TOPIC 7: The Binomial Theorem

Pascal's triangle

Consider the following expansions:

$(a + b)^0 = 1$

$(a + b)^1 = 1a + 1b$

$(a + b)^2 = 1a^2 + 2ab + 1b^2$

$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a + b)^4 = 1a^4 + 4a^3 + 6a^2b^2 + 4ab^3 +1d^4$

If the coefficients are written alone in a form of a triangle we obtain,

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

This form of triangle is called **Pascal's triangle.**

The reader should note that every coefficient in the table is obtained from the two on either side of it in the row above. In this way the next line can be obtained.

For example,

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

There are points to note about the expansion of $(a + b)^n$ and the reader should verify them for the cases n = 2,3,4 in the expansions obtained so far.

(a) Reading from either end of each row, the coefficients are the same.
(b) There are (n+1) terms.

(c) Each term is of degree n.
(d) The coefficients are obtained from the row in Pascal's triangle.

Example 1

Expand $(a + b)^7$ in descending powers of a.

Solution

We shall get (7+1) terms, i.e. 8 terms. The coefficients are picked from the 8th row of Pascal's triangle and each term is of degree 7, i.e.

$a^7, a^6b^1, a^5b^2, a^4b^3, a^3b^4, a^2b^5, a^1b^6, b^7$

From			1	5	10	10	5	1
		1	6	15	20	15	6	1
	1	7	21	35	35	21	7	1

The last line gives the coefficients for degree 7.

$(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

Example 2.

Expand $(x + 2y)^4$ in descending powers of x.

Solution

Let a = x and b = 2y.

There will be 5 terms, i.e.

$x^4, x^3(2y)^1, x^2(2y)^2, x(2y)^3, (2y)^4$

The coefficient, obtained from Pascal's triangle are respectively;

Therefore

$(x+2y)^4 = x^4 + 4x^3(2y) + 6x^2(2y)^2 + 4x(2y)^3 + (2y)^4$.

$= x^4 + 8x^3y + 24x^2y + 32xy^3 + 16y^4$

Example 3

Expand $(2x + \frac{1}{3})^3$ in descending powers of x.

Solution

Let a = 2x and b = $\frac{1}{3}$.

There will be 4 terms, each of degree three, that is

$(2x)^3$, $(2x)^2(\frac{1}{3})^1$, $(2x)^1\,(\frac{1}{3})^2$, $(\frac{1}{3})^3$

The coefficients from Pascal's triangle are 1, 3, 3, 1 respectively.

Therefore,

$$\left(2x + \frac{1}{3}\right)^3 = (2x)^3 + 3(2x)^2 \text{ x } \frac{1}{3} + 3\,(2x)\text{x}\frac{1}{9} + \frac{1}{27}$$

$$= 8x^3 + 4x^2 + \frac{2}{3}x + \frac{1}{27}.$$

Example 4

Use Pascal's triangle to obtain the value of $(1.002)^4$, correct to six places of decimals.

Solution

1.002 = 1 + 0.002

The expansion of $(a + b)^4$, may be used

Let a = 1 and b = 0.002.

The coefficients will be,

1, 4, 6, 4, 1

$(1+0.0002)^4 = 1^4 + 4(1)^3(0.0002)^1 + 6(1)^2(0.002)^2 + 4(1)^1(0.002)^3 + (0.002)^4$

$$= \ 1 + 0.008 + 6\text{x}0.000004 + 4\text{x}0.000000008$$

$$= 1 + 0.0008 + 0.000024 + 0.000000032$$

$\therefore\ (1.002)^4 = 1.008024$ (6d.p)

The reader should not that the last two terms in the expansion could be ignored because they don't make any difference to the answer, correct to six places of decimals.

Exercise 1.

This exercise is intended to give the reader practice in using Pascal's triangle, therefore calculators should not be used in the numerical questions.

1. Expand the following;

   (a) $(x + y)^5$, (b) $(x+2y)^3$ (c) $(2x+3y)^4$, (d) $(4x+1)^3$. (e) $\left(\frac{x}{2}+\frac{2}{x}\right)^4$

2. Write down the expansion of $(2 + x)^5$ in a ascending powers of x. Taking the first three terms of the expansion, put x = 0.001, and find the value of $(2.001)^5$ correct to five places of decimals.

3. Write down the expansion of $\left(1+\frac{1}{4}x\right)^4$. Taking the first three terms of the expansion, put x = 0.1, and find the value of $(1.025)^4$, correct to three places of decimals.

4. Expand $(2-x)^6$ in ascending powers of x. Taking x=0.002, and using the first three terms of the expansion, find the value of $(1.998)^6$ as accurately as you can.

