Discrete probability distribution

A probability density function (p.d.f) if it takes on specific values

Properties of discrete probability density functions

(i) $\sum P(X = x) = 1 \text{ or } \sum f(x) = 1$
(ii) P(X=x)≥0

Examples 1

A discrete random variable has a probability function $P(X = x) = \begin{cases} cx^2 & x = 0, 1, 2, 3, 4 \\ 0 & otherwise \end{cases}$

Find the value of c and draw the graph of P(X = x)

Solution

$\sum P(X = x) = 1$

$c(0^2) + c(1^2) + c(2^2) + c(3^2) + c(4^2) = 1$

c + 4c + 9c + 16c = 2

c = $\frac{1}{30}$

Example 2

A discrete random variable has probability function

$f(x) = \begin{cases} kx, & x = 1, 2,3,4 \\ 0, & otherwise \end{cases}$, find the value of k and draw the graph of f(x)

Solution

$\sum f(x) = 1$

k + 2k + 3k + 4k = 1

k = $\frac{1}{10}$

Example 3

A random variable X of a discrete probability distribution given by

P(X=1) = 0.2, P(X=2) = P(X=3) = 0.1, P(X=4) = P(X=5) = c

Find the value of the constant c and draw the graph of P(X = x)

Solution

$\sum P(X = x) = 1$

0.2 + 0.1 + 0.1 + c + c = 2; c = 0.3

Example 4

A discrete random variable has a probability function

$$P(X = x) = \begin{cases} k\left(\frac{2}{3}\right)^x, & x = 1, 2, 3, \ldots \\ 0, & otherwise \end{cases}$$

Find the value of k

Solution

$$k\left(\frac{2}{3}\right)^0 + k\left(\frac{2}{3}\right)^1 + k\left(\frac{2}{3}\right)^2 + k\left(\frac{2}{3}\right)^3 + \cdots. = 1$$

$$k\left(1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots.\right) = 1$$

Sum to infinity = $S_\infty = \frac{a}{1-r}$

⇨ $k\left(\frac{1}{1-\frac{2}{3}}\right)$ = 1; k = $\frac{1}{3}$

Finding probabilities

Example 5

A discrete random variable has a probability distribution

y	-3	-2	-1	0	1
P(Y=y)	0.1	0.25	0.3	0.15	a

Find

(i) value of a (ii) P(-3$\leq Y < 0$) (iii) P(Y>-1) (iv) P(-1<Y<1) (v) mode

Solution

(i) $\sum P(Y = y) = 1$

0.1 + 0.25 + 0.3 + 0.15 + a = 1; a = 0.2

(ii) P(-3$\leq Y < 0$) = P(Y= -3) + P(Y = -2) + P(Y = -1) = 0.1 + 0.25 + 0.3 = 0.65

(iii) P(Y> -1) = P(Y = 0) + P(Y = 1) = 0.15 + 0.2 = 0.35

(iv) P(-1<Y<1) = P(Y = 0) = 0.15

(v) mode is the value y with the highest probability, mode = -1

Example 6

A discrete random variable X has a probability distribution

X	1	2	3	4	5
P(X = x)	0.15	0.20	0.15	c	0.1

Find

(i) the value of x (ii) P(X<4) (iii) P(X≤4 (iv) P(2≤ $X \le 4$) (v) P$\left(\frac{X>2}{X\le 4}\right)$ (vi) mode

Solution

(i) $\sum P(X = x) = 1$

0.15 + 0.20 + 0.15 + c + 0.1 = 1; c = 0.4

(ii) P(X<4) = P(X=1) + P(X= 2) + P(X = 3) = 0.15 + 0.20 + 0.15 = 0.5

(iii) P(X<4) = P(X=1) + P(X= 2) + P(X = 3) + P(X= 4) = 0.15 + 0.20 + 0.15 + 0.4 = 0.9

