.** Let  $ABC$  be a triangle. A circle  $\omega$  is internally tangent to the circumcircle of  $\triangle ABC$  and also to the sides  $AB$  and  $AC$  at  $P$  and  $Q$ , respectively. Prove that the midpoint of  $PQ$  is the incenter of  $\triangle ABC$ .

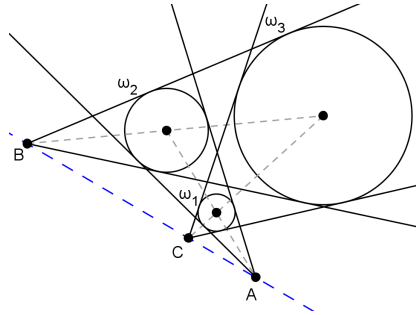
*Proof.* Let the tangent point of  $(ABC)$  and  $\omega$  be  $T$ . Let  $\mathcal{H}$  be a homothety that sends  $\omega$  to  $(ABC)$ , i.e.  $\mathcal{H}_T : \omega \rightarrow (ABC)$ . Let  $TP \cap (ABC) = M$ . Since  $AB$  is tangent to  $\omega$  at  $P$ , the parallel line to  $AB$  through  $M$  should be tangent to  $(ABC)$  at  $M$ . This means that  $M$  must be the midpoint of the arc  $\widehat{AB}$  in  $(ABC)$ . So  $CM$  is an angle bisector in  $\triangle ABC$ , i.e.  $C - I - M$  are collinear, where  $I$  is the incenter in  $\triangle ABC$ . Let  $TQ \cap (ABC) = N$ . Similarly,  $B - I - N$  are collinear. By applying Pascal's Theorem to the points  $T, M, C, A, B$  and  $N$ , we get that the points  $P - I - Q$  are collinear. Also,  $\overline{AP} = \overline{AQ}$  as tangent segment, and  $AI$  is the angle bisector of  $\angle BAC \equiv \angle PAQ$ , so  $I$  must be the midpoint of  $PQ$ . ■



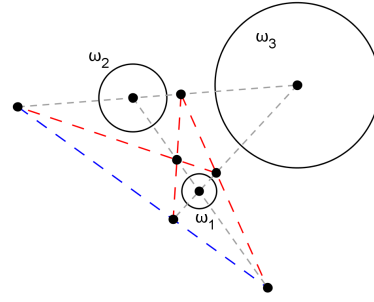
**Example 14.4** (Monge's Theorem). The exsimilicenters of three circles are collinear.

*Proof.* Let  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  be three circles. Let  $A$ ,  $B$  and  $C$  be the intersections of the external tangents of  $\omega_1$  and  $\omega_2$ ;  $\omega_2$  and  $\omega_3$ ; and  $\omega_1$  and  $\omega_3$ , respectively. One of the two homotheties that sends  $\omega_1$  to  $\omega_2$  is centered at  $A$  and has a coefficient  $k_1 > 0$ , i.e.  $\mathcal{X}_{A,k_1} : \omega_1 \rightarrow \omega_2$ . Similarly,  $\mathcal{X}_{B,k_2} : \omega_2 \rightarrow \omega_3$ , where  $k_2 > 0$ . Therefore, the composition homothety  $\mathcal{X}_{comp} = \mathcal{X}_B \circ \mathcal{X}_A$  sends  $\omega_1$  to  $\omega_3$ . By the properties of composition of homotheties, we know that the center of  $\mathcal{X}_{comp}$  lies on the line  $AB$  and the coefficient is positive (as it is equal to  $k_1 \cdot k_2$ ). But the center of the homothety that sends  $\omega_1$  to  $\omega_3$  with positive coefficient is found as the intersection of the common external tangents, so it is  $C$ . In conclusion,  $C \in AB$ , i.e. the points  $A$ ,  $B$  and  $C$  are collinear. ■

*Remark.* Can you prove this using Menelaus' Theorem? What about using Desargues' Theorem?



(a) Monge's Theorem

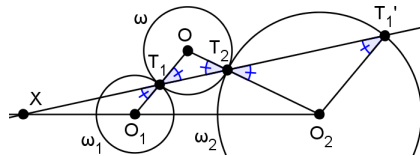


(b) Monge-d'Alembert Theorem

**Example 14.5** (Monge-d'Alembert Theorem). Given three circles, the insimilicenters of any two pairs of circles and the exsimilicenter of the third one are collinear.

*Proof.* The proof is analogous to the proof of [Monge's Theorem](#). ■

**Example 14.6.** Let  $\omega$  be a circle that is tangent to  $\omega_1$  and  $\omega_2$  at  $T_1$  and  $T_2$ , respectively. Prove that the line  $T_1T_2$  passes through one of the homothetic centers of  $\omega_1$  and  $\omega_2$ .



*Proof 1.* We will discuss the case where  $\omega$  is externally tangent to both  $\omega_1$  and  $\omega_2$ . The other cases should be analogous. Let  $O$ ,  $O_1$  and  $O_2$  be the centers of  $\omega$ ,  $\omega_1$  and  $\omega_2$ , respectively.

$$\angle OT_1T_2 = \angle OT_2T_1 \quad (\because \overline{OT_1} = \overline{OT_2})$$

Let  $X = T_1T_2 \cap O_1O_2$ . Because of the tangency of  $\omega$  and  $\omega_1$ , we know that  $T_1 \in OO_1$ , so

$$\angle OT_1T_2 = \angle O_1T_1X.$$

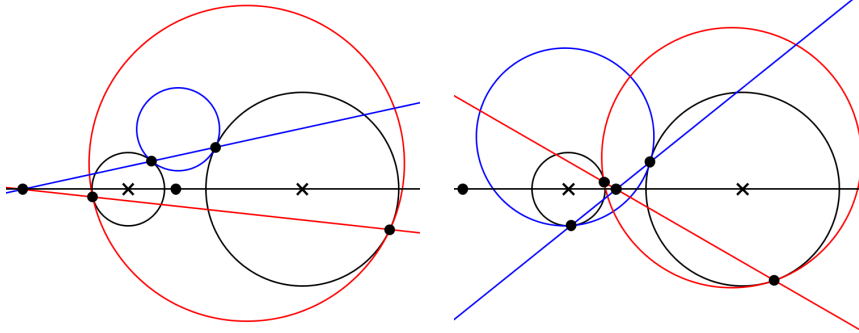
Let  $T'_1 \in T_1T_2 \cap \omega_2$ . From the tangency of  $\omega$  and  $\omega_2$ , we know that  $T_2 \in OO_2$ , so

$$\angle OT_2T_1 = \angle O_2T_2T'_1 = \angle O_2T'_1T_2 \equiv \angle O_2T'_1X.$$

Combining the three equations, we get  $\angle O_1T_1X = \angle O_2T'_1X$ , so  $X$  is a homothetic center of  $\omega_1$  and  $\omega_2$ . ■

*Proof 2.* Using the same case and the same notations as in the previous proof, the tangent point  $T_1$  is the insimilicenter of the tangent circles  $\omega$  and  $\omega_1$ . Similarly,  $T_2$  is the insimilicenter of  $\omega$  and  $\omega_2$ . Therefore, by [Monge-d'Alembert Theorem](#),  $T_1T_2$  passes through the exsimilicenter of  $\omega_1$  and  $\omega_2$ . ■

*Remark.* The line  $T_1T_2$  passes through the external homothetic center of  $\omega_1$  and  $\omega_2$  when  $\omega$  is either internally or externally tangent to both  $\omega_1$  and  $\omega_2$ . Otherwise, when  $\omega$  is internally tangent to one of  $\omega_1$  and  $\omega_2$  and externally tangent to the other one, then the line  $T_1T_2$  passes through the internal homothetic center of  $\omega_1$  and  $\omega_2$ .




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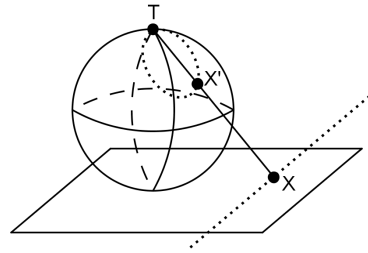
**Related problems:** 40, 46, 53 and 56.

# Chapter 15

## Inversion

Inversion, like homothety, is a function that sends a point to another point. However, before continuing, let's firstly introduce the term "extended plane", because inversion is defined there.

The *extended plane* is a set of all the points in a plane together with one special point that we will call *the point at infinity* ( $P_\infty$ ). We also imagine that all the lines pass through this point. To make this a little bit clearer, let's imagine a sphere sitting on a horizontal plane. Let the top-most point of the sphere be  $T$ . Then, for every point  $X$  on the plane, the line  $TX$  will intersect the sphere at a unique point  $X'$ . If we move a point  $X$  on a line, the image points  $X'$  on the sphere will make a circle. However, as we go towards infinity on *any* line on the plane, the image circle will pass through the *same* image, i.e. the point  $T$ . Thus, in our scenario, it is OK to imagine that all the lines pass through the same point at infinity.



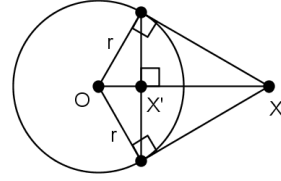
Now, back to the formal definition of inversion. Inversion with center  $O$  and radius  $r$  is a function on the extended plane that sends a point  $X$  to a point  $X'$  on the ray  $OX$ , such that  $\overline{OX} \cdot \overline{OX'} = r^2$ . If  $X$  is the center of inversion, then it is sent to the point at infinity and vice versa. The circle with center  $O$  and radius  $r$  is called the *circle of inversion*. From the definition, we can easily check that  $(X')' = X$ .

$$\mathcal{I}_{O,r} : X \leftrightarrow \begin{cases} X' \in OX, \overline{OX} \cdot \overline{OX'} = r^2 & O \neq X \neq P_\infty \\ P_\infty & X \equiv O \\ O & X \equiv P_\infty \end{cases}$$

### A point

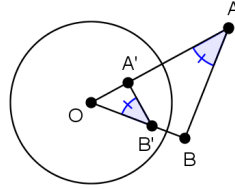
It is easy to see that if a point lies on the circle of inversion, since  $\overline{OX} = r$ , the image will be the same point, i.e.  $X' \equiv X$ . If a point is inside the circle of inversion, then its image will be outside and vice versa. How can we construct those images?

Well, if the point  $X$  is outside the circle of inversion, we draw the tangents from  $X$  to the circle of inversion. The image  $X'$  is found as the intersection of the line connecting the tangent points and the line  $OX$ . It can be easily checked, by similarity of triangles, that the equation in the definition is satisfied. Since  $(X')' = X$ , it can be easily figured out what the image is if  $X$  is inside the circle of inversion.



## Two points

Let  $A'$  and  $B'$  be the images of the points  $A$  and  $B$  under inversion with center  $O$  and radius  $r$ .



Then,  $\overline{OA} \cdot \overline{OA'} = r^2 = \overline{OB} \cdot \overline{OB'}$ .

$$\therefore \frac{\overline{OA}}{\overline{OB}} = \frac{\overline{OB'}}{\overline{OA'}}$$

Keeping in mind that  $O - A - A'$  and  $O - B - B'$  are collinear, i.e.  $\angle AOB = \angle A'OB'$ , we get that  $\triangle OAB \sim \triangle OB'A'$ . From this similarity, we conclude two important properties that will be further used when solving problems:

**Property 15.1.** Let  $A'$  and  $B'$  be the images of the points  $A$  and  $B$  under inversion with center  $O$  and radius  $r$ . Then,

$$\angle OB'A' = \angle OAB \quad (15.1)$$

**Property 15.2.** Let  $A'$  and  $B'$  be the images of the points  $A$  and  $B$  under inversion with center  $O$  and radius  $r$ . Then,

$$\overline{A'B'} = \overline{AB} \cdot \frac{r^2}{\overline{OA} \cdot \overline{OB}} \quad (15.2)$$

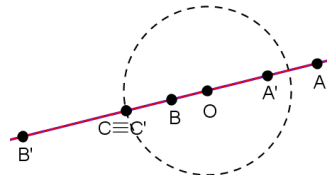
*Proof.* From the similarity  $\triangle OAB \sim \triangle OB'A'$ , we get

$$\frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{OA'}}{\overline{OB}} = \frac{\overline{OA'} \cdot \overline{OA}}{\overline{OB} \cdot \overline{OA}} = \frac{r^2}{\overline{OA} \cdot \overline{OB}} \quad \blacksquare$$

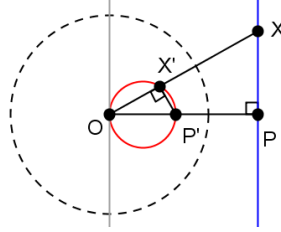
## A line

A line that passes through the center is sent to itself. (Each point is not sent to itself, obviously, but the line as a figure is sent to itself.)

Let's see what happens when a line doesn't pass through the center. Let  $P$  be the foot of the



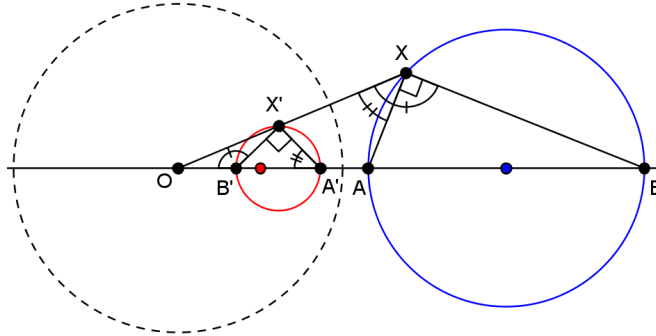
perpendicular from the center  $O$  to the line, with inverse  $P'$ , and let  $X$  be any point on the line, with inverse  $X'$ . Then, by [Property 15.1](#),  $\angle OX'P' = \angle OPX = 90^\circ$ . Therefore,  $X'$  lies on a circle with diameter  $OP'$ . In conclusion, a line that doesn't pass through the center is sent to a circle through the center. Moreover, this circle is tangent to the line through  $O$  parallel to the original line.



## A circle

From the previous case, it's obvious that a circle that passes through the center is sent to a line.

Let's see what happens when a circle doesn't pass through the center. Let  $A$  and  $B$  be the points on the original circle that are closest and furthest, respectively, to the center of inversion. Then  $AB$  is diameter of the original circle. Let  $A'$  and  $B'$  be the images of  $A$  and  $B$ , respectively.



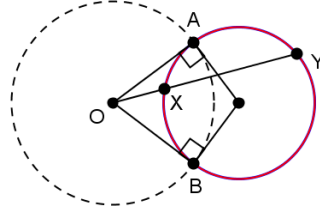
Let  $X$  be another point on the original circle. Then  $\angle OXB - \angle OXA = \angle AXB = 90^\circ$ . By [Property 15.1](#), we get

$$\begin{aligned} \angle OB'X' - \angle OA'X' &= 90^\circ \quad [\angle OB'X' \text{ is an exterior angle for } \triangle A'B'X'] \\ \angle B'A'X' + \angle A'X'B' - \angle OA'X' &= 90^\circ \quad [\angle B'A'X' \equiv \angle OA'X'] \\ \angle A'X'B' &= 90^\circ \end{aligned}$$

So,  $X'$  lies on a circle with diameter  $A'B'$ . In conclusion, a circle that doesn't pass through the center of inversion is sent to a circle. Moreover, the center of the original circle and the center of the image circle are collinear with the center of inversion. However, the center of the original circle *is not sent* to the center of the image circle.

In the case when the original circle is outside the circle of inversion, the image circle is inside and vice versa. When the original circle intersects the circle of inversion, it shares two common points with the image circle (the ones that are on the circle of inversion). So, is it possible, under any conditions, that

a circle is sent to itself? Let's assume it is and see if we can understand the conditions when that happens. Let  $\omega$  be a circle that intersects the circle of inversion at  $A$  and  $B$ . Let  $X$  be any other point on  $\omega$  and let  $Y$  be the second intersection of  $OX$  and  $\omega$ . Since we assumed that  $\omega$  is sent to itself under the inversion,  $Y$  must be the image of  $X$ . Therefore,  $\overline{OX} \cdot \overline{OY} = r^2$ . But  $A$  lies on the circle of inversion, so  $\overline{OA} = r$ . Therefore,  $\overline{OX} \cdot \overline{OY} = \overline{OA}^2$ , so by the [Secant-Tangent Theorem](#), we get that  $OA$  is tangent to  $\omega$ , i.e. the circle of inversion and  $\omega$  are orthogonal. In conclusion, a circle orthogonal to the circle of inversion is sent to itself.

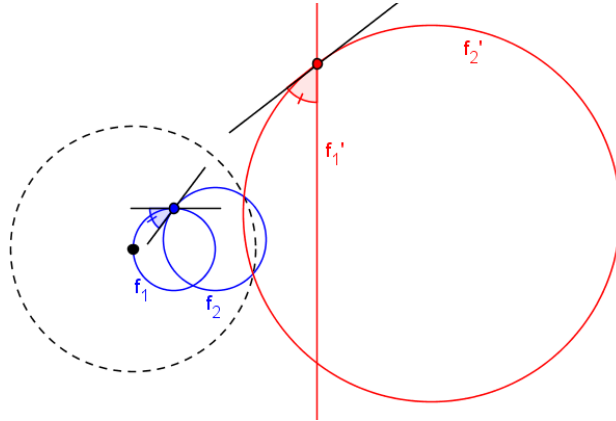


**Property 15.3.** Let  $f_1$  and  $f_2$  be two figures (line or circle). Let  $\mathcal{J} : f_1 \leftrightarrow f'_1$  and  $\mathcal{J} : f_2 \leftrightarrow f'_2$ . Then, inversion preserves angles between figures<sup>1</sup>, i.e.

$$\angle(f_1, f_2) = \angle(f'_1, f'_2)$$

As a consequence,  $f_1$  is tangent to  $f_2$  if and only if  $f'_1$  is tangent to  $f'_2$ .

