University of Northern Colorado Mathematics Contest 2022-2023

Solutions of First Round

1. Consider the subtraction problem with answer 1111 below. Each letter represents a different digit from 0 to 9, where the letter *O* represents the digit zero, and $G > U > N > C$. Find the value of the four-digit number *BRAG*.

$$\begin{array}{r} GOUNC \\ -\,BEARS \\ \hline 1\,1\,1\,1 \end{array}$$

Answer: 7358

Solution:

E is 8 or 9.

If *E* is 8, *A* must be 9 and *U* must be 1. Then *R* must be 9 and *N* must be 1. It contradicts.

So $E = 9$:

$$\begin{array}{r} G\,0\,UNC \\ -\,B\,9\,ARS \\ \hline 1\,1\,1\,1 \end{array}$$

Then we must have $C = S + 1$, $N = R + 1$, and $U = A + 1$, and $G = B + 1$.

With $G > U > N > C$ we obtain the only possible filling:

$$\begin{array}{r} 8\,0\,6\,4\,2 \\ -\,7\,9\,5\,3\,1 \\ \hline 1\,1\,1\,1 \end{array}$$

Therefore, $BRAG = 7358$.

2. 100 frogs and 100 toads are sitting on a long line of 200 lily pads in alternating order: frog, toad, frog, toad, $\cdots$, frog, toad. Each minute, one frog and one toad on neighboring lily pads switch places. What is the shortest amount of time (in minutes) before all the frogs could be end up on lily pads to the right of all the toads?

Answer: 5050

Solution:

We use the numbers from 1 to 200 inclusive to number the lily pad positions from the rightmost to the leftmost.

At the beginning the frogs have positions 2, 4, 6, $\cdots$, 200.

The sum of the position numbers for all frogs is $2 + 4 + 6 + \cdots + 200 = 2 \cdot (1 + 2 + 3 + \cdots + 100)$.

At the end the frogs have positions 1, 2, 3, $\cdots$, 100.

The sum of the position numbers for all frogs is $1 + 2 + 3 + \cdots + 100$.

In one minute, the sum of the position numbers is decreased by at most one.

Therefore, the number of minutes is at least

$$2\cdot(1+2+3+\cdots+100)-(1+2+3+\cdots+100)=1+2+3+\cdots+100=\frac{100\cdot 101}{2}=5050.$$

It can be achieved by moving the frogs only right one by one from the rightmost frog.

The answer is 5050.

3. Determine the sum of the prime divisors of $2^{16}-1$.

Answer: 282

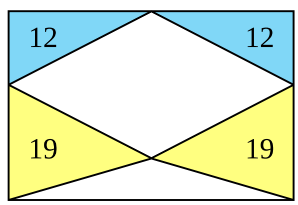
Solution:

$$2^{16}-1=(2^8-1)(2^8+1)=(2^4-1)(2^4+1)(2^8+1)=(2^2-1)(2^2+1)(2^4+1)(2^8+1)=3\cdot 5\cdot 17\cdot 257.$$

All 3, 5, 17, and 257 are primes.

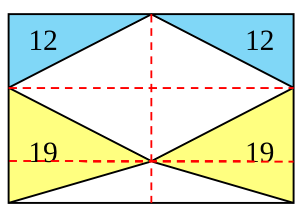
Their sum is $3+5+17+257=282$.

4. The rectangular stained-glass window shown below is missing a rhombus and an isosceles triangle of glass. The areas of the four remaining glass pieces are given. What is the total combined area of the two missing regions?



Answer: 62

Solution:



We see the total area of the two missing regions equal to

$$12+12+19+19=62.$$

5. Let m and n be positive integers. Suppose $\gcd(m,n)=18$ and $\operatorname{lcm}(m,n)=4050$. Determine the value of $m+n$.

Answer: 4068 or 612

Solution:

Let $m = 18a$ and $n = 18b$ where $\gcd(a, b) = 1$.

