TOPIC 4: FURTHER DIFFERENTIATION

The Chain rule

The expanding of some functions would be so complex for example $y=(2x+3)^{20}$. There is method which is used to differentiate such complex functions is known as chain rule.

Example

Differentiate $y = (3x+2)^{20}$

Let u = 3x +2

$$\frac{du}{dx} = 3$$

But $y = u^{20}$

$$\frac{dy}{du} = 20u^{19} = 20(3x+2)^{19}$$

$$\frac{dy}{du} = \frac{dy}{du}.\frac{du}{dx} = 20(3x+2)^{19}.3$$

$$=60(3x+2)^{19}$$

Example

Differentiate

$$y = \sqrt{x^2 - \frac{1}{x^2}}$$

Let $x^2 - \frac{1}{x^2} = u$

$$\frac{du}{dx} = (2x + \frac{2}{x^3}) =$$

$$y = u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = y = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{x^2 - \frac{1}{x^2}}}$$

$$= \frac{1}{2\sqrt{x^2 - \frac{1}{x^2}}} . \left(2x + \frac{2}{x^2}\right)$$

$$= \frac{(2x^4 + 2)}{x^3} . \frac{1}{2\sqrt{\frac{x^4 - 1}{x}}}$$

$$= \frac{2x^4 + 2}{x^3} . \frac{x}{2\sqrt{x^4 - 1}}$$

$$= \frac{2(x^4 + 1)}{2x^2\sqrt{x^4 - 1}}$$

$$= \frac{(x^4 + 1)}{x^2\sqrt{x^4 - 1}}$$

Rates Of Change

The chain rule can also be used to investigate related rates of changes .

A learner is expected to identify this use the units given

Cubic units per time taken its $\frac{dv}{dt}$

Its square units per time taken its $\frac{dA}{dt}$ and

If it is units per time taken $\frac{dl}{dt}$

Example

A container in a shape of a right circular cone of height 10cmm and base radius 1cm is catching the drips from a tap leaking at a rate of $0.1cm^3s^{-1}$. Find the rate at which the surface area of water is increasing when the water is half way up. The cone .

Note that because of the units used

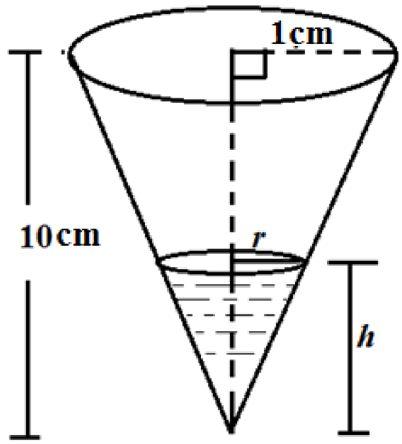
$\frac{dv}{dt}$ **=0.1 and what is required is** $\frac{dA}{dt}$ **at the end they are mentioning when it is ½ way up therefore our variable to use must be terms of height(h)**

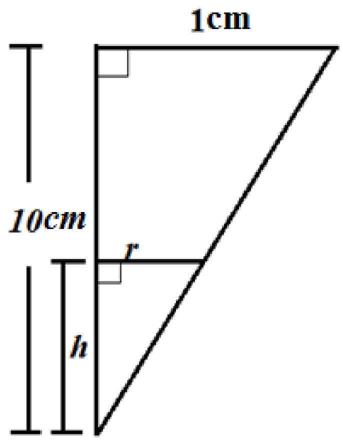
From chain rule

$$\frac{dv}{dt} = \frac{dv}{dt} \cdot \frac{du}{dt} \quad and \quad \frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$$

From o'level volume of a cone is given by expression $V = \frac{1}{3}\pi r^2 h$

We have two variables r and h so there is need to change r in terms of h





Comparing similar sides of the triangle base and height

$$\frac{r}{1} = \frac{h}{10}$$

$$r = \frac{1}{10}h$$

Substituting for r in expression for volume

$$V = \frac{1}{3}\pi\left(\frac{1}{10}h\right)^2 h = \frac{1}{300}\pi h^3$$

$$\frac{dV}{dh} = \frac{3}{300}\pi h^2 = \frac{\pi h^2}{300}$$

When the cone is half way up h =5cm

$$\frac{dV}{dh} = \frac{\pi(5)^2}{100} = \frac{1}{4}\pi$$

$$from\ \frac{dV}{dt} = \frac{dV}{dh}.\frac{du}{dt}$$

$$0.1 = \frac{1}{4}\pi.\frac{dV}{dt}$$

$$\frac{0.4}{\pi} = \frac{dh}{dt}$$

The surface of the water in the cone is circular therefore formula for area of a circle