*Proof.* This can be proven using [Property 15.1](#) in the intersection point. ■



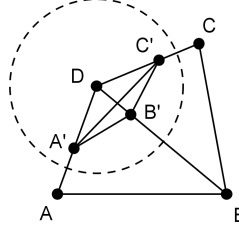
## Summary

1. A point on the circle of inversion is sent to itself.
2. A line passing through the center is sent to itself.
3. A line not passing through the center is sent to a circle through the center.
4. A circle passing through the center is sent to a line.
5. A circle not passing through the center is sent to a circle.

5.1. A circle orthogonal to the circle of inversion is sent to itself.

<sup>1</sup>The angle between two circles is defined as the angle between the tangents to the circles at a point of intersection. Analogously, the angle between a circle and a line is defined as the angle between the line and the tangent to the circle at a point of intersection with the line.

**Example 15.1** (Ptolemy's Theorem). Let  $ABCD$  be a quadrilateral. Prove that  $\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{BD}$  and that equality holds iff  $ABCD$  is a cyclic quadrilateral.



*Proof.* Let's invert with center  $D$  and any radius  $r$ . Let the images of  $A$ ,  $B$  and  $C$  be  $A'$ ,  $B'$  and  $C'$ , respectively. By the [Triangle Inequality](#) for  $\triangle A'B'C'$ , we have  $\overline{A'B'} + \overline{B'C'} \geq \overline{A'C'}$ . By [Property 15.2](#), this is equivalent to

$$\overline{AB} \cdot \frac{r^2}{\overline{DA} \cdot \overline{DB}} + \overline{BC} \cdot \frac{r^2}{\overline{DB} \cdot \overline{DC}} \geq \overline{AC} \cdot \frac{r^2}{\overline{DA} \cdot \overline{DC}}.$$

Multiplying by  $\overline{DA} \cdot \overline{DB} \cdot \overline{DC}$  and dividing by  $r^2$  on both sides, we get:

$$\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{BD}.$$

Equality holds iff the points  $A'$ ,  $B'$  and  $C'$  are collinear, i.e. iff the points  $A$ ,  $B$  and  $C$  are concyclic with the center of inversion  $D$ . ■

**Example 15.2** (IMO 1996/2). Let  $P$  be a point inside triangle  $ABC$  such that  $\angle APB - \angle ACB = \angle APC - \angle ABC$ . Let  $D$  and  $E$  be the incenters of  $\triangle APB$  and  $\triangle APC$ , respectively. Show that  $AP$ ,  $BD$  and  $CE$  are concurrent.

*Proof.* Here, we see that there are many angles in the condition that are in the form  $\angle XAY$  with fixed  $A$ . That's why we will try to invert through  $A$ .

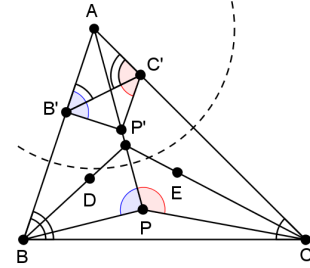
By [Property 15.1](#), the condition now becomes

$$\angle AB'P' - \angle AB'C' = \angle AC'P' - \angle AC'B'$$

$$\angle P'B'C' = \angle P'C'B'$$

$$\therefore \overline{P'B'} = \overline{P'C'}$$

(\*)



If we want to prove that the angle bisectors  $BD$  and  $CE$  intersect the line segment  $AP$  at the same point, then by the [Angle Bisector Theorem](#), we need to prove that

$$\frac{\overline{AB}}{\overline{BP}} = \frac{\overline{AC}}{\overline{CP}}$$

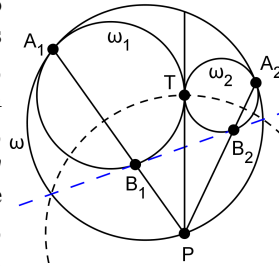
From the similarity of the triangles  $\triangle APB \sim \triangle AB'P'$  and  $\triangle APC \sim \triangle AP'C'$  we get

$$\frac{\overline{AB}}{\overline{BP}} = \frac{\overline{AP'}}{\overline{P'B'}} \stackrel{(*)}{=} \frac{\overline{AP'}}{\overline{P'C'}} = \frac{\overline{AC}}{\overline{CP}}$$

■

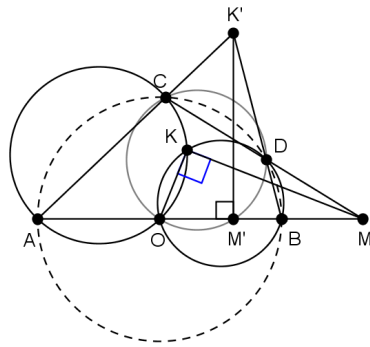
**Example 15.3.** Two circles  $\omega_1$  and  $\omega_2$  touch each other externally at  $T$ . They also touch a circle  $\omega$  internally at  $A_1$  and  $A_2$ , respectively. Let  $P$  be one point of intersection of  $\omega$  with the common tangent to  $\omega_1$  and  $\omega_2$  at  $T$ . The line  $PA_1$  meets  $\omega_1$  again at  $B_1$  and the line  $PA_2$  meets  $\omega_2$  again at  $B_2$ . Prove that  $B_1B_2$  is a common tangent to  $\omega_1$  and  $\omega_2$ .

*Proof.* In many problems, we tend to invert through a "busy" point, i.e. a point through which many lines or circles pass. Consider the inversion  $\mathcal{J}(P, PT)$ . In this way, since  $PT$  is tangent to  $\omega_1$ , i.e.  $\omega_1$  is orthogonal to the circle of inversion,  $\mathcal{J} : \omega_1 \leftrightarrow \omega_1$ . Let's see what is  $A_1$  sent to. The image of  $A_1$  must be on  $PA_1$ . Also, since  $A_1$  lies on  $\omega_1$ , then the image of  $A_1$  must lie on the image of  $\omega_1$ . Therefore,  $\mathcal{J} : A_1 \leftrightarrow B_1$ . Similarly,  $\mathcal{J} : \omega_2 \leftrightarrow \omega_2$  and  $\mathcal{J} : A_2 \leftrightarrow B_2$ . Now, the circle  $\omega$  passes through the center of inversion  $P$ , so it will be sent to a line. The points  $A_1$  and  $A_2$  lie on this circle, so their images will lie on the image line. Therefore,  $\mathcal{J} : \omega \leftrightarrow B_1B_2$ . Finally, since  $\omega$  is tangent to  $\omega_1$  and  $\omega_2$ , then by [Property 15.3](#), its image will be tangent to their images, i.e.  $B_1B_2$  will be tangent to  $\omega_1$  and  $\omega_2$ . ■



**Example 15.4.** A semicircle with diameter  $AB$  is centered at  $O$ . A line intersects the semicircle at  $C$  and  $D$  and the line  $AB$  at  $M$ , such that  $\overline{MB} < \overline{MA}$  and  $\overline{MD} < \overline{MC}$ . Let  $K$  be the second point of intersection of the circumcircles of  $\triangle AOC$  and  $\triangle BOD$ . Prove that  $\angle MKO = 90^\circ$ .

*Proof.* The point  $O$  is one of the busy points in this diagram and also the angle  $\angle MKO$  which is of interest for us is in the form  $\angle OXY$ , so it is wise to try to invert through  $O$ . Consider the inversion  $\mathcal{J}(O, OA)$ . By [Property 15.1](#), we need to prove that  $\angle OM'K' = 90^\circ$ . The points  $A, B, C$  and  $D$  are sent to themselves since they lie on the circle of inversion. The circles  $(OAC)$  and  $(OBD)$  pass through the center of inversion, so they are sent to the lines  $AC$  and  $BD$ , respectively. Since  $K$  lies on these circles, then  $K'$  must lie on their images, i.e.  $AC \cap BD = K'$ . The line  $AB$  passes through the center of inversion, so it is sent to itself. The line  $CD$  doesn't pass through the center, so it is sent to the circle  $(OCD)$ . Since  $M$  lies on the lines  $AB$  and  $CD$ , its image  $M'$  will lie on their images, i.e.  $M' = AB \cap (OCD)$ . Now, since  $AB$  is the diameter of  $(ABCD)$ ,  $C$  and  $D$  are feet of the altitudes in  $\triangle ABK'$  and  $O$  is a midpoint in the same triangle. So,  $(OCD)$  is the nine point circle of  $\triangle ABK'$ . Since  $M'$  lies on  $AB$  and the nine point circle, then it must be the feet of the altitude from  $K'$  to  $AB$  and therefore  $\angle OM'K' = 90^\circ$ . ■

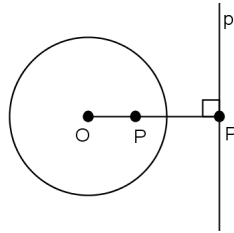


**Related problems:** 18, 22 and 49. [Ptolemy's Theorem](#): 17 and 42.

# Chapter 16

## Pole & Polar

Let the image of the point  $P$  under inversion with respect to the circle with center  $O$  and radius  $r$  be  $P'$ . The *polar* of  $P$  is the line  $p$  perpendicular to the line  $OP$  at  $P'$ . In this case, the point  $P$  is called the *pole* of  $p$ .



We will now present some properties that will be useful when solving problems.

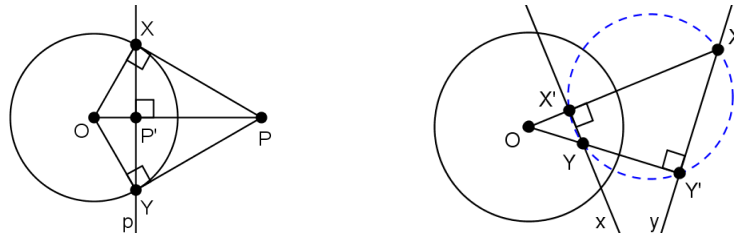
**Property 16.1.** If  $P$  is outside the circle  $\omega$ , and  $X$  and  $Y$  are points on  $\omega$ , such that  $PX$  and  $PY$  are tangents, then the polar  $p$  of  $P$  is the line  $XY$ .

*Proof.* Recall that the image point  $P'$  can be found as the intersection of  $XY$  and  $OP$ , i.e.  $P' \in XY$ . By symmetry,  $XY \perp OP$ . Therefore, by the definition of polar,  $p \equiv XY$ . ■

**Property 16.2** (La Hire's Theorem). Let  $x$  and  $y$  be the polars of  $X$  and  $Y$ , respectively. Then,  $X \in y \iff Y \in x$ .

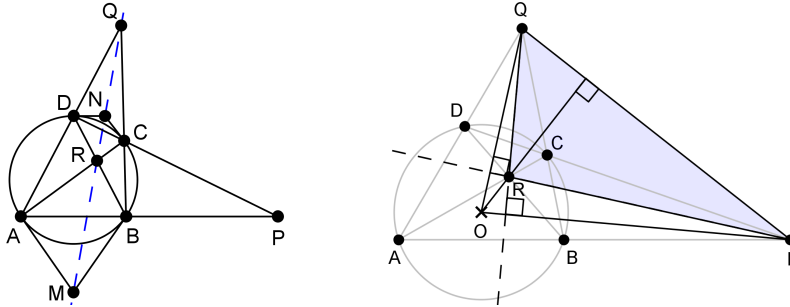
*Proof.* Let  $X'$  and  $Y'$  be the images of  $X$  and  $Y$  under the inversion. Then  $\overline{OX} \cdot \overline{OX'} = r^2 = \overline{OY} \cdot \overline{OY'}$ , which means that the points  $X, Y, X'$  and  $Y'$  are concyclic. Therefore,

$$X \in y \iff \angle XY'Y = 90^\circ \iff \angle XX'Y = 90^\circ \iff Y \in x \quad \blacksquare$$



**Property 16.3** (Brocard's Theorem). Let  $ABCD$  be a cyclic quadrilateral centered at  $O$ . Let  $AB \cap CD = P$ ,  $BC \cap AD = Q$  and  $AC \cap BD = R$ . Then, the polar of  $P$  is  $QR$ . Moreover, the triangle  $\triangle PQR$  is autopolar and  $O$  is the orthocenter of  $\triangle PQR$ .

*Proof.* Let the intersection of the tangents at  $A$  and  $B$  be  $M$ . Then,  $AB \equiv m$ . Let the intersection of the tangent at  $C$  and  $D$  be  $N$ . Then,  $CD \equiv n$ . By La Hire's Theorem, since  $P \in m$  and  $P \in n$ , then  $M \in p$  and  $N \in p$ , i.e.  $MN \equiv p$ . By applying Pascal's Theorem on the points  $AACBBD$ , we get that  $M - R - Q$  are collinear. By applying Pascal's Theorem on the points  $CCADDB$ , we get that the points  $N - R - Q$  are collinear. Therefore  $QR \equiv MN \equiv p$ .

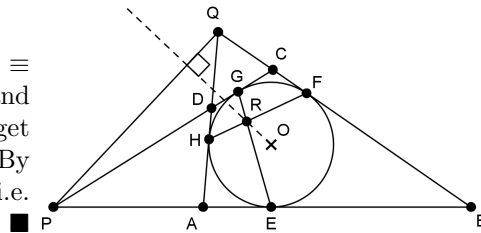


Similarly, we can get that  $PR \equiv q$ . Also, since  $R$  lies on the polars of  $P$  and  $Q$ , then the polar of  $R$ ,  $r \equiv PQ$ . Therefore, the triangle  $\triangle PQR$  is autopolar, i.e. the polar of each of the vertices is the opposite side. So, by the definition of polar, it also follows that  $O$  is the orthocenter of  $\triangle PQR$ . ■

We will now solve a few examples to see how these properties of polars can be used in problems. In these examples, we will use lowercase letters to denote the polars of the points in uppercase (e.g.  $p$  is the polar of  $P$ ).

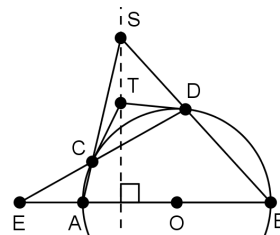
**Example 16.1.** The quadrilateral  $ABCD$  has an inscribed circle  $\omega$  which is tangent to the sides  $AB$ ,  $BC$ ,  $CD$  and  $DA$  at  $E$ ,  $F$ ,  $G$  and  $H$ , respectively. Let  $AB \cap CD = P$ ,  $AD \cap BC = Q$  and  $EG \cap FH = R$ . If  $O$  is the center of  $\omega$ , then prove that  $OR \perp PQ$ .

*Proof.* By [Property 16.1](#), we get  $p \equiv EG$  and  $q \equiv FH$ . Since  $R \in p$  and  $R \in q$ , by [La Hire's Theorem](#), we get  $P \in r$  and  $Q \in r$ , i.e.  $r \equiv PQ$ . By the definition of polar,  $OR \perp r$ , i.e.  $OR \perp PQ$ . ■



**Example 16.2.** Let  $AB$  be a diameter of a semicircle.  $C$  and  $D$  are two points on the semicircle such that  $\widehat{AC} < \widehat{AD}$ . The tangents to the semicircle at  $C$  and  $D$  meet at  $T$ . If  $S = AC \cap BD$ , prove that  $ST \perp AB$ .

*Proof.* Let  $E = CD \cap AB$ . By [Property 16.1](#),  $t \equiv CD$ . Since  $E \in t$ , by [La Hire's Theorem](#),  $T \in e$ . On the other hand, by [Property 16.3](#),  $S \in e$ . Therefore,  $ST \equiv e$ , so by the definition of polar  $OE \perp ST$ , i.e.  $AB \perp ST$ . ■

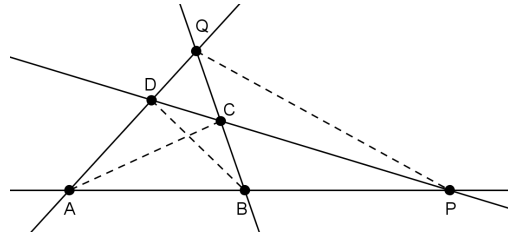


**Related problems:** 28 and 34.

## Chapter 17

# Complete quadrilateral

A **complete quadrilateral** is a system of four lines (no three of which pass through the same point) and the six points of intersection of these lines.



Among the six points of a complete quadrilateral there are three pairs of points that are not already connected by lines. The line segments connecting these pairs are called *diagonals* of the complete quadrilateral.

In all of the following properties, let  $ABCD$  be a quadrilateral such that the rays  $AB$  and  $DC$  intersect at  $P$  and the rays  $BC$  and  $AD$  intersect at  $Q$ .

