TOPIC 4: PROBABILITY THEORY

Probability simply means chance or possibility.

Terms used:

Sample space(S):

It is a set of all possible outcomes of an experiment. Each possible outcome is called a sample point. For example if a die is tossed once, the sample space S= $S = \{1,2,3,4,5,6\}$. This sample space has 6 sample points.

Event (E):

An event is a subset of a sample space. If E is the event of obtaining an odd number then, E= $\{1,3,5\}$

Probability of an event:

If a sample space, S, consists of a finite number of equally likely outcomes, then the probability that an event E occurs is given by

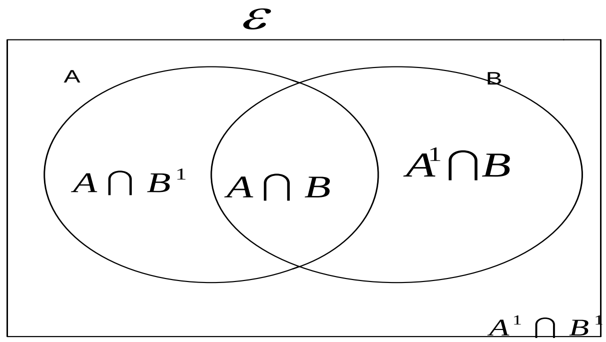
$$P(E) = \frac{n(E)}{n(S)}$$

Where $n(E)$ = number of times an event occurs

$n(S)$ = total sample points

Interaction of probability with set theory

Here we relate probability theory with set theory



From the set above we can be able to deduce the following

$$P(A) = P(A \cap B) + P(A \cap B^1)$$
$$P(A^1) = P(A^1 \cap B^1) + P(A^1 \cap B)$$
$$P(A^1) + P(A) = 1$$

$$P(B) = P(A \cap B) + P(A^1 \cap B)$$
$$P(B^1) = P(A \cap B^1) + P(A \cap B^1)$$
$$P(B) + P(B^1) = 1$$

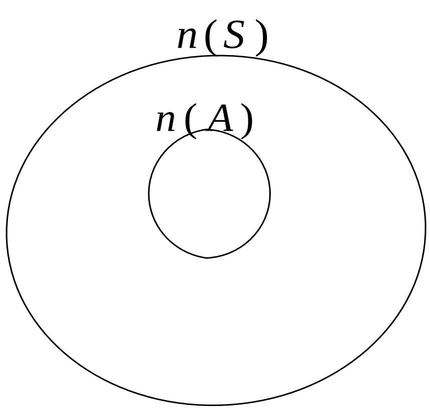
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

PROBABILITY LAWS

a) For an event A in a sample space S, then

$$0 \le P(A) \le 1$$

Proof



$0 \le n(A) \le n(S)$ ....(i)

Divide (i) by n(S), we get

$$\frac{0}{n(S)} \le \frac{n(A)}{n(S)} \le \frac{n(S)}{n(S)}$$
$$\Rightarrow 0 \le P(A) \le 1$$

Show that $P(A) + P(A') = 1$

$$P(A) = \frac{n(A)}{n(S)}, P(A') = \frac{n(A')}{n(S)} = \frac{n(S) - n(A)}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(A)}{n(S)} = 1 - P(A)$$

From $P(A') = 1 - P(A)$.

$P(A) + P(A') = 1$

b) The Demorgan's law

$P(A'UB') = P(A \cap B)'$ and $P(A' \cap B') = P(AUB)'$

Types of events

a) Independent events: Two events A and B are said to be independent if the occurrence of one does not affect the other. If two events A and B are independent then $P(A \cap B) = P(A).P(B)$. Also $P(A' \cap B) = P(A').P(B), P(A' \cap B') = P(A').P(B')$

b) Mutually exclusive events: Two events A and B are said to be mutually exclusive if they cannot happen at the same time. Therefore A and B are said to be mutually exclusive if $P(A \cap B) = 0$

c) Exhaustive events: Two events A and B are said to be exhaustive if $P(A \cup B) = 1$