$$A = \pi r^2$$

$$A = \pi\left(\frac{1}{10}h\right)^2 = \frac{\pi}{100}h^2$$

$$\frac{dA}{dh} = \frac{2\pi h}{100} = \frac{\pi h}{50}$$

But h =5

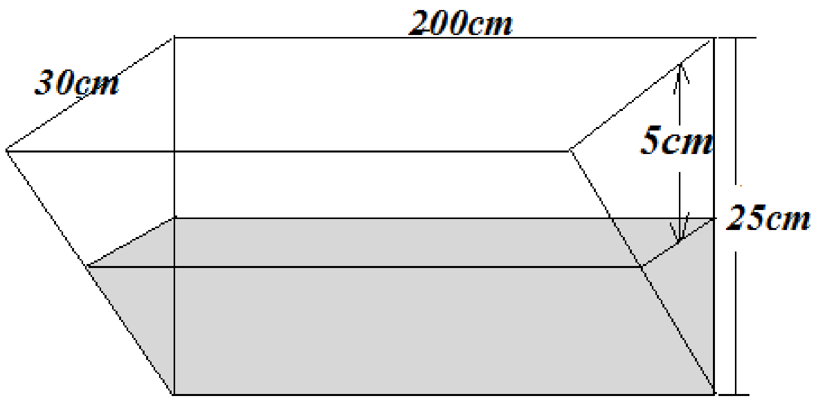
$$\frac{dA}{dh} = \frac{5\pi}{50} = \frac{\pi}{10}$$

$$\frac{dA}{dt} = \frac{dA}{dh}.\frac{dh}{dt} = \frac{\pi}{10}.\frac{0.4}{\pi} = 0.04cm^2s^{-1}$$

Example

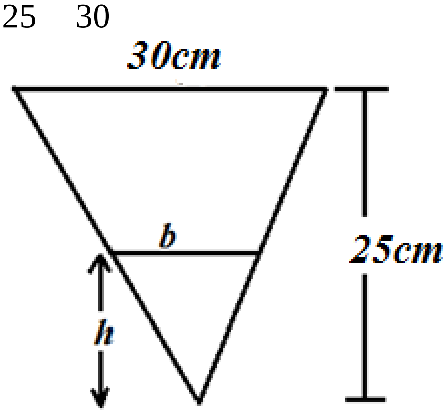
A horse trough has a triangular cross section of height 25cm, base 30cm and 2m long . A horse is drinking steadily and when water is 5cm below the top it is being lowered at a rate of 1cm per Minute . Find the rate of consumption in litres per minute