(iv) P(2≤ $X \le 4$) = P(X = 2) + P(X = 3) + P(X = 4) = 0.20 + 0.15 + 0.4 = 0.75

(v) P$\left(X > 2/_{X \le 4}\right)$ = $\frac{P(X>2,\ X\le 4)}{P(X\le 4)}$ = $\frac{P(X=3)+P(X=4)}{\mathrm{P(X=1) + P(X= 2) + P(X = 3) + P(X= 4)}} = \frac{0.15 + 0.4}{0.9}$ = 0.6111

(vi) the mode is a value with highest probability = 4

Example 7

A discrete random variable X has a probability function

$$f(x) = \begin{cases} kx, & x = 1, 2,3,4,5 \\ 0, & otherwise \end{cases}$$

Find (i) the value of k (ii) P(X = 3) (iii) P(X≥3) (iv) P(X≤ 3) (v) P(1< X ≤3) (vi) P$\left(\frac{X\ge 1}{X<4}\right)$

Solution

(i) $\sum P(X = x) = 1$

k + 2k + 3k + 4k + 5k = 1; k = $\frac{1}{15}$

(ii) P(X=3) = 3k = $\frac{3}{15} = \frac{1}{5}$

(iii) P(X≥3)= P(X=3) + P(X=4) + P(X = 5) = 3k + 4k + 5k = 12k = $\frac{12}{15} = \frac{4}{5}$

(iv) P(X≤ 3)= P(X= 1) + P(X=2) + P(X = 3) = k + 2k + 3k =$\frac{6}{15} = \frac{2}{5}$

(v) P(1< X ≤3) = P(X=2) + P(X = 3) =2k + 3k =$\frac{5}{15} = \frac{1}{3}$

(vi) P$\left(X \ge 2/_{X < 4}\right) = \frac{P(X\ge 2,X<4)}{P(X<4)} = \frac{P(X=2)+P(X=3)}{P(X=1)+P(X=2)+P(X=3)} = \frac{2k+3k}{k+2k+3k} = \frac{5k}{6k} = \frac{5}{6}$

Revision exercise 1

1. A discrete random variable X has probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	a	0.05

Find (i) value of a = 0.1 (ii) P(1≤ $x \le 3$)=0.85 (iii) P(X>2) =0.55 (iv) P(2<X<5) = 0.5 (v) mode =3

2. A random variable x of a discrete pdf is given by P(X x) = kx, x = 12, 13, 14
Write the probability distribution and find the value of k

x	12	13	14
P(X=x)	12k	13k	14k

k = $\frac{1}{39}$

3. A random variable Y of discrete probability distribution is given by
P(Y = 3) = 0.1, P(Y = 5) = 0.05, P(Y=6) = 0.45 P(Y = 8) = 3P(Y = 10). Find P(Y=10) = 0.1
4. A discrete random variable has a distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	k	0.2	0.05

Find
(i) value of k = $\frac{7}{20}$ (ii) P(X≥4) = 0.25 (iii) P(X<1)=0 (iv) P(2≤ $x < 4$) = $\frac{13}{20}$

5. Write out the probability distribution for each of these variables
(a) The number of heads X obtained when two fair coins are tossed

x	0	1	2
P(X=x)	0.25	0.5	0.25

(b) The number of tails, X obtained when three fair coins are tossed.

x	0	1	2	3
P(X=x)	0.125	0.375	0.375	0.125

6. A drawer contains 8 brown socks and 4 blue shocks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. The procedure is performed twice more. X is the random variable for the number of brown socks taken. Find the probability distribution for X.

x	0	1	2	3
P(X=x)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

7. The discrete random variable R has a p.d.f is given by P(R=r) = c(3 – r), r = 0, 1, 2, 3
Find (i) value of c = $\frac{1}{6}$ (ii) P(1≤R<3) = 0.5
8. A discrete random variable has probability function

$$P(X = x) = \begin{cases} k\left(\frac{4}{5}\right)^x, & x = 1, 2, 3, \ldots. \\ 0, & otherwise \end{cases}$$, find the value k = 0.2.