By taking any three of the four lines of a complete quadrilateral, we can get four triangles. For the complete quadrilateral  $ABCDPQ$ , those triangles are  $\triangle ABQ$ ,  $\triangle BCP$ ,  $\triangle CDQ$  and  $\triangle DAP$ .

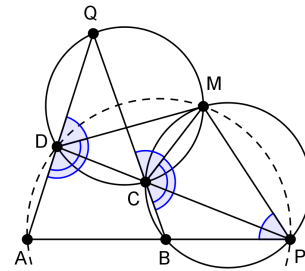
**Property 17.1** (Miquel Point). The circumcircles of the four triangles mentioned above pass through a common point, called the Miquel point of the quadrilateral.

*Proof 1.* Observe that this is a different wording of [Example 8.3](#) in the direction when it is given that the points are collinear, which we already proved. ■

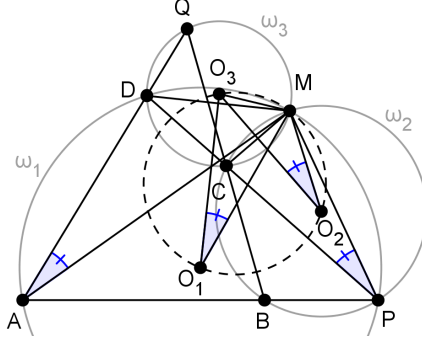
*Proof 2.* Let  $M$  be the second intersection of  $(BCP)$  and  $(CDQ)$ . Then,

$$\begin{aligned}\angle APM &\equiv \angle BPM = 180^\circ - \angle BCM = \\ &= \angle QCM = \angle QDM = \\ &= 180^\circ - \angle ADM\end{aligned}$$

Therefore,  $ADMP$  is cyclic, i.e.  $M \in (DAP)$ . In exactly the same manner, we can prove that  $M \in (ABQ)$ . ■



**Property 17.2.** The circumcenters of the four triangles mentioned above, and the Miquel point are concyclic.



*Proof.* Let  $M$  be the Miquel point of  $ABCDPQ$ . Let the circles  $(DAPM)$ ,  $(BCMP)$ ,  $(CDQM)$  and  $(ABMQ)$  be  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  and  $\omega_4$ , respectively and let  $O_i$  be the center of  $\omega_i$ . We will firstly prove that  $O_1O_2MO_3$  is cyclic.

$MD$  is the radical axis of  $\omega_1$  and  $\omega_3$ , so  $O_1O_3$  is the bisector of  $MD$  and therefore the angle bisector of  $\angle MO_1D$ . Similarly,  $O_2O_3$  is the bisector of  $MC$  and the angle bisector of  $\angle MO_2C$ .

$$\angle MO_1O_3 = \frac{\angle MO_1D}{2} \stackrel{\omega_1}{=} \angle MAD \stackrel{\omega_1}{=} \angle MPD \equiv \angle MPC \stackrel{\omega_2}{=} \frac{\angle MO_2C}{2} = \angle MO_2O_3$$

Therefore, the quadrilateral  $O_1O_2MO_3$  is cyclic. Similarly,  $O_2MO_3O_4$  is cyclic. ■

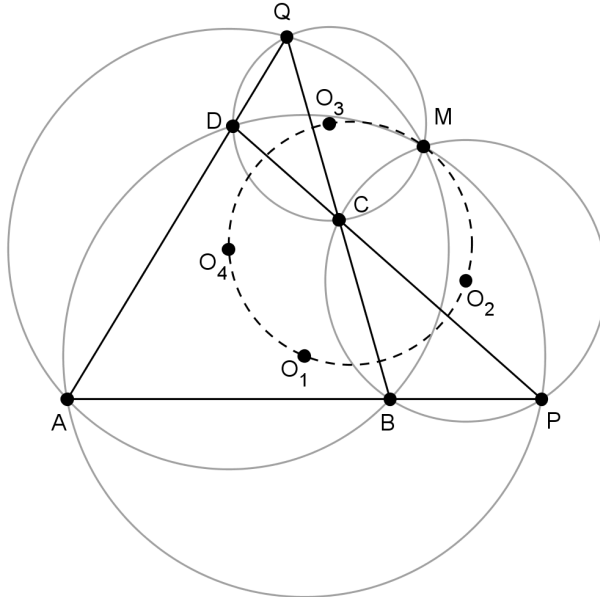
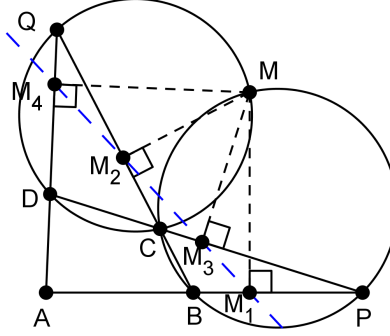


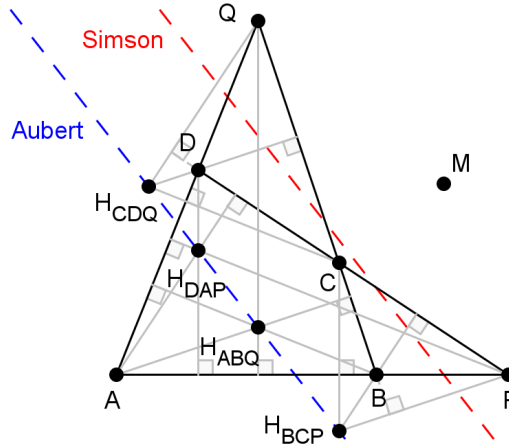
Figure 17.1: The circumcircles of  $\triangle ABQ$ ,  $\triangle BCP$ ,  $\triangle CDQ$  and  $\triangle DAP$  pass through the Miquel point  $M$ . Their circumcenters and  $M$  are concyclic.

**Property 17.3** (Simson's Line). The feet of the perpendiculars from the Miquel point to the sides of the complete quadrilateral lie on a line, called the *Simson's line* of the complete quadrilateral.



*Proof.* Let the feet of the perpendiculars from  $M$  to  $AB$ ,  $BC$ ,  $CD$  and  $DA$  be  $M_1$ ,  $M_2$ ,  $M_3$  and  $M_4$ . Using the [Simson Line Theorem](#) for  $\triangle PBC$  and the point  $M$  which lies on its circumcircle, we get that  $M_1$ ,  $M_2$  and  $M_3$  are collinear. Similarly, by using the [Simson Line Theorem](#) for  $\triangle CQD$  and the point  $M$ , we get that the points  $M_2$ ,  $M_3$  and  $M_4$  are collinear. ■

**Property 17.4** (Aubert's Line). The orthocenters of the four triangles mentioned above lie on a line, called the *Aubert's line*, which is parallel to the Simson's line of the complete quadrilateral.



*Proof.* By [Example 8.10](#), we know that the Simson line from  $M$  bisects the line segment  $MH$ , where  $H$  is the orthocenter of the triangle. Therefore, the homothety  $\mathcal{X}_{M,2}$  sends the Simson line of a triangle, to a line through its orthocenter which is parallel to the Simson line. Since in [Property 17.3](#), we proved that the Simson lines of all four triangles coincide, we get that the orthocenters of all four triangles lie on a line parallel to Simson's line. ■

**Property 17.5.** The three circles with diameters the diagonals of the complete quadrilateral have a common chord.

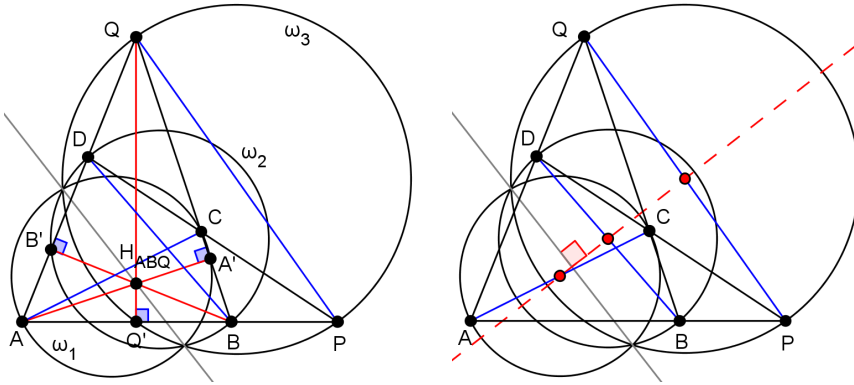
*Proof.* We will prove that the common chord lies on Aubert's line. Let  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  be the circles with diameters the diagonals  $AC$ ,  $BD$  and  $PQ$ , respectively. Let  $\omega_1 \cap BQ = A'$ ,  $\omega_2 \cap AQ = B'$  and  $\omega_3 \cap AB = Q'$ . Since the inscribed angles over the diameter are right angles, we get that  $AA' \perp BQ$ ,  $BB' \perp AQ$  and  $QQ' \perp AB$ . Therefore,  $AA'$ ,  $BB'$  and  $QQ'$  pass through the orthocenter of  $\triangle ABQ$ ,  $H_{ABQ}$ . From [Property 6.7](#), we know that

$$\overline{AH_{ABQ}} \cdot \overline{H_{ABQ}A'} = \overline{BH_{ABQ}} \cdot \overline{H_{ABQ}B'} = \overline{QH_{ABQ}} \cdot \overline{H_{ABQ}Q'},$$

which is equivalent to

$$\text{pow}(H_{ABQ}, \omega_1) = \text{pow}(H_{ABQ}, \omega_2) = \text{pow}(H_{ABQ}, \omega_3).$$

Therefore,  $H_{ABQ}$  has equal powers to all three circles. Similarly, we can get that the other three orthocenters also have equal powers to the three circles. Therefore, there isn't a single radical center of the three circles, but all the points on the line containing the orthocenters have equal powers to all three circles. Therefore, Aubert's line is the common chord of the circles with diameters the diagonals of the complete quadrilateral. ■



**Property 17.6 (Gauss' Line).** The midpoints of the diagonals of the complete quadrilateral lie on a line, called the *Gauss' line*, which is perpendicular to Simson's and Aubert's line.

*Proof.* Since the circles  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  defined in [Property 17.5](#) have a common chord and their centers are the midpoints of the diagonals of the complete quadrilateral, we can conclude that the midpoints of the diagonals are collinear. We also know that the line joining the centers of two circles is perpendicular to their common chord ([Property 10.3](#)), thus Gauss's Line is perpendicular to Aubert's Line. ■

## 17.1 Cyclic Quadrilateral

**Property 17.7.** The Miquel point of  $ABCD$  lies on the line  $PQ$  if and only if  $ABCD$  is cyclic.

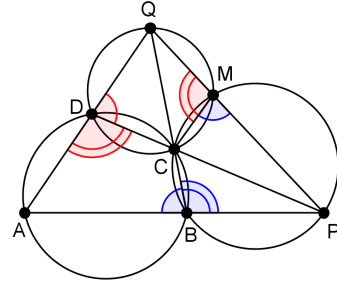
*Proof.* Let  $M$  be the Miquel point of the quadrilateral  $ABCD$ . Then  $MCBP$  and  $MQDC$  are cyclic quadrilaterals.

$$\angle PMC = 180^\circ - \angle PBC = \angle ABC$$

$$\angle QMC = 180^\circ - \angle QDC = \angle ADC$$

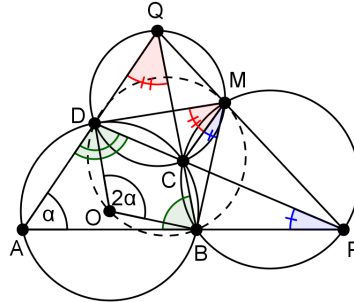
$$\therefore \angle PMC + \angle QMC = \angle ABC + \angle ADC$$

The Miquel point  $M$  lies on  $PQ$  iff the left-hand side is  $180^\circ$ . The right-hand side is  $180^\circ$  iff  $ABCD$  is a cyclic quadrilateral. ■



**Property 17.8.** Let  $ABCD$  be a cyclic quadrilateral, inscribed in a circle  $\omega$  centered at  $O$ . Let the intersection of the diagonals  $AC$  and  $BD$  be  $R$ . Let  $M$  be the Miquel point of  $ABCD$ . Then,

- The point  $M$  lies on the circumcircles of  $\triangle AOC$  and  $\triangle BOD$
- The point  $M$  is the image of the point  $R$  under the inversion with respect to  $\omega$
- The point  $M$  lies on the line  $OR$
- The line  $O - R - M$  bisects  $\angle AMC$  and  $\angle BMD$
- The line  $O - R - M$  is perpendicular to  $PQ$
- $PQ$  is the polar of  $R$



*Proof.*

$$\angle BMC \stackrel{(MCBP)}{=} \angle BPC \equiv \angle APD \stackrel{\triangle ADP}{=} 180^\circ - \alpha - \delta$$

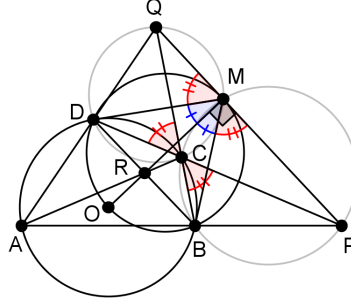
$$\angle CMD \stackrel{(MCDQ)}{=} \angle CQD \equiv \angle BQA \stackrel{\triangle ABQ}{=} 180^\circ - \alpha - \beta$$

$$\therefore \angle BMD = \angle BMC + \angle CMD = 360^\circ - (\beta + \delta) - 2\alpha = 180^\circ - 2\alpha$$

$$\therefore \angle BOD + \angle BMD = 2\alpha + 180^\circ - 2\alpha = 180^\circ$$

So,  $M \in (BOD)$ . In a similar way, we can prove that  $M \in (AOC)$ .

Now, let's consider the inversion with respect to  $\omega$ . The points  $A, B, C$  and  $D$  are sent to themselves. Therefore, the line  $AC$  is sent to the circle  $(OAC)$  and the line  $BD$  is sent to the circle  $(OBD)$ . The lines  $AC$  and  $BD$  intersect at  $R$ , so their images will intersect at the image of  $R$ . Since we proved that  $M$  lies on  $(OAC)$  and  $(OBD)$ , we can conclude that  $M$  is the image of  $R$ , i.e.  $R' \equiv M$ . Since the center of inversion, the original and the image are collinear, we can also conclude that the point  $M$  lies on the line  $OR$ .



From the cyclic quadrilateral  $OBMD$ , since  $\overline{OB} = \overline{OD}$ , we get that

$$\angle OMB = \angle OMD.$$

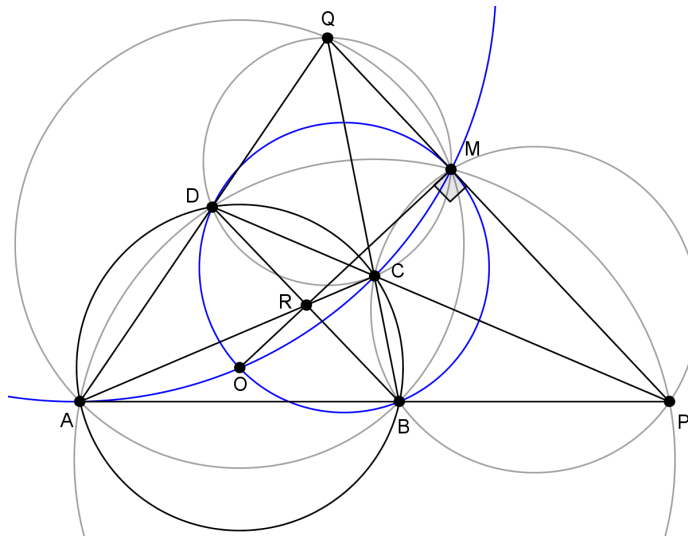
Therefore,  $OM$  bisects  $\angle BMD$ . Similarly, we get that  $OM$  bisects  $\angle AMC$ . On the other hand,

$$\angle BMP = \angle BCP = \angle DCQ = \angle DMQ.$$

Adding these last two equations side by side, we get that

$$\angle OMP = \angle OMB + \angle BMP = \angle OMD + \angle DMQ = \angle OMQ.$$

Since  $ABCD$  is cyclic, by [Property 17.7](#), we know that  $M \in PQ$ . Therefore,  $OM \perp PQ$ . By the definition of polar, since  $R' \equiv M$  and  $OM \perp PQ$ , we get that  $PQ$  is the polar of  $R$ . ■



**Property 17.9.** Let  $ABCD$  be a cyclic quadrilateral that is inscribed in a circle  $\omega$  centered at  $O$ . Let the intersection of the diagonals  $AC$  and  $BD$  be  $R$ . Let  $M$  be the Miquel point of  $ABCD$ . Let  $P'$  and  $Q'$  be the images of  $P$  and  $Q$ , respectively, under inversion with respect to  $\omega$ . Then,

- The points  $P - R - Q'$  and  $Q - R - P'$  are collinear.
- The quadrilaterals  $ABRQ'$ ,  $BCRP'$ ,  $CDQ'R$  and  $DAP'R$  are cyclic.
- The quadrilaterals  $ABP'O$ ,  $BCQ'O$ ,  $CDOP'$  and  $DAOQ'$  are cyclic.
- The quadrilaterals  $AP'CQ$ ,  $BP'DQ$ ,  $AQ'CP$  and  $BQ'DP$  are cyclic.

