University of Northern Colorado Mathematics Contest 2022-2023

Solutions of Final Round

1. Determine the sum of all 3-digit numbers that contain only the digits 2, 3, 4, and 5.

Answer: $24,864$

Solution:

There are $4^3 = 64$ numbers.

In every digit position, any of digits 2, 3, 4, 5 appears $\frac{64}{4} = 16$ times.

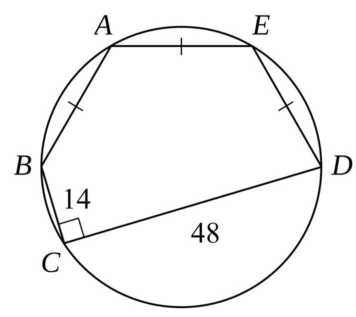
The sum of digits in a positive is

$$16 \cdot (2+3+4+5) = 224 .$$

The sum of all 64 3-digit numbers is

$$111 \cdot 224 = 24,864 .$$

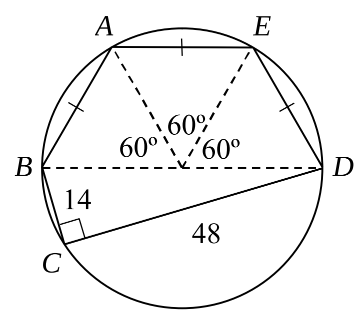
2. A pentagon inscribed in a circle has three equal length sides, a side of length 14, and a side of length 48. The sides of lengths 14 and 48 meet in a right angle, as shown in the image below. Find the perimeter of the pentagon?



Answer: 137

Solution:

Draw *BD*.



Since $\angle BCD = 90°$, *BD* is a diameter.

ΔBCD is a 7-24-25 triangle with $BC = 14 = 2 \cdot 7$ and $CD = 48 = 2 \cdot 24$.

So $BD = 2 \cdot 25 = 50$.

AB, *AE*, and *DE* are chords corresponding to arcs of $60°$.

So $AB = AE = DE = \frac{1}{2} \cdot 50 = 25$.

Therefore, the perimeter of the pentagon is

$$14 + 48 + 3 \cdot 25 = 137.$$

3. Write the numbers 1 to 100 on index cards, stacked in order. Now mix up these cards as follows: your friend flips a coin, and if it lands heads, deal a card from the top of your stack; if it lands tails, deal a card from the bottom of your stack. For example, if the coin landed HTTH your new stack would start 1, 100, 99, 2.

   Once you have dealt all 100 cards, look at the cards in the new order, taking five at a time, and compute the sum of each set of five. This will give you 20 sums. What is the largest number that you can be sure every sum is divisible by?

Answer: 5

Solution:

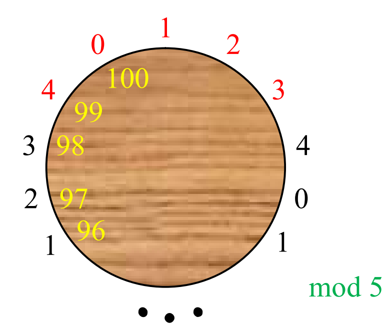
Let *L* be the largest number by which every sum is divisible.

We claim that *L* is 5.

The smallest possible sum of 5 numbers is $1 + 2 + 3 + 4 + 5 = 15 = 3 \cdot 5$. *L* is divisible by at most two prime factors 3 and 5. Actually, *L* is at most $3 \cdot 5$.

We may have the sum to be $1 + 2 + 3 + 4 + 100 = 110$ not divisible by 3. Therefore, the largest value for *L* is at most 5.

In fact, the sum of 5 numbers is always divisible by 5. It can be seen by placing numbers around a circle.



Every time we are taking 5 numbers that have all values 0, 1, 2, 3, and 4 mod 5 once.

Note that

$$0 + 1 + 2 + 3 + 4 = 10 = 0 \mod 5.$$

Therefore, all the 20 sums of 5 numbers are divisible by 5.

4. Place each digit 1 through 6 exactly once in the grid below, to form two 3-digit numbers (reading left-to-right) and three 2-digit numbers (reading top-to-bottom). If you do so satisfying the following rules, what will the sum of the two 3-digit numbers be?

Out of the five 2- and 3-digit numbers:

- The number of even numbers is even.
- The smallest number is the only multiple of 3.
- The median number is the only multiple of 5.
- The largest number is the only multiple of 11.
- No number is a multiple of 7.

Answer: 417

Solution:

There are two 3-digit numbers and three 2-digit numbers. The median is a 2-digit number. The median is a multiple of 5. Then 5 is in the bottom row.