5. Expand $\left(2x+\frac{1}{2x}\right)^5$ in descending powers of x.

6. Simplify $\left(\sqrt{5}+\sqrt{3}\right)^4-\left(\sqrt{5}-\sqrt{3}\right)^4$.

7. Write down the expansion of $(a - b)^5$ and use the result to find the value of $\left(9\frac{1}{2}\right)^5$ correct to the nearest 100.

8. Expand $(x + 2)^5$ and $(x - 2)^4$. Obtain the coefficient of x^5 in the product of the expansions.

The Binomial Theorem

A binomial is a mathematical expression with two unlike terms. e.g. (a + b)

If n is a positive integer

$(a + b)^n = a^n + nC_1\, a^{n-1}b^1 + nC_2 a^{n-2}b^2 + \text{----} + nCr\, a^{n-r}b^r + \text{----} + b^n$, where

$nCr = \frac{n!}{(n-r)!r!}$. *This is the binomial theorem.*

When a few terms of an expansion are required the theorem is used in the form:

$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}\ a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}\ a^{n-3}b^3 + \text{-----------} + b^n$, since

$nC_1 = \frac{n!}{(n-1)!1!} = \frac{n(n-1)!}{(n-1)!\,1\,!} = n$

$nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)(n-2)!}{(n-2)!\,2\,!} = \frac{n(n-1)}{2!}$

$nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\,3\,!} = \frac{n(n-1)(n-2)}{3!}$

Qn: Show that $nC_{n-r} = nCr$

Example 5

Find the coefficient of x^{12} in the expansion of $(3x - 2)^{15}$

Solution

The required terms is in $(3x)^{12}(-2)^3$ by the binomial theorem it is $15c_3\ (3x)^{12}(-2)^3$

The coefficient of x^{12} is

$= \frac{15!}{(15-3)!3!} \text{ x } 3^{12} \text{ x } (-2)^3$

$= \frac{15x14x13}{3x2x1} \text{ x } 3^{12} \text{ x } (-2)^3$

= -1, 934,445,240.

Example 6

Write down and simplify the 5th term in the expansion of $\left(2 - \frac{x}{2}\right)^{12}$.

Solution

The term in $(2)^8 \left(-\frac{x}{2}\right)^4$ is the 5th term by binomial theorem,

5th term = $12C4\ (2^8) \left(-\frac{x}{2}\right)^4$

$\frac{12!}{8!3!} \text{ x} 2^8 \frac{(-x)4}{24}$

$= \frac{12x11x10x9x8!}{8!x43x2x1} x24\ x4$

$= 11x10x9x2^3x^4$

$= 7920x^4$

Qn: Obtain the four terms of the expansion of (1.005)10, correct to four decimal places.

Example 7

Obtain the expansion of $(1+x+x^2)^6$, as far as the term in x^3

Solution

$(1+x+x^2)^6 = \{1+(x+x^2)\}^6$, as a binomial expression.

By the binomial theorem

$\{1+(x+x^2)\}^6 = 1 + 6(x+x^2) + \frac{6x5}{2}(x+x^2)^2 + \frac{6x5x4}{3!}(x+x^2)^3 + ----$

$= 1 + 6x + 6x^2 + 15(x^2 + 2x^3) + 20x^3 + -----$

(Ignoring terms in x^4 and higher powers)

$:. (1+x+x^2)^6 = 1 + 6x + 21x^2 + 50x^3$

Exercise 2

Calculators should not be used in this exercise.

1.Write down the terms indicated, in the expansions of the following, and simplify your answers.

(a) $(x+3)^7$, 5th term, (b) $(2t - \frac{1}{2})^{12}$ term in t^7, (c)$(2-\frac{x}{2})^{12}$, 4th term, (d) $(2x+y)^{11}$, term in x^3

2. Write down, and simplify, the coefficients of the terms indicated, in the expansions of the following:

(a) $\left(4+\frac{3}{4}x\right)^6$, term in x^3, (b) $(2x-3)^7$, term in x^5, (c) $\left(\frac{1}{2}t-\frac{1}{2}\right)^{10}$, term in t^4.

3. Write down the coefficients of the terms indicated, in the expansions of the following in ascending powers of x:

(a) $(1+x)^{16}$, 3rd term, (b) $\left(2+\frac{3}{2}x\right)^8$, 5th term, (c) $(5-4x)^6$, 3rd term.

4. Write down the constant terms in the expansions of the following

(a) $\left(x-\frac{1}{x}\right)^8$, (b)$\left(2x^2-\frac{1}{2x}\right)^6$, (d) $\left(2x+\frac{1}{x}\right)^{10}$

5. Find the ratio of the term in x^5 to the term in x^6, in the expansion of $(2x+3)^{20}$

6. Write down the first four terms of the expansions of the following, in ascending powers of x:

(a) $\left(x+\frac{1}{2}x\right)9$, (b) $(1-x)^{11}$.