*Proof.* Let's recall [Brocard's Theorem](#), where we proved that the triangle  $\triangle PQR$  is autopolar and that  $O$  is its orthocenter.

The point  $P'$ , by definition, must lie on the line  $OP$ . On the other hand, it must lie on the polar of  $P$ , which is  $QR$ . Therefore,  $P' = OP \cap QR$ . Similarly,  $Q' = OQ \cap PR$ .

We know, from [Property 17.8](#), that  $\mathcal{J}_\omega : R \leftrightarrow M$ . We also know, from [Property 17.1](#), that the circumcircle of  $\triangle ABQ$  passes through  $M$ . Therefore, the image of the circle  $(ABMQ)$  under inversion with respect to  $\omega$  is the circle  $(ABRQ')$ , i.e. the quadrilateral  $ABRQ'$  is cyclic. Similarly, the quadrilaterals  $BCRP'$ ,  $CDQ'R$  and  $DAP'R$  are cyclic, too.

Since  $AB$  is a line that doesn't pass through  $O$  and  $A, B \in \omega$ , we get  $\mathcal{J}_\omega : AB \leftrightarrow (ABO)$ . Since  $P \in AB$ , we get that  $P' \in (ABO)$ , i.e.  $ABP'O$  is cyclic. Similarly,  $BCQ'O$ ,  $CDOP'$  and  $DAOQ'$  are cyclic, too.

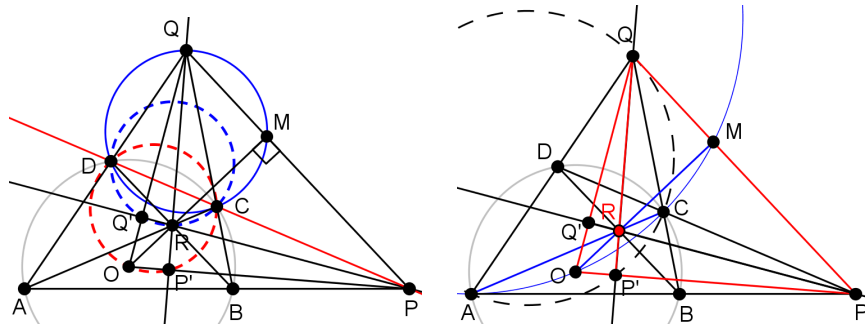
Earlier, in [Property 17.8](#), we proved that  $M$  lies on the circumcircle of  $\triangle AOC$  and also, that  $O - R - M$  are collinear. Therefore, by the [Intersecting Chords Theorem](#) we get

$$\overline{AR} \cdot \overline{RC} = \overline{OR} \cdot \overline{RM}. \quad (1)$$

$O$  is the orthocenter of  $PQR$ , so  $R$  is the orthocenter of  $OPQ$ . Therefore, by [Property 6.7](#), we get

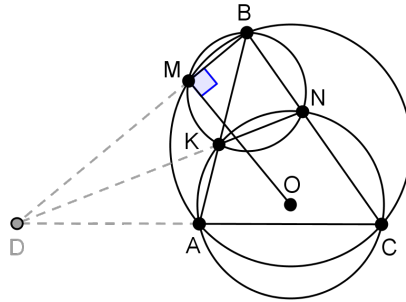
$$\overline{OR} \cdot \overline{RM} = \overline{QR} \cdot \overline{RP'}. \quad (2)$$

By combining (1) and (2), by the converse of the [Intersecting Chords Theorem](#), we get that  $AP'CQ$  is cyclic. Similarly,  $BP'DQ$ ,  $AQ'CP$  and  $BQ'DP$  are cyclic, too. ■



It is very important to learn to recognize these configurations because they show up in many olympiad problems. However, the configuration is not always complete, so sometimes you have to draw additional points, lines or circles in order to come to these "well-known" configurations.

**Example 17.1** (IMO 1985/5). A circle with center  $O$  passes through the vertices  $A$  and  $C$  of the triangle  $ABC$  and intersects the segments  $AB$  and  $BC$  again at distinct points  $K$  and  $N$ , respectively. Let  $M$  be the point of intersection of the circumcircles of triangles  $ABC$  and  $KBN$  (apart from  $B$ ). Prove that  $\angle OMB = 90^\circ$ .



*Proof.* Let  $AC \cap KN = D$ . Let's take a look at the complete quadrilateral  $ACNKBD$ . The triangles  $\triangle ACB$  and  $\triangle KNB$  are two of the four triangles formed by the lines of the complete quadrilateral, so their circumcircles intersect at the [Miquel Point](#) of the complete quadrilateral, i.e.  $M$  is the Miquel point of  $ACNKBD$ . Since  $ACNK$  is cyclic, by [Property 17.7](#),  $M \in BD$ . Finally, by [Property 17.8](#), we get that  $OM \perp BD$ , i.e.  $\angle OMB = 90^\circ$ . ■

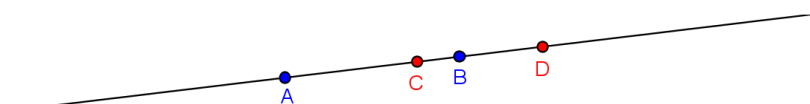
**Related problems:** 45 and 48.

# Chapter 18

## Harmonic Ratio

If  $A, B, C$  and  $D$  are collinear points, then their *cross-ratio* is defined as:

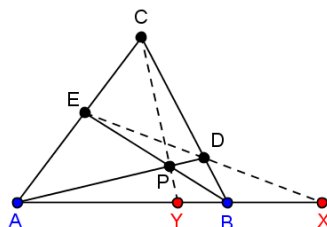
$$(A, B; C, D) = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}.$$



If  $(A, B; C, D) = 1$  and the order of the points on the line is such that the line segments  $AB$  and  $CD$  partially overlap (e.g.  $A - C - B - D$ ), then the ratio is called *harmonic ratio* and the four-tuple  $(ACBD)$  is called a *harmonic division*, or simply *harmonic*. The points  $C$  and  $D$  are *harmonic conjugates* with respect to the points  $A$  and  $B$  and vice versa.

Notice, by the definition, that if  $(ACBD)$  is a harmonic, then  $(DBCA)$  is also harmonic.

**Property 18.1.** Let  $X$  be a point on the extension of the side  $AB$  in  $\triangle ABC$ . A line which passes through  $X$  meets the sides  $BC$  and  $CA$  at points  $D$  and  $E$ , respectively. Let  $P$  be the intersection of  $AD$  and  $BE$ . The line  $CP$  meets  $AB$  at  $Y$ . Then,  $X$  and  $Y$  are harmonic conjugates with respect to the points  $A$  and  $B$ .



*Proof.* We need to prove that  $\frac{\overline{AX}}{\overline{BX}} = \frac{\overline{AY}}{\overline{BY}}$ . By using Menelaus Theorem for  $\triangle ABC$  and the collinear points  $D - E - X$  and Ceva's Theorem for  $\triangle ABC$

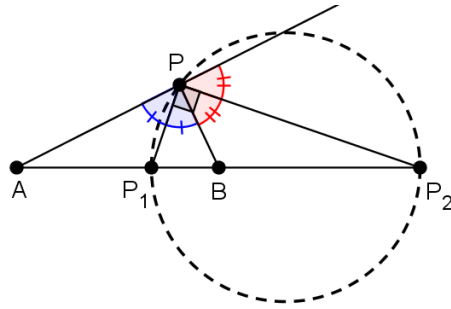
and the concurrent cevians  $AD$ ,  $BE$  and  $CY$ , we get:

$$\frac{\overline{AX}}{\overline{XB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1 = \frac{\overline{AY}}{\overline{YB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}}.$$

By canceling, we get the needed ratio.  $\blacksquare$

**Property 18.2.** Given two points  $A$  and  $B$ , find the locus of the points  $P$  such that

$$\frac{\overline{AP}}{\overline{PB}} = \lambda, \lambda > 0.$$



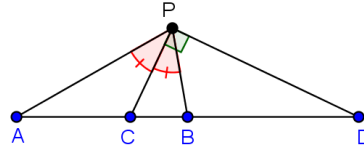
*Proof.* If  $\lambda = 1$ , then the locus of the points  $P$  is the side bisector of  $AB$ . Let's investigate the case when  $\lambda \neq 1$ . WLOG, let  $\lambda > 1$ . Obviously, there is a point  $P_1$  between  $A$  and  $B$  (in this case, closer to  $B$ ) that satisfies the condition. Also, there is another point,  $P_2$ , on the extension of the line (in this case, beyond  $B$ ), that also satisfies the condition. Note that we know how to construct the latter using [Property 18.1](#). Now let  $P$  be a point that doesn't lie on the line  $AB$ , but satisfies the condition. Then,

$$\frac{\overline{AP}}{\overline{PB}} = \lambda = \frac{\overline{AP_1}}{\overline{P_1B}} = \frac{\overline{AP_2}}{\overline{P_2B}},$$

so by the [Angle Bisector Theorem](#), we get that  $PP_1$  is the internal angle bisector of  $\angle APB$ . By the [External Angle Bisector Theorem](#), we get that  $PP_2$  is the external angle bisector of  $\angle APB$ . Therefore  $PP_1 \perp PP_2$ , because  $\angle P_1PP_2 = \frac{1}{2} \cdot 180^\circ = 90^\circ$ , so by [Thales' Theorem](#)  $P$  lies on the circle with diameter  $P_1P_2$ .  $\blacksquare$

**Property 18.3.** Let  $A$ ,  $C$ ,  $B$  and  $D$  be four collinear points lying on a line  $l$  in this order. Let  $P$  be a point not lying on  $l$ . Then, if any two of the following propositions are true, then the third is also true:

1. The division  $(ACBD)$  is harmonic.
2.  $PC$  is the angle bisector of  $\angle APB$ .
3.  $PC \perp PD$ .



*Proof.* This is a direct consequence of the result of [Property 18.2](#). See its proof for details.  $\blacksquare$

**Property 18.4.** Let  $A, C, B$  and  $D$  be four collinear points lying on a line in this order. Then, the division  $(ACBD)$  is harmonic if and only if  $C$  is the image of  $D$  under inversion with respect to the circle with diameter  $AB$ .

*Proof.* Let  $O$  be the midpoint of  $AB$  and  $r = \frac{1}{2}\overline{AB}$ .

$(ACBD)$  is harmonic

$$\iff \frac{\overline{CA}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$$

$$\iff \frac{r + \overline{OC}}{r - \overline{OC}} = \frac{\overline{DO} + r}{\overline{DO} - r}$$

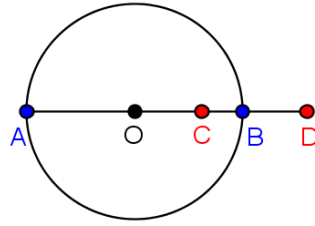
$$\iff (r + \overline{OC}) \cdot (\overline{DO} - r) = (\overline{DO} + r) \cdot (r - \overline{OC})$$

$$\iff r \cdot \overline{OD} - r^2 + \overline{OC} \cdot \overline{OD} - r \cdot \overline{OC} = r \cdot \overline{OD} - \overline{OC} \cdot \overline{OD} + r^2 - r \cdot \overline{OC}$$

$$\iff \overline{OC} \cdot \overline{OD} = r^2$$

$$\iff \mathcal{J}_{O,r} : C \leftrightarrow D$$

■



**Property 18.5.** Let  $A, C, B$  and  $D$  be four collinear points lying on a line in this order. Let  $O$  be the midpoint of  $AB$ . Then, the division  $(ACBD)$  is harmonic if and only if  $\overline{DA} \cdot \overline{DB} = \overline{DC} \cdot \overline{DO}$ .

*Proof.* Let  $r = \frac{1}{2}\overline{AB}$ .

$$\overline{DA} \cdot \overline{DB} = \overline{DC} \cdot \overline{DO}$$

$$\iff (\overline{OD} + r) \cdot (\overline{OD} - r) = (\overline{OD} - \overline{OC}) \cdot \overline{OD}$$

$$\iff \overline{OD}^2 - r^2 = \overline{OD}^2 - \overline{OC} \cdot \overline{OD}$$

$$\iff r^2 = \overline{OC} \cdot \overline{OD}$$

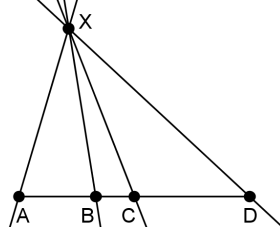
$$\iff \mathcal{J}_{O,r} : C \leftrightarrow D$$

$$\stackrel{\text{Property 18.4}}{\iff} (ACBD) \text{ is harmonic}$$

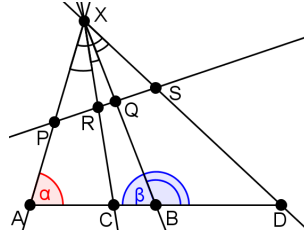
■

## 18.1 Harmonic Pencil

Four points  $A, B, C$  and  $D$ , are given on a line  $l$  in this order. If  $X$  is a point not lying on  $l$ , then the pencil  $X(ABCD)$ , which consists of the four lines  $XA, XB, XC$  and  $XD$ , is harmonic if  $(ABCD)$  is harmonic.



**Property 18.6.** If any pencil  $X(ABCD)$  is intersected with another line at points  $P, Q, R$  and  $S$ , then  $(A, B; C, D) = (P, Q; R, S)$ . As a consequence, if a harmonic pencil is intersected with a line, the intersection points form a harmonic division.



*Proof.* WLOG let  $A, C, B$  and  $D$  (and  $P, R, Q$  and  $S$ ) be collinear in this order. Let  $\angle XAC = \alpha$  and  $\angle XBC = \beta$ . By using the [Law of Sines](#) in the triangles  $\triangle CXA, \triangle CXB, \triangle DXA$  and  $\triangle DXB$ , we get:

$$\begin{aligned} \frac{\overline{CA}}{\sin(\angle CXA)} &= \frac{\overline{CX}}{\sin(\alpha)} \\ \frac{\overline{CB}}{\sin(\angle CXB)} &= \frac{\overline{CX}}{\sin(\beta)} \\ \frac{\overline{DA}}{\sin(\angle DXA)} &= \frac{\overline{DX}}{\sin(\alpha)} \\ \frac{\overline{DB}}{\sin(\angle DXB)} &= \frac{\overline{DX}}{\sin(180^\circ - \beta)}. \end{aligned}$$

By rearranging and using that  $\sin(\beta) = \sin(180^\circ - \beta)$ , we get

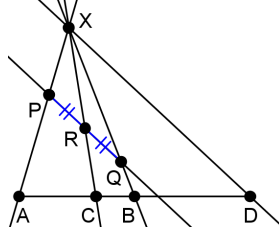
$$(A, B; C, D) = \frac{\sin(\angle CXA)}{\sin(\angle CXB)} : \frac{\sin(\angle DXA)}{\sin(\angle DXB)} \quad (18.1)$$

Since  $\angle CXA \equiv \angle RXP, \angle CXB \equiv \angle RXQ, \angle DXA \equiv \angle SXP$  and  $\angle DXB \equiv \angle SXQ$ , it follows that  $(A, B; C, D) = (P, Q; R, S)$ .  $\blacksquare$

*Remark.* Since the cross ratio is not dependent on the line intersecting the pencil, we can define the cross ratio of a pencil  $X(ABCD)$  to be

$$(XA, XB; XC, XD) = (A, B; C, D).$$

**Property 18.7.** Given a pencil  $X(ABCD)$  and a line parallel to  $XD$  that intersect  $XA$ ,  $XB$  and  $XC$  at points  $P$ ,  $Q$  and  $R$ , respectively, then  $X(ABCD)$  is a harmonic pencil if and only if  $\overline{PR} = \overline{RQ}$ .



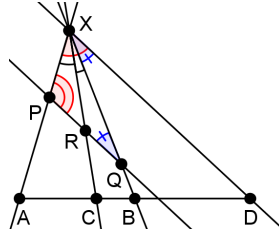
*Proof 1.* We will firstly give a not so Euclidean proof :)  
 Since  $XD \parallel PQ$ ,  $XD \cap PQ = P_\infty$ . Then, by [Property 18.6](#)

$$1 = (A, B; C, D) = (P, Q; R, P_\infty) = \frac{\overline{RP}}{\overline{RQ}} : \frac{\overline{PP_\infty}}{\overline{QP_\infty}}$$

Since  $P_\infty$  is the point at infinity, then we can take  $\overline{PP_\infty} = \overline{QP_\infty}$ , giving us  $\overline{PR} = \overline{RQ}$ . ■

*Proof 2.* For the more skeptical readers, here is a valid Euclidean proof. Let  $A$ ,  $C$ ,  $B$  and  $D$  be collinear in this order. From [Equation 18.1](#), we know that

$$(A, B; C, D) = \frac{\sin(\angle CXA)}{\sin(\angle CXB)} : \frac{\sin(\angle DXA)}{\sin(\angle DXB)} \quad (1)$$



By using the [Law of Sines](#) in the triangles  $\triangle PRX$  and  $\triangle QRX$ , we get

$$\frac{\overline{PR}}{\sin(\angle RXP)} = \frac{\overline{XR}}{\sin(\angle XPR)} \quad \text{and} \quad \frac{\overline{QR}}{\sin(\angle RXQ)} = \frac{\overline{XR}}{\sin(\angle XQR)},$$

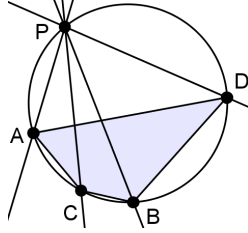
$$\text{i.e. } \frac{\overline{PR}}{\overline{QR}} = 1 \iff \frac{\sin(\angle RXP)}{\sin(\angle RXQ)} : \frac{\sin(\angle XPR)}{\sin(\angle XQR)} = 1. \quad (2)$$

We have  $\angle CXA \equiv \angle RXP$  and  $\angle CXB \equiv \angle RXQ$ . Since  $XD \parallel PQ$ , we also have  $\angle DXA \equiv \angle DXP = 180^\circ - \angle XPR$  and  $\angle DXB \equiv \angle DXQ = \angle XQR$ . Combining with (1) and (2), we get that

$$\overline{PR} = \overline{QR} \iff (A, B; C, D) = 1 \quad \blacksquare$$

## 18.2 Harmonic Quadrilateral

Let  $ABCD$  be a cyclic quadrilateral and  $P$  be a point on the circle. Then,  $ABCD$  is called *harmonic quadrilateral* if the pencil  $P(ABCD)$  is harmonic, i.e. if  $(PA, PB; PC, PD) = 1$  and  $AB$  and  $CD$  intersect inside the circle (the order of the points on the circle is  $A - C - B - D - A$ , in any direction).