The largest number is a 3-digit number.

Note these for the 3-digit number to be divisible by 11:

$$6 = 1 + 5 = 2 + 4,\ 5 = 1 + 4 = 2 + 3,\ 4 = 1 + 3, \text{ and } 3 = 1 + 2.$$

Case 1: $6 = 1 + 5$

Since the number divisible by 11 is the largest one, we have

5	6	1

The number above 5 is 4 to make the median.

However, 45 is not the smallest to be divisible by 3.

Case 2: $6 = 2 + 4$

We have

4	6	2

or

2	6	4

Then we have only one even number in the five numbers.

Case 3: $5 = 1 + 4$

Since the number divisible by 11 is the largest one, we have

4	5	1

The number above 5 is 6 to make the median.

The number above 1 cannot be 2 since 21 is divisible by 7.

Then we have the filling

2	6	3
4	5	1

However, we have only one even number in the 5 numbers.

Case 4: $5 = 2 + 3$

Subcase a: the number divisible by 11 is 253

2	5	3

The number above 5 is 6 to make the median.

Since the smallest number is divisible by 3, the number above 2 is 1.

We have the filling:

1	6	4
2	5	3

All conditions are satisfied:

- The number of even numbers is 2.
- The smallest number 12 is the only multiple of 3.
- The median number 65 is the only multiple of 5.
- The largest number 253 is the only multiple of 11.
- No number is a multiple of 7.

The problem indicates that there is only one solution. This is the solution.

The two 2-digit numbers are 164 and 253.

Their sum is 417 as the answer.

Do we have more solutions? Let us finish the analysis.

Subcase b: the number divisible by 11 is 352

3	5	2

The number above 5 is 6 to make the median.

Since the smallest number is divisible by 3, the number above 2 is 1.

We have the filling:

4	6	1
3	5	2

However, the number divisible by 11 is not the largest.

Case 5: $4=1+3$

Since the number divisible by 11 is the largest one, we have

3	4	1

The number under 4 is 5 to make the median.

The number under 1 is 2 for the smallest number to be divisible by 3.

We have the filling:

3	4	1
6	5	2

However, we have another number 36 to be divisible by 3.

Case 6: $3=1+2$

Note that $1+2+3+4+5+6=21=0 \mod 3$.

The two 3-digit numbers are divisible by 3. There is no filling.

Therefore, there is only one filling:

1	6	4
2	5	3

5. Consider the sequence $a_1=1$, $a_2=2+3$, $a_3=4+5+6$, $a_4=7+8+9+10$, $\cdots$. Find the value of a_{20}.

Answer: 4010

Solution 1:

Note that

$$1+2+\cdots+19=\frac{19\cdot 20}{2}=190.$$

So

$$a_{20}=191+192+\cdots+210=10\cdot(191+210)=4010.$$

Solution 2:

We derive a general formula for a_n.

Let

$$S_n=1+2+\cdots+n.$$

a_n is the sum of n numbers:

$$S_{n-1}+1,\ S_{n-1}+2,\ \cdots,\ S_{n-1}+n.$$

So

$$a_n = nS_{n-1} + 1 + 2 + \cdots + n = nS_{n-1} + S_{n-1} + n = (n+1)S_{n-1} + n$$
$$= (n+1) \cdot \frac{(n-1)n}{2} + n = n\left(\frac{n^2-1}{2} + 1\right) = \frac{n(n^2+1)}{2} \quad .$$

Therefore,

$$a_{20} = \frac{20(20^2+1)}{2} = 4010 .$$

6. Lefty McGraw, a new promising right-handed pitcher, can throw curve balls or fastballs (C or F). However, he never throws three curve balls in a row. How many different sequences of 10 pitches could he throw?

Answer: 504

Solution:

We are building strings with C and F without three C's to be consecutive.

Let f_n be the number of strings of length n.

Let a_n be the number of strings of length n ending with F.

Let b_n be the number of strings of length n ending with one C only.

Let c_n be the number of strings of length n ending with CC.

We have

$$f_n = a_n + b_n + c_n .$$

We build the recursions:

$$a_n = a_{n-1} + b_{n-1} + c_{n-1} \tag{1}$$

$$b_n = a_{n-1} \tag{2}$$

$$c_n = b_{n-1} \tag{3}$$

From (2)

$$b_{n-1} = a_{n-2} \tag{4}$$

From (3)

$$c_{n-1} = b_{n-2} = a_{n-3} \tag{5}$$

Plugging (4) and (5) into (1) we obtain

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} .$$

Since all equations are linear, we have the same recursion for f_n:

$$f_n = f_{n-1} + f_{n-2} + f_{n-3} .$$

Calculate three initial values:

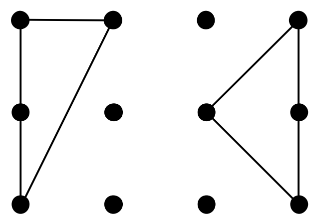
$$f_0 = 1 , \ f_1 = 2 , \ f_2 = 4 .$$

Running the recursion we have the sequence:

$$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, \cdots.$$

We see $f_{10} = 504$.