The contingency table

	B	B'	
A	P(AnB)	P(AnB')	P(A)
A'	P(A'nB)	P(A'nB')	P(A')

Sum	P(B)	P(B')	1

We can use to deduce the identities above

Conditional probabilities

For two events A and B such that $P(A) \neq 0, P(B) \neq 0$, then the conditional probability of A given B denoted by P(A/B) is given by

$$P\left(A/B\right) = \frac{P(A \cap B)}{P(B)}$$

BAYE'S THEORY

This is an extension of conditional probability. Let $A_1, A_2, \ldots\ldots\ldots A_n$ be a set of mutually exclusive events that together for a sample space, S. let B be an event within the same sample space such that P(B)>0 then

$$P\left(A_K/B\right) = \frac{P(A_k \cap B)}{P(A_1 \cap B) + P(A_2 \cap B) + \ldots\ldots\ldots\ldots + P(A_k \cap B)}$$

the expression above is quite intimidating. However it is easy to use.

Example 1:

Two events A and B are such that $P(A) = \frac{2}{3}, P(AUB) = \frac{7}{12}, and, P(A \cap B) = \frac{5}{12}$

Find P(B).

Solution

P (AUB) =P (A) +P (B)-P (AnB)

$$\frac{7}{12} = \frac{2}{5} + P(B) - \frac{5}{12}$$

$$P(B) = \frac{1}{3}$$

Example 2:

The probability that a student passes math is $\frac{2}{3}$ and the probability that he passes Physics is $\frac{4}{9}$. If the probability that he passes at least one of them is $\frac{4}{5}$. Find the probability that he passes both papers.

Solution:

P (MUP)=P(M)+P(P)-P(MnP)

$$\frac{4}{5} = \frac{4}{9} + \frac{2}{3} - P(M \cap P)$$

$$P(M \cap P) = \frac{14}{45}$$

Example 3:

Events A and B are such that $P(A) = \frac{1}{2}, P(B) = \frac{3}{8}$ and $P\left(A/B\right) = \frac{7}{12}$

Find i) P(AnB) ii) P(B/A')

Solution

i) $P(A \cap B)$

$$P\left(A/B\right) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{7}{12}=\frac{P(A\cap B)}{\frac{3}{8}}\Rightarrow P(A\cap B)=\frac{21}{96}$$

ii) P(B/A')

$$P\left(B/_{A'}\right)=\frac{P(B\cap A')}{P(A')}$$ But $P(A'\cap B)=P(B)-P(A\cap B)=\frac{3}{8}-\frac{21}{96}=\frac{5}{32}$

And $P(A)+P(A')=1,\Rightarrow P(A')=1-\frac{1}{2}=\frac{1}{2}$

$$P\left(B/_{A'}\right)=\left(\frac{5}{32}\right)\div\frac{1}{2}=\frac{5}{16}$$

PROBABILITY SITUATIONS

a) The ‘AND’ situation.

Here we are interested in joint occurrence of events and in probability ‘and’ means intersection

b) The ‘OR’ situation

Here we are interested in either one, two or more events occurring. In probability theory ‘OR’ means union of events/ at least one.

Note: ‘Either A or B or both’ means union

c) The ‘one and only one’ situation

Here we are interested in only one event occurring, and other events do not occur/fail.

Note: ‘Either A or B but not both’ means $(AUB)-(A\cap B)$

Example 4:

Two events A and B are such that P(A)=$\frac{8}{15}$, $P(B)=\frac{1}{3}$, $P\left(A/_{B}\right)=\frac{1}{5}$. Calculate the probabilities that

i) Both events occur

ii) At least one of the events occur

iii) Only one of the events occur

iv) Neither event occurs.

