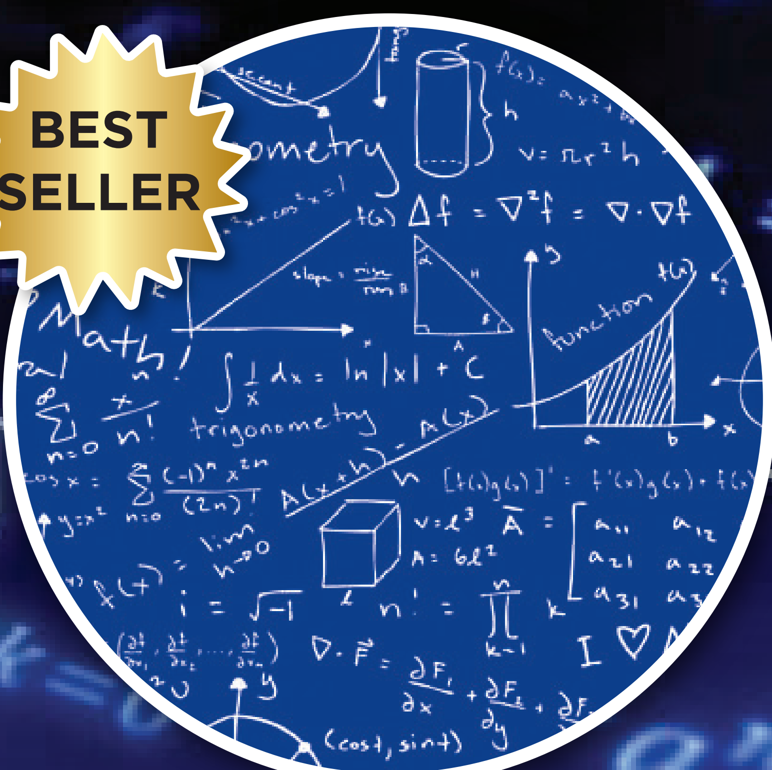
A Holistic Approach to A-Level

SUBSIDIARY MATHEMATICS

THIRD EDITION

KAWUMA FAHAD



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BASED ON THE NEW NCDC SYLLABUS

A book that guarantees you a point in Sub-Math

A HOLISTIC APPROACH TO A-LEVEL

Subsidiary Mathematics

THIRD EDITION

BASED ON THE NEW NCDC
SYLLABUS

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Preface

Am happy to introduce the thoroughly revised edition of A Holistic Approach to A-Level Subsidiary Mathematics, Third Edition, which has been restructured and redesigned strictly in accordance with the new syllabus. The National Curriculum Development Centre made changes in the syllabus for the UNEB examinations to be held starting from the year 2023. The complete section of Mechanics has been eliminated from the syllabus and one new chapter, Linear Programming has been introduced. The examination format has also been restructured as I will discuss it later. I have also used this edition as an opportunity to incorporate the invaluable suggestions and feedback received from different examiners and Subsidiary Mathematics teachers all over the country from time to time.

All chapters in this book start from the basic level and there is a gradual increase in the difficult level which ultimately matches the level expected from the students. Most of the concepts have been explained in multiple ways, so that the students can understand with or without any external assistance. This will help the learners develop the ability to prepare for the final examinations by themselves

Salient features of this book are:

- This book is designed strictly according to the latest NCDC syllabus for A-Level
- The chapters have been sequenced in an order that is easy to follow as released by the NCDC
- The problems have been carefully selected and well graded keeping in mind their difficulty levels
- Some chapters have been thoroughly revised and rehashed
- Important questions from the examination past papers have been included appropriately at the end of each chapter.
- Self-Evaluation Exercises have been included at the end of each chapter to help the learner acquaint themselves with the skills involved
- A Sample examination paper based on the new examination format has been included

Now I will discuss the new examination format
There will be one paper of 2 hours 40 minutes. The Paper will consist of two sections A and B.

Section A will comprise **Eight (8)** short compulsory questions. **Four (4)** from Pure Mathematics and **Four (4)** from Statistics. Each question in this section will carry 5 marks giving a total of 40 marks.

Section B will consist of two parts; **PURE** in part 1 and **STASTICS** in part 2, each with four questions. Candidates will be required to answer FOUR (4) questions from this section, selecting at least **ONE** question from each part. Each of the questions attempted in this Section will carry 15 marks, giving a total of 60 marks.

Contents

Chapters

Chapter 1 Matrices and their Applications

Introduction

Suppose in a college, there were 400 first year, 350 second year and 300 third year students in 1998. In 1999, the number of students in different years were respectively 410, 365 and 315. We can express this information in tabular form as below.

No. of Students			
Year	1st Year	2nd Year	3rd Year
1998	400	350	300
1999	410	365	315

This arrangement of numbers can be expressed as

$$\begin{bmatrix} 400 & 350 & 300 \\ 410 & 365 & 315 \end{bmatrix} \text{ or } \begin{pmatrix} 400 & 350 & 300 \\ 410 & 365 & 315 \end{pmatrix}$$

The above arrangement is known as a matrix.

Definition

A matrix is defined as a rectangular arrangement of numbers in rows and columns.

$$\text{e.g,} A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & \sqrt{2} & 0 \end{bmatrix}, \qquad B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 7 \end{pmatrix} \begin{matrix} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \leftarrow \text{row 3} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{col. 1} & \text{col. 2} & \text{col. 3} \end{matrix}$$

Note:

1. A matrix is always denoted by a capital letter, A, B, C, .., etc.
2. The numbers which are listed within brackets are known as elements or entries or members of the matrix.
3. Generally [ ] or ( ) brackets are used to denote a matrix.
4. The horizontal lines are known as rows and vertical lines as columns. Thus the matrix A has two rows and three columns. We say A to be a rectangular (since, the number of rows is different from the number of columns) matrix of order 2 × 3 and B to be a square matrix of order 3 × 3 (since the number of rows is same as the number of columns)

Some Definitions

1. Row matrix: This is a matrix which has only one row e.g. $[2 \quad -3 \quad 1]$
2. Column matrix: This is a matrix which has only one column e.g. $\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$
3. Null or zero matrix: This is a matrix whose each element is zero, and is denoted by O. e.g.
$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
4. Diagonal matrix: This is a square matrix whose each element except those in the principal diagonal is zero.
$$\text{e.g. } \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Note: A square matrix has two diagonals. The diagonal extending from left-hand top corner to right-hand bottom corner is known as principal or main leading diagonal. The other diagonal is known as a second diagonal.

5. **Scalar matrix:** This is a diagonal matrix whose each element in the principal diagonal is the same.

$$\text{e.g.} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

6. Upper triangular matrix: This is a square matrix whose each element below the principal diagonal is zero.

$$\text{e.g} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

7. Lower triangular matrix: This a square matrix whose each element above the principal diagonal is zero

$$\text{e.g} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

8. Transpose of a matrix A: This is the matrix obtained from the given matrix A by interchanging its rows and columns and it is denoted by A' or A^T

$$\text{e.g. Let A} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \end{bmatrix} \text{ then } A' = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

A is of order 2×3 and A' is of order 3×2

9. Symmetric matrix: A square matrix A is said to be symmetric if $A' = A$

$$\text{e.g. } A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 4 \\ 1 & 4 & 6 \end{bmatrix}$$

Equality of Two Matrices

Two matrices are said to be equal to each other only if they are of same order and their corresponding elements are equal to each other.

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $B = \begin{bmatrix} p & q & r \\ x & y & z \end{bmatrix}$,

then $A = B$ implies $a = p, b = q, c = r, d = x, e = y$ and $f = z$

Operations on Matrices

1. Addition and Subtraction of two matrices

Two matrices can be added or subtracted only if they are of the same order and this is done by adding or subtracting the corresponding elements of two matrices.

Example

$$\begin{bmatrix} 3 & -1 & 7 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3+2 & -1-2 & 7-1 \\ 0+1 & 1+2 & 2-3 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 6 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 7 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3-2 & -1+2 & 7+1 \\ 0-1 & 1-2 & 2+3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 8 \\ -1 & -1 & 5 \end{bmatrix}$$

2. Multiplication of a matrix by a number

If a matrix is multiplied by a number k, then each element becomes k times the original value.

Example

Example 10

Given that $A = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -4 \\ 2 & 1 \end{pmatrix}$.

Show that $A + B$ is a singular matrix.

Solution:

$$A + B = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$$

$$\det(A + B) = 3 \times 2 - 2 \times 3 = 6 - 6 = 0$$

Since the $det\ (A + B) = 0, A + B$ is a singular matrix

Example 11

Given that matrix $A = \begin{pmatrix} 4 & 2 \\ a & 3 \end{pmatrix}$ is a singular matrix, find the value of a.

Solution:

$$\det A = 4 \times 3 - 2 \times a = 12 - 2a$$

For a singular matrix, $\det A = 0$

$$\Rightarrow 12 - 2a = 0$$

$$2a = 12$$

$$a = 6$$

Example 12

Given that $M = \begin{pmatrix} 3a & a-6 \\ -6 & a+2 \end{pmatrix}$, find the values of a for which the matrix M is singular

Solution:

$$\det M = 3a(a+2) - (-6)(a-6)$$

$$= (3a^2 + 6a) - (-6a + 36)$$

$$= 3a^2 + 6a + 6a - 36$$

$$= 3a^2 + 12a - 36$$

Since matrix M is singular, then $det\ M = 0$

$$\Rightarrow \quad 3a^2 + 12a - 36 = 0$$

$$a^2 + 4a - 12 = 0$$

$$a^2 - 2a + 6a - 12 = 0$$

$$a(a-2) + 6(a-2) = 0$$

$$(a-2)(a+6) = 0$$

$$\therefore a = 2 \text{ or } a = -6$$

Solution of two linear equations in two unknowns by Cramer's Rule

We consider two equations

$a_1x + b_1y = d_1$ and $a_2x + b_2y = d_2$

Then according to Cramer's rule

$$x = \frac{\Delta_1}{\Delta} \text{ and } y = \frac{\Delta_2}{\Delta}, (\Delta \neq 0)$$

where $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

Steps for applying Cramer's Rule:

1. Form Δ by taking the coefficients of unknowns and find its value
2. Form Δ_1 by replacing the first column of Δ by d_1, d_2 and find its value.
3. Form Δ_2 by replacing the second column of Δ by d_1, d_2 and find its value
4. Apply the formula: $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}$

Example 13

Using Cramer's rule, solve the following system of equations

(a) $3x - y = 7$ and $2x + 5y = -1$

(b) $\frac{2}{x} - \frac{3}{y} = 0$ and $\frac{1}{x} + \frac{2}{y} = 7$

Solution:

(a) $\Delta = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = 17$, $\Delta_1 = \begin{vmatrix} 7 & -1 \\ -1 & 5 \end{vmatrix} = 34$

$\Delta_2 = \begin{vmatrix} 3 & 7 \\ 2 & -1 \end{vmatrix} = -17$

$x = \frac{\Delta_1}{\Delta} = \frac{34}{17} = 2$

$y = \frac{\Delta_2}{\Delta} = -\frac{17}{17} = -1$

(b) Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$

The equations become

$$2a - 3b = 0$$

$$a + 2b = 7$$

Now, $\Delta = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7$, $\Delta_1 = \begin{vmatrix} 0 & -3 \\ 7 & 2 \end{vmatrix} = 21$

$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 1 & 7 \end{vmatrix} = 14$

$a = \frac{\Delta_1}{\Delta} = \frac{21}{7} = 3$; $b = \frac{\Delta_2}{\Delta} = \frac{14}{7} = 2$

$$\therefore x = \frac{1}{3}; y = \frac{1}{2}$$

Adjoint of a Square Matrix A [denoted by adj A]

This is the transpose of the matrix obtained from the given matrix by interchanging the elements of the major diagonal and multiplying the elements of the minor diagonal by -1

Let $B = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$, then

$$\text{adj } B = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$$

An important note:

$$A \,.\, \text{adj } A = \text{adj } A \,.\, A = \det A \,.\, I$$

Inverse of a Non-Singular Square Matrix A

Let A be a square matrix such that $\det A \neq 0$. If there exists a matrix B such that $AB = BA = I$ where A, B and I are of the same order, then B is known as the inverse of A and is written as $B = A^{-1}$. Thus,

$$A.A^{-1} = A^{-1}.A = I$$

Steps to obtain A^{-1}

1. Find $\det A$
2. Find adj A
3. Finally use the formula

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$

Note:

1. If A and B are two square matrices of same order such that $AB = BA = I$, then
$$A^{-1} = B \quad \text{and also } B^{-1} = A$$
Again, $A = B^{-1} = (A^{-1})^{-1}$
$$\therefore (A^{-1})^{-1} = A$$
2. For two matrices A and B of same order
$$(AB)^{-1} = B^{-1}A^{-1}$$
3. For a square matrix A
$$(A^{-1})^T = (A^T)^{-1}$$

Example 14

Find the inverse of the matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$. Also verify that $A \,.\, A^{-1} = I$

Solution:

$\det A = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 8 - (-3) = 11$

adj $A = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A} \text{adj } A = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$

Verification:

$$A \,.\, A^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} . \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$
$$= \frac{1}{11}\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$
$$= \frac{1}{11}\begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example 15

Find the matrix A, if $A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$

Solution:

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$\det A^{-1} = \left(\frac{3}{5}\right)\left(\frac{2}{5}\right) - \left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right) = \frac{6}{25} - \frac{1}{25} = \frac{1}{5}$$

$$\text{adj } A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = \frac{1}{\det A^{-1}} \text{. adj } A^{-1} = 5 \,. \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Example 16

Given the matrix $A = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$, find $(AB)^{-1}$

Solution:

$$AB = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \times -1 + 1 \times 2 & 4 \times 1 + 1 \times 3 \\ 5 \times -1 + 2 \times 2 & 5 \times 1 + 2 \times 3 \end{pmatrix}$$
$$= \begin{pmatrix} -4 + 2 & 4 + 3 \\ -5 + 4 & 5 + 6 \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ -1 & 11 \end{pmatrix}$$
$$\det(AB) = (-2 \times 11) - (-1 \times 7) = -22 + 7$$
$$= -15$$
$$\text{Adj } AB = \begin{pmatrix} 11 & -7 \\ 1 & -2 \end{pmatrix}$$
$$(AB)^{-1} = \frac{1}{-15}\begin{pmatrix} 11 & -7 \\ 1 & -2 \end{pmatrix}$$

Example 17

Find the 2×2 matrix $A = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$ if $\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} A = \begin{pmatrix} -16 & -6 \\ 7 & 2 \end{pmatrix}$

Solution:

Let $B = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} -16 & -6 \\ 7 & 2 \end{pmatrix}$

$$BA = C$$
$$\Longrightarrow A = B^{-1}C$$
$$\det B = \begin{vmatrix} 5 & -7 \\ -2 & 3 \end{vmatrix} = 5 \times 3 - (-7)(-2) = 1$$
$$\text{adj } B = \begin{pmatrix} 3 & -(-2) \\ -(-7) & 5 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$$
$$B^{-1} = \frac{1}{\det B} \text{adj } B = \frac{1}{1}\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$$

Thus,

$$A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -16 & -6 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 3 & -2 \end{pmatrix}$$

Solution of a system of Linear Equations by Matrix Inversion method

We consider the following two linear equations in two unknowns x, y

$$a_1x + b_1y = d_1$$
$$a_2x + b_2y = d_2$$

The equations can be written as

$$AX = B \quad \text{…. (i)}$$

where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

We now premultiply equation (i) by A^{-1}

$$\therefore A^{-1}(AX) = A^{-1}B \quad \text{or} \quad (A^{-1}A)X = A^{-1}B$$
$$\text{or } IX = A^{-1}B \quad \text{or } X = A^{-1}B$$

Thus, the solution of the system of equations is obtained from

$$\boldsymbol{X = A^{-1}B}$$

Steps for the solution of linear equations by matrix method

Step 1. Write the system of equations in the form $AX = B$, where A is the matrix obtained from the coefficients, X is the column matrix of the variables and B is the column matrix containing the values.

Step 2. Find A^{-1}

Step 3. Find $X = A^{-1}B$

Example 18

Using matrices, solve the following equations

$3x + y = 5$ and $x + 2y = 3$

Solution:

The given equations can be written as $AX = B$

where $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$$\det A = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$$

$$adj\ A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A}\text{adj } A = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7/5 \\ 4/5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7/5 \\ 4/5 \end{pmatrix}$$

$$\therefore x = \frac{7}{5}, \qquad y = \frac{4}{5}$$

Example 19

Solve the simultaneous equations using the matrix method

$$2x + y = 3$$
$$4x - 2y = 10$$

Solution:

$$\begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$$

let $A = \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix}$, $B = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$

$$\det A = (2 \times -2) - (4 \times 1) = -4 - 4 = -8$$

$$A^{-1} = -\tfrac{1}{8}\begin{pmatrix} -2 & -1 \\ -4 & 2 \end{pmatrix}$$

But $B = A^{-1}C \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\tfrac{1}{8}\begin{pmatrix} -2 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 10 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} -2 \times 3 + -1 \times 10 \\ -4 \times 3 + 2 \times 10 \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} -16 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$x = 2 \text{ and } y = -1$$

Example 20

Solve the simultaneous equations below using the matrix method.

$$x + 2y = 4$$
$$x + y = 3$$

Solution:

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Now $AB = C \quad \Rightarrow B = \frac{C}{A}$ which gives $B = A^{-1}C$

$$B = A^{-1}C$$

$$\det A = (1 \times 1) - (2 \times 1) = 1 - 2 = -1$$

$$A^{-1} = \tfrac{1}{-1}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

But $B = A^{-1}C \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \times 4 + 2 \times 3 \\ 1 \times 4 + -1 \times 3 \end{pmatrix} = \begin{pmatrix} -4 + 6 \\ 4 + -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

From the equality of matrices, $x = 2$ and $y = 1$

Solving mathematical problems using the matrix method

In day-to-day life, we are obligated to solving problems which require the ideas of finding total expenditures, the would be amount of profit got after transacting a business and others. These mathematical problems can be solved using the matrix approach

2016, No. 7

Given the matrices $A = \begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & -3 \\ -4 & 7 \end{pmatrix}$, find

(a) matrix C such that $3A - 2C + B = I$, where I is a 2×2 identity matrix

(b) the determinant of C

Solution:

(a) Let matrix $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$3A - 2C + B = I$$

$$3\begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix} - 2\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 8 & -3 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 15 \\ -6 & 12 \end{pmatrix} - \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} + \begin{pmatrix} 8 & -3 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9 - 2a + 8 & 15 - 2b - 3 \\ -6 - 2c - 4 & 12 - 2d + 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 17 - 2a & 12 - 2b \\ -10 - 2c & 19 - 2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$17 - 2a = 1 \Longrightarrow a = 8$$

$$12 - 2b = 0 \Longrightarrow b = 6$$

$$-10 - 2c = 0 \Longrightarrow c = -5$$

$$19 - 2d = 1 \Longrightarrow d = 9$$

$$\therefore C = \begin{pmatrix} 8 & 6 \\ -5 & 9 \end{pmatrix}$$

(b) $\det C = 8 \times 9 - (-5) \times 6 = 72 + 30 = 102$

2017, No. 7

Using the matrix method, solve the simultaneous equations

$$3x - y = 16$$
$$x + 2y = 3$$

Solution:

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 3 \end{pmatrix}$$

Let $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

$$\det A = (3 \times 2) - (-1 \times 1) = 6 + 1 = 7$$

$$\text{Adj } A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 16 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 \times 16 + 1 \times 3 \\ -1 \times 16 + 3 \times 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 35 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$x = 5$ and $y = -1$

2020, No. 12

(a) Given that $M = \begin{pmatrix} 4x & 6 \\ -5 & -2x \end{pmatrix}$, $N = \begin{pmatrix} -1 & -2 \\ 3 & 3y \end{pmatrix}$, $K = \begin{pmatrix} y & 4 \\ -2 & 12 \end{pmatrix}$ and $K = M + N$, find the values of x and y

(b) In a football tournament, three teams Arsenal, Chelsea and Liverpool had the following results:

- Arsenal won two matches, drew once and lost one match
- Chelsea won two matches and lost two matches
- Liverpool won 1 match, drew twice and lost one match

The teams are awarded 3 points for a win, 1 point for a draw and no point for a loss.

(i) Write a 3×3 matrix for the results and a column matrix for the points

(ii) By matrix multiplication, determine the winner of the tournament

Solution:

(a) $K = M + N$

$$\begin{pmatrix} y & 4 \\ -2 & 12 \end{pmatrix} = \begin{pmatrix} 4x & 6 \\ -5 & -2x \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ 3 & 3y \end{pmatrix}$$

$$\begin{pmatrix} y & 4 \\ -2 & 12 \end{pmatrix} = \begin{pmatrix} 4x - 1 & 4 \\ -2 & -2x + 3y \end{pmatrix}$$

$$y = 4x - 1 \quad \ldots \text{(i)}$$

$$12 = -2x + 3y \quad \ldots \text{(ii)}$$

Substitute y from (i) into (ii);

$$12 = -2x + 3(4x - 1)$$

$$12 = -2x + 12x - 3$$

$$15 = 10x$$

$$x = \frac{3}{2}$$

Using (i) ; $y = 4\left(\frac{3}{2}\right) - 1 = 5$

(b) (i) Result matrix $= \begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

Points matrix $= \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 1 \times 1 + 1 \times 0 \\ 2 \times 3 + 0 \times 1 + 2 \times 0 \\ 1 \times 3 + 2 \times 1 + 1 \times 0 \end{pmatrix}$

$$= \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix}$$

The teams with the most points, Arsenal (7 points) won the tournament

Chapter 2 Linear Programming

Introduction

In a military operation, the effort is to inflict maximum damage to the enemy at minimum cost and loss. In an industry, the management always tries to utilize its resources in the best possible manner. The industrialist would like to have unlimited profits, but he is constrained by limited manpower, capital and market demand. A salaried person tries to make investments in such a manner that the returns on investment are high but at the same time income tax liability is also kept low.

In all the above cases, if constraints are represented by linear equations/inequations (in one, two or more variables), and a particular plan of action from several alternatives is to be chosen, we use linear programming. The word linear means that all inequations used and the function to be maximised or minimized are linear, and the word programming refers to planning (choosing amongst alternatives) rather than the computer programming sense.

Thus, linear programming is a method of determining optimum values of a linear function subject to constraints expressed as linear equations and inequalities.

A practical problem may involve dozens of variables, and is usually solved by using Simplex Method and a computer. In this chapter, we will restrict ourselves to linear functions involving two variables, so that they can be solved by drawing a graph on x-y plane.

Common terms

Objective function: A linear function $Z = ax + by$, where a and b are constants which has to be maximized or minimized according to a set of given conditions, is referred to as linear objective function.

Decision Variables: In the objective function $Z = ax + by$, the variables x, y are said to be decision variables.

Constraints: The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \geq 0$, $y \geq 0$ are known as non-negative restrictions.

Feasible region: The common region determined by all the constraints including non-negative constraints x,$y \geq 0$ of a linear programming problem is known as the feasible region.

Feasible solution: Points within and on the boundary of the feasible region represent feasible solutions of constraints. In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.

Theorem 1: Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner p into (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function R has both maximum and minimum values of R and each of these occurs at a corner point (vertex) of R.

Find the maximum profit if each unit of commodity A earns a profit of 2 dollars and each unit of B earns a profit of 3 dollars.

Solution:

Let x be the no. of units of commodity A produced and y be the no. of units of commodity B produced, then

$x \geq 0$, $y \geq 0$

$x + 2y \leq 8$ … (i)

$3x + 2y \leq 12$ … (ii)

$x + y \leq 9$ …. (iii)

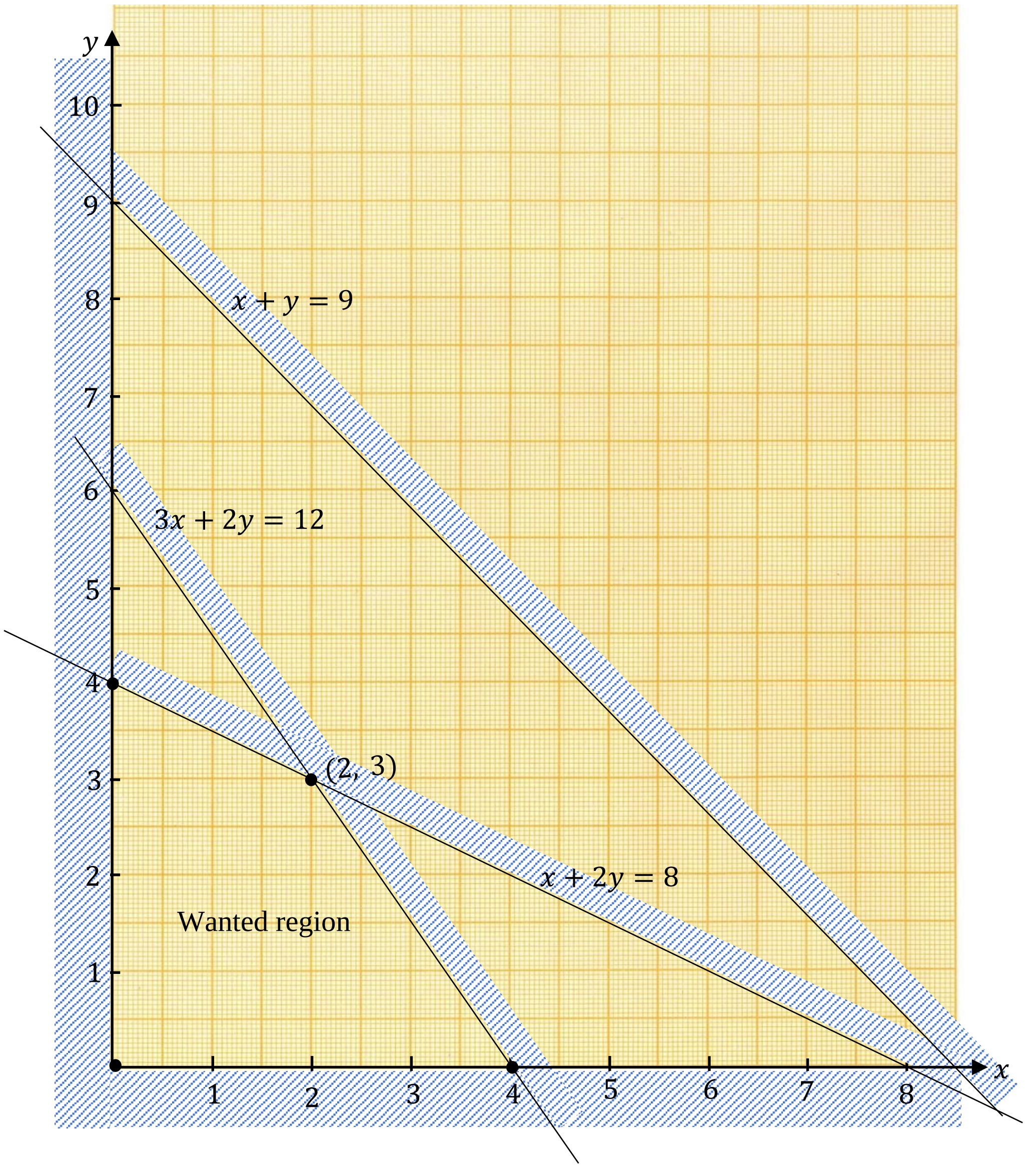
Consider the equations $x + 2y = 8$, $3x + 2y = 12$ and $x + y = 9$

$x + 2y = 8$		
x	0	8
y	4	0

$3x + 2y = 12$		
x	0	4
y	6	0

$x + y = 9$		
x	0	9
y	9	0

Plot a graph to locate these points



Let z be the profit, then

$$z = 2x + 3y$$

Now we can determine the maximum value of Z by evaluating the value of Z at the vertices.

Points	$z = 2x + 3y$
(0, 0)	1
(4, 0)	8
(2, 3)	13 (maximum)
(0, 4)	12

Hence the maximum profit is 13 dollars.

Example 2

A factory manufactures two types of radio sets (a) ordinary radio and (b) deluxe radio. 20 machine hours are needed to manufacture one ordinary radio and 60 machine hours are spent on one deluxe radio. Cost of an ordinary radio is 100 dollars and that of a delux radio is 600 dollars. According to the government regulation, at least 3 ordinary and 2 delux radios have to be manufactured daily. Also due to the law of labour board not more than 320 machine hours are available in one day. If the capacity of the factory is to produce in all not more than 10 radios a day, find how many radios of each type should be manufactured to spend minimum cost.

Solution:

Let the number of ordinary and delux radios to be produced per day be respectively x and y. Therefore

Total cost, $C = 100x + 600y$

According to government regulation $x \geq 3$ and $y \geq 2$

According to law of labour board, $20x + 60y \leq 320$

According to capacity of factory, $x + y \leq 10$

Again the number of radios cannot be negative, i.e. $x \geq 0$ and $y \geq 0$

Thus we have to maximise the objective function

$$C = 100x + 600y$$

Subject to constraints: $x \geq 3$

$y \geq 2$

$20x + 60y \leq 320$ i.e. $x + 3y \leq 16$

$x + y \leq 10$

and non-negative restrictions: $x \geq 0$ and $y \geq 0$

Consider the equations $x + 3y = 16$ and $x + y = 10$

$x + 3y = 16$		
x	1	16
y	5	0

$x + y = 10$		
x	0	10
y	10	0

The feasible region is the shaded area ABCD. The extreme points are A(3, 2), B(8, 2), C(7, 3) and D(3, 13/3)

The values of the objective function C at different extreme points are

$C = 100 \times 3 + 600 \times 2 = 1500$ at A(3, 2)

$= 100 \times 8 + 600 \times 2 = 2000$ at B(8, 2)

$= 100 \times 7 + 600 \times 3 = 2500$ at C(7, 3)

$= 100 \times 3 + 600 \times \frac{13}{2} = 2900$ at D(3, 13/3)

Thus the cost is minimum at A(3, 2) i.e. when 3 ordinary radios and 2 deluxe radios are manufactured.

Item	No. of aids	Time on fabricating (in hrs)	Time on finishing (in hrs)	Profit (in Shs)
A	x	$9x$	x	$80x$
B	y	$12y$	$3y$	$120y$
Total	$x+y$	$9x+12y$	$x+3y$	$80x+120y$
Availability		180	30	

The profit on type A is Shs 80 and type B is Shs 120. Thus, the required LPP is

$$\text{Max } Z = 80x + 120y \quad .... \text{(i)}$$

Subject to constraints

$$9x + 12y \leq 180 \text{ or } 3x + 4y = 60 \quad ... \text{(ii)}$$
$$x + 3y \leq 30 \quad ... \text{(iii)}$$
$$x \geq 0, y \geq 0$$

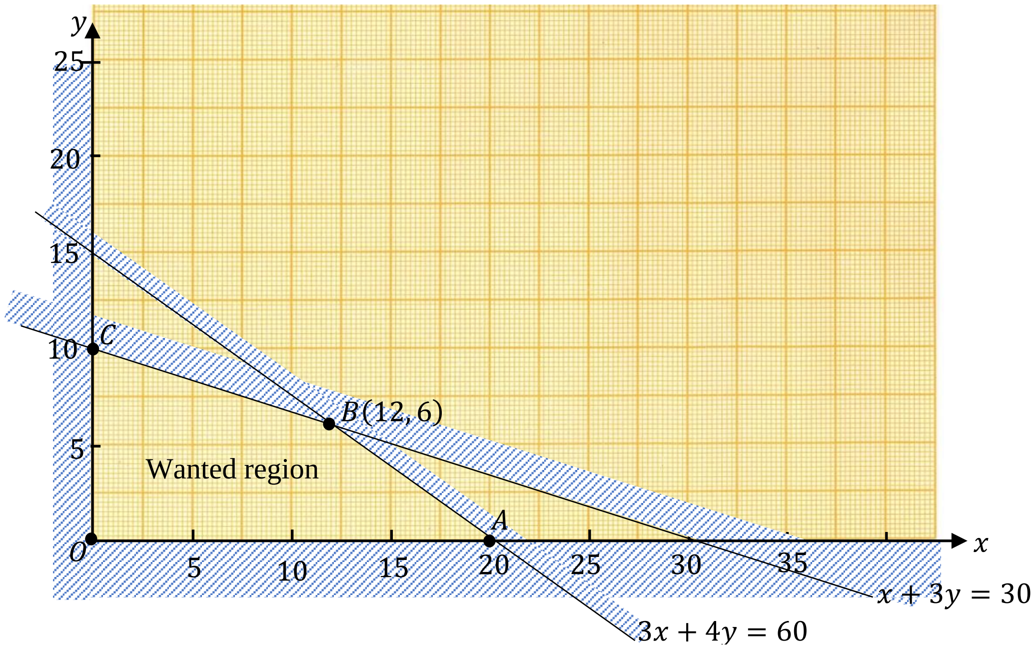
Consider the equations, $3x + 4y = 60$

x	0	20
y	15	0

and $x + 3y = 30$

x	0	30
y	10	0

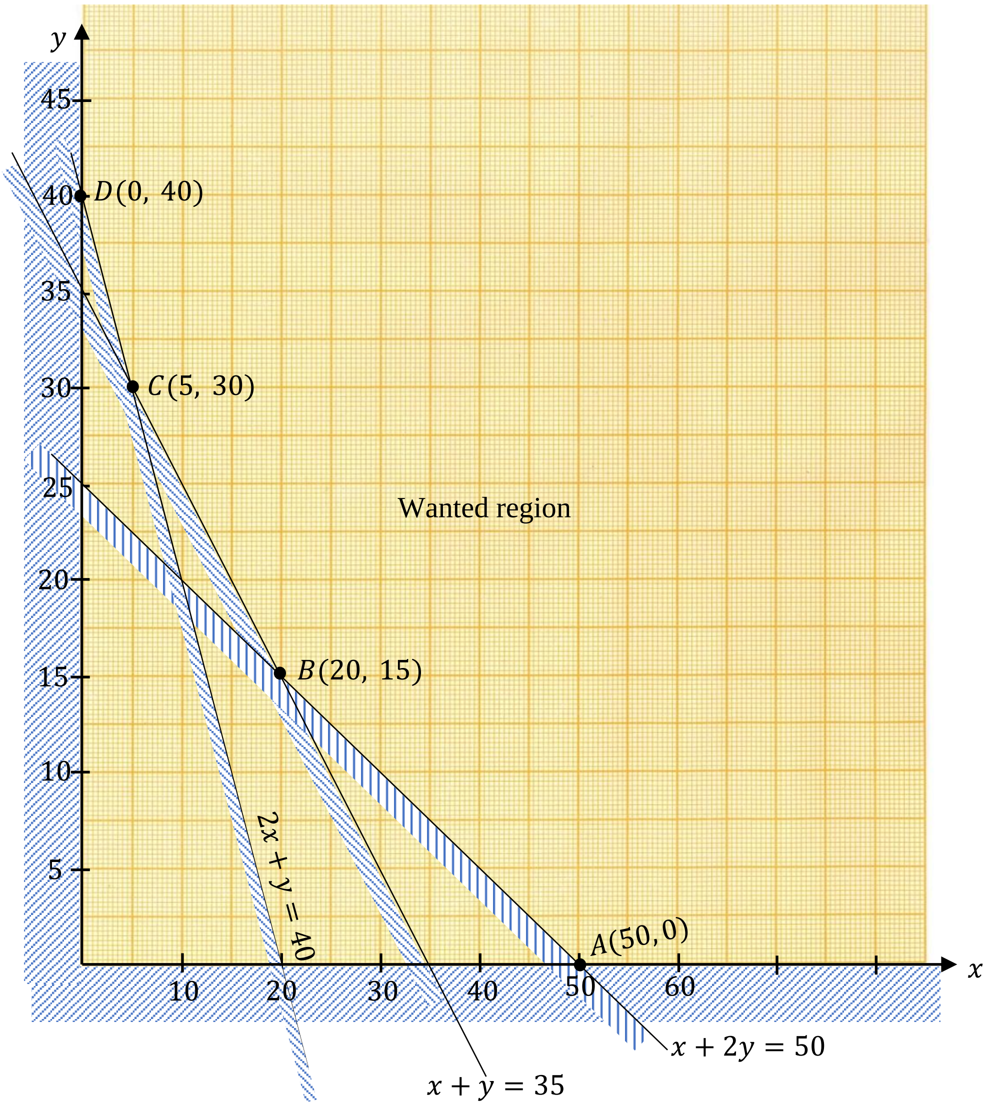
The graphical representation of the system of inequalities is shown below.



The feasible region is OABCO whose corner points are (0, 0), A (20, 0), B (12, 6) and C(0, 10).
The values of Z at the corner points are as follows.

Corner Points	Value of $Z = 80x + 120y$
O(0, 0)	$Z = 80 \times 0 + 120 \times 0 = 0$
A(20, 0)	$Z = 80 \times 20 + 120 \times 0 = 1600$
B(12, 6)	$Z = 80 \times 12 + 120 \times 6 = 1680$
C(0, 10)	$Z = 80 \times 0 + 120 \times 10 = 1200$

From the table, the maximum value of Z is Shs 1680.
Hence, 12 pieces of type A and 6 pieces of type B should be manufactured per week to get maximum profit of Shs 1680 per week.



Self-Evaluation Exercise

1. Formulate the following problems as Linear Programming Problems

   (a) A manufacturer of furniture makes two products: chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on Machine A and 6 hours on Machine B. A table requires 5 hours on machine A and 1 hour on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is US $ 1 and US $ 5 respectively. What should be the daily production of each of the two production?
   [Ans: No. of chairs $= x$, No. of tables $= y$. Maximise $Z = x + 5y$ subject to $2x + 5y \leq 16$, $6x + y \leq 30$, $x \geq 0$; $y \geq 0$.]

   (b) A company sells two different products A and B. The company makes a profit of shs. 40 and shs 30 per unit on products A and B respectively. The two products are produced by a common production process and are sold in two different markets. The production process has a capacity of 3000 man hours. It takes 3 hours to produce one unit of A and 1 hour to produce one unit of B. The market has been surveyed and the company official feels that the maximum number of units A that can be sold

[Ans: Max. profit is 40 million when he stocks 20 Toyota Hiace and no Nissan caravan]

13. Gonda Clock company produces clocks of two types regular and deluxe. The electrical components necessary supplied by another company whose supply is limited to 600 components per day. Each regular clock requires 5 components and each deluxe clock requires 6 components. The production of one regular clock requires 1 man day of labour, while 2 man days are required for deluxe, the production labour force has on a daily basis 160 man days. The production takes place in 2 different departments for each type of clock. The daily departmental capacity for regular is 80 and for deluxe is 60. A profit of Shs 5,000 is possible for regular and Shs 8,000 for each deluxe.
    (a) Using graphical method, determine how many regular and deluxe clocks should be produced on a daily basis so as to maximise profit
    (b) Determine the maximum profit expected on a daily basis.

[Ans: Maximum profit is Shs 700,000]

14. A local SACCO owns two types of vehicles; buses and coasters. A bus can carry 80 passengers and 40 boxes of cargo while the coaster can carry 30 passengers and 50 boxes of cargo. The SACCO has been contracted to move at least 480 passengers and at least 400 boxes of cargo per trip. In terms of fuel consumption, per trip, a bus needs Shs 120,000 while a coaster needs Shs 80,000.
    (a) Formulate a linear programming model for the above data
    (b) Using the graphical method, advise the SACCO on the best choice of vehicles to use in order to minimize the total cost.
15. A district agriculture officer is planning a trip of farmers to an agricultural exhibition. There are two types of buses available to be hired. Type A buses can carry 40 people and 1,000 kg of baggage and cost Shs 2,000,000. Type B can carry 50 people and 750 kg of baggage at a cost of Shs 2,400,000. A total of 800 farmers have registered for the exhibition and they have 18,000 kg of baggage.

    Taking x to be the number of type A buses, y to be number of type B buses and taking cost to be in thousands of shillings,
    (a) Formulate a standard linear programming problem for minimizing the transportation cost
    (b) Find how many buses of each type should be hired to minimize the cost of the trip and the associated minimum cost.
16. A small scale industrialist produces three types of machine components M_1, M_2 and M_3 made of steel and brass. The amounts of steel and brass required for each component and the number of man-weeks of labour required to manufacture and assemble one unit of each component are as follows.

	M_1	M_2	M_3	Availability
Steel	6	5	3	100 kg
Brass	3	4	9	75 kg
Man-weeks	1	2	1	20 weeks

    The labour is restricted to 20 man-weeks, steel is restricted to 100 kg per week and the brass to 75 kg per week. The industrialist's profit on each unit of M_1, M_2 and M_3 is ¥ 6, ¥ 4 and ¥ 7 respectively. Give its mathematical formulation as a linear programming problem such that the total profit is maximum.
17. An airline agrees to charter a plane for a group. The group needs at least 160 first class seats and at least 300 tourist class seats. The airline must use at least two of its model 314 planes which have at least 20 first class seats and 30 tourist class seats. The airline will also use some of its model 535 planes which have 20 first class seats and 60 tourist class seats. Each flight of a model 314 plane costs the company UGX 1 million, and each flight of a model 535 plane costs UGX 1.5 million. How many of each type of plane should be used to minimize the flight cost? Formulate this as a LPP.
18. During a mountain climbing season, World Tour Travel Service (WTTS) offers one month vacation in Aspen Hotel, including a round trip fare and accommodation at the hotel. The minimum number of rooms available at the hotel for each of the booking periods are as follows

Chapter 3 Quadratics

Introduction

An equation of the form $ax^2 + bx + c = 0$ where a, b, c are three constants and $a \neq 0$ is known as a quadratic equation. The equation has two roots, namely $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$

In general

For a quadratic equation $ax^2 + bx + c = 0$, the roots are obtained from the formula;

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Example 1

Solve $x^2 + 3x - 1 = 0$

Solution:

Comparing with the general equation $ax^2 + bx + c = 0$; $a = 1,\ b = 3,\ c = -1$

Substituting in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 - 1}}{2 \times 1}$$

$$x = \frac{-3 \pm \sqrt{9+4}}{2} \implies x = \frac{-3+\sqrt{13}}{2}$$

$$\text{Or } x = \frac{-3-\sqrt{13}}{2}$$

$$\therefore x = 0.30 \text{ or } x = -3.30$$

Solution of a quadratic equation that factorizes

Example 2

Find the roots of the equation $x^2 - 5x + 6 = 0$

Solution:

$$x^2 - 2x - 3x + 6 = 0$$

$$x(x-2) - 3(x-2) = 0$$

$$(x-2)(x-3) = 0$$

Either $x - 2 = 0, x = 2$ or $x - 3 = 0\ , x = 3$

Solution of a quadratic equation by completing the square

This method uses the expansion $(x + b)^2 = x^2 + 2bx + b^2$. It is important to note that the last term b^2, is the square of half the coefficient of x, $(2b)$

Note that the coefficient of the highest term x^2 should be 1

Example 3

Find the roots of the equation $2x^2 - 5x + 1 = 0$

Solution

Dividing through by 2 gives;

$$x^2 - \frac{5}{2}x + \frac{1}{2} = 0$$

$$x^2 - \frac{5}{2}x = -\frac{1}{2}$$

Adding the square of half the coefficient of x to both sides of the equation;

$$x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = -\frac{1}{2} + \left(\frac{5}{4}\right)^2$$

$$\left(x - \frac{5}{4}\right)^2 = -\frac{1}{2} + \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{17}{16}$$

$$\sqrt{\left(x - \frac{5}{4}\right)^2} = \sqrt{\frac{17}{16}}$$

$$x - \frac{5}{4} = \frac{\sqrt{17}}{4}$$

$$x = \frac{5 \pm \sqrt{17}}{4}$$

Either $x = \frac{5+\sqrt{17}}{4} = 2.281$

Or $x = \frac{5-\sqrt{17}}{4} = 0.219$

Example 4

Solve $2x^2 - 6x + 4 = 0$

Solution:

$$2x^2 - 6x + 4 = 0$$

$$x^2 - 3x + 2 = 0$$

$$x^2 - 3x = -2$$

Adding the square of half the coefficient of x to each side of the equation

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = -2 + \left(\frac{3}{2}\right)^2$$

$$\left(x - \frac{3}{2}\right)^2 = -2 + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\sqrt{\left(x-\frac{3}{2}\right)^2} = \sqrt{\frac{1}{4}}$$

$$x-\frac{3}{2} = \pm\frac{1}{2}$$

$$x = \frac{3\pm 1}{2}$$

$$\therefore \quad x = 2 \text{ or } x = 1$$

Example 5

Solve $x^2+3x-1=0$

Solution:

$$x^2+3x=1$$

Adding the square of half the coefficient of x to each side of the equation gives;

$$x^2+3x+\left(\frac{3}{2}\right)^2 = 1+\left(\frac{3}{2}\right)^2$$

$$\left(x+\frac{3}{2}\right)^2 = 1+\frac{9}{4}$$

$$\left(x+\frac{3}{2}\right)^2 = \frac{13}{4}$$

$$x+\frac{3}{2} = \pm\frac{\sqrt{13}}{2}$$

$$x = \frac{-3\pm\sqrt{13}}{2}$$

$$x = \frac{-3+\sqrt{13}}{2} = 0.30 \text{ or } x = \frac{-3-\sqrt{13}}{2} = -3.30$$

Quadratic Expression or Function

An expression of the form ax^2+bx+c involving an unknown x and constants a, b, c $(a \neq 0)$ is known as a quadratic expression or function. For different values of x, the expression takes different values. Any value of x which makes the expression vanish is known as a root of the equation $ax^2+bx+c=0$.

Maximum and Minimum Value of a Quadratic function

The method of completing the square, used to solve any equation in the form $ax^2+bx+c=0$ can be used to find the maximum or minimum value of the expression ax^2+bx+c

Example 6

Find the maximum value of $5-2x-4x^2$

Solution:

Let us first rewrite $5-2x-4x^2$ as $-4x^2-2x+5$

$$-4x^2-2x+5 = -4\left(x^2+\frac{1}{2}x\right)+5$$

$$= -4\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+\frac{4}{16}+5$$

$$= -4\left(x+\frac{1}{4}\right)^2+\frac{21}{4}$$

$$= \frac{21}{4}-4\left(x+\frac{1}{4}\right)^2$$

Now $4\left(x+\frac{1}{4}\right)^2 \geq 0$

Thus $5-2x-4x^2$ has a maximum value of $\frac{21}{4}$ when $x=-\frac{1}{4}$

Example 7

Find for real values of x, the maximum value of

$$3-20x-25x^2$$

Solution:

$$3-20x-25x^2 = -(25x^2+20x-3)$$

$$= -(25x^2+20x+4-7)$$

$$= -[(5x+2)^2-7]$$

$$= 7-(5x+2)^2 \leq 7$$

since for all real values of x, $(5x+2)^2 \geq 0$

$\therefore$ The maximum value of the expression is 7 and this occurs when $5x+2=0$ i.e. when $x=-\frac{2}{5}$

Example 8

Find by completing the square, the greatest value of the function $f(x)=1-6x-x^2$

Solution:

$$1-6x-x^2 = -x^2-6x+1$$

$$= -[x^2+6x]+1$$

$$= -[x^2+6x+3^2-3^2]+1$$

$$= -[x^2+6x+9-9]+1$$

$$= -[x^2+6x+9]+9+1$$

$$= -(x+3)^2+10$$

$$= 10-(x+3)^2$$

Since $(x+3)^2$ is the square of a real number, it cannot be negative, it is zero when $=-3$, otherwise it is positive. $10-(x+3)^2$ is therefore always less than or equal to 10. Thus, the greatest value is 10

Example 9

Find the minimum value of the expression x^2+3x+4

Solution:

By completing the square;

Example 19

Given that the equation $(5a+1)x^2 - 8ax + 3a = 0$ has equal roots, find the possible values of a

Solution:

We identify a, b and c from the above equation and then apply the condition for equal roots

$a = (5a+1)$, $b = -8a$ and $c = 3a$

For equal roots, $b^2 - 4ac = 0$

$$(-8a)^2 - 4(5a+1)(3a) = 0$$
$$64a^2 - 12a(5a+1) = 0$$
$$64a^2 - 60a^2 - 12a = 0$$
$$4a^2 - 12a = 0$$
$$4a(a-3) = 0$$

Either $4a = 0$ or $a - 3 = 0$

$$\therefore a = 0 \text{ or } a = 3$$

Examination questions

2014, No. 1

The roots of the equation $2x^2 + 4x - 1 = 0$ are α and β. Find the value of $\alpha^2 + \beta^2$

Solution:

$$2x^2 + 4x - 1 = 0$$
$$x^2 + 2x - \tfrac{1}{2} = 0$$

Sum of roots $\alpha + \beta = -2$

product of roots $\alpha\beta = -\frac{1}{2}$

From $(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

It follows that $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$= (-2)^2 - 2\left(-\tfrac{1}{2}\right) = 4 + 1 = 5$$

2015, No. 12

The roots of the equation $2x^2 - 6x + 7 = 0$ are α and β. Determine the

(a) values of $(\alpha-\beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

(b) quadratic equation with integral coefficient whose roots are $(\alpha-\beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

Solution:

$$2x^2 - 6x + 7 = 0$$
$$x^2 - 3x + \frac{7}{2} = 0$$
$$\alpha + \beta = 3, \qquad \alpha\beta = \frac{7}{2}$$

(a) $(\alpha-\beta)^2 = (\alpha-\beta)(\alpha-\beta)$

$$= \alpha^2 - \alpha\beta - \alpha\beta + \beta^2$$
$$= \alpha^2 + \beta^2 - 2\alpha\beta$$

But $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$\Rightarrow (\alpha-\beta)^2 = (\alpha+\beta)^2 - 2\alpha\beta - 2\alpha\beta$$
$$= (\alpha+\beta)^2 - 4\alpha\beta$$
$$= 3^2 - 4\left(\frac{7}{2}\right) = 9 - 14 = -5$$

$$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\beta+\alpha}{\alpha^2\beta^2} = \frac{\alpha+\beta}{(\alpha\beta)^2} = \frac{3}{\left(\frac{7}{2}\right)^2} = 3 \div \frac{49}{4}$$
$$= 3 \times \frac{4}{49} = \frac{12}{49}$$

(b) Sum of roots $= -5 + \frac{12}{49} = \frac{12}{49} - 5 = -\frac{233}{49}$

Product of roots $= -5 \times \frac{12}{49} = -\frac{60}{49}$

$x^2 -$(sum of roots)x +product of roots$= 0$

$$x^2 - \left(-\tfrac{233}{49}\right)x - \tfrac{60}{49} = 0$$

or $49x^2 + 233x - 60 = 0$

2018, No. 1

The roots of the equation $4x^2 + 9x - k = 0$ are α and 2. Find the values of α and k.

Solution:

$$x^2 + \frac{9}{4}x - \frac{k}{4} = 0$$

Sum of roots, $\alpha + 2 = -\frac{9}{4}$

$$\alpha = -\frac{9}{4} - 2$$
$$\alpha = -\frac{17}{4}$$

Product of roots, $2\alpha = -\frac{k}{4}$

$$2 \times -\frac{17}{4} = -\frac{k}{4}$$
$$k = 34$$

2019, No. 3

Determine the possible values of a for which the equation $2x^2 + (a+2)x + (a+2) = 0$ has equal roots

Solution:

For equal roots, $b^2 - 4ac = 0$

$$(a+2)^2 - 4(2)(a+2) = 0$$
$$(a+2)(a+2) - 8(a+2) = 0$$
$$a^2 + 4a + 4 - 8a - 16 = 0$$
$$a^2 - 4a - 12 = 0$$
$$a^2 - 6a + 2a - 12 = 0$$
$$a(a-6) + 2(a-6) = 0$$
$$(a-6)(a+2) = 0$$

Chapter 4 Surds, Logarithms and Indices

SURDS

Expressions such as $\sqrt{4}, \sqrt{25}$ have exact numerical values i.e. $\sqrt{4} = 2, \sqrt{25} = 5$. However expressions such as $\sqrt{2}, \sqrt{3}, \sqrt{5} \ldots$ ..can not be written numerically as exact quantities i.e. $\sqrt{2} = 1.414$ and $\sqrt{3} = 1.732$. Such numbers are called irrational and it's often more convenient to leave them in the form $\sqrt{2}, \sqrt{3}, \sqrt{5} \ldots\ldots\ldots\ldots\ldots$ hence called **surds**. They are thus irrational numbers, which can be expressed as powers.

Examples

1. Write the following as the simplest possible surds

(i) $\sqrt{8}$ (ii) $\sqrt{12}$ (iii) $\sqrt{50}$ (iv) $\sqrt{48}$

Solution:

(i) $\sqrt{8} = \sqrt{2 \times 4} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$

(ii) $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

(iii) $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

(iv) $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$

2. Simplify:

(i) $\sqrt{75} + \sqrt{108} + \sqrt{27}$

Solution:

$$\sqrt{75} + \sqrt{108} + \sqrt{27} = \sqrt{25 \times 3} + \sqrt{36 \times 3} + \sqrt{9 \times 3}$$
$$= \sqrt{25} \times \sqrt{3} + \sqrt{36} \times \sqrt{3} + \sqrt{9} \times \sqrt{3} = 5\sqrt{3} + 6\sqrt{3} + 3\sqrt{3} = 14\sqrt{3}$$

(ii) $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution:

$$\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8} = \sqrt{25 \times 2} + \sqrt{2} - 2\sqrt{9 \times 2} + \sqrt{4 \times 2}$$
$$= \sqrt{25} \times \sqrt{2} + \sqrt{2} - 2\sqrt{9} \times \sqrt{2} + \sqrt{4} \times \sqrt{2}$$
$$= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2}$$
$$= 8\sqrt{2} - 6\sqrt{2} = 2\sqrt{2}$$

3. Expand and simplify

(a) $(3 - 3\sqrt{3})(3 + 2\sqrt{3})$

(b) $(5 - 2\sqrt{7})(5 + 2\sqrt{7})$

Solution:

(a) $(3 - 3\sqrt{3})(3 + 2\sqrt{3}) = 6 - 9\sqrt{3} + 4\sqrt{3} - 6(\sqrt{3})^2$
$$= 6 - 5\sqrt{3} - 6 \times 3 = -12 - 5\sqrt{3}$$

(b) $(5 - 2\sqrt{7})(5 + 2\sqrt{7}) = 25 - 10\sqrt{7} + 10\sqrt{7} - 4(\sqrt{7})^2$
$$= 25 - 4 \times 7 = 25 - 28 = -3$$

4. Rationalize the denominator of $\frac{3}{\sqrt{2}}$

Solution:

Multiply numerator and denominator by $\sqrt{2}$

$$\frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

5. Express $\frac{\sqrt{2}}{2\sqrt{3}}$ in the form $\sqrt{\frac{a}{b}}$ where a and b are real numbers.

Solution:

$$\frac{\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{4} \times \sqrt{3}} = \frac{\sqrt{2}}{\sqrt{12}} = \sqrt{\frac{2}{12}} = \sqrt{\frac{1}{6}}$$

Alternatively:

$$\frac{\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{6} = \frac{\sqrt{6}}{\sqrt{36}} = \sqrt{\frac{6}{36}} = \sqrt{\frac{1}{6}}$$

6. Rationalize the denominator of $\frac{3-\sqrt{5}}{1+3\sqrt{5}}$

Solution:

Multiply numerator and denominator by the denominator with sign of $3\sqrt{5}$ changed (conjugate of the denominator)

$$\frac{3-\sqrt{5}}{1+3\sqrt{5}} = \frac{3-\sqrt{5}}{1+3\sqrt{5}} \times \frac{1-3\sqrt{5}}{1-3\sqrt{5}} = \frac{(3-\sqrt{5})(1-3\sqrt{5})}{(1+3\sqrt{5})(1-3\sqrt{5})}$$

$$= \frac{3-9\sqrt{5}-\sqrt{5}+3\sqrt{25}}{1^2-(3\sqrt{5})^2} = \frac{3+15-10\sqrt{5}}{1-45}$$

$$= \frac{18-10\sqrt{5}}{-44} = \frac{18}{-44} - \frac{10\sqrt{5}}{-44}$$

$$= \frac{-9}{22} + \frac{5}{22}\sqrt{5}$$

7. Rationalize the denominator of $\frac{1}{3-\sqrt{2}}$

Solution:

$$\frac{1}{3-\sqrt{2}} = \frac{1}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3+\sqrt{2}}{3^2-(\sqrt{2})^2}$$

$$= \frac{3+\sqrt{2}}{9-2} = \frac{1}{7}(3+\sqrt{2})$$

8. Express $\frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}}$ in the form $a + b\sqrt{c}$

Solution:

$$\frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}} = \frac{(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}+3\sqrt{2})}{(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})} = \frac{2\sqrt{3}\times2\sqrt{3}+2\sqrt{3}\times3\sqrt{2}+3\sqrt{2}\times2\sqrt{3}+3\sqrt{2}\times3\sqrt{2}}{2\sqrt{3}\times2\sqrt{3}+2\sqrt{3}\times3\sqrt{2}-3\sqrt{2}\times2\sqrt{3}-3\sqrt{2}\times3\sqrt{2}}$$

$$= \frac{4\sqrt{9}+6\sqrt{6}+6\sqrt{6}+9\sqrt{4}}{4\sqrt{9}+6\sqrt{6}-6\sqrt{6}-9\sqrt{4}}$$

$$= \frac{12+12\sqrt{6}+18}{12-18}$$

$$= \frac{30+12\sqrt{6}}{-6} = \frac{30}{-6} + \frac{12\sqrt{6}}{-6} = -5 - 2\sqrt{6}$$

INDICES

Index is another word to mean power i.e. for $a^3 = a \times a \times a$, here a is the base and 3 is a power or an index or exponent.