Solutions to revision exercise 1

5 Write out the probability distribution for each of these variables
(a) The number of heads X obtained when two fair coins are tossed
S =(TT, TH, HT, HH)
P(X= 0) = $\frac{1}{4} = 0.25$, P(X= 1) = $\frac{2}{4} = 0.50$, P(X= 2) = $\frac{1}{4} = 0.25$
Probability distribution table

x	0	1	2
P(X=x)	0.25	0.5	0.25

(b) The number of tails, X obtained when three fair coins are tossed.

S = (TTT, TTH, THT, HTH, THH, HTH, HHT, HHH)

Probability distribution table

number of heads, x	0	1	2	3
P(X=x)	$\frac{1}{8}$ = 0.125	$\frac{3}{8}$ = 0.375	$\frac{3}{8}$=0.375	$\frac{1}{8}$ = 0.125

6. A drawer contains 8 brown socks and 4 blue shocks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. The procedure is performed twice more. X is the random variable for the number of brown socks taken. Find the probability distribution for X.

   Let X' represent blue shocks

   P(X = 0) = P(X'∩X'∩X') = $\frac{4}{12}x\frac{4}{12}x\frac{4}{12} = \frac{1}{27}$

   P(X=1) = P(X∩X'∩X')+ P(X'∩X∩X') + P(X'∩X'∩X) = $\frac{8}{12}x\frac{4}{12}x\frac{4}{12}+\frac{4}{12}x\frac{8}{12}x\frac{4}{12}+\frac{4}{12}x\frac{4}{12}x\frac{8}{12}$ = $\frac{2}{9}$

   P(X=2) = P(X∩X∩X')+ P(X'∩X∩X) + P(X∩X'∩X) =$\frac{8}{12}x\frac{8}{12}x\frac{4}{12}+\frac{4}{12}x\frac{8}{12}x\frac{8}{12}+\frac{8}{12}x\frac{4}{12}x\frac{8}{12}$ =$\frac{4}{9}$

   P(X = 3)= P(X∩X∩X)= $\frac{8}{12}x\frac{8}{12}x\frac{8}{12}$=$\frac{8}{27}$

   Probability distribution table

x	0	1	2	3
P(X=x)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

7. The discrete random variable R has a p.d.f is given by P(R=r) = c(3 – r), r = 0, 1, 2, 3

   Find (i) value of c

   $\sum P(X = x) = 1$

   3c +2c + c = 3

   c= $\frac{1}{6}$

   (ii) P(1≤R<3) = 2c + c = 3c = 3 x $\frac{1}{6}$ =0.5

8. A discrete random variable has probability function

   $P(X = x) = \begin{cases} k\left(\frac{4}{5}\right)^x, & x = 1, 2,3, \dots. \\ 0, & otherwise \end{cases}$, find the value k.

Solution

$k\left(\frac{4}{5}\right)^0 + k\left(\frac{4}{5}\right)^1 + k\left(\frac{4}{5}\right)^2 + k\left(\frac{4}{5}\right)^3 + \cdots. = 1$

$k\left(1 + \left(\frac{4}{5}\right)^1 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^3 + \cdots.\right) = 1$

Sum to infinity = $S_\infty = \frac{a}{1-r}$

⇨ $k\left(\frac{1}{1-\frac{4}{5}}\right)$ = 1; k = $\frac{1}{5}$ =0.2

Expectation of x, E(x) or mean

The expected value of x is given by E(x) = $\sum xP(X = x)$

Example 8

A discrete random variable has a probability distribution

x	-2	-1	0	1	2
P(X = x)	0.3	0.1	0.15	0.4	0.05

Find expectation, E(x)

Solution

E(X) = (-2 x 0.3) + (-1 x 0.1) + (0 x 0.15) + (1 x 0.4) + (2 x 0.05) = -0.2

Example 9

The discrete random variable Y has a probability distribution is given by

P(Y = y) = cy, y = 1 2,3,

P(Y = y) = c(8-y), y = 4, 5, 6, 7

Find (i) the value of c (ii) mean, μ

Solution

y	1	2	3	4	5	6	7
P(Y = y)	c	2c	3c	4c	3c	2c	c

(i) $\sum P(Y = y) = 1$
c + 2c + 3c + 4c + 3c +2c + c = 1
c = $\frac{1}{16}$

(ii) E(Y) = $\sum y(Y = y) =$ 1 x c + 2 x 2c + 3 x 3c + 4 x 4c + 5 x 3c + 6 x 2c + 7x c = 64c
= 64 x $\frac{1}{16}$ = 4