**Property 18.8.** Let  $A, C, B$  and  $D$  be points on a circle in this order. Let  $P$  be any point on that circle. Then the cross ratio  $(PA, PB; PC, PD)$  does not depend on  $P$ .

*Proof.* Let the radius of the circle be  $R$ . By the Sine Law for  $\triangle CPA$ , we get

$$\frac{\overline{CA}}{\sin(\angle CPA)} = 2R.$$

We can get similar equations for the triangles  $\triangle CPB$ ,  $\triangle DPA$  and  $\triangle DPB$ . Therefore, by the definition of a cross ratio of a pencil and by Equation 18.1, we get

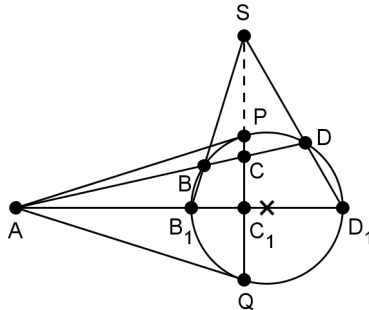
$$(PA, PB; PC, PD) = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)} = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$$

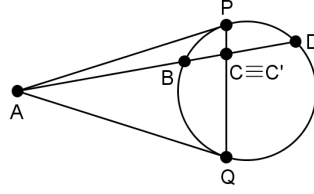
which doesn't depend on the point  $P$ .

*Remark.* As a consequence, for a harmonic quadrilateral  $ACBD$ , the products of the opposite sides are equal, i.e.  $\overline{AC} \cdot \overline{BD} = \overline{BC} \cdot \overline{AD}$ . ■

**Property 18.9.** Let  $A$  be a point outside of a circle  $\omega$ . A line  $l$  which passes through  $A$ , meets  $\omega$  at points  $B$  and  $D$ .  $C$  is a point on the line segment  $BD$ . Prove that the division  $(ABCD)$  is harmonic if and only if  $C$  lies on the polar of  $A$ .

*Proof 1.* Let  $P$  and  $Q$  be points on  $\omega$  such that  $AP$  and  $AQ$  are tangents. Then,  $PQ$  is the polar of  $A$ . Let's prove the first direction. Let  $C = BD \cap PQ$ . We need to prove that  $(ABCD)$  is harmonic. Let the secant through  $A$  that passes through the center of  $\omega$  intersect  $\omega$  at  $B_1$  and  $D_1$  and the line  $PQ$  at  $C_1$ , such that the points  $A - B_1 - C_1 - D_1$  are collinear in that order. Then  $C_1$  is the image of  $A$  under inversion with respect to  $\omega$ . By Property 18.4, we know that  $(AB_1C_1D_1)$  is harmonic. Let  $B_1B \cap D_1D = S$ . By Property 16.3,  $S \in a \equiv PQ$ . Therefore,  $B_1B$ ,  $C_1C$  and  $D_1D$  concur at  $S$ . Since  $(AB_1C_1D_1)$  is harmonic, then the pencil  $S(AB_1C_1D_1)$  is harmonic. When it is intersected by another line, by Property 18.6, the intersection points form a harmonic division, i.e.  $(ABCD)$  is harmonic. □





For proving the other direction, let  $C'$  be a point on the segment  $BC$  such that  $(ABC'D)$  is harmonic. We need to prove that  $C' \in PQ$ . From above, we know that  $(ABCD)$  is harmonic, where  $C = BD \cap PQ$ . Since three of the points in the cross ratio coincide and the cross ratio is equal, then the fourth point must also coincide. We will prove this property once here, but remember it because it is often used in problems.

$$\frac{\overline{BA}}{\overline{BC'}} : \frac{\overline{DA}}{\overline{DC'}} = 1 = \frac{\overline{BA}}{\overline{BC}} : \frac{\overline{DA}}{\overline{DC}}$$

$$\frac{\overline{DC'}}{\overline{BC'}} = \frac{\overline{DC}}{\overline{BC}}$$

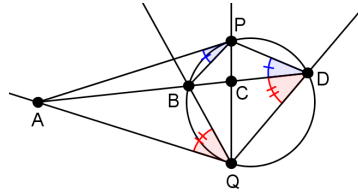
By adding 1 on both sides, we get

$$\frac{\overline{DB}}{\overline{BC'}} = \frac{\overline{DB}}{\overline{BC}}$$

$$\overline{BC'} = \overline{BC}$$

Since we know that both  $C$  and  $C'$  are between  $B$  and  $D$ , we get that  $C' \equiv C$ , i.e.  $C' \in PQ$ . ■

*Proof 2.* Again, let  $P$  and  $Q$  be points on  $\omega$  such that  $AP$  and  $AQ$  are tangents and let  $C = BD \cap PQ$ . We will give an alternate proof of the first direction, i.e. that  $(ABCD)$  is harmonic.



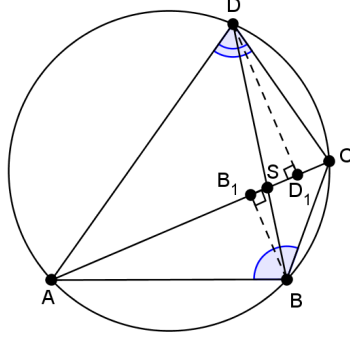
It's obvious that  $\triangle ABP \sim \triangle APD$  and  $\triangle ABQ \sim \triangle AQD$ . Since  $\overline{AP} = \overline{AQ}$ :

$$\frac{\overline{BP}}{\overline{PD}} = \frac{\overline{AB}}{\overline{AP}} = \frac{\overline{AB}}{\overline{AQ}} = \frac{\overline{BQ}}{\overline{QD}}$$

So,  $\overline{BP} \cdot \overline{QD} = \overline{PD} \cdot \overline{BQ}$ , which by [Property 18.8](#), means that  $QBPD$  is a harmonic quadrilateral. Then, the pencil  $Q(QBPD)$  is a harmonic pencil. By [Property 18.6](#), we know that if we intersect it by the line  $AB$ , then the intersection points  $QQ \cap AB = A$ ,  $QB \cap AB = B$ ,  $QP \cap AB = C$  and  $QD \cap AB = D$  will form a harmonic division, i.e.  $(ABCD)$  is a harmonic.

The other direction is the same as in the previous proof. ■

**Property 18.10.** Given a cyclic quadrilateral  $ABCD$ , let  $S$  be the intersection of the diagonals  $AC$  and  $BD$ . Then,  $ABCD$  is a harmonic quadrilateral if and only if  $AS$  is a symmedian in  $\triangle ABD$ .



*Proof.* Firstly, let's investigate something that is true in any cyclic quadrilateral. Let  $B_1$  and  $D_1$  be the feet of the perpendiculars from  $B$  and  $D$ , respectively, to  $AC$ . Then,  $\triangle BB_1S \sim \triangle DD_1S$ . Also,  $\sin \beta = \sin(180^\circ - \delta) = \sin \delta$ .

$$\frac{\overline{BA} \cdot \overline{BC}}{\overline{DA} \cdot \overline{DC}} = \frac{\frac{1}{2} \overline{BA} \overline{BC} \sin \beta}{\frac{1}{2} \overline{DA} \overline{DC} \sin \delta} = \frac{P_{\triangle ABC}}{P_{\triangle ADC}} = \frac{\frac{1}{2} \overline{AC} \overline{BB_1}}{\frac{1}{2} \overline{AC} \overline{DD_1}} = \frac{\overline{BB_1}}{\overline{DD_1}} = \frac{\overline{BS}}{\overline{SD}} \quad (*)$$

Then,

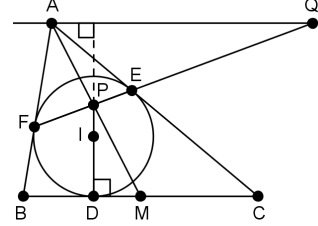
$$\frac{\overline{BS}}{\overline{SD}} = \left( \frac{\overline{AB}}{\overline{AD}} \right)^2 \stackrel{(*)}{\iff} \frac{\overline{BA} \cdot \overline{BC}}{\overline{DA} \cdot \overline{DC}} = \left( \frac{\overline{AB}}{\overline{AD}} \right)^2 \iff \frac{\overline{BC}}{\overline{CD}} = \frac{\overline{AB}}{\overline{AD}}$$

By [Property 13.1](#), the left-hand side is true iff  $AS$  is a symmedian in  $\triangle ABD$ . By [Property 18.8](#), the right-hand side is true iff  $ABCD$  is a harmonic quadrilateral.  $\blacksquare$

### 18.3 Useful Lemmas

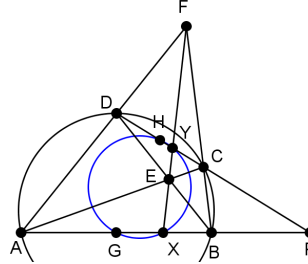
**Example 18.1.** In  $\triangle ABC$ , the incircle  $\omega$  centered at  $I$  touches the sides  $BC$ ,  $CA$  and  $AB$  at  $D$ ,  $E$  and  $F$ , respectively. Let  $DI \cap EF = P$  and let  $AP \cap BC = M$ . Prove that  $\overline{BM} = \overline{MC}$ .

*Proof.* We need to prove that  $\overline{BM} = \overline{MC}$ , so our main idea, by [Property 18.7](#), is to prove that the pencil  $A(BMCQ)$  is harmonic, where  $AQ$  is some line parallel to  $BC$ . Let  $Q$  be a point such that  $AQ \parallel BC$  and  $Q$  lies on the line  $EF$ . We will use polars, so let  $x$  denote the polar of a point  $X$  with respect to  $\omega$ .  $AF$  and  $AE$  are tangents to  $\omega$ , so by [Property 16.1](#),  $EF \equiv a$ .  $P \in a$ , so by [La Hire's Theorem](#),  $A \in p$ . Also, since  $IP \perp AQ$  (because  $ID \perp BC$  and  $BC \parallel AQ$ ),  $AQ \equiv p$ . Since  $Q \in a$  and  $Q \in p$ , then by [La Hire's Theorem](#),  $AP \equiv q$ .



Since  $P \in q$ , by [Property 18.9](#), the division  $(QEPF)$  is harmonic. Then, the pencil  $A(QEPF)$  is harmonic. By [Property 18.7](#), by intersecting the harmonic pencil  $A(QEPF)$  with the line  $BC$  which is parallel to  $AQ$ , we get that  $(P_\infty CMB)$  is harmonic, i.e.  $\overline{CM} = \overline{MB}$ . ■

**Example 18.2.** Given a cyclic quadrilateral  $ABCD$ , let the diagonals  $AC$  and  $BD$  meet at  $E$  and the lines  $AD$  and  $BC$  meet at  $F$ . The midpoints of  $AB$  and  $CD$  are  $G$  and  $H$ , respectively. The line  $EF$  intersects  $AB$  and  $CD$  at  $X$  and  $Y$ , respectively. Prove that  $GXYH$  is a cyclic quadrilateral.



*Proof.* Let  $AB \cap CD = P$ . In  $\triangle ABF$ , the cevians  $AC$ ,  $BD$  and  $FX$  are concurrent, so by [Property 18.1](#), we get that  $(AXB P)$  is a harmonic. Since  $\overline{GA} = \overline{GB}$ , by [Property 18.5](#), we get that

$$\overline{PA} \cdot \overline{PB} = \overline{PX} \cdot \overline{PG}. \quad (1)$$

Since  $(AXB P)$  is harmonic, then  $F(AXB P)$  is a harmonic pencil. If we intersect it with the line  $CD$ , by [Property 18.6](#), the intersection points  $(DYCP)$  form a harmonic division. Again, since  $\overline{DH} = \overline{HC}$ , by [Property 18.5](#), we get that

$$\overline{PC} \cdot \overline{PD} = \overline{PY} \cdot \overline{PH}. \quad (2)$$

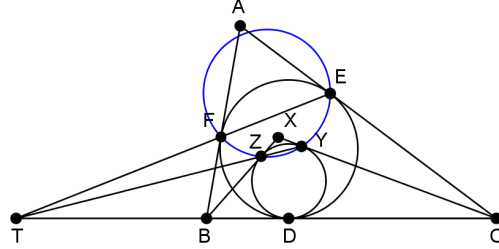
Since  $ABCD$  is cyclic, by the [Intersecting Secants Theorem](#), we have

$$\overline{PX} \cdot \overline{PG} \stackrel{(1)}{=} \overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD} \stackrel{(2)}{=} \overline{PY} \cdot \overline{PH}.$$

Therefore, by the converse of the [Intersecting Secants Theorem](#),  $GXYH$  is a cyclic quadrilateral. ■

Now, let's solve some problems.

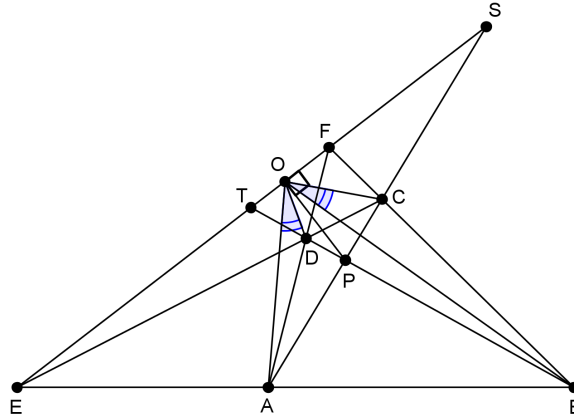
**Example 18.3** (IMO Shortlist 1995/G3). The incircle of  $\triangle ABC$  touches the sides  $BC$ ,  $CA$  and  $AB$  at  $D$ ,  $E$  and  $F$ , respectively.  $X$  is a point inside  $\triangle ABC$  such that the incircle of  $\triangle XBC$  touches  $BC$  at  $D$  and touches  $CX$  and  $XB$  at  $Y$  and  $Z$ , respectively. Show that  $E$ ,  $F$ ,  $Z$  and  $Y$  are concyclic.



*Proof.* Let  $EF$  intersect  $BC$  at  $T_1$ . Since  $AD$ ,  $BE$  and  $CF$  are concurrent at the [Gergonne Point](#) of  $\triangle ABC$ , by [Property 18.1](#), we get that  $(T_1BDC)$  is a harmonic. Similarly, if  $YZ \cap BC = T_2$ , then  $(T_2BDC)$  is a harmonic. Since three of the points are fixed, then the fourth one must also be fixed, i.e.  $T_1 \equiv T_2 \equiv T$ .

Now, by the [Secant-Tangent Theorem](#) for the circle  $(DEF)$  and then for the circle  $(DYZ)$ , we get  $\overline{TF} \cdot \overline{TE} = \overline{TD}^2 = \overline{TZ} \cdot \overline{TY}$ , which by the converse of the [Intersecting Secants Theorem](#) means that  $E$ ,  $F$ ,  $Z$  and  $Y$  are concyclic. ■

**Example 18.4** (China TST 2002). Let  $E$  and  $F$  be the intersections of opposite sides of a convex quadrilateral  $ABCD$ . The two diagonals meet at  $P$ . Let  $O$  be the foot of the perpendicular from  $P$  to  $EF$ . Show that  $\angle BOC = \angle AOD$ .

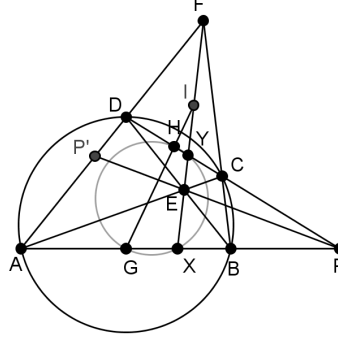


*Proof.* Let  $E = BA \cap CD$  and  $F = BC \cap AD$ . Also, let  $AC \cap EF = S$  and  $BD \cap EF = T$ . Since the cevians  $EC$ ,  $FA$  and  $BT$  in  $\triangle EFB$  are concurrent (at  $D$ ), by [Property 18.1](#), we get that the division  $(ETFS)$  is harmonic. Therefore, the pencil  $B(ETFS)$  is a harmonic pencil. If we intersect it with the line  $AC$ , by [Property 18.6](#), the intersection points also form a harmonic division, i.e.  $(APCS)$  is harmonic. Since  $OP \perp OS$ , by [Property 18.3](#),  $\angle AOP = \angle POC$ .