Laws of indices

1. $\boldsymbol{a^m \times a^n = a^{m+n}}$ **i.e.** $\boldsymbol{2^3 \times 2^2 = 2^{3+2} = 2^5 = 32}$
2. $\boldsymbol{a^m \div a^n = a^{m-n}}$ $\boldsymbol{ie\ 3^3 \div 3^2 = 3^{3-2} = 3^1 = 3}$
3. $\boldsymbol{(a^m)^n = a^{mn}\ ie\ \left(4^2\right)^3 = 4^{2\times3} = 4^6}$
4. $\boldsymbol{a^0 = 1\ ie\ 5^0 = 1,\ \left(\frac{2}{7}\right)^0 = 1, 1000^0 = 1}$ **etc.**
5. $\boldsymbol{a^{-n} = \frac{1}{a^n}\ ie\ 2^{-1} = \frac{1}{2^1},\ 5^{-2} = \frac{1}{5^2} = \frac{1}{25}}$
6. $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ **i.e.** $\boldsymbol{4^{\frac{1}{2}} = \sqrt[2]{4} = 2}$
7. $\boldsymbol{a^n \times b^n = (ab)^n}$ **i.e.** $\boldsymbol{2^2 \times 3^2 = (2\times3)^2 = 6^2 = 36}$
8. $\boldsymbol{a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{(a^m)}}$

Examples

1. Simplify (i) $27^{\frac{1}{3}}$ (ii) $4^{\frac{-1}{2}}$ (iii) $100^{1\frac{1}{2}}$ (iv) $(625)^{\frac{-1}{4}}$ (v) $\left(\frac{27}{1000}\right)^{\frac{-1}{3}}$

Solution:

(i) $27^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = 3^{3\times\frac{1}{3}} = 3^1 = 3$

(ii) $4^{\frac{-1}{2}} = (2^2)^{\frac{-1}{2}} = 2^{2\times\frac{-1}{2}} = 2^{-1} = \frac{1}{2}$

(iii) $100^{1\frac{1}{2}} = 100^{\frac{3}{2}} = (10^2)^{\frac{3}{2}} = 10^{2\times\frac{3}{2}} = 10^3 = 1000$

(iv) $(625)^{\frac{-1}{4}} = (5^4)^{\frac{-1}{4}} = 5^{4\times\frac{-1}{4}} = 5^{-1} = \frac{1}{5}$

(v) $\left(\frac{27}{1000}\right)^{\frac{-1}{3}} = \frac{(27)^{\frac{-1}{3}}}{(1000)^{\frac{-1}{3}}} = \frac{(3^3)^{\frac{-1}{3}}}{(10^3)^{\frac{-1}{3}}} = \frac{3^{3\times\frac{-1}{3}}}{10^{3\times\frac{-1}{3}}} = \frac{3^{-1}}{10^{-1}}$

$$= 3^{-1} \div 10^{-1} = \frac{1}{3} \div \frac{1}{10} = \frac{1}{3} \times \frac{10}{1} = \frac{10}{3}$$

2. Simplify

(i) $\frac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}}$

Solution:

$$\frac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}} = \frac{(3^3)^{\frac{1}{2}} \times (3^5)^{\frac{1}{2}}}{(3^5)^{\frac{4}{5}}} = \frac{3^{\frac{3}{2}} \times 3^{\frac{5}{2}}}{3^4} = \frac{3^{\left(\frac{3}{2}+\frac{5}{2}\right)}}{3^4} = \frac{3^4}{3^4} = 1$$

(ii) $\frac{a^{\frac{1}{n}} \div a^{-n}}{a^{\left(\frac{n+1}{n}\right)}}$

Solution:

$$\frac{a^{\frac{1}{n}} \div a^{-n}}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\frac{1}{n}} \div \frac{1}{a^n}}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\frac{1}{n}} \times a^n}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\frac{1}{n}+n}}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\left(\frac{n^2+1}{n}\right)}}{a^{\left(\frac{n+1}{n}\right)}}$$

$$= a^{\left(\frac{n^2+1}{n}\right)-\left(\frac{n+1}{n}\right)} = a^{\left(\frac{n^2-n}{n}\right)} = a^{n-1}$$

4. $2^{2x+3} + 1 = 9(2^x)$

Solution:

$$2^{2x} \times 2^3 + 1 = 9(2^x)$$
$$8(2^{2x}) - 9(2^x) + 1 = 0$$
$$8(2^x)^2 - 9(2^x) + 1 = 0$$

since $2^{2x} = 2^x \times 2^x = (2^x)^2$

Let $2^x = y$, then;

$$8y^2 - 9y + 1 = 0$$
$$8y^2 - 8y - y + 1 = 0$$
$$(8y - 1)(y - 1) = 0 \quad ,$$

either $8y - 1 = 0 \Rightarrow y = \frac{1}{8}$

or $y - 1 = 0 \Rightarrow y = 1$

when $y = \frac{1}{8}, 2^x = \frac{1}{8}$

$$2^x = \frac{1}{2^3}$$
$$2^x = 2^{-3}$$
$$x = -3$$

when $y = 1, 2^x = 1$

$$2^x = 2^0$$
$$x = 0$$
$$\therefore x = 0 \text{ or } x = -3$$

5. $3(3^{2x}) + 26(3^x) - 9 = 0$

Solution:

$$3(3^x)^2 + 26(3^x) - 9 = 0$$

Let $y = 3^x$, then

$$3y^2 + 26y - 9 = 0$$
$$3y^2 - y + 27y - 9 = 0$$
$$y(3y - 1) + 9(3y - 1) = 0$$
$$(3y - 1)(y + 9) = 0$$

Either $3y - 1 = 0$

$$y = \frac{1}{3}$$

or $y + 9 = 0, \ y = -9$

Either $3^x = \frac{1}{3}$

$$3^x = 3^{-1}$$
$$\therefore x = -1$$

Or $3^x = -9$ and value of x does not exist

$$\therefore x = -1$$

6. $2^{2x+1} + 15(2^x) = 8$

Solution:

$$2^{2x} \times 2^1 + 15(2^x) - 8 = 0$$
$$2(2^x)^2 + 15\,(2^x) - 8 = 0$$

Let $2^x = y$, then $2y^2 + 15y - 8 = 8$

$$2y^2 - y + 16y - 8 = 0$$
$$y(2y - 1) + 8(2y - 1) = 0$$
$$(2y - 1)(y + 8) = 0$$

Either $2y - 1 = 0$ or $y + 8 = 0$

$\Rightarrow y = \frac{1}{2}$ or $y = -8$

Either $2^x = \frac{1}{2} = 2^{-1} \Rightarrow x = -1$

or $2^x = -8$ and value of x does not exist

$$\therefore x = -1$$

7. $4^{(2t+1)} + 4^{(t+3)} = 16\frac{1}{4}$

Solution:

$$4^{2t} \times 4^1 + 4^t \times 4^3 = \frac{65}{4}$$
$$4(4^{2t}) + 64(4^t) = \frac{65}{4}$$
$$4(4^t)^2 + 64(4^t) = \frac{65}{4}$$

Let $4^t = y \Rightarrow 4y^2 + 64y = \frac{65}{4}$

$$16y^2 + 256y - 65 = 0$$

(on multiplying through by 4)

$16 \times -65 = -1040$ whose factors are 260 and-4 that add up to 256

$$16y^2 + 260y - 4y - 65 = 0$$
$$4y(4y + 65) - (4y + 65) = 0$$
$$(4y + 65)(4y - 1) = 0$$

Either $4y + 65 = 0 \Rightarrow 4y = -65$ which gives $y = -\frac{65}{4}$

$4^t = -\frac{65}{4}$ and here value of t does not exist

or $4y - 1 = 0 \Rightarrow 4y = 1$ which gives $y = \frac{1}{4}$

$$4^t = \frac{1}{4} \Rightarrow 4^t = 4^{-1}$$
$$\therefore t = -1$$

8. $2^{4(x-1)} = (4 \times 8^x)^3$

Solution:

$$2^{4x-4} = [2^2 \times (2^3)^x]^3$$
$$2^{4x-4} = [2^2 \times 2^{3x}]^3$$
$$2^{4x-4} = \left[2^{(2+3x)}\right]^2$$
$$2^{4x-4} = 2^{2(2+3x)}$$
$$\Rightarrow 4x - 4 = 4 + 6x$$
$$4x = 8 + 6x$$
$$-2x = 8$$
$$\therefore x = -4$$

LOGARITHMS

Logarithm is another word to mean index or power i.e. if $y = a^x$, then we define x as logarithm of y to base a $(\log_a y)$

If $y = a^x$, then $x = \log_a y$

Operating rules for logarithms

1.$\boldsymbol{\log_a b + \log_a c = \log_a bc}$ ***ie*** $\boldsymbol{\log_2 3 + \log_2 5 = \log_2 3 \times 5 = \log_2 15}$

2.$\boldsymbol{\log_a b - \log_a c = \log_a \left(\frac{b}{c}\right)}$ ***ie*** $\boldsymbol{\log_3 6 - \log_3 8 = \log_3 \left(\frac{6}{3}\right) = \log_3 2}$

3.$\boldsymbol{\log_a b^n = n \log_a b}$ ***ie*** $\boldsymbol{\log_3 7^2 = 2 \log_3 7}$

4.$\boldsymbol{\log_a b = \frac{\log_c b}{\log_c a}}$ **i.e.** $\boldsymbol{\log_2 3 = \frac{\log_4 3}{\log_4 2}}$ **,** $\boldsymbol{\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2}}$**.** This is known as the change of base rule

5.$\boldsymbol{\log_a b = \frac{1}{\log_b a}}$

6.$\boldsymbol{\log_a 1 = 0}$ **since** $\boldsymbol{a^0 = 1}$

7.$\boldsymbol{\log_a a = 1}$ **since** $\boldsymbol{a^1 = a}$

Examples

1. Express the following statements in logarithm notation

   (i) $16 = 2^4$ (ii) $27 = 3^3$

   Solution:

   (i) Introducing $\log_2$ on both sides (ii) Introducing $\log_3$ on both sides

   $\log_2 2^4 = \log_2 16$ $\quad\quad$ $\log_3 27 = \log_3 3^3$

   $\log_2 16 = \log_2 2^4$ $\quad\quad$ $\log_3 27 = 3$

   $\log_2 16 = 4$

2. Express the following in index notation

   (i) $\log_2 32 = 5$ (ii) $7 = \log_2 128$

Solution:

   (i) $2^5 = 32$ (ii) $2^7 = 128$

3. Simplify $\log_4 9 + \log_4 21 - \log_4 7$

Solution:

$$\log_4 9 + \log_4 21 - \log_4 7 = \log_4(9 \times 21) - \log_4 7$$
$$= \log_4 \left(\frac{9\times21}{7}\right) = \log_4 27 = \log_4 3^3 = 3\log_4 3$$

4. Simplify $\log_5 125 - \log_5 50 + \log_5 2$

Solution:

$$\log_5 125 - \log_5 50 + \log_2 2 = \log_5 5^3 - \log_5(2 \times 25) + \log_5 2$$
$$= 3\log_5 5 - [\log_5 2 + \log_5 25] + \log_5 2$$
$$= 3 - \log_5 2 - \log_5 25 + \log_5 2$$
$$= 3 - \log_5 5^2 = 3 - 2\log_5 5 = 3 - 2 = 1$$

5. If $\log_7 2 = 0.356$ and $\log_7 3 = 0.566$. Find the value of $2\log_7 \left(\frac{7}{15}\right) + \log_7 \left(\frac{25}{12}\right) - 2\log_7 \left(\frac{7}{3}\right)$

Solution:

Examination questions

2013, No. 1

Given that $p = \log_a(a^3y^{-2})$ and $\log_a(ay^2)$, find the value of $p + q$

Solution:

$$p + q = \log_a(a^3y^{-2}) + \log_a(ay^2)$$
$$= \log_a[a^3y^{-2} \times ay^2]$$
$$= \log_a[(a^3 \times a) \times (y^{-2} \times y^2)]$$
$$= \log_a\left[a^{(3+1)} \times y^{(-2+2)}\right]$$
$$= \log_a[a^4 \times y^0] \quad \text{but } y^0 = 1$$
$$= \log_a a^4 = 4\log_a a$$
$$\therefore p + q = 4 \times 1 = 4$$

2015, No. 1

Evaluate $\frac{\log_6 216+\log_2 64}{\log_3 243-\log_{10} 0.1}$

Solution:

$$\frac{\text{Log}_6 216+\log_2 64}{\log_3 243-\log_{10} 0.1} = \frac{\log_6 6^3+\log_2 2^6}{\log_3 3^5-\log_{10} 10^{-1}}$$
$$= \frac{3\log_6 6 + 6\log_2 2}{5\log_3 3 + \log_{10} 10} = \frac{3+6}{5+1} = \frac{9}{6} = \frac{3}{2}$$

2017, No.1

Given that $\log_3 x = 2\log_3 4 - \log_3 5 + \log_3 9$, find the value of x

Solution:

$$\log_3 x = \log_3 4^2 - \log_3 5 + \log_3 9$$
$$\log_3 x = \log_3 16 + \log_3 9 - \log_3 5$$
$$\log_3 x = \log_3(16 \times 9) - \log_3 5$$
$$\log_3 x = \log_3\left(\frac{16\times9}{5}\right)$$

Since logarithms to both sides are the same,

$$x = \frac{16\times9}{5} = 28.8$$

2018, No. 5

Express $\frac{4}{\sqrt{3}+\sqrt{2}} + \frac{4}{\sqrt{3}-\sqrt{2}}$ in the form $b\sqrt{c}$ where b and c are integers.

Solution:

$$\frac{4}{\sqrt{3}+\sqrt{2}} + \frac{4}{\sqrt{3}-\sqrt{2}} = \frac{4(\sqrt{3}-\sqrt{2})+4(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$
$$= \frac{4\sqrt{3}-4\sqrt{2}+4\sqrt{3}+4\sqrt{2}}{3-2}$$
$$= 8\sqrt{3}$$

2019, No. 1

Show that

$$\sqrt{\frac{25^3+5^6}{5^7-5^6}} = \frac{\sqrt{2}}{2}$$

Solution:

$$\sqrt{\frac{25^3+5^6}{5^7-5^6}} = \sqrt{\frac{(5^2)^3+5^6}{5(5^6)-5^6}}$$
$$= \sqrt{\frac{5^6+5^6}{5^6(5-1)}}$$
$$= \sqrt{\frac{2(5^6)}{5^6(4)}}$$
$$= \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

2020, No. 1

Without using a calculator, evaluate

$$\frac{6\sqrt{10}+2\sqrt{40}}{\sqrt{2}+\sqrt{20}}$$

Solution:

$$\frac{6\sqrt{10}+2\sqrt{40}}{\sqrt{2}+\sqrt{20}} = \frac{6\sqrt{10}+2\sqrt{4\times10}}{\sqrt{2}+\sqrt{2\times10}}$$
$$= \frac{6\sqrt{10}+2\sqrt{4}\times\sqrt{10}}{\sqrt{2}+\sqrt{2}\times\sqrt{10}}$$
$$= \frac{6\sqrt{10}+4\sqrt{10}}{\sqrt{2}+\sqrt{2}\times\sqrt{10}}$$
$$= \frac{\sqrt{10}(6+4)}{\sqrt{2}\left(1+\sqrt{10}\right)}$$
$$= \frac{10\sqrt{5}}{1+\sqrt{10}}$$

Now rationalizing the denominator;

$$\frac{10\sqrt{5}}{1+\sqrt{10}} \times \frac{1-\sqrt{10}}{1-\sqrt{10}} = \frac{10\sqrt{5}-10\sqrt{50}}{1-10}$$
$$= \frac{10\sqrt{5}-50\sqrt{2}}{-9}$$
$$= \frac{-10\sqrt{5}+50\sqrt{2}}{9} \text{ or}$$
$$= \frac{10}{9}\left(5\sqrt{2}-\sqrt{5}\right)$$

Chapter 5 Series – A.Ps and G.Ps

Sequence

A sequence is a set of numbers called terms, arranged in a definite order and formed according to a definite law. A sequence may be finite or infinite.

Illustration 1: 3, 8, 13, 23

This is a sequence of 5 terms whose each term except the first is formed by adding 5 to the preceding term.

Illustration 2: 2, 6, 18, 54, 162, ..

This is an infinite sequence whose each term except the first, is formed by multiplying 3 to the preceding term.

The terms of a sequence are denoted by u_1, u_2, u_3, …..u_n i.e. nth term is known as the general term of the sequence.

Series

When the terms of a sequence are connected by plus or minus signs, then the expression is known as a series. If $\{u_n\}$ is a sequence, then $u_1 + u_2 + u_3 + \cdots$ i.e. $\sum u_n$ is known as a series. A series may be finite of infinite. u_1 , u_2 , u_3, ... Are respectively known as first, second, third, … terms of the series.

Illustration 3: Finite series

(a) $1 + 2 + 3 + \cdots + 10$

(b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

(c) $1 + 4 + 9 + 16 + \cdots + 100$

Illustration 4: Infinite series

(a) $1^3 + 2^3 + 3^3 + \cdots$

(b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

Arithmetic Progression

Consider the series $3 + 5 + 7 + 9 + \cdots$. Here we see that the difference between any term and its previous term is constant, namely 3. Such a series where the difference between any term and the previous term is constant is known as an Arithmetic Series or Arithmetic Progression.

Standard Notations

First term $= a$, Common difference $= d$

Sum of terms $= S_n$ Last term $= l$

nth term $= t_n$

General term (nth term) of an A.P

First term $(t_1) = a$,   Second term $(t_2) = a + d$

Third term $(t_3) = a + d + d = a + 2d$

Fourth term $= a + 2d + d = a + 3d$

Similarly, tenth term $(t_{10}) = a + 9d$

Thus we have the general term,

$$t_n = a + (n - 1)d$$

Example 1

Find the tenth term of the A.P whose first term is 5 and common difference is 2.

Solution:

Here $a = 5$ and $d = 2$

$$t_n = a + (n - 1)d$$
$$t_{10} = 5 + 9 \times 2 = 23$$

Example 2

The 20th term of an A.P is 79. If the first term is 3, find the tenth term.

Solution:

First term, $a = 3$. Let the common difference be d

Now, $t_{20} = 79$

$$a + (20 - 1)d = 79$$
$$3 + 19d = 79$$
$$19d = 76$$
$$d = 4$$
$$\therefore t_{10} = a + (10 - 1)d = a + 9d = 3 + 9 \times 4$$
$$= 39$$

Example 3

Which term of the A.P $2 + 5 + 8 + \cdots$ is 92?

Solution:

Here $a = 2$, $d = 3$. Let 92 be the nth term of the A.P

$$t_n = 92$$
$$a + (n - 1)d = 92$$
$$2 + (n - 1)(3) = 92$$

Example 8

The fourth term of an AP is 13 and the tenth term is 31. Find the sum of the first ten terms.

Solution

$$T_4 = a + 3d = 13 \ldots\ldots\ldots\ldots..(i)$$
$$T_{10} = a + 9d = 31 \ldots\ldots\ldots\ldots..(ii)$$

Solving (i) and (ii) simultaneously i.e. (ii)$-(i)$

$$6d = 18 \quad \therefore d = 3$$
$$a + 3 \times 3 = 13$$
$$a = 4$$
$$\text{For} \quad n = 10,$$
$$S_{10} = \frac{10}{2}(2 \times 4 + (10 - 1)3)$$
$$= 5(35) = 175$$

$\therefore$ The sum of the ten terms of the AP is 175

Example 9

A man is employed in a company on 800 USD per month and is promised an increment of 25 USD per year. Find the total amount which he receives in 13 years and his pay in the last year.

Solution:

The monthly salary of the man forms an A.P with first term = 800 and common difference = 25

Total amount received by the man in 13 years

= Sum of 13 terms of the A.P × 12

(Since there are 12 months in a year)

$$= \frac{13}{2}[2 \times 800 + (13 - 1) \times 25] \times 12$$
$$= \frac{13}{20} \times 1900 \times 12$$
$$= 148200 \text{ USD}$$

The pay of the man in the last year = 13th term of the A.P

$$= 800 + (13 - 1) \times 25$$
$$= 1100 \text{ USD}$$

Example 10

A man borrows USD 1200 at the total interest of USD 168. He repays the entire amount in 12 instalments, each instalment being less than the preceding one by USD 20. Find the first instalment.

Solution:

The payments of the man form an A.P with first term, a and common difference -20

Total amount the man paid in 12 instalments

$= 1200 + 168 = 1368$ USD

$$S_{12} = 1368$$
$$\frac{12}{2}[2a + (12 - 1)(-20)] = 1368$$
$$6(2a - 220) = 1368$$
$$2a - 220 = 228$$
$$2a = 448$$
$$a = 224$$

Hence, the first instalment is USD 224

Example 11

A man arranges to pay off a debt of USD 7200 by 20 instalments which form an A.P. When 15 of the instalments are paid, he finds that one-third of his debt still remains unpaid. Find the amount of his 16th installment.

Solution:

Let the first term and common difference be a and d respectively.

$S_{20} = 7200$ and $S_{15} = \frac{2}{3} \times 7200 = 4800$

Thus,

$$\frac{20}{2}[2a + (20 - 1)d] = 7200$$
$$2a + 19d = 720 \quad \ldots\text{(i)}$$

Also,

$$\frac{15}{2}[2a + (15 - 1)d] = 4800$$
$$2a + 14d = 640 \quad \ldots\text{(ii)}$$

Subtracting (ii) from (i);

$$5d = 80$$
$$d = 16$$

From (i) we get;

$$2a + 19 \times 16 = 720$$
$$2a = 416$$
$$a = 208$$

Therefore, the 16th instalment is

$t_{16} = a + 15d = 208 + 15 \times 16 = 448$ USD

Example 12

A sum of USD 6240 is paid off in 30 instalments, such that each instalment is USD 10 more than the preceding instalment. Calculate the value of the first instalment.

Solution:

Here the instalments form an A.P of 30 terms with common difference 10.

$$S_{30} = 6240$$
$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\frac{30}{2}[2a + (30-1)\times 10] = 6240$$
$$15(2a + 290) = 6240$$
$$2a + 290 = 416$$
$$2a = 126$$
$$a = 63$$

Thus, the first instalment is USD 63

Geometric progression

Consider the series $2 + 6 + 18 + 54 + \cdots$. Here we see that the ratio between any term and its immediately preceding term is constant i.e. 3. Such a series where the ratio of any term to its immediately preceding term is constant is known as a **Geometric Series** or **Geometric Progression**. The constant ratio is known as the common ratio of the progression.

Illustration: The following series are in G.P.

(a) $2 + 2\sqrt{2} + 4 + 4\sqrt{2} + \cdots$
(b) $3 - 6 + 12 - 24 + 48 - \cdots$
(c) $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \cdots$

Standard Notations

First term $= a$, common ratio $= r$
nth term $= t_n$, Sum of n terms $= S_n$, Last term $= l$

Observation. If the first term a and common ratio r are known, then the G.P can be determined completely. Each term is obtained by multiplying r to the preceding term.

Example 13

Determine the G.P whose first term is 3 and common ratio is -2.

Solution:

First term = 3, Second term $= 3 \times -2 = -6$
Third term $= -6 \times -2 = 12$,
Fourth term $= 12 \times -2 = -24$ and so on.
Thus, the G.P is $3, -6, 12, -24, ...$

General term (nth term) of a G.P

First term $t_1 = a$, Second term $t_2 = ar$
Third term $t_3 = ar.r = ar^2$
Fourth term $t_4 = ar^2.r = ar^3$
Similarly, tenth term, $t_{10} = ar^9$
Thus, we note that in any term the index of r is one less than the position of the term. Thus

$$t_n = ar^{n-1}$$

Example 14

Find the tenth term of the G.P whose first term is 3 and common ratio is -2.

Solution:

Here $a = 3$ and $r = -2$

$$t_n = ar^{n-1}$$
$$t_{10} = 3(-2)^9 = 3(-512) = -1536$$

Example 15

The fifth term of a G.P is 162 and the first term is 12. Find the common ratio.

Solution:

First term $= a = 2$. Let the common ratio be r.
Now,

$$t_5 = 165$$
$$ar^4 = 162$$
$$2r^4 = 162$$
$$r^4 = 81$$
$$r = \pm 3$$

$\therefore$ The common ratio is 3 or -3

Example 16

The 10th term of a G.P is -2560 and the first term is 5. Find the 5th and nth term of the G.P.

Solution:

$$ar^9 = -2560$$
$$5r^9 = -2560$$
$$r^9 = -512$$
$$r^9 = (-2)^9$$
$$r = -2$$
$$\therefore t_5 = ar^4 = 5\cdot(-2)^4 = 80$$
$$t_n = ar^{n-1} = 5(-2)^{n-1}$$

Example 17

Which term of the series $4 + 12 + 36 + 108 + \cdots$ is 2916?

Solution:

The series is in G.P with first term $a = 4$ and the common ratio $r = 12 \div 4 = 36 \div 12 = 3$
Let 2916 be the nth term of the series

$$t_n = 2916$$
$$ar^{n-1} = 2916$$
$$4\,.\,3^{n-1} = 2916$$
$$3^{n-1} = 729$$
$$3^{n-1} = 3^6$$
$$n - 1 = 6 \Rightarrow n = 7$$

So, 2916 is 7th term of the series.

Example 18

It is given

$t_4 = 24$ and $t_7 = 192$

$$ar^3 = 24 \quad \text{.. (i)}$$

$$ar^6 = 192 \quad \text{... (ii)}$$

Dividing (ii) by (i) gives;

$$r^3 = 8$$

$$r = 2$$

Substituting the value of r in (i), we get

$$a\,.(2)^3 = 24$$

$$a = 3$$

Thus, first term $a = 3$,

second term $ar = 3 \times 2 = 6$,

third term $(ar^2) = 3(2)^2 = 12$

ninth term $(ar^8) = 3(2)^8 = 768$

Example 19

If $3x + 1$, $7x$ and $10x + 8$ are in a G.P, find the value of x

Solution:

Since $3x + 1$, $7x$ and $10x + 8$ are in G.P, we have

$$\frac{7x}{3x+1} = \frac{10x+8}{7x}$$

$$49x^2 = 30x^2 + 10x + 24x + 8$$

$$19x^2 - 34x - 8 = 0$$

$$19x^2 - 38x + 4x - 8 = 0$$

$$19x(x-2) + 4(x-2) = 0$$

$$(x-2)(19x+4) = 0$$

$$x = 2 \text{ or } x = -\frac{4}{19}$$

Sum of first n terms of a G.P

Let S_n be the sum of first n terms of a G.P whose first term is a and common ratio r, then

$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$ .. (i)

Multiplying both sides by r gives

$r.S_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$ ... (ii)

Subtracting (ii) from (i),

$$S_n(1-r) = a - ar^n$$

cancelling intermediate terms

$$\therefore S_n = \frac{a(1-r^n)}{1-r} \text{ or } \frac{a(r^n-1)}{r-1}$$

provided $r \neq 1$

Example 20

Find the sum of the following series

(a) $2 + 6 + 18 + 54 + \cdots$ to 10 terms

(b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ up to 12 terms

Solution:

(a) Here $a = 2$, $r = 6 \div 2 = 3$, $n = 10$

$$\text{Sum} = \frac{a(r^n-1)}{r-1} = \frac{2(3^{10}-1)}{3-1} = 59048$$

(b) Here $a = 1$, $r = -\frac{1}{2} \div 1 = -\frac{1}{2}$

$$\text{Sum} = \frac{a(r^n-1)}{r-1} = \frac{1\left[1-\left(-\frac{1}{2}\right)^{12}\right]}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{1-\frac{1}{2^{12}}}{\frac{1}{2}} = \frac{4096-1}{4096} \times 2 = \frac{4095}{2048}$$

Note: When $r > 1$, we generally use the formula $S_n = \frac{a(r^n-1)}{r-1}$ and when $r < 1$ we generally use the formula $S_n = \frac{a(1-r^n)}{1-r}$

Example 21

Find the sum of the series $3 + 6 + 12 + \cdots + 3072$

Solution:

The series is in G.P with first term $a = 3$ and common ratio $r = 2$.

Let the series contain n terms.

$$t_n = 3072$$

$$ar^{n-1} = 3072$$

$$3\,.\,2^{n-1} = 3072$$

$$2^{n-1} = 1024$$

$$2^{n-1} = 2^{10}$$

$$n - 1 = 10$$

$$n = 11$$

The required sum is

$$S_{11} = \frac{a(r^{11}-1)}{r-1} = \frac{3(2^{11}-1)}{2-1}$$

$$= 3 \times 2047 = 6141$$

Example 22

A man borrows Shs 16,380 without interest and repays the loan in 12 monthly instalments, each instalment (beginning with the second) being twice the preceding one. Find the amount of the last instalment.

Solution:

Here the successive instalments form a G.P of 12 terms whose common ratio is 2 and sum 16380.

2020, No. 3

The sum of the first 16 terms of an arithmetic progression (A.P) is 1088. The 16th term is twice the 8th term. Determine the value of the first term of the A.P.

Solution:

$$t_n = a + (n-1)d$$
$$t_8 = a + 7d$$
$$t_{16} = a + 15d$$

Given that $t_{16} = 2t_8$

$$a + 15d = 2(a + 7d)$$
$$a + 15d = 2a + 14d$$
$$d = a \text{ .. (i)}$$

Given $S_{16} = 1088$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$1088 = \frac{16}{2}[2a + 15d\ ]$$
$$1088 = 8(2a + 15d]$$
$$136 = 2a + 15d$$

From (i);

$$136 = 2a + 15a$$
$$136 = 17a$$
$$a = \frac{136}{17} = 8$$

2022, No. 3

A man's monthly salary in his first year of work was Shs 250,000. He got an increment of 5% every year. Calculate;

(a) the man's total earnings at the end of the year
(b) his total earnings after 5 years

Solution:

(a) At the end of the year, man's earnings is

$$= 12 \times 250000$$
$$= \text{Shs } 3{,}000{,}000$$

(b) At the beginning of 2nd year, man's monthly earning $= 250{,}000 + \frac{5}{100}(250{,}000)$

$$= 250{,}000\ (1.05)$$

At the end of 2nd year, total earnings

$$= 12 \times 250{,}000\ (1.05)$$

At the beginning of 3rd year, man's monthly earning

$$= 250{,}000(1.05) + \frac{5}{100} \times 250{,}000(1.05)$$
$$= 250{,}000\ (1.1025)$$

At the end of 3rd year, total earnings

$$= 12 \times 250{,}000\ (1.1025)$$

Total earnings after 3 years

$$= 12 \times 250000 + 12 \times 250{,}000\ (1.05) + 12 \times 250{,}000\ (1.1025)$$
$$= 3{,}000{,}000\ [1 + 1.05 + 1.1025]$$

The term in the brackets forms a G.P with $a = 1$, $r = 1.05$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

At the end of 5 years, total earnings

$$= 3{,}000{,}000 \frac{1(1.05^5 - 1)}{1.05 - 1}$$
$$= \text{Shs } 16{,}576{,}893.75$$

Self-Evaluation Exercise

1. Find the number of terms in each of the following A.Ps

(a) $5 + 8 + 11 + 14 + \cdots \ldots. +62$

(b) $1 + 6 + 11 + \cdots \ldots \ldots + 501 + 506$

[Ans: (a) 20 (b) 102]

2. Find the sum to infinity of the series;

(a) $16 + 12 + 9 + \cdots$

(b) $16 + 8 + 4 + 2 + 1 + \cdots.$

(c) $84 - 42 + 21 - 10\frac{1}{2} + \cdots \ldots$

[Ans: (a) 64 (b) 32 (c) 56]

3. Find the sum of each of the following A.Ps

(a) $2 + 4 + 6 + 8 + 10 + \cdots. +146$

(b) $100 + 95 + 90 + 85 + \cdots \ldots \ldots \ldots. -20$

(c) $4 + 10 + 16 + 22 + 28 + \cdots \ldots. . +334$

[Ans: (a) 5402 (b) 1000 (c) 9464]

4. The 5^{th} term of an arithmetic progression is 12 and the sum of the first 5 terms is 80. Determine the 1^{st} term and common difference.

[Ans: $a = 20,\ d = -2$]

5. What is the number of terms of a geometric progression $5, 10, 20 \ldots \ldots$ that can give a sum greater than 500,000?

[Ans: 17]

6. The 10^{th} term of an arithmetic progression is 20 and the 15^{th} term is 44. Find the value of the first term and the common difference, hence find the sum of the first 60 terms

[Ans:5430]

7. The 8^{th} term of an AP is twice the third term and the sum of the first eight terms is 39. Find the sum of the of the first 21 terms of the AP.

[Ans: 204.75]

Chapter 6 Permutations and Combinations

Introduction

A permutation means arrangement of things taken one or more at a time. For example, let us consider three different things a, b, c. The permutations of these things

(i) taken one at a time are a, b, c

(ii) taken two at a time are ab, ba, bc, cb, ca, ac; and

(iii) taken three at a time are abc, acb, bca, bac, cab, cba

A combination means selections or grouping of things taken one or more at a time. Thus, the combinations of three different things a, b, c

(i) taken one at a time are a, b, c

(ii) taken two at a time are ab, bc, ca; and

(iii) taken three at a time is abc

Thus, in a combination we only consider those things which are included in the group. We do not consider the order in which they appear. Thus, ab and ba both are taken as the same combination. In the same way, no matter in what order a, b, c are written, we get, only one combination $'abc'$ of three letters taken all at a time.

In permutation, we always consider the order in which they appear. Thus, ab and ba both are two different permutations. Likewise abc, acb, bca, bac, cab, cba are six different permutations. Actually from a single combination, we can get many permutations. As for example, from a single combination $'abc'$, we get six different permutations.

Permutation

Some Notations

When n and r are positive integers and $r \leq n$, then the product of r consecutive integers in decreasing order beginning from n is denoted ${}^nP_r$. Thus

$${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1)$$

e.g. ${}^7P_3 = 7 \times 6 \times 5 = 210$ (Product of 3 integers in decreasing order beginning from 7)

$${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$$

The notation $n!$ (read as n factorial) denotes the product of the first n positive integers

$$n! = 1 \times 2 \times 3 \times \ldots\ldots \times (n-2) \times (n-1) \times n$$

$$n! = n(n-1)(n-2)\ldots.2 \times 1$$

We note that $n! = {}^nP_n$ i.e. $5! = {}^5P_5$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120,\ 4! = 4 \times 3 \times 2 \times 1 = 24$$

Presentation of ${}^n\boldsymbol{P}_r$ in terms of factorial

$${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)\ldots.2\,.\,1}{(n-r)(n-r-1)\ldots\ldots2\,.\,1} = \frac{n!}{(n-r)!}$$

Thus

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\text{e.g. } {}^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!}$$

Fundamental principle of permutation

Let an operation be operated in m ways. After it has been performed in any one of these m ways, let the second operation be performed in n ways and after this second operation has been performed in any one of the n ways, let the third operation be performed in p ways. Then the three operations can be performed in $m \times n \times p$ ways. The result can be extended to any number of operations.

Example 1

A community hall has 5 doors. In how many ways can a person enter the hall and come through a different door?

Solution:

The person can enter the hall through any one of the 5 doors, i.e. in 5 ways. After entering the hall in any one of the 5 ways, he can come out of the hole through any one of the remaining 4 doors, i.e. in 4 ways. Therefore, by the fundamental principle of permutation, the required number of ways is $5 \times 4 = 20$

Example 2

There are 4 different routes between the places A and B, 3 different routes between B and C and 6 different routes between C and D. In how many ways can a person go from A to D through B and C?

Solution:

The person can go from A to B by any one of the 4 different routes i.e. by 4 different ways. After reaching B in any one of the 4 ways, he can go to C by any one of the three different routes i.e. by 3 different ways. After reaching C, he can then go to D by any one of 6 different routes i.e. by 6 different ways. Thus, by the fundamental principle of permutation, the required number of ways $= 4 \times 3 \times 6 = 72$

Example 3

In how many ways can 4 students be arranged in a row of (a) 4 seats (b) 6 seats.

Solution:

(a) The first seat can be filled by any one of the 4 students in 4 ways. After filling the first place in any one of the 4 ways, the second seat can be filled in by any one of the remaining 3 students in 3 ways. Then the third seat can be filled in by any one of the remaining 2 students in 2 ways and the last seat can be filled in by the remaining 1 student in 1 way. Thus, by the fundamental principle of permutation,

$$\text{Required number of ways} = 4 \times 3 \times 2 \times 1 = 24$$

(b) The first student can be arranged in any one of the 6 seats in 6 ways. After arranging the first in any one of the 6 ways, the second student can be arranged in any one of the remaining 5 seats in 5 ways. Then the third student can be arranged in remaining 4 seats in 4 ways and the last student can be arranged in any one of the remaining 3 seats in 3 ways. Thus, by the fundamental principle of permutation, the required number of ways $= 6 \times 5 \times 4 \times 3 = 360$

Permutations of Different Things

The number of permutations of n different things taken r at a time is ${}^{n}P_{r}$

Example 4

In how many ways can the letters of the word **MEASURING** be arranged or permuted?

Solution:

Number of letters = 9

So we are arranging 9 letters out of 9

$${}^{9}P_{9} = \frac{9!}{(9-9)!} = \frac{9!}{0!} = 9! = 362{,}880 \text{ ways}$$

things in space is $\frac{1}{2}(n-1)!$. As for example, the number of different necklaces that can be formed with 10 different beads is $\frac{1}{2}(10-1)! = \frac{1}{2}9!$

Example 12

In how many ways can 6 boys form a ring?

Solution:

We first the position one boy. The remaining 5 boys can be arranged among themselves in $5!$ Ways.

$$\therefore \text{The required number of different ways} = 5! = 120$$

Example 13

In how many different ways can 7 different coloured stones from a necklace?

Solution:

We first fix the position of any one of 7 stones. The remaining 6 stones can be arranged among themselves in 6! Ways. But for corresponding to each clock-wise arrangement there is one anti-clockwise arrangement which are essentially the same arrangement. Thus, the required number of arrangements

$$= \frac{1}{2}6! = 360$$

Example 14

Find the number of ways in which the letters of the word '**HISTORY**' can be arranged? How many of them

(a) begin with H?

(b) end with Y?

(c) begin with H and end with Y?

(d) begin H but do not end with Y?

Solution:

The word 'HISTORY' consists of 7 different letters. These 7 letters can be arranged among themselves in 7! ways. Thus, the number of all possible arrangements of the letters

$$= 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 1 = 5040$$

(a) We keep H fixed at the first place. The remaining 6 letters can be arranged in the remaining 6 places in 6! ways. Thus, the number of arrangements beginning with H is $6! = 720$ (H ××××××)

(b) We keep Y fixed at the last place. Then the first 6 places can be filled in by remaining 6 letters in 6! ways. Thus, the number of arrangements ending with Y is $6! = 720$. (××××××Y)

(c) We now keep H fixed at the first place and Y fixed at the last place. We have 5 more letters and 5 more spaces. These 5 letters can be placed in 5 places in 5! ways. Thus, the number of arrangements beginning with H and ending with Y is $5! = 120$. (H×××××Y)

(d) Number of arrangements beginning with H and not ending with Y = Number of arrangements beginning with $H -$ Number of arrangements beginning with H and ending with Y

$$= 6! - 5! = 720 - 120 = 600 \text{ [(H ××××××) − (H×××××Y)]}$$

Example 15

In how many ways can 3 girls and 5 boys be arranged in a row so that all the 3 girls are together?

Solution:

We consider 3 girls together to be a single person. Thus, with 5 other boys we have $(5+1)$ i.e. 6 persons. These 6 persons can be arranged among themselves in 6! ways. Again, for each of these 6 arrangements the 3 girls among themselves can be arranged in 3 ways. Thus, the required number of arrangements

$$= 6! \times 3! = 720 \times 6 = 4320$$

Example 21

How many words can be performed from the letters of the word **DAUGHTER** so that

(i) the vowels always come together

(ii) the vowels are never together

Solution

(i) The given word contains 8 different letters. When the vowels AUE are always, they can be treated as an entity i.e. DGHTR (AUE)

There are six letters which can be arranged = $6! = 720$

But the three vowels can also be arranged in 3! = 6

Total number of ways = $720 \times 6 = 4320$

(ii) The total number of ways of arranging the word DAUGHTER = 8! = 40320

$$\begin{pmatrix}\text{number of ways when the}\\ \text{vowels are never together}\end{pmatrix} = \begin{pmatrix}\text{total number}\\ \text{of ways}\end{pmatrix} - \begin{pmatrix}\text{number of ways when the}\\ \text{vowels are always together}\end{pmatrix}$$

$$= 40320 - 4320 = 36000$$

Example 22

Find the number of ways in which the letters of the word **SHALLOW** can be arranged (a) if the two $L's$ must not come together (b) if the two $L's$ must always be together

Solution:

Leaving out the two $L's$, the letter **SHAOW** can be arranged in 5! ways

$$\uparrow\ \boldsymbol{S}\ \uparrow\ \boldsymbol{H}\ \uparrow\ \boldsymbol{A}\ \uparrow\ \boldsymbol{O}\ \uparrow\ \boldsymbol{W}\ \uparrow$$

(a) With each of these ways the first L can be inserted in any one of the places.
When this is done, there are then 5 possible places for the second L not next to the first. Hence the total number of arrangements with the two L s separated is $5! \times 6 \times 5$ provided the L s can be distinguished. They cannot and so the number of arrangements is

$$\frac{5! \times 6 \times 5}{2} = 5! \times 15 = 1800$$

(b) In this case take the two $L's$ (LL) as one object. There are then six places for it in each of the 5! arrangements of the letters SHAOW. Hence the number of arrangements is

$$6 \times 5! = 6! = 720$$

Combination of Different Things

The number of combinations of n different things taken r at a time is ${}^{n}C_{r}$.

$${}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

Example 23

A question paper contains 8 questions. In how many ways can a student answer any 5 questions?

Solution:

The number of ways is equal to the number of selections of 5 things from 8 different things.

Thus, the required number of ways

$$= {}^{8}C_{5} = \frac{8!}{3!\,5!} = \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} = 56$$

Restricted Combination

The number of combinations of n different things taken r at a time such that m particular things are

(a) never included is ${}^{n-m}C_{r}$ and

(b) always included is ${}^{n-m}C_{r-m}$

Example 24

Find the number of selections of 15 persons taken 9 at a time in which 3 particular persons are

(a) never included

(b) always included

Solution:

(a) When 3 particular persons are excluded then we have to select 9 persons from remaining $(15-3)$ i.e. 12 persons.

∴ The required number of selections = $^{12}C_9 = \frac{12!}{3!9!} = \frac{12\times11\times10\times9!}{3\times2\times1\times9!} = 220$

(b) When 3 particular persons are always included then we have to select (-3) i.e. 6 more persons from remaining $(15-3)$ i.e. 12 persons.

∴ The required number of selections = $^{12}C_6 = \frac{12!}{6!6!} = \frac{12\times11\times10\times9\times8\times7\times6!}{6\times5\times4\times3\times2\times1\times6!} = 924$

Example 25

In an election, a voter may vote for any number of candidates not greater than the number to be chosen. There are seven candidates and four members are to be chosen. In how many ways can a person vote?

Solution:

There are 7 candidates from which 4 are to be chosen. A voter may vote for any number of candidates not more than 4.

Thus, a voter can vote (i.e. select) for

(i) any 1 candidate which can be done in $^7C_1$ ways

(ii) any 2 candidates which can be done in $^7C_2$ ways

(iii) any 3 candidates which can be done in $^7C_3$ ways

(iv) any 4 candidates which can be done in $^7C_4$ ways

∴ Total number of ways = $^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 = 7 + 21 + 35 + 35 = 98$

Example 26

There are 7 men and 3 ladies. Find the number of ways in which a committee of 6 persons can be formed if the committee is to have at least 2 ladies.

Solution:

The committee can be formed in 2 ways i.e. (i) selecting 2 ladies and 4 men and (ii) selecting 3 ladies and 3 men.

From 7 men any 4 and from 3 ladies any 2 can be selected in $^7C_4 \times {}^3C_2$ ways $= 35 \times 3 = 105$

From 7 men any 3 and from 3 ladies any 3 can be selected in $^7C_3 \times {}^3C_3$ ways $= 35 \times 1 = 35$

∴ Total number of ways $= 105 + 35 = 140$

Example 27

From six gentlemen and four ladies, a committee of five is to be formed. In how many ways can this be done if the committee is to include at least one lady?

Solution:

There are $(6+4)$ i.e. 10 persons altogether

From these 10 person, 5 persons can be selected in $^{10}C_5 = 252$ ways

If all 5 persons selected are men then the number of ways = $^6C_5 = 6$

Number of ways in which at least one lady is selected

= Number of all possible ways – Number of ways in which all selected persons are men

$= 252 - 6 = 246$

Alternative method:

From 10 consonants any 4 and from 5 vowels any 2 can be selected in ${}^{10}C_4 \times {}^{5}C_2$ ways. For each way of selecting 6 letters, they can be arranged among themselves in 6! ways to form different words.

$$\therefore \text{Total number of words} = {}^{10}C_4 \times {}^{5}C_2 \times 6! = 210 \times 10 \times 720 = 1512000$$

Example 32

A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw.

Solution:

From $(2 + 3 + 4)$ i.e. 9 balls, 3 balls can be drawn in ${}^{9}C_3$ ways.

3 non-black balls can be drawn from $(2 + 4)$ i.e. 6 non-black balls in ${}^{6}C_3$ ways.

Therefore, the number of ways in which at least one black ball will be included in the draw

= Total numbers of ways in which 3 balls can be drawn – Number of ways in which no black ball is drawn

$$= {}^{9}C_3 - {}^{6}C_3 = 64$$

Example 33

A group consists of 4 boys and 7 girls. In how many ways can a team of five be selected if it is to contain

(a) no boys (b) 2 boys and 3 girls (c) at least 3 boys?

Solution:

(a) No boys are selected, so the team is chosen from the 7 girls number of ways of choosing 5 girls from 7 $= {}^{7}C_5 = 21$

(b) 2 boys can be chosen from 4 in ${}^{4}C_2 = 6$ ways

3 girls can be chosen from 7 in ${}^{7}C_3 = 35$ ways

Number of teams = $6 \times 35 = 210$

(c) If the team is to have at least 3 boys, then there must be either 3 or 4 boys

Number of teams with 3 boys and 2 girls = ${}^{4}C_3 \times {}^{7}C_2 = 84$

Number of teams with 4 boys and 1 girl = ${}^{4}C_4 \times {}^{7}C_1 = 7$

These are mutually exclusive events,

so, number of teams with at least 3 boys $= 84 + 7 = 91$

Example 34

A group consists of 6 men and 5 women. If a committee of five members is to be formed, in how many ways can this be done if it must contain

(a) at least one woman (b) not more than three men?

Solution:

(a) If the committee is to have at least one woman, then it can have $1, 2, 3, 4$ or 5 women

With 1 woman and 4 men, number of ways = ${}^{5}C_1 \times {}^{6}C_4 = 75$

With 2 women and 3 men, number of ways = ${}^{5}C_2 \times {}^{6}C_3 = 200$

With 3 women and 2 men, number of ways = ${}^{5}C_3 \times {}^{6}C_2 = 150$

With 4 women and 1 man, number of ways = ${}^{5}C_4 \times {}^{6}C_1 = 30$

With 5 women and no man, number of ways = ${}^{5}C_5 \times {}^{6}C_0 = 1$

Total number of ways $= 75 + 200 + 150 + 30 + 1 = 456$ ways

(b) If the committee is not to have more than 3 men, then it can have 3, 2, 1 or no man

With 3 men and 2 women, number of ways = ${}^{6}C_3 \times {}^{5}C_2 = 200$

With 2 men and 3 women, number of ways = ${}^{6}C_2 \times {}^{5}C_3 = 150$

Number of arrangements = $\frac{7!}{3!} = 840$

Number of arrangements of N and G $= 2! = 2$

Total number of arrangements = $840 \times 2 = 1680$

2015, No. 4

A committee of 5 people is to be formed from a group of 6 men and 7 women.

(a) Find the number of possible committees

(b) What is the probability that there are only 2 women on the committee?

Solution:

(a)

Men/6	Women/7	Number of committees
0	5	$6C_0 \times 7C_5 = 21$
1	4	$6C_1 \times 7C_4 = 210$
2	3	$6C_2 \times 7C_3 = 525$
3	2	$6C_3 \times 7C_2 = 420$
4	1	$6C_4 \times 7C_1 = 105$
5	0	$6C_5 \times 7C_0 = 6$
Total		1287

Number of possible committees = 1287

(b) Number of committees with two women = 525

$$\text{P (2 women)} = \frac{525}{1287} = 0.408$$

2017, No. 2

A father and a mother with their five children are to sit on a bench. What is the probability that the father and mother will sit next to each other?

Solution:

Total number of family members = 7

Number of ways of sitting on a bench

$$= 7! = 5040 \text{ ways}$$

If the father and mother are always seated next to each other, we can take them as one/ a couple.

Number of ways = $6!$

But the father and mother can sit in two ways i.e. Father – Mother or Mother – Father

Total number of ways in which mother and father are seated next to each other

$$= 6! \times 2 = 1440 \text{ ways}$$

Now, $n(E) = 1440$

$n(S) = 5040$

$$\text{Probability} = \frac{n(E)}{n(S)} = \frac{1440}{5040} = \frac{2}{7} \text{ or } 0.286$$

2020 No. 11

(a) Find the number of all possible arrangements of all the letters of the word DISAPPEAR

(b) In a school, there are nine A-level teachers. In the Science department, there is a teacher for each of the following subjects: Mathematics, Physics, Chemistry and Biology. In the Arts department, there is a teacher for each of the following subjects: Economics, Geography, History, Literature and Fine Art. Three teachers are to be sent for a workshop.

(i) Find the number of all possible combinations of teachers that may be sent for the workshop

(ii) What is the probability that at least two teachers from the science department are sent for the workshop?

(iii) If a Mathematics teacher must attend the workshop, determine the number of possible combinations of teachers to be sent.

Solution:

(a) DISAPPEAR has 9 letters with 2A's and 2P's

Number of possible arrangements $= \frac{9!}{2!2!} = 90720$

(b) The School has 4 Science teachers and 5 Arts teachers to make a total of 9

(i) Number of all possible combinations $= {}^9C_3 = 84$

(ii) At least two teachers from the Science department means either 2 science teachers and 1 Arts teacher or 3 science teachers and no Arts teacher

Number of combinations $= {}^4C_2 \times {}^5C_1 + {}^4C_3 \times {}^5C_0 = 6 \times 5 + 4 \times 1 = 34$

$n(S) = 84, n(E) = 34$

$$\text{Probability} = \frac{n(E)}{n(S)} = \frac{34}{84} = \frac{17}{42}$$

(iii) If the mathematics teacher is not to attend, we chose 3 teachers out of the remaining 8 in

$${}^8C_3 = 56 \text{ ways}$$

Number of ways in which mathematics teacher must attend

= Total number of ways – Number of ways in which mathematics teacher never attends

$$= 85 - 56 = 28 \text{ ways}$$

Alternatively:

If a Mathematics teacher must attend, then 3 possible scenarios are possible i.e. 1 math teacher and 2 other science teachers or 1 math teacher and 2 Arts teachers or 1 math teacher, 1 other science teacher and 1 Arts teacher

Number of combinations $= {}^3C_2 + {}^5C_2 + {}^3C_1 \times {}^5C_1 = 3 + 10 + 15 = 28$

2022, No.4

Three letters are chosen at random from the word **CLOTHINGS**. Determine the probability that two of these three letters chosen are consonants.

Solution:

The word 'CLOTHINGS' has 9 letters 2 of which are vowels and 7 are consonants.

Number of ways of choosing 3 letters out 9 is

$${}^9C_3 = 84$$

If two of the letters chosen are consonants, then the third letter will be a vowel i.e. we select 2 letters out of the 7 consonants and 1 letter out of the 2 vowels

$$\text{Number of ways} = {}^7C_2 \times {}^2C_1 = 21 \times 2 = 42$$

$n(S) = 84, n(E) = 42$

$$\text{Probability} = \frac{n(E)}{n(S)} = \frac{42}{84} = \frac{1}{2}$$

Self-Evaluation Exercise

1. Evaluate without using a calculator

   (a) $\frac{8!}{6!}$ (b) $\frac{9!}{3\times5!}$ (c) $\frac{5!\times4!}{6!}$

   [Ans: (a) 56 (b) 1008 (c) 4]
2. Find the number of permutations of two different letters taken from the letters A, B, C, D, E, F

   [Ans: 30]
3. In how many ways can six books be arranged on a shelf when the books are selected from ten different books?

   [Ans: 151200]
4. How many code words each consisting of five different letters, can be formed from the letters A, B, C, D, E, F, G and H?

   [Ans: 6720]
5. In how many ways can the letters of the word **MEDIAN** be arranged? [Ans: 720]
6. How many different teams of 7 players can be chosen from 10 girls? [Ans: 120]
7. Three students are to be promoted from a particular class. If five students are under consideration for promotion, in how many ways can the group to be promoted be chosen?

   [Ans: 10]
8. A librarian has to select 5 newspapers and 7 magazines from the 8 newspapers and 9 magazines which are available. In how many ways can she make her selection?

   [Ans: 2016]
9. Find the number of different selections of 3 letters from the word **METHOD**

   [Ans: 20]
10. A group consists of 5 boys and 8 girls. In how many ways can a team of four be chosen, if it contains (a) no girls (b) not more than one girl (c) at least two boys?

    [Ans: (a) 5 (b) 85 (c) 365]
11. In how many ways can a committee of five people be selected from 7 men and 3 women if it must contain (a) 3 men and 2 women (b) 3 women and 2 men (c) at least 1 woman?