According to the question the variable considered here is $\frac{dh}{dt} = 1$



Comparing similar side

$$\frac{h}{25} = \frac{b}{30}$$



$$\frac{30}{25}h = b$$

$$\frac{6}{5}h = b$$

Volume of a trough = Area of cross section x distance in between

$$V = \frac{1}{2}bh \times 200 = 100bh$$

But $b = \frac{6}{5}h$

$$V = 120h^2$$

$$\frac{dv}{dh} = 240h$$

When water is 5cm blow them the height =20cm

$$\frac{dV}{dh}=\frac{500}{300}\times 20=\frac{10000}{3}=240\times 20=4800$$

From chain rule

$$\frac{dV}{dt}=\frac{dV}{dh}.\frac{dh}{dt}$$

$= 4800 \ \text{x} \ 1$

$= 4800 \text{ cm}^3\text{Min}^{-1}$

But $1000\text{cm}^3 = 1$litre

$$\frac{dV}{dt}=4.8l\min^{-1}$$

Product And Quotients Rule

Products

If y = uV   where u and V are functions of x

$$\frac{dy}{dx}=u\frac{dV}{dx}+V\frac{du}{dx}$$

Example

Differentiate $y = (x^2 +1)^2(x+2)^3$

let $u =(x^2+1)^2$

$$\frac{du}{dx}=2(x^2+1)2x=4x(x^2+1)$$

Let $V= (x+2)^3$

$$\frac{dv}{dx}=3(x^2+1)^2$$

$$\frac{dy}{dx} = (x^2+1)^2.3(x^2+2)^2 + (x+2)^3.4(x^2+1)$$

$$= (x^2+1)(x+2)^2\left[3(x^2+1)+4x(x+2)\right]$$

$$= (x^2+1)(x+2)^2\left[3x^2+3+4x^2+8x)\right]$$

$$= (x^2+1)(x+2)^2(7x^2+8x+3)$$

Ensure to simplify up to the end

Example

$$y = (x^2-1)\sqrt{x+1}$$

Let $u = x^2 - 1$ $\quad \frac{du}{dx} = 2x$

$$V = (x-1)^{1/2}$$

$$\frac{dV}{dx} = \frac{1}{2}(x-1)^{1/2}(1) = \frac{1}{2\sqrt{x+1}}$$

$$\frac{dy}{dx} = (x^2-1)\frac{1}{2}\left(\frac{1}{\sqrt{x+1}}\right) + (x+1)^{1/2}(2x)$$

$$\frac{dy}{dx} = \left(\frac{(x^2-1)}{2\sqrt{x+1}}\right) + 2x\sqrt{(x+1)}$$

$$\frac{dy}{dx} = \frac{(x^2-1)}{2\sqrt{x+1}} + 2x\sqrt{(x+1)}$$

$$= \frac{x^2-1+4x^2+4x}{2\sqrt{x+1}}$$

$$= \frac{3x^2 + 4x - 1}{2\sqrt{x+1}}$$

Quotient

Given $y = \frac{u}{V}$

$$\frac{dy}{dx} = \frac{V\frac{du}{dx} - u\frac{dv}{dx}}{V^2}$$

Example

Differentiate

$$y = \frac{(x-3)^2}{(x+2)^2}$$

$$Let\ \ u = (x-3)^2, \frac{du}{dx} = 2(x-3)$$

$$Let\ \ V = (x+2)^2, \frac{dv}{dx} = 2(x+2)$$

$$= \frac{(x+2)^2.2(x-3) - (x+2)^2.2(x+2)}{(x+2)^4}$$

$$= \frac{2(x-3)(5)}{(x+2)^3}$$

$$= \frac{10(x-3)}{(x+2)^3}$$

Ensure to simplify up to the end

Example

Differentiate

$$y=\sqrt{\frac{(x-3)^3}{(x+2)}}=\frac{(x-1)^{3/2}}{(x+2)^{1/2}}$$

Let u= $(x+1)^{3/2}$

$$\frac{du}{dx}=\frac{3}{2}(x+1)^{1/2}(1)$$

$$v=(x+2)^{1/2}$$

$$\frac{du}{dx}=\frac{3}{2}(x+1)^{1/2}(1)$$

Substituting in the formula

$$\frac{dy}{dx}=\frac{(x+1)^{1/2}.\frac{3}{2}(x+1)^{1/2}-(x+1)^{3/2}\frac{1}{2}(x+2)^{-1/2}}{(x+2)}$$

Multiplying the numerator and denominator with $(x+2)^{½}(x+1)^{½}$ in order to remove the fraction powers on the numerator

$$\frac{dy}{dx}=\frac{\left((x+1)^{1/2}.\frac{3}{2}(x+1)^{1/2}-(x+1)^{3/2}\frac{1}{2}(x+2)^{-1/2}\right)(x+1)^{1/2}(x+2)^{1/2}}{(x+2)(x+2)^{1/2}(x+1)^{1/2}}$$

$$\frac{dy}{dx}=\frac{\frac{3}{2}(x+2)(x+1)-(x+2)^2\frac{1}{2}(x+2)^0}{(x+2)^{3/2}(x+1)^{1/2}}$$

$$\frac{dy}{dx}=\frac{\frac{1}{2}(x+1)[3(x+2)-(x+1)]}{(x+1)^{1/2}(x+2)^{3/2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{(x+1)}(2x+5)}{2\sqrt{(x+2)}}$$

$$or\, \frac{dy}{dx} = \sqrt{\frac{(x+1)}{(x+2)^3}}\left(\frac{2x+2}{2}\right)$$

Implicit Functions

These are functions where two variables are mixed up

Example

Differentiate

$x^2+2xy-2y^2+x=2$

$$\frac{d}{dx}(x^2 + 2xy - 2y^2 + x - 2)$$

$$\Rightarrow (2x + 2x\frac{dy}{dx} + 2y.1 - 4y\frac{dy}{dx} + 1 - 0 = 0$$

$$\Rightarrow (2x + 2y + 1) - (4y - 2x)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x + 2y + 1}{2(2y - x)}$$