On the other hand, since  $(APCS)$  is harmonic, the pencil  $E(APCS)$  is harmonic, so by intersecting it with the line  $BD$ , we get that  $(BPDT)$  is harmonic. Again, since  $OP \perp OT$ , we get  $\angle BOP = \angle POD$ .

Finally,  $\angle BOC = \angle POC - \angle POB = \angle AOP - \angle POD = \angle AOD$ . ■

**Example 18.5** (IMO Shortlist 2009/G4). Given a cyclic quadrilateral  $ABCD$ , let the diagonals  $AC$  and  $BD$  meet at  $E$  and the lines  $AD$  and  $BC$  meet at  $F$ . The midpoints of  $AB$  and  $CD$  are  $G$  and  $H$ , respectively. Show that  $EF$  is tangent at  $E$  to the circle through the points  $E$ ,  $G$  and  $H$ .



*Proof 1.* Let the line  $EF$  intersect the lines  $AB$ ,  $CD$  and  $GH$  at  $X$ ,  $Y$  and  $I$ , respectively. By [Property 17.6](#), the midpoints of the diagonals of the complete quadrilateral  $FDECAB$  are collinear, so  $I$  is the midpoint of  $EF$ . Let  $AB \cap CD = P$  and  $PE \cap AD = P'$ . In  $\triangle ADP$ , the cevians  $AC$ ,  $DB$  and  $PP'$  are concurrent, so by [Property 18.1](#),  $(FDP'A)$  is a harmonic. Therefore,  $P(FDP'A)$  is a harmonic pencil. If we intersect it with the line  $FE$ , by [Property 18.6](#), the intersection points will form a harmonic division, i.e.  $(FYEX)$  is a harmonic. By [Property 18.4](#),  $\mathcal{J}_{I, \overline{FE}} : X \leftrightarrow Y$ , i.e.

$$\overline{IE}^2 = \overline{IX} \cdot \overline{IY}. \quad (1)$$

From [Example 18.2](#), we know that  $GXYH$  is a cyclic quadrilateral, so

$$\overline{IX} \cdot \overline{IY} = \overline{IH} \cdot \overline{IG}. \quad (2)$$

By combining (1) and (2), by the converse of the [Secant-Tangent Theorem](#), we get that  $IE \equiv FE$  is tangent to  $(EHG)$ .  $\blacksquare$

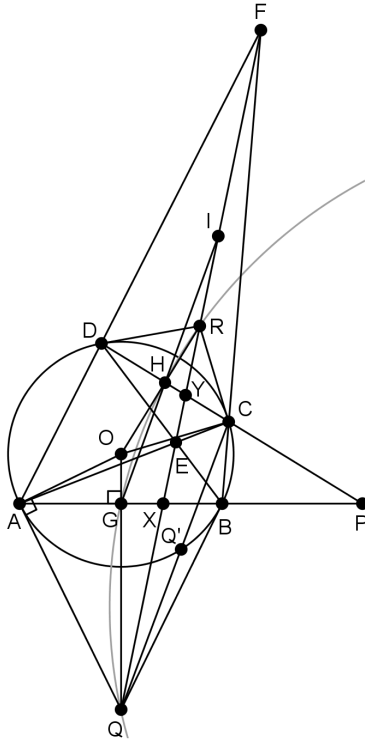
*Proof 2.* Let  $AB \cap CD = P$  and  $GH \cap FE = I$ . Let  $\omega \equiv (ABCD)$  and let  $O$  be its center. Let the tangents to  $\omega$  at  $A$  and  $B$  intersect at  $Q$ . Let the tangents to  $\omega$  at  $C$  and  $D$  intersect at  $R$ .

Since  $G$  is the midpoint of the  $AB$ ,  $G \in OQ$  and  $OG \perp AB$ . By the [Euclid's laws](#) for  $\triangle OAQ$ , we get  $\overline{OA}^2 = \overline{OG} \cdot \overline{OQ}$ . Similarly,  $\overline{OC}^2 = \overline{OH} \cdot \overline{OR}$ . Since  $\overline{OA} = \overline{OC}$  as radii in  $\omega$ , we have  $\overline{OG} \cdot \overline{OQ} = \overline{OH} \cdot \overline{OR}$ , so by the converse of the [Intersecting Secants Theorem](#)  $GQRH$  is a cyclic quadrilateral. Therefore, for the secants through the point  $I$ , by the [Intersecting Secants Theorem](#), we get

$$\overline{IG} \cdot \overline{IH} = \overline{IQ} \cdot \overline{IR} \quad (1)$$

By [Property 16.1](#),  $AB$  is the polar of  $Q$ , i.e.  $AB \equiv q$ . Since  $P \in q$ , then by [La Hire's Theorem](#),  $Q \in p$ . On the other hand, from [Brocard's Theorem](#) we know that  $FE \equiv p$ . Therefore,  $Q \in FE$ . Similarly,  $R \in FE$ .

Since  $QA$  and  $QB$  are tangents to  $(ABC)$ , then by [Property 13.4](#),  $CQ$  is a symmedian in  $\triangle ABC$ . If  $Q' = CQ \cap (ABC)$ , then by [Property 18.10](#),  $AQ'BC$



is a harmonic quadrilateral. Therefore,  $C(A, B; Q', C)$  is a harmonic pencil. By intersecting it with the line  $FE$ , by [Property 18.6](#), we get that the intersection points form a harmonic division, i.e.  $(E, F; Q, R)$  is harmonic. By [Property 17.6](#), the midpoints of the diagonals of the complete quadrilateral  $FDECAB$  are collinear, so  $I$  is the midpoint of  $EF$ . Therefore, by [Property 18.4](#),  $\mathcal{J}_{I, \overline{IE}} : Q \leftrightarrow R$ , i.e.

$$\overline{IE}^2 = \overline{IQ} \cdot \overline{IR}. \quad (2)$$

By combining (1) and (2), by the converse of the [Secant-Tangent Theorem](#), we get that  $IE \equiv FE$  is tangent to  $(EHG)$ . ■

**Example 18.6** (BMO Shortlist 2007, Cosmin Pohoata). Let  $\omega$  be a circle centered at  $O$  and  $A$  be a point outside it. Denote by  $B$  and  $C$  the points where the tangents from  $A$  with respect to  $\omega$  meet the circle. Let  $D$  be the point on  $\omega$ , for which  $O$  lies on the line segment  $AD$ . Let  $X$  be the foot of the perpendicular from  $B$  to  $CD$ ,  $Y$  be the midpoint of the line segment  $BX$  and  $Z$  be the second intersection of  $DY$  with  $\omega$ . Prove that  $ZA \perp ZC$ .

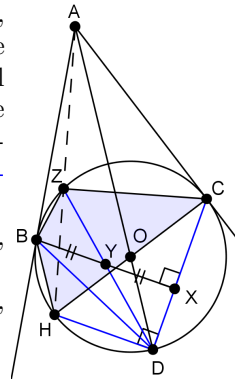
*Proof.* Let  $CO \cap \omega = H$ . Then, by [Thales' Theorem](#),  $DH \perp DC$ . Since  $XB \perp DC$ , we get  $DH \parallel XB$ . Since  $\overline{XY} = \overline{YB}$ , by [Property 18.7](#), we get that the pencil  $(DX, DY, DB, DH)$  harmonic. Therefore, by definition, the cyclic quadrilateral formed by the intersections of the pencil with  $\omega$ ,  $CZBH$ , is a harmonic quadrilateral. By [Property 18.10](#),  $HZ$  is symmedian in  $\triangle HBC$ .

Since  $BA$  and  $CA$  are tangents to  $\omega$ , then by [Property 13.4](#),  $HA$  is a symmedian in  $\triangle HBC$ .

Finally,  $HA \equiv HZ$ , i.e.  $H - Z - A$  are collinear. Therefore,

$$\angle AZC = 180^\circ - \angle CZH = 180^\circ - 90^\circ = 90^\circ,$$

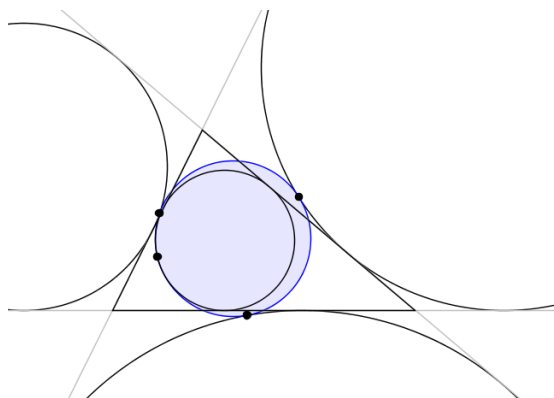
$$\text{i.e. } ZA \perp ZC \quad \blacksquare$$



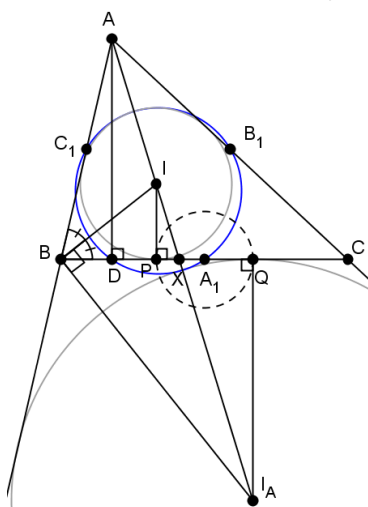
## Chapter 19

# Feuerbach's Theorem

**Example 19.1** (Feuerbach's Theorem). The nine point circle of a triangle is internally tangent to its incircle and externally tangent to its three excircles.



*Proof.* Let  $A_1$ ,  $B_1$  and  $C_1$  be the midpoints of  $BC$ ,  $CA$  and  $AB$ , respectively. Let  $\omega$  be the incircle of the triangle, centered at  $I$ . Let  $\omega_A$  be the  $A$ -excircle, centered at  $I_A$ . Let  $P$  and  $Q$  be the tangent points of  $\omega$  and  $\omega_A$ , respectively to



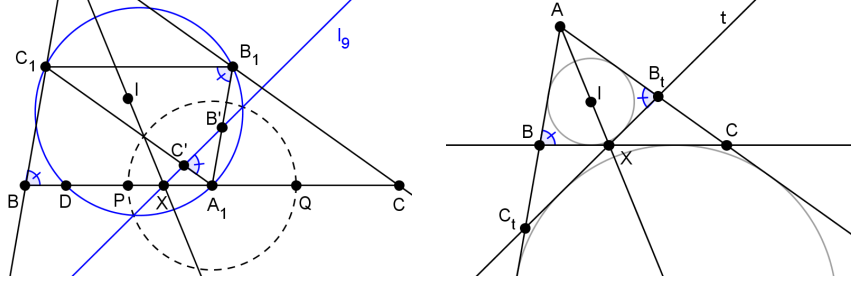
$BC$ . From [Example 8.5](#), we know that  $\overline{A_1P} = \overline{A_1Q}$ . Let  $\mathcal{J}$  be the inversion with center  $A_1$  and radius  $\overline{A_1P}$ . Since  $A_1P$  is tangent to  $\omega$ ,  $\mathcal{J} : \omega \leftrightarrow \omega$ . Similarly  $\mathcal{J} : \omega_A \leftrightarrow \omega_A$ . Since the nine point circle,  $C_9$ , passes through the center of inversion  $A_1$ , it will be sent to a line, i.e.  $\mathcal{J} : C_9 \leftrightarrow l_9$ . Because we want to prove that the nine point circle is tangent to the incircle and  $A$ -excircle, we will prove that its image,  $l_9$ , is tangent to their images,  $\omega$  and  $\omega_A$ .

Let  $AI \cap BC = X$ . Since  $BI$  is the internal angle bisector of  $\angle ABC$  and  $BI \perp BI_A$ , by [Property 18.3](#), the division  $(AIXI_A)$  is harmonic. Let  $D$  be the foot of the altitude from  $A$  to  $BC$ . Then  $AD \parallel IP \parallel I_AQ$  (because they are all perpendicular to  $BC$ ). So, by Thales'

Proportionality Theorem, the division  $(DPXQ)$  is also harmonic. By [Property 18.4](#), since  $PQ$  is the diameter of the circle of inversion,  $\mathcal{J} : D \leftrightarrow X$ . Since  $D \in C_9$ , then  $X \in l_9$ . Let  $\mathcal{J} : B_1 \leftrightarrow B'$  and  $\mathcal{J} : C_1 \leftrightarrow C'$ . Then by [Property 15.1](#)

$$\angle A_1C'B' = \angle A_1B_1C_1 = \angle ABC. \quad (1)$$

Also, since  $B_1, C_1 \in C_9$ , then  $B', C' \in l_9$ .



Since  $X$  is the intersection of the line connecting the centers of  $\omega$  and  $\omega_A$ ,  $II_A$ , and one of the common internal tangents,  $BC$ , then the other common internal tangent,  $t$ , must also pass through  $X$ . We want to prove that  $l_9 \equiv t$ . Let  $t \cap AB = C_t$  and  $t \cap AC = B_t$ . By symmetry with respect to the line  $AI$ ,

$$\angle AB_tC_t = \angle ABC. \quad (2)$$

From (1), we know that  $\angle(A_1C_1, l_9) = \angle ABC$ . From (2), we know that  $\angle(AC, t) = \angle ABC$ . Since  $A_1C_1 \parallel AC$ , it means that  $l_9 \parallel t$ . But we already know that  $X \in l_9$  and  $X \in t$ , so  $l_9 \equiv t$ .

Therefore, the nine point circle is tangent to the incircle and the  $A$ -excircle. Similarly, it is tangent to the other two excircles. ■

*Remark.* The tangent point of the incircle and the nine point circle is called the *Feuerbach point* of the triangle.

## Chapter 20

# Apollonius' Problem

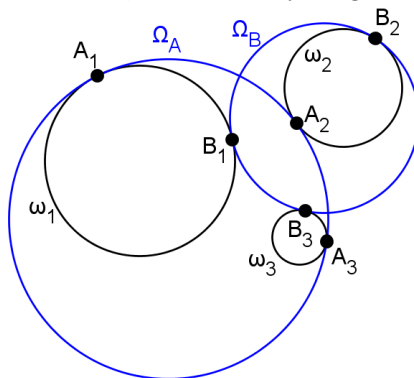
This topic is not directly related to Olympiad geometry problems, but it is a nice collection of properties that we already visited during our journey combined in a beautiful result.

**Example 20.1** (Apollonius' problem). Construct circles that are tangent to three given circles in a plane,  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ .

We will examine the case where the three circles are in general position, i.e. none of them intersect and all of them have different radii.

Firstly, let's find the number of solution circles that are tangent to all three circles. The solution circles can be tangent either internally or externally to any of the three circles. So the number of solution circles is  $2^3 = 8$ .

We will explain Gergonne's approach to solving this problem. It considers the solution circles in pairs such that if one of the solution circles is internally tangent to a given circle, then the other solution circle is externally tangent to that circle and vice versa. For example, if a solution circle is internally tangent to  $\omega_3$ , but externally tangent to  $\omega_1$  and  $\omega_2$ , then the paired solution circle is externally tangent to  $\omega_1$  and  $\omega_3$ , but internally tangent to  $\omega_2$ .



Let  $\Omega_A$  and  $\Omega_B$  be a pair of solution circles. Let  $\Omega_A$  be tangent to  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  at  $A_1$ ,  $A_2$  and  $A_3$ . Let  $\Omega_B$  be tangent to  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  at  $B_1$ ,  $B_2$  and  $B_3$ .

So, we somehow need to find these 6 tangent points. Then, the circumcircles of  $\triangle A_1A_2A_3$  and  $\triangle B_1B_2B_3$  would be the solution circles. Gergonne's approach was to construct a line  $l_1$  such that  $A_1$  and  $B_1$  must always lie on it. Then,  $A_1$

and  $B_1$  could be obtained as the intersection points of  $l_1$  and  $\omega_1$ . Similarly, by finding lines  $l_2$  and  $l_3$  that contained  $A_2$  and  $B_2$ , and  $A_3$  and  $B_3$ , respectively, we would find all 6 tangent points.