    [Ans: (a) 105 (b) 21 (c) 231]
12. In how many ways can a committee of 7 people be selected from 4 men and 6 women if the committee must have at least 4 women on it?

    [Ans: 100]
13. A group consists of 5 boys and 8 girls. In how many ways can a team of five be chosen if it is to contain (a) no girls (b) no boys (c) at least one boy ?

    [Ans:(a) 1 (b) 56 (c) 1231]
14. A tennis club has to select two mixed double pairs from a given group of 5 men and 4 women. In how many ways can this be done?

    [Ans: 120]
15. A circular ring has ten different beads. In how many ways can the beads be arranged along the ring?

    [Ans: 181440]
16. Find the number of arrangements of the letters of the word **COMMITTEE**

    [Ans: 60480]
17. A combination of five vehicles is to be chosen from six saloon cars and seven vans. If at least three saloon cars must be chosen. In how many ways can the combination be done?

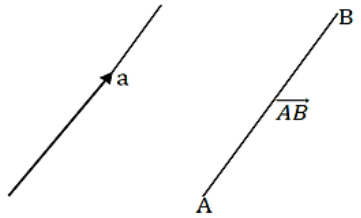
    [Ans: 531]
18. Determine the number of different arrangements of the letters in the word **ARRANGE**?

    [Ans: 1260]

Chapter 7 Vectors

Introduction

A vector quantity is one that has magnitude and it is related to a definite direction in space i.e.



Equal vectors

For any two vectors to be equal, they must have the same magnitude and direction

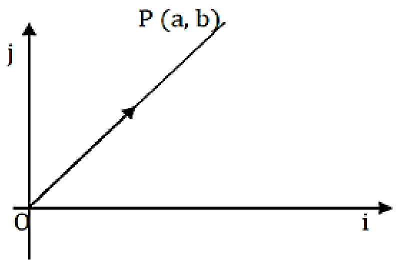
Parallel vectors

Two vectors a and b are parallel if one is a scalar multiple of the other i.e.

$$a = \lambda b$$

Position vectors

A position vector whose distance and direction from the origin is specific. Consider a vector $ai + bj$



The position vector $\overrightarrow{OP}$ is given by $\overrightarrow{OP} = ai + bj$

Addition and subtraction of vectors

Example 1

If $a = 3i + 4j$ and $b = 2i + 8j$. Find

(a) $a + b$

(b) $a - 2b$

Solution:

(a) $a + b = 3i + 4j + 2i + 8j = 5i + 12j$

(b) $a - 2b = 3i + 4j - 2(2i + 8j)$

$= 3i + 4j - 4i - 16j$

$= -i - 12j$

Modulus of a vector

The modulus of a vector a is the magnitude of a i.e. the length of the line representing a. The modulus of a vector a is denoted by $|a|$

$$|ai + bj| = \sqrt{a^2 + b^2}$$

Note: The vector $ai + bj$ can be denoted by $\begin{pmatrix} a \\ b \end{pmatrix}$ which is a column vector.

Example 2

Given that $a = 3i + 4j$ and $b = 2i + 8j$. Find (a) $|a|$ (b) $|b|$ (c) $|a + b|$

Solution:

(a) $|a| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

(b) $|b| = \sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17}$

(c) $a + b = 3i + 4j + 2i + 8j$

$|a + b| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

Unit vectors

A unit vector is a vector whose magnitude or length is one. It is usually written as $\tilde{a}$. The unit vector of a is given by $\tilde{a} = \frac{a}{|a|}$

Example 3

Find the unit vector of $2i - j$

Solution:

$|2i - j| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$ units

The unit vector will be $\frac{1}{\sqrt{5}}(2i - j)$

Example 4

Find the unit vector of a if $a = 3i + 2j$

Solution:

$$|a| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\tilde{a} = \frac{a}{|a|}$$

$$\tilde{a} = \frac{1}{\sqrt{13}}(3i + 2j)$$

Example 5

In a parallelogram $OABC$, $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = c$. The point D lies on AB such that $AD: DB = 1: 2$.

$$\frac{PN}{NS} = 2 \Rightarrow PN = 2NS$$

Since PN is a scalar multiple of NS, then PN is parallel to NS.

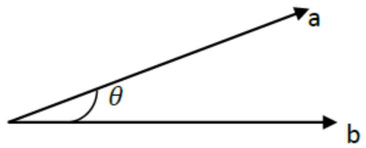
But since both vectors contain a common point N, N and S are collinear (lie on a straight line) and so N lies on PS as required.

Now $\frac{PN}{NS} = \frac{2}{1}$ so $PN : NS = 2 : 1$

Hence $\overline{PN} : \overline{SN} = 2 : 1$

The scalar product

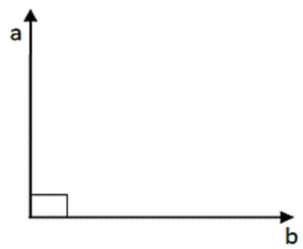
The scalar dot product of two vectors is defined as the product of the magnitude of the two vectors and the cosine of the angle between the two vectors



$$\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \boldsymbol{\theta}$$

Properties of the scalar product

From definition $a.b = |a||b| \cos \theta$, it follows that two perpendicular vectors have a scalar product of zero



$$a.b = |a||b| \cos 90°$$
$$\Rightarrow a.b = 0$$

1. $a.b = b.a$
2. $a.(b + c) = a.b + a.c$
3. $\lambda(a.b) = a.(\lambda b) = (\lambda a).b = \lambda |a||b| \cos \theta$

Consider the vectors $a = x_1 i + y_1 j$ and $b = x_2 i + y_2 j$

Now $a.b = (x_1 i + y_1 j).(x_2 i + y_2 j)$

$= x_1 x_2 i.i + x_1 y_2 i.j + y_1 x_2 j.i + y_1 y_2 j.j$

But $i.i = j.j = 1$ and $i.j = j.i = 0$

$\therefore a.b = x_1 x_2 + y_1 y_2 = |a||b| \cos \theta$ where θ is the acute angle between a and b.

Example 10

Find the angle between the vectors a and b given that $a = 3i + 4j$ and $b = 5i - 12j$

Solution:

Let θ be the required angle

$$a.b = |a||b| \cos \theta$$
$$|a| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$
$$|b| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$$
$$a.b = \begin{pmatrix} 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 5 \\ -12 \end{pmatrix} = (3 \times 5) + (4 \times -12)$$
$$= 15 - 48 = -33$$
$$-33 = 5 \times 13 \cos \theta$$
$$-\frac{33}{65} = \cos \theta$$
$$\cos \theta = -0.50769$$
$$\theta = \cos^{-1}(-0.50769) = 120.51°$$

The angle between the vectors is $120.51°$

Example 11

The points A, B, C and D have position vectors $-2i + 3j$, $3i + 8j$, $7i + 6j$ and $7i - 4j$ respectively. Show that AC is perpendicular to BD.

Solution:

To show that two vectors are perpendicular, we must get their dot product and it should be equal to zero.

$$\overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \overrightarrow{OD} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$$
$$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} 7 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ -12 \end{pmatrix}$$
$$AC.BD = \begin{pmatrix} 9 \\ 3 \end{pmatrix} . \begin{pmatrix} 4 \\ -12 \end{pmatrix} = (9 \times 4) + (3 \times -12)$$
$$= 36 - 36 = 0$$

Therefore since $AC.BD = 0, AC$ is perpendicular to BD

Example 12

The position vectors $OP = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $OQ = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ are joined to form triangle OPQ. Determine

(i) the lengths of triangle OPQ

(ii) the angle between OP and OQ

(iii) the area of the triangle OPQ

Solution:

$$-12 = \sqrt{29} \times \sqrt{5} \cos\theta$$
$$\cos\theta = \frac{-12}{\sqrt{29}\times\sqrt{5}} = -0.9965$$
$$\theta = \cos^{-1}(-0.9965) = 175.24°$$

The angle between the two vectors OP and OQ is 175.24°

(b) (i) $PQ = OQ - OP = \begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}-2\\-5\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

(ii) $OR = OP + \lambda PQ = \begin{pmatrix}-2\\-5\end{pmatrix} + \lambda\begin{pmatrix}3\\3\end{pmatrix}$

$$= \begin{pmatrix}-2+3\lambda\\-5+3\lambda\end{pmatrix}$$

(iii) The dot product of perpendicular vectors is zero i.e. $OR.OQ = 0$

$$\begin{pmatrix}-2+3\lambda\\-5+3\lambda\end{pmatrix}.\begin{pmatrix}3\\3\end{pmatrix} = 0$$
$$3(-2+3\lambda) + 3(-5+3\lambda) = 0$$
$$-6 + 9\lambda - 15 + 9\lambda = 0$$
$$18\lambda - 21 = 0$$
$$\lambda = \frac{21}{18} = \frac{7}{6}$$

2017, No. 3

The vector $\boldsymbol{a} = \boldsymbol{3i} + \boldsymbol{2j}$ and $\boldsymbol{b} = \boldsymbol{4i} - \boldsymbol{5j}$

Determine

(a) $|\boldsymbol{b}|$

(b) $\boldsymbol{a}\,.\,\boldsymbol{b}$

Solution:

$a = \begin{pmatrix}3\\2\end{pmatrix}$ and $b = \begin{pmatrix}4\\-5\end{pmatrix}$

(a) $|b| = \sqrt{4^2 + (-5)^2} = \sqrt{16+25}$

$$= \sqrt{41} = 6.4$$

(b) $a\,.\,b = \begin{pmatrix}3\\2\end{pmatrix}\cdot\begin{pmatrix}4\\-5\end{pmatrix} = 3\times4 + 2\times-5$

$$= 12 - 10 = 2$$

2018, No. 10

Points A, B and C have position vectors, $2j$, $4i$ and $2i - 2j$ respectively in the $x - y$ plane.

(a) Find $2OA + 3OB - 4OC$

(b) Determine;

(i) $\boldsymbol{AB}$ and $\boldsymbol{AC}$

(ii) $\boldsymbol{AB}\cdot\boldsymbol{AC}$

(iii) angle $\boldsymbol{BAC}$

Solution:

$$OA = \begin{pmatrix}0\\2\end{pmatrix}; OB = \begin{pmatrix}4\\0\end{pmatrix}; OC = \begin{pmatrix}2\\-2\end{pmatrix}$$

(a) $2OA + 3OB - 4OC$

$$= 2\begin{pmatrix}0\\2\end{pmatrix} + 3\begin{pmatrix}4\\0\end{pmatrix} - 4\begin{pmatrix}2\\-2\end{pmatrix}$$
$$= \begin{pmatrix}0\\4\end{pmatrix} + \begin{pmatrix}12\\0\end{pmatrix} - \begin{pmatrix}8\\-8\end{pmatrix}$$
$$= \begin{pmatrix}4\\12\end{pmatrix}$$

(b) (i) $AB = OB - OA = \begin{pmatrix}4\\0\end{pmatrix} - \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$

$AC = OC - OA = \begin{pmatrix}2\\-2\end{pmatrix} - \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}2\\-4\end{pmatrix}$

(ii) $AB\cdot BC = \begin{pmatrix}4\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-4\end{pmatrix} = 8 - 8 = 0$

(iii) Let angle $BAC = \theta$

$$AB\cdot AC = |AB||AC|\cos\theta$$
$$0 = \sqrt{20}\times\sqrt{20}\cos\theta$$
$$\cos\theta = 0$$
$$\theta = 90°$$
$$\therefore \text{angle } BAC = 90°$$

Alternatively, since $AB\cdot AC = 0$, AB is perpendicular to AC, thus angle $BAC = 90°$

2019, No. 12

If $OA = \begin{pmatrix}6\\5\end{pmatrix}$, $OB = \begin{pmatrix}9\\2\end{pmatrix}$ and $OC = \begin{pmatrix}7\\0\end{pmatrix}$,

(a) find the vectors;

(i) BC

(ii) AB

(b) Show that the vectors AB and BC are perpendicular

(c) Determine the magnitude of the vector $2BC - 3AB$

Solution:

(a) (i) $BC = OC - OB = \begin{pmatrix}7\\0\end{pmatrix} - \begin{pmatrix}9\\2\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix}$

(ii) $AB = OB - OA = \begin{pmatrix}9\\2\end{pmatrix} - \begin{pmatrix}6\\5\end{pmatrix} = \begin{pmatrix}3\\-3\end{pmatrix}$

(b) $AB\cdot BC = \begin{pmatrix}3\\-3\end{pmatrix}\cdot\begin{pmatrix}-2\\-2\end{pmatrix} = -6 + 6 = 0$

Since $AB\cdot BC = 0$, the vectors AB and BC are perpendicular.

(c) $2BC - 3AB = 2\begin{pmatrix}-2\\-2\end{pmatrix} - 3\begin{pmatrix}3\\-3\end{pmatrix}$

$$= \begin{pmatrix}-4\\-4\end{pmatrix} - \begin{pmatrix}9\\-9\end{pmatrix}$$
$$= \begin{pmatrix}-13\\5\end{pmatrix}$$
$$|2BC - 3AB| = \sqrt{(-13)^2 + 5^2} = \sqrt{194}$$
$$= 13.93$$

2020, No. 7

Given $a = \begin{pmatrix}5\\-12\end{pmatrix}$ and $b = \begin{pmatrix}-3\\4\end{pmatrix}$, find the;

(a) dot product of a and b

(b) angle between the vectors a and b

Solution:

(a) $a.b = \begin{pmatrix} 5 \\ -12 \end{pmatrix} . \begin{pmatrix} -3 \\ 4 \end{pmatrix} = (5)(-3) + (-12)(4)$

$= 63$

(b) $|a| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$

$|b| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$

Let the angle between the vectors be θ

$$a.b = |a||b| \cos\theta$$

$$\cos\theta = \frac{a.b}{|a||b|} = \frac{63}{(13)(5)} = \frac{63}{65}$$

$$\theta = \cos^{-1}\left(\frac{63}{65}\right) = 14.25°$$

Self-Evaluation exercise

1. If the point P has position vector $7i - 3j$ and point Q has position vector $5i + 5j$.
   Find (a) $\overrightarrow{PQ}$ (b) $\overrightarrow{QP}$
   [Ans: $(a) - 2i + 8j$ $(b)\ 2i - 8j$]
2. The point P has position vector $3i - 2j$ and Q is a point such that $\overrightarrow{QP} = 2i - 3j$. Find the position vector of Q. [Ans: $i + j$]
3. Given that $a = 3i - j$ and $b = 2i + j$, find (a) $|a|$ $(b)\ |b|$ $(c)\ a + b$ $(d) |a + b|$
   [Ans: (a) $\sqrt{10}$ $(b)\ \sqrt{5}$ $(c) 5i$ $(d)\ 5$]
4. Given that $a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$; find (a) $a + 2b$ $(b) |a + 2b|$ (c) $2a + 3b$ (d) $|2a + 3b|$
   [Ans: (a) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ $(b)\ 5$ (c) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ $(d)\ \sqrt{58}$]
5. The three points A, B and C have position vectors $i - j, 5i - 3j$ and $11i - 6j$ respectively. Show that A, B and C are collinear.
6. Find the angle between each of the following pairs of vectors
   (a) $a = 3i + 4j$ and $b = 5i + 12j$
   (b) $c = 5i - j$ and $d = 2i + 3j$
   [Ans: $(a) 14°$ $(b) 68°$]
7. The points A, B, C and D have position vectors $5i + j, -3i + 2j, -3i - 3j$ and $i - 6j$ respectively. Show that AC is perpendicular to BD
8. The points E, F and G have position vectors $2i + 2j$, $i + 6j$ and $-7i + 4j$. Show that the triangle EFG is right angled at F.
9. The points A, B and C have position vectors $4i - j, i + 3j$ and $-5i + 2j$ respectively.
   Find (a) $\overrightarrow{AB}$ $(b)\overrightarrow{BC}$ $(c)\ \overrightarrow{CA}$ $(d)$the angles of a triangle ABC
   [Ans: (a) $-3i + 4j$ (b) $-6i - j$ (c) $9i - 3j$ $(d) 35°, 117°, 28°$]
10. E is the centre of the rectangle ABCD and $\overrightarrow{AB} = a, \overrightarrow{BC} = b$, Express in terms of a and b the vectors
    (i)$\overrightarrow{AC}$ (ii)$\overrightarrow{CD}$ (iii) $\overrightarrow{BD}$ (iv) $\overrightarrow{EB}$ (v) $\overrightarrow{EA}$
    [Ans: (i) $a + b$ (ii) $- a$ (iii) $b - a$ (iv)$\frac{1}{2}(a - b)$ (v)$-\frac{1}{2}(a + b)$]
11. The position vectors of three points A, B and C relative to the origin O are p , $3q - p$ and $9q - 5p$ respectively. Show that the points A, B and C lie on the same straight line, state the ratio $AB{:}BC$. Given that $OBCD$ is a parallelogram and that E is the point such that $DB = \frac{1}{3}DE$, find the position vectors of D and E relative O
    [Ans: $1{:}2$, $6q - 4p$, $5p - 3q$]
12. The position vectors of the points A and B with respect to the origin O are $2i + 3j$, $-i + 5j$ respectively. Find the position vector C such that $\overrightarrow{AC} = 2\overrightarrow{AB}$. Calculate the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{OB}$ [Ans: $-4i + 7j$, $45°$]
13. The position vectors of points A, B and C relative to the origin O are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ respectively. Write down the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. Use the vector methods to calculate (i) angle BAC (ii) angle ABC. State the special property of triangle ABC and deduce its area.
    [Ans: $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $45°, 90°$, right angled isosceles , 5 sq units]
14. Given the vectors $a = 2i - 4j$ and $b = 3i + 5j$. Find the
    (a) modulus of the vector $5a + 2b$
    (b) angle between the vectors a and b
    [Ans: (a) 18.87 (b) 122.47°]
15. If $a = 2i + j$ and $b = i - 2j$. Express in terms of i and j
    (i) $2a + b$ (ii) $-3a + 4b$, hence find the angle between the vectors $2a + b$ and $-3a + 4b$
    [Ans: (i) $5i$ (ii) $-2i - 11j$; $100.3°$]

Chapter 8 Trigonometry

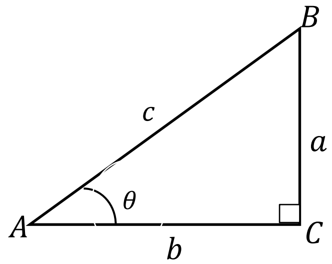
Introduction

Trigonometry is a branch of mathematics that studies the relationship between the three sides and the three angles of a right-angled triangle in terms of ratios and representing them as trigonometric ratios; sine, cosine and tangent.

Trigonometric ratios for the general angle

The trigonometric ratios include the main three mentioned above and the others include secant, cosecant and cotangent abbreviated as sec, cosec and cot respectively.

If we consider a right-angled triangle ABC



then $\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$

Also $\tan\theta = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin\theta}{\cos\theta}$

It can also be observed that the remaining angle, $B = 90° - \theta$ and that $\sin(90° - \theta) = \frac{b}{c} = \cos\theta$ and $\cos(90° - \theta) = \frac{a}{c} = \sin\theta$

Therefore, $\sin\theta = \cos(90° - \theta)$ and $\cos\theta = \sin(90° - \theta)$

Now using the Pythagoras theorem i.e.

$$a^2 + b^2 = c^2$$

Dividing through by c^2 gives;

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\mathbf{\sin^2\theta + \cos^2\theta = 1}$$

Though we have derived these relationships using an acute angle, they are identities i.e. true for any angle and should be memorized.

The three remaining trigonometric ratios are reciprocals of sine, cosine and tangent.

They are;

$$\text{secant} = \frac{1}{\text{cosine}}$$

$$\text{cosecant} = \frac{1}{\text{sine}}$$

$$\text{cotangent} = \frac{1}{\text{tangent}} = \frac{\text{cosine}}{\text{sine}}$$

Further trigonometric identities

Using $sin^2\theta + cos^2\theta = 1$

Dividing through by $cos^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

But $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\frac{1}{\cos\theta} = \sec\theta$

$$\therefore \mathbf{\tan^2\theta + 1 = \sec^2\theta}$$

Now if we divide the original identity by $sin^2\theta$, we obtain;

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

But $\frac{\cos\theta}{\sin\theta} = \cot\theta$ and $\frac{1}{\sin\theta} = cosec\ \theta$

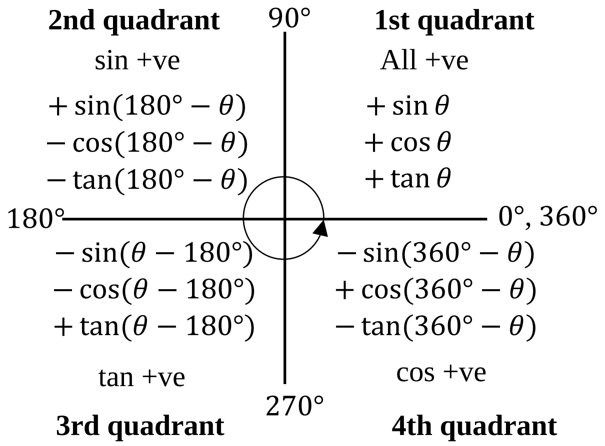
$$\therefore \mathbf{1 + \cot^2\theta = cosec^2\theta}$$

These three identities will be found useful later when solving equations and also proving other identities.

Trigonometric ratios for general angles

The relationship between the ratios of the general angles and the corresponding acute angles depends on which quadrant the basic angle lies in. The angles can lie in four quadrants following in the anti-clockwise direction. In the 1st quadrant, all are positive, in the 2nd quadrant only sine is positive, in the 3rd quadrant only tan is positive and in the 4th quadrant only cosine is positive. Note that the positive angles are measured in the anticlockwise direction and the negative angles are measured in the clockwise direction.

The relationships can be summarized as shown below.



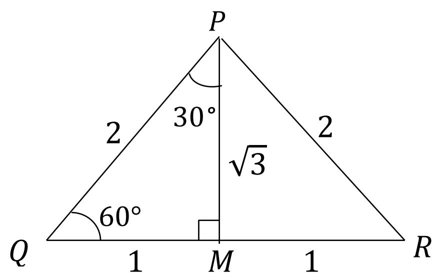
It can also be remembered by a student saying "All Scientist Take Chemistry" in the anticlockwise direction.

Trigonometric ratios for special angles

The trigonometric ratios of the angles 0°, 30°, 45°, 60° and 90° are used often in mechanics and other branches of mathematics and so it is useful to have their values in surd form.

30° and 60°

Suppose ΔPQR *is* equilateral, with sides 2 units and that PM is the perpendicular bisector of QR



Using the Pythagoras theorem, $MP^2 + MQ^2 = PQ^2$

$$MP = \sqrt{2^2 - 1^2} = \sqrt{3}$$

Since ΔPQR *is* equilateral,

$$PQM = 60° \text{ and } QPM = 30°$$

From ΔPQM;

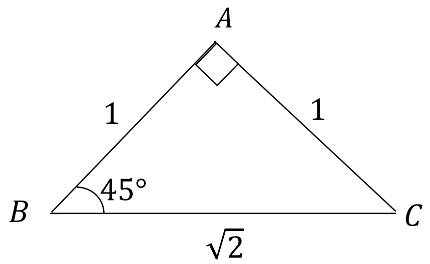
$$\sin 30° = \tfrac{1}{2}; \cos 30° = \tfrac{\sqrt{3}}{2}; \tan 30° = \tfrac{1}{\sqrt{3}} \text{ or } \tfrac{\sqrt{3}}{3}$$

and

$$\sin 60° = \tfrac{\sqrt{3}}{2}\ ;\ \cos 60° = \tfrac{1}{2}\ ;\tan 60° = \tfrac{\sqrt{3}}{1} = \sqrt{3}$$

45°

Consider a right-angled triangle which is isosceles and in which the equal sides are 1 unit in length. The equal sides will each be 45°



Using the Pythagoras theorem;

$$BC^2 = 1^2 + 2^2 \text{ or } BC = \sqrt{2}$$

Hence

$$\sin 45° = \tfrac{1}{\sqrt{2}} \text{ or } \tfrac{\sqrt{2}}{2}; \cos 45° = \tfrac{\sqrt{2}}{2}; \tan 45° = \tfrac{1}{1} = 1$$

0° and 90°

$$\sin 0° = 0, \cos 0° = 1 \text{ and } \tan 0° = 0$$

$$\sin 90° = 1, \quad \cos 90° = 0$$

$$\text{and } \tan 90° = \tfrac{\sin 90°}{\cos 90°} = \tfrac{1}{0} = \infty \text{ (undefined)}$$

The results can be summarized in the table below

Angle	sin	cos	tan
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

Example 1

Show that $\cos^2 30° + \cos 60° \sin 30° = 1$

Solution:

The left-hand side is $\cos^2 30° + \cos 60° \sin 30°$

$$L.H.S = (\cos 30°)(\cos 30°) + \cos 60° \sin 30°$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$$

$$= R.H.S \text{ as required}$$

Obtuse angles

Trigonometric ratios of obtuse angles cannot be defined by means of a right-angled triangle. The sine, cosine or tangent of an obtuse angle is the sine, cosine or tangent of a supplement angle, with the appropriate sign

If θ is an obtuse angle;

$\theta = \cos^{-1}(1) = 0^0 \Rightarrow \theta = \{180^0\}$

Or $3\cos\theta - 4 = 0 \Rightarrow \cos\theta = \frac{4}{3}$

$\cos\theta = \cos^{-1}\left(\frac{4}{3}\right)$ (undefined)

$\therefore$for $0^0 \leq \theta \leq 180^0 = \{180^0\}$

2016, No. 12

(a) Solve the equation $1 + \cos\theta = 2\sin^2\theta$ for values of θ between 0° and 360°

(b) By eliminating θ from the equations $x = a\sec\theta$ and $y = b + C\cos\theta$, show that $x(y-b) = Ca$

Solution:

(a) $1 + \cos\theta = 2\sin^2\theta$

$$1 + \cos\theta = 2(1 - \cos^2\theta)$$
$$1 + \cos\theta = 2 - 2\cos^2\theta$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$2\cos^2\theta + 2\cos\theta - \cos\theta - 1 = 0$$
$$2\cos\theta\,(\cos\theta + 1) - (\cos\theta + 1) = 0$$
$$(\cos\theta + 1)(2\cos\theta - 1) = 0$$

Either $\cos\theta + 1 = 0 \Rightarrow \cos\theta = -1$

$$\cos^{-1} 1 = 0°$$

Since cos is negative, it lies in the 2nd and 3rd quadrant

$$\theta = 180°$$

Or $2\cos\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{2}$

$$\cos^{-1}\frac{1}{2} = 60°$$

Since cos is positive, it lies in the 1st and 4th quadrants

$$\theta = 60°, 300°$$

For $0° \leq \theta \leq 360°, \theta = \{60°, 180°, 300°\}$

(b) $x = a\sec\theta$ … (i), $y + b + C\cos\theta$ .. (ii)

From (i); $x = \frac{a}{\cos\theta}$

$$\cos\theta = \frac{a}{x}$$

Substituting for $\cos\theta$ in (ii) gives;

$$y = b + C\frac{a}{x}$$
$$y - b = \frac{Ca}{x}$$
$$x(y-b) = Ca$$

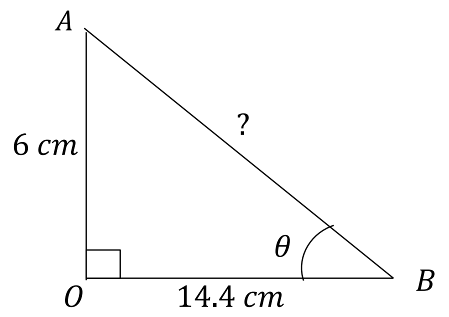
2017, No. 12

(a) Triangle OAB is such that angle $AOB = 90°$, angle $ABO = \theta$, $\overline{OB} = 14.4\ cm$, $\overline{OA} = 6\ cm$. Find $\sin\theta + \cot\theta$

Solve: $2\cos^2 x = \sin x + 1$ for $0° \leq x \leq 360°$

Solution:

(a)



$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2$$
$$\overline{AB}^2 = 14.4^2 + 6^2$$
$$AB = \sqrt{243.36} = 15.6\ cm$$
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{OA}{AB} = \frac{6}{15.6} = 0.385$$
$$\cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{OA}{OB} = \frac{6}{14.4}$$
$$\cot\theta = 1 \div \frac{6}{14.4} = 1 \times \frac{14.4}{6} = 2.4$$
$$\therefore \sin\theta + \cot\theta = 0.385 + 2.4 = 2.785$$

(b) $2\cos^2 x = \sin x + 1$

$$2(1 - \sin^2 x) = \sin x + 1$$
$$2 - 2\sin^2 x = \sin x + 1$$
$$2\sin^2 x + \sin x - 1 = 0$$
$$2 \times -1 = -2 \leftrightarrow (-1, 2)$$
$$2\sin^2 x + 2\sin x - \sin x - 1 = 0$$
$$2\sin x\,(\sin x + 1) - (\sin x + 1) = 0$$
$$(\sin x + 1)(2\sin x - 1) = 0$$

Either $\sin x + 1 = 0$

$$\sin x = -1$$
$$\sin^{-1}(1) = 90°$$
$$x = \{270°\}$$

Or $2\sin x - 1 = 0$

$$\sin x = \frac{1}{2}$$
$$\sin^{-1}\left(\frac{1}{2}\right) = 30°$$
$$x = \{30°, 150°\}$$

For $0° \leq x \leq 360°, x = \{30°, 150°, 270°\}$

Chapter 9 Differentiation

Introduction

Differentiation is the process of obtaining the gradient of the curve. The gradient function $\frac{dy}{dx}$ (pronounced as " dee y dee x ") or the differential coefficient of y with respect to x.

The form ax^n

If $y = ax^n$, then $\frac{dy}{dx} = nax^{n-1}$ where a is a constant

(Multiply by the power and then decrease the power by one)

Example 1

Differentiate the following with respect to x

(i) x^8 (ii) $3x^5$ (iii) $\frac{3}{x^2}$ (iv) $\sqrt{x}$

Solution:

(i) let $y = x^8$

$$\Rightarrow \frac{dy}{dx} = 8 \times x^{8-1} = 8x^7$$

(ii) let $y = 3x^5$

$$\Rightarrow \frac{dy}{dx} = 5 \times 3x^{5-1} = 15x^4$$

(iii) let $y = \frac{3}{x^2} = 3x^{-2}$

$$\Rightarrow \frac{dy}{dx} = -2 \times 3x^{-2-1} = -6x^{-3} = -\frac{6}{x^3}$$

(iv) let $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Differentiating a constant (k)

If $y = k$ and k is a constant, $\frac{dy}{dx} = 0$

$\Rightarrow$ The derivative of a constant is zero

If $y = 6$, then $\frac{dy}{dx} = 0$

Differentiating a sum or difference

When differentiating a sum or difference, we differentiate separately i.e.

If $y = f(x) + g(x) - h(x)$

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) - \frac{d}{dx}(h(x))$$

$$\frac{dy}{dx} = f'(x) + g'(x) - h'(x)$$

Example 2

If $y = ax^2 + bx + c$. Find $\frac{dy}{dx}$ if a, b and c are constants

Solution:

$$\frac{dy}{dx} = 2ax + b$$

Example 3

If $y = 3x^2 - 6x + \frac{2}{x^2}$, find $\frac{dy}{dx}$

Solution:

$$y = 3x^2 - 6x + \frac{2}{x^2} = 3x^2 - 6x + 2x^{-2}$$

$$\frac{dy}{dx} = 6x - 6 + (-2 \times x^{-2-1})$$

$$= 6x - 6 - 4x^{-3} = 6x - 6 - \frac{4}{x^3}$$

Example 4

Find $\frac{dy}{dx}$ if $y = x^3 + 7x^2 - 2x + 12$

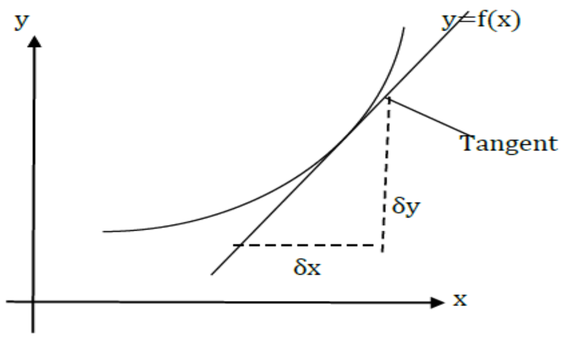
Solution:

$$\frac{dy}{dx} = 3x^2 + 14x - 2$$

Gradient of a function / curve

The gradient of a curve is not a constant therefore the gradient of a curve is determined at a particular point. The gradient of the curve at any point is defined as the gradient of the tangent to the curve at that point and measures the rate of increase of y with respect to x

The gradient of the curve at a point is equal to the gradient of the tangent at that same point



Gradient of tangent = $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

At $x = 1$, $\frac{dy}{dx} = 4 - 2(1) = 2$

when $x = 1$, $y = 4(1) - 1^2 = 3$

Gradient of tangent = 2,

Gradient of normal = $\frac{-1}{2}$ at a point $(1, 3)$

The equation of the normal is thus given by;

$$\frac{y-3}{x-1} = \frac{-1}{2}$$

$$2(y-3) = -1(x-1)$$

$$2y - 6 = -x + 1$$

$\therefore 2y + x = 7$ is the equation of normal to the curve

Example 11

Find the equation of the tangent and normal to the curve $y = x^2 - 4x + 1$ at the point $(-2, 13)$

Solution:

$y = x^2 - 4x + 1, \frac{dy}{dx} = 2x - 4$

At $(-2, 13)$, $\frac{dy}{dx} = 2(-2) - 4 = -8$

Thus gradient of the tangent is -8

Equation of tangent is given by; $\frac{y-13}{x+2} = -8$

$$\Rightarrow y - 13 = -8(x+2)$$

$$y - 13 = -8x - 16$$

$$y = -8x + 3$$

Gradient of the normal = $\frac{-1}{-8} = \frac{1}{8}$

Equation of normal is given by; $\frac{y-13}{x+2} = \frac{1}{8}$

$$\Rightarrow 8(y - 13) = x + 2$$

$$8y - 104 = x + 2$$

$$8y = x + 106$$

Second derivative

We can repeat the differentiation process to obtain the second derivative, which is written as $\frac{d^2y}{dx^2}$

Or if $y = f(x)$, it is written as $f''(x)$

Example 12

Find $\frac{d^2y}{dx^2}$ if $y = 3x^3 - 6x + 4$

Solution:

$$\frac{dy}{dx} = 9x^2 - 6$$

$$\frac{d^2y}{dx^2} = 18x$$

Example 13

Find $\frac{d^2y}{dx^2}$ if $y = 3x^2 + 45x - 75 - x^3$

Solution:

$$y = 3x^2 + 45x - 75 - x^3$$

$\frac{dy}{dx} = 6x + 45 - 3x^2$

$\frac{d^2y}{dx^2} = 6 - 6x$

Techniques of differentiation

Example 14

Differentiate with respect to (w.r.t) x

(a) $(2x - 1)^2$

(b) $(3x - 1)(2x + 4)$

(c) $\frac{3x^4 + 2x^2 - 1}{2x^2}$

Solution

(a) Let $y = (2x - 1)^2$

$$y = (2x-1)(2x-1) = 4x^2 - 4x + 1$$

$$\frac{dy}{dx} = 8x - 4$$

(b) let $y = (3x - 1)(2x + 4)$

$$y = 6x^2 + 10x - 4$$

$$\frac{dy}{dx} = 12x + 10$$

(c) Let $y = \frac{3x^4 + 2x^2 - 1}{2x^2}$

$$= \frac{3x^4}{2x^2} + \frac{2x^2}{2x^2} - \frac{1}{2x^2} = \frac{3}{2}x^2 + 1 - \frac{1}{2}x^{-2}$$

$$\frac{dy}{dx} = 3x + x^{-3} = 3x + \frac{1}{x^3}$$

Applications of differentiation

Stationary points/Turning points

A point on a curve at which $\frac{dy}{dx} = 0$ is called a stationary point. At such point, the tangent to the curve is parallel to the x-axis

Example 15

Find the stationary points of the curve $y = 4x^3 + 15x^2 - 18x + 7$

Solution:

$\frac{dy}{dx} = 12x^2 + 30x - 18$

For stationary values $\frac{dy}{dx} = 0$

$$\Longrightarrow 12x^2 + 30x - 18 = 0$$

$2x^2 + 5x - 3 = 0$ (on dividing through by 6)

$$2x^2 + 2x + 3x - 3 = 0$$

$$(2x - 1)(x + 3) = 0$$

Either $x = -3$ or $x = \frac{1}{2}$

when $x = -3$,

$$y = 4(-3)^3 + 15(-3)^2 - 18(-3) + 7 = 88$$

$$x(4-x)=0$$

Either $x=0$ or $x=4$,

$\Rightarrow (0,0)$ and $(4,0)$ are the x −intercepts

Turning point

$$y=4x-x^2$$

$$\frac{dy}{dx}=4-2x$$

$$4-2x=0\Rightarrow x=2$$

when $x=2, y=4(2)-2^2=4$

$(2,4)$ is a turning point

We now investigate for the nature of the turning point of the curve

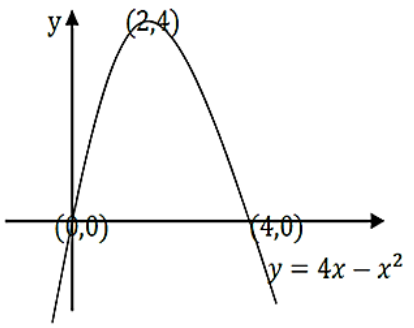
	L	2	R
Sign of $\frac{dy}{dx}$	+	0	−
	/	—	\

We observe that $(2,4)$ is a maximum turning point

Alternatively; if we would wish to investigate the nature of the turning point using the second derivative, we find out that $\frac{d^2y}{dx^2}=-2$ which is less than 0 ($\frac{d^2y}{dx^2}<0$)

Hence, the curve has a maximum turning point

We can now sketch the curve



Example 21

Sketch the graph of the function $y=5+4x-x^2$

Solution:

Intercepts

when $y=0, 5+4x-x^2=0$

$$5+5x-x-x^2=0$$

$$5(1+x)-x(1+x)=0$$

$$(1+x)(5-x)=0$$

either $x=5$ or $x=-1$

The curve cuts the x-axis at $(5,0)$ and $(-1,0)$

when $x=0, y=5+4(0)-0^2=5$

The curve cuts the y-axis at $(0,5)$

Turning point

$$y=5+4x-x^2$$

$$\frac{dy}{dx}=4-2x$$

For turning points, $\frac{dy}{dx}=0 \quad \Rightarrow 4-2x=0$ which gives $x=2$

Now we need to find the y-value corresponding to the x-value obtained above

When $x=2$,

$$y=5+4(2)-(2)^2=5+8-4=9$$

Thus $(2,9)$ is the turning point

Nature of the turning point

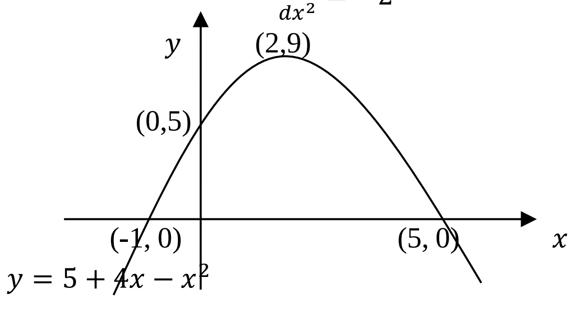
	L	2	R
Sign of $\frac{dy}{dx}$	+	0	−
	/	—	\

Thus $(2,9)$ is a maximum turning point

Alternatively, we can find the nature of the turning point using the second derivative

From $\frac{dy}{dx}=4-2x$

$$\frac{d^2y}{dx^2}=-2$$



Example 22

Sketch the curve $y=x^2+2x-3$

Solution

Intercepts

when $y=0, x^2+2x-3=0$

$$x^2+3x-x-3=0$$

$$x(x+3)-(x+3)=0$$

$$(x+3)(x-1)=0$$

either $x=-3$ or $x=1$

The curve cuts the x-axis at $(1,0)$ and $(-3,0)$

when $x=0, y=(0)^2+2(0)-3=-3$

The curve cuts the y-axis at $(0,-3)$

Turning point

$$y=x^2+2x-3$$

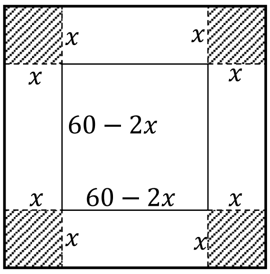
$$\frac{dy}{dx}=2x+2$$

For turning points, $\frac{dy}{dx}=0$

length of the side of the square to be cut out so that the box has maximum value.

Solution:

Let the length of the side of the square to be cut out be x cm.



$\therefore$ Length of the box $= (60 - 2x)$ cm

breadth of the box $= (60 - 2x)$ cm

Height of the box $= x$ cm

Volume of the box $= (60 - 2x)^2 x$

$$= 4(30 - x)^2 x$$
$$= 4(900 - 60x + x^2)x$$
$$v = 4(900x - 60x^2 + x^3)$$
$$\frac{dv}{dx} = 4(900x - 120x + 3x)$$

Now for v to be maximum or minimum we have

$$\frac{dv}{dx} = 0$$
$$4(900x - 120x + 3x) = 0$$
$$3x^2 - 120x + 900 = 0$$
$$x^2 - 40x + 300 = 0$$
$$(x - 10)(x - 30) = 0$$

$x = 10$ or $x = 30$

$$\frac{d^2v}{dx^2} = 4(-120 + 6x) = 24(x - 20)$$

At $x = 10, \frac{d^2v}{dx^2} = 24(10 - 20) < 0$

Thus v is maximum when $x = 10$

At $x = 30, \frac{d^2v}{dx^2} = 24(30 - 20) > 0$

Thus v is minimum when $x = 30$

Hence the required length is 10 cm

Displacement, Velocity and Acceleration

If $y = f(x), \frac{dy}{dx}$ is the rate of change of y with respect to x

Similarly, if $u = f(v)$, then $\frac{du}{dv}$ is the rate of change of u with respect to v

Now the velocity v of a body is defined as the rate of displacement s of a body from some fixed origin, with respect to time i.e. $v = \frac{ds}{dt}$

The acceleration a of a body is defined as the rate of the velocity of a body with respect to time i.e. $a = \frac{dv}{dt}$

So, displacement, velocity and acceleration are linked up with the process of differentiation with respect to time.

Example 30

The displacement s metres of a body from an origin O at a time t seconds is given by $s = 2t^2 - 3t + 6$. Find (a) the displacement (b) the velocity (c) the acceleration of the body when $t = 1$

Solution:

Given $s = 2t^2 - 3t + 6$

(a) When $t = 1, s = 2(1)^2 - 3(1) + 6 = 5\ m$

(b) Since $v = \frac{ds}{dt}$

$$v = \frac{ds}{dt} = 4t - 3$$

when $t = 1, v = 4(1) - 3 = 1\ ms^{-1}$

(c) From $a = \frac{dv}{dt}$

$$v = 4t - 3,\ a = \frac{dv}{dt} = 4\ ms^{-2}$$

Example 31

If $v = t^2 - 4t + 3$, find (a) the values of t when the body is at rest (b) the acceleration when $t = 5$

Solution:

(a) At rest, $v = 0$

$$t^2 - 4t + 3 = 0$$
$$t^2 - t - 3t + 3 = 0$$
$$t(t - 1) - 3(t - 1) = 0$$
$$(t - 1)(t - 3) = 0$$

$t = 1$ or $t = 3$

(b) Using $a = \frac{dv}{dt}$

$$v = t^2 - 4t + 3 \Rightarrow \frac{dv}{dt} = 2t - 4$$
$$a = 2t - 4$$

when $t = 5, a = 2(5) - 4 = 6\ ms^{-2}$

Example 32

If $s = 5t^3 - t$, find the expressions of v and a in terms of t

Solution:

$$v = \frac{ds}{dt} = 15t^2 - 1$$
$$a = \frac{dv}{dt} = 30t$$

Differentiation of natural log (ln x)

It is important to take note of the differentiation of natural logarithms i.e. $\log_e x$ (written as $\ln x$).

If $y = \log_e x = \ln x$, then $e^y = x$

If $y = e^x$, then $\ln y = x$

Similar to other logarithms, $\ln 1 = 0$, $\ln e = 1$ and other laws of logarithms apply.

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

This a very important result (proof not required) that will be used in related problems.

Example 33

Differentiate the following with respect to x

(a) $\ln(ax+b)$ (b) $x^2 + \ln x$ (c) $x^3 + 3x^2 - \ln x$

Solution:

(a) Let $y = \ln(ax+b)$

$$\frac{dy}{dx} = \frac{a}{ax+b}$$

(b) Let $y = x^2 + \ln x$

$$\frac{dy}{dx} = 2x + \frac{1}{x}$$

(c) Let $y = x^3 + 3x^2 - \ln x$

$$\frac{dy}{dx} = 3x^2 + 6x - \frac{1}{x}$$

Examination questions

2013, No. 10

(a) Sketch the curve$y = 5 + 4x - x^2$

(b) Find the area enclosed between the curve and the x-axis from $x = -1$ to $x = 5$

Solution:

(a) The requirements to sketch a curve are; find the intercepts i.e. where curve cuts the axes and the turning point and its nature i.e. where $\frac{dy}{dx} = 0$

Intercepts

when $x = 0, y = 5 + 4(0) - (0)^2 = 5$

$\Rightarrow (0,5)$ is the y −intercept

when $y = 0, 5 + 4x - x^2 = 0$

Thus $5 + 5x - x - x^2 = 0$

$5(1+x) - x(1+x) = 0$

$(1+x)(5-x) = 0$

Either $1 + x = 0 \Rightarrow x = -1$

or $5 - x = 0 \Rightarrow x = 5$

Thus $(-1,0)$ and $(5,0)$ are the x – intercepts

Turning point

$$y = 5 + 4x - x^2$$

$$\frac{dy}{dx} = 4 - 2x$$

For turning points, $\frac{dy}{dx} = 0 \quad \Rightarrow 4 - 2x = 0$ which gives $x = 2$

When $x = 2$,

$$y = 5 + 4(2) - (2)^2 = 5 + 8 - 4 = 9$$

Thus $(2,9)$ is the turning point

Nature of the turning point

	L	2	R
Sign of $\frac{dy}{dx}$	+ /	0 —	− \

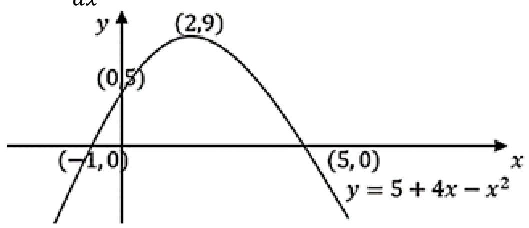
Thus $(2,9)$ is a maximum turning point

Alternatively,

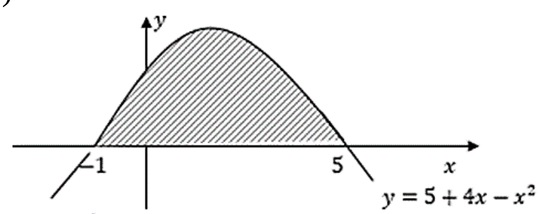
From $\frac{dy}{dx} = 4 - 2x$

$$\frac{d^2y}{dx^2} = -2$$

Since $\frac{d^2y}{dx^2} < 0$, it is a maximum turning point



(b)



$A = \int_a^b y\,dx$

$= \int_{-1}^{5}(5 + 4x - x^2)dx$

$= \left[5x + 2x^2 - \frac{x^3}{3}\right]_{-1}^{5}$

$= \left(5(5) + 2(5)^2 - \frac{5^3}{3}\right)$

$- \left(5(-1) + 2(-1)^2 - \frac{(-1)^3}{3}\right)$

$= \left(25 + 50 - \frac{125}{3}\right) - \left(-5 + 2 + \frac{1}{3}\right)$

$= \left(75 - \frac{125}{3}\right) - \left(-3 + \frac{1}{3}\right)$

$= \frac{100}{3} + \frac{8}{3} = \frac{108}{3} = 36$ sq. units

2014, No. 12

Given the curve $y = 3x^3 - 4x^2 - x$

(a) Find the turning points of the curve

(b) Distinguish between the nature of the turning points

Solution:

(a) For turning points; $\frac{dy}{dx} = 0$

$$y = 3x^3 - 4x^2 - x$$
$$\frac{dy}{dx} = 9x^2 - 8x - 1$$
$$9x^2 - 8x - 1 = 0$$
$$9x^2 - 9x + x - 1 = 0$$
$$9x(x-1) + (x-1) = 0$$
$$(x-1)(9x+1) = 0$$

Either $x - 1 = 0 \Rightarrow x = 1$

or $9x + 1 = 0 \Rightarrow x = -\frac{1}{9}$

When $x = 1, y = 3(1)^3 - 4(1)^2 - (1) = -2$

$(1, -2)$ is a turning point

When $x = -\frac{1}{9}$,

$$y = 3\left(-\frac{1}{9}\right)^3 - 4\left(-\frac{1}{9}\right)^2 - \left(-\frac{1}{9}\right) = \frac{14}{243}$$

$(-\frac{1}{9}, \frac{14}{243})$ is a turning point

(b) Nature of the turning points

$$\frac{d^2y}{dx^2} = 18x - 8$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 18(1) - 8 = 10 > 0$$

$\Rightarrow (1, -2)$ is a minimum turning point

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{1}{9}} = 18\left(-\frac{1}{9}\right) - 8 = -10 < 0$$

$\Rightarrow \left(-\frac{1}{9}, \frac{14}{243}\right)$ is a maximum turning point

2015, No. 5

Find the gradient of the curve $y = 4x^2(3x + 2)$ at the point $(1, 20)$

Solution:

$$y = 4x^2 \times 3x + 4x^2 \times 2 = 12x^3 + 8x^2$$
$$\frac{dy}{dx} = 36x^2 + 16x$$

At $(1, 20)$, $x = 1$

$$\frac{dy}{dx} = 36(1)^2 + 16(1) = 52$$

The gradient of the curve is 52

2016, No. 5

Determine the coordinates of the stationary point of the curve $y = \frac{1}{4}x^2 - 2x - 5$

Solution:

$$y = \frac{1}{4}x^2 - 2x - 5$$
$$\frac{dy}{dx} = \frac{1}{2}x - 2$$

For stationary points, $\frac{dy}{dx} = 0 \Rightarrow \frac{1}{2}x - 2 = 0$

$$x = 4$$

When $x = 4$, $y = \frac{1}{4}(4)^2 - 2(4) - 5 = -9$

Therefore the stationary point is $(4, -9)$

2019, No. 10

The equation of a curve is $y = 3 + 2x - x^2$

(a) Determine the;

(i) coordinates and nature of the turning point of the curve

(ii) y- and x- intercept of the curve

(b) (i) Sketch the curve

(ii) Find the area encosed by the curve and the $x -$ axis

Solution:

(a) (i) $y = 3 + 2x - x^2$

$$\frac{dy}{dx} = 2 - 2x$$

For turning point, $\frac{dy}{dx} = 0$

$$2 - 2x = 0$$
$$2 = 2x$$
$$x = 1$$

When $x = 1$, $y = 3 + 2(1) - (1)^2 = 4$

Point $(1, 4)$ is a turning point

$$\frac{d^2y}{dx^2} = -2 < 0$$

$(1, 4)$ is a maximum turning point (maxima)

(ii) $y -$intercept, when $x = 0$

$y = 3 + 2(0) - (0)^2 = 3$

$(0, 3)$ is the y-intercept

x-intercepts, when $y = 0$

$$0 = 3 + 2x - x^2$$
$$x^2 - 2x - 3 = 0$$
$$x^2 - 3x + x - 3 = 0$$
$$x(x-3) + (x-3) = 0$$
$$(x-3)(x+1) = 0$$
$$x = 3 \text{ or } x = -1$$

$(3, 0)$ and $(-1, 0)$ are the $x -$intercepts

(b) (i)

Chapter 10 Integration

Introduction

Integration is the process of obtaining an original function from a given gradient function; hence, it is the reverse of differentiation.

If $\frac{dy}{dx} = x^n$, then $y = \frac{x^{n+1}}{n+1}$ where $n \neq -1$

$\Rightarrow \int ax^n\,dx = \frac{x^{n+1}}{n+1} + C$ where C is an arbitrary constant

The general rule when integrating a power of x is that we add one onto the exponent/power and then divide by the new exponent/power. It is clear (hopefully) that we will need to avoid $n = -1$ in this formula because we will end up with division by zero, which is undefined.

Indefinite integrals

We call $\int f(x)\,dx$ an indefinite integral because it does not give a definite answer and we add an arbitrary constant after integrating.

We know that $y = x^3, y = x^3 + 5$, $y = x^3 - 6$ all satisfy $\frac{dy}{dx} = 3x^2$, for this reason, when we integrate $3x^2$, we write $y = x^3 + C$ because we are not certain whether the original function had a constant or not as when we differentiate a constant we get zero.

Note: We always integrate with respect to a certain variable i.e. $\int f(x)\,dx$ means integrating the function with respect to x and $\int f(t)\,dt$ means integrating the function with respect to t.

Example 1

Integrate the following with respect to x

(a) 5

Solution:

$$\int 5\,dx = \int 5x^0\,dx = \frac{5x^{0+1}}{0+1} + C$$
$$= 5x + C$$

Hence $\int k\,dx = kx + C$, where k and C are constants

(b) x^3

Solution:

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$$

(c) $x^{\frac{3}{2}}$

Solution:

$$\int x^{\frac{3}{2}}\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$
$$= \frac{2x^{\frac{5}{2}}}{5} + C$$

(d) $4\sqrt{x}$

Solution:

$$\int 4\sqrt{x}\,dx = \int 4x^{1/2}\,dx = 4\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$
$$= \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{8x^{\frac{3}{2}}}{3} + C$$

(e) $7x^5$

Solution:

$$\int 7x^5\,dx = \frac{7x^{5+1}}{6} + C = \frac{7x^6}{6} + C$$

(f) $\frac{1}{\sqrt{x}}$

Solution:

$$\int \frac{1}{\sqrt{x}}\,dx = \int \frac{1}{x^{\frac{1}{2}}}\,dx = \int x^{-\frac{1}{2}}\,dx$$
$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= 2x^{\frac{1}{2}} + C = 2\sqrt{x} + C$$

Integrating a sum or difference

When integrating a sum or difference, just like differentiating, we integrate the terms separately.