Example

Find $\frac{dy}{dx}$ of the function

$X^2+y^2-6xy+3x-2y+5=0$

$$2x + 2y\frac{dy}{dx} - 6x\frac{dy}{dx} - 6y.1 + 3 - 2\frac{dy}{dx} + 0 = 0$$

$$(2x-6y+3)-(6x+2-2y)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{2x-6y+3}{2(3x+1-y)}$$

Parametric Equations

If both *x* and *y* are given out in a different variable

Say *x* is in terms of *t* and *y* is in terms of *t*

Example

Find the gradient of the curve

$$x=\frac{2t}{t+2}\ \ and\ \ y=\frac{3t}{t+3}$$

$$\frac{dx}{dt}=\frac{(t+2)(2)-(2t)(1)}{(t+3)^2}=\frac{-3t+9-3t}{(t+3)^2}=\frac{9}{(t+3)^2}$$

$$\frac{dx}{dt}=\frac{dx}{dt}.\frac{dt}{dx}=\frac{9}{(t+3)^2}.\frac{(t+2)^2}{4}=\frac{9(t+2)^2}{4(t+3)^2}$$

Differentiating parametric equations

Given that $x=\frac{t^2}{1+t^3}$, $y=\frac{t^3}{1+t^3}$, find $\frac{dy}{dx}$.

$$\frac{dx}{dt}=\frac{(1+t^3)(2t)-t(3t^3)}{\left(1+t^3\right)^2}$$

$$= t\frac{(2+2t^3-3t^3)}{(1+2t^3)^2}$$

$$= \frac{t(2-t^3)}{(1+t3^3)^2}$$

$$\frac{dy}{dx} = \frac{(1+t^3)(3t^2) - t^3(3t^3)}{(1+t^3)^2}$$

$$= \frac{3t^3 + 3t^5 - 3t^5}{(1+t^3)^2}$$

$$= \frac{3t^2}{(1+t^3)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{3t^2}{(1+t^3)^2} \text{ x } = \frac{(1+t^3)^2}{t(2-t^3)} = \frac{3t}{2-t^3}$$

Small changes

From above we already seen that $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$ and Δx tends to zero

$$\therefore \Delta y \approx \frac{dy}{dx}.\Delta x$$

Example

This side of a square is 5cm. Find the increase in the area of the square when the side expands 0.01cm

A= x^2

$$\frac{dA}{dx} = 2x$$

When x =5

$$\frac{dA}{dx} = 2 \times 5 = 10$$

And $\Delta x = 0.01$

$$\frac{\Delta A}{\Delta x} \approx \frac{dA}{dx}$$

$$\Delta A = \frac{dA}{\Delta x}.\Delta x$$

$$\Delta A \approx 10 \times 0.01 = 0.1$$
$$\therefore Increase\, in\ Area = 0.1$$

Example

A 2% error is made in measuring the radius of a sphere. Find the percentage error in surface area.

$$S = 4\pi r^2$$
$$\frac{ds}{dr} = 8\pi r$$
$$\frac{\Delta s}{\Delta r} = \frac{ds}{dr}$$
$$\Delta s = \frac{ds}{dr}.\Delta r$$

$$\Delta s = (8\pi r)\Delta r$$

And $\Delta r = {}^{2}\!/_{100}\, r = 0.02r$

$$\Delta s = (8\pi r)(0.2r)$$

$$\Delta s \approx 0.16\pi r^2$$

$$\frac{\Delta s}{s} \times 100 \ \ is \ \ percentage \, error$$

$$\frac{0.16\pi r^2}{4\pi r^2} \times 100 = 4\%$$

Example

Find the approximation for

$$\sqrt{9.01}$$

Let $\sqrt{x}$ where x=9

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

When x=9

$$\frac{dy}{dx} = \frac{1}{6}$$

$$But \ \ \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

$$But \ \ \Delta y \approx \frac{dy}{dx} . \Delta x = \frac{1}{6} \times 0.01 = \frac{0.01}{6}$$