Then

$$4050 = 18ab.$$

We have

$$ab = 225 = 3^2 \cdot 5^2.$$

Since a and b are relatively primes, we have two pairs: $\{a, b\} = \{1, 225\}$ or $\{a, b\} = \{9, 25\}$.

We obtain

$$m + n = 18(a + b) = 18 \cdot (1 + 225) = 4068 \text{ or } m + n = 18(a + b) = 18 \cdot (9 + 25) = 612.$$

6. Your teacher has left the digits 1 through 9 in order on the board. With nothing better to do, you decide to erase each number and replace it with one more than its double (so n becomes $2n+1$). Then, if any number is more than a single digit, you replace it with the sum of its digits (possibly doing this twice). You repeat this some number of times before you are shocked to see that you once again have the digits 1 through 9 in order. What is the smallest number of times you could have repeated the process?

Answer: 6

Solution 1:

After k operations n becomes

$$2^k n + 2^k - 1.$$

It can by proved by induction:

$$2\left(2^{k-1} n + 2^{k-1} - 1\right) + 1 = 2^k n + 2^k - 1.$$

We need

$$2^k n + 2^k - 1 = n \mod 9.$$

So

$$\left(2^k - 1\right)(n+1) = 0 \mod 9.$$

If $n + 1 \neq 0 \mod 3$, $2^k - 1 = 0 \mod 9$. The smallest value of k is 6.

If $n + 1 = 0 \mod 3$, $\left(2^k - 1\right) \cdot \frac{n+1}{3} = 0 \mod 3$. The smallest value of k is 1 ($n + 1 = 0 \mod 9$) or 2 ($n + 1 \neq 0 \mod 9$).

The least common multiple of 6, 2, and 1 is 6.

The smallest number of processes is 6.

Solution 2:

If we write n in base 2, the operation is to concatenate 1 to the right of n.

For 1 we have

$$1 \to 11 \to 111 \to 1111 - 1001 = 110 \to 1101 - 1001 = 100 \to 1001 - 1001 = 0 \to 1$$

Taking two red digits away we get the remainder upon division by $1001_2 = 9$.

The cycle includes 1, 3(11_2), 7(111_2), 6(110_2), 4(100_2), and 9(0). This confirms 6 processes for each of these 6 numbers.

For 2 we see

$$10 \quad \to \quad 101 \quad \to \quad 1011 - 1001 = 10$$

The cycle includes 2 and 5(101_2). Two processes are needed for each of these 2 numbers.

For 8 we have

$$1000 \quad \to \quad 10001 - 1001 = 1000$$

One process is needed for 8.

The least common multiple of 6, 2, and 1 is 6.

The smallest number of processes is 6.

Solution 3:

For 1 we see

$$1 \quad \to \quad 3 \quad \to \quad 7 \quad \to \quad 15(6) \quad \to \quad 13(4) \quad \to \quad 9 \quad \to \quad 19(1)$$

The cycle includes 1, 3, 7, 6, 4, and 9. This confirms 6 processes for each of these 6 numbers.

For 2 we have

$$2 \quad \to \quad 5 \quad \to \quad 11(2)$$

The cycle includes 2 and 5. Two processes are needed for each of these 2 numbers.

For 8 we see

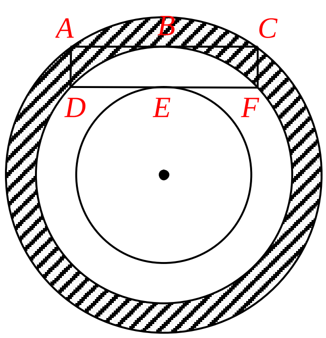
$$8 \quad \to \quad 17(8)$$

One process is needed for 8.

The least common multiple of 6, 2, and 1 is 6.

The smallest number of processes is 6.