Example 2

Integrate $3x^3 - 4x^2 + 5x - 1$ with respect to x

Solution:

$= \frac{3}{2} - \left(-\frac{165}{2}\right) = \frac{168}{2} = 84$

(d) $\int_1^3 (x^2 - 4x + 1)\, dx$

Solution:

$$\int_1^3 (x^2 - 4x + 1)\, dx = \left[\frac{x^3}{3} - 2x^2 + x\right]_1^3$$

$$= \left(\frac{(3)^3}{3} - 2(3)^2 + 3\right) - \left(\frac{(1)^3}{3} - 2(1)^2 + 1\right)$$

$$= (9 - 18 + 3) - \left(\frac{1}{3} - 2 + 1\right)$$

$$= (-6) - \left(-\frac{2}{3}\right) = -\frac{16}{3}$$

(e) $\int_1^2 y^2 + y^{-2}\, dy$

Solution:

$$\int_1^2 y^2 + y^{-2}\, dy = \left[\frac{y^3}{3} + \frac{y^{-1}}{-1}\right]_1^2 = \left[\frac{y^3}{3} - \frac{1}{y}\right]_1^2$$

$$= \left(\frac{(2)^3}{3} - \frac{1}{2}\right) - \left(\frac{(1)^3}{3} - \frac{1}{1}\right)$$

$$= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 = \frac{17}{6}$$

(f) $\int_0^4 \sqrt{t}(t-2)\, dt$

Solution:

We need to first multiply the integral before integrating.

$$\sqrt{t}(t-2) = t^{\frac{1}{2}}(t-2) = t^{\frac{3}{2}} - 2t^{\frac{1}{2}}$$

Thus $\int_0^4 \sqrt{t}(t-2)\, dt$

$$= \int_0^4 t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\, dt = \left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= \left[\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right]_0^4$$

$$= \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(4)^{\frac{3}{2}}\right) - (0)$$

$$= \frac{64}{5} - \frac{32}{3} = \frac{32}{15}$$

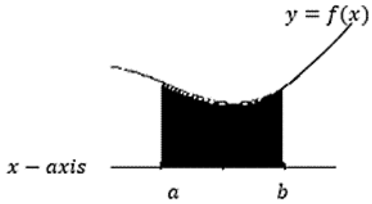
In the evaluation process, recall that;

$$(4)^{\frac{5}{2}} = \left((4)^{\frac{1}{2}}\right)^5 = 2^5 = 32$$

$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

Area under the curve

The area between the graph of $y = f(x)$ and the x-axis is given by the definite integral of the function of the curve.



$$\text{Area} = \int_a^b f(x)\, dx$$

This formula gives a positive result for the area above the x-axis and a negative result for the area below the x-axis

Note: If asked to find the area under the curve, it is a requirement to first sketch the curve i.e. by getting the intercepts and knowing the nature of the turning point.

The nature of the turning point can be known by mere looking at the equation of the curve i.e. the coefficients of the x^2 [This is very important]

If first asked to sketch the curve, then go smoothly through the processes of curve sketching.

Example 7

Find the area enclosed by the curve $y = 4x - x^2$

Solution:

The x-intercepts are when $y = 0$, i.e.

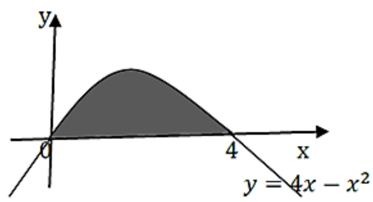
$$4x - x^2 = 0$$

$$x(4 - x) = 0$$

$$x = 0 \text{ and } x = 4$$

The curve has a maximum turning point [Recall concept]

We can now sketch the curve as follows;



$$A = \int_a^b y\, dx = \int_0^4 (4x - x^2)\, dx$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \left(2(4)^2 - \frac{(4)^3}{3}\right) - 0$$

$$= \left(32 - \frac{64}{3}\right) - 0 = \frac{32}{3} \text{ sq. units}$$

$$1 = 2 - \frac{2^2}{2} + C$$
$$C = 1$$

Therefore $v = t - \frac{t^2}{2} + 1$

Similarly, $s = \int v\,dt = \int \left(t - \frac{t^2}{2} + 1\right) dt$

$s = \frac{t^2}{2} - \frac{t^3}{6} + 2t + C$

But when $t = 1$, s = $\frac{13}{3}$

$$\frac{13}{3} = \frac{1}{2} - \frac{1}{6} + 2 + C$$
$$\frac{13}{3} = \frac{1}{3} + 2 + C$$
$$C = 2$$

Thus $s = \frac{t^2}{2} - \frac{t^3}{6} + 2t + 2$

Example 13

A body moves in a straight line. At time t seconds, its acceleration is given by $a = 6t + 1$ When $t = 0$, the velocity of the body is $2\ m/s$ and its displacement is $1\ m$. Find the expressions of v and s in terms of t.

Solution:

$$a = \frac{dv}{dt} = 6t + 1$$
$$v = \int a\,dt = \int 6t + 1\,dt$$
$$v = 3t^2 + t + C$$

But when $t = 0, v = 2$, thus $2 = 0 + C$

$$\Rightarrow C = 2$$

Substituting for C; $v = 3t^2 + t + 2$

Now using, $v = \frac{ds}{dt} = v = 3t^2 + t + 2$

$s = \int v\,dt = \int (3t^2 + t + 2)\,dt$

$s = t^3 + \frac{t^2}{2} + 2t + C$

But when $t = 0, s = 1$,

$$\Rightarrow 1 = 0 + C \text{ thus } C = 1$$

Substituting; $s = t^3 + \frac{t^2}{2} + 2t + 1$

Integral of $\frac{1}{x}$

$$\int \frac{1}{x} dx = \ln x + c$$

In the same way

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$$

Example 14

Integrate the following with respect to x

(a) $\int_1^2 \frac{1}{x} dx$ (b) $\int x^2 + \frac{1}{x} dx$

Solution:

(a) $\int_1^2 \frac{1}{x} dx = [\ln x]_1^2$

$$= \ln 2 - \ln 1$$
$$= \ln 2$$

(b) $\int x^2 + \frac{1}{x} dx = \int x^2\,dx + \int \frac{1}{x} dx$

$$= \frac{x^3}{3} + \ln x + c$$

Examination questions

2017, No. 10

A particle moves with velocity $V = 2t^2 - 9t + 10$ where t is time. The particle is at the origin when $t = 0$. Determine the

(a) expressions for the distance and the acceleration in terms of t

(b) distances of the particle from the origin when the particle is at rest.

Solution:

Velocity, V =rate of change of displacement

i.e. $V = \frac{dS}{dt}$ where S = displacement

Acceleration, a = rate of change of velocity

i.e. $a = \frac{dV}{dt}$

(a) From $a = \frac{dV}{dt}$

$$a = \frac{d}{dt}(2t^2 - 9t + 10)$$
$$a = 4t - 9$$

From $V = \frac{dS}{dt}$

$$dS = Vdt$$
$$S = \int V\ dt$$
$$S = \int 2t^2 - 9t + 10\ dt$$
$$S = \frac{2t^3}{3} - \frac{9t^2}{2} + 10t + C$$

At $t = 0, S = 0$

$$= (12(2) - (2)^3) - (12(-2) - (-2)^3)$$
$$= 16 + 16$$
$$= 32 \text{ sq. units}$$

Method II:

Area of the rectangle formed by the line and the x −axis between -2 and 2

$$= 14 \times 4 = 56$$

Area enclosed by the curve and the x −axis is given by

$$A = \int_{-2}^{2} 3x^2 + 2\,dx$$
$$= [x^3 + 2x]_{-2}^{2}$$
$$= \left(2^3 + 2(2)\right) - \left((-2)^3 + 2(-2)\right)$$
$$= 12 + 12$$
$$= 24$$

Area enclosed by the line and the curve is given by

$$56 - 24 = 32 \text{ sq. units}$$

2018, No. 14

A body of mass 4 kg is initially at rest at a point P whose position vector is $(3i + 4j)m$. A constant force $F = (8i + 4j)\ N$ acts on the body causing it to move. The body passes through another point Q after 4seconds.

Find the;

(a) acceleration of the body
(b) velocity of the body as it passes through Q
(c) kinetic energy of the body after 4 seconds
(d) distance between P and Q

Solution:

(a) $F = ma$

$$a = \frac{F}{m} = \frac{1}{4}\begin{pmatrix}8\\4\end{pmatrix}$$
$$a = \begin{pmatrix}2\\4\end{pmatrix}\ ms^{-2}$$

(b) $a = \frac{dv}{dt}$

$$v = \int a\,dt$$
$$v = \int \begin{pmatrix}2\\4\end{pmatrix} dt$$
$$v = \begin{pmatrix}2t\\4t\end{pmatrix} + c$$

At $t = 0, v = 0$

$$c = \begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore v = \begin{pmatrix}2t\\4t\end{pmatrix}\ ms^{-1}$$

As it passes Q, $t = 4$

$$v = \begin{pmatrix}8\\16\end{pmatrix}\ ms^{-1}$$

(c) $K.E = \frac{1}{2}mv^2$

$$v = \sqrt{8^2 + 16^2} = \sqrt{320}$$
$$K.E = \frac{1}{2} \times 4 \times 320 = 640\,J$$

(d) $v = \frac{ds}{dt}$

$$s = \int v\ dt$$
$$s = \int \begin{pmatrix}2t\\4t\end{pmatrix} dt$$
$$s = \begin{pmatrix}t^2\\2t^2\end{pmatrix} + c$$

At $t = 0$, $s = \begin{pmatrix}3\\4\end{pmatrix}$

$$c = \begin{pmatrix}3\\4\end{pmatrix}$$
$$s = \begin{pmatrix}t^2\\2t^2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}t^2 + 3\\2t^2 + 4\end{pmatrix}$$
$$s = \begin{pmatrix}t^2 + 3\\2t^2 + 4\end{pmatrix}$$

At $t = 4$,

$$s = \begin{pmatrix}4^2 + 3\\2(4)^2 + 4\end{pmatrix} = \begin{pmatrix}19\\36\end{pmatrix}$$
$$\overline{PQ} = |s| = \sqrt{19^2 + 36^2} = 40.71\,m$$

2019, No. 5

Evaluate $\int_1^2 \frac{x^4 - 1}{x^2}\,dx$

Solution:

$$\int_1^2 \frac{x^4 - 1}{x^2}\,dx = \int_1^2 \frac{x^4}{x^2} - \frac{1}{x^2}\,dx$$
$$= \int_1^2 x^2 - x^{-2}\,dx$$
$$= \left[\frac{x^3}{3} - \left(\frac{x^{-1}}{-1}\right)\right]_1^2$$
$$= \left[\frac{x^3}{3} + \frac{1}{x}\right]_1^2$$
$$= \left(\frac{2^3}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{1}\right)$$
$$= \frac{19}{6} - \frac{4}{3}$$
$$= \frac{11}{6}$$

Chapter 11 Differential Equations

A differential equation is an equation that contains a differential coefficient e.g.

$$\frac{dy}{dx} = 3x$$

$$\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} = 6$$

Order of a differential equation

The order of a differential equation is the highest derivative, which appears in it for instance;

The equation $\frac{dy}{dx} - 4x = 3$ is a first order differential equation because it contains only a first differential coefficient i.e. $\frac{dy}{dx}$

The equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 9$ is a second order differential equation because it contains a second differential coefficient, $\frac{d^2y}{dx^2}$

Note: Any differential equation represents a relationship between two variables say x and y and the same relationship can often be expressed in a form that does not contain a differential coefficient e.g. $y = x^2 + c$ and $\frac{dy}{dx} = 2x$

Solution to a differential equation

The solution of a differential equation is an equation relating the variables involved but containing no differential coefficient like $\frac{dy}{dx}$. There are two types of solutions i.e.

(i) The general solution

This contains an arbitrary constant

(ii) Particular solution

It may be obtained if the " x-value" and the corresponding "y-value" are given. These are called initial conditions are used to calculate the value of the constant.

Consider $\frac{dy}{dx} = 3x^2$; this is a first order differential equation

By separating the variables

$$dy = 3x^2dx$$

Integrating on both sides

If $x = 1$, when $y = 2$ (these are initial conditions)

$$2 = 1 + c$$

$$\Rightarrow c = 1$$

$\therefore y = x^3 + 1$ (This is a particular solution)

Separable differential equations

A separable differential equation is any differential equation that we can write in the following form $N(y)\frac{dy}{dx} = M(x)$

Note that in order for a differential equation to be separable, all the $y's$ in the differential equation must be multiplied by the derivative and the $x's$ in the differential equation must be on the other side of the equal sign.

Solving separable differential equations is fairly easy. We first rewrite the differential equation as;

$$N(y)dy = M(x)dx$$

Then you integrate on either sides

$$\int N(y)\,dy = \int M(x)\,dx$$

Example 1

Find the general solutions to the following differential equations

(a) $3y\frac{dy}{dx} = 5x^2$

Solution:

By separating variables i.e. by separating dy from dx and collecting on one side all terms involving y together with dy, while all the x terms with dx, it gives;

$$3ydy = 5x^2dx$$

We now integrate both sides of the equation

$$\int 3ydy = \int 5x^2dx$$

$$\frac{3y^2}{2} = \frac{5x^3}{3} + c$$

(b) $u\frac{du}{dv} = v + 2$

Solution:

$$u\,du = (v + 2)dv$$

Integrating on both sides gives;

$$\int u\,du = \int (v + 2)dv$$

$$\frac{u^2}{2} = \frac{v^2}{2} + 2v + c$$

(c) $\frac{dy}{dx} = y^2$

Solution:

$$\frac{1}{y^2}dy = dx$$
$$\int y^{-2}\, dy = \int dx$$
$$-y^{-1} = x + c$$
$$-\frac{1}{y} = x + c$$

(d) $\frac{1}{x}\frac{dy}{dx} = \frac{1}{y^2-2}$

Solution:

By separating the variables;

$$(y^2 - 2)\, dy = x\, dx$$

Integrating on both sides;

$$\int (y^2 - 2)\, dy = \int x\, dx$$
$$\frac{y^3}{3} - 2y = \frac{x^2}{2} + c$$

(e) $\frac{dy}{dx} = x^2 - 2x$

Solution:

$$dy = (x^2 - 2x)dx$$
$$\int dy = \int (x^2 - 2x)dx$$
$$y = \frac{x^3}{3} - x^2 + c$$

Example 2

Find the particular solutions of the following differential equations

(a) $y^2\frac{dy}{dx} = x^2 + 1$ if $y = 1$ when $x = 2$

Solution:

$$y^2 dy = (x^2 + 1)dx$$
$$\int y^2 dy = \int (x^2 + 1\ )dx$$
$$\frac{y^3}{3} = \frac{x^3}{3} + x + c$$

This is a general solution but since the initial condition is given, we can find the value of c

$$\frac{(1)^3}{3} = \frac{(2)^3}{3} + 2 + c$$
$$c = -\frac{13}{3}$$

Therefore $\frac{y^3}{3} = \frac{x^3}{3} + x - \frac{13}{3}$ is the particular solution

(b) $\frac{dy}{dx} = 6y^2x$ if $y = \frac{1}{25}$ when $x = 1$

Solution:

$$y^{-2}dy = 6x\, dx$$
$$\int y^{-2}dy = \int 6x\, dx$$
$$\frac{y^{-1}}{-1} = \frac{6x^2}{2} + c$$
$$-\frac{1}{y} = 3x^2 + c$$

So, we now have the general solution. Let us apply the initial condition and find the value of c

$\frac{1}{\frac{1}{25}} = 3(1)^2 + c$ which gives $c = -28$

Substitute this value in the general solution to get the particular solution

$$-\frac{1}{y} = 3x^2 - 28$$

(c) $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$ if $y = 3$ when $x = 1$

Solution:

This differential equation is clearly separable, so let us put it in the proper form and then integrate both sides

$$(2y - 4)dy = (3x^2 + 4x - 4)dx$$
$$\int (2y - 4)dy = \int (3x^2 + 4x - 4)dx$$
$$y^2 - 4y = x^3 + 2x^2 - 4x + c$$

when $x = 1, y = 3$

$$(3)^2 - 4(3) = (1)^3 + 2(1)^2 - 4(1) + c$$
$$c = -2$$

Therefore, $y^2 - 4y = x^3 + 2x^2 - 4x - 2$

Example 3

Solve the differential equation

$$\frac{dy}{dx} = 2xy\,;\ \ y(2) = 1$$

Solution:

$$\frac{dy}{y} = 2x\, dx$$
$$\int \frac{1}{y}dy = \int 2x\, dx$$
$$\ln y = x^2 + c$$

when $x = 2, y = 1$

$$\ln 1 = 2^2 + c$$
$$0 = 4 + c$$
$$c = -4$$
$$\therefore \ln y = x^2 - 4$$

$$t = 40\,.\frac{\ln 3}{\ln 2} = 63.4$$

Thus, the population will treble itself in 63.4 years

Example 9

The rate of increase of a bacteria in a culture is proportional to the number of bacteria present and it is found that the number doubles in 5 hours. Express this mathematically using rate of increase of bacteria with respect to time. Hence, calculate how many times the bacteria may be expected to grow at the end of 15 hours.

Solution:

Let x be the number of bacteria at the end of t hours

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx$$

where k is a constant

$$\frac{dx}{x} = k\,dt$$

Integrating on both sides

$$\int \frac{1}{x}dx = \int k\,dt$$

$$\ln x = kt + c \quad \ldots.(1)$$

Let the original number of bacteria be m, i.e. $x = m$ when $t = 0$

$$\ln m = k\,(0) + c$$

$$c = \ln m$$

Thus, from (1) we get

$$\ln x = kt + \ln m \quad \ldots.(2)$$

Again, $x = 2m$ when $t = 5$

$$\ln 2m = k(5) + \ln m$$

$$k = \frac{\ln 2}{5}$$

Thus, from (2) we get

$$\ln x = \frac{\ln 2}{5}t + \ln m \quad \ldots(3)$$

Now when $t = 15$, from (3) we get

$$\ln x = \frac{\ln 2}{5} \times 15 + \ln m$$

$$\ln x = 3\ln 2 + \ln m$$

$$\ln x = \ln 2^3 + \ln m$$

$$\ln x = \ln 8m$$

$$x = 8m$$

Thus, the number of bacterial will be 8 times in 15 years

Example 10

The engine of a motor boat moving at $10\ \text{ms}^{-1}$ is shut off given that the retardation at any subsequent time (after shutting off the engine) equals the velocity of that time. Find the velocity after 2 seconds of switching off the engine (leave yours answers in terms of e)

Solution:

Let v m/sec be the velocity and x m the distance travelled at time t secs, after shutting off the engine

$$\frac{dv}{dt} = -v \ \left[\frac{dv}{dt} \text{ being retardation}\right]$$

$$\frac{dv}{v} = -dt$$

Integrating

$$\ln v = -t + c \quad ..(1)$$

when $t = 0, v = 10$

$$\ln 10 = c$$

Thus from (1), we get

$$\ln v = -t + \ln 10$$

$$t = \ln 10 - \ln v$$

$$t = \ln\frac{10}{v}$$

$$t = \log_e\frac{10}{v}$$

$$e^t = \frac{10}{v}$$

$$e^{-t} = \frac{v}{10}$$

$$v = 10e^{-t} \quad \ldots.(2)$$

when $t = 2$; $v = 10e^{-2} = \frac{10}{e^2}$ m/sec

Example 11

The temperature T of a cooling object drops at a rate proportional to the difference $T - S$, where S is the constant temperatures of the surrounding medium. Thus

$$\frac{dT}{dt} = -k(T - S)$$

where k is a constant and t is the time. Solve the differential equation if it is given $T(0) = 150$.

Solution:

We have $dT = -k(T - S)dt$

$$\frac{1}{T - S}dT = -kdt$$

Integrating both sides

$$\int \frac{1}{T - S}dT = \int -kdt$$

$$\ln(T - S) = -kt + c$$

where c is the constant of integration

To evaluate c, put $T = 150$ and $t = 0$

$$\ln(150 - S) = -c$$

Hence

$$\ln(T - S) = -kt + \ln(150 - S)$$

$$\ln(T - S) - \ln(150 - S) = -kt$$

$$\ln\left(\frac{T - S}{150 - S}\right) = -kt$$

$$\frac{T - S}{150 - S} = e^{-kt}$$

Example 12

The population of a villages increases at a rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what was the population in 2009?

Solution:

Let the population at time t be x

Given, rate of population increase ∝ number of inhabitants

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx$$

where k is the constant of proportionality

$$\frac{dx}{x} = k\,dt$$

$$\int \frac{dx}{x} = \int k\,dt$$

$$\ln x = kt + c \quad \ldots (1)$$

In 1999, $t = 0$ and population, $x = 20{,}000$

$$\ln 2000 = c$$

putting the value of c in (1), we have

$$\ln x = kt + \ln 20000$$

$$\ln x - \ln 20000 = kt$$

$$\ln \frac{x}{20000} = kt \quad \ldots (2)$$

In 2004, $t = 5$, $x = 25{,}000$

$$\ln \frac{25000}{20000} = k(5)$$

$$k = \frac{1}{5}\ln 1.2$$

Equation (2) may be written as

$$\ln \frac{x}{20000} = \left(\frac{1}{5}\ln 1.2\right)t$$

In 2009, $t = 10$

$$\ln \frac{x}{20000} = \left(\frac{1}{5}\ln 1.2\right) \times 10$$

$$\ln \frac{x}{20000} = 2\ln 1.2$$

$$\ln \frac{x}{20000} = \ln 1.2^2$$

$$\frac{x}{20000} = 1.44$$

$$x = 31250$$

Example 13

Hot tea in a cup has a temperature T ℃ at a time t minutes and it is left to cool in a room of constant temperature T_0. Newton's Law of cooling asserts that the rate at which a body cools is directly proportional to the excess temperature of the body and the temperature of its immediate surroundings.

(a) Assuming the tea cooling in the cup obeys this law, form a differential equation in terms of T, T_0, t and a proportionality k

(b) Show clearly that $T = T_0 + Ae^{-kt}$ where k is a constant

Initially the temperature of the tea is 80 ℃ and 10 minutes later is 60 ℃. The room temperature remains constant at 20 ℃

(c) Find the value of t when the tea reaches a temperature of 40 ℃

Solution:

(a)

$$\frac{dT}{dt} = -k(T - T_0)$$

(b)

$$\frac{1}{T - T_0}dT = -k\,dt$$

$$\int \frac{1}{T - T_0}dT = \int -k\,dt$$

$$\ln(T - T_0) = -kt + c$$

$$T - T_0 = e^{-kt+c}$$

$$T - T_0 = e^c.e^{-kt}$$

$$T - T_0 = Ae^{-kt}\ (A = e^c)$$

$$T = T_0 + Ae^{-kt}$$

(c) when $t = 0, T = 80,\ T_0 = 20$

$$80 = 20 + Ae^0$$

$$80 = 20 + A$$

$$A = 60$$

$$\therefore T = 20 + 60e^{-kt}$$

when $t = 10, T = 60$

$$60 = 20 + 60e^{-10k}$$

$$40 = 60e^{-10k}$$
$$\frac{2}{3} = e^{-10k}$$
$$\ln\frac{3}{2} = 10k$$
$$k = \frac{1}{10}\ln\frac{3}{2} = 0.04055$$
$$T = 20 + 60e^{-0.04055t}$$

Thus when $T = 40$,

$$40 = 20 + 60e^{-0.04055t}$$
$$20 = 60e^{-0.04055t}$$
$$\frac{1}{3} = 60e^{-0.04055t}$$
$$3 = e^{0.04055t}$$
$$\ln 3 = 0.04055t$$
$$t = 27.1 \text{ minutes}$$

Examination questions

2013, No. 5

Solve the differential equation $8y\frac{dy}{dx} = 9x^2$. Hence find the equation given that $y = 2$ and $x = 1$.

Solution:

$$8y\frac{dy}{dx} = 9x^2$$

By separating the variables

$$8y\,dy = 9x^2\,dx$$

$\int 8y\,dy = \int 9x^2\,dx$ …. Integrating on both sides

$$\frac{8y^2}{2} = \frac{9x^3}{3} + c$$
$$4y^2 = 3x^3 + c$$

This is a general solution and since the initial conditions are given, we can find the value of the constant c

when $x = 1,\ y = 2$

$$4(2)^2 = 3(1)^3 + c$$
$$16 = 3 + c$$
$$\Rightarrow \quad c = 13$$

$\therefore 4y^2 = 3x^3 + 13$ is the particular solution

2014, No. 4

Solve the differential equation $\frac{dy}{dx} = 2x + 5$, given that $y = -1$ and $x = 3$

Solution:

$$\frac{dy}{dx} = 2x + 5$$
$$dy = (2x + 5)dx$$
$$\int dy = \int(2x + 5)dx$$
$$y = \frac{2x^2}{2} + 5x + c$$
$$y = x^2 + 5x + c$$

when $x = 3, y = -1$

$$(-1) = (3)^2 + 5(3) + c$$
$$c = -25$$
$$\therefore\ y = x^2 + 5x - 25$$

2015, No. 10

The rate of decay of a radioactive material is proportional to the amount x grams of the material present at any time t. Initially there was 60 grams of the material. After 8 years the material had reduced to 15 grams.

(a) Form a differential equation for the rate of decay of the material

(b) Solve the differential equation formed in (a) above

(c) Find the time taken for the material to reduce to 10 grams.

Solution:

(a) $\frac{dx}{dt} \propto x$

$$\frac{dx}{dt} = -kx$$

(b) $\frac{dx}{x} = -kdt$

$$\int\frac{dx}{x} = \int -k\,dt$$
$$\ln x = -kt + C$$

at $t = 0, x = x_0 \Rightarrow \ln x_0 = C$

$$\ln x = -kt + \ln x_0$$
$$\ln x - \ln x_0 = -kt$$
$$\ln\left(\frac{x}{x_0}\right) = -kt$$

$x_0 = 60,\ at\ t = 8, x = 15$

$$\ln\left(\frac{15}{60}\right) = -k(8) \Rightarrow k = \frac{1}{8}\ln 4$$
$$\ln\left(\frac{x}{x_0}\right) = -\left(\frac{1}{8}\ln 4\right)t$$

(c) at $t = ?,\quad x = 10$

$$\ln\left(\frac{10}{60}\right) = -\left(\frac{1}{8}\ln 4\right)t$$
$$t = 10.34 \text{ years}$$

2016, No. 10

Chemical A is converted into another chemical by a chemical reaction. The rate at which chemical A is being converted is directly proportional to the amount present at any time. Initially 100g of chemical A was present. After 5 minutes, 90g of A is present.

(a) Form a differential equation for the chemical reaction

(b) By solving the differential equation in (a), determine the
   (i) amount of chemical A present after 20 minutes
   (ii) time taken for the amount of chemical A to be reduced to 20g

Solution:

Let the amount of chemical present at any time be x

(a) $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = -kx$

(b) $\frac{dx}{x} = -kdt$

$\int \frac{dx}{x} = \int -k\,dt$

$\ln x = -kt + C$

at $t = 0, x = 100$

$\ln 100 = C$

$\ln x = -kt + \ln 100$

$\ln x - \ln 100 = -kt$

$\ln\left(\frac{x}{100}\right) = -kt$

at $t = 5, x = 90$

$\ln\left(\frac{90}{100}\right) = -k(5)$

$k = -\frac{1}{5}\ln 0.9$

$\ln\left(\frac{x}{100}\right) = \frac{t}{5}\ln 0.9$

(i) at $t = 20$

$\ln\left(\frac{x}{100}\right) = \frac{20}{5}\ln 0.9$

$\frac{x}{100} = e^{4\ln 0.9}$

$x = 100e^{4\ln 0.9} = 65.61$

After 20 minutes, 65.61g of A is present

(ii) For $x = 20$g

$\ln\left(\frac{20}{100}\right) = \frac{t}{5}\ln 0.9$

$\ln 0.2 = \frac{t}{5}\ln 0.9$

$t = \frac{5\ln 0.2}{\ln 0.9} = 76.62$ min

2017, No. 5

Solve the differential equation $3y\frac{dy}{dx} = \frac{1}{x^2}$ given that $y = 2$ when $x = 1$

Solution:

By separating the variables;

$$3y\,dy = \frac{1}{x^2}dx$$

$$\int 3y\ dy = \int x^{-2}\ dx$$

$$\frac{3y^2}{2} = \frac{x^{-1}}{-1} + c$$

$$\frac{3y^2}{2} = -\frac{1}{x} + c$$

When $x = 1, y = 2$.

Thus; $\frac{3(2)^2}{2} = -\frac{1}{1} + c$

$$c = 7$$

$$\therefore \frac{3y^2}{2} = -\frac{1}{x} + 7$$

2022, No. 10

The rate of cooling of a body is proportional to the difference in temperature T of the body at any time t and that of the surroundings. If the temperature of the surroundings is 25℃ and the body cools from 85℃ to 70℃ in 15 minutes;

(a) (i) form a differential equation for the cooling of the body
   (ii) solve the differential equation formed in (i)

(b) determine the temperature of the body after 30 minutes

Solution:

(a) (i) Difference in temperature $= T - 25$

$$\frac{dT}{dt} \propto T - 25$$

$$\frac{dT}{dt} = -k(T - 25)$$

(ii) Seperating the variables

$$\frac{dT}{T - 25} = -k\ dt$$

$$\int \frac{1}{T - 25}dT = -\int k\,dt$$

$$\ln(T - 25) = -kt + c$$

at $t = 0, T = 85$

$$\ln(85 - 25) = -k(0) + c$$

$$c = \ln 60$$

Thus, $\ln(T - 25) = -kt + \ln 60$

$$\ln(T - 25) - \ln 60 = -kt$$

$$\ln\left(\frac{T - 25}{60}\right) = -kt$$

at $t = 15, T = 70$

$$\ln\left(\frac{70 - 25}{60}\right) = -k(15)$$

$$k = -\frac{1}{15}\ln\frac{3}{4}$$

Substituting for k

$$\ln\left(\frac{T - 25}{60}\right) = -\left(-\frac{1}{15}\ln\frac{3}{4}\right)t$$

$$\ln\left(\frac{T - 25}{60}\right) = \left[\frac{1}{15}\ln\frac{3}{4}\right]t$$

(b) at $t = 30, T = ?$

$$\ln\left(\frac{T-25}{60}\right) = \left[\frac{1}{15}\ln\frac{3}{4}\right] \times 30$$

$$\ln\left(\frac{T-25}{60}\right) = 2\ln\frac{3}{4}$$

$$\ln\left(\frac{T-25}{60}\right) = \ln\left(\frac{3}{4}\right)^2$$

$$\frac{T-25}{60} = \frac{9}{16}$$

$$T = 25 + 60 \times \frac{9}{16} = 58.75\ ℃$$

Self-Evaluation exercise

1. Solve the differential equation $\frac{dy}{dx} = 3x^2y^2$ given that $y = 1$ when $x = 0$

   [Ans: $x^3y = y - 1$]

2. Find the general solution to the differential equation $6t\frac{dt}{ds} + 1 = 0$, and the particular solution given by the conditions $s = 0$ when $t = -2$.

   [Ans: $s = 12 - 3t^2$]

3. Find the general solutions of the following differential equations

   (a) $\frac{dy}{dx} = 3x$ (b) $2y\frac{dy}{dx} = 3$ (c) $\frac{dy}{dx} = \frac{x-4}{4y^3}$ (d) $\frac{dy}{dx} = -\frac{x}{y}$ (e) $\frac{dy}{dx} = y^{\frac{4}{5}}$

   [Ans: (a) $y = \frac{3x^2}{2} + c$ (b) $y^2 = 3x + c$ $(c)\ y^4 = \frac{x^2}{2} - 4x + c$ (d) $\frac{y^2}{2} = \frac{-x^2}{2} + c$ (e) $5y^{\frac{1}{5}} = x + c$]

4. Find the particular solution of the differential equation $\frac{dx}{dt} = t$, where $x = 3$ when $t = 1$.

   [Ans: $x = \frac{t^2}{2} + \frac{5}{2}$]

5. The rate of change of y with respect to x is proportional to the square of x. Write a differential equation that models this statement.

   [Ans: $\frac{dy}{dx} = kx^2$]

6. Given that $\frac{dy}{dx} = x^2 + kx$ where k is a constant. If y has a turning point at the point $(3, -2)$, calculate the value of (i) k (ii) y when $x = 4$

   [Ans: (i) $k = -3$ (ii) $y = -\frac{1}{6}$]

7. An entomologist believes that the population P of insects in a colony, t weeks after it was first observed, obeys the differential equation

   $$\frac{dP}{dt} = kP^2$$

   where k is a positive constant

   Initially 1000 insects were observed, and this population doubled after 4 weeks. Find a solution of the differential equation

   [Ans: $P = \frac{8000}{8-t}$]

8. At time t hours, the rate of decay of the mass, x kg, of a radioactive substance is directly proportional to the mass present at that time. Initially the mass is x_0

   (a) By forming and solving a suitable differential equation, show that

   $$x = x_0e^{-kt}$$

   where k is a positive constant

   When $t = 5$, $x = \frac{1}{4}x_0$

   (b) Find the value of t when $x = \frac{1}{2}x_0$

   [Ans: $t = \frac{5}{2}$]

9. The mass m grams, of a burning candle, t hours after it was lit up, satisfies the differential equation

   $$\frac{dm}{dt} = -k(m - 10)$$

   where k is a positive constant

   (a) Solve the differential equation to show that

   $$m = 10 + Ae^{-kt}$$

   where A is a non-zero constant

   The initial mass of the candle was 120 grams and 3 hours later its mass was halved

   (b) Find the value of A and show that

   $$k = \frac{1}{3}\ln\frac{11}{5}$$

   (c) Calculate the mass of the candle after a further period of 3 hours has elapsed.

   [Ans: (b) $A = 110$ (c) $m = 32.7$]

10. A radioactive isotope decays in such a way so that the number N of the radioactive nuclei present at time t days, satisfies the differential equation

    $$\frac{dN}{dt} = -kN$$

    where k is a positive constant

    (a) Show clearly that $N = Ae^{-kt}$ where A is a non zero constant

    Initially there were 6.00×10^{24} radioactive nuclei and 10 days later this number reduced to 6.25×10^{22}

    (b) Show further that $k = 0.45643$

    (c) Calculate the number of the radioactive nuclei after a further period of 10 days has elapsed.

    [Ans: 6.51×10^{20}]

Chapter 12 Statistics

Introduction

Descriptive statistics is the term given to the analysis of data that helps describe, show or summarize data in a meaningful way such that, for example, patterns might emerge from the data. Descriptive statistics do not, however allow us to make conclusions beyond the data we have analysed or reach conclusions regarding any hypotheses we might have made. They are simply a way to describe our data.

Descriptive statistics are very important because if we simply presented our raw data, it would be hard visualize what the data was showing, especially if there was a lot of it. Descriptive statistics therefore enables us to present the data in a more meaningful way, which allows simpler interpretation of the data.

When we use descriptive statistics, it is useful to summarize our group of data using a combination of tabulated description (i.e. tables), graphical description (i.e. graphs and charts).

Data classification

In order to present and analyse data in a logical and meaningful way, it is necessary to understand some of the natural forms that they can take. There are various ways of classifying data and are as follows

a) By source: Data can be described as either primary or secondary, depending on their source.
b) By measurement; Data can be measurement in either numeric (or quantitative) or non-numeric (qualitative) terms.
c) By preciseness: Data can either be measured precisely (described as discrete) or only ever be approximated to (described as continuous)
d) By number of variables: Data can consist of measurements of one or more variable for each subject or item. Univariate is the name given to a set of data consisting of measurements of just one variable, bivariate is used for two variables, and for two or more variables the data is described as multivariate.

Discrete data

Discrete data can be described as data that can be measured precisely. One way of obtaining discrete data is by counting. For example:

i. the number of components produced from an assembly line over a number of consecutive shifts:

$$45, 51, 44, 44, 43, 50, 46, 43, \dots \text{ etc.}$$

ii. the number of employees working in various offices of a company

$$12, 32, 8, 13, 8, 6, 11, 24, \dots \text{ etc.}$$

Discrete data can also be obtained from situations where counting is not involved.

For example:

iii. shoe sizes of a sample of people:

$$8, 10, 10, 6\tfrac{1}{2}, 9, 9, 9\tfrac{1}{2}, 8\tfrac{1}{2}, \dots \text{ etc.}$$

iv. weekly wages in thousands of shillings for a set of workers

$$121.45, 162.85, 133.37, 108.32, \dots \text{ etc.}$$

A particular characteristic of discrete data is that possible data values progress in definite steps, i.e. shoe sizes are measured as 6 or $6\frac{1}{2}$ or 7 or $7\frac{1}{2}$ … etc. or there are 1 or 2 or 3 … etc people (and not 3.5 or 4.67)

Continuous data

The most significant characteristic of continuous data is the fact that they cannot be measured precisely; their values can only be approximated to. Examples of continuous data are dimensions (lengths, heights); weights; areas and volumes; temperatures; times.

Although continuous values cannot be identified exactly, they are often recorded as if they were precise and this is normally acceptable. For example:

i. diameters (in mm) of a sample of screws from a production run:

$4.11, 4.10, 4.10, 4.10, 4.15, 4.09, 4.12, \ldots$ etc.

ii.weights (in gm) of the contents of a selection of cans of baked beans:

$446.8, 447.0, 446.8, 447.2, 447.1,$ .. etc.

Frequency distributions

This is concerned with the organisation and presentation of 'numeric (univariate)' data. It describes how numeric data can be organized into frequency distributions of various types.

Raw statistical data

Before the data obtained from a statistical survey or investigation have been worked on, they are known as raw data. The table below gives an example of a set of raw data.

Hours worked in one week by employees in a company's production department

46.3	45.1	45.6	46.1	45.0	43.5	39.2	39.2
39.2	42.3	39.6	38.9	44.4	43.4	43.8	43.2
44.2	43.5	42.0	42.4	42.4	42.8	42.9	42.9
41.3	40.0	39.6	42.1	39.8	44.3	46.2	46.2

Simple frequency distributions

Some sets raw data contain a limited number of data values, even though there may be many occurrences of each value. In this type of situation, the standard form into which the data is organized is known as a simple frequency distribution.

A simple frequency distribution consists of a list of data values, each showing the number of items having that value (called the frequency). This type of structure is normally applicable to discrete raw data (i.e. where values have usually been obtained by counting), since data values are quite likely to be repeated many times. A simple frequency distribution is not normally suitable for continuous data, since the likelihood of repeated values is small.

Example 1

The simple frequency distribution resulted from examining the time cards of the employees of a firm for one complete month.

Number of days late for work	Number of employees
0	45
1	52
2	18
3	11
4	5
5	2
6	4
7	0
8	1

In this case, 'number of days late for work' is describing the data and 'number of employees' is the frequency of occurrence of each value. This distribution gives certain useful information about the nature of absences for the employees. For example:

i. the majority of employees were late for work at most on only one day during the month;

ii. very few employees were late more than four days in the month;

Miles travelled	Number of salesmen
100 and up to 200	3
200 and up to 300	5
300 and up to 400	2
400 and up to 500	8
500 and up to 600	2

Cumulative frequency distribution:

Miles travelled	Number of salesmen
Less than 200	3
Less than 300	8
Less than 400	10
Less than 500	18
Less than 600	20

Note carefully, in the case of transformed distribution, how:

i. the description of classes need to be changed

ii.the frequencies are accumulated

b) More than distributions

Here, a set of item values is listed (normally class 'lower boundaries'), with each one showing the number of items in the distribution having values greater than this. The table below shows the distribution from a) transformed into a cumulative (more than) distribution.

Miles travelled	Number of salesmen
More than 100	20
More than 200	17
More than 300	12
More than 400	10
More than 500	2

Note again how:

i. the distributions of classes need to be changed
ii. the frequencies are formed by accumulating 'in reverse'

Formation of grouped frequency distributions

Given a set of raw statistical data, there is no single grouped frequency distribution which is uniquely correct in representing them; many different structures of classes could be set up to describe the data. However governing principles such as the class width and starting class can be important to specify the frequency distribution we are supposed to arrive at.

Example 1

Thirty *AA* batteries were tested to determine how long they would last. The results, to the nearest minute were as follows:

423	369	387	411	393	394
371	377	389	409	392	408
431	401	363	391	405	382
400	381	399	415	428	422
396	372	410	419	386	390

Construct a grouped frequency distribution with class width 10, starting with the class $360 - 369$

Solution:

Battery life (minutes)	Tally	Frequency (f)
$360 - 369$	//	2
$370 - 379$	///	3
$380 - 389$	~~////~~	5
$390 - 399$	~~////~~ //	7
$400 - 409$	~~////~~	5
$410 - 419$	////	4
$420 - 429$	///	3
$430 - 439$	/	1
Total		**30**

Statistical measures

Statistical measures describe the basic analysis of univariate data (data obtained from measuring just one attribute). The measures themselves are split into various groups. We shall discuss two groups of these measures.

a) Measures of central tendency
b) Measures of variation (or spread)

a) Measures of central tendency

There are three common measures of central tendency which include: 1 Arithmetic mean or mean, Median and mode.

Arithmetic mean

We have the following techniques to evaluate the arithmetic mean of a given data.

Case 1: Simple distribution or Individual Observation

The arithmetic mean or the mean of a set of numbers x_1, x_2, x_3, …., x_n is denoted by $\bar{x}$ and is defined as

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

The above formula can be stated in simple form as $\bar{x} = \frac{\sum x_i}{n}$

Example 2

Calculate the arithmetic mean of the following scores

17, 39, 21, 50, 27, 19, 24, 23, 10, 35

Solution:

$$\text{Mean, } \bar{x} = \frac{\sum x_i}{n}$$

$$\bar{x} = \frac{17 + 39 + 21 + 50 + 27 + 19 + 24 + 23 + 10 + 35}{10} = \frac{265}{10} = 26.5$$

Example 3

The arithmetic mean of the marks obtained by 10 students of class X in mathematics in a certain examination is 30. The marks obtained are 25, 30, 21, 55, 47, 10, 15, x, 45, 35. Find the value of x

Case 3. Arithmetic mean of a Continuous or Grouped Frequency Distribution

In this case also, the mean is computed by applying any formulas given for discrete frequency distribution. The values $x_1, x_2, \ldots, x_n$ are taken as the mid-points or class marks of the various class intervals. If the given frequency distribution is inclusive, then it should be first converted into an exclusive distribution.

Example 5

Following is the distribution of earnings of 200 workers in a flour mill.

Monthly wages (in thousand shillings)	80 – 100	100 – 120	120 – 140	140 – 160	160 – 180
No. of workers	20	30	20	40	90

Find the average earning of the workers

Solution:

Method 1:

Monthly wages (in thousand shillings)	Number of workers f	Class marks x	fx
80 – 120	20	90	1800
100 – 120	30	110	3300
120 – 140	20	130	2600
140 – 160	40	150	6000
160 – 180	90	170	15300
Total	200		29000

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{29000}{200} = 145$$

Hence, average earning of the workers is 145 thousand shillings per worker

Method 2:

Since the values of x are large, we can use the method of assumed mean. Let $A = 130$

Monthly wages (in thousand shillings)	Number of workers f	Class marks x	$d = x - 130$	fd
80 – 120	20	90	-40	-800
100 – 120	30	110	-20	-600
120 – 140	20	130	0	0
140 – 160	40	150	20	800
160 – 180	90	170	40	3600
Total	200			3000

$$\bar{x} = A + \frac{\sum fd}{\sum f} = 130 + \frac{3000}{200} = 130 + 15 = 145$$

Hence, average earning of the workers is 145 thousand shillings per worker

Locating the mode graphically

The mode can be obtained graphically using the following procedure of 3 steps

Step 1: Draw a histogram of the data and locate three rectangles i.e. for the modal, pre-modal and post-modal classes.

Step 2: Draw two lines diagonally inside the modal class rectangle, starting from each upper corner of the modal rectangle to the upper corner of the adjacent rectangle.

Step 3: From the point of intersection of both these lines, draw a perpendicular to the x-axis; the point where this perpendicular meets the x-axis is the value of the mode.

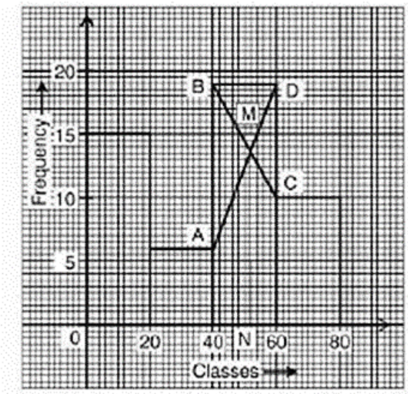
Example 7

Obtain graphically, the mode of the distribution given below.

Classes	0 – 20	20 – 40	40 – 60	60 – 80
Frequency	15	6	18	10

Solution:

We draw the histogram by taking 2 cm = 20 units on horizontal axis and 2 cm = 5 units on the vertical axis.



Now, inside the highest rectangle, we join A to D and B to C. AD and BC meet at M. From M, we draw the perpendicular MN to the horizontal axis meeting it at N. We estimate ON approximately equal to 52 units

Hence the required mode $= 52$

Median in a discrete distribution (Simple distribution)

If a set of numbers is given, they can be arranged in order of magnitudes, preferably the highest score at the top, then the middle point of the array gives the position of the median and the score at that point is called the median or the median score. There will be an equal number of items above and below the median.

For a set of observations arranged in order size, the median is the value 50% of the way through the distribution. Described often as the middle value, the median is an average that is unaffected by extreme values.

For instance, if seven boys of different heights are made to stand in a row, the tallest first, the next tallest and so on, the median height is of the fourth boy from either end. If there is an even number of boys, say eight, it would be natural to take as median the height midway between that of the fourth and that of the fifth boy i.e. the position given by $\frac{4+5}{2}$ from either end.

Example 8

A student secured the following marks in seven subjects: 50, 53, 61, 49, 45, 63, 48.

Solution:

Arranging the marks in ascending order, we have

45, 48, 49, 50, 53, 61, 63

The median is 50 marks

Example 9

The weights (in kg) of 15 students are as follows: 31, 35, 27, 29, 32, 43, 37, 41, 34, 28, 36, 44, 45, 42, 30. Find the median.

Solution:

Writing the given weights in increasing order, we have

27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 41, 42, 43, 44, 45

The median is the 8th item i.e. 35 kg

Example 10

Find the median of the following set of numbers: 10, 75, 3, 81, 18, 27, 4, 48, 12, 47, 9, 15

Solution:

Arranging the numbers in ascending order, we have

3, 4, 9, 10, 12, 15, ↓, 18, 27, 47, 48, 75, 81

The median, lies between the 6th and 7th items

$$\text{Median } = \frac{15 + 18}{2} = 16.5$$

Median in a grouped distribution

For grouped data, we can use the following formula to calculate the median i.e.

$$\text{Median} = L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right)c$$

where;

L_1= lower class boundary of the median class

N = Total number of observations or the total frequency

F_b = Cumulative frequency before median class

f_m= Frequency of the median class

c = Class width

Example 11

The marks (out of 75) obtained by 60 students in a certain examination are given below. Find the median marks.

Marks	15 – 20	20 – 25	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50	50 – 55	55 – 60	60 – 65
No. of Students	4	5	11	6	5	8	9	6	4	2

Solution:

Marks	Frequency, f	Cumulative frequency, F
15 – 20	4	4
20 – 25	5	9
25 – 30	11	20
30 – 35	6	26
35 – 40	5	31
40 – 45	8	39
45 – 50	9	48
50 – 55	6	54
55 – 60	4	58
60 – 65	2	60

Here, $\frac{N}{2} = \frac{60}{2} = 30$

The cumulative frequency just greater than 30 is 31 and the corresponding class is 35 – 40. Therefore the median class is 35 – 40.

$$\text{Median} = L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right)c = 35 + \left(\frac{30 - 26}{5}\right) \times 5 = 39 \text{ marks}$$

Example 12

The following table gives the population of males in different age groups in a certain town

Age group (in years)	5 – 14	15 – 24	25 – 34	35 – 44	45 – 54	55 – 64	65 – 74
No. of males	447	307	279	220	157	91	39

Calculate their median age.

Solution:

First it is necessary to convert the given inclusive distribution into the exclusive distribution.

Age group (in years)	Class boundaries	Number of males f	F
5 – 14	4.5 – 14.5	447	447
15 – 24	14.5 – 24.5	307	754
25 – 34	24.5 – 34.5	279	1033
35 – 44	34.5 – 44.5	220	1253
45 – 54	44.5 – 54.5	157	1410
55 – 64	54.5 – 64.5	91	1501
65 – 74	64.5 – 74.5	39	1540

Here $\frac{N}{2} = \frac{1540}{2} = 770$

The cumulative frequency just greater than 770 is 1033 and the corresponding class is 24.5 – 34.5 which is the median class. Using the formula

$$\text{Median} = L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right)c = 24.5 + \left(\frac{770 - 754}{279}\right) \times 10 = 25.07 \text{ years}$$

Example 13

The following table shows the age distribution of persons in a particular region.

Age (years)	No. of persons
Below 10	20
Below 20	50
Below 30	90
Below 40	120
Below 50	140
Below 60	150
Below 70	155
70 and above	156

Find the median age

Solution:

The frequencies are given as cumulative frequencies. First, we convert them into absolute frequencies

$$Q_1 = L + \left(\frac{\frac{N}{4} - F}{f}\right)c, \quad Q_3 = L + \left(\frac{\frac{3N}{4} - F}{f}\right)$$

For deciles, we use $\frac{N}{10}$ for the first decile, $\frac{2N}{10}$ for the second decile, $\frac{3N}{10}$ for the third decile, and so on. For percentiles, we use $\frac{N}{100}$ for the first percentile, $\frac{2N}{100}$ for the second percentile, $\frac{3N}{100}$ for the third percentile, and so on.

Example 14

Find lower quartile, median, upper quartile, seventh decile, sixtieth percentile for the following frequency distribution.

Wages	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of persons	1	3	11	21	43	32	9

Solution:

Wages	f	F
10 – 20	1	1
20 – 30	3	4
30 – 40	11	15
40 – 50	21	36
50 – 60	43	79
60 – 70	32	111
70 – 80	9	120

$\frac{N}{4} = \frac{120}{4} = 30$, $\therefore Q_1$ lies in the class 40 – 50

$$Q_1 = L + \left(\frac{\frac{N}{4} - F}{f}\right)c = 40 + \left(\frac{30 - 15}{21}\right) \times 10 = 40 + 7.14 = 47.14$$

$\frac{N}{2} = \frac{120}{2} = 60$, $\therefore$ Median, M or Q_2 lies in the class 50 – 60

$$M = L + \left(\frac{\frac{N}{2} - F}{f}\right)c = 50 + \left(\frac{60 - 36}{43}\right) \times 10 = 50 + 5.58 = 55.58$$

$\frac{3N}{4} = \frac{3\times120}{4} = 90$, $\therefore Q_3$ lies in the class 60 – 70

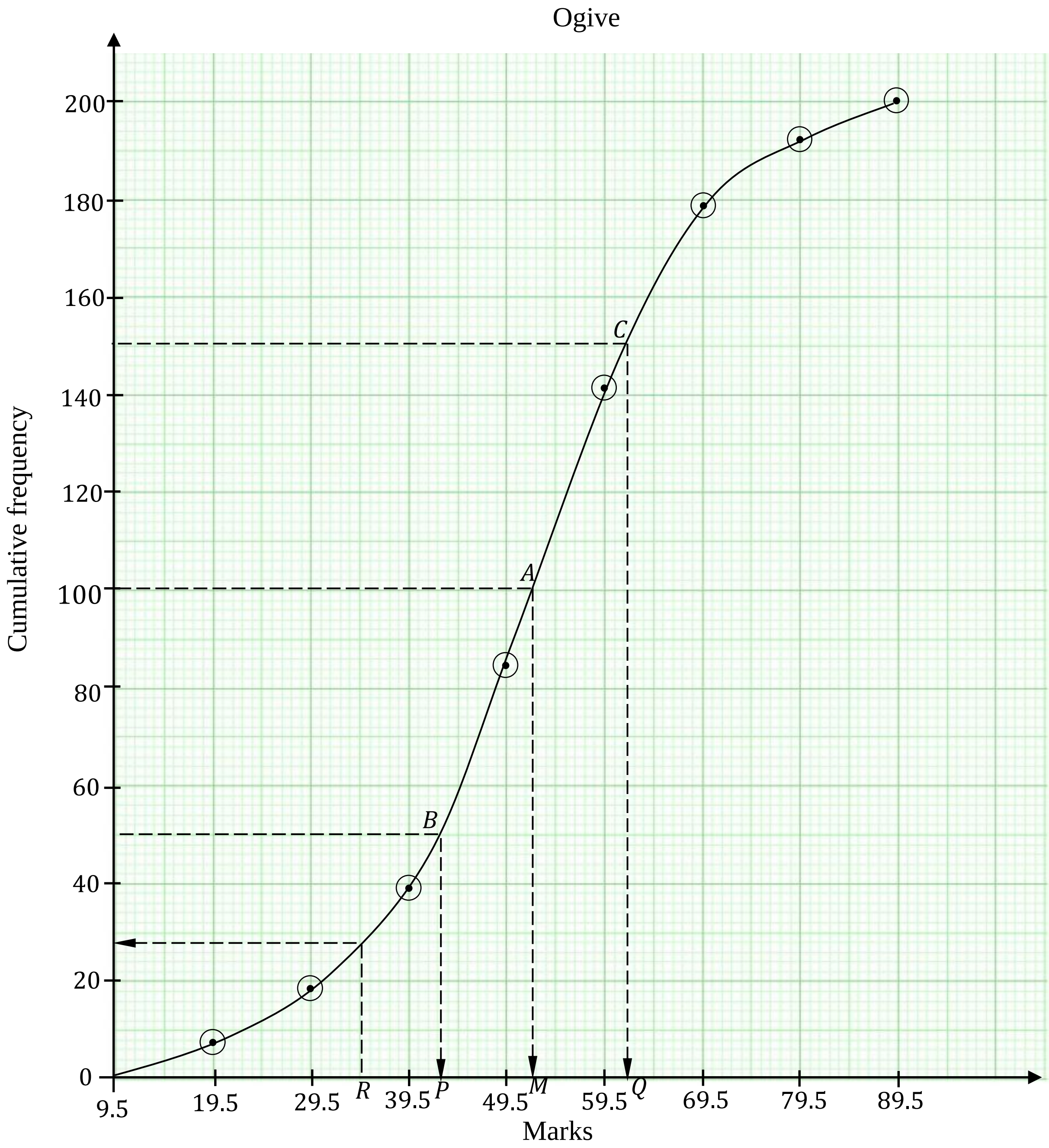
$$Q_3 = L + \left(\frac{\frac{3N}{4} - F}{f}\right)c = 60 + \left(\frac{90 - 79}{32}\right) \times 10 = 60 + 3.44 = 63.44$$

$\frac{7N}{10} = \frac{7\times120}{10} = 84$, $\therefore D_7$ lies in the class 60 – 70

$$D_7 = L + \left(\frac{\frac{7N}{10} - F}{f}\right)c = 60 + \left(\frac{84 - 79}{32}\right) \times 10 = 60 + 1.56 = 61.56$$

$\frac{60N}{100} = \frac{60\times120}{100} = 72$, $\therefore P_{60}$ lies in the class 50 – 60

$$P_{60} = L + \left(\frac{\frac{60N}{100} - F}{f}\right)c = 50 + \left(\frac{72 - 36}{43}\right) \times 10 = 50 + 8.37 = 58.37$$



Note: When drawing the Histogram or Ogive, the horizontal axis should be labelled with the specific variable e.g. marks, mass, weight, height, etc.

b) Measures of Dispersion

The measures of central tendency reveal only one characteristic of the data i.e. the point of central tendency. They fail to give us an idea about the extent to which the items of a distribution deviate from a central value. We get this information by the study of dispersion. Dispersion, thus, measures the variability of a series.

Definition: The degree to which numerical data tend to spread about an average value is called the variation or dispersion of the data.

Various measures of dispersion are available, the most common being the range, semi-interquartile range and the standard deviation:

Range

This is the difference between the largest and the smallest values of the data.

i.e. for the data about lengths of leaves in garden tree, 5,6,7,7,4,5,3,2,9,8,8,6,5,3

$$\text{Range} = 9 - 2 = 7$$

The standard deviation (S.D)

This is the most important measure of dispersion and is widely used in many statistical formulae. It is defined as the positive square root of the arithmetic mean of the squares of the deviations of the given observations from their arithmetic mean. It is denoted by the Greek letter σ (sigma).

Computation of S.D

The following methods are used to calculate standard deviation.