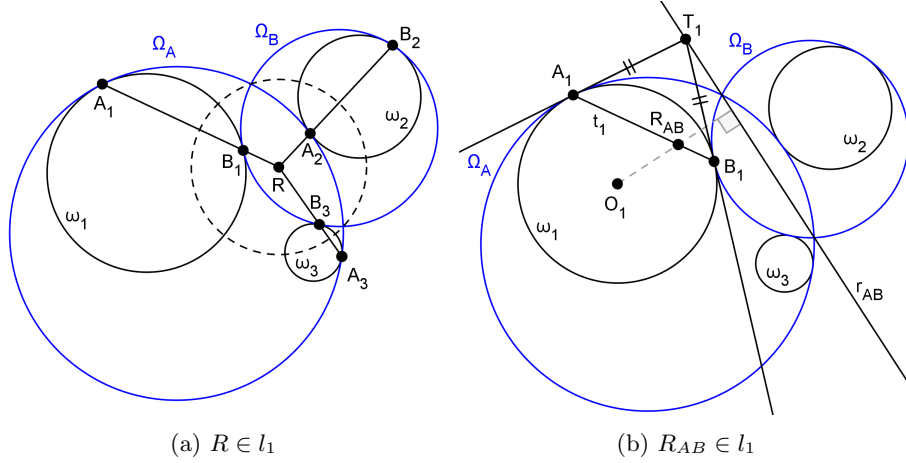
Let's recall, from [section 10.2](#), that the radical center of three circles is the center of the unique circle (called the radical circle) that intersects the three given circles orthogonally. Let  $R$  be the radical center of  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . Now, consider the inversion  $\mathcal{J}$  with the radical circle as the circle of inversion. Since the radical circle is orthogonal to the three given circles, each of them will be sent to itself. Since the solution circles are tangent to the three given circles, their images need to be tangent to the images of the given circles (which happen to be the three circles themselves), so the solution circles will be sent one into the other, i.e.

$$\begin{aligned}\mathcal{J} : \omega_1 &\leftrightarrow \omega_1 & \mathcal{J} : \omega_2 &\leftrightarrow \omega_2 & \mathcal{J} : \omega_3 &\leftrightarrow \omega_3 \\ \mathcal{J} : \Omega_A &\leftrightarrow \Omega_B\end{aligned}$$

Therefore, the tangent point  $A_1 = \omega_1 \cap \Omega_A$  will be sent to a point

$$A'_1 = \omega'_1 \cap \Omega'_A = \omega_1 \cap \Omega_B = B_1, \text{ i.e. } \mathcal{J} : A_1 \leftrightarrow B_1.$$

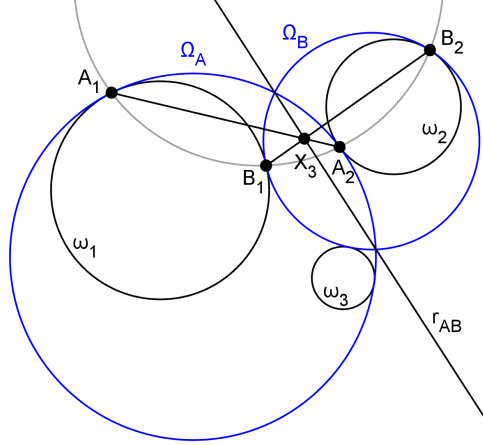
Since the center of inversion, the original point and the image point are collinear, we get that the radical center lies on the line  $A_1B_1$ , i.e.  $R \in l_1$ . Thus, we found one point on the line  $l_1$ . Now we need to find another one in order to be able to construct it.



Let the tangents to  $\omega_1$  at  $A_1$  and  $B_1$  intersect at  $T_1$ . Then,  $\overline{T_1A_1} = \overline{T_1B_1}$  and also, by [Property 16.1](#),  $A_1B_1$  is the polar of  $T_1$  with respect to  $\omega_1$ , i.e.  $A_1B_1 \equiv t_1$ . Notice that  $T_1A_1$  and  $T_1B_1$  are also tangents to the circles  $\Omega_A$  and  $\Omega_B$ . By [Property 10.2](#) and since  $\overline{T_1A_1} = \overline{T_1B_1}$ , the power of the point  $T_1$  with respect to  $\Omega_A$  and  $\Omega_B$  is equal, which means that  $T_1$  lies on their radical axis  $r_{AB}$ . By [La Hire's Theorem](#), the pole of  $r_{AB}$  with respect to  $\omega_1$  lies on  $t_1$ . So, here is our second point on the line  $t_1 \equiv A_1B_1 \equiv l_1$ .

But in order to construct the pole of  $r_{AB}$  with respect to  $\omega_1$ , we firstly need to construct  $r_{AB}$ . How do we construct the radical axis of two circles  $\Omega_A$  and  $\Omega_B$  if we don't have them yet? Well, we will find points that should lie on the radical axis and then we will construct the radical axis as the line through those points.

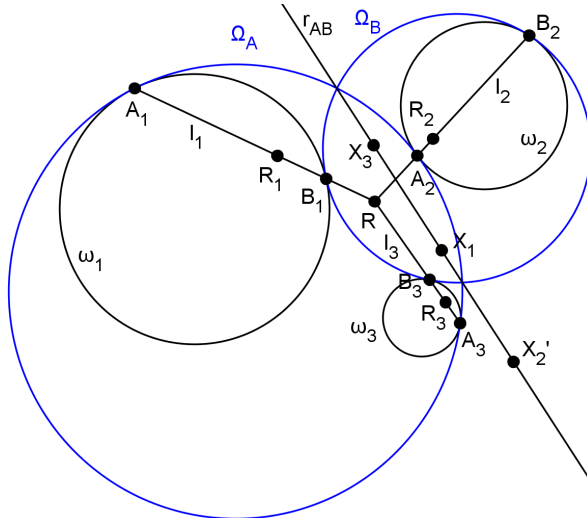
Let's recall, from [Example 14.6](#) that if a circle is tangent to two other circles, then the line through the tangent points passes through one of the homothetic centers of the two circles. Let  $X_3$  be a homothetic center of  $\omega_1$  and  $\omega_2$ , which can be constructed as the intersection of their common tangents. Then  $X_3 \in A_1A_2$  and  $X_3 \in B_1B_2$ .



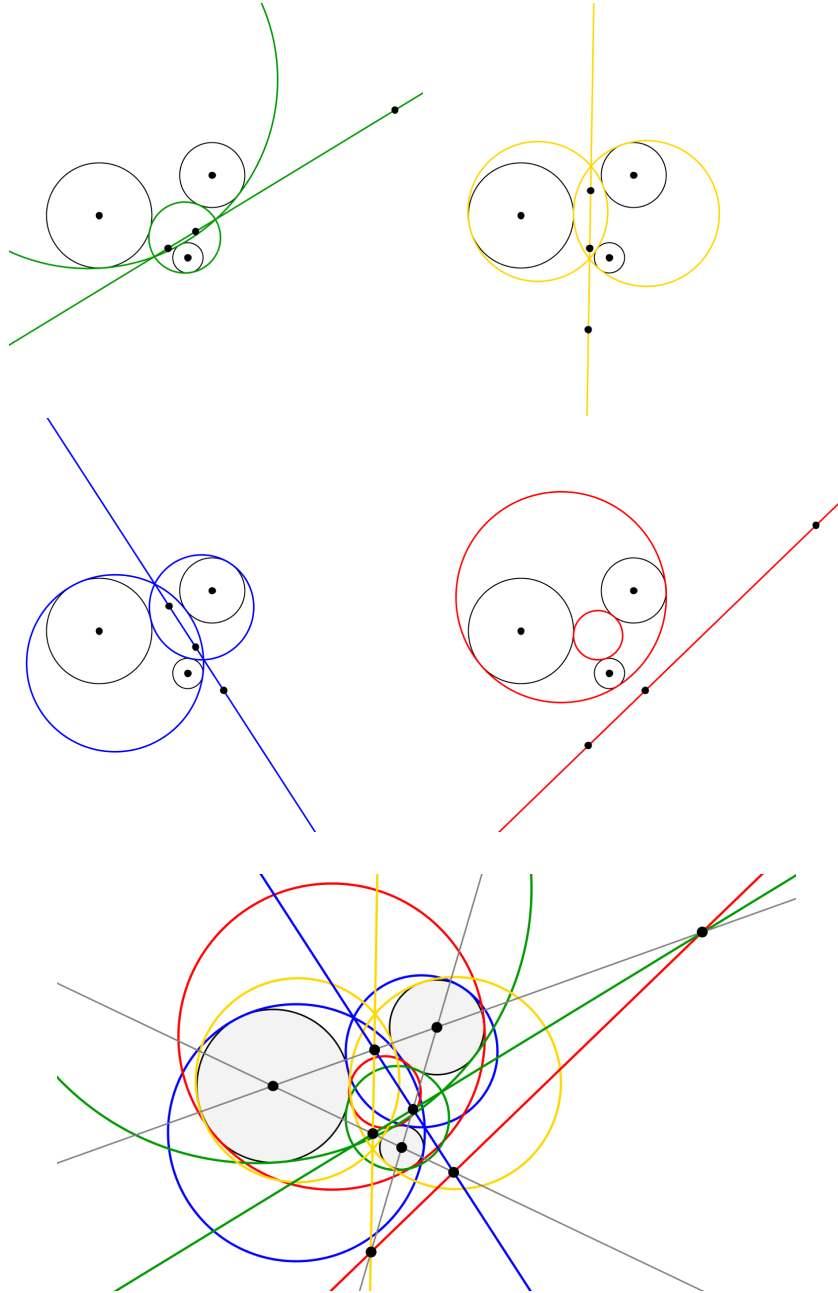
Recall also, from [section 14.1](#), the definition and the properties of antihomologous points. This means that in our case, since  $\Omega_A$  is tangent to  $\omega_1$  and  $\omega_2$ ,  $A_1$  and  $A_2$  are a pair of antihomologous points. Similarly,  $B_1$  and  $B_2$  are antihomologous points. Because, we know that two pairs of antihomologous points are concyclic, then:

$$\overline{X_3A_1} \cdot \overline{X_3A_2} = \overline{X_3B_1} \cdot \overline{X_3B_2}$$

But since  $A_1, A_2 \in \Omega_A$  and  $B_1, B_2 \in \Omega_B$ , this means that the power of the point  $X_3$  with respect to  $\Omega_A$  and  $\Omega_B$  is the same, i.e.  $X_3$  lies on their radical axis  $r_{AB}$ . Similarly, the homothetic centers  $X_1$  and  $X_2$  (which can be constructed as the intersection of the common tangents of  $\omega_2$  and  $\omega_3$ , and  $\omega_1$  and  $\omega_3$ , respectively) also lie  $r_{AB}$ . Thus, we know how to construct the radical axis  $r_{AB}$  of the solution circles  $\Omega_A$  and  $\Omega_B$ .



In summary, the desired line  $l_1$  is defined by two points: the radical center  $R$  of the three given circles and the pole with respect to  $\omega_1$  of the line connecting the homothetic centers. Depending on whether we choose all three external homothetic centers (1 possibility), or we choose one external and the other two internal homothetic centers (3 possibilities), we have 4 ways of defining "the line connecting the homothetic centers"<sup>1</sup>. Each of these 4 lines generates a different pair of solution circles, so that's how we can get all 8 solution circles.




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<sup>1</sup>Recall Monge-d'Alembert Theorem, page 88



**Part II**

**Mixed Problems**



**Problem 1.** Let  $BD$  be a median in  $\triangle ABC$ . The points  $E$  and  $F$  divide the median  $BD$  in three equal parts, such that  $\overline{BE} = \overline{EF} = \overline{FD}$ . If  $\overline{AB} = 1$  and  $\overline{AF} = \overline{AD}$ , find the length of the line segment  $CE$ .

**Problem 2.** Let  $I$  be the incenter of  $\triangle ABC$ . Let  $l$  be a line parallel to  $AB$ , that intersects the sides  $CA$  and  $CB$  at  $M$  and  $N$ , respectively. Prove that

$$\overline{AM} + \overline{BN} = \overline{MN}.$$

**Problem 3.** Let  $ABCD$  be a convex quadrilateral with area 1. Let  $A_1$  be a point on the ray  $AB$  beyond  $B$ , such that  $\overline{AB} = \overline{BA_1}$ . Similarly define the points  $B_1$ ,  $C_1$  and  $D_1$ . Prove that the area of  $A_1B_1C_1D_1$  is 5.

**Problem 4.** Let  $ABCD$  be a convex quadrilateral with area 3. The points  $M$  and  $N$  divide the line segment  $AB$  in three equal parts, such that  $\overline{AM} = \overline{MN} = \overline{NB}$ . The points  $P$  and  $Q$  divide the line segment  $CD$  in three equal parts, such that  $\overline{CP} = \overline{PQ} = \overline{QD}$ . Prove that the area of  $MNPQ$  is 1.

**Problem 5.** Let  $ABCD$  be a trapezoid ( $AB \parallel CD$ ). Let its diagonals  $AC$  and  $BD$  intersect at  $P$ . Let the areas of the triangle  $\triangle ABP$  and  $\triangle CDP$  be  $m$  and  $n$ , respectively. Prove that  $P_{ABCD} = (\sqrt{m} + \sqrt{n})^2$ .

**Problem 6.** Let  $ABCD$  be a parallelogram. Let  $M$  and  $N$  be the midpoints of the sides  $BC$  and  $DA$ , respectively. Prove that the lines  $AM$  and  $CN$  divide the diagonal  $BD$  in three equal parts.

**Problem 7.** Let  $ABCD$  be a parallelogram with area 1. Let  $M$  be the midpoint of the side  $AD$ . Let  $BM \cap AC = P$ . Find the area of  $MPCD$ .

**Problem 8.** Let  $P$  be a point on the side  $AB$  in  $\triangle ABC$ , such that  $\overline{AP} = 3 \cdot \overline{PB}$ . Let  $Q \in AC$ , such that  $\overline{AQ} = 4 \cdot \overline{QC}$ . Prove that  $BQ$  bisects the line segment  $CP$ .

**Problem 9.** Let  $\triangle ABC$  be a right triangle ( $\gamma = 90^\circ$ ). The angle bisector of  $\angle ABC$  intersect  $AC$  at  $D$ . If  $\overline{AD} = 5$  and  $\overline{CD} = 3$ , find  $\overline{AB}$ .

**Problem 10** (Stefan Lozanovski). In the triangle  $ABC$ ,  $\gamma = 60^\circ$ . Let  $O$  be the circumcenter of  $\triangle ABC$ .  $AO$  intersects  $BC$  at  $M$  and  $BO$  intersects  $AC$  at  $N$ . Prove that  $\overline{AN} = \overline{BM}$ .

**Problem 11.** Let  $ABC$  be an isosceles triangle, such that  $\overline{AC} = \overline{BC}$ . Let  $P$  be a point on the side  $AC$ . The tangent to  $(ABP)$  at the point  $P$  intersects  $(BCP)$  at  $D$ . Prove that  $CD \parallel AB$ .

**Problem 12.** Two circles intersect at  $A$  and  $B$ . One of their common tangents touches the circles at  $P$  and  $Q$ . Let  $A'$  be the reflection of  $A$  across the line  $PQ$ . Prove that  $A'PBQ$  is a cyclic quadrilateral.

**Problem 13** (Bosnia and Herzegovina TST 2013). Triangle  $ABC$  is right angled at  $C$ . Lines  $AM$  and  $BN$  are internal angle bisectors.  $AM$  and  $BN$  intersect the altitude  $CD$  at points  $P$  and  $Q$ , respectively. Prove that the line which passes through the midpoints of the segments  $QN$  and  $PM$  is parallel to  $AB$ .

**Problem 14.** Let  $P$  be a point outside a circle  $\omega$ . Let  $A$  and  $B$  be points on  $\omega$ , such that  $PA$  and  $PB$  are tangents to  $\omega$ . On the minor arc  $\widehat{AB}$  lies an arbitrary point  $C$ . Let  $D$ ,  $E$  and  $F$  be the feet of the perpendiculars from  $C$  to  $AB$ ,  $PA$  and  $PB$ , respectively. Prove that  $\overline{CD}^2 = \overline{CE} \cdot \overline{CF}$ .

**Problem 15.** Two circles intersect at  $A$  and  $B$ . One of their common tangents touches the circles at  $P$  and  $Q$ . Prove that the line  $AB$  bisects the line segment  $PQ$ .

**Problem 16.** Let  $C$  be a point on a semicircle with diameter  $AB$  and let  $D$  be the midpoint of arc  $AC$ . Let  $E$  be the projection of  $D$  onto the line  $BC$  and  $F$  the intersection of line  $AE$  with the semicircle. Prove that  $BF$  bisects the line segment  $DE$ .

**Problem 17.** Let  $ABC$  be an equilateral triangle. Let  $S$  be a point on the arc  $\widehat{AB}$  of  $(ABC)$  that doesn't contain  $C$ . Prove that  $\overline{SA} + \overline{SB} = \overline{SC}$ .

**Problem 18.** Let  $k_1$  and  $k_2$  be two circles intersecting at  $A$  and  $B$ . Let  $t_1$  and  $t_2$  be the tangents to  $k_1$  and  $k_2$  at point  $A$  and let  $t_1 \cap k_2 = \{A, C\}$ ,  $t_2 \cap k_1 = \{A, D\}$ . If  $E$  is a point on the ray  $AB$ , such that  $\overline{AE} = 2 \cdot \overline{AB}$ , prove that  $ACED$  is concyclic.

**Problem 19.** Let  $A_1$ ,  $B_1$  and  $C_1$  be the second intersections of the angle bisectors of  $\triangle ABC$  with its circumcircle. Prove that the incenter of  $\triangle ABC$  is the orthocenter of  $\triangle A_1B_1C_1$ .

**Problem 20.** Let  $A_1$ ,  $B_1$  and  $C_1$  be the second intersections of the altitudes of  $\triangle ABC$  with its circumcircle. Prove that the orthocenter of  $\triangle ABC$  is the incenter of  $\triangle A_1B_1C_1$ .