7. Determine how many right triangles can be made on a 4 × 3 geoboard with a rubber band. That is, how many right triangles can be formed using 3 of the 12 dots as vertices arranged in a 3 × 4 rectangle? The dots are equally spaced horizontally and vertically. Two samples are drawn.

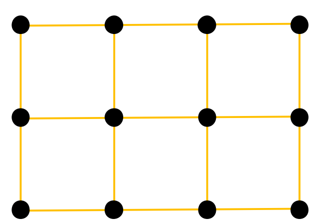


Answer: 94

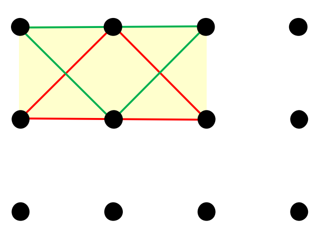
Solution:

In a rectangle we have four right triangles by choosing three vertices of the rectangle.

There are $\binom{3}{2} \cdot \binom{4}{2} = 3 \cdot 6 = 18$ rectangles with two horizontal lines and two vertical lines as their sides.

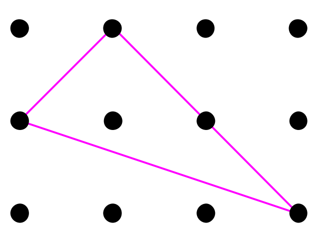


In a 1 by 2 rectangle, there are two extra right triangles:

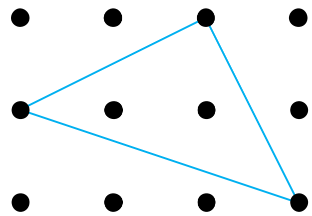


There are 7 1 by 2 rectangles.

There are 4 $\sqrt{2}$-$\sqrt{8}$-$\sqrt{10}$ right triangles:



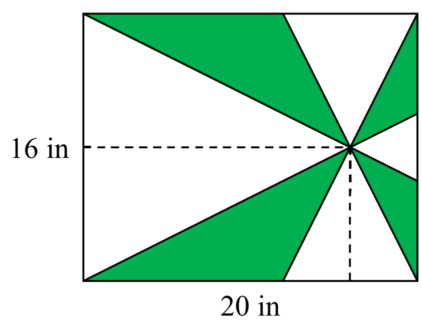
There are 4 $\sqrt{5}$-$\sqrt{5}$-$\sqrt{10}$ right triangles:



The total number of right triangles is

$$18\cdot 4+7\cdot 2+4+4=94.$$

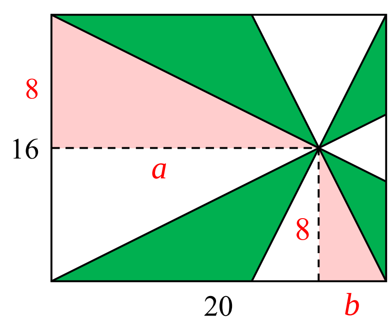
8. The rectangular stained-glass window shown here is 16 inches tall and 20 inches wide. The window consists of four triangles made of colored glass and four triangles made of clear glass. The four clear glass triangles are isosceles and are similar to one another. Find the total area, in square inches, of the four colored triangles.



Answer: 120

Solution:

Look at two similar triangles shaded by light red:



Mark the lengths as shown where $a>10$.

We have $\frac{a}{8}=\frac{8}{b}$. So $b=\frac{64}{a}$.

With $a+b=20$ we have $a+\frac{64}{a}=20$.

It is a quadratic equation in *a*. We observe two roots to be 4 and 16.

Since $a>10$, we take $a=16$.

We see one white triangle with base 16 and height 16, two white triangles with base 8 and height 8, and one white triangle with base 4 and height 4.

The total area of the four colored triangles is

$$16\cdot 20-\frac{1}{2}\cdot\left(16^2+2\cdot 8^2+4^2\right)=120.$$

9. Let T_n be the number of ways you can write n as a sum of one or more consecutive positive integers. For example, T_{15} is 4, since,

$$1+2+3+4+5=4+5+6=7+8=15.$$

Find the value of T_{45}.