Solution:

i) P(both events occur)= $P(A\cap B)=P(B).P\left(A/B\right)=\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$

ii) P(at least one)=P(AUB)= $\frac{8}{15}+\frac{1}{3}-\frac{1}{15}=\frac{4}{5}$

iii) P(only one)= $P(A'\cap B)+P(A\cap B')=P(A\cup B)-P(A\cap B)=\frac{4}{5}-\frac{1}{15}=\frac{11}{15}$

iv) $P(neither)=P(A\cup B)^{1}=1-P(A\cup B)=1-\frac{4}{5}=\frac{1}{5}$

Example 5:

Abel, Bob and Charles applied for a job in a certain company. The probability that Abel will take the job is $\frac{3}{4}$, the probability that Bob will take the job is $\frac{1}{`2}$ and while the probability that Charles will not take the job is $\frac{1}{3}$

What will be the probability that

i) None of them will take the job
ii) Exactly two will take the job

Solution:

$P(A)=\frac{3}{4}, P(A')=\frac{1}{4}, P(B)=\frac{1}{2}, P(B')=\frac{1}{2}, P(C)=\frac{2}{3}, P(C')=\frac{1}{3}$

i) $P(A'\cap B'\cap C')=\frac{1}{4}\times\frac{1}{2}\times\frac{2}{3}=\frac{1}{24}$

ii) $P(A'\cap B\cap C)+P(A\cap B'\cap C)+P(A\cap B\cap C')$

$=\frac{1}{4}\times\frac{1}{2}\times\frac{2}{3}+\frac{3}{4}\times\frac{1}{2}\times\frac{2}{3}+\frac{3}{4}\times\frac{1}{2}\times\frac{1}{3}=\frac{11}{24}$

Assignment 4.1.4:

1. Events A and C are such that $P(C)=0.39$, $P(CUD)=0.75$ and $P\left(C/D\right)=\frac{17}{33}$.

   Find the

   i) $P(D)$

ii) $P(C \cap D^1)$

2. If events X and Y are independent events.
   a) Show that the events X^1 and Y^1 are also independent
   b) Find $P(X \cap Y^1)$, given that $P(X^1) = 0.75$ and $P(Y) = 0.4$
3. Events A and B are independent such that $P(A) = \frac{3}{8}$ and $P(A^1 \cup B) = \frac{3}{4}$

   Find the;

   i) $P(B)$ ii) $P(A \cup B)$
4. Events A and B are such that $P(A) = 0.7$, $P(B) = 0.7$ and $P(A/B) = 0.1$, find the probability that;
   i) Both of the events will occur
   ii) Exactly one of the events occur
5. (a) A and B are events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B^1) = \frac{1}{4}$ and $P(B \cap A^1) = \frac{1}{6}$
   i) Represent the information on the Venn diagram
   ii) Hence or otherwise find $P(A)$ and $P(B)$
   iii) Obtain $P(A/B^1)$
6. X and Y are two independent events such that $P(X) = 0.8$ and $P(Y) = 0.5$.
   Obtain $P(X \cup Y)$ and $P(X \cup Y^1)$
7. In a class of science Students, if a student does not take chemistry the probability that he takes Mathematics is $\frac{4}{13}$. The probability that a student takes neither of these subjects is $\frac{3}{8}$. Find the probability that a student picked at random

   i) Takes Chemistry

   ii) Takes Chemistry or does not take mathematics

8. A student offers two subsidiaries ICT and GP. The probability that he will fail both ICT and GP is 0.18. The probability that he will pass G.p is

0.52 and the probability that he will pass ICT is 0.5. Find the probability that he will pass either ICT or GP but not both.