**Problem 21** (St. Petersburg City MO 1996). Let  $BD$  be the angle bisector of angle  $ABC$  in  $\triangle ABC$  with  $D$  on the side  $AC$ . The circumcircle of  $\triangle BDC$  meets  $AB$  at  $E$ , while the circumcircle of  $\triangle ABD$  meets  $BC$  at  $F$ . Prove that  $\overline{AE} = \overline{CF}$ .

**Problem 22** (Serbia State Competition 2016). In  $\triangle ABC$ , the angle bisector of  $\angle BAC$  intersects  $BC$  at  $D$ . Let  $M$  be the midpoint of  $BD$ . Let  $k$  be a circle through  $A$  that is tangent to  $BC$  at  $D$  and let the second intersections of  $k$  with the lines  $AM$  and  $AC$  be  $P$  and  $Q$ , respectively. Prove that the points  $B$ ,  $P$  and  $Q$  are collinear.

**Problem 23** (USAMO 1990). An acute-angled triangle  $ABC$  is given in the plane. The circle with diameter  $AB$  intersects altitude  $CE$  and its extension at points  $M$  and  $N$ , and the circle with diameter  $AC$  intersects altitude  $BD$  and its extension at points  $P$  and  $Q$ . Prove that the points  $M$ ,  $N$ ,  $P$  and  $Q$  lie on a common circle.

**Problem 24** (India MO 2010). Let  $ABC$  be a triangle with circumcircle  $\Gamma$ . Let  $M$  be a point in the interior of  $\triangle ABC$  which is also on the bisector of  $\angle BAC$ . Let  $AM$ ,  $BM$  and  $CM$  meet  $\Gamma$  in  $A_1$ ,  $B_1$  and  $C_1$ , respectively. Let  $P$  be the point of intersection of  $A_1C_1$  with  $AB$  and  $Q$  be the point of intersection of  $A_1B_1$  with  $AC$ . Prove that  $PQ \parallel BC$ .

**Problem 25** (IMO 2013/4). Let  $ABC$  be an acute triangle with orthocenter  $H$ , and let  $W$  be a point on the side  $BC$ , lying strictly between  $B$  and  $C$ . The points  $M$  and  $N$  are the feet of the altitudes from  $B$  and  $C$ , respectively. Denote by  $\omega_1$  the circumcircle of  $\triangle BWN$ , and let  $X$  be the point on  $\omega_1$  such that  $WX$  is a diameter of  $\omega_1$ . Analogously, denote by  $\omega_2$  the circumcircle of triangle  $\triangle CWM$ , and let  $Y$  be the point such that  $WY$  is a diameter of  $\omega_2$ . Prove that  $X$ ,  $Y$  and  $H$  are collinear.

**Problem 26** (Poland MO 2000). Let a triangle  $ABC$  satisfy  $\overline{AC} = \overline{BC}$ . Let  $P$  be a point inside the triangle  $ABC$  such that  $\angle PAB = \angle PBC$ . Denote by  $M$  the midpoint of the segment  $AB$ . Show that  $\angle APM + \angle BPC = 180^\circ$ .

**Problem 27** (Macedonia MO 2015). Let  $k_1$  and  $k_2$  be two circles that intersect at points  $A$  and  $B$ . A line through  $B$  intersects  $k_1$  and  $k_2$  at  $C$  and  $D$ , respectively, such that  $C$  doesn't lie inside of  $k_2$  and  $D$  doesn't lie inside of  $k_1$ . Let  $M$  be the intersection point of the tangent lines to  $k_1$  and  $k_2$  that pass through  $C$  and  $D$ , respectively. Let  $P$  be the intersection of the lines  $AM$  and  $CD$ . The tangent line to  $k_1$  passing through  $B$  intersects  $AD$  in point  $L$ . The tangent line to  $k_2$  passing through  $B$  intersects  $AC$  in point  $K$ . Let  $KP$  intersect  $MD$  at  $N$  and  $LP$  intersect  $MC$  at  $Q$ . Prove that  $MNPQ$  is a parallelogram.

**Problem 28** (China MO 1997). Let  $ABCD$  be a cyclic quadrilateral. Let  $AB \cap CD = P$  and  $AD \cap BC = Q$ . Let the tangents from  $Q$  meet the circumcircle of  $ABCD$  at  $E$  and  $F$ . Prove that  $P$ ,  $E$  and  $F$  are collinear.

**Problem 29** (Serbia State Competition 2016). Let  $\triangle ABC$  be an acute-angled triangle with  $\overline{AB} < \overline{AC}$ . Let  $D$  be the midpoint of  $BC$  and let  $p$  be the reflection of the line  $AD$  with respect to the angle bisector of  $\angle BAC$ . If  $P$  is the foot of the perpendicular from  $C$  to the line  $p$ , prove that  $\angle APD = \angle BAC$ .

**Problem 30** (IMO 2014/4). Let  $P$  and  $Q$  be on segment  $BC$  of an acute triangle  $ABC$  such that  $\angle PAB = \angle BCA$  and  $\angle CAQ = \angle ABC$ . Let  $M$  and  $N$  be the points on  $AP$  and  $AQ$ , respectively, such that  $P$  is the midpoint of  $AM$  and  $Q$  is the midpoint of  $AN$ . Prove that the intersection of  $BM$  and  $CN$  is on the circumcircle of  $\triangle ABC$ .

**Problem 31** (Vietnam TST 2001). Two circles intersect at  $A$  and  $B$  and a common tangent intersects the circles at  $P$  and  $Q$ . Let the tangents at  $P$  and  $Q$  to the circumcircle of  $\triangle APQ$  intersect at  $S$  and let  $H$  be the reflection of  $B$  across the line  $PQ$ . Prove that the points  $A$ ,  $S$  and  $H$  are collinear.

**Problem 32** (JBMO 2013, Stefan Lozanovski). Let  $ABC$  be an acute-angled triangle and let  $O$  be the centre of its circumcircle  $\omega$ . Let  $D$  be a point on the line segment  $BC$  such that  $\angle BAD = \angle CAO$ . Let  $E$  be the second point of intersection of  $\omega$  and the line  $AD$ . If  $M$ ,  $N$  and  $P$  are the midpoints of the line segments  $BE$ ,  $OD$  and  $AC$ , respectively, show that the points  $M$ ,  $N$  and  $P$  are collinear.

**Problem 33** (JBMO 2014). Consider an acute triangle  $ABC$  of area  $S$ . Let  $CD \perp AB$  ( $D \in AB$ ),  $DM \perp AC$  ( $M \in AC$ ) and  $DN \perp BC$  ( $N \in BC$ ). Denote by  $H_1$  and  $H_2$  the orthocentres of the triangles  $MNC$  and  $MND$ , respectively. Find the area of the quadrilateral  $AH_1BH_2$  in terms of  $S$ .

**Problem 34** (EMC 2012, Senior). Let  $ABC$  be an acute triangle with orthocenter  $H$ .  $AH$  and  $CH$  intersect  $BC$  and  $AB$  in points  $A_1$  and  $C_1$ , respectively.  $BH$  and  $A_1C_1$  meet at point  $D$ . Let  $P$  be the midpoint of the segment  $BH$ . Let  $D'$  be the reflection of the point  $D$  with respect to  $AC$ . Prove that the quadrilateral  $APCD'$  is cyclic.

**Problem 35** (BMO 2010). Let  $ABC$  be an acute triangle with orthocentre  $H$ , and let  $M$  be the midpoint of  $AC$ . The point  $C_1$  on  $AB$  is such that  $CC_1$  is an altitude of the triangle  $ABC$ . Let  $H_1$  be the reflection of  $H$  in  $AB$ . The orthogonal projections of  $C_1$  onto the lines  $AH_1$ ,  $AC$  and  $BC$  are  $P$ ,  $Q$  and  $R$ , respectively. Let  $M_1$  be the point such that the circumcentre of triangle  $PQR$  is the midpoint of the segment  $MM_1$ . Prove that  $M_1$  lies on the segment  $BH_1$ .

**Problem 36** (Macedonia MO 2009, corrected). Let  $I$  be the incenter of  $\triangle ABC$ . Points  $K$  and  $L$  are the intersection points of the circumcircles of  $\triangle BIC$  and  $\triangle AIC$  with the bisectors of  $\angle BAC$  and  $\angle ABC$ , respectively ( $K, L \neq I$ ). Let  $P$  be the midpoint of the segment  $KL$ . Let  $M$  be the reflection of  $I$  with respect to  $P$  and  $N$  be the reflection of  $I$  with respect to  $C$ . Prove that the points  $K$ ,  $L$ ,  $M$  and  $N$  lie on the same circle.

**Problem 37** (Hong Kong TST 2003). In the triangle  $ABC$ , the point  $M$  is the midpoint of  $AC$  and  $D$  is a point on  $AB$ .  $BM$  and  $CD$  meet at  $O$ , with  $\overline{AB} = \overline{CO}$ . Prove that  $AB$  is perpendicular to  $BC$  if and only if  $ADOM$  is a cyclic quadrilateral.

**Problem 38** (Stefan Lozanovski). Let  $D$  be a point on the side  $AB$  in  $\triangle ABC$ . Let  $F$  be a point on  $CD$  such that  $\overline{AB} = \overline{CF}$ . The circumcircle of  $\triangle BDF$  intersects  $BC$  again at  $E$ . Assume that  $A$ ,  $F$  and  $E$  are collinear. If  $\angle ACB = \gamma$ , find the measurement of  $\angle ADC$ .

**Problem 39** (Stefan Lozanovski). Let  $AA'$  be a median in the triangle  $ABC$ . Let  $D$  be a point on  $AA'$  and let the intersection of  $BD$  and  $AC$  be  $E$ . The circumcircle of  $\triangle BCE$  intersects  $AB$  again at  $F$ . If  $C$ ,  $D$  and  $F$  are collinear, prove that  $\triangle ABC$  is isosceles.

**Problem 40** (Russia 2003). Let  $ABC$  be a triangle with  $AB \neq AC$ . Point  $E$  is such that  $\overline{AE} = \overline{BE}$  and  $BE \perp BC$ . Point  $F$  is such that  $\overline{AF} = \overline{CF}$  and  $CF \perp BC$ . Let  $D$  be the point on line  $BC$  such that  $AD$  is tangent to the circumcircle of  $\triangle ABC$ . Prove that  $D$ ,  $E$  and  $F$  are collinear.

**Problem 41** (BMO 2009). Let  $MN$  be a line parallel to the side  $BC$  of a triangle  $ABC$ , with  $M$  on the side  $AB$  and  $N$  on the side  $AC$ . The lines  $BN$  and  $CM$  meet at point  $P$ . The circumcircles of  $\triangle BMP$  and  $\triangle CNP$  meet at two distinct points  $Q$  and  $R$ . Prove that  $\angle BAQ = \angle CAP$ .

**Problem 42** (JBMO 2002).  $ABC$  is an isosceles triangle ( $\overline{CA} = \overline{CB}$ ). Let  $P$  be a point on the arc  $\widehat{AB}$  on  $(ABC)$  that doesn't contain  $C$ . Let  $D$  be the foot of the perpendicular from  $C$  to  $PB$ . Show that  $\overline{PA} + \overline{PB} = 2 \cdot \overline{PD}$ .

**Problem 43** (RMM 2015/4). Let  $ABC$  be a triangle, and let  $D$  be the point where the incircle touches the side  $BC$ . Let  $I_B$  and  $I_C$  be the incentres of the triangles  $\triangle ABD$  and  $\triangle ACD$ , respectively. Prove that the circumcentre of  $\triangle AI_B I_C$  lies on the angle bisector of  $\angle BAC$ .

**Problem 44** (Stefan Lozanovski). Let  $S$  be a point on  $AC$ , such that  $BS$  is an angle bisector in the triangle  $ABC$ . Let  $O_1$  and  $O_2$  be the circumcenters of  $\triangle ABS$  and  $\triangle BSC$ , respectively. The median  $AM$  in  $\triangle ABC$  intersects  $BS$  at  $X$ . Prove that the lines  $AB$ ,  $O_1O_2$  and  $CX$  are concurrent.

**Problem 45** (China MO 1992). A convex quadrilateral  $ABCD$  is inscribed in a circle with center  $O$ . The diagonals  $AC$ ,  $BD$  of  $ABCD$  meet at  $P$ . Circumcircles of  $\triangle ABP$  and  $\triangle CDP$  meet at  $P$  and  $Q$  ( $O$ ,  $P$  and  $Q$  are pairwise distinct). Show that  $\angle OQP = 90^\circ$ .

**Problem 46** (IMO 1983/2). Let  $A$  be one of the two points of intersection of the circles  $\omega_1$  and  $\omega_2$  with centers  $O_1$  and  $O_2$ , respectively. One of the common tangents to the circles touches  $\omega_1$  at  $P_1$  and  $\omega_2$  at  $P_2$ , while the other touches  $\omega_1$  at  $Q_1$  and  $\omega_2$  at  $Q_2$ . Let  $M_1$  be the midpoint of  $P_1Q_1$  and  $M_2$  be the midpoint of  $P_2Q_2$ . Prove that  $\angle O_1AO_2 = \angle M_1AM_2$ .

**Problem 47** (Macedonia MO 2008).  $ABC$  is an acute-angled triangle ( $AB \neq BC$ ). Let  $AV$  and  $AD$  be the angle bisector and the altitude from vertex  $A$ , respectively. The circumcircle of  $\triangle AVD$  intersects  $CA$  and  $AB$  in points  $E$  and  $F$ , respectively. Prove that  $AD$ ,  $BE$  and  $CF$  are concurrent.

**Problem 48** (Russia MO 1999). A circle through vertices  $A$  and  $B$  of triangle  $ABC$  meets the side  $BC$  again at  $D$ . A circle through  $B$  and  $C$  meets the side  $AB$  at  $E$  and the first circle again at  $F$ . Prove that if the points  $A$ ,  $E$ ,  $D$  and  $C$  lie on a circle with center  $O$  then  $\angle BFO = 90^\circ$ .

**Problem 49** (Israel MO 1995). Let  $\omega$  be a semicircle with diameter  $PQ$ . A circle  $k$  is tangent internally to  $\omega$  and to the segment  $PQ$  at  $C$ . Let  $AB$  be the tangent to  $k$  perpendicular to  $PQ$ , with  $A$  on  $\omega$  and  $B$  on the segment  $CQ$ . Show that  $AC$  bisects  $\angle PAB$ .

**Problem 50** (IMO 2007/4). In triangle  $ABC$ , the bisector of  $\angle BCA$  intersects the circumcircle of  $\triangle ABC$  again at  $R$ , the perpendicular bisector of  $BC$  at  $P$  and the perpendicular bisector of  $AC$  at  $Q$ . The midpoint of  $BC$  is  $K$  and the midpoint of  $AC$  is  $L$ . Prove that the triangles  $\triangle RPK$  and  $\triangle RQL$  have the same area.

**Problem 51** (IMO 2003/4). Let  $ABCD$  be a cyclic quadrilateral. Let  $P$ ,  $Q$  and  $R$  be the feet of the perpendiculars from  $D$  to the lines  $BC$ ,  $CA$  and  $AB$ , respectively. Show that  $\overline{PQ} = \overline{QR}$  if and only if the bisectors of  $\angle ABC$  and  $\angle ADC$  are concurrent with  $AC$ .

**Problem 52** (IMO Shortlist 2012/G2). Let  $ABCD$  be a cyclic quadrilateral whose diagonals  $AC$  and  $BD$  meet at  $E$ . The extensions of the sides  $AD$  and  $BC$  beyond  $A$  and  $B$  meet at  $F$ . Let  $G$  be the point such that  $ECGD$  is a parallelogram, and let  $H$  be the image of  $E$  under reflection in  $AD$ . Prove that  $D$ ,  $H$ ,  $F$  and  $G$  are concyclic.

**Problem 53** (IMO Shortlist 2007/G3). The diagonals of a trapezoid  $ABCD$  intersect at point  $P$ . Point  $Q$  lies between the parallel lines  $BC$  and  $AD$  such that the line  $CD$  separates the points  $P$  and  $Q$  and  $\angle AQD = \angle CQB$ . Prove that  $\angle BQP = \angle DAQ$ .

**Problem 54** (Serbia MO 2016). Let  $ABC$  be a triangle and  $I$  its incenter. Let  $M$  be the midpoint of  $BC$  and  $D$  the tangent point of the incircle and  $BC$ . Prove that the perpendiculars from  $M$ ,  $D$  and  $A$  to  $AI$ ,  $IM$  and  $BC$ , respectively are concurrent.

**Problem 55** (Serbia MO 2016). Let  $ABC$  be a triangle and  $O$  be its circumcentre. A line tangent to the circumcircle of the triangle  $BOC$  intersects sides  $AB$  at  $D$  and  $AC$  at  $E$ . Let  $A'$  be the image of  $A$  with respect to the line  $DE$ . Prove that the circumcircle of  $\triangle A'DE$  is tangent to the circumcircle of  $\triangle ABC$ .

**Problem 56** (IMO 2008/6). Let  $ABCD$  be a convex quadrilateral ( $\overline{BA} \neq \overline{BC}$ ). Denote the incircles of triangles  $\triangle ABC$  and  $\triangle ADC$  by  $\omega_1$  and  $\omega_2$ , respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .

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