(a) Deviation taken from actual mean
(b) Making direct use of the variable without calculating deviations
(c) Deviation from assumed mean

a. By taking deviations from the actual mean

If $x_1, x_2, \ldots, x_n$ are set of numbers, then

$$\sigma = \sqrt{\frac{\sum(x_i - \bar{X})^2}{N}}$$

$$\text{or } \sigma = \sqrt{\frac{\sum d^2}{N}}$$

where d represents the deviations of each number x_i from the mean $\bar{X}$.

b. Making direct use of the variable without calculating deviations

For a simple distribution,

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{\sum x^2}{n} - \bar{X}^2}$$

For a frequency distribution

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{X}^2}$$

c. Deviation from assumed mean

If any arbitrary number A is chosen as the assumed mean instead of the true mean $\bar{X}$ and if the deviation of x from A is represented by d, then

Method 3: By taking deviations from the assumed mean

Let the assumed mean be 10

Values, x	$d = x - 10$	d^2
8	-2	4
9	-1	1
15	5	25
23	13	169
5	-5	25
11	1	1
19	9	81
8	-2	4
10	0	0
12	2	4
$\sum$	20	314

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

$$= \sqrt{\frac{314}{10} - \left(\frac{20}{10}\right)^2}$$

$$= \sqrt{31.4 - 4}$$

$$= \sqrt{27.4} = 5.23$$

Example 17

Calculate the standard deviation for the following distribution by taking deviations from the assumed mean

x	8	11	17	20	25	30	35
f	2	3	4	1	5	7	3

Solution:

Let the assumed mean A be 20

Variable (x)	Frequency (f)	$d = x - A$	d^2	fd	fd^2
8	2	-12	144	-24	288
11	3	-9	81	-27	243
17	4	-3	9	-12	36
20	1	0	0	0	0
25	5	5	25	25	125
30	7	10	100	70	700
35	3	15	225	45	675
				77	2067

$$\text{S.D}, \sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} = \sqrt{\frac{2067}{25} - \left(\frac{77}{25}\right)^2} = 8.56$$

Example 18

Compute the standard deviation for the following distribution

Variable (x)	10	15	18	20	25
Frequency (f)	3	2	5	8	2

Solution:

Variable (x)	Frequency (f)	fx	$d = x - \bar{X}$	d^2	fd^2
10	3	30	-8	64	192
15	2	30	-3	9	18
18	5	90	0	0	0
20	8	160	2	4	32
25	2	50	7	49	98
$\sum$	20	360			340

Arithmetic mean $\bar{X} = \frac{\sum fd}{\sum f} = \frac{360}{20} = 18$

$$\text{S.D}, \sigma = \sqrt{\frac{\sum fd^2}{n}} = \sqrt{\frac{340}{20}} = \sqrt{17} = 4.1$$

Example 19

Calculate the standard deviation of the following distribution

Age	20 – 25	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50
Number of persons	170	110	80	45	40	35

Solution:

Let the assumed mean be 32.5

Age	Mid-values x	Number of persons f	$d = x - 32.5$	d^2	fd	fd^2
20 – 25	22.5	170	-10	100	-1700	17000
25 – 30	27.5	110	-5	25	-550	2750
30 – 35	32.5	80	0	0	0	0
35 – 40	37.5	45	5	25	225	1125
40 – 45	42.5	40	10	100	400	4000
45 – 50	47.5	35	15	225	525	7875
$\sum$		480			-1100	32750

$$S.D, \sigma = \sqrt{\frac{\sum fd^2}{n} - \left(\frac{\sum fd}{n}\right)^2} = \sqrt{\frac{32750}{480} - \left(\frac{-1100}{480}\right)^2} = \sqrt{68.23 - 5.25} = 7.936$$

Example 20

The mean of 200 items is 48 and their standard deviation is 3. Find the sum, and the sum of squares of all items.

Solution:

It is given that $n = 300$, $\bar{x} = 48$ and $\sigma = 3$ and it is required to find the value of $\sum x$ and $\sum x^2$

$$\bar{x} = \frac{\sum x}{n} \Rightarrow \sum x = n\bar{x} = 200 \times 48 = 9600$$

Also, $\sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$

Substituting the values, we get

$$9 = \frac{\sum x^2}{200} - \left(\frac{9600}{200}\right)^2$$

$$\frac{\sum x^2}{200} = 9 + (48)^2$$

$$\sum x^2 = 200 \times 2313 = 462{,}600$$

Examination questions

2013, No.2

The table below shows the age in years of mothers at the time they had their first child.

Age in years	$15\ -$	$20-$	$25-$	$30-$	$35-$	$40-45$
Number of numbers	2	14	29	43	33	9

Calculate the modal age of the mothers

Solution:

The class boundaries are provided in this case and not the class limits

$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$$

$\Delta_1 = 43 - 29 = 14$ and $\Delta_2 = 43 - 33 = 10$, class width $= 5$

$$\text{Mode} = 30 + \left(\frac{14}{14+10}\right) \times 5 = 30 + 2.917 = 32.917$$

The modal age is 32.917 years

2014, No. 5

A class of n students sat for a mathematics test. Given that $\sum fx = 400, \sum fx^2 = 6500$ and the mean $\bar{x} = 16$, where x is the mark and f is the frequency; determine the value of

(a) n

(b) the standard deviation

Solution:

(a) Mean, $\bar{x} = \frac{\sum fx}{n}$

$$16 = \frac{400}{n}$$

$$16n = 400 \Rightarrow n = \frac{400}{16} = 25$$

(b) Standard deviation $= \sqrt{\frac{\sum fx^2}{n} - \left(\frac{\sum fx}{n}\right)^2}$

$$= \sqrt{\frac{6500}{25} - 16^2} = \sqrt{260 - 256} = \sqrt{4} = 2$$

2015, No. 9

The table below shows the number of students and the marks scored in a test

MARKS	NUMBER OF STUDENTS
$0-4$	10
$5-9$	7
$10-14$	5
$15-19$	3
$20-24$	7
$25-29$	11
$30-34$	37
$35-39$	20

(a) (i) Draw a cumulative frequency curve (Ogive) for the data

(ii) Use the Ogive to estimate the median mark

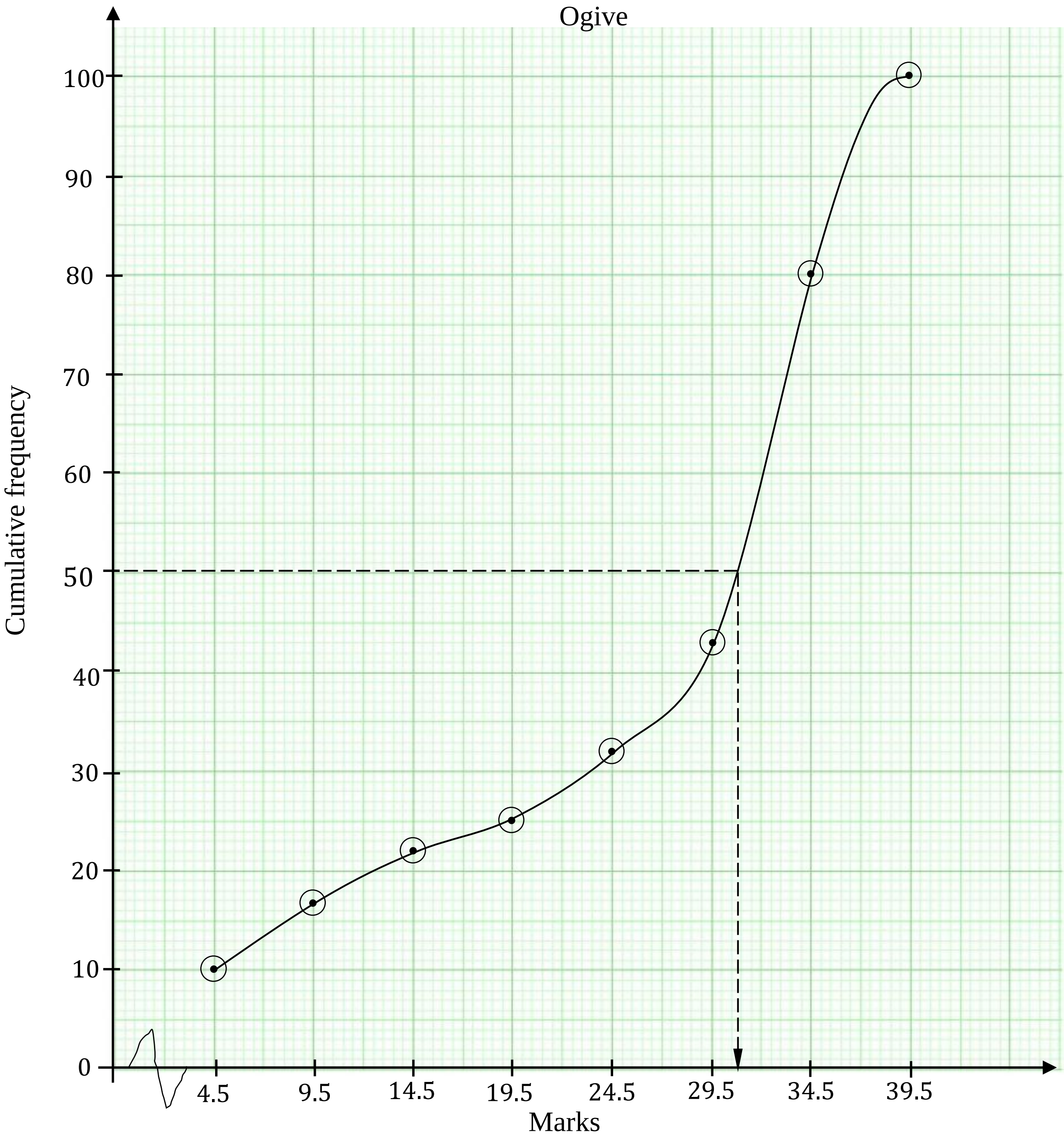
(b) Calculate the;

(i) mean mark

(ii) standard deviation

Solution:

(a)(i)



(ii) From the graph, median = 31 marks

Class	f	x	fx	fx^2	F	Upper class boundaries
$0-4$	10	2	20	40	10	4.5
$5-9$	7	7	49	343	17	9.5
$10-14$	5	12	60	720	22	14.5
$15-19$	3	17	51	867	25	19.5
$20-24$	7	22	154	3388	32	24.5
$25-29$	11	27	297	8019	43	29.5
$30-34$	37	32	1184	37888	80	34.5
$35-39$	20	37	740	27380	100	39.5
$\sum$	100		2555	78645		

(b)(i) Mean mark = $\frac{\sum fx}{\sum f} = \frac{2555}{100} = 25.55$

(ii) Standard deviation = $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{78645}{100} - (25.55)^2} = \sqrt{133.6475} = 11.56$

2016, No. 9

The table below shows a frequency distribution of marks scored by 55 students in a test.

Marks	$10-$	$20-$	$30-$	$40-$	$50-$	$60-$	$70-$	$80-\leq 90$
Number of students	2	6	12	15	10	6	3	1

(a) Draw a histogram for the data and use it to estimate the modal mark

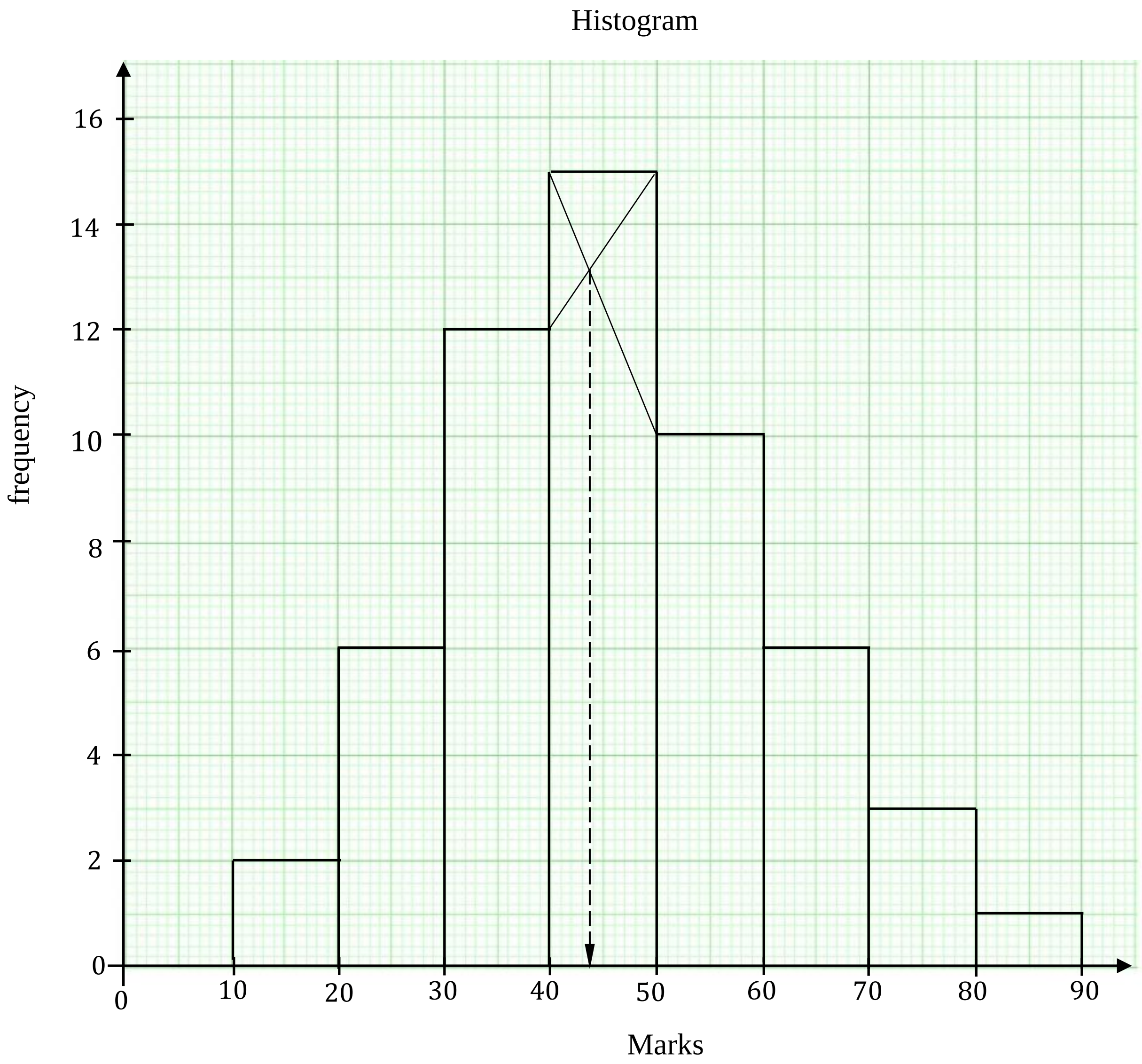
(b) Calculate the

(i) mean mark

(ii) standard deviation

Solution:

(a)



Estimated modal mark is 44 marks

(b)

Class	Freq (f)	x	fx	fx^2
$10-\leq 20$	2	15	30	450
$20-\leq 30$	6	25	150	3750
$30-\leq 40$	12	35	420	14700
$40-\leq 50$	15	45	675	30375
$50-\leq 60$	10	55	550	30250
$60-\leq 70$	6	65	390	25350
$70-\leq 80$	3	75	225	16875
$80-\leq 90$	1	85	85	7225
$\sum$	55		2525	128975

(i) Mean $= \frac{\sum fx}{\sum f} = \frac{2525}{55} = 45.91$ marks

(ii) Standard deviation, $\sigma = \sqrt{\frac{128975}{55} - (45.91)^2} = \sqrt{237.2719} = 15.4$

2017, No. 9

The data below shows the weights in kg of 50 cattle on a farm.

60	81	76	68	84	112	76	102	86	67
65	98	107	110	72	99	87	92	76	77
94	102	87	86	73	118	98	120	62	87
65	92	104	116	91	93	78	122	102	92
80	111	73	120	106	123	94	109	80	96

(a) Form a grouped frequency table for the data with classes of equal intervals, starting with the class $60-69$

(b) Draw a cumulative frequency curve (O give) for the given data

(c) Use your Ogive to estimate the

  (i) lower and upper quartiles

  (ii) median weight

  (iii) number of cattle which weigh 118 and above

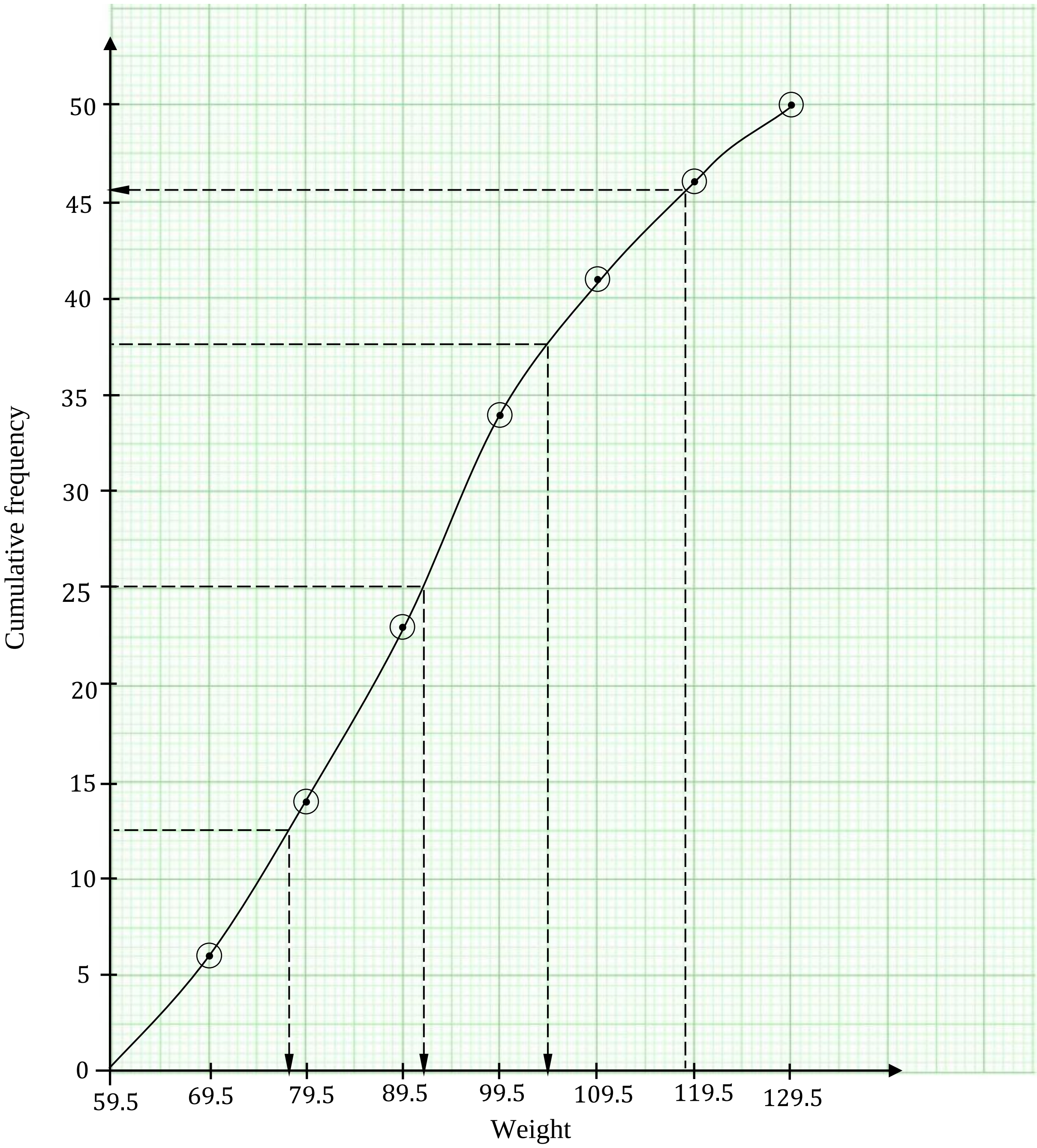
Solution:

(a)

Class	Frequency, f	F	Upper Class boundaries
$60-69$	6	6	69.5
$70-79$	8	14	79.5
$80-89$	9	23	89.5
$90-99$	11	34	99.5
$100-109$	7	41	109.5
$110-119$	5	46	119.5
$120-129$	4	50	129.5

(b) .

Ogive



(c) (i) $Q_1 = \left(\frac{1}{4}N\right)^{th}$ value $= 12.5^{th}$ value $= 78.5\ kg$

$Q_3 = \left(\frac{3}{4}N\right)^{th}$ value $= 37.5^{th}$ value $= 105\ kg$

(ii) Median $= \left(\frac{1}{2}N\right)^{th}$ value $= 25^{th}$ value $= 91.5\ kg$

(iii) Number of cattle weighing above $118\ kg$ is given by

$$50 - 45 = 5$$

2018, No. 6

The marks scored in the test by 8 students are: 5, 9, 11, 15, 19, 15, 10, 14. Determine the:

(a) mean mark
(b) variance

Solution:

x	$(x-\mu)$	$(x-\mu)^2$
5	-7.25	52.5625
9	-3.25	10.5625
11	-1.25	1.5625
15	2.75	7.5625
19	6.75	45.5625
15	2.75	7.5625
10	-2.25	5.0625
14	1.75	3.0625
$\sum x = 98$		$\sum(x-\mu)^2 = 133.5$

(a) Mean mark, $\mu = \frac{\sum x}{n} = \frac{98}{8} = 12.25$

(b) Variance, $(\sigma^2) = \frac{\sum(x-\mu)^2}{n} = \frac{133.5}{8} = 16.6875$

2019, No. 6

The ages of eight students in a class are 12, 13, 14, 15, 12, 17, 13, 16. Find the

(a) mean age
(b) variance

Solution:

x	$(x-\mu)$	$(x-\mu)^2$
12	-2	4
13	-1	1
14	0	0
15	1	1
12	-2	4
17	3	9
13	-1	1
16	2	4
$\sum x = 112$		$\sum(x-\mu)^2 = 24$

(a) Mean age, $\mu = \frac{\sum x}{n} = \frac{112}{8} = 14$

Variance, $(\sigma^2) = \frac{\sum(x-\mu)^2}{n} = \frac{24}{8} = 3$

2022, No. 9

The frequency distribution table below shows the marks of 50 students scored in a test.

Marks	Number of Students
50 – 52	3
53 – 55	16
56 – 58	14
59 – 61	13
62 – 64	2
65 – 67	2

(a) Calculate the:
(i) mean mark
(ii) standard deviation

(b) (i) Plot a cumulative frequency curve (Ogive) for the given data

(ii) Use the Ogive to estimate the median mark

Solution:

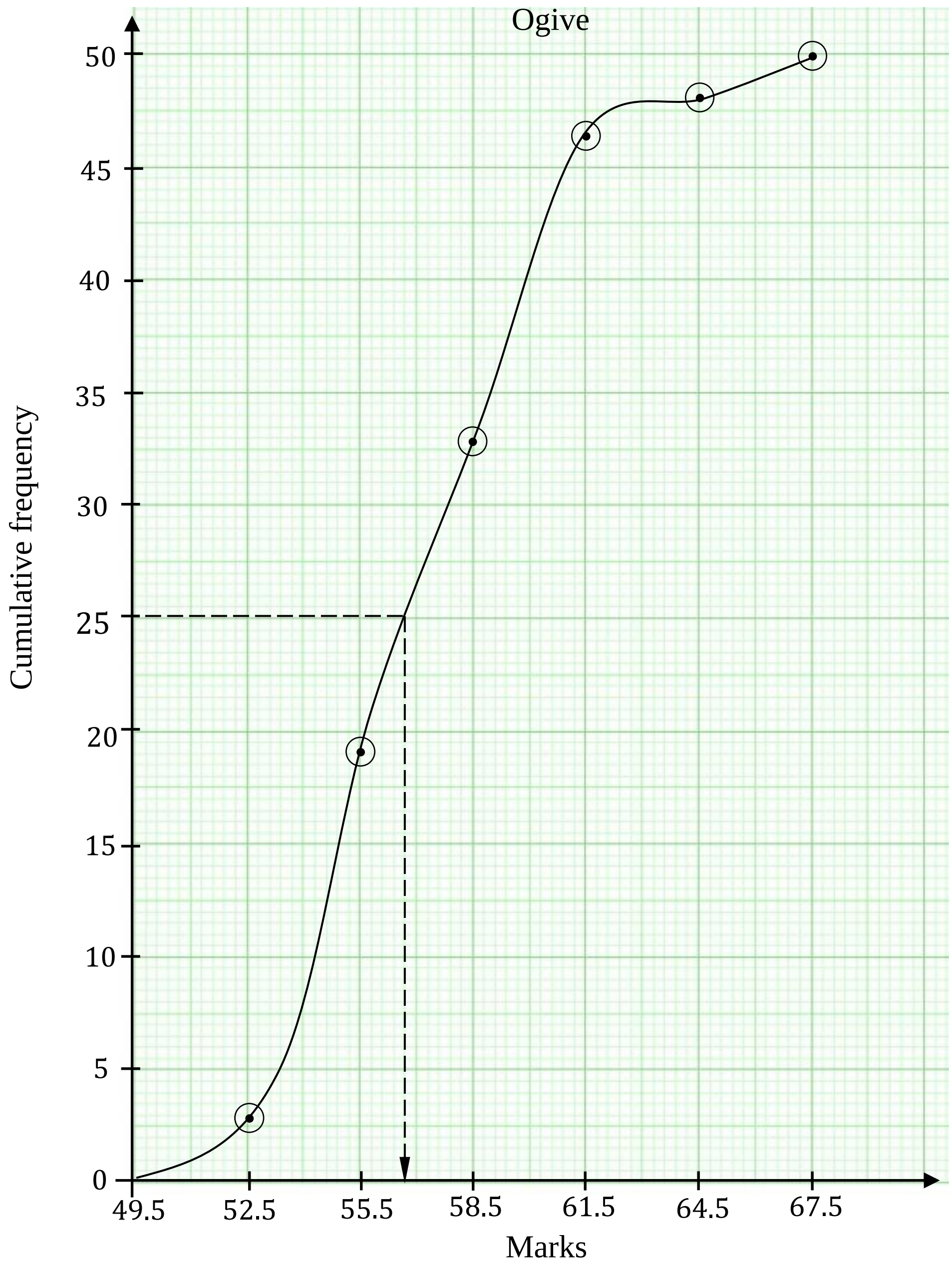
(a)

Marks	x	f	fx	fx^2	F	Class boundaries
50 – 52	51	3	153	7803	3	49.5 – 52.5
53 – 55	54	16	864	46656	19	52.5 – 55.5
56 – 58	57	14	798	45486	33	55.5 – 58.5
59 – 61	60	13	780	46800	46	58.5 – 61.5
62 – 64	63	2	126	7938	48	61.5 – 64.5
65 – 67	66	2	132	8712	50	64.5 – 67.5
$\sum$		50	2853	163395		

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{2853}{50} = 57.06$$

$$\text{Standard deviation} = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{163395}{50} - 57.06^2} = \sqrt{12.0564} = 3.472$$

(b) (i)



(ii) From the Ogive, the median is 56.5 marks

Self-Evaluation exercise

1. Below are heights, measured to the nearest cm of 50 pupils

157	167	165	162	160	157	160	152	157	162
157	165	152	162	155	160	157	160	162	160
157	152	167	157	160	160	162	165	157	160
157	157	157	160	157	162	155	157	160	157
150	162	152	160	157	157	165	160	162	150

a) Make a frequency distribution table by dividing them into class intervals of 5 starting with the class $148 - 152$

b) Draw a cumulative frequency curve and use it to estimate

(i) The median (ii) Interquartile range

2. The table below shows marks obtained by students of mathematics in a certain school.

Marks	$30-<40$	$40-<50$	$50-<60$	$60-<70$	$70-<80$
No. of students	2	15	10	11	27

(i) Calculate the mean, median and standard deviation for the above data

(ii) Draw an Ogive for the above data

3. Sixty pupils were asked to draw a free hand line of length $20\ cm$. The lengths of the lines were measured to nearest cm, and were recorded as shown in the table.

Length(cm)	$11-13$	$13-15$	$15-17$	$17-19$	$19-21$	$21-23$	$23-25$
Frequency	3	6	11	15	13	10	2

a) Calculate the mean length

b) Draw a cumulative frequency graph and estimate the median, the upper and the lower quartiles.

4. Calculate the mean and the standard deviation of the following distribution of scores

Scores	$1-5$	$6-10$	$11-15$	$16-20$	$21-25$	$26-30$	$31-35$
Frequency	3	19	38	69	45	21	5

5. The numbers of the eggs collected from a poultry farm for 40 consecutive days were as follows.

138	145	145	157	150	142	154	140
146	135	128	149	164	147	152	138
168	142	135	125	158	135	148	176
146	150	165	144	126	153	136	163
161	156	144	132	176	140	147	130

a) Construct a frequency distribution table with classes of equal interval width 5, starting from $125 - 129$.

b) Draw a cumulative frequency curve (Ogive) and use it to estimate the (i) Interquartile range

(i) Median number of eggs

6. The times taken by a group of students to solve a mathematical problem are given below.

Time(min)	$5-9$	$10-14$	$15-19$	$20-24$	$25-29$	$30-34$
No. of students	5	14	30	17	11	3

(a) Draw a histogram for the data. Use it to estimate the modal time for solving a problem.

(b) Calculate the mean time and standard deviation of solving a problem.

7. The table below shows the weights (in kg) of 150 patients who visited a certain health unit during a certain week.

Weight (kg)	$0-19$	$20-29$	$30-39$	$40-49$	$50-59$	$60-69$	$70-79$
No. of patients	30	16	24	32	28	12	8

a) Calculate the appropriate mean and modal weights of the patients.

b) Plot an Ogive for the above data. Use the Ogive to estimate the median and semi interquartile for the weights of patients.

Chapter 13 Moving Averages

Introduction

The simplest method of smoothing out fluctuations and obtaining the trend values with fair degree of accuracy is the moving average method.

Moving averages are a number of arithmetic averages calculated from the time series data, each based on a fixed number of consecutive observations. If the time series data are yearly, then moving averages of period, say, r years are a series of arithmetic averages each of r consecutive observations. The first moving average is the average of the first r observations and is placed at the time point midway between the time points of the first and the r-th observations of the series. For second moving average, drop the first observation and include the $(r+1)$th observation in the calculation of the average. The second moving average is the average of the second to $(r+1)$th observations of the series and is placed at the middle of the period covering second to $(r+1)$ years. Similarly, the third moving average is the average of the third to $(r+2)$th observations of the series and is placed at the midpoint of the time interval covering third to $(r+2)$ years, and so on.

Since each moving average is placed at the time point midway between the time points of the first and the last observations included in the calculation of average, the moving average values do not correspond to any of the original periods when there are even number of periods and hence two-item moving averages of the moving averages already obtained, have to be calculated to correspond them to any of the original periods. This process is called **recentering**.

The purpose of the moving average method is to smooth out cyclical, seasonal and irregular variations of the time series data in order to isolate the trend. It is observed that moving average will completely eliminate a fluctuation if the period of moving average is equal to the period of the fluctuation or its integral multiple.

When a series of yearly figures are given, the seasonal fluctuations, whose period is, generally, a year is automatically excluded from the series. The other fluctuation to be removed now is the cyclic fluctuation. If the period of the cyclical fluctuation is known, this can be eliminated by calculating moving averages taking the period of the moving average equal to or an integral multiple of the period over which the cyclical fluctuations occur. When the period of the cyclical fluctuation is not obvious then a graph of the actual data is to be drawn and the distance between two 'peaks' or two 'depressions' of the graph will be taken as period when the cycle is regular and this period or an integral multiple of this period will be taken as the period of the moving average to smooth out the cyclic fluctuations. When the period of the cycle is not uniform, the average duration of the cycles or an integral multiple of it may be taken as the period.

When the monthly or quarterly figures are given, then a twelve month or four-year quarter moving average is called for to smooth out seasonal fluctuations. Now if the cyclical fluctuations have a period of, say, four years then the moving average with a period of 48 months or 16 quarters will smooth out both the seasonal and cyclical fluctuations. But if the period of the cyclical fluctuation be, say 30 months then the moving average with a period of 60 months (the least common multiple of both the period of the seasonal fluctuation and the period of cyclical fluctuation) is required to smooth out both the seasonal and the cyclical fluctuations.

Moving average, in general, cannot eliminate irregular fluctuations but it only reduces them.

Thus the moving averages with period same as the period of the cycle or its integral multiple will smooth out the seasonal and the cyclical fluctuations and give and estimate of the trend.

The moving average values, sometimes, do not follow the data which describe a curve unless some weighing schemes are used. The usual type of weighing is, however, binomial, which employ binomial coefficients as weights. There are other systems of weighing also. The main point in favour of weighted moving average is that they are both relatively smooth and sufficiently sensitive.

Merits and Demerits

- This method is flexible and not subjective. It is simple to understand and easy to adopt
- This method is appropriate only when the trend is linear. If the trend is not linear the moving average will over-estimate or under-estimate the trend value.
- Cyclical fluctuation may be eradicated completely if the cycles are regular and the period of moving average be equal to or an integral multiple of the period fluctuation. In all other cases moving averages will reduce them.
- Trend values cannot be determined for some periods at the beginning and at the end.
- The method cannot be used for forecasting future trend as the moving averages assume no definite mathematical law of change.
- This method is very sensitive to a few very high and low values which the series may contain.

Example 1

The production of cement by a firm in years 1 to 9 is given below.

Year	1	2	3	4	5	6	7	8	9
Production (Tonnes)	4	5	5	6	7	8	9	8	10

Calculate the trend values for the above series by a 3-yearly moving average method.

Solution:

Year	Production (Tonnes)	3-year moving totals	3-year moving average i.e. trend (÷3)
1	4	-	
2	5	4+5+5 = 14	4.67
3	5	5+5+6 = 16	5.33
4	6	5+6+7 = 18	6
5	7	6+7+8 = 21	7
6	8	7+8+9 = 24	8
7	9	8+9+8 = 25	8.33
8	8	9+8+10 = 27	9
9	10	-	-

Example 2

Using 3-year moving average method, determine the trend and short term fluctuations for the following data.

Year	1961	1962	1963	1964	1965	1966	1967
Values	21	34	45	28	40	57	73

Solution:

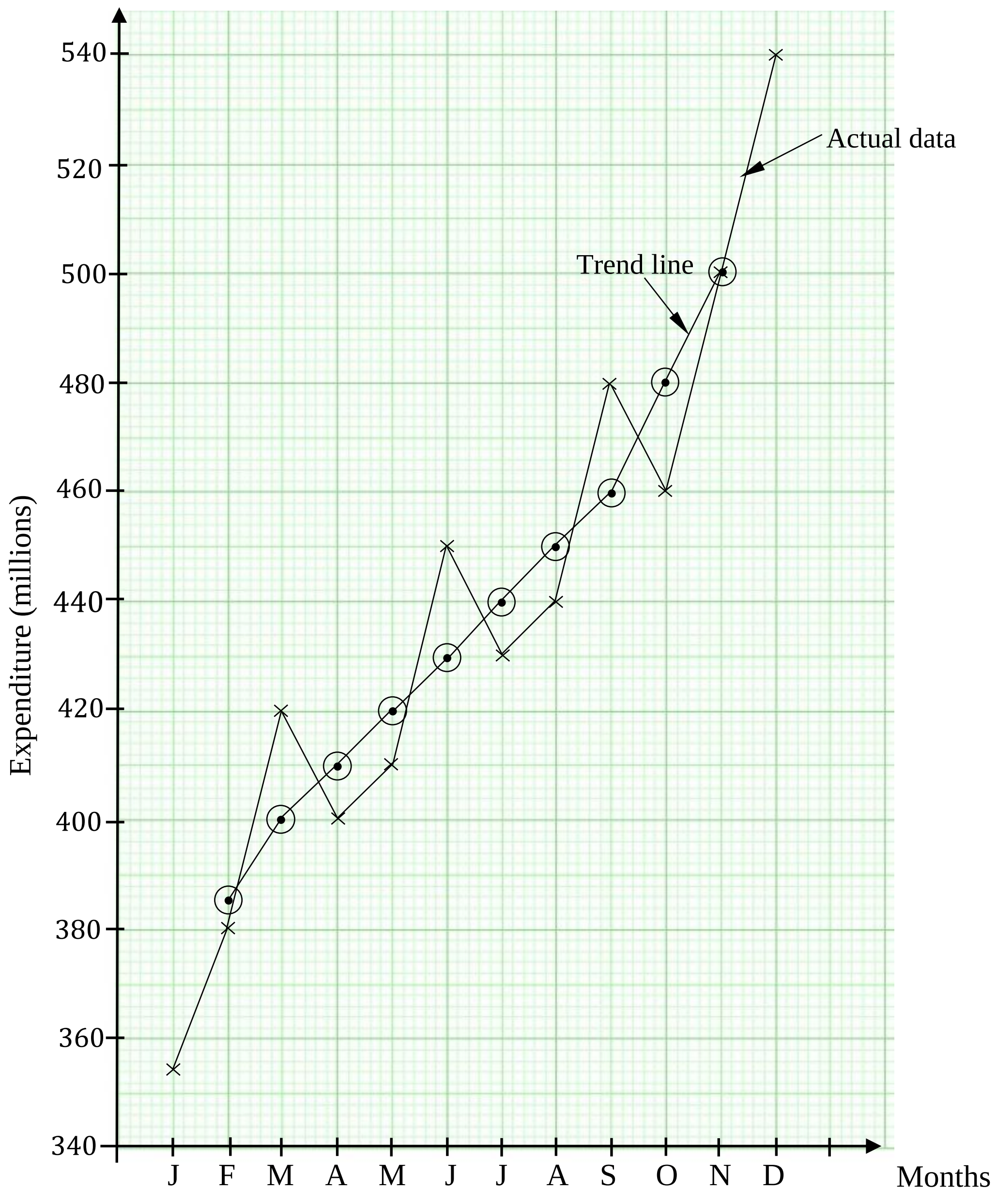
Year	Values (Y)	3-year moving total	3-year moving average i.e. trend (T)	Short-term fluctuations ($Y - T$)
1961	21	-	-	-
1962	34	100	33.33	0.67
1963	45	107	35.67	9.33
1964	28	113	38.33	- 10.33
1965	40	125	41.67	- 1.67
1966	57	170	56.67	0.33
1967	73	-	-	-

Solution:

(a)

Month	Shs (in millions)	Moving Totals	Moving averages
January	355		
February	380	1155	385
March	420	1200	400
April	400	1230	410
May	410	1260	420
June	450	1290	430
July	430	1320	440
August	440	1350	450
September	480	1380	460
October	460	1440	480
November	500	1500	500
December	540		

(b) Graph of expenditure and 3 Monthly Moving Averages



(c) The moving average line (trend line) indicates a positive trend i.e. a general increase in expenditure.

Example 7

The table below shows the average termly marks scored by a student in the end of term exams from senior one in 2017 upto the time he sat UNEB exams in 2020.

Year	Termly marks		
	1st term	2nd term	3rd term
2017	36	50	54
2018	40	45	60
2019	39	46	70
2020	49	50	-

(a) Compute the 5 point moving averages for the marks

(b) (i) Draw a graph to represent 5 point moving averages superimposed on the same graph with the termly average scores

(ii) Estimate from your graphs the average marks scored by the student in the UNEB exams sat in third term 2020.

Solution:

(a)

Year	Term	Marks	Moving total	Moving Average
2017	1st term	36		
	2nd term	50		
	3rd term	54	225	45
2018	1st term	40	249	49.8
	2nd term	45	238	47.6
	3rd term	60	230	46
2019	1st term	39	260	52
	2nd term	46	264	52.8
	3rd term	70	254	50.8
2020	1st term	49		**47**
	2nd term	50		
	3rd term	x		

(b) (i) Graph on next page

(ii) From the moving average corresponding to 1st term 2019, the mark scored = 47.

Let the mark scored in 3rd term 2020 be x

Then

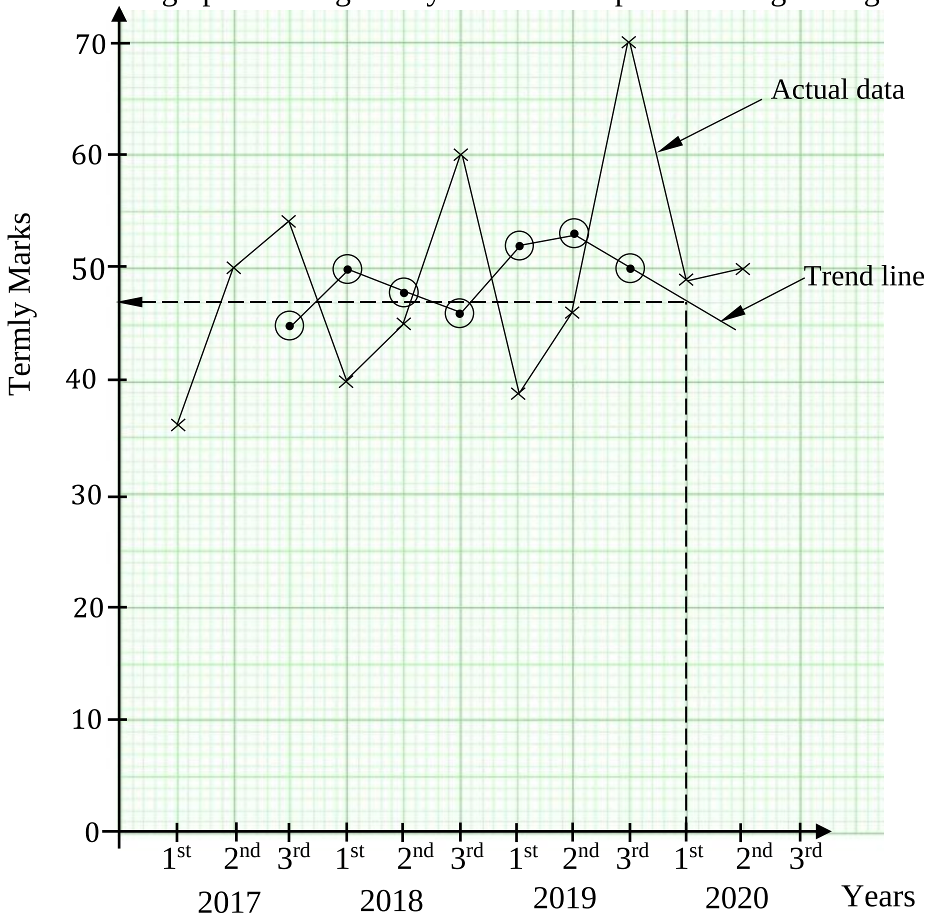
$$\frac{46 + 70 + 49 + 50 + x}{5} = 47$$

$$\frac{215 + x}{5} = 47$$

$$x = 47 \times 5 - 215 = 20$$

Hence estimated mark in the UNEB exams in 3rd term 2020 = 20%

A graph showing termly marks and 5 point moving averages



Example 8

The manager of Berikito Abattoir received the document with the information from the records officer at the end of the first quarter of 2015 as in the table below.

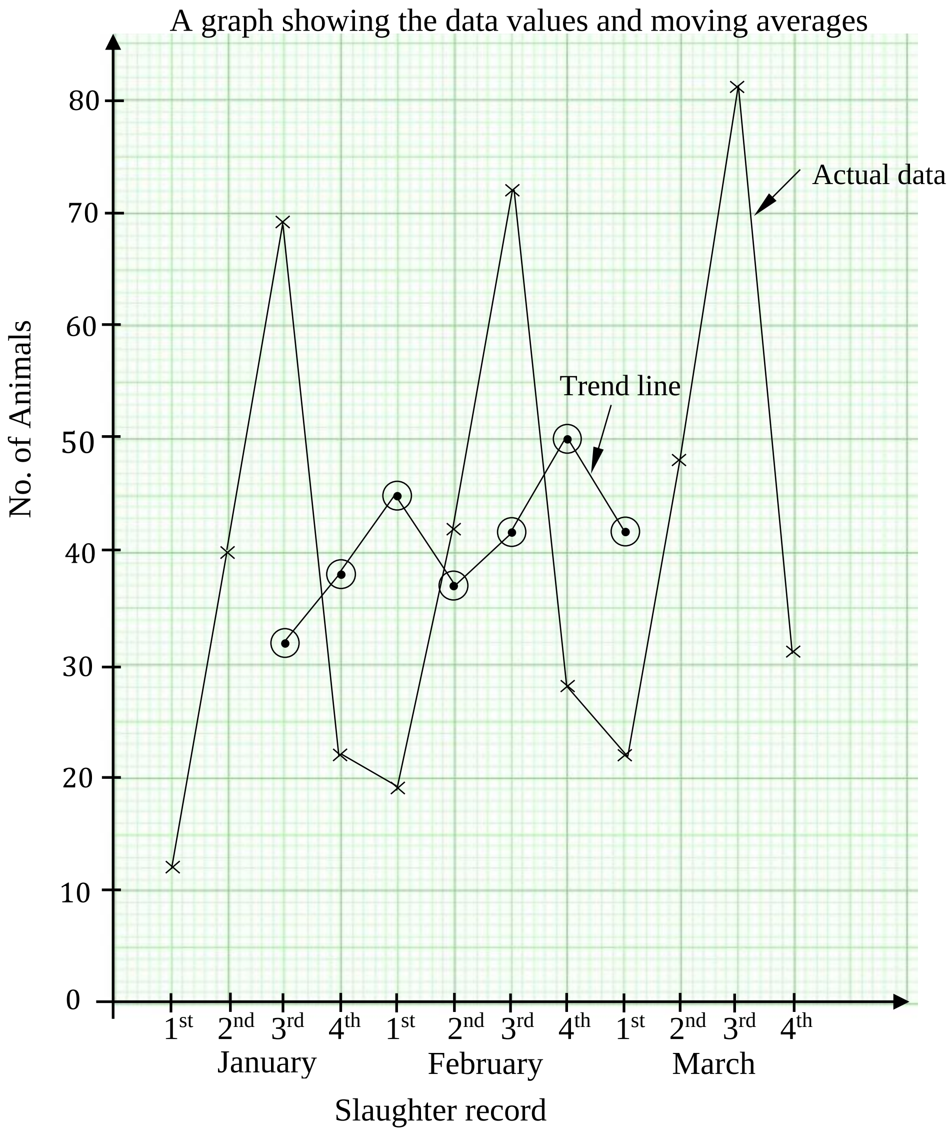
	Slaughter record											
	January				February				March			
Week	1	2	3	4	1	2	3	4	1	2	3	4
No. of Animals	12	40	69	22	19	42	72	28	22	48	81	31

Compute the five weekly moving averages of Berikito Abattoir

Plot the data values and the moving averages on the same graph and hence determine the trend

Solution:

Year	Term	Marks	Moving total	Moving Average
2017	1	12		
	2	40		
	3	69	162	32.4
	4	22	192	38.4
2018	1	19	224	44.8
	2	42	183	36.6
	3	72	183	36.6
	4	28	212	42.4
2019	1	22	251	50.2
	2	48	210	42.0
	3	81		
	4	31		



On average the data vales follow a positive trend

Example 9

The following table shows quarterly sales of items by a company over three years in "00,000" units

Year	Quarter			
	1	2	3	4
2010	70	104	142	87
2011	79	120	163	94
2012	84	131	200	97

(a) Compute the 4-centred quarterly moving averages

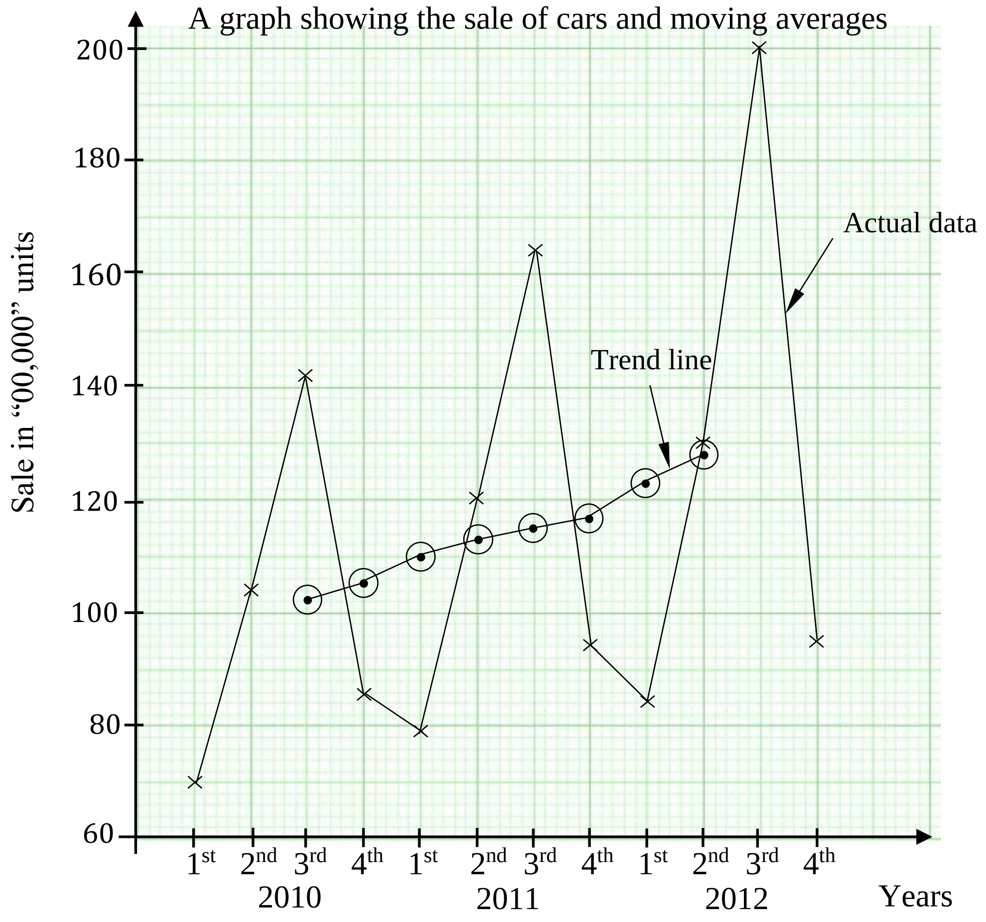
(b) Plot the original data and the 4-centred moving averages on the same diagram.

Solution:

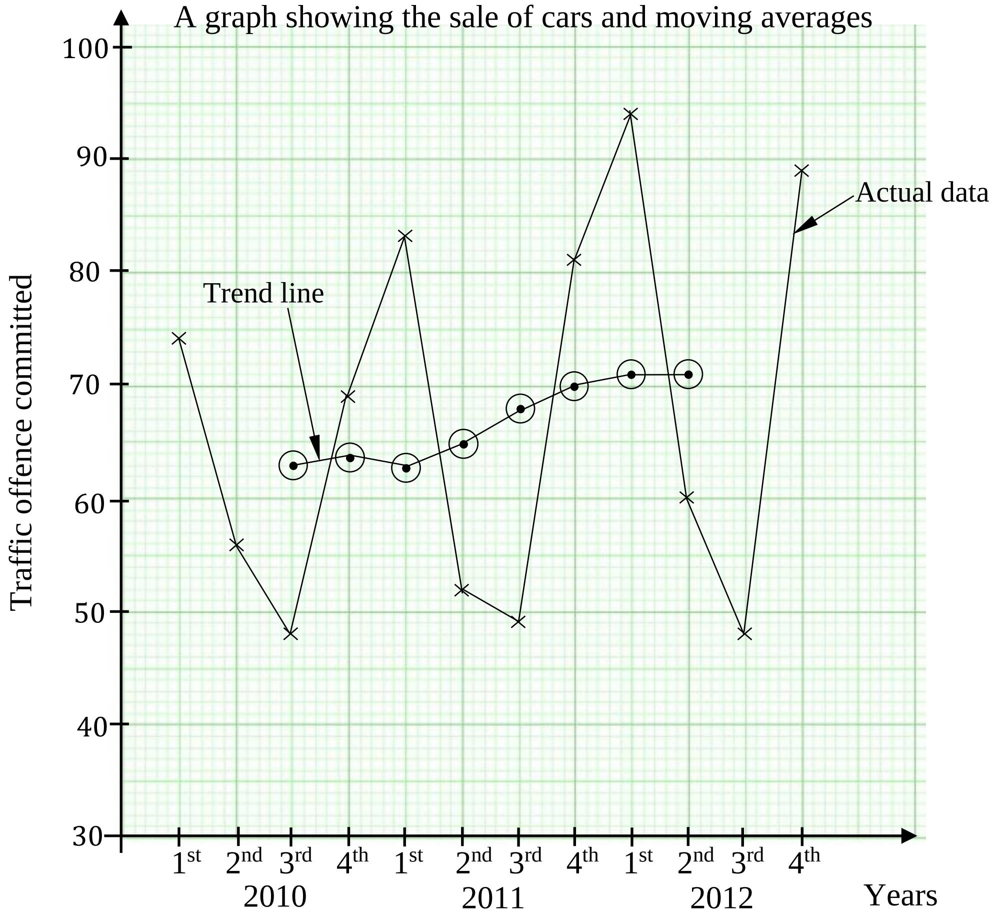
To get moving totals (M.T), add the first four values and position the total halfway the positions used as illustrated in the table, then drop the first value and add the next four values to get the next M.T, and hence follow through to get the next etc.

To get centered averages, get the sum of the first two moving averages (M.A) and divide that sum by 2 i.e. $\frac{98.25+103}{2}$ =100.625, then the next $\frac{103+107}{2}$ = 105, etc. and position them in between the moving averages as shown in the table

Year	Quarter	Sales	4 Moving totals (M.T)	Moving averages $M.A = \frac{M.T}{4}$	4 centered M.A
2010	1	70			
	2	104			
			403	100.75	
	3	142			101.875
			412	103	
	4	87			105
			428	107	
2011	1	79			109.625
			449	112.25	
	2	120			113.125
			456	114	
	3	163			114.625
			461	115.25	
	4	94			116.625
			472	118	
2012	1	84			122.625
			509	127.25	
	2	131			127.625
			512	128	
	3	200			
	4	97			



Year	Quarter	Traffic offences	4 quarterly moving total	4 quarterly moving average	4 quarterly centered moving average
2000	1	74			
	2	56			
			247	61.75	
	3	48			62.875
			256	64	
	4	69			63.5
			252	63	
2001	1	83			63.125
			253	63.25	
	2	52			64.750
			265	66.25	
	3	49			67.625
			276	69	
	4	81			70.000
			284	71	
2002	1	94			70.875
			283	70.75	
	2	60			70.500
			281	70.25	
	3	48			
	4	79			



The moving averages show a long term increasing trend in the number of traffic offences

Examination questions

2013, No. 11

The table below shows the number of bags of sugar sold by a certain wholesale shop from 2009 to 2012.

Year	QUARTER			
	1st	2nd	3rd	4th
2009	192	280	320	260
2010	300	360	380	270
2011	342	420	430	320
2012	424	480	510	412

(a) Calculate the four-point moving averages for the data

(b) (i) on the same axes, plot the original data and the four-point moving averages

(ii) Comment on the trend of the number of bags of sugar sold over the four-year period

(iii) Use your graph to estimate the number of bags to be sold in the first quarter of 2013.