$$y + \Delta y = \sqrt{9} + \frac{0.01}{6}$$

=3.00167

Example

Using small changes

Show that $(244)^{1/5} = 3\frac{1}{405}$

Let $y = x^{1/5}$ $\quad x = 243$

$\Delta x = 1$

$$\frac{dy}{dx} = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{-4/5}}$$

$$= \frac{1}{5(243)^{4/5}} = \frac{1}{5(3^5)^{4/5}} = \frac{1}{5(3^4)}$$

$$= \frac{1}{5 \times 81}$$

$$= \frac{1}{405}$$

$$But \ \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

$$But \ \Delta y \approx \frac{dy}{dx}.\Delta x$$

$$\Delta y = \left(\frac{0.01}{6}\right)(1)$$

$$= \frac{1}{405}$$

$$\mathrm{y} + \Delta \mathrm{y} = (243)^{1/5} + \frac{1}{405}$$

$$= (3^5)^{1/5} + \frac{1}{405}$$

$$=3\frac{1}{405}$$

Second Derivative

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$ differentiating twice even $\frac{dv}{dt}=\frac{dv}{ds}.\frac{ds}{dt}=V\frac{dv}{ds}$

Also if x and y are in different variable say t

$$\frac{d^2y}{dx^2}=\left(\frac{d}{dx}\left(\frac{dy}{dt}\right)\right).\frac{dt}{dx}$$

Example

Given $y=4x^3-6x^2-9x+1$. Find

$$\frac{dy}{dx}\,and\,\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx}=4(x^2)-6(2x^1)-9(1x^{1-1})\ \ to\ \ \frac{dy}{dx}=12x^2-12x-9$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dy}{dx}(12x^2 12x-9=24x-12$$

Example

If $x=a(t^2-1)$, $y=2a(t+1)$ find $\frac{d^2y}{dx^2}$

$$\frac{dx}{dt}=a(2t)=2at,\ \frac{dy}{dt}=2a(1)$$

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{29}{2at}=\frac{1}{t}=t^{-1}$$

$$\frac{d^2y}{dx^2}=\left(\frac{d}{dt}\left(\frac{dy}{dx}\right)\right)\frac{dt}{dx}$$

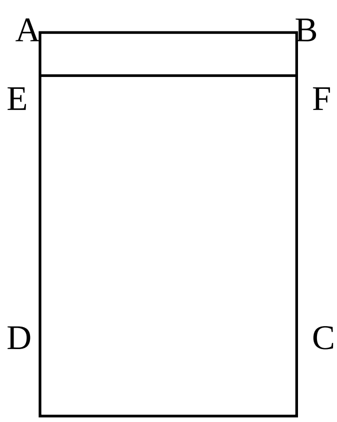
$$=\left(\frac{d}{dt}\left(t^{-1}\right)\right)\frac{1}{2at}$$

$$=\frac{-1}{t^2}\cdot\frac{1}{2at}$$

$$=\frac{-1}{2at^3}$$

Exercise 1

1. Find the derivative of f(x) $=\dfrac{x^4+3x^2}{2x^2}$
2. Find the derivative of $f(x)=(x^2+2)(x-4)$
3. Find the equation of the tangent at point P(3,9) to the curve y= $x^3+6x^2+15x-9$ .If O is the origin and N is the foot of the perpendicular from P to the x-axis . Prove that the tangent at P passes through the mid point of ON . Find the coordinates of another point on the curve , the tangent at which is parallel to the tangent at the point (3,9)
4. The figure below represents the end view of the outer cover of a match box AB and EF being C gummed together and assumed to be of the same length. If the total length of the edge (ABCDEF) is 12cm . Calculate the lengths of AB and BC which will give the maximum possible area .



5. Sketch the curve $y = 4x^3 - 3x^4$ Showing clearly the turning points and points where the curve crosses the axes.
6. Differentiate with respect to x

$$y = (1 - x^2)(1 - 2x)^{1/3}$$

7. Differentiate

$$y = \sqrt{\frac{(x+2)^3}{(x-1)}}$$

8. Find $\frac{dy}{dx}$ of $x^2-3xy+y^2-2y+4x=0$
9. Find $\frac{dy}{dx}$, Given $x = \frac{t}{1-t}$ *and* $y = \frac{1-2t}{1-t}$
10. Find the approximation of Find $\sqrt[3]{65}$
11. Given $y = \frac{x^2}{\sqrt{x+1}}$ find $\frac{d^2y}{dx^2}$
12. If x $=(t^2-1)^2$ and y $=t^3$ find $\frac{d^2y}{dx^2}$