7. Use the binomial theorem to find the values of $(2.001)^{10}$, correct to six significant figures,

8. Expand the following as far as the terms in x^3

(a) $(1 - x + x^2)^4$, (b) $(2 + x + x^2)^5$

Convergent Series

Consider a geometrical progression with first term 1 and common ratio x, that is,

$1 + x + x^2 + x^3 + \text{---------} + x^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_n = 1 + x + x^2 + x^3 + \text{-----} + x^{n-1} = \frac{1-x^n}{1-x}$

If x lies between -1 and +1, as x tends to ∞, x^n *tends to* 0

$\therefore 1+x+x^2+x^3+\text{-----}x^{n-1} = \frac{1}{1-x}$

A series of terms, whose sum approaches a finite value as the number of terms is increased indefinitely is called a **convergent** series and the finite value is called its **sum to infinity**.

Thus $1+ x + x^2 + x^3 +\text{---}+ x^{n-1}$ is a convergent series, provided $-1<x<1$ and its sum to infinity is $\frac{1}{1-x}$

The binomial theorem for any index

the binomial theorem states that,

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \text{-----}$ for any rational value of n, provided

$|x| < 1\ i.e. -1 < x < 1.$

In the expansion of $(1+ x)^n$, the coefficient of x^r is nC_r

nC_r may be written as $\binom{n}{r}$.

Thus $\binom{n}{r} = \frac{n(n-1)(n-2)(n-3)---(n-r+1)}{rx(r-1)x-------x3x2x1..}$

Example 8

Use the binomial theorem to expand $(1 - 2x)^{\frac{1}{2}}$ in ascending powers of x, as far as the term in x^3, and state the values of x for which the expansion is valid.

Solution

$$(1-2x)^{\frac{1}{2}} = 1+\frac{1}{2}(-2x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(-2x)^2}{2!}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(-2x)^3}{3!}+\text{------}$$

$$=1-x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}4x^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-8x3)+---}{6}$$

$$= 1- x -\frac{1}{2}x^2-\frac{1}{2}x^3+----$$

Exercise 3

Calculators should not be used in this exercise

1. Evaluate the following binomial coefficients;

   (a) $\binom{5}{3}$, $(b)\binom{-2}{6}$, $(c)\binom{\frac{1}{2}}{4}$, $(d)\binom{-\frac{1}{4}}{5}$

2. Expand the following in ascending powers of x, as far as the term in x^3, and state the values of x for which the expansions are valid;

(a) $\sqrt{(1-x^2)}$, (b) $\sqrt[3]{(1-x)}$,

(c) $\frac{3}{3\sqrt{(3-x^2)}}$, (d) $\frac{2x-3}{x+2}$, (e) $\frac{x+3}{3\sqrt{(1-3x)}}$

3. Use the binomial theorem to find the values of the $\sqrt{0.998}$, correct to six places of decimals,

4. Find the first four terms of the expansion of $(1\text{-}8x)^{½}$ in ascending powers of x. Substitute x $=\frac{1}{100}$ and obtain the value of $\sqrt{23}$ correct to five significant figures.

5. Expand $(1\text{-}x)^{1/3}$ in ascending powers of x as far as the fourth term. By taking the first two terms of the expansion and substituting x = $\frac{1}{1000}$ find the value of $\sqrt[3]{37}$, correct to 6s.f.

6. Find the middle term in the expansion of $(6x+\frac{1}{3}y)10$,

7. (a) Obtain the first four terms in the expansion of $(1+2x+3x^2)^6$ in ascending powers of x.

   (b) Find the first four terms in the expansion of (i)$(1\text{-}x+2x^2)^5$, (ii) (1+x)-4 , in ascending powers of x.

8. Find the first four terms in the expansion of the following in ascending powers of x.

(a) $\frac{x+2}{(1+x)2,}$ (b) $\frac{1-x}{\sqrt{(1+x)}}$ (c) $\sqrt{\left(\frac{1+x}{1-x}\right)}$

9(a) Write down the first four terms of the expansion of $(2+\frac{1}{4}x)^{10}$ in ascending powers of x. hence find the values of 2.025^{10}, correct to the nearest whole number.

(b) Obtain the first four terms of the expansion of $(1+8x)^{½}$ in ascending powers of x. By putting x = $\frac{1}{100}$, obtain the value of $\sqrt{3}$, correct to five places of decimals.

10. Show that, if x is small enough for its cube and higher powers to be neglected,

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = 1 - x + \frac{x^2}{2}$$

By putting x = $\frac{1}{8}$, show that $\sqrt{7} \approx 2\frac{83}{128}$.