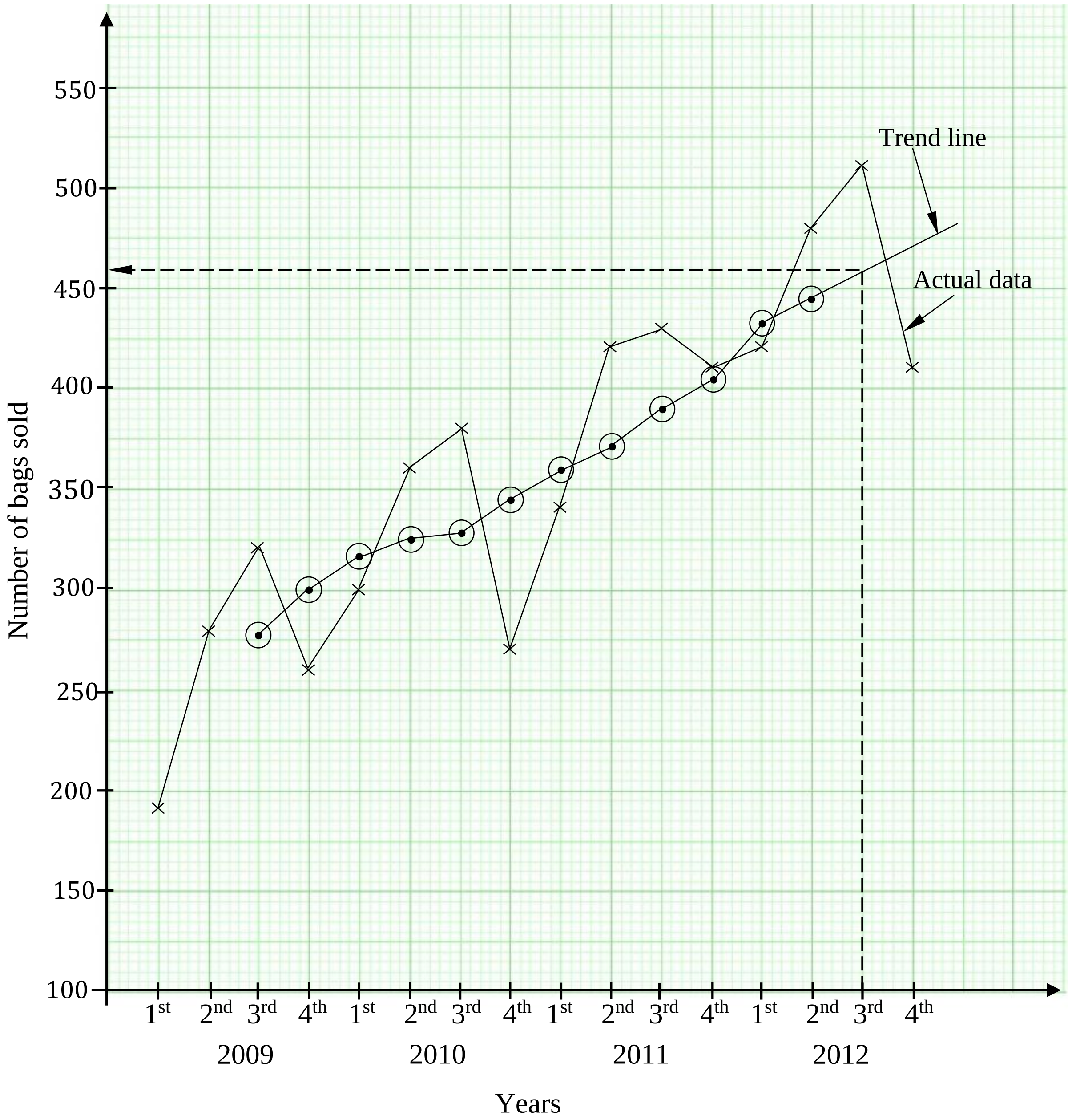
Solution:

(a)

YEAR	Quarter	Sales	4-point M. T	4-point M.A	4-point centered M.A
	1	192			
	2	280			
2009			1052	263	
	3	320			276.5
			1160	290	
	4	260			300
			1240	310	
	1	300			317.5
			1300	325	
2010	2	360			326.25
			1310	327.5	
	3	380			332.75
			1352	338	
	4	270			345.5
			1412	353	
	1	342			359.25
			1462	365.5	
	2	420			371.75
2011			1512	378	
	3	430			388.25
			1594	398.5	
	4	412			406
			1654	413.5	
	1	424			423.5
			1734	433.5	
	2	480			445
2012			1826	456.5	
	3	510			z
				y	
	4	412	$480 + 510 + 412 + x$		
2013					
	1	x			

(b) (i)

A graph showing the number of bags sold and the moving averages



(ii) There is a general increase in the number of bags of sugar sold over the given period
Or there is an upward trend in the number of bags of sugar sold over the period

(iii) Let the number of bags to be sold in the first quarter of 2013 be x and the next 4-point moving average be y and the next 4-point centered moving average be z

From the graph, we estimate $z = 460$

$$\frac{456 + y}{2} = 460$$

$$y = 464$$

$$\frac{480+510+412+x}{4} = 464$$

$$1402 + x = 4 \times 464$$

$$x = 1856 - 1402 = 454$$

The number of bags to be sold would be 454

2017, No. 11

The table below shows quarterly sales of cars for the years 2000, 2001 and 2002 by a company.

YEAR	QUARTER			
	1st	**2nd**	**3rd**	**4th**
2000	390	310	280	355
2001	420	320	305	410
2002	460	350	315	425

(a) Calculate a four-point moving average for the data

(b) (i) Plot the original data and the four-point moving averages on the same axes

(ii) Comment on the trend of sales of the cars

(iii) Use your graph to estimate the number of cars sold in the first quarter of 2003

Solution:

(a)

YEAR	Quarter	Sales	4-point M. T	4-point M. A	4-point centered M. A
2000	1	390			
	2	310			
			1335	333.75	
	3	280			337.5
			1365	341.25	
	4	355			342.5
2001			1375	343.75	
	1	420			346.875
			1400	350	
	2	320			356.875
			1455	363.75	
	3	305			368.75
			1495	373.75	
	4	410			377.5
2002			1525	381.25	
	1	460			382.5
			1535	383.75	
	2	350			385.625
			1550	387.5	
	3	315			z
				y	
	4	425	$350+315+425+x$		
2003					
	1	x			

(b) (i) Check graph on the next page

(ii) The sales of cars generally increase for the period of three years

Or There is an upward trend in the sales of the cars over the period

(iii) Let the number of sales in the first quarter of 2003 be x and the next 4-point moving average be y and the next 4-point centered moving average be z. From the graph, $z = 388$

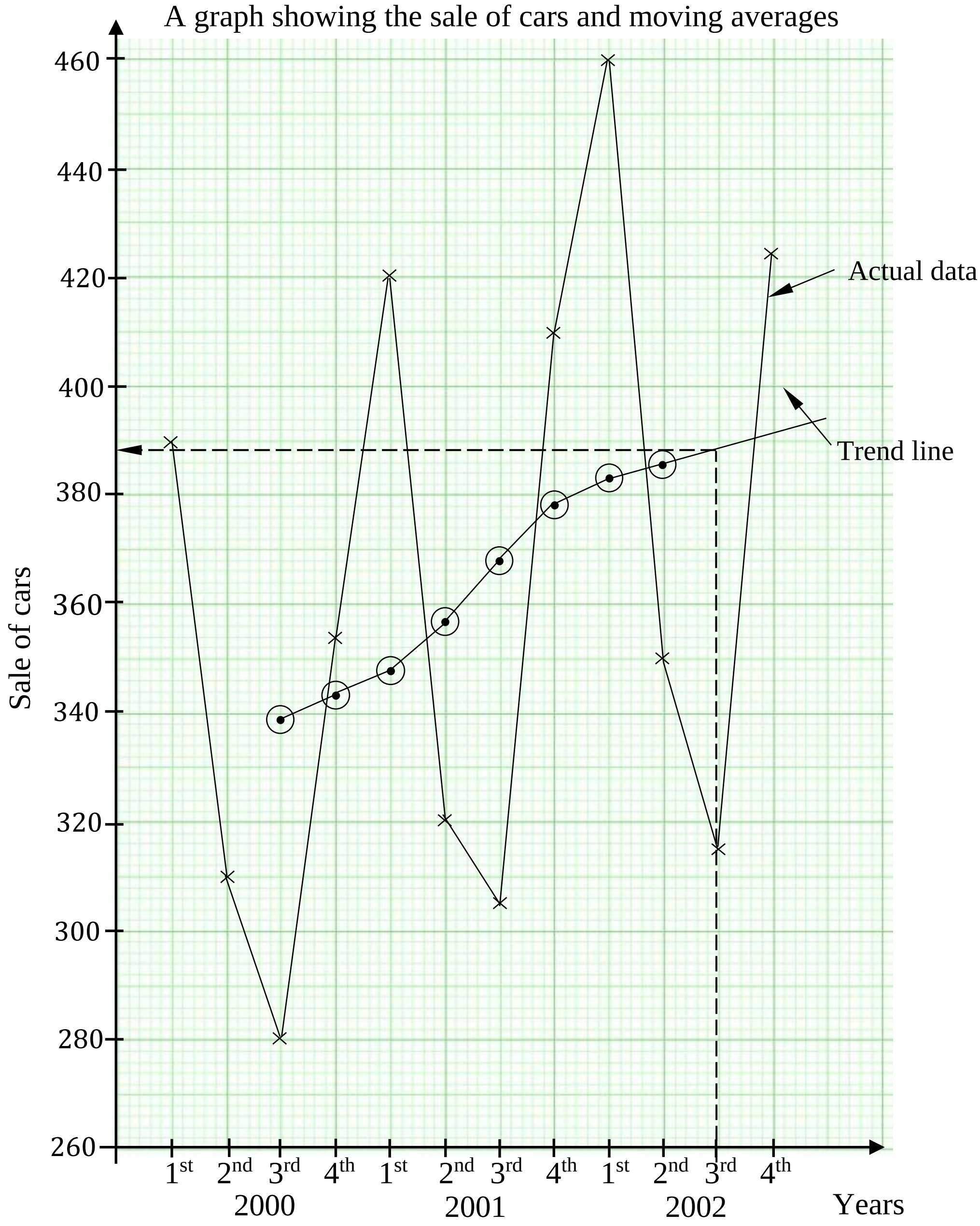
$$\frac{387.5 + y}{2} = 388 \Rightarrow y = 388.5$$

$$\frac{350 + 315 + 425 + x}{4} = 388.5$$

$$1090 + x = 1554$$

$$x = 464$$

Therefore, 464 cars are estimated to be sold in the 1st quarter of the year 2003.

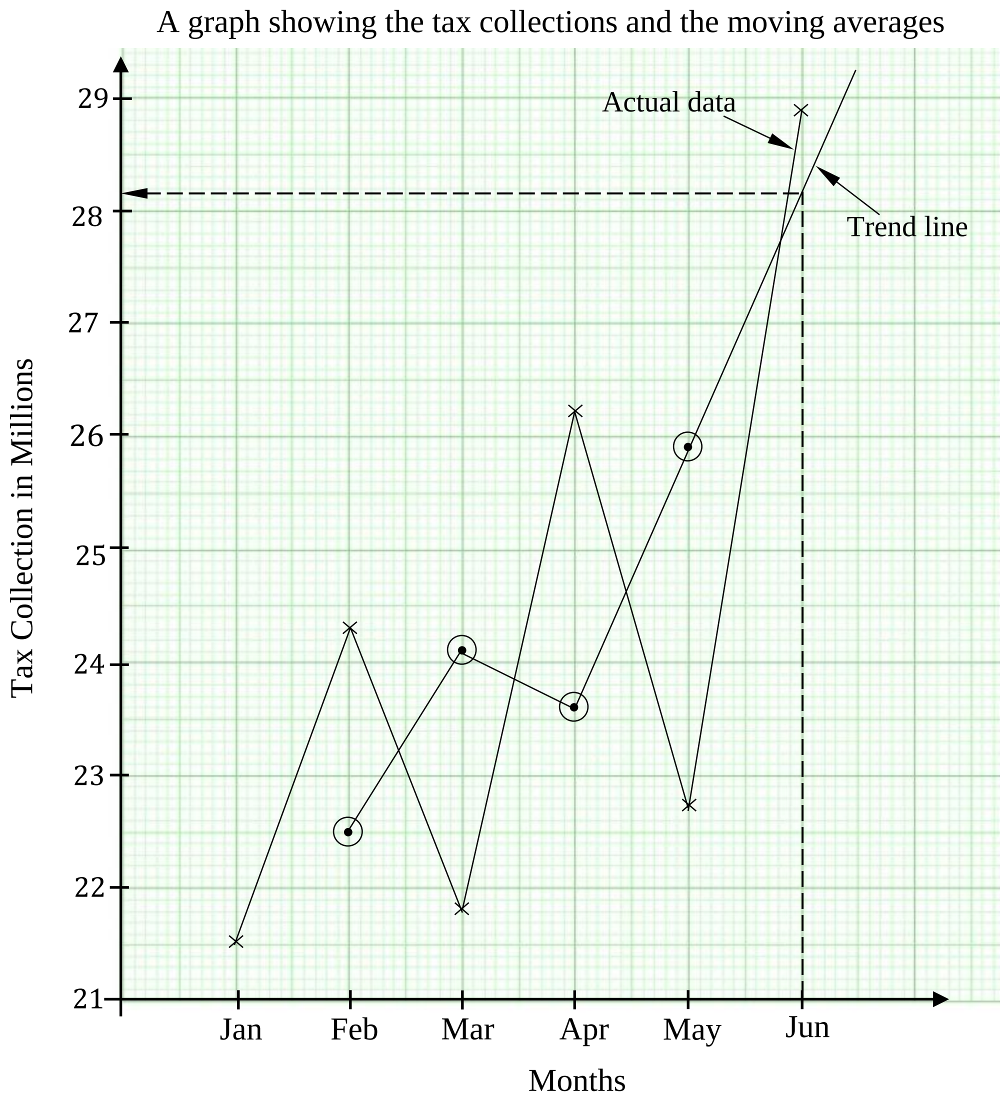


2018, No. 13

The table below shows the sales in thousands of copies by a local Newspaper over a period of 12 weeks.

Week	Number of copies sold
1	318
2	378
3	490
4	430
5	510
6	580
7	565
8	595
9	640
10	660
11	628
12	670

(b)



(c) Let the tax collection for July be x

From the graph, June's moving average = 28.1

$$\frac{22.7 + 28.9 + x}{3} = 28.1$$

$$51.6 + x = 84.3$$

$$x = 32.7$$

The town's tax collection in July will be 32.7 millions.

2022, No. 2

The table below shows the 3-months moving averages for the quantity of goods (in tonnes) manufactured by a certain company from January to August of 2019.

Month	February	March	April	May	June	July
3-month Moving Average (tonnes)	15	17.5	19	20	21.5	22.5

(a) Find the moving totals

(b) If 20 tonnes and 10 tonnes of goods were manufactured in February and March respectively, calculate the quantity that was manufactured in January.

Draw a graph, illustrating these figures. Calculate 5 day moving averages and plot them on the same graph paper.

8. Plot the following data and their four quarterly moving averages of the following quarterly electricity bill (in hundreds) paid in three years.

Quarter / Year	1	2	3	4
1987	39	47	20	56
1988	68	59	66	72
1989	88	60	60	67

9. The sales (in thousands of shillings) of a computer accessories company for the period 2002 to 2004 are given in the table below.

Year	**QUARTERS**			
	1st	**2nd**	**3rd**	**4th**
2002	1235	1242	1410	1400
2003	1275	1270	1450	1480
2004	1302	1280	1510	1500

(a) Calculate the four point moving averages
(b) On the same axes, plot graphs of the sales and the moving averages against time. Comment on the general trend of the sales for the three years period
(c) Use your graph to estimate the sales of computer accessories in the first quarter of 2005.

10. The table below shows the quarterly cost (in 1000's Uganda shillings) of electricity for a house hold over a period of 3 years 1992 – 1994

Year	**QUARTER**			
	1st	**2nd**	**3rd**	**4th**
1992	68	60	59	65
1993	82	80	80	92
1994	94	78	90	105

(a) Calculate the four point moving averages
(b) On the same axes, plot both the raw data and the moving averages
(c) Comment on the trend of electricity over the period of 3 years

11. The table below shows the electricity supplied (in million kilowatt hours) to a company on a quarterly basis between 1988 and 1991

Year	**QUARTER**			
	1st	**2nd**	**3rd**	**4th**
1988	8.9	7.1	6.7	9.3
1989	10.1	7.5	7.1	10.5
1990	11.7	7.5	8.3	16.9
1991	12.5	8.3	9.5	17.7

(a) Calculate the quarterly moving averages
(b) On the same axes, represent the data above and the quarterly moving averages
(c) Comment on the trend of power supply to the company over the four years period

12. The average prices of a bunch of matooke in each third of a year over a period of $3\frac{1}{3}$ years are given in Uganda shillings in the table below.

Year	**1st third**	**2nd third**	**3rd third**
1998	4500	5000	5200
1999	5500	5700	6000
2000	6200	6500	6800
2001	7000	***X***	

(iii) Comment on the trend of the school fees over the given period.

18. The table below shows the monthly sales of a certain product (in kg) in the year 1995

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales	672	636	680	704	744	700	756	784	828	800	840	880

(a) Calculate the 6-point moving averages

(b) Plot on the same axes the actual sales and the moving averages. Comment on the trend of sales during the year.

19. The table below shows the annual production of oil in millions of litres by a company for a period 1998 – 2006

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006
Production	195	145	170	177	154	152	132	155	167

(a) Construct a 5-year moving averages for the oil production

(b) On the same axes, plot the graphs of annual production and moving averages

(c) Comment on the general trend of oil production over the year period

20. The table below shows the amount of milk (in thousands of litres) produced by a certain exotic farm in yearly quarters for the 1986 – 1989 period.

Year	**QUARTER**			
	1st	**2nd**	**3rd**	**4th**
1986	19.5	30.0	32.5	25.0
1987	30.5	37.0	38.5	26.5
1988	36.5	44.5	46.6	35.0
1989	45.5	50.5	52.5	42.5

(a) Calculate the four point moving averages for the data

(b) On the same axes, plot the four point moving averages and the original data

(c) (i) comment on the trend of milk production over the period of 4 years

(ii) Use your graph to estimate the amount of milk that will be plotted in the 1st quarter of 1990.

21. The table below indicates the quarterly variation in a certain school earnings in millions of shillings from 1950 – 1952.

Year	**QUARTER**			
	1st	**2nd**	**3rd**	**4th**
1950	11.0	10.0	10.0	9.4
1951	10.5	9.7	9.4	9.3
1952	9.9	9.3	9.0	8.6

(a) Calculate the quarterly moving averages for the data

(b) On the same axes, plot the four point moving averages and the original data

(c) (i) Comment on the trend of the school earnings over the period

(ii) Estimate the earning that will be recorded in the first quarter of 1953

22. The number of traffic offences committed in a certain city over a period of 3 years is given in the following table.

Year	Jan – March	April – June	July – Sept	Oct – Dec
1980	74	56	48	69
1981	83	52	49	81
1982	94	60	48	79

Calculate four quarterly centered moving averages

Chapter 14 Index Numbers

Introduction

Index numbers are statistical economic indicators which provide a measure of the relative change in some variable or group of variables. They are widely used in almost all areas of economic activities.

The most common type of index number is price index number which measures variation in prices over a span of time. It enables us to know how the price level of a group of commodities has changed at certain period of time as compared to another period, called **base period**. When the price of one commodity rises while the price of another falls and the prices of various commodities all react in different degrees, the index number shall not give here any indication of changes in the values of the individual commodity but will reveal the average net effect of all the changes. Retail price index, wholesale price index, index of wage rates (wages being the price of labour) are some of price indices

Similarly, a quantity index enables us to know the average changes in the quantities of the items belonging to a group of commodities. An index of industrial production or an index of the volume of exports are some of the examples of quantity index.

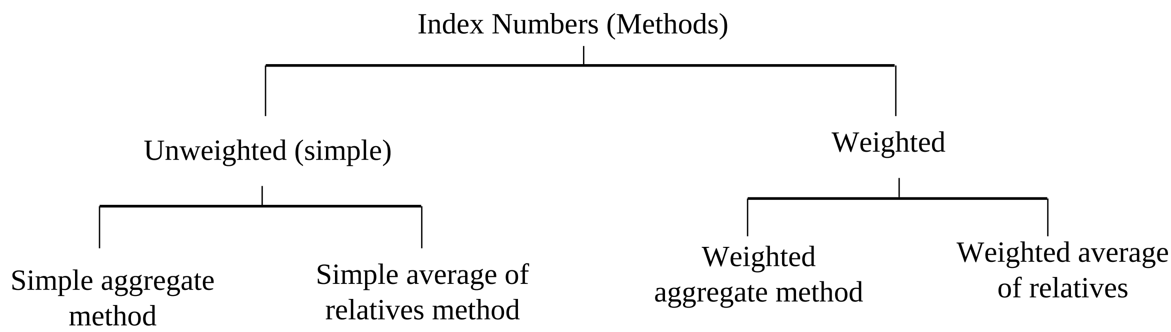
Uses of index numbers

Index numbers are today one of the most important statistical tools and are widely used in a number of economical activities especially in measuring relative changes. Some of the uses are as follows:

- They measure the relative change. Index numbers help in getting better idea of changes in the level of prices, production, employment, etc. and many of the policies are guided by index numbers
- They reveal trends and tendencies
- They help in framing suitable policies
- They help in making comparisons by reducing the changes to percentages which are easily comparable.
- They act as indicators of inflationary or deflationary tendencies. They act as a sort of economic barometers, indicating the state of affairs in various sectors of economy.

Construction of index numbers

The various methods of constructing index numbers can be grouped under two heads:
(a) Unweighted indices, and (b) Weighted indices.
In the unweighted indices, weights are not expressly assigned to various items. The different methods available are depicted in the diagram below.



Before taking up the weighted method of constructing index numbers, we will explain the term Price Relative.

Price Relative

A price relative or simply a relative is the ratio of the price of a certain commodity in a given period to its price in an earlier fixed period called the base period or reference period expressed as a percentage.

$$\text{Price relative} = \frac{\text{current price}}{\text{base price}} \times 100$$

$$\text{or Price relative, } I = \frac{p_1}{p_0} \times 100$$

where p_0 and p_1 denote the commodity price during the base period and given period respectively.

For example, let sugar be sold at Ug. Shs 1,000 per kg in the year 1998 and Ug. Shs 1,500 per kg in the year 2000, then

Relative price in 2000 with 1998 as base year = $\frac{1500}{1000} \times 100 = 150$

It means that if the price was Ug. Shs 10,000 in the year 1998, it would have been 15,000 in the year 2000

Note: The statement "$2012 = 100$" can be used to identify the base period in this case which means that 2012 is the base year.

Unweighted methods of constructing index numbers

There are two methods of constructing unweighted index numbers

1. Simple aggregate method
2. Simple average of price relative method

Simple aggregate method

This is the simplest method of constructing index numbers and is obtained as follows:

Consider the following table of prices of commodities consumed in a working class family in 1949 and 1950.

Commodity	Price per unit (Ug. Shs)	
	1949	1950
Bread and Flour	40	50
Meat	240	300
Potatoes	20	30
Tea	360	360
Sugar	50	60
Butter	180	270
Margarine	120	150
Eggs	300	450
Total	1310	1670

If the total prices in 1949 is represented by 100, then the total of prices in 1950 will be represented by $\frac{167}{131} \times 100$ or 127.5 which is called the simple aggregate price index for 1950, taking 1949 as the base year.

Procedure

Step 1: Add the prices of the different commodities of the current year

Step 2: Add the prices of these commodities of the base year

Step 3: Divide the total obtained in step 1 by the total obtained in step 2 and multiply the quotient by 100

The result gives the required price index number

Symbolically;

$$\textbf{Simple Aggregate Price Index} = \frac{\sum p_1}{\sum p_0} \times 100$$

where

$\sum p_1$ denotes the total of current year prices of all commodities under consideration.

$\sum p_0$ denotes the total of base year prices of the same commodities as considered for $\sum p_1$

Example 1

A small industrial firm used three raw materials A, B and C in its manufacturing process. The prices in USD of the materials are as shown below.

Commodity	Price in the year 1995	Price in the year 2005
A	4	5
B	60	57
C	36	42

Using 1995 as the base year, calculate a simple aggregate price index for 2005.

Solution:

Commodity	p_0	p_1
A	4	5
B	60	57
C	36	42
$\sum$	100	104

$$\text{Simple Aggregate Price Index} = \frac{\sum p_1}{\sum p_0} \times 100 = \frac{104}{100} \times 100 = 104$$

Example 2

Compute a price index for the following by simple aggregate method.

Commodity	A	B	C	D	E	F
Price in 1986	20	30	10	25	40	50
Price in 1991	25	30	15	35	45	55

Solution:

$\sum p_0 = 20 + 30 + 10 + 25 + 40 + 50 = 175$

$\sum p_1 = 25 + 30 + 15 + 35 + 45 + 55 = 205$

$$\text{Simple Aggregate Price Index} = \frac{\sum p_1}{\sum p_0} \times 100 = \frac{205}{175} \times 100 = 117.143$$

Example 3

From the data given below, calculate the index number of prices for the years 1981 to 1983 with reference to 1980 as the base year using the Simple aggregate method.

Items	Prices			
	1980	1981	1982	1983
Rice	56.4	58.7	57.2	60.4
Wheat	48.2	50.3	51.7	53.3
Posho	121.3	124.6	130.3	133.5

Solution:

Let p_0, p_1, p_2 and p_3 represent prices in 1980, 1981, 1982 and 1983 respectively

Simple average of price relative method

In this method, the price relative of each commodity is calculated separately and then averaged. The average maybe determined by using any measure of central value. If n is the number of items (commodities), then

$$\textbf{Simple price index} = \frac{\sum\left(\frac{p_1}{p_0} \times 100\right)}{n}$$

Example 5

Construct by simple average of price relative method the price index of 2004, taking 1999 as the base year from the following data

Item	A	B	C	D	E	F
Price (in 1999)	6000	5000	6000	5000	2500	2000
Price (in 2004)	8000	6000	7200	7500	3750	3000

Solution:

Item	Price in 1999 (p_0)	Price in 2004 (p_1)	Price relatives $\left(\frac{p_1}{p_0} \times 100\right)$
A	6000	8000	133.33
B	5000	6000	120.00
C	6000	7200	120.00
D	5000	7500	150.00
E	2500	3750	150.00
F	2000	3000	150.00
			823.33

$$\text{Simple price index} = \frac{\sum \frac{p_1}{p_0} \times 100}{n} = \frac{823.33}{6} = 137.22$$

Example 6

Find the simple price index for 2001 taking 1996 as the base year from the following data

Commodity	Wheat	Rice	Sugar	Ghee	Meat
Price (1996)	1200	2000	1200	4000	8000
Price (2001)	1600	2500	1600	6000	9600

Solution:

Commodity	Price in 1996 (p_0)	Price in 2001(p_1)	Price relatives
Wheat	1200	1600	133.33
Rice	2000	2500	125.00
Sugar	1200	1600	133.33
Ghee	4000	6000	150.00
Meat	8000	9600	120.00
$\sum$			661.66

$$\text{Simple price index} = \frac{\sum \frac{p_1}{p_0} \times 100}{n} = \frac{661.66}{6} = 132.33$$

Example 7

Calculate the index number for the year 2005 with 2000 as the base year by weighted aggregate method from the following data.

Commodity	Price in Ug. Shs		Weights
	2000	2005	
A	140	180	10
B	400	550	7
C	100	250	6
D	125	150	8
E	200	300	4

Solution:

Commodity	p_0	p_1	w	p_1w	p_0w
A	1400	1800	10	18000	14000
B	4000	5500	7	38500	28000
C	1000	2500	6	15000	6000
D	1250	1500	8	12000	10000
E	2000	3000	4	12000	8000
$\sum$				95500	66000

$$\text{Weighted Aggregate Price Index} = \frac{\sum p_1 w}{\sum p_0 w} \times 100 = \frac{95500}{66000} \times 100 = 144.69$$

Hence, the index number for the year 2005, with 2000 as the base year, is 144.69

Example 8

Construct index number for price for the year 2007 with 2005 as the base year from the following data by taking quantities in the base year as weights.

Commodity	2005		2007	
	Price	Quantity	Price	Quantity
A	200	8	400	6
B	500	10	600	5
C	400	14	500	10
D	200	19	200	3

Solution:

Commodity	2005		2007			
	Price (p_0)	Quantity (q_0)	Price (p_1)	Quantity (q_1)	p_1q_0	p_0q_0
A	200	8	400	6	3200	1600
B	500	10	600	5	6000	5000
C	400	14	500	10	7000	5600
D	200	19	200	3	3800	3800
$\sum$					20000	16000

$$\text{Price index} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100 = \frac{20000}{16000} \times 100 = 125$$

Weighted average of price relative method

In this method, we make use of price relatives. When the base and current prices of a number of items along with weights or quantities are given, their weighted average of price relatives is given by

$$\textbf{Weighted Average Price Index} = \frac{\sum\left(\frac{p_1}{p_0}\times 100\right)\times w}{\sum w} = \frac{\sum(\text{price relative} \times \text{weights})}{\sum(\text{weight})} = \frac{\sum wI}{\sum w}$$

$$\text{or } \textbf{Weighted Average Price Index} = \frac{\sum \frac{p_1}{p_0} w}{\sum w} \times 100$$

If p_0 and q_0 denote respectively the price and quantity of a commodity in the base year, then its weight (w) is given by $p_0 q_0$. So, we can rewrite the above formula as

$$\text{Weighted Average Price Index} = \frac{\sum \frac{p_1}{p_0} \times p_0 q_0}{\sum p_0 q_0} \times 100$$

Note: Compute base year values $p_0 q_0$ for each commodity and consider them as weights if weights are not given.

Example 9

The quotations for four different commodities for the years 2000 and 2005 are given below. Calculate the index number for 2005, with 2000 as the base year, by using the weighted average of price relatives method and comment on your result.

Commodity	Weight	Price in the year 2000	Price in the year 2005
A	5	200	450
B	7	250	320
C	6	300	450
D	2	100	180

Solution:

Commodity	Weight	p_0	p_1	$I = \frac{p_1}{p_0} \times 100$	wI
A	5	200	450	225	1125
B	7	250	320	128	896
C	6	300	450	150	900
D	2	100	180	180	360
$\sum$	20	200	450		3281

$$\text{Weighted Average Price Index} = \frac{\sum wI}{\sum w} = \frac{3281}{20} = 164.05$$

From the above result, there is an increase of 64.05% in the prices of the commodities.

Example 10

Taking 1975 as the year, with an index number 100, calculate the index number for 1979, based on weighted average of price relatives derived from the table given below.

Commodity	A	B	C	D
Weight	30	15	25	30
Price per unit in 1975	20	10	5	40
Price per unit in 1979	24	20	30	40

Solution:

Items	p_0	p_1	w	$\frac{p_1}{p_0}$	$\frac{p_1}{p_0}w$
Rice	180	200	11	1.111	12.221
Wheat	140	170	5	1.214	6.070
Pulse	480	500	4	1.042	4.168
Fish	140	160	1	1.142	1.142
Total			21	4.509	23.601

Simple average of price relative index number

$$= \frac{\sum \frac{p_1}{p_0}}{n} \times 100 = \frac{4.509}{4} \times 100 = 112.7$$

Weighted average of Price Relative Index Number

$$= \frac{\sum \frac{p_1}{p_0} w}{\sum w} \times 100 = \frac{23.601}{21} \times 100 = 112.4$$

Example 13

Taking 1975 as the base year, with an index number 100, calculate an index number for 1979, based on weighted average of price relatives from the table given below:

Commodity	A	B	C	D
Weight	30	15	25	30
Price per unit in 1975	20	10	5	40
Price per unit in 1979	24	20	30	40

Solution:

Commodity	w	p_0	p_1	$\frac{p_1}{p_0}$	$\frac{p_1}{p_0}w$
A	30	20	24	1.2	36
B	15	10	20	2.0	30
C	25	5	30	6.0	150
D	30	40	40	1.0	30
$\sum$	100				246

$$\therefore \text{Price Index Number} = \frac{\sum \frac{p_1}{p_0} w}{\sum w} \times 100 = \frac{246}{100} \times 100 = 246$$

Cost of living index

A cost-of-living index would measure changes over time in the amount that consumers need to spend to reach a certain "utility level" or "standard of living.

Cost of living data includes the expenses incurred for food, shelter, transportation, energy, clothing, education, health care, childcare, entertainment, etc. A cost of living index tracks how much basic expenses rise over time and among different regions.

The cost of living index is calculated using the weighted average of price relative method.

Example 14

Find the cost of living index based on the following data

Examination Questions

2013, No.4

The table below shows the prices of items and their corresponding weights in years 2000 and 2004.

Items	Price (Ushs)		Weight
	2000	2004	
Food	55,000	60,000	4
Housing	48,000	52,000	2
Transport	16,000	20,000	1

Using 2000 as the base year, calculate the weighted price index for the items in 2004.

Solution:

Weighted aggregate price index = $\frac{\sum P_1 W}{\sum P_0 W} \times 100$

$$= \frac{60000\times4+52000\times2+20000\times1}{55000\times4+48000\times2+16000\times1} \times 100$$

$$= \frac{364000}{332000} \times 100 = 109.64$$

Alternatively;

Weighted average price index = $\frac{\sum \frac{P_1}{P_0} \times W}{\sum W} \times 100$

$$= \frac{\frac{60000}{55000}\times4+\frac{52000}{48000}\times2+\frac{20000}{16000}\times1}{4+2+1} \times 100 = 111.11$$

2014, No. 11

(a) the table below shows the price (U Shs) of flour and eggs in the years of 2000 and 2010

COMMODITY	PRICE (U Shs)	
	2000	2010
Flour (kg)	3000	5000
Eggs (1 tray)	5000	7000

Taking 2000 as the base year, calculate the

(i) price relative of each commodity

(ii) simple aggregate price index

Comment on your result

(b) the data below shows items with their corresponding price relatives and weights

ITEM	PRICE RELATIVE	WEIGHT
Food	120	172
Clothing	124	160
Housing	125	170
Transport	135	210
Others	104	140

(i) Find the cost of living index

(ii) Comment on your result

Solution:

(a) (i) Price relative for flour = $\frac{5000}{3000} \times 100 = 166.67$

Price relative for eggs = $\frac{7000}{5000} \times 100 = 140$

(ii) Simple aggregate price index = $\frac{\sum P_1}{\sum P_0} \times 100$

Total price in 2010 i.e. $\sum P_1 = 5000 + 7000 = 12000$

Total price in 2000 i.e. $\sum P_0 = 3000 + 5000 = 8000$

Simple aggregate price index = $\frac{12000}{8000} \times 100 = 150$

Solution:

(a)

Component	p_0	p_1	w	$\frac{p_1}{p_0}\times 100$	p_0w	p_1w
A	35	60	6	171.4	175	360
B	70	135	5	192.9	350	675
C	43	105	3	244.2	129	315
D	180	290	2	161.1	360	580
E	480	800	1	166.7	480	800
$\sum$	808	1390			1494	2730

$$\text{Simple aggregate price index} = \frac{\sum p_1}{\sum p_0}\times 100 = \frac{1390}{808}\times 100 = 172.03$$

$$\text{Weighted aggregate price index} = \frac{\sum p_1 w}{\sum p_0 w}\times 100 = \frac{2730}{1494}\times 100 = 182.73$$

(b) Let the cost of an engine in 1998 be x

$$\frac{1600}{x}\times 100 = 182.73$$

$$x = \frac{160000}{182.73} = 875.61$$

∴ The cost of an engine in 1998 was 875.61 US Dollars

2017, No. 4

The table below shows the expenditures in shillings of a University student for the years 2005 and 2006.

ITEM	**EXPENDITURE (Shs)**		**WEIGHT**
	2005	2006	
Text books	100,000	120,000	3
Pocket money	50,000	70,000	2
Research	40,000	50,000	1

Using the year 2005 as the base year, calculate the weighted aggregate price index.

Solution

$$W.A.P.I = \frac{\sum P_1 W}{\sum P_0 W}\times 100$$

$$\sum P_1 W = 120000(3) + 70000(2) + 50000(1) = 550000$$

$$\sum P_0 W = 100000(3) + 50000(2) + 40000(1) = 440000$$

$$W.A.P.I = \frac{550000}{440000}\times 100 = 120$$

2022, No. 13

The table below shows the expenditure of a family for the months of January and July in a certain year.

ITEM	EXPENDITURE (Shs)		WEIGHT
	JANUARY	JULY	
Food	150,000	174,000	8
Rent	50,000	60,000	2
Clothing	100,000	125,000	6
Power	20,000	25,000	1
Water	60,000	90,000	4

(a) Calculate the

(i) Price relative for each item

(ii) Simple aggregate index

(b) (i) Find the weighted aggregate price index

(ii) Comment on your result in (b) (i)

Solution:

Item	p_0	p_1	w	$\frac{p_1}{p_0} \times 100$	$p_0 w$	$p_1 w$
Food	150,000	174,000	8	116	1,200,000	1,392,000
Rent	50,000	60,000	2	120	100,000	120,000
Clothing	100,000	125,000	6	125	600,000	750,000
Power	20,000	25,000	1	125	20,000	25,000
Water	60,000	90,000	4	150	240,000	360,000
$\sum$	380,000	474,000			2,160,000	2,647,000

$$\text{Simple aggregate price index} = \frac{\sum p_1}{\sum p_0} \times 100 = \frac{474000}{380000} \times 100 = 124.74$$

$$\text{Weighted aggregate price index} = \frac{\sum p_1 w}{\sum p_0 w} \times 100 = \frac{2647000}{2160000} \times 100 = 122.55$$

There is an increase of 22.55% in the price of items from January to July

Self-Evaluation Exercise

1. The average price of mustard oil per quintal in the year 1984 to 1988 are given below:

Year	1984	1985	1986	1987	1988
Price	295	275	300	225	250

Find the index numbers for all the years taking 1986 as the base year.

[Ans: 98.3, 91.7, 100, 75, 75, 83.3]

2. Find the index number by method of relatives from the following data

Commodity	Base Price	Current price
Rice	35	42
Wheat	30	35
Pulse	40	38
Fish	105	120

[Ans: 111.5]

3. The following table gives the average whole sale price in US $ for three commodities during the years 1985-90. Construct index numbers for all the three years using price relative method taking 1985 as the base year.

Commodity	1985	1986	1987	1988	1989	1990
A	25.3	30.8	33.4	35.5	35.3	36.0
B	17.3	14.5	4.9	5.7	17.1	11.6
C	7.8	5.4	6.7	5.6	7.2	10.2

[Ans: 100, 92, 82.82, 111, 113]

9. The cost of making a cake is calculated from the cost of baking flour, sugar, milk and eggs. The following table gives the cost of these items in 1985 and 1986

Item	Price 1985	Price 1986	Weight
Flour per kg	60	78	12
Sugar per kg	50	40	5
Milk per litre	25	30	2
Egg per egg	10	15	1

Using 1985 as the base year,

(i) Calculate the price index for each item hence find the simple price index of making a cake

(ii) Find the weighted aggregate price index for the cost of a cake

(iii) If the cost of making a cake in 1986 was shs 30, find the cost in 1985 using the two indices in (i) and (ii)

[Ans: (i) 120 (ii) 117.57 (iii) 26/=]

10. The prices per unit (in U shs) of four food stuffs A, B, C and D in December 2004 and December 2005 are shown in the following table

Food stuff	Price (U shs) in December	
	2004	2005
A	635	887.5
B	720	815
C	730	1045
D	362	503

The weights of the food stuffs A, B, C and D are 6, 4, 3 and 7 respectively. Taking 2004 as the base year, calculate for 2005 the

(a) price relative for each food stuff hence the simple price index

(b) (i) weighted price index

(ii) price of food stuff costing shs 500 in December 2004 using the weighted aggregate index.

[Ans: (a) (ii) 133.7 (b) (i) 133.53 (ii) 667.64/=]

11. A family's monthly shopping list in 2015 and 2016 included the following

		2015	2016
Item	Quantity	Unit price (Shs)	
Beans	10 kg	2,600	3,000
Rice	15 kg	3,500	4,200
Eggs	3 trays	9,000	10,000
Sugar	20 kg	3,600	5,000
Soap	4 bars	4,000	4,500

(a) Compute the cost of living index using 2015 as base year

(b) Comment on the result obtained in (a) above [Ans: 124.6]

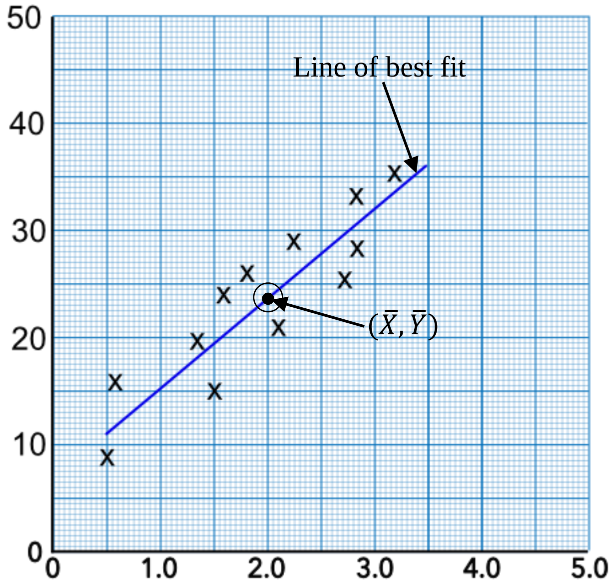
12. Construct an index number of price from the following data by applying (i) Laspeyres' Method (ii) Paasche's Method

Items	Base year		Current year	
	Price	Quantity	Price	Quantity
A	200	8	400	6
B	500	10	600	5
C	400	14	500	10
D	200	19	200	13

[Ans: Laspeyres' index = 125, Paasche's index = 126.2]

The line of best fit

Most often there is not a straight line which often passes through all the points but we can still draw a straight line which comes closest to finding all the points. We can estimate the position of the line by the eye. This line must pass through the mean point i.e. $(\bar{X}, \bar{Y})$ where $\bar{X} = \frac{\sum x}{n}$ and $\bar{Y} = \frac{\sum y}{n}$



Spearman's rank correlation coefficient

The correlation coefficient between two series of ranks is called "Rank Correlation coefficient". The formula for coefficient of rank correlation is given as

$$r = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

where d is the difference between the corresponding ranks of the two series and n is the number of individuals in each series.

Note:

1. Instead of assigning ranks 1, 2, 3, … from the highest to the lowest, we can also assign these ranks from the lowest to the highest i.e. rank 1 to the least intelligent, rank 2 to the next more intelligent, next rank 3, and so on.
2. Remember that the algebraic sum of the rank differences is always 0 i.e. $\sum d = 0$. If it is not so, then some mistake has been committed at the time of assigning ranks.

Interpretation of correlation coefficient

1. The coefficient of correlation shall always be between -1 and $+1$
2. When r is $+1$, there is perfect correlation between the variables
3. When r is -1, there is perfect negative correlation between the variables
4. When r is between 0.7 to 0.999, there is a high degree of correlation between the variables
   The correlation shall be positive if the sign of r is $(+)$ and negative if the sign of r is $(-)$
5. When r is between 0.5 and 0.699, there is a moderate degree of correlation between the variables
6. When r is less than 0.5, there is a low degree of correlation between the variables
7. When r is zero, there is no correlation between the variables

Solved examples

We may come across two types of problems i.e. (i) when ranks are given (ii) when ranks are not given

Example 1

Following are the ranks obtained by 10 students in two subjects, Statistics and Mathematics. To what extent is the knowledge of the students in the two subjects related.?

Statistics	1	2	3	4	5	6	7	8	9	10
Mathematics	2	4	1	5	3	9	7	10	6	8

Solution:

Rank of Statistics (R_x)	Rank of Mathematics (R_y)	$d = R_x - R_y$	d^2
1	2	-1	1
2	4	-2	4
3	1	2	4
4	5	-1	1
5	3	2	4
6	9	-3	9
7	7	0	0
8	10	-2	4
9	6	3	9
10	8	2	4
			$\sum d^2 = 40$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 40}{10(10^2-1)} = 1 - \frac{240}{990} = 0.76$$

Caution: When the ranks are already given as in the above example, do not commit the mistake of assigning new ranks.

Example 2

The marks obtained by students in Physics and Mathematics are as follows:

Marks in Physics	35	23	47	17	10	43	9	6	28
Marks in Mathematics	30	33	45	23	8	49	12	4	31

Compute the ranks in the two subjects and coefficient of rank correlation. Interpret the result.

Solution:

When no ranks are given, but actual data are given, then we should assign ranks. We can give ranks by taking the highest as 1 or the lowest value as 1, next to the highest (lowest) as 2 and follow the same procedure for both variables

x	y	R_x	R_y	$d = R_x - R_y$	d^2
35	30	3	5	-2	4
23	33	5	3	2	4
47	45	1	2	-1	1
17	23	6	6	0	0
10	8	7	8	-1	1
43	49	2	1	1	1
9	12	8	7	1	1
6	4	9	9	0	0
28	31	4	4	0	0
					$\sum d^2 = 12$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 12}{9(9^2-1)} = 1 - \frac{72}{720} = 0.9$$

The high value of r indicates a very high correlation. This means that the students who are good in Physics are also good in Mathematics and vice-versa.

Note: We could have started by assigning ranks from lowest value, i.e. rank 1 to 54, rank 2 to 55, then rank $\frac{3+4}{2}$ = 3.5 to each of the values 69, rank 5 to 72, rank 6 to 78, then rank $\frac{7+8+9+10}{4}$ = 8.5 to each of the values 80, rank 11 to 85 and lastly rank 12 to 90. Then, the ranks would have been as below.

Series (X)	90	85	80	80	80	80	78	72	69	69	55	54
Rank assigned	12	11	8.5	8.5	8.5	8.5	6	5	3.5	3.5	2	1

Example 4

Find the rank correlation coefficient between the heights of fathers and sons from the following data:

Height of fathers in inches	65	66	67	67	68	69	70	72
Height of sons in inches	67	68	65	68	72	72	69	71

Solution:

x	y	R_x	R_y	$d = R_x - R_y$	d^2
65	67	8	7	1	1
66	68	7	5.5	1.5	2.25
67	65	5.5	8	−2.5	6.25
67	68	5.5	5.5	0	0
68	72	4	1.5	2.5	6.25
69	72	3	1.5	1.5	2.25
70	69	2	4	−2	4
72	71	1	3	−2	4
				$\sum$	26

In the x-series, 67 occurs twice and its rank is $\frac{5+6}{2} = 5.5$

In the y-series, 72 occurs twice and its rank is $\frac{1+2}{2}$ = 1.5

68 occurs twice and its rank is $\frac{5+6}{2}$ = 5.5

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 26}{8(8^2-1)} = 1 - \frac{156}{504} = 0.69$$

Example 5

Find the spearman's rank correlation coefficient from the following data

X	48	33	40	9	16	16	65	25	16	57
Y	13	13	24	6	15	4	29	9	6	19

Solution:

x	y	R_x	R_y	$d = R_x - R_y$	d^2
48	13	8	5.5	2.5	6.25
33	13	6	5.5	0.5	0.25
40	24	7	10	−3	9
9	6	1	2.5	−1.5	2.25
16	15	3	7	−4	16
16	4	3	1	2	4
65	29	10	9	1	1
25	9	5	4	1	1
16	6	3	2.5	0.5	0.25
57	19	9	8	1	1
				$\sum$	41

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 41}{10(10^2-1)} = 1 - \frac{246}{990} = 0.752$$

Examination questions

2013, No. 9

Eight candidates seeking admission to a university course sat for written and oral tests. The scores were as shown in the table below:

Written(X)	55	54	35	62	87	53	71	50
Oral(Y)	57	60	47	65	83	56	74	63

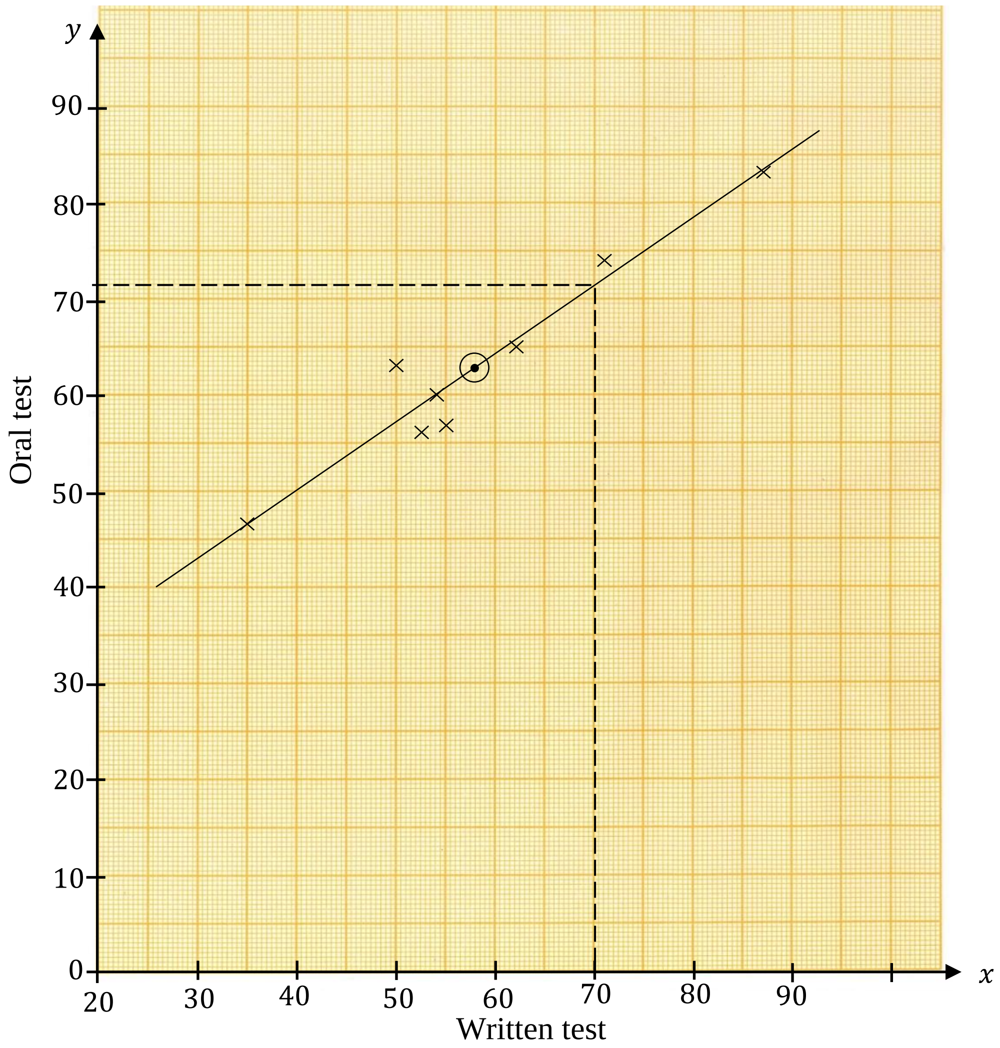
(a) (i) Draw a scatter diagram for the data
(ii) Draw a line of best fit on your scatter diagram
(iii) Use the line of best fit to find the value of Y when $X = 70$

(b) Calculate Spearman's rank correlation co-efficient. Comment on your result.

Solution:

(a) (i)

A scatter diagram showing the scores in the written and oral tests



(ii) The line of best fit should pass through the point $(\bar{X}, \bar{Y})$ where $\bar{X} = \frac{\sum X}{n}$ and $\bar{Y} = \frac{\sum Y}{n}$

$$\bar{X} = \frac{467}{8} = 58.4 \quad \text{and} \quad Y = \frac{505}{8} = 63.1$$

The line of best fit passes through $(58.4, 63.1)$

(iii) From the graph, it is estimated that when $X = 70, Y = 71$

(b)

Written(X)	Oral (Y)	R_X	R_Y	$d = R_X - R_Y$	d^2
55	57	4	6	−2	4
54	60	5	5	0	0
35	47	8	8	0	0
62	65	3	3	0	0
87	83	1	1	0	0
53	56	6	7	−1	1
71	74	2	2	0	0
50	63	7	4	3	9
					$\sum d^2 = 14$

Spear man's rank correlation coefficient, $r = 1 - \frac{6\sum d^2}{n(n^2-1)}$

$$= 1 - \frac{6\times 14}{8(8^2-1)} = 1 - \frac{84}{8(63)} = 1 - \frac{84}{504} = 1 - 0.167 = 0.833$$

Comment: There is a high positive correlation between the two tests

2014, No. 9

The table below shows the marks of 8 students in the mid-term test and end of term test in economics.

Mid-term tests (x)	99	71	50	67	77	81	96	72
End of term test (y)	99	55	35	60	75	70	99	50

(a) (i) draw a scatter diagram for the data

(ii) on the same diagram draw a line of best fit

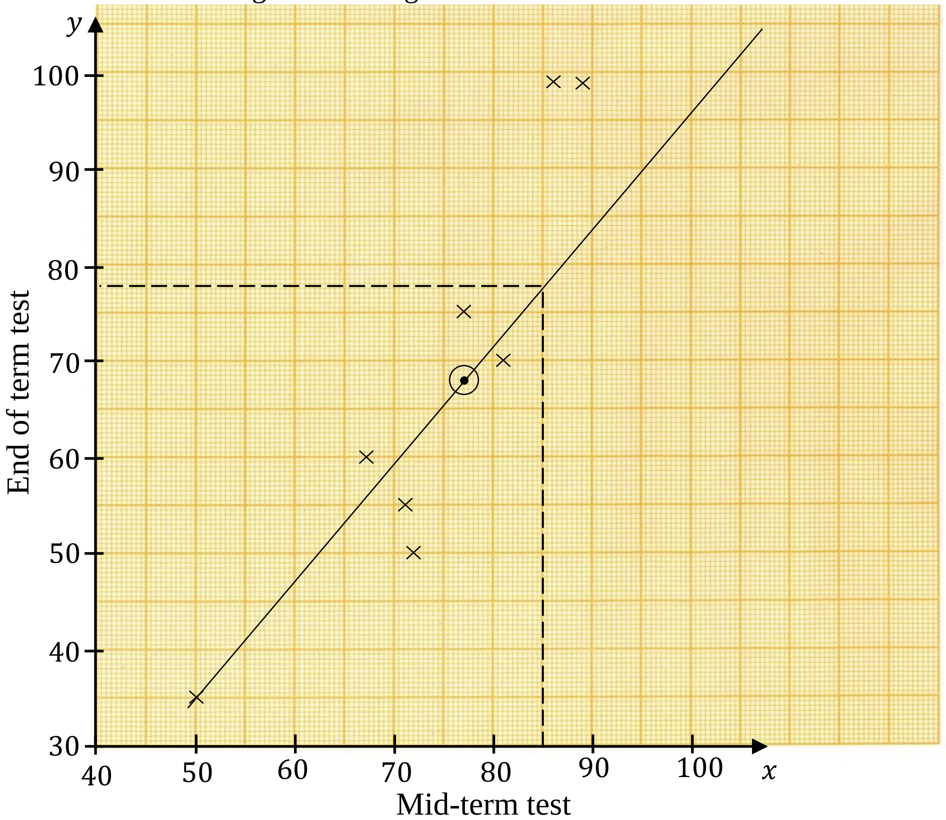
(iii) Use the line of best fit to find the value of y when $x = 85$

(b) Calculate the spearman's rank correlation coefficient. Comment on your result

Solution:

(a) (i)

A scatter diagram showing the scores in Mid-term and End of term tests



2018, No. 9

The table below shows scores by 10 students (A to J) in Physics and Mathematics tests

Student	Mathematics (x)	Physics (y)
A	28	30
B	20	20
C	40	40
D	28	28
E	21	22
F	31	35
G	36	35
H	29	27
I	33	31
J	24	23

(a) (i) Plot a scatter diagram for the given data

(ii) Draw a line of best fit on the scatter diagram

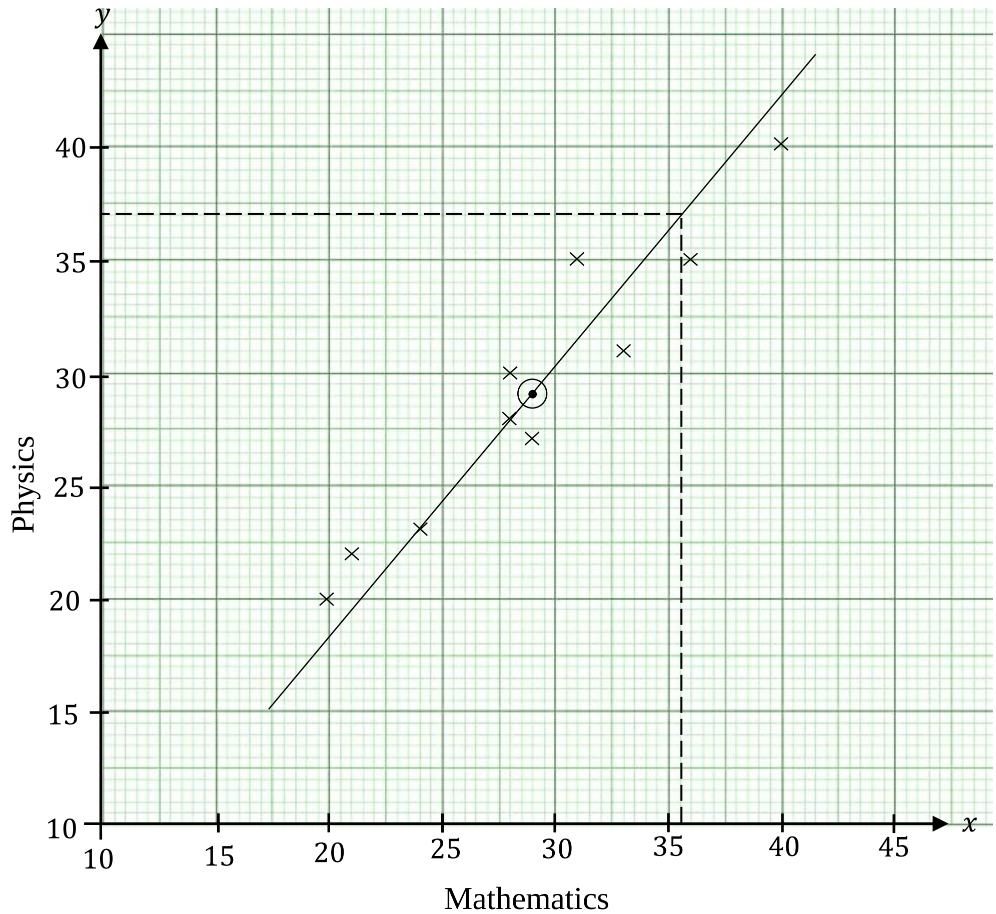
(iii) Estimate the score in Mathematics for a student who scored 37 in Physics

(b) Calculate the rank correlation coefficient for the data and comment on your result.