7. The diagram below shows three concentric circles (not necessarily drawn to scale). Rectangle *ACFD* is tangent to the two inner circles at *E* and *B*. The area of the innermost circle is 3, while the area of the middle circle is 5. Find the shaded area between the middle and outermost circle

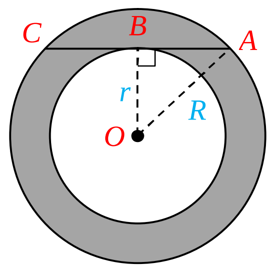


Answer: 2

Solution:

Lemma:

Two concentric circles have the common center O. AC is a chord of the larger circle and is tangent to the smaller circle at B. Then the area of the shaded ring depends on the length of chord AC only.



Proof of the lemma:

Let R and r be the radii of the larger circle and the smaller circle respectively.

Let S be the area of the shaded ring. Then

$$S = R^2\pi - r^2\pi = \pi\left(R^2 - r^2\right).$$

Draw OA and OB. Then $\angle OBA = 90°$ and $AB = \dfrac{AC}{2}$.

In right triangle OBA, $AB^2 = R^2 - r^2$. That is, $R^2 - r^2 = \dfrac{1}{4}AC^2$.

We obtain $S = \dfrac{\pi}{4}AC^2$.

That is, the area of the shaded ring depends on the length of chord AC only.

The two rings have an equal area because $AC = DF$ in the original diagram.

The shaded area in the original diagram is $5 - 3 = 2$.

8. In what number base is 217 equal to 574 in base 9?

   Hint: In base 10, 342 means $3 \cdot 10^2 + 4 \cdot 10 + 2$ and in base 8, $342 = 3 \cdot 8^2 + 4 \cdot 8 + 2$, which is the same as 226 in base 10.

Answer: 15

Solution:

Let b be the base. Then

$$2b^2 + b + 7 = 5 \cdot 9^2 + 7 \cdot 9 + 4.$$

We have the quadratic equation in b:

$$2b^2 + b - 465 = 0.$$

That is,

$$(b-15)(2b+31)=0.$$

We obtain $b=15$ since b is a positive integer.

The answer is 15.

9. The numbers on 3 doors leading away from a circular room are being replaced by a new numbering system, and an unhelpful bellhop has posted signs on the doors describing what the new numbers should be based on the original numbers. All the numbers, original and new, are positive single digits, and all distinct. The signs read:

   A. The average of this door's original number and the new number on the door that originally had the smallest number.

   B. The sum of the three original numbers.

   C. This door is the only one whose number has decreased. In fact, this number will be less than any original number.

   What is the sum of the three digits that were never used (as original or new numbers)?

Answer: 20

Solution:

From C:

The smallest number must be at least 2.

From B:

Three original numbers must be 2, 3, 4 in some order.

The new number for B is $2+3+4=9$.

The new number for C is 1.

Let a be the original number for A. We need $\frac{a+9}{2}$ or $\frac{a+1}{2}$ to be an integer.

So a is odd. Then a is 3.

Since $\frac{3+1}{2}=2$, it contradicts.

The new number for A is $\frac{3+9}{2}=6$.

We have three numbers for A, B, C respectively:

$$3,\ 2,\ 4 \quad \rightarrow \quad 6,\ 9,\ 1$$

Three digits 5, 7, and 8 are not used.

Their sum is $5+7+8=20$.

10. What is the smallest integer n such that the product $88n$ is made up of just the two digits 6 and 7?

Answer: 77

Solution:

Let $N = 88n$.

N must be divisible by 8 and 11.

N is divisible by 2. Then the ones digit is 6.

N is divisible by 4. Then the last two digits are 76.

N is divisible by 8. Then the last three digits are 776.

Now 776 is not divisible by 11. The smallest value of N has at least 4 digits.

Let the thousands digit be 6. 6776 is divisible by 11.

The smallest value of N is 6776.

The smallest value of n is $\frac{6776}{88} = 77$.