Answer: 6

Solution:

Let n be the number of consecutive positive integers with the first number to be a ($a\ge 1$).

The following n numbers have 45 as the sum:

$$a,\ a+1,\ a+2,\ \cdots,\ a+n-1.$$

We have

$$\frac{a+a+n-1}{2}\cdot n=45.$$

That is,

$$n(2a+n-1)=2\cdot 3^2\cdot 5.$$

Note that $2a+n-1\ge 2+n-1=n+1>n$, and n and $2a+n-1$ are of different parities.

$2\cdot 3^2\cdot 5$ has $(1+1)(2+1)(1+1)=12$ factors making 6 pairs with each pair to have one even factor and one odd factor.

We just assign the smaller factor to n and the larger factor to $2a+n-1$ for each pair.

Therefore,

$$T_{45}=6.$$

Let us make a table to find the six expressions:

n	$2a+n-1$	a	Sum
1	90	45	$45=45$
2	45	22	$22+23=45$
3	30	14	$14+15+16=45$
5	18	7	$7+8+9+10+11=45$
6	15	5	$5+6+7+8+9+10=45$
9	10	1	$1+2+\cdots+9=45$

10. Oscar and Nat like to play the game "Number Guesser." The way they play is as follows: Nat thinks of a whole number between 1 and 100, and Oscar has to figure out what number Nat is thinking of. He can only ask questions of the form "Is your number divisible by n?", where n is a whole number. When Oscar knows what number Nat is thinking of, he rings a bell and announces the number.

    One time they were playing this game, and Oscar figured out Nat's number after asking 9 questions. After announcing the answer, he correctly observed, "That felt like it took a lot of questions, but actually, I could not have figured out your number without asking at least that many." What was Nat's number?

Answer: 5

Solution:

Oscar checks all possible prime factors and the times these primes appear.

If Nat's number is 2, Oscar needs to check 2, 4, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. The least number of questions by Oscar is at least 16.

If Nat's number is 3, Oscar needs to check 3, 9, 2, 5, 7, 11, 13, 17, 19, 23, 29, 31. The least number of questions by Oscar is at least 12.

If Nat's number is 4, Oscar needs to check 4, 8, 3, 5, 7, 11, 13, 17, 19, 23. The least number of questions by Oscar is at least 10.

If Nat's number is 5, Oscar needs to check 5, 25, 2, 3, 7, 11, 13, 17, 19. The least number of questions by Oscar is at least 9.

If Nat's number is 6, Oscar needs to check 2, 4, 3, 9, 5, 7, 11, 13. The least number of questions by Oscar is at least 8.

If Nat's number is larger than 6, Oscar needs to check primes up to 13. The least number of questions by Oscar is less than 9.

For the least number of questions by Oscar to be 9, Nat's number is 5 only.

The answer is 5.

11. Prove that your answer to question 3, the shuffled numbers question, is correct.

Solution:

See the solution in problem 3.

12. Explain your reasoning that led to your answer to question 7, the geoboard triangle question. Then if possible, give a generalization and justified solution.

See the solution in problem 7.

13. Find a formula for T_{2^n}, the number of ways to write 2^n as a sum of consecutive positive integers, as in question 9. Prove your formula is correct.

Answer: $T_{2^n} = 1$

Solution:

We find the general solution for T_k.

Let $k = 2^m t$ where $m \geq 0$, t is odd, and t has f factors.

Let n be the number of consecutive positive integers with the first number to be a ($a \geq 1$).

The following n numbers have k as the sum:

$$a,\ a+1,\ a+2,\ \cdots,\ a+n-1.$$

We have

$$\frac{a+a+n-1}{2} \cdot n = k.$$

That is,

$$n(2a+n-1) = 2^{m+1} t.$$

Note that $2a+n-1 > n$, and n and $2a+n-1$ are of different parities.

Let u and v be a pair of factors of t with $uv = t$ and $u \neq v$. There are two ways to assign values: $\left(2^{m+1}u, v\right)$ and $\left(u, 2^{m+1}v\right)$ for n and $2a+n-1$.

If t is a square with $u^2 = t$, there is one way to assign values: $\left(2^{m+1}u, u\right)$ for n and $2a+n-1$ for this factor u.

Anyway, the number of ways to assign two values to n and $2a+n-1$ is f.

Therefore, the number of ways to write $k = 2^m t$ ($m \geq 0$ and $t \neq 0$ mod 2) as a sum of one or more consecutive positive integers is the number of factors of t.

Therefore, $T_{2^n} = 1$.