Solution:

(a) (i)

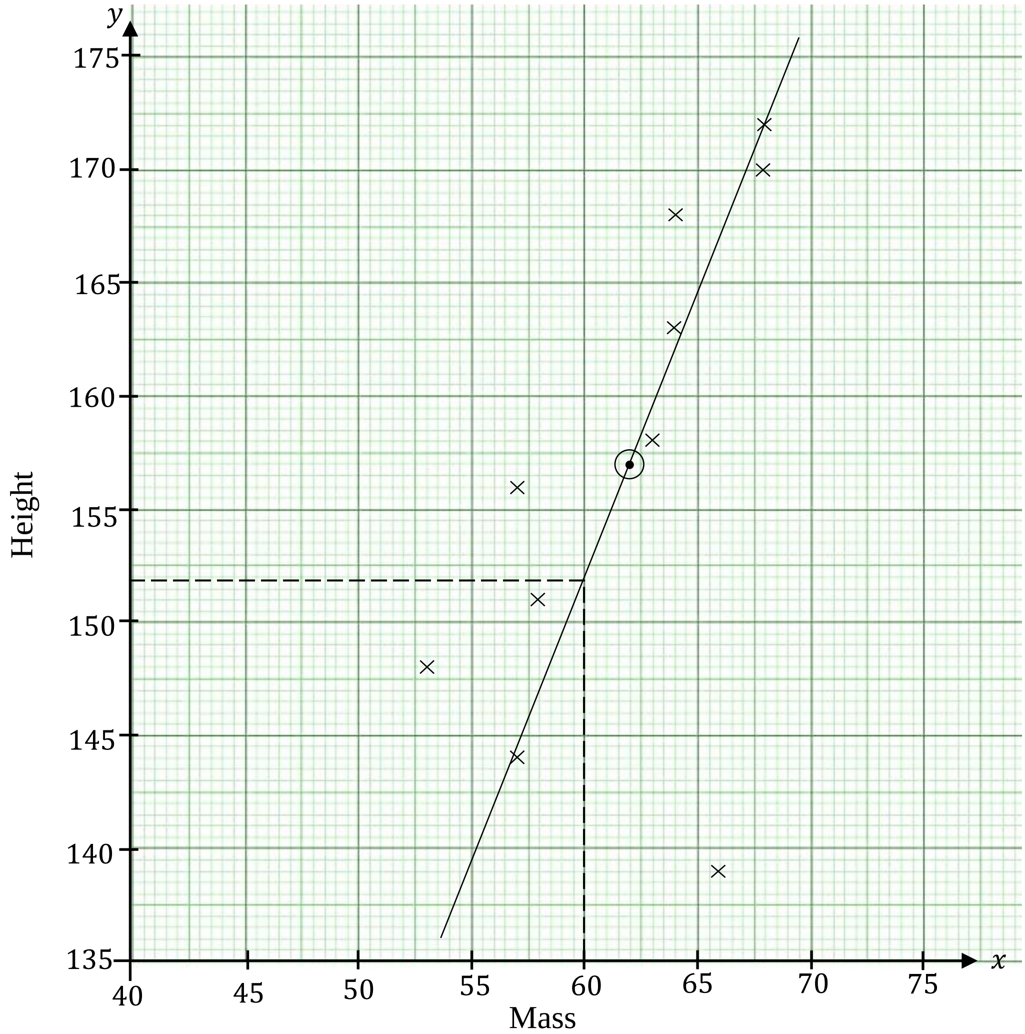
A scatter diagram showing the scores in Mathematics and Physics



$\bar{X} = \frac{\sum x}{n} = \frac{290}{10} = 29$ and $\bar{Y} = \frac{\sum y}{n} = \frac{291}{10} = 29.1$. The line of best fit passes through the point $(29, 29)$

(iii) From the graph, a student who scored 37 in Physics scored 36 in Mathematics.

A scatter diagram showing the Heights and Masses of 10 students



(iii) From the graph, a student with mass 60 kg has a height of 152 cm

(b)

Student	R_x	R_y	d	d^2
A	10	8	−2	4
B	1.5	1	0.5	0.25
C	7.5	6	1.5	2.25
D	9	10	−1	1
E	3	4	−1	1
F	4	5	−1	1
G	5	3	2	4
H	6	7	−1	1
I	7.5	9	−1.5	2.25
J	1.5	2	−0.5	0.25
			$\sum$	17

Spearman's rank correlation coefficient is given by

$$r = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6(17)}{10(10^2 - 1)} = 1 - \frac{102}{990} = 0.897$$

There is a high positive correlation between the height and weight of the students.

2020, No. 9

The table below shows marks obtained in Sub-Math and Physics by nine students

Sub-Math (X)	51	62	64	47	54	44	68	61	56
Physics (Y)	45	54	58	46	49	43	59	56	53

(a) (i) Draw a scatter diagram for the data

(ii) On your scatter diagram, draw a line of best fit

(iii) Use the line of best fit to estimate the value of X when $Y = 55$

(b) Calculate the Spearman's rank correlation coefficient and comment on the result.

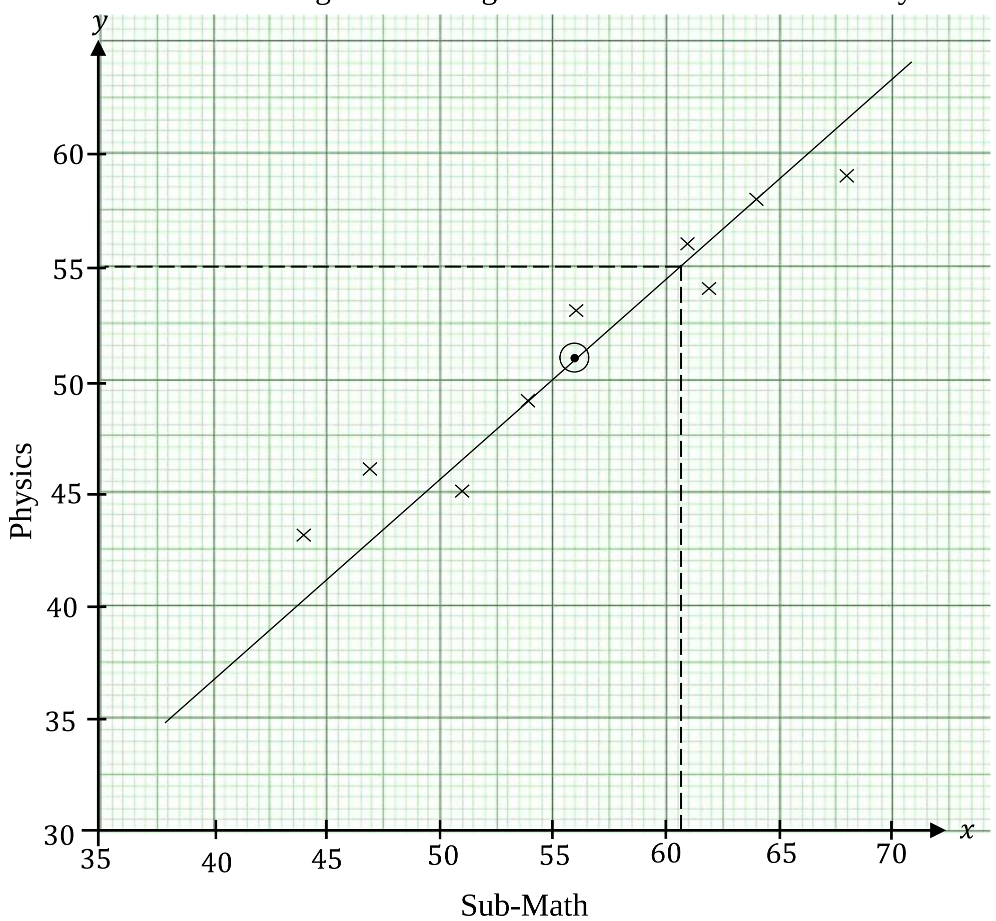
Solution:

(a)

$\bar{X} = \frac{\sum x}{n} = \frac{507}{9} = 56.3$ and $\bar{Y} = \frac{\sum y}{n} = \frac{463}{9} = 51.5.$

The line of best fit passes through the point $(56, 51)$

A scatter diagram showing the scores in Sub-Math and Physics



when $Y = 55$, $X = 61$

Chapter 16 Probability Theory

Probability (or chance) is a way of describing the likelihood of different possible outcomes occurring as a result of some experiment.

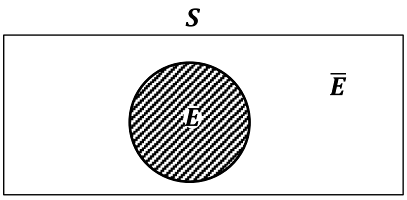
Terminologies used

An experiment has a finite number of outcomes, called the outcome set S.

An event E of an experiment is defined to be a subset of the outcome set S.

The complement of E, E', is the subset of S where E does not occur.

Venn diagram showing outcome set S and its subsets E and $\bar{E}$



Two events of the same experiment are mutually exclusive if they cannot occur simultaneously.

Two events are independent if the occurrence of one has no effect on the occurrence of the other.

For example, if a die is thrown once, then two different scores, e.g. 2 and 3, cannot occur simultaneously, so they are mutually exclusive events. If the die is thrown again, the second score is independent of the first.

If an experiment has $n(S)$ equally likely outcomes and $n(E)$ of them are the event E, then the theoretical probability of event E occurring is

$$P(E) = \frac{n(E)}{n(S)}$$

Note: $0 \leq P(E) \leq 1$

If the outcomes S has only n different possible events E_1, E_2, ….., E_n, then

$$P(E_1) + P(E_2) + \cdots + P(E_n) = \sum_{i=1}^{n} P(E_i) = 1$$

and $P(E') = 1 - P(E)$

Probabilities of 0 and 1

The two extremes of probability are certainty at one end of the scale and impossibility at the other. Here are examples of certain and impossible events.

Experiment	Certain event	Impossible event
Rolling a single die	The result is in the range 1 to 6 inclusive	The result is a 7
Tossing a coin	Getting either heads or tails	Getting neither heads nor tails

Certainty

As you can see from the table above, for events that are certain, the number of ways that the event can occur, $n(E)$ in the formula, is equal to the number of possible events, $n(S)$.

$$\frac{n(E)}{n(S)} = 1$$

So the probability of an event which is certain is one.

Impossibility

For impossible events, the number of ways that the event can occur, $n(E)$, is zero.

$$\frac{n(E)}{n(S)} = \frac{0}{n(S)} = 0$$

So the probability of an event which is impossible is zero.

Typical values of probabilities might be something like 0.3 or 0.9. If you arrive at probability values of, say, -0.4 or 1.7, you will know that you have made a mistake since these are meaningless.

Example 1

A fair die is thrown. List the possible outcomes.

What is the probability of scoring:

(a) a multiple of 3, (b) not a multiple of 3?

Solution:

S ={possible scores with a die} = {1, 2, 3, 4, 5, 6}

(a) E = {multiple of 3 scores} = {3, 6}

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

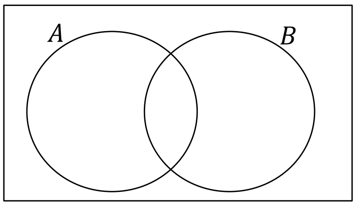
(b) $P(E')$ ={not multiple of 3 scores}

$$P(E') = 1 - P(E) = 1 - \frac{1}{3} = \frac{2}{3}$$

Intersection of events

If A and B are two events of the sample space S, then the intersection of events is denoted as $A \cap B$ and contains sample points common to both A and B.

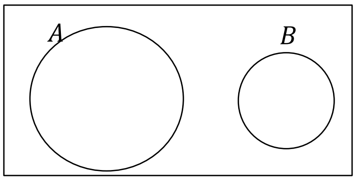
Addition rule



If A and B are two events of the same experiment, then the probability of A or B or both occurring is $P(A \text{ or } B)$ or $P(A \cup B)$ given by

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive, then $P(A \cap B) = 0$



So $P(A \cup B) = P(A) + P(B)$

If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

So $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

Conditional probability

If A and B are two events (not necessarily from the same experiment), then the conditional probability that A will occur given that B has occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If A and B are mutually exclusive, then $P(A|B) = 0$

Two events A and B are independent, if

$$P(A) = P(A|B) \text{ and } P(B) = P(B|A)$$

Multiplication rule

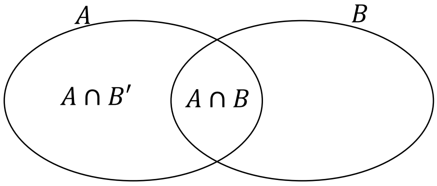
If A and B are any two events, then the probability that both A and B occur is

$$P(A \cap B) = P(A) \times P(A|B)$$
$$= P(B) \times P(B|A)$$

Interaction with the set theory

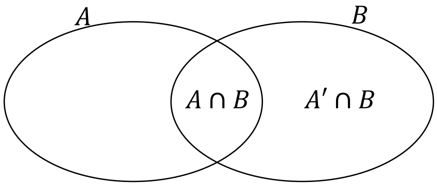
The following results can be deduced from the set theory for any two events A and B

(a) Result 1



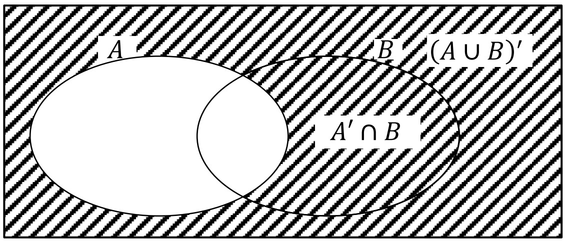
$$P(A) = P(A \cap B) + P(A \cap B')$$

(b) Result 2



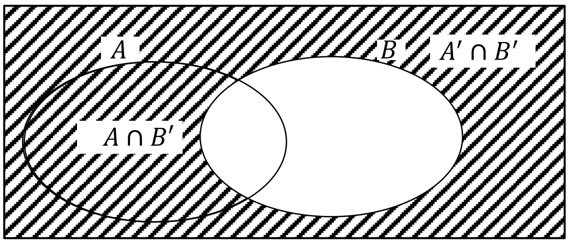
$$P(B) = P(A \cap B) + P(A' \cap B)$$

(c) Result 3



$$P(A') = P(A' \cap B) + P(A' \cap B')$$
$$P(A' \cap B') = P(A \cup B)'$$

(d) Result 4



$$P(B') = P(A \cap B') + P(A' \cap B')$$

(e) Result 5

$$P(A' \cup B') = P(A \cap B)'$$

The contingency table

The alternative way of recalling the results is by using the contingency table as shown below.

Event	Event		Total
	$\boldsymbol{B}$	$\boldsymbol{B'}$	
$\boldsymbol{A}$	$P(A \cap B)$	$P(A \cap B')$	$\boldsymbol{P(A)}$
$\boldsymbol{A'}$	$P(A' \cap B)$	$P(A' \cap B')$	$\boldsymbol{P(A')}$
Total	$\boldsymbol{P(B)}$	$\boldsymbol{P(B')}$	**1**

$$P(A) = P(A \cap B) + P(A \cap B')$$

(a) $P(A \cup B)$ (b) $P(A' \cap B)$ (c) $P(B|A')$

Solution:

(a) Since the events are independent,

$$P(A \cap B) = P(A) \times P(B) = 0.3 \times 0.5 = 0.15$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.3 + 0.5 - 0.15 = 0.65$$

(b) Since the events are independent

$$P(A' \cap B) = P(A') \times P(B) = 0.7 \times 0.5$$
$$= 0.35$$

(c) As the events are independent

$$P(B|A') = P(B) = 0.5$$

Example 7

The events A and B satisfy $P(A) = 0.5$, $P(B) = 0.2$ and $P(A|B) = 0.3$. Determine

(a) $P(A \cap B)$ (b) $P(A \cup B)$ (c) $P(B|A)$

Solution:

(a) Using conditional probability formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$0.3 = \frac{P(A \cap B)}{0.2}$$
$$P(A \cap B) = 0.06$$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.5 + 0.2 - 0.06 = 0.64$$

(c) $P(B|A) = \frac{P(B \cap A)}{P(A)}$

$$P(B|A) = \frac{0.06}{0.5} = 0.12$$

Example 8

The events A and B satisfy $P(A) = 0.6$, $P(B) = 0.52$ and $P(A \cup B) = 0.88$.

(a) Find the value of $P(A \cap B)$

(b) Determine

(i) $P(B|A)$

(ii) $P(A'|B')$

(c) State, giving a reason, whether A and B are

(i) independent

(ii) mutually exclusive

Solution:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$0.88 = 0.6 + 0.52 - P(A \cap B)$$
$$P(A \cap B) = 0.24$$

(b) (i) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.24}{0.6} = 0.4$

(ii) $P(A'|B') = \frac{P(A' \cap B')}{P(B')}$

$$P(A' \cap B') = P(A \cup B)'$$
$$= 1 - P(A \cup B)$$
$$P(A' \cap B') = 1 - 0.88 = 0.12$$
$$P(B') = 1 - P(B) = 1 - 0.52 = 0.48$$
$$P(A'|B') = \frac{0.12}{0.48} = 0.25$$

(c) (i) $P(B|A) = 0.4$

$P(B) = 0.52$

$P(B|A) \neq P(B)$, implying not independent

OR

$P(A) \times P(B) = 0.6 \times 0.52 = 0.312$

$P(A \cap B) = 0.24$

$P(A \cap B) \neq P(A) \times P(B)$, not independent

(ii) The events are not mutually exclusive since $P(A \cap B) \neq 0$

Example 9

The events A and B are independent such that $P(A) = 0.2$ and $P(A \cup B) = 0.68$.

(a) Determine $P(B)$

(b) Find the probability that exactly one of the two events occur.

Solution:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

$$0.68 = 0.2 + P(B) - 0.2P(B)$$
$$0.48 = 0.8P(B)$$
$$P(B) = 0.6$$

(b) $P(\text{Exactly one}) = P(A \text{ only}) + P(B \text{ only})$

$$= P(A \cap B') + P(A' \cap B)$$
$$= P(A) \cdot P(B') + P(A') \cdot P(B)$$
$$= 0.2(0.4) + 0.8(0.6)$$
$$= 0.56$$

Example 10

The events A and B satisfy $P(A) = P(B) = p$ and $P(A \cup B) = \frac{11}{36}$. Given that A and B are independent, determine the value of p.

Solution:

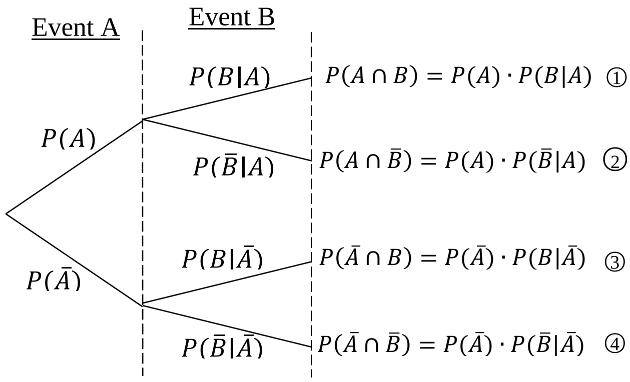
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$$
$$\frac{11}{36} = p + p - p^2$$
$$p^2 - 2p + \frac{11}{36} = 0$$
$$36p^2 - 72p + 11 = 0$$
$$36p^2 - 6p - 66p + 11 = 0$$
$$6p(6p - 1) - 11(6p - 1) = 0$$
$$(6p - 1)(6p - 11) = 0$$
$$p = \frac{1}{6} \text{ or } p = \frac{11}{6}$$
$$\therefore p = \frac{1}{6}$$

Probability tree diagrams

Tree diagrams are useful for organizing and visualizing the different possible outcomes of a sequence of events. For each possible outcome of the first event, we draw a line where we write down the outcome and state of the world if that outcome happened. Then, for each possible outcome of the second event, we do the same thing.

Tree diagrams are very helpful for analyzing dependent events. A tree diagram allows you to show how each possible outcome of one event affects the probability of the other events.

Tree diagrams are not so useful for independent events since we can multiply the probabilities of separate events to get the probability of the combined event.



Remember:

(a) The total probability for any one set of 'branches' $= 1$

(b) The sum of final probabilities (intersections) $= 1$

(c) The tree shows conditional probabilities for $(B|A)$ etc.

If $P(A|B)$ is required, then:

(i) Look in the final column for the intersections containing B; in this case $P(B) = 1 + 3$

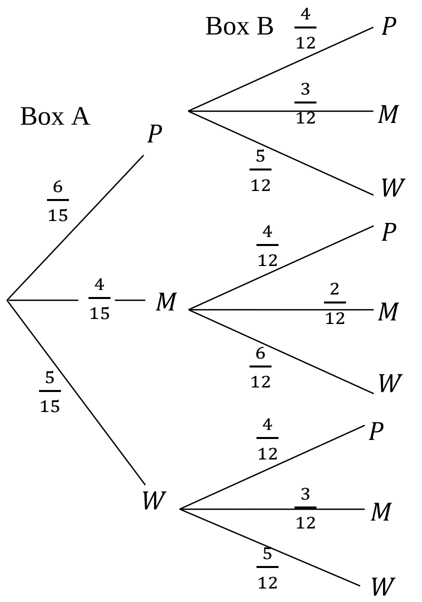
(ii) Find the term giving $A \cap B$, in this case 1,

$$\text{Then } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{1+3}$$

Example 11

Two boxes A and B contain chocolates. Box A contains 6 plain chocolates, 4 milk chocolates and 5 white chocolates. Box B contains 4 plain chocolates, 3 milk chocolates and 5 white chocolates. One chocolate is selected from each box at random. Determine the probability that the two chocolates will be of different type

Solution:



Easier to work: P(both the same type)

$$= P_1P_2 + M_1M_2 + W_1W_2$$

$$= \frac{6}{15} \times \frac{4}{12} + \frac{4}{15} \times \frac{3}{12} + \frac{5}{13} \times \frac{5}{12}$$

$$= \frac{24 + 12 + 25}{180} = \frac{61}{180}$$

$\therefore P(\text{both different}) = 1 - \frac{61}{180} = \frac{119}{180}$

Example 12

Three boxes, X, Y and Z contain coloured balls. X contains 5 black and 4 white balls, Y contains 7 black and 5 white balls and Z contains 3 black and 5 white balls.

(a) If balls are withdrawn from box Z, with replacement, find the probability that the third ball drawn is the second white ball

(b) One of the boxes is selected at random and a ball is withdrawn from it. Find the probability that

(i) box X was chosen, and the ball was black,

(ii) a white ball was chosen

(iii) the ball was selected from box Z, given that it was black.

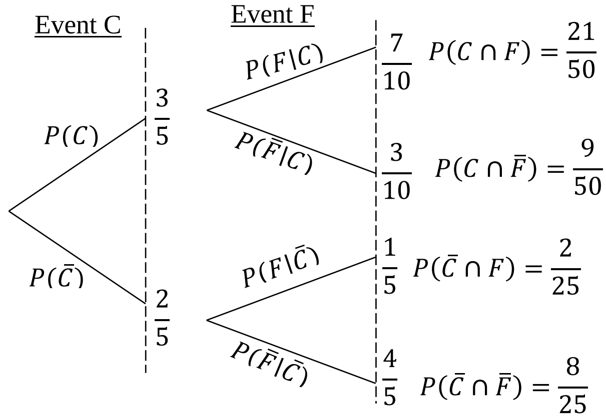
Solution:

(a) Box Z. $P(\text{black drawn}) = \frac{3}{8}$, $P(\text{white drawn}) = \frac{5}{8}$

For the third ball to be the second white one the possible are BWW and WBW

The balls are replaced after selection so

$$P(BWW) = \left(\frac{3}{8}\right)\left(\frac{5}{8}\right)\left(\frac{5}{8}\right) = \frac{75}{512}$$



To find $P(C|F')$:

$P(\text{he did not catch fish}) = P(F') = P(C \cap F') + P(C' \cap F')$

$$= \frac{9}{50} + \frac{8}{25} = \frac{1}{2}$$

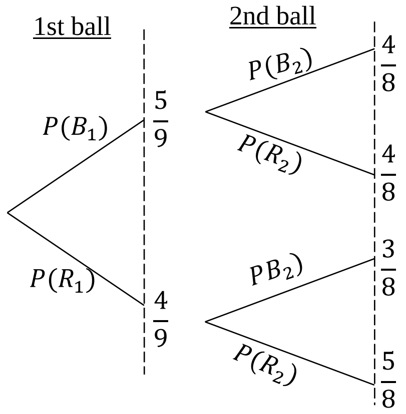
$$P(C|F') = \frac{P(C \cap F')}{P(F')} = \frac{9}{50} \div \frac{1}{2} = \frac{9}{25}$$

Examination questions

2013, No. 7

A bag contains 5 black pens (B) and 4 red pens (R). Two pens are picked at random, one after the other without replacement. Find the probability that both pens are of the same colour.

Solution:



P(Same colour)

$$= P(B_1 \cap B_2) + P(R_1 \cap R_2)$$
$$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{3}{8}$$
$$= \frac{20}{72} + \frac{12}{72}$$
$$= \frac{32}{72}$$
$$= \frac{4}{9}$$

2015, No. 3

Three events A, B and C are such that $P(A) = 0.6$, $P(B) = 0.8$, $P(B/A) = 0.45$ and $P(B \cap C) = 0.28$. Find

(a) $P(A \cap B)$

(b) $P(C/B)$

Solution:

(a) From $P(B/A) = \frac{P(B \cap A)}{P(A)}$

$P(A \cap B) = P(A) \times P(B/A)$

$= 0.6 \times 0.45 = 0.27$

(b) $P(C/B) = \frac{P(C \cap B)}{P(B)} = \frac{0.28}{0.8} = 0.35$

2016, No. 6

Two independent events A and B are such that $P(A) = \frac{1}{4}$ and $P(B) = \frac{3}{5}$. Find $P(A \cup B)$.

Solution:

From independent events,

$P(A \cap B) = P(A) \times P(B)$

$$= \frac{1}{4} \times \frac{3}{5} = \frac{3}{20}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{4} + \frac{3}{5} - \frac{3}{20} = \frac{5 + 12 - 3}{20} = \frac{14}{20} = \frac{7}{10}$$

2018, No. 4

Events A and B are such that $P(A) = \frac{6}{13}$, $P(B) = \frac{2}{5}$ and $P(A/B) = \frac{1}{4}$. Find:

(a) $P(A \cap B)$

(b) $P(A \cup B)$

Solution:

(a) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$\frac{1}{4} = \frac{P(A \cap B)}{2/5}$$

$$P(A \cap B) = \frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{6}{13} + \frac{2}{5} - \frac{1}{10}$$
$$= \frac{99}{130} \text{ or } 0.762$$

2019, No. 2

Two events A and B are such that $P(A) = \frac{19}{30}$, $P(B) = \frac{2}{5}$ and $P(A \cap \bar{B}) = \frac{2}{5}$. Find

(a) $P(A \cap B)$

(b) $P(A \cup B)$

Solution:

From the contigency table

(a) $P(A) = P(A \cap B) + P(A \cap B')$

$$\frac{19}{30} = P(A \cap B) + \frac{2}{5}$$

$$P(A \cap B) = \frac{19}{30} - \frac{2}{5} = \frac{7}{30}$$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{19}{30} + \frac{2}{5} - \frac{7}{30} = \frac{4}{5}$$

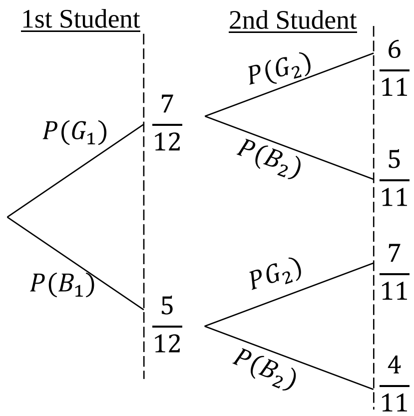
2020, No. 4

A school students' council consists of 7 girls and 5 boys. Two students are selected at random from the council. Find the probability that;

(a) both are girls

(b) the first is a boy and the second is a girl

Solution:



(a) P(both girls) $= P(G_1 \cap G_2) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$

(b) $P(B_1 \cap G_2) = \frac{5}{12} \times \frac{7}{11} = \frac{35}{132}$

2022, No. 6

Given that $P(A) = \frac{3}{4}$ and $P(B/A) = \frac{8}{15}$, find $P(A \cap B')$

Solution:

$$P(A \cap B') = P(A) - P(A \cap B)$$

Given $P(B/A) = \frac{8}{15}$

$$\Rightarrow \frac{P(B \cap A)}{P(A)} = \frac{8}{15}$$

$$P(B \cap A) = \frac{8}{15} \times P(A) = \frac{8}{15} \times \frac{3}{4} = \frac{2}{5}$$

$$P(A \cap B') = P(A) - P(A \cap B) = \frac{3}{4} - \frac{2}{5} = \frac{7}{20}$$

Self-Evaluation exercise

1. The events A and B are such that $P(A \cap B') = 0.25$, $P(A) = 2P(B)$ and $P(A \cup B) = 0.45$. Determine
   (a) $P(A \cap B)$
   (b) $P(A \cup B')$
   [Ans: (a) 0.15 (b) 0.95]
2. The chances of winning of two race-horses are $1/3$ and $1/6$ respectively. What is the probability that at least one will win when the horses are running
   (a) in different races, and
   (b) in the same race?
   [Ans: (a) $8/18$ (b) $1/2$]
3. The events E and F are such that $P(E) = 0.5$, $P(F|E) = 0.6$ and $P(E' \cap F') = 0.4$. Determine the value of
   (a) $P(E \cap F)$
   (b) $P(E|F)$
   (c) $P(F'|E')$
   [Ans: (a) 0.3 (b) 0.75 (c) 0.8]
4. The events A and B are such that $P(A) = P(B) = p$ and $P(A \cup B) = 0.84$. Given that A and B are independent events, determine the value of p.
   [Ans: $p = 0.6$]
5. The events A and B satisfy $P(A) = x$, $P(B) = y$, $P(A \cup B) = 0.6$ and $P(B|A) = 0.2$.
   (a) Show clearly that $4x + 5y = 3$
   (b) The events B and C are mutually such that $P(B \cup C) = 0.9$ and $P(C) = x + y$. Find the value of x and y
   (c) Show that A and B are independent events
   [Ans: (b) $x = 0.5, y = 0.2$]
6. The events A and B are such that $P(A) = \frac{1}{2}$, $P(A'|B) = \frac{1}{3}$, $P(A \cup B) = \frac{3}{5}$, where A' is the event 'A does not occur'.
   (a) Using a Venn diagram, or otherwise, determine $P(B|A')$, $P(B \cap A)$ and $P(A/B')$.
   (b) The event C is independent of A and $P(A \cap C) = \frac{1}{8}$. Determine $P(C|A')$
   (c) State, with a reason in each case, whether
   (i) A and B are independent
   (ii) A and C are mutually exclusive
   [Ans: (a) $1/5$, $1/5$, $3/7$ (b) $1/4$ (c) (i) not independent (ii) not mutually exclusive]
7. Three boxes A, B and C contain coins. Box A contains 3 gold coins, Box B contains 2 gold

20. If A and B are two events such that $P(A) = \frac{5}{8}$ and $P(B/A) = \frac{3}{7}$, find $P(A \cap B)$

[Ans: $\frac{15}{56}$]

21. If A and B are two events such that $P(A) = \frac{2}{5}$ and $P(A \cap B) = \frac{3}{10}$, find $P(B/A)$

[Ans: $\frac{3}{4}$]

22. Two events A and B are such that $P(A) = 0.2$, $P(A' \cap B) = 0.22, P(A \cap B) = 0.18$. Evaluate (i) $P(A \cap B')$ (ii) $P(A/B)$

[Ans: (i) 0.02 (ii) 0.45]

23. Two events A and B are such that $P(A) = \frac{1}{2}$, $P\left(\frac{A}{B'}\right) = \frac{2}{3}, P\left(\frac{A}{B}\right) = \frac{3}{7}$, where B' is the event B does not occur. Find
(i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(B)$ (iv) $P(B/A)$

[Ans: (i) $\frac{3}{10}$ (ii) $\frac{9}{10}$ (iii) $\frac{7}{10}$ (iv) $\frac{3}{5}$]

24. A bag contains 4 white balls, 3 black balls and 1 red ball. Two balls are picked at random in succession without replacement. Find the probability that
(i) both are of the same colour
(ii) at least one black ball is picked

[Ans: (i) $\frac{9}{28}$ (ii) $\frac{9}{14}$]

25. A box contains 7 red balls and 6 blue balls. Two balls are selected at random without replacement. Find the probability that;
(i) they are of the same colour
(ii) at least one is blue

[Ans: (i) $\frac{6}{13}$ (ii) $\frac{19}{26}$]

26. Two boxes P and Q contain white and brown cards. P contains 6 white cards and 4 brown cards. Q contains 2 white cards and 3 brown cards. A box is selected at random and a card selected. Find the probability that;
(i) a brown card selected
(ii) box Q is selected given that the card is white

[Ans: (i) $\frac{1}{2}$ (ii) $\frac{2}{5}$]

27. Bag A contains 3 green and 2 red balls. Bag B contains 4 green and 3 red balls. If a ball is picked at random from a bag chosen at random, find the probability that a red ball is
(i) picked
(ii) not picked

[Ans: (i) $\frac{29}{70}$ (ii) $\frac{41}{70}$]

28. Two independent events A and B are such that $P(A) = 0.40$, P(B) $= a$, $P(A \cup B) = 0.70$. Find
(i) $P(A \cup B)'$ (ii) the value of a (iii) $P(A \cap B)$ (iv) $P(A \cap B')$

[Ans: (i) 0.3 (ii) 0.5 (iii) 0.2 (iv) 0.2]

29. Given that A and B are two events such that $P(A) = 0.5$, $P(B) = 0.7$ and $P(A \cup B) = 0.8$.
Find (i) $P(A \cap B)$ (ii) $P(A \cap B')$

[Ans: (i) 0.4 (ii) 0.1]

30. Two events A and B are independent such that $P(A) = 0.2$ and $P(A \cup B) = 0.8$. Find (i) $P(B)$ (ii) $P(A' \cup B')$

[Ans: (i) $\frac{3}{4}$ (ii) $\frac{17}{20}$]

31. In a school canteen, the probability that a child has chips with their meal is 0.9 and the probability that they have baked beans is 0.6. Find the probability that a child;
(i) has both chips and beans
(ii) has chips but not beans
(iii) neither chips nor beans

[Ans: (i) 0.54 (ii) 0.36 (iii) 0.04]

32. In a certain city suburb 30% of the residents read New Vision paper only, 55% read both New Vision and Monitor. If 10% do not read any paper, find the probability that a person picked at random reads;
(i) Monitor
(ii) Monitor or New Vision but not both

[Ans: (i) 0.6 (ii) 0.9]

33. On average, Maurice comes to tea on 2 days out of every 5. If comes to tea, the probability that we have jam tarts is 0.7. If he does not come for tea, the probability that we have jam tarts is 0.4. What is the probability that we have jam tarts for tea tomorrow?

[Ans: 0.52]

34. A die is thrown twice. Find the probability that;
(a) two odd numbers are obtained
(b) the same two numbers are obtained

[Ans: (a) $\frac{1}{4}$ (b) $\frac{1}{6}$]

35. Given that A and B are mutually exclusive events such that $P(A) = 0.4, P(A \cup B) = 0.7$. Find (i) $P(A' \cap B')$ (ii) $P(A' \cup B)$

[Ans: (i) 0.3 (ii) 0.6]

Similarly, $P(X = 1) = 0.6$ and $P(X = 2) = 0.3$

When defining variables, the variable is usually denoted by a capital letter (X, Y, R, etc.) and a particular value that variable takes by a small letter (x, y, r etc.), so that $P(X = x)$ means "the probability that the variable X takes the value x "

The probability distribution for X can be summarized in the table below

x	0	1	2
$P(X = x)$	0.1	0.6	0.3

If the sum of the probabilities is 1, the variable is said to be random

In this example; $P(X = 0) + P(X = 1) + P(X = 2) = 0.1 + 0.6 + 0.3 = 1$

So, X is a discrete random variable.

For a discrete random variable, the sum of the probabilities is 1,

i.e. $\sum_{all\ x} P(X = x) = 1$

also $P(X = x) \geq 0$ for all values of x

The function responsible for allocating probabilities, $P(X = x)$is known as the probability density function of X, sometimes abbreviated as p.d.f of X. The probability density function can either list the probabilities individually or summarize them in a formula

Example 1

The discrete random variable X has the following probability distribution

x	1	2	3	4	5
$P(X = x)$	0.2	0.25	0.4	a	0.05

(a) Find the value of a

(b) Find (i) $P(1 \leq X \leq 3)$ (ii) $P(X > 2)$ (iii) $P(2 < X < 5)$

Solution

(a) Using the property $\sum_{all\ x} P(X = x) = 1$

$$0.2 + 0.25 + 0.4 + a + 0.05 = 1$$

$$0.9 + a = 1$$

$$a = 0.1$$

(b) (i) $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$

$= 0.2 + 0.25 + 0.4 = 0.85$

(i) $P(X > 2) = P(X = 3) + P(X = 4) + P(X = 5)$

$= 0.4 + a + 0.05 = 0.4 + 0.1 + 0.05 = 0.55$

(ii) $P(2 < X < 5) = P(X = 3) + P(X = 4)$

$= 0.4 + a = 0.4 + 0.1 = 0.5$

Example 2

The p.d.f of a discrete random variable X is given by $P(X = x) = kx^2$ for $x = 0, 1, 2, 3, 4$. Given that k is a constant, find the value of k

Solution:

By drawing the table, it would help us write out the probability distribution of X

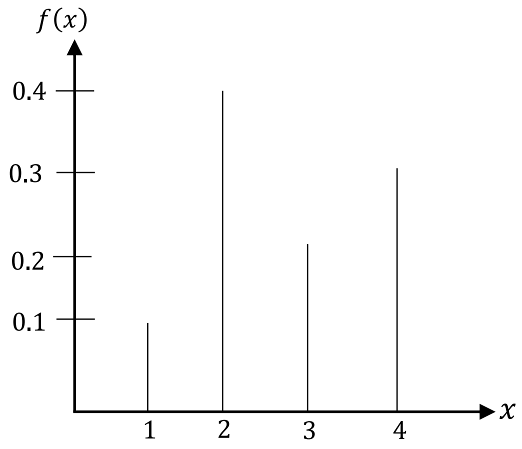
x	0	1	2	3	4
$P(X = x)$	0	k	$4k$	$9k$	$16k$

Since X is a discrete random variable, $\sum_{all\ x} P(X = x) = 1$

So, $0 + k + 4k + 9k + 16k = 1$

$30k = 1 \Rightarrow k = \frac{1}{30}$

(iii) A graph pf $f(x)$



Expectation of X, E(X)

E(X) is read as 'E of X' and it gives an average or typical value of X, known as the expected value or expectation of X. This is comparable with the mean in descriptive statistics.

The expectation of X (expected value or mean), written as E(X) is given by;

$$\boldsymbol{E(X) = \sum_{all\ x} xP(X = x)}$$

The symbol μ, pronounced 'mew' is often used for the expectation, where $\mu = E(X)$

Example 9

A random variable X has the following probability distribution

x	-2	-1	0	1	2
$P(X = x)$	0.3	0.1	0.15	0.4	0.05

Find the expectation E(X)

Solution

x	-2	-1	0	1	2
$P(X = x)$	0.3	0.1	0.15	0.4	0.05
$xP(X = x)$	-0.6	-0.1	0	0.4	0.1

$E(X) = \sum_{all\ x} xP(X = x)$

$= -0.6 + -0.1 + 0 + 0.4 + 0.1 = -0.2$

Example 10

X is the number of heads obtained when two coins are tossed. Find the expected number of heads.

Solution

$P(HH) = \frac{1}{4}$, $P(HT) = \frac{1}{4}$, $(TH) = \frac{1}{4}, P(TT) = \frac{1}{4}$

$P(X = 0) = P(TT) = \frac{1}{4}, P(X = 1) = (HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(X = 2) = P(HH) = \frac{1}{4}$

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$xP(X = x)$	0	$\frac{1}{2}$	$\frac{1}{2}$

$E(X) = 0 + \frac{1}{2} + \frac{1}{2} = 1$

Example 13

The random variable X has a probability distribution as shown in the table

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	0.2	0.3	0.1

Find (a) $E(X)$ (b) $E(X^2)$ (c) $Var\ (X)$ (d) the standard deviation of X

Solution:

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	0.2	0.3	0.1
$xP(X=x)$	0.1	0.6	0.6	1.2	0.5
x^2	1	4	9	16	25
$x^2P(X=x)$	0.1	1.2	1.8	4.8	2.5

(a) $E(X) = 0.1 + 0.6 + 0.6 + 1.2 + 0.5 = 3$

(b) $E(X^2) = 0.1 + 1.2 + 1.8 + 4.8 + 2.5 = 10.4$

(c) Var (X) = $E(X^2) - [E(X)]^2$

$= 10.4 - 3^2 = 10.4 - 9 = 1.4$

(d) standard deviation of X, $\sigma = \sqrt{Var(X)} = \sqrt{1.4} = 1.18$ (2 d.p)

Example 14

The discrete random variable X has p.d.f $P(X = x)\ \ for\ x = 1, 2, 3$

x	1	2	3
$P(X=x)$	0.2	0.3	0.5

Find (a) $E(X)$ $(b)\ E(X^2)$ $(c)\ \ Var\ (X)$ (d) the standard deviation of X

Solution:

x	1	2	3
$P(X=x)$	0.2	0.3	0.5
$xP(X=x)$	0.2	0.6	1.5
x^2	1	4	9
$x^2P(X=x)$	0.2	1.2	4.5

(a) $E(X) = 0.2 + 0.6 + 1.5 = 2.3$

(b) $E(X^2) = 0.2 + 1.2 + 4.5 = 5.9$

(c) Var (X) = $E(X^2) - [E(X)]^2$

$= 5.9 - (2.3)^2 = 5.9 - 5.29 = 0.61$

(d) standard deviation of X, $\sigma = \sqrt{Var(X)} = \sqrt{0.61} = 0.781\ (3\ d.p)$

Examination questions

2014, No. 13

The table below shows the probability distribution of the number of Compact Discs (CDs) sold.

Number of CD's sold	0	1	2	3	4
Probability, $P(X=\ x)$	0.05	0.28	c	0.22	0.99

Determine the

(a) value of c

(b) probability that at least 2 CD's are sold

(c) expectation E(X)

(d) standard deviation

Solution:

(a) $\sum_{all\ x} P(X = x) = 1$

$0.05 + 0.28 + c + 0.22 + 0.09 = 1$

$c + 0.64 = 1 \Rightarrow c = 0.36$

(b) $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$= c + 0.22 + 0.09$

2020, No. 13

A discrete random variable W has a probability distribution shown below

w	-3	-2	-1	0	1
$P(W=w)$	0.1	0.25	0.3	0.15	d

Find;

(a) the value of d

(b) $P(-3 \le W \le -1)$

(c) $P(W > -1)$

(d) (i) the mode

(ii) the median

(iii) the variance of the distribution

Solution:

(a) $\sum_{all\ x} P(X=x) = 1$

$$0.1 + 0.25 + 0.3 + 0.15 + d = 1$$
$$d + 0.8 = 1$$
$$d = 0.2$$

(b) $P(-3 \le W \le -1) = P(W=-3) + P(W=-2) + P(W=-1)$

$= 0.1 + 0.25 + 0.3 = 0.65$

(c) $P(W > -1) = P(W=0) + P(W=1) = 0.15 + d = 0.15 + 0.2 = 0.35$

(d) (i) the mode $= -2$

(ii) Find $F(m) \ge 0.5$ where m is the median

w	-3	-2	-1	0	1
$P(W=w)$	0.1	0.25	0.3	0.15	0.2
$F(W)$	0.1	0.35	0.65	0.80	1.0

$$F(-1) = 0.65$$

$\therefore$ The median is -1

(iii)

w	-3	-2	-1	0	1	$\sum$
$P(W=w)$	0.1	0.25	0.3	0.15	0.2	1
$wP(W=w)$	-0.3	-0.5	-0.3	0	0.2	$E(W) = -0.9$
$w^2P(W=w)$	0.9	1	0.3	0	0.2	$E(W^2) = 2.4$

$$Var\,(W) = E(W^2) - [E(W)]^2 = 2.4 - (-0.9)^2 = 2.4 + 0.81 = 3.21$$

Self-Evaluation exercise

1. A random variable X has the probability distribution as shown in the table

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	a	0.2	0.05

Find (a) the value of a (b) $P(X \ge 4)$ $(c) P(X < 1)$ (d) $P(2 \le X \le 4)$

[Ans: (a) 0.35 (b) 0.25 (c) 0 (d) 0.65]

2. The probability distribution of a random variable X is as shown in the table below

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	y	0.2	0.1

Find (a) the value of y (b) $E(X)$ [Ans: (a) 0.3 (b) 2.9]

3. Find the expected number of heads when two fair coins are tossed [Ans: 1]
4. The discrete random variable X has p.d.f; $P(X=0) = 0.05$, $P(X=1) = 0.45$, $P(X=2) = 0.5$. Find (a) $E(X)$ (b) $E(X^2)$ [Ans: (a) 1.45 (b) 2.45]
5. The discrete random variable X has a p.d.f $P(X=x) = kx$ for $x = 1, 2, 3, 4, 5$ where k is a costant. Find $E(X)$ [Ans: $\frac{11}{3}$]
6. A discrete random variable X can take on the values 0, 1, 2 or 3 and its probability distribution is given by $P(X=0) = k$, $P(X=1) = 3k$, $P(X=2) = 4k$, $P(X=3) = 5k$, where k is a constant. Find

(a) the value of k

Find (a) $E(X)$ (b) $E(X^2)$ (c)Var (X) (d) the standard deviation, σ of X

Solution

(a) $E(X) = \int_{all\ x} xf(x)\,dx$

$$= \int_0^4 x \times \frac{1}{8}x\,dx = \frac{1}{8}\int_0^4 x^2\,dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{8}\left(\frac{4^3}{3} - 0\right) = \frac{1}{8} \times \frac{64}{3} = 2.7$$

(b) $E(X^2) = \int_{all\ x} x^2 f(x)\,dx$

$$= \int_0^4 x^2 \times \frac{1}{8}x\,dx$$

$$= \frac{1}{8}\int_0^4 x^3\,dx = \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4$$

$$= \frac{1}{8}\left(\frac{4^4}{4} - 0\right) = \frac{1}{8} \times \frac{64}{3} = 8$$

(c) Var (X) = $E(X^2) - [E(X)]^2$

$= 8 - (2.7)^2 = 8 - 7.29 = 0.71\ (2\ d.p)$

(d) Standard deviation $= \sqrt{Var(X)}$

$$= \sqrt{0.71} = 0.8439$$

Example

As an experiment, a temporary roundabout is installed at cross roads. The time X in minutes which vehicles have to wait before entering the roundabout has a probability density function

$$f(x) = \begin{cases} 0.8 - 0.32x & 0 \le x \le 2.5 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and standard deviation of X

Solution:

$E(X) = \int_{all\ x} xf(x)\,dx$

$$= \int_0^{2.5} x(0.8 - 0.32x)\,dx = \int_0^{2.5} (0.8x - 0.32x^2)\,dx$$

$$= \left[0.8\frac{x^2}{2} - 0.32\frac{x^3}{3}\right]_0^{2.5}$$

$$= \left(\left(0.8\frac{(2.5)^2}{2} - 0.32\frac{(2.5)^3}{3}\right) - (0)\right) = 0.833$$

$$E(X^2) = \int_{all\ x} x^2 f(x)\,dx$$

$$= \int_0^{2.5} (0.8x^2 - 0.32x^3)\,dx$$

$$= \left[0.8\frac{x^3}{3} - 0.32\frac{x^4}{4}\right]_0^{2.5}$$

$$= \left(\left(0.8\frac{(2.5)^3}{3} - 0.32\frac{(2.5)^4}{4}\right) - (0)\right)$$

$$= 4.167 - 3.125 = 1.042$$

Var (X) $= E(X^2) - [E(X)]^2 = 1.042 - 0.833^2 = 0.348$

Standard deviation of X = $\sqrt{0.348} = 0.59$ minutes

Examination questions

2015, No. 13

A continuous random variable X has a probability density function given by,

$$f(x) = \begin{cases} \frac{kx}{6}, & 1 \le x \le 2, \\ 0 & otherwise, \end{cases}$$

where k is a constant

(a) Find

(i) the value of k

(ii) $P(X \ge 1.5)$

(iii) the mean of X, E(X)

(b) Sketch the graph of $f(x)$

Solution:

(a) (i) $\int_{all\ x} f(x)\,dx = 1$

$$\int_1^2 \frac{kx}{6}\,dx = 1$$

$$\frac{k}{6}\int_1^2 x\,dx = 1$$

$$\frac{k}{6}\left[\frac{x^2}{2}\right]_1^2 = 1$$

$$\frac{k}{6}\left(2 - \frac{1}{2}\right) = 1$$

$$k = 4$$

(ii) $f(x) = \begin{cases} \frac{2x}{3}, & 1 \le x \le 2, \\ 0 & \text{otherwise,} \end{cases}$

$$P(X \ge 1.5) = \int_{1.5}^2 \frac{2x}{3}\,dx = \left[\frac{x^2}{3}\right]_{1.5}^2$$

$$= \left(\frac{2^2}{3} - \frac{1.5^2}{3}\right)$$

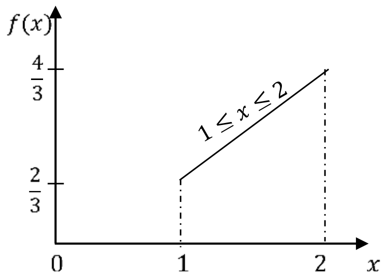
$$= \frac{4}{3} - \frac{2.25}{3} = 0.583$$

(iii) Mean, $E(X) = \int_{all\ x} xf(x)\,dx$

$$= \int_1^2 \frac{2x^2}{3}\,dx$$

$$= \left[\frac{2x^3}{9}\right]_1^2 = \left(\frac{16}{9} - \frac{2}{9}\right) = \frac{14}{9} = 1.56$$

(b)



2016, No. 13

A random variable X has a probability density function $f(x)$, defined by

$$f(x) = \begin{cases} \frac{kx}{6}, & 1 \le x \le 2, \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant

Determine the

(a) value of k

(b) $P(1 \le X \le 1.5)$

(c) Expectation, $E(X)$

(d) Variance, $\text{Var}(X)$

Solution:

(a) $\int_{all\ x} f(x)\, dx = 1$

$$\int_0^2 kx(x+2)\, dx = 1$$

$$k\int_0^2 x^2 + 2x\, dx = 1$$

$$k\left[\frac{x^3}{3} + x^2\right]_0^2 = 1$$

$$k\left(\frac{2^2}{3} + 2^2 - 0\right) = 1$$

$$\frac{20}{3}k = 1$$

$$k = \frac{3}{20}$$

(b) $P(1 \le X \le 1.5) = \int_1^{1.5} f(x)\, dx$

$$= \frac{3}{20}\int_1^{1.5} x^2 + 2x\, dx$$

$$= \frac{3}{20}\left[\frac{x^3}{3} + x^2\right]_1^{1.5}$$

$$= \frac{3}{20}\left[\left(\frac{1.5^3}{3} + 1.5^2\right) - \left(\frac{1^3}{3} + 1^2\right)\right]$$

$$= \frac{3}{20} \times 2.042 = 0.30625$$

(c) $E(X) = \int_{\text{all } x} xf(x)\, dx$

$$= \frac{3}{20}\int_0^2 x^2(x+2)dx$$

$$= \frac{3}{20}\int_0^{20} x^3 + 2x\, dx$$

$$= \frac{3}{20}\left[\frac{x^4}{4} + \frac{2x^3}{3}\right]_0^2$$

$$= \frac{3}{20}\left[\left(\frac{2^4}{4} + \frac{2(2)^3}{3}\right) - 0\right]$$

$$= \frac{3}{20}\left(4 + \frac{16}{3}\right)$$

$$= \frac{3}{20} \times \frac{28}{3} = 1.4$$

(d) $Var\ (X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_{all\ x} x^2 f(x)\ dx$$

$$= \frac{3}{20}\int_0^2 x^3(x+2)\, dx$$

$$= \frac{3}{20}\int_0^2 x^4 + 2x^3\, dx$$

$$= \frac{3}{20}\left[\frac{x^5}{5} + \frac{x^4}{2}\right]_0^2$$

$$= \frac{3}{20}\left[\left(\frac{2^5}{5} + \frac{2^4}{4}\right) - 0\right]$$

$$= \frac{3}{20}\left(\frac{32}{5} + 8\right)$$

$$= \frac{3}{20} \times \frac{72}{5} = 2.16$$

$$Var\ (X) = 2.16 - 1.4^2 = 0.2$$

2017, No. 13

A continuous random variable X has a probability density function (pdf) given by

$$f(x) = \begin{cases} k(x^2+6), & 0 \le x \le 3 \\ 0 & , \quad \text{Otherwise} \end{cases}$$

where k is a constant

Determine the

(a) value of k

(b) $P(X > 1)$

(c) Expectation, $E(X)$

(d) Variance, $Var\ (X)$

Solution:

(a) $\int_{\text{all } x} f(x)\, dx = 1$

$$\int_0^3 k(x^2+6)\, dx = 1$$

$$k\int_0^3 (x^2+6)\, dx = 1$$

Chapter 19 Binomial Distribution

Introduction

There are some probability situations that may result into only two outcomes, or even be reduced to only two. Such situations may include:

(i) When a baby is born, it may be either male or female

(ii) In a final football match, a team either wins or loses.