PROBABILITY TREES

The easiest way of solving many probability problems is by use of probability trees

Note:

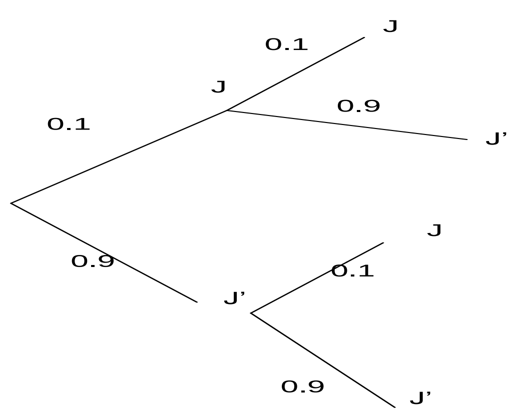
1. The probabilities along the branch are independent
2. The sum of the probabilities at each junction of the branch is one
3. The probabilities after the first branch are conditional
4. Sampling can be done with or without replacement

Example1:

The probability that James is late for work is 0.1 on a given day is 0.1. What is the probability that on two consecutive morning, James is late for work once

Solution: Let J be the event that James is late for work;

P(J)=0.1, P(J')=0.9



$$P(J \cap J') + P(J' \cap J) = 0.1 \times 0.9 + 0.9 \times 0.1 = 0.18$$

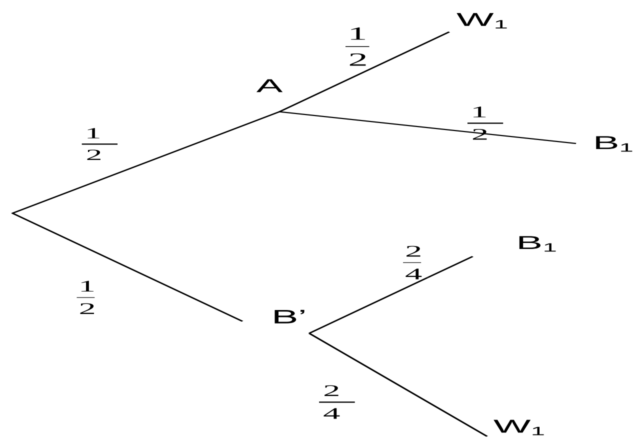
Example 2:

A box A contains 1 white ball and 1 blue ball. Box B contains only two white balls and 2 blue balls. If a box is picked at random and a ball is picked at random from it, find the probability that it is:

i) White

ii) From box A given that it is white

Solution:



i) $P(W) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{4} = \frac{1}{2}$

ii). $P\left(A/W\right) = \frac{P(A \cap W)}{P(W)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}$

Assignment 4.2.5

1. Box A contains 4 red and 3 green sweets and box B contains 7 red and 4 green sweets. A box is randomly selected and 2 sweets randomly selected from it one at a time without replacement. If A is twice as likely to be picked ass B, find the probability that both sweets are;

i) Of different colors

ii) From A given that they are of different colors

2. Don has a bag containing 15 balls. 10 balls are red and 5 are green. Terry has a bag containing 7 red and (x-3) green balls. A ball is randomly picked from Don's bag and put in Terry's bag. A ball is then

randomly picked from terry's bag. If the probability of the ball being green is $\frac{16}{39}$, find the value of x.

3. A box A contains 4 white and 2 red balls. Another box B contains 3 white and 3 red balls. A box is selected at random and two balls are picked one after the other without replacement
   a) Find the probability that the two balls picked are red.
   b) Given that two white balls are picked, what is the probability that they are from box B

4. A certain flight is planned to take less than five hours depending on weather conditions. The probabilities of wind W, rainy R and Dry weather D are estimated to be 0.3, 0.5 and 0.2 respectively. The probabilities of the flight being on schedule under these conditions are o.25, 0.4 and 0.9 respectively.
   a) Determine the probability that the flight delayed
   b) If the flight is not on schedule, find the probability of dry weather

5. In certain inter- University tournament; 35% watched football but not cricket, 10% watched cricket but not football and 40% did not watch either game. Find the probability that they watched foot ball, given that they did not watch cricket