Other situations that are reduced to only two possible outcomes may include:

(i) A person taking a Pioneer bus may arrive either on time or not on time.

(ii) A company producing items that are either defective or not defective.

(iii) A drug administered to a patient may be either effective or ineffective.

All the above-mentioned situations are called binomial or Bernoulli experiments and the outcomes of a binomial experiment are classified as successes or failures.

For a situation to be described using a binomial model,

- a finite number, n, trials are carried out
- the trials are independent
- the outcome of each trial is deemed either a success or a failure
- the probability, p, of a successful outcome is the same for each trial

The discrete random variable, X, is **the number of successful outcomes in n trials.**

If the above conditions are satisfied, X is said to follow a binomial distribution. This is written

$$\boldsymbol{X \sim B(n, p)}$$

Note: The number of trials, n, and the probability of success, p , are both needed to describe the distribution completely. They are known as the parameters of the binomial distribution.

Writing P(failure) as q where $\boldsymbol{q = 1 - p}$

If $X \sim B(n, p)$, the probability of obtaining r successes in n trials is $P\ (X = r)$ where;

$$\boldsymbol{P(X = r) = {}^nC_r p^r q^{n-r}}$$
$$\textbf{for } \boldsymbol{r = 0, 1, 2\ 3, \ldots\ldots..n}$$

Example 1

A coin is tossed three times. Find the probability of getting exactly three heads

Solution 1:

The problem can be obtained by looking at the sample space, there are three ways of getting 2 heads out of 8 i.e.

$\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

The answer is $\frac{3}{8} = 0.375$

Solution 2:

Looking at the problem above from the stand point of a binomial experiment, one can show that it meets the four requirements i.e.

1. There are only two outcomes for each trial, head or tail

2. There is a fixed number of trials, three

3. The outcomes are independent of each other (the outcome of one toss in no way affects the outcome of another toss)

4. The probability of success (heads) is ½ in each case.

In this case; $n = 3$, $X = 2$, $p = ½$, $q = ½$

Hence substituting in the formula $P(X = r) = {}^nC_r p^r q^{n-r}$ gives;

P(2 heads)$= P(X = 2) = {}^3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1$

$$= 3 \times \frac{1}{4} \times \frac{1}{2} = \frac{3}{8}$$

Which is the same answer obtained by using the sample space

Example 2

In a particular population, 10% of the people have blood type B. If three people are selected at random from the population, what is the probability that exactly two of them have blood type B?

Solution:

Let X be the random variable people with blood type B

From the table when $n = 15, r = 5, p = 0.25$

$$P(X = 5) = 0.1651$$

(ii) P(five incorrect answers) = P(ten correct answers) $= P(X = 10)$

Using the tables, $n = 15, r = 10, p = 0.25$

$$P(X = 10) = 0.0007$$

P(five incorrect answers) = 0.0007

Mean, Variance and Standard deviation of a binomial distribution

The mean, variance and standard deviation of a variable that has the binomial distribution can be found by using the formulas

$$\textbf{Mean, } \boldsymbol{\mu = np}$$

$$\textbf{Variance, } \boldsymbol{\sigma^2 = npq}$$

$$\textbf{Standard deviation, } \boldsymbol{\sigma = \sqrt{Var(X)} = \sqrt{npq}}$$

Example 13

A coin is tossed four times. Find the mean, variance and standard deviation of the number of heads that will be obtained.

Solution:

$$n = 4, \ p = \frac{1}{2}, q = \frac{1}{2}$$

$$\text{mean} = np = 4 \times \frac{1}{2} = 2$$

$$\text{variance} = npq = 4 \times \frac{1}{2} \times \frac{1}{2} = 1$$

$$\text{Standard deviation} = \sqrt{Var(X)} = \sqrt{1} = 1$$

Example 14

In Makerere University, it is known that $\frac{1}{3}$ of the students play volleyball. In a sample of 12 students, what is the expected value and the standard deviation of the number of volley ballers?

Solution:

Let X = number of volley ballers

$$X \sim B(n, p)$$

$$n = 12, \ p = \frac{1}{3} \ , q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{mean} = np = 12 \times \frac{1}{3} = 4$$

$$\text{Standard deviation} = \sqrt{12 \times \frac{1}{3} \times \frac{2}{3}} = \sqrt{\frac{8}{3}} = 1.633$$

Example 15

Given that 25% of city dwellers use a particular product. A random sample of fifteen city dwellers are asked whether they use the product.

(a) What is the probability that:

(i) exactly three of the fifteen use the product?

(ii) fewer than three people use the product?

(b) What is the mean and variance of the number of people out of the fifteen who use the product?

Solution:

The random variable, X, is the number of city dwellers, out of fifteen, who use this particular product. The distribution of X is thus binomial with $n = 15$ and $p = \frac{1}{4}$

(a) (i) $P(X = 3) = \binom{15}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{12} = 0.2252$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \binom{15}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{15} + \binom{15}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{14} + \binom{15}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{13}$$

$$= 0.0134 + 0.0668 + 0.1559 = 0.2361$$

(b) $E(X) = np = 15 \times \frac{1}{4} = 3.75$

$Var(X) = npq = 15 \times \frac{1}{4} \times \frac{3}{4} = 2.8125$

Examination questions

2014, No. 7

In a bimodal experiment, the probability of a success for n trials is 0.6. If the mean is 7.2, find the

(a) value of n

(b) probability of obtaining 7 successes

Solution:

(a) Mean $= np$

$$7.2 = 0.6n$$

$$n = \frac{7.2}{0.6} = 12$$

(b) $P(X = r) = {}^nC_r p^r q^{n-r}$

$$P(X = 7) = {}^{12}C_7 (0.6)^7 (0.4)^5 = 0.227$$

2017, No. 6

It was observed that 3 seeds in every 4 germinate. If 16 seeds were planted, calculate the

(a) expected number of seeds that will germinate

(b) probability that exactly 14 seeds will germinate

Solution:

Let X be the random variable number of seeds that germinate

$p = \frac{3}{4}, q = \frac{1}{4}, n = 16$

$$X \sim B\left(16, \frac{3}{4}\right)$$

(a) $E(X) = np = 16 \times \frac{3}{4} = 12$ seeds

(b) $P(X = 14) = {}^{16}C_{14}\left(\frac{3}{4}\right)^{14}\left(\frac{1}{4}\right)^{2} = 0.1336$

Self-Evaluation exercise

1. 30% of students in a school travel to school by bus. From a sample of ten students chosen at random, find the probability that
   (a) only three travel by bus
   (b) less than half travel by bus

   [Ans:(a) 0.267 (b) 0.850]
2. In a survey on washing powder, it is found that the probability that a shopper chooses Omo is 0.25. Find the probability that in a random sample of nine shoppers
   (a) exactly three choose Omo
   (b) more than seven chose Omo

   [Ans: (a) 0.234 (b) 0.000107]
3. The random variable X is $B(6, 0.42)$. Find
   (a) $P(X = 6)$
   (b) $P(X = 4)$
   (c) $P(X \leq 2)$

   [Ans: (a) 0.00549 (b) 0.157 (c) 0.503]
4. An unbiased die is thrown seven times. Find the probability of throwing at least 5 sixes

   [Ans: 0.002]
5. A fair coin is tossed six times. Find the probability of throwing at least four heads.

   [Ans: 0.344]
6. In a test, there are ten multiple choice questions. For each, there is a choice of four answers, only one of which is correct. A student guesses each of the answers
   (a) Find the probability that he gets more than seven correct
   (b) If he needs to obtain over half marks to pass and each question carries equal weight, find the probability that he passes the test

   [Ans:(a) 0.000416 (b) 0.0197]
7. The probability that it will be a fine day is 0.4. Find the
   (a) expected number of fine days in a week
   (b) the standard deviation in the week

   [Ans: (a) 2.8 (b) 1.3]
8. The probability of a football team winning a match is 0.75. If the team has five matches to play, find the probability that it will win at least three of these matches

   [Ans: 0.8965]
9. Of 1000 patients who visited a health center, 250 of them were diagnosed of malaria. If a sample of 5 was drawn at random from the patients, what is the probability that
   (i) 2 of the patients had malaria
   (ii) 4 of the patients did not have malaria

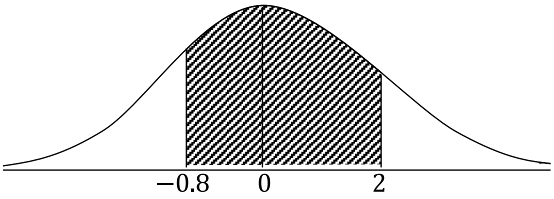
   [Ans: (i) 0.2637 (ii) 0.3955]
10. In a large city, one person in five is left handed. Find the probability that in a random sample of 10 people;
    (a) exactly three will be left handed
    (b) more than a half are left handed

    [Ans:(i)0.2013 (ii) 0.0064]
11. A man's chance of hitting a target with each of his shots is $\frac{1}{5}$.
    (a) If he has to fire five shots, calculate the probability that
    (i) exactly 3 shots hit the target
    (ii) at least two shots hit the target
    (b) Given that he has 20 shots to fire, determine the mean number and variance of his shots.at the target.

    [Ans: (a)(i) 0.0512 (ii) 0.2627 (b) 4, 3.2]
12. The probability that a student guesses the answer correctly to a multiple-choice question is $\frac{1}{4}$. If a quiz has 15 multiple choice questions, determine the probability that a student guesses correctly the answers to
    (i) exactly six questions
    (ii) at most three questions
    (iii) between three and eight questions

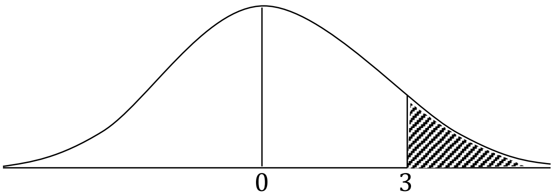
    [Ans: (i) 0.0917 (ii) 0.4613 (iii) 0.5213]
13. It was found out that 20% of a sample of chicken recovered from a rare disease after treatment. In a random sample of 5 of such treated chicken, find the probability that
    (i) there is more than one that recovered
    (ii) either 3 or 4 recovered
    (iii) less than 4 recovered

    [Ans:(i)0.2627 (ii) 0.0567(iii) 0.9933]



$P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$
$= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$
$= P(0 \le Z \le 0.8) + P(0 \le Z \le 2)$
$= 0.2881 + 0.4772$
$= 0.7653$

(b) When $X = 45$, $Z = \frac{45-30}{5} = 3$



$P(X \ge 45) = P(Z \ge 3) = 0.5 - P(0 \le Z \le 3)$
$= 0.5 - 0.4987$
$= 0.0013$

(c) $P(|X - 30| \le 5) = P(25 \le X \le 35)$
$$= P\left(\frac{25-30}{5} \le Z \le \frac{35-30}{5}\right)$$
$= P(-1 \le Z \le 1)$
$= 2P(0 \le Z \le 1)$
$= 2 \times 0.3413$
$= 0.6826$
$P(|X - 30| > 5) = 1 - P(|X - 30| \le 5)$
$= 1 - 0.6826$
$= 0.3174$

Example 2

The time taken by the milk man to deliver to Kampala Market Street is normally distributed with a mean of 12 minutes and standard deviation of 2 minutes. He delivers milk every day. Estimate the number of days during the year when he takes

(a) longer than 17 minutes
(b) less than 10 minutes
(c) between 9 and 13 minutes

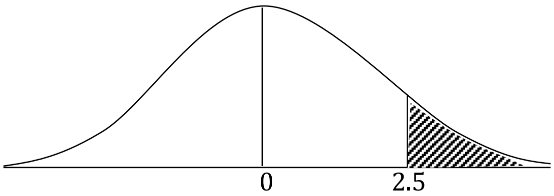
Solution:

Let X be the time in minutes, taken to deliver milk to Market Street

$$X \sim N(12, 2^2)$$

Standardizing X using $Z = \frac{X-\mu}{\sigma}$ i.e. $\frac{X-12}{2}$

(a) $P(X > 17) = P(Z > \frac{17-12}{2})$
$= P(Z > 2.5)$

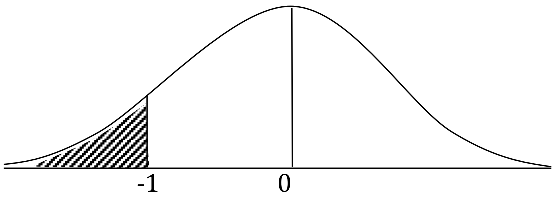


$P(Z > 2.5) = 0.5 - P(0 < Z < 2.5)$
$= 0.5 - 0.4938 = 0.0062$

To find the number of days, multiply by 365
$365 \times 0.0062 = 2.263 \approx 2$

On 2 days in the year, he takes longer than 17 minutes

(b) $P(X < 10) = P(Z < \frac{10-12}{2})$
$= P(Z < -1)$

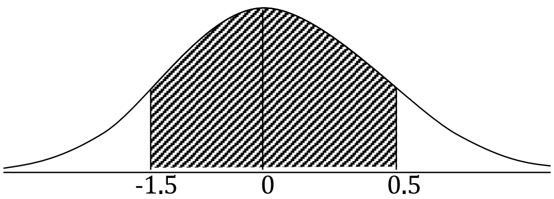


$P(Z < -1) = 0.5 - P(-1 < Z < 0)$
$= 0.5 - 0.3413 = 0.1587$

Now $365 \times 0.1587 = 57.92 \approx 58$

On 58 days in the year, he takes less than 10 minutes

(c) $P(9 < X < 13) = P(\frac{9-12}{2} < Z < \frac{13-12}{2})$
$= P(-1.5 < Z < 0.5)$



$P(-1.5 < Z < 0.5)$
$= P(-1.5 < Z < 0) + P(0 < Z < 0.5)$
$= 0.1915 + 0.4332 = 0.6247$

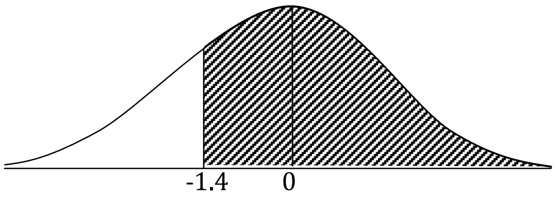
Now $365 \times 0.6247 = 228.01 \approx 228$

On 228 days in the year, he takes between 9 and 13 minutes

Example 3

A product sold in packets whose masses are normally distributed with a mean of 1.42 kg and a standard deviation of 0.025 kg

(a) Find the probability that the mass of a packet picked at random lies between 1.37 kg and 1.45 kg



$$P(Z > -1.4) = 0.5 + P(0 < Z < -1.4)$$
$$= 0.5 + 0.4192 = 0.9192$$

Calculating from probabilities

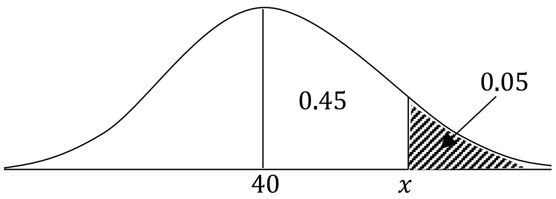
Example 7

The scores in a test are normally distributed with a mean of 40 and a standard deviation of 8. Find the score exceeded by 5% of the tests.

Solution:

Let X be the random variable, score in a test

$$X \sim N(40, 8^2)$$
$$P(X > x) = 5\% = 0.05$$
$$P\left(Z < \frac{x - 40}{8}\right) = 0.05$$



From the critical tables, $z_{0.05} = 1.645$

$$\frac{x - 40}{8} = 1.645$$
$$x - 40 = 13.16$$
$$x = 53.16$$

∴ The scores exceeded by 5% of the tests is 53.16

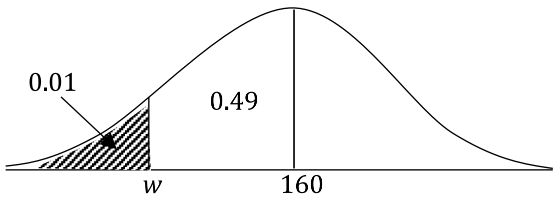
Example 8

The weights of fairy cakes, in grams, produced by a local baker are assumed to be normally distributed with mean of 160 and standard deviation of 5. Find the weight **not** exceeded by 1% of these fairy cakes.

Solution:

Let W be random variable, weight of a fairy cake.

$$W \sim N(160, 5^2)$$
$$P(W < w) = 0.01$$
$$P\left(Z < \frac{w - 160}{5}\right) = 0.01$$



$$\frac{w - 160}{5} = -2.326$$
$$w - 160 = -11.63$$
$$w = 148.37 \text{ grams}$$

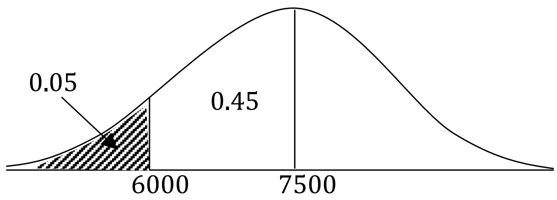
Example 9

The lifetimes of a certain brand of car tyres, in km, are normally distributed with a mean of 7500. Find the standard deviation, if 5% of these tyres last less than 6000 km.

Solution:

Let T = tyre lifetime (in km)

$$T \sim N(7500, \sigma^2)$$
$$P(T < 6000) = 0.05$$
$$P\left(Z < \frac{6000 - 7500}{\sigma}\right) = 0.05$$
$$P\left(Z < -\frac{1500}{\sigma}\right) = 0.05$$



$$-\frac{1500}{\sigma} = -1.645$$
$$\sigma = 912 \text{ km}$$

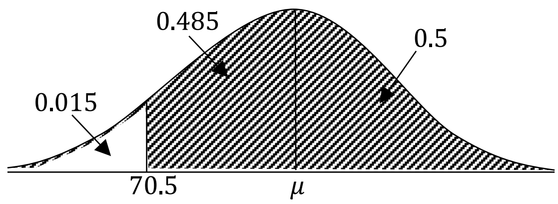
Example 10

The weights of eggs, in grams, classed as medium in size are thought to be normally distributed with a standard deviation of 6. Determine the mean weight of a medium size egg if 98.5% of the medium size eggs weigh more than 70.5 grams.

Solution:

Let W = weight of a medium size egg

$$W \sim N(\mu, 6^2)$$
$$P(W > 70.5) = 0.985$$
$$P\left(Z > \frac{70.5 - \mu}{6}\right) = 0.985$$



From the normal probability tables, $z_{0.485} = 2.170$

$$\frac{70.5 - \mu}{6} = -2.170$$
$$70.5 - \mu = -13.02$$
$$\mu = 83.02 \text{ grams}$$

$= 0.3362$

Therefore, 33.62% candidates got second division in the examination.

Examination questions

2013, No. 13

A bakery produces loaves of bread whose weight is normally distributed with mean 1000 g and standard deviation 40 g

(a) Find the probability that a randomly selected loaf has a weight of utmost 1020 g.

(b) Assuming that the bakery makes 10500 loaves, find the approximate number of loaves with a weight greater than 950 g.

Solution:

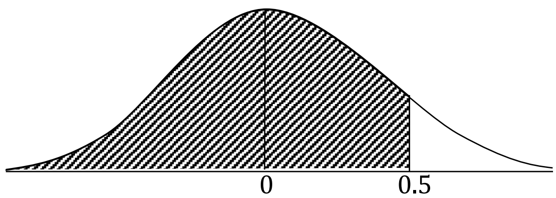
Let X be the random variable weight of loaves of bread

$$X \sim N(1000, 40^2)$$

(a) Utmost 1020 g means a value that does not exceed 1020 i.e. it must be below or equal to.

$$P(X \leq 1020) = P\left(Z \leq \frac{1020 - 1000}{40}\right)$$

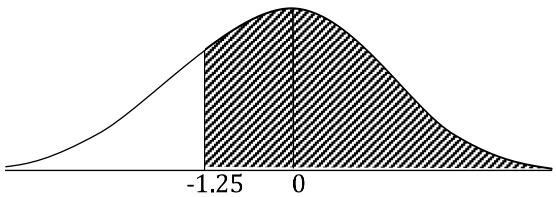
$$= P\left(Z \leq \frac{20}{40}\right) = P(Z \leq 0.5)$$



$$P(Z \leq 0.5) = 0.5 + P(0.5 \leq Z \leq 0)$$
$$= 0.5 + 0.1915 = 0.6915$$

(b) $P(X > 950) = P\left(Z > \frac{950-1000}{40}\right)$

$$= P\left(Z > -\frac{50}{40}\right) = P(Z > -1.25)$$



$$P(Z > -1.25) = 0.5$$
$$+ P(-1.25 < Z < 0)$$
$$= 0.5 + 0.3944 = 0.8944$$

Approximate number of loaves

$= 10500 \times 0.8944 = 9391.2 \approx 9391$

The number of loaves with a weight greater than 950 g is 9391

2018, No. 11

A factory sells animal food in bags. The weights of the bags are normally distributed with mean weight 50 kg and standard deviation 2.8 kg.

(a) Find the probability that the weight of any bag selected at random;

(i) is more than 52 kg

(ii) lies between 46 and 55 kg

(b) Determine the percentage of bags whose weights are less than 54 kg

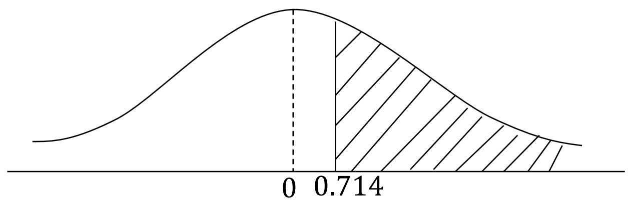
Solution:

Let X be the random variable weight of bags

$$X \sim N(\mu, \delta^2)$$

(a) (i) $P(X > 52)$

$$P(X > 52) = P\left(Z > \frac{52 - 50}{2.8}\right)$$
$$= P(Z > 0.714)$$

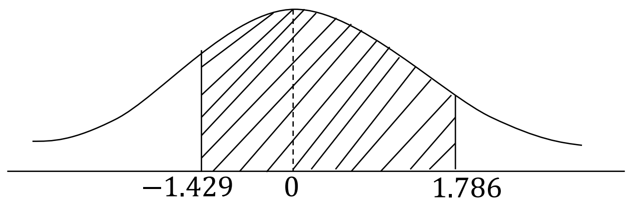


$$P(X > 52) = 0.5 - P(0 < Z < 0.714)$$
$$= 0.5 - 0.2623$$
$$= 0.2377$$

(ii) $P(46 < X < 55)$

$P(46 < X < 55)$

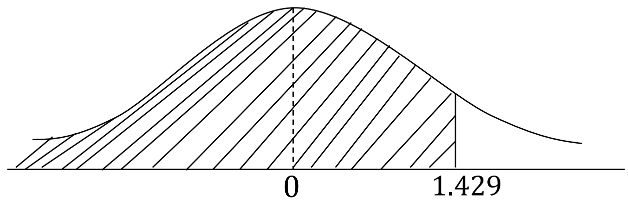
$$= P\left(\frac{46-50}{2.8} < Z < \frac{55-50}{2.8}\right)$$
$$= P(-1.429 < Z < 1.786)$$



$$= P(-1.429 < Z < 0) + P(0 < Z < 1.786)$$
$$= 0.4235 + 0.4630$$
$$= 0.8865$$

(b) $P(X < 54) = P\left(Z < \frac{54-50}{2.8}\right)$

$$= P(Z < 1.429)$$



$$P(Z < 1.429) = 0.5 + P(0 < Z < 1.429)$$
$$= 0.5 + 0.4235$$

$= 0.9235$

$\therefore$ 92.35% weighs less than 54 kg

2019, No. 11

The marks scored by candidates in an examination are normally distributed with a mean score of μ and stardard deviation of σ. Given that 37.5% of the candiadates scored below 40 and 12.5% scored above 60, find the;

(a) values of μ and σ

(b) probability that a candidate scored between 46 and 55

Solution:

Let X the random variable, marks scored

$$X \sim N(\mu, \sigma^2)$$

$P(X < 40) = 0.375$ and $P(X > 60) = 0.125$

(a) $P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.375$

$$\frac{40-\mu}{\sigma} = -0.319$$

$$40 - \mu = -0.319\sigma \quad \text{.... (i)}$$

$$P\left(Z > \frac{60-\mu}{\sigma}\right) = 0.125$$

$$\frac{60-\mu}{\sigma} = 1.15$$

$$60 - \mu = 1.15\sigma \quad \text{... (ii)}$$

(ii) – (i); $20 = 1.469\sigma$

$$\sigma = 13.6$$

Substituing for μ in (ii);

$$60 - \mu = 1.15\sigma$$

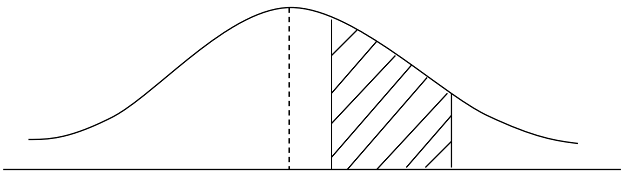
$$\mu = 60 - 1.15(13.6)$$

$$\mu = 44.4$$

(b) $P(46 < X < 55)$

$$= P\left(\frac{46-44.4}{13.6} < Z < \frac{55-44.4}{13.6}\right)$$

$$= P(0.118 < Z < 0.779)$$



$$= P(0 < Z < 0.779) - P(0 < Z < 0.118)$$

$$= 0.2822 - 0.0470$$

$$= 0.2352$$

2022, No. 11

The time taken for a bus to make a journey is normally distributed with mean 3½ hours and standard deviation ¾ hours.

(a) Determine the probability that the bus makes a journey

(i) in less than 2 hours

(ii) between 3¼ and 3¾ hours

(b) If the bus made two hundred journeys, how many of these journeys did it take less than 2 hours?

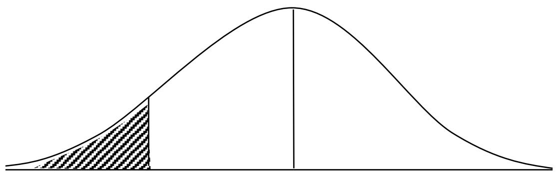
Solution:

Let X be the random variable, the time taken for a bus to make a journey.

$$X \sim N(3.5, 0.75^2)$$

(a) (i) $P(X < 2)$

$$= P\left(Z < \frac{2-3.5}{0.75}\right) = P(Z < -2)$$



$$P(Z < -2) = 0.5 - P(0 \le Z \le -2)$$

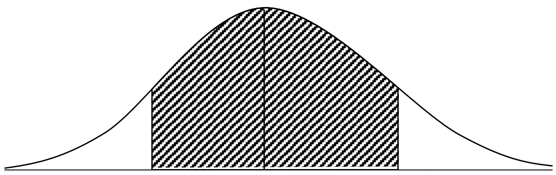
$$= 0.5 - 0.4772 = 0.0228$$

$$\therefore P(X < 2) = 0.0228$$

(ii) $P(3.25 < X < 3.75)$

$$= P\left(\frac{3.25-3.5}{0.75} < Z < \frac{3.75-3.5}{0.75}\right)$$

$$= P(-0.25 < Z < 0.25)$$



$P(-0.25 < Z < 0.25)$

$$= P(-0.25 \le Z \le 0) + P(0 \le Z \le 0.25)$$

$$= 0.0987 + 0.0987 = 0.1974$$

$$\therefore\ P(3.25 < X < 3.75) = 0.1974$$

(b) $P(X < 2) = 0.228$

$n = 200$

Number of journeys that took the bus less than two hours = $200 \times 0.0228 = 4.56$

Self-Evaluation exercise

1. The masses of packages from a particular machine are normally distributed with a mean of 200 g and standard deviation 2 g. Find the probability that a randomly selected package from the machine weighs
   (a) less than 197 g
   (b) more than 200.5 g
   (c) between 198.5 g and 199.5 g

   [Ans:(a) 0.0668 (b) 0.4013 (c) 0.1747]
2. The heights of boys at a particular age follow a normal distribution with mean 150.3 and variance 25 cm. Find the probability that a boy chosen at random from his age group has a height
   (a) Less than 153 cm
   (b) More than 158 cm
   (c) Between 150 cm and 158 cm

   [Ans: (a) 0.7054 (b) 0.0618 (c) 0.4621]
3. The random variable X is distributed normally such that $X \sim N(50, 20)$. Find
   (a) $P(X > 60.3)$ (b) $P(X < 59.8)$

   [Ans : (a) 0.0106 (b) 0.9857]
4. The masses of a certain type of cabbage are normally distributed with a mean of 1000 g and standard deviation of 150 g. In a batch of 800 cabbages, estimate how many have a mass between 750 g and 1290 g

   [Ans: 740]
5. The lifetime of a certain make of electric bulbs is known to be normally distributed with a mean life of 2000 hours and standard deviation of 120 hours. Estimate the probability that the life of such a bulb will be
   (a) greater than 2150 hours
   (b) greater than 1910 hours
   (c) between 1850 hours and 2090 hours

   [Ans: (a) 0.1056 (b) 0.7734 (c) 0.6678]
6. The manufacturers of a new model of a car state
7. that, when travelling at 56 miles per hour, the petrol consumption has a mean value of 32.4 miles per gallon with standard deviation of 1.4 miles per gallon. Assuming a normal distribution, calculate the probability that a randomly chosen car of that model will have petrol consumption greater than 30 miles per gallon when travelling at 56 miles per hour.

   [Ans: 0.957]
8. The processing time of a newly manufactured product is normally distributed with mean 110.5 minutes and standard deviation 12 minutes. Find the probability that the product is processed between 108 and 119 minutes

   [Ans: 0.3429]
9. In an orange plantation, the weights of oranges are normally distributed with a mean of 210 g and variance 30 g. Find the percentage of oranges that
   (i) weigh between 201 g and 221 g
   (ii) weigh 197 g and below

   [Ans: (i) 0.9276 (ii) 0.0088]
10. The time taken by Sam to pray is normally distributed with a mean of 24 minutes and a standard deviation of 4 minutes
    (a) If he prays every day, find the probability that his prayers take
    (i) more than 34 minutes
    (ii) at most 20 minutes
    (b) In 1000 days, estimate the number of days in which he prays between 34 and 36 minutes.

    [Ans: (a)(i) 0.0062 (ii) 0.1587 (b) 5]
11. The marks obtained in an aptitude test are normally distributed with mean 54 and standard deviation 14.2 . Determine the probability that an examinee scored
    (i) between 60 and 70 marks
    (ii) at least 40 marks

    [Ans: (i) 0.2063 (ii) 0.3379]
12. The marks obtained by UACE candidates were found to be normally distributed with mean 50 and standard deviation 10
    (i) Determine the percentage of candidates who obtained more than 70 marks
    (ii) What percentage of the candidates obtained between 40 and 60 marks?
    (iii) What is the probability that a candidate selected at random from those who scored well above the average, scored more than 65?

    [Ans:(i) 2.28%(ii) 68.3%(iii) 0.0688]
13. The mean lifetime of a certain make of dry cells is 150 days and standard deviation 32 days. Their duration is normally distributed.
    (a) Find the probability that the cells will last between 125 and 210 days

SAMPLE EXAMINATION PAPER

S475/1
SUBSIDIARY MATHEMATICS
Paper 1
2⅔ hours

Uganda Advanced Certificate of Education

SUBSIDIARY MATHEMATICS

Paper 1

2 hours 40 minutes

INSTRUCTIONS TO CANDIDATES:

Answer **all** *the* **eight** *questions in section* **A** *and only* **four** *questions from section* **B** *with* **at least one** *question from each part.*

Any addition question (s) answered will not be marked

Each question is section **A** *carries* **5** *marks and each question in section* **B** *carries* **15** *marks*

All *necessary working* **must** *be shown clearly*

Begin each answer on a fresh sheet of paper.

Graph paper is provided

Silent non-programmable scientific calculators and mathematical tables with a list of formulae may be used

SECTION A (40 MARKS)

Answer **all** *the questions in this section*

1. The marks scored in a test by 8 students are 3, 4, -1, 22, 14, 0, 9, 18. Determine the
   (a) mean mark (*02 marks*)
   (b) variance (*03 marks*)

2. Evaluate $\int_{-1}^{2} \frac{2x^4 - 6x^3}{2x^2} dx$ (*05 marks*)

3. A random variable x has a probability distribution given by

$$P(X = x) = \begin{cases} \frac{x}{5k}, & x = 1, 2, 3, 4 \\ 0 & \text{Elsewhere} \end{cases}$$

   Calculate the:
   (a) value of k (*02 marks*)
   (b) mean of X, $E(X)$ (*03 marks*)

4. The table below shows the current and base year prices together with the weights for items A, B, C, D, E, F

Item	Weight	Base year price (shs)	Current Price (shs)
A	25	3500	4650
B	10	800	1200
C	35	600	900
D	20	2500	3000
E	1	7500	10500
F	9	3000	3150

   Calculate the weighted average price related index number and comment on your result (*05 marks*)

5. Solve the equation $2\sin\theta\cos\theta = \tan\theta$ for values $0° < \theta < 180°$ (*05 marks*)

6. Express $\frac{2}{\sqrt{5}+\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}}$ in the form $a\sqrt{b}$, where a and b are integers (*05 marks*)

7. Using matrix method to solve the simultaneous equations

$$3x^2 + 5y = 2$$

$$2x^2 - 3y = 14$$

   (*05 marks*)

8. For the function $f(x) = ax^3 + bx^2 + 4x - 3$, given that $f'(2) = 0$ and $f''(2) = 10$, find the values of a and b (*05 marks*)

SECTION B (60 MARKS)

Answer only **four** questions from this section including at **least one** equation from each part

PART I: PURE MATHEMATICS

9. The equation of a curve is $y = 4x - x^2$
   (a) (i) Determine the turning point of the curve
   (ii) Find the nature of the turning point
   (iii)Sketch the graph of the curve

   (*07 marks*)

   (b) The curve and the line $y = 3$ intersect at the point (1, 3) and (3, 3). Calculate the area of the region enclosed between the line and the curve.

   (*08 marks*)

10. The table below shows the processing time of products 1 and 2 on two machines A and B manufactured by a certain company together with the selling price per unit and the maximum possible sales (units).

	Product 1	**Product 2**
Processing time for A (hours per unit)	4	4
Processing time for B (hours per unit)	2	4
Selling price (shillings per unit)	2000	2500
Maximum possible sales (unit)	260	300

Given that the amount of time available on the machine A is 720 hours, machine B is 520 hours and X units and Y units of products of 1 and 2 produced are all sold.

(a) Write down linear inequalities which when solved would indicate the number of product 1 and 2 if sold will maximise the total revenue of the company.

(*04 marks*)

(b) Illustrate the inequalities in (a) above on the graph indicating constraints on both machines

(*06 marks)*

(c) By shading the unwanted regions, determine the maximum revenue for the company

(*05 marks*)

11. Points A, B and C have position vectors $4i - j$, $i + 3j$ and $5i + 2j$ respectively in the x-y plane
   (a) Find the value of $3OA + 4OB - 2OC$ (*04 marks*)
   (b) Determine
   (i) AB and AC (*04 marks*)
   (ii) $AB\,.\,AC$ (*02 marks*)
   (iii) angle ABC (*05 marks*)

12. (a) A farmer sells four of his farm products maize (M), potatoes (P), carrots (C) and tomatoes (T) in bags in each of the towns Masaka and Mbarara to three classes of customers: wholesalers, retailers, consumers as shown in the table below

Town	**MASAKA**				**MBARARA**			
Product	M	P	C	T	M	P	C	T
Wholesalers	40	60	70	40	40	50	30	60
Retailers	30	20	10	60	70	80	40	40
Consumers	20	40	0	10	40	30	50	30

Given that the selling price in thousands of shillings per bag of maize, potatoes, carrots and tomatoes were 50, 40, 60 and 35 respectively, using matrix algebra

(i) find the total sales in bags by product and customer type

(ii) calculate the total sales in shillings for each town

(*10 marks*)

(b) A certain section of an examination has two parts with four questions from each a part to make a total of eight questions. A candidate is required to attempt four questions taking at least one from each part. In how many ways can the candidate make the selection? (*05 marks*)

PART II: STATICSTICS

13. Weights of bags of cement manufactured by a certain factory are normally distributed with a mean of 50 kg and a standard deviation of 2 kg. 2% of the bags are rejected for being underweight and 1% of the bags are rejected for being overweight.

(a) Determine the range of values the weight of a bag should lie if it has to be accepted

(*10 marks*)

(b) Calculate the probability that a bag picked at random from a large consignment will weigh less than 47.5 kg (*05 marks*)

14. The following table shows the prices in (shs 000”) of a certain commodity for three consecutive years.

Year	**Quarters**			
	1st	**2nd**	**3rd**	**4th**
2005	85	63	70	92
2006	82	61	64	84
2007	71	56	62	78

(a) Calculate four quarterly moving averages (*06 marks*)

(b) On the same axes, plot the original data and the four quarterly moving averages. Comment on your results (*05 marks*)

(c) Draw a trend line through the points and use it to predict the price of the commodity in the 1st quarter of 2008. (*04 marks*)

15. The following distribution represents the height of 160 students of a school.

Height (in cm)	No. of students
140 – 145	12
145 – 150	20
150 – 155	30
155 – 160	38
160 – 165	24
165 – 170	16
170 – 175	12
175 – 180	8

(a) Calculate the mean height of the students (*05 marks*)

(b) Draw an ogive for the given distribution. Using the graph, determine

(i) the median height

(ii) the interquartile range

(iii) the number of students whose height is above 172 cm

(*10 marks*)

16. A psychologist selected a random sample of 22 students. He grouped them in 11 pairs so that the students in each pair have nearly equal scores in an intelligence test. In each pair, one student was taught by method A and the other by method B and examined after the course. The marks obtained by them after the course are as follows:

Pairs	1	2	3	4	5	6	7	8	9	10	11
Method A	24	29	19	14	30	19	27	30	20	28	11
Method B	37	35	16	26	23	27	19	20	16	11	21

(a) Draw a scatter diagram and a line of best fit to illustrate the data. Comment on your results (*07 marks*)

(b) If a certain student who used method B had a score of 25 in the intelligence test, find the score of the corresponding student who used method A (*02 marks*)

(c) Calculate a rank correlation coefficient and comment on your results. (*06 marks*)

END

BINOMIAL PROBABILITIES (DISTRIBUTION) B(*n*,*x*), INDIVIDUAL TERMS *Pr*

							x					
n	*r*	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
2	0	0.9801	9025	8100	7225	6400	5625	4900	4225	3600	3025	2500
	1	0.0198	0950	1800	2550	3200	3750	4200	4550	4800	4950	5000
	2	0.0001	0025	0100	0225	0400	0625	0900	1225	1600	2025	2500
3	0	0.9703	8574	7290	6141	5120	4219	3430	2746	2160	1664	1250
	1	0.0294	1354	2430	3251	3840	4219	4410	4436	4320	4084	3750
	2	0.0003	0071	0270	0574	0960	1406	1890	2389	2880	3341	3750
	3		0001	0010	0034	0080	0156	0270	0429	0640	0911	1250
4	0	0.9606	8145	6561	5220	4096	3164	2401	1785	1296	0915	0625
	1	0.0388	1715	2916	3685	4096	4219	4116	3845	3456	2995	2500
	2	0.0006	0135	0486	0975	1536	2109	2646	3105	3456	3675	3750
	3		0005	0036	0115	0256	0469	0756	1115	1536	2005	2500
	4			0001	0005	0016	0039	0081	0150	0256	0410	0625
5	0	0.9510	7738	5905	4437	3277	2373	1681	1160	0778	0503	0312
	1	0.0480	2036	3280	3915	4096	3955	3602	3124	2592	2059	1562
	2	0.0010	0214	0729	1382	2048	2637	3087	3364	3456	3369	3125
	3		0011	0081	0244	0512	0879	1323	1811	2304	2757	3125
	4			0004	0022	0064	0146	0284	0488	0768	1128	1562
	5				0001	0003	0010	0024	0053	0102	0185	0312
6	0	0.9415	7351	5314	3771	2621	1780	1176	0754	0467	0277	0156
	1	0.0571	2321	3543	3993	3932	3560	3025	2437	1866	1359	0938
	2	0.0014	0305	0984	1762	2458	2966	3241	3280	3110	2780	2344
	3		0021	0146	0415	0819	1318	1852	2355	2765	3032	3125
	4		0001	0012	0055	0154	0330	0595	0951	0382	1861	2344
	5			0001	0004	0015	0044	0102	0205	0369	0609	0938
	6					0001	0002	0007	0018	0041	0083	0156
7	0	0.9321	6983	4783	3206	2097	1335	0824	0490	0280	0152	0078
	1	0.0659	2573	3720	3960	3670	3115	2471	1848	1306	0872	0547
	2	0.0020	0406	1240	2097	2753	3115	3177	2985	2613	2140	1641
	3		0036	0230	0617	1147	1730	2269	2679	2903	2918	2734
	4		0002	0026	0109	0287	0577	0972	1442	1935	2388	2734
	5			0002	0012	0043	0115	0250	0466	0774	1172	1641
	6				0001	0004	0013	0036	0084	0172	0320	0547
	7						0001	0002	0006	0016	0037	0078
8	0	0.9227	6634	4305	2725	1678	1001	0576	0319	0168	0084	0039
	1	0.0746	2793	3826	3847	3355	2670	1977	1373	0896	0548	0312
	2	0.0026	0515	1488	2376	2936	3115	2965	2587	2090	1569	1094
	3	0.0001	0054	0331	0839	1468	2076	2541	2786	2787	2568	2188
	4		0004	0046	0185	0459	0865	1361	1875	2322	2627	2734
	5			0004	0026	0092	0231	0467	0808	1239	1719	2188
	6				0002	0011	0038	0100	0217	0413	0703	1094
	7					0001	0004	0012	0033	0079	0164	0312
	8							0001	0002	0007	0017	0039
9	0	0.9135	6302	3874	2316	1342	0751	0404	0207	0101	0046	0020
	1	0.0830	2985	3874	3679	3020	2253	1556	1004	0605	0339	0176
	2	0.0034	0629	1722	2597	3020	3003	2668	2162	1612	1110	0703
	3	0.0001	0077	0446	1069	1762	2336	2668	2716	2508	2119	1641
	4		0006	0074	0283	0661	1168	1715	2194	2508	2600	2461
	5			0008	0050	0165	0389	0735	1181	1672	2128	2461
	6			0001	0006	0028	0087	0210	0424	0743	1160	1641
	7					0003	0012	0039	0098	0212	0407	0703
	8						0001	0004	0013	0035	0083	0176
	9								0001	0003	0008	0020
10	0	0.9044	5987	3487	1969	1074	0563	0282	0135	0060	0025	0010
	1	0.0914	3151	3874	3474	2684	1877	1211	0725	0403	0207	0098
	2	0.0042	0746	1937	2759	3020	2816	2335	1757	1209	0763	0439
	3	0.0001	0105	0574	1298	2013	2503	2668	2522	2150	1665	1172
	4		0010	0112	0401	0881	1460	2001	2377	2508	2384	2051
	5		0001	0015	0085	0264	0584	1029	1536	2007	2340	2461
	6			0001	0012	0055	0162	0368	0689	1115	1596	2051
	7				0001	0008	0031	0090	0212	0425	0746	1172

Where a space in the table is empty the probability is less than 0.00005.

BINOMIAL PROBABILITIES (DISTRIBUTION) B(*n*,*x*), INDIVIDUAL TERMS *Pr*

n	*r*	*x* = 0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
10	8					0001	0004	0014	0043	0106	0229	0439
	9							0001	0005	0016	0042	0098
	10									0001	0003	0010
11	0	0.8953	5688	3138	1673	0859	0422	0198	0088	0036	0014	0005
	1	0.0995	3293	3835	3248	2362	1549	0932	0518	0266	0125	0054
	2	0.0050	0867	2131	2866	2953	2581	1998	1395	0887	0513	0269
	3	0.0002	0137	0710	1517	2215	2581	2568	2254	1774	1259	0806
	4		0014	0158	0536	1107	1721	2201	2428	2365	2060	1611
	5		0001	0025	0132	0388	0803	1321	1830	2207	2360	2256
	6			0003	0023	0097	0268	0566	0985	1471	1931	2256
	7				0003	0017	0064	0173	0379	0701	1128	1611
	8					0002	0011	0037	0102	0234	0462	0806
	9						0001	0005	0018	0052	0126	0269
	10								0002	0007	0021	0054
	11										0002	0005
12	0	0.8864	5404	2824	1422	0687	0317	0138	0057	0022	0008	0002
	1	0.1074	3413	3766	3012	2062	1267	0712	0368	0174	0075	0029
	2	0.0060	0988	2301	2924	2835	2323	1678	1088	0639	0339	0161
	3	0.0002	0173	0852	1720	2362	2581	2397	1954	1419	0923	0537
	4		0021	0213	0683	1329	1936	2311	2367	2128	1700	1208
	5		0002	0038	0193	0532	1032	1585	2039	2270	2225	1934
	6			0005	0040	0155	0401	0792	1281	1766	2124	2256
	7				0006	0033	0115	0291	0591	1009	1489	1934
	8				0001	0005	0024	0078	0199	0420	0762	1208
	9					0001	0004	0015	0048	0125	0277	0537
	10							0002	0008	0025	0068	0161
	11								0001	0003	0010	0029
	12										0001	0002
15	0	0.8601	4633	2059	0874	0352	0134	0047	0016	0005	0001	
	1	0.1303	3658	3432	2312	1319	0668	0305	0126	0047	0016	0005
	2	0.0092	1348	2669	2856	2309	1559	0916	0476	0219	0090	0032
	3	0.0004	0307	1285	2184	2501	2252	1700	1110	0634	0318	0139
	4		0049	0428	1156	1876	2252	2186	1792	1268	0780	0417
	5		0006	0105	0449	1032	1651	2061	2123	1859	1404	0916
	6			0019	0132	0430	0917	1472	1906	2066	1914	1527
	7			0003	0030	0138	0393	0811	1319	1771	2013	1964
	8				0005	0035	0131	0348	0710	1181	1647	1964
	9				0001	0007	0034	0116	0298	0612	1048	1527
	10					0001	0007	0030	0096	0245	0515	0916
	11						0001	0006	0024	0074	0191	0417
	12							0001	0004	0016	0052	0139
	13								0001	0003	0010	0032
	14										0001	0005
20	0	0.8179	3585	1216	0388	0115	0032	0008	0002			
	1	0.1652	3774	2702	1368	0576	0211	0068	0020	0005	0001	
	2	0.0159	1887	2852	2293	1369	0669	0278	0100	0031	0008	0002
	3	0.0010	0596	1901	2428	2054	1339	0716	0323	0123	0040	0011
	4		0133	0898	1821	2182	1897	1304	0738	0350	0139	0046
	5		0022	0319	1028	1746	2023	1789	1272	0746	0365	0148
	6		0003	0089	0454	1091	1686	1916	1712	1244	0746	0370
	7			0020	0160	0545	1124	1643	1844	1659	1221	0739
	8			0004	0046	0222	0609	1144	1614	1797	1623	1201
	9			0001	0011	0074	0271	0654	1158	1597	1771	1602
	10				0002	0020	0099	0308	0686	1171	1593	1762
	11					0005	0030	0120	0336	0710	1185	1602
	12					0001	0008	0039	0136	0355	0727	1201
	13						0002	0010	0045	0146	0366	0739
	14							0002	0012	0049	0150	0370
	15								0003	0013	0049	0148
	16									0003	0013	0046
	17										0002	0011
	18											0002

If the probability of success in a single trial is x the probability Pr of exactly r successes in n independent trials is given by the binomial or Bernoulli distribution $B(n, x_p)$:

$$Pr = \binom{n}{r} x^r (1-x)^{n-r}$$

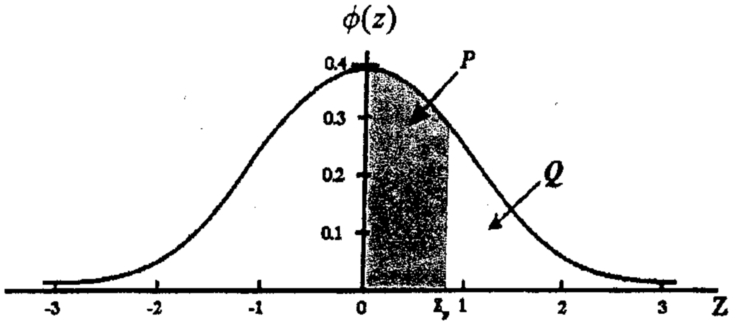
CUMULATIVE NORMAL DISTRIBUTION $P(z)$

											ADD								
Z	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.0	0.0000	0040	0080	0120	0160	0199	0239	0279	0319	0359	4	8	12	16	20	24	28	32	36
0.1	0.0398	0438	0478	0517	0557	0596	0636	0675	0714	0753	4	8	12	16	20	24	28	32	36
0.2	0.0793	0832	0871	0910	0948	0987	1026	1064	1103	1141	4	8	12	15	19	22	27	31	35
0.3	0.1179	1217	1255	1293	1331	1368	1406	1443	1480	1517	4	8	11	15	19	22	26	30	34
0.4	0.1554	1591	1628	1664	1700	1736	1772	1808	1844	1879	4	7	11	14	18	22	25	29	32
0.5	0.1915	1950	1985	2019	2054	2088	2123	2157	2190	2224	3	7	10	14	17	21	24	27	31
0.6	0.2257	2291	2324	2357	2389	2422	2454	2486	2517	2549	3	6	10	13	16	19	23	26	29
0.7	0.2580	2611	2642	2673							3	6	9	12	15	19	22	25	28
					2704	2734	2764	2794	2823	2852	3	6	9	12	15	18	21	24	27
0.8	0.2881	2910	2939	2967	2995	3023					3	6	8	11	14	17	20	22	25
							3051	3078	3106	3133	3	5	8	11	13	16	19	22	24
0.9	0.3159	3186	3212	3238	3264	3289					3	5	8	10	13	16	18	21	23
							3315	3340	3365	3389	2	5	7	10	12	15	17	20	22
1.0	0.3413	3438	3461	3485	3508						2	5	7	10	12	14	17	19	22
						3531	3554	3577	3599	3621	2	4	7	9	11	13	15	18	20
1.1	0.3643	3665	3686	3708							2	4	6	8	11	13	15	17	19
					3729	3749	3770	3790	3810	3830	2	4	6	8	10	12	14	16	18
1.2	0.3849	3869	3888	3907	3925						2	4	6	8	10	11	13	15	17
						3944	3962	3980	3997	4015	2	4	5	7	9	11	13	14	16
1.3	0.4032	4049	4066	4082	4099	4115	4131	4147	4162	4177	2	3	5	6	8	10	11	13	14
1.4	0.4192	4207	4222	4236	4251	4265	4279	4292	4306	4319	1	3	4	6	7	8	10	11	13
1.5	0.4332	4345	4357	4370	4382	4394	4406	4418	4429	4441	1	2	4	5	6	7	8	10	11
1.6	0.4452	4463	4474	4484	4495	4505	4515	4525	4535	4545	1	2	3	4	5	6	7	8	9
1.7	0.4554	4564	4573	4582	4591	4599	4608	4616	4625	4633	1	2	3	3	4	5	6	7	8
1.8	0.4641	4649	4656	4664	4671	4678	4686	4693	4699	4706	1	1	2	3	4	4	5	6	6
1.9	0.4713	4719	4726	4732	4738	4744	4750	4756	4761	4767	1	1	2	2	3	4	4	5	5
2.0	0.4772	4778	4783	4788	4793	4798	4803	4808	4812	4817	0	1	1	2	2	3	3	4	4
2.1	0.4821	4826	4830	4834	4838	4842	4846	4850	4854	4957	0	1	1	2	2	2	3	3	4
2.2	0.4861	4864	4868	4871	4875	4878	4881	4884	4887	4890	0	1	1	1	2	2	2	3	3
2.3	0.4893	4896	4898	4901	4904	4906	4909	4911	4913	4916	0	0	1	1	1	2	2	2	2
2.4	0.4918	4920	4922	4925	4927	4929	4931	4932	4934	4936	0	0	1	1	1	1	1	2	2
2.5	0.4938	4940	4941	4943	4945	4946	4948	4949	4951	4952									
2.6	0.4953	4955	4956	4957	4959	4960	4961	4962	4963	4964									
2.7	0.4965	4966	4967	4968	4969	4970	4971	4972	4973	4974									
2.8	0.4974	4975	4976	4977	4977	4978	4979	4979	4980	4981									
2.9	0.4981	4982	4982	4983	4984	4984	4985	4985	4986	4986									
3.0	0.4987	4990	4993	4995	4997	4998	4998	4999	4999	5000									

The table gives $P(z) = \int_0^z \phi(z)dz$

If the random variable Z is distributed as the standard normal distribution N(0,1) then:

1. $P(0 < Z < z_p) = P(\text{Shaded Area})$
2. $P(Z > Z_p) = Q = ½ - P$
3. $P(Z > |Z_p|) = 1 - 2P = 2Q$



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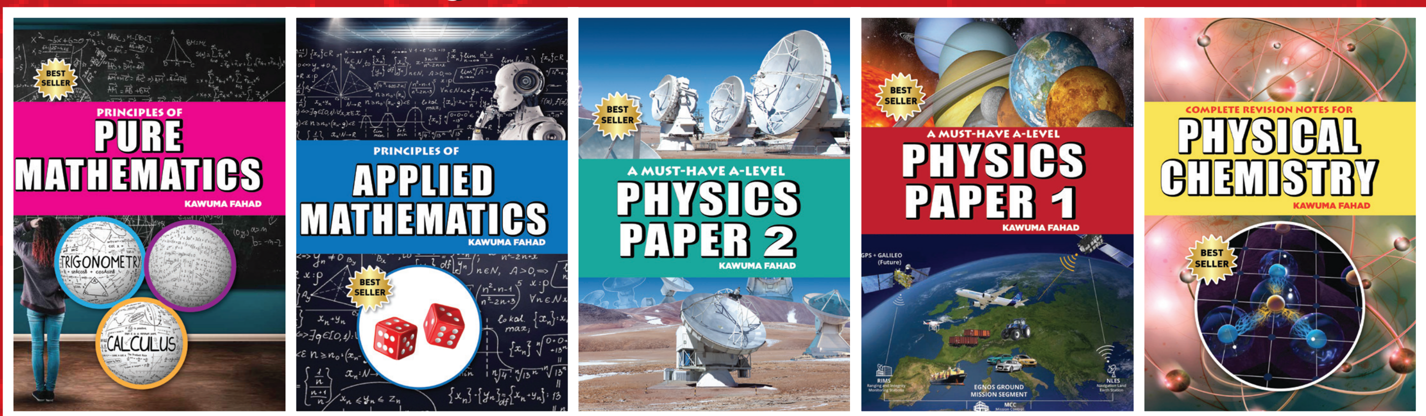

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