Example 10

A fair coin is tossed three times write out the probability distribution for the number of heads, X, obtained and hence obtain the expected number of heads

Solution

S = (TTT, TTH, THT, HTH, THH, HTH, HHT, HHH)

Probability distribution table

number of heads, x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

E(X) = $\sum x(X = x)$ = (0 x $\frac{1}{8}) + 1x\ \frac{3}{8} + 2\ x\ \frac{3}{8} + 3\ x\ \frac{1}{8} = \frac{12}{8} = 1.5$

Example 11

A family plans to have 4children. Given that X is the number of girls in the family. Find the expected number of girls

Solution

S = (BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGGB, BGBG, GGBB, BGGB, GBBG, BGGG, GBGG, GGBG, GGGB GGGG)

Probability distribution table

Number of girls, x	0	1	2	3	4
P(X=x)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

E(X) $= \sum x(X = x) = \left(0\ x\ \frac{1}{16} + 1\ x\ \frac{4}{16} + 2\ x\ \frac{6}{16} + 3\ x\ \frac{4}{16} + 4\ x\ \frac{1}{16}\right) = 2$

Example 12

A box A contains 4 red sweets and 3 green sweets. Box B contains 5 red sweets and 6 green sweets. Box A is twice more likely to be picked as Box B. If a box is chosen at random and two sweets are removed from it, one at a time without replacement.

(a) Find the probability that two sweets removed are of the same colour.

P(same colour) = $P(A\cap R_1\cap R_2) + P(A\cap G_1\cap G_2) + P(B\cap R_1\cap R_2) + P(B\cap B_1\cap B_2)$

$= \frac{2}{3} x \frac{4}{7}\ x \frac{3}{6} + \frac{2}{3} x \frac{3}{7}\ x \frac{2}{6} + \frac{1}{3} x \frac{5}{11}\ x \frac{4}{10} + \frac{1}{3} x \frac{6}{11}\ x \frac{5}{10}$

$= \frac{24}{126} + \frac{12}{126} + \frac{20}{330} + \frac{30}{330} = \frac{42}{126} + \frac{50}{330} = 0.4372$

(b) (i) construct a probability distribution table for the number of red sweets removed

Let x = number of red sweets removed

$P(X=0) = P(A\cap G_1\cap G_2) + P(B\cap G_1\cap G_2) = \frac{2}{3} x \frac{3}{7}\ x \frac{2}{6} + \frac{1}{3} x \frac{6}{11}\ x \frac{5}{10} = 0.1861$

$P(X=1) = P(A\cap R_1\cap G_2) + P(A\cap G_1\cap R_2) + P(B\cap R_1\cap G_2) + P(B\cap G_1\cap R_2)$

$= \frac{2}{3} x \frac{4}{7}\ x \frac{3}{6} + \frac{2}{3} x \frac{3}{7}\ x \frac{4}{6} + \frac{1}{3} x \frac{5}{11}\ x \frac{6}{10} + \frac{1}{3} x \frac{6}{11}\ x \frac{5}{10}$

$= \frac{24}{126} + \frac{24}{126} + \frac{30}{330} + \frac{30}{330} = \frac{48}{126} + \frac{60}{330}$

= 0.5628

$P(X=2) = P(A\cap R_1\cap R_2) + P(B\cap R_1\cap R_2) = \frac{2}{3} x \frac{4}{7}\ x \frac{3}{6} + \frac{1}{3} x \frac{5}{11}\ x \frac{4}{10} = \frac{24}{126} + \frac{20}{330} = 0.2511$

Probability distribution table

x	0	1	2
P(X= x)	0.1861	0.5628	0.2511

(ii) find the mean number of red sweets removed

Mean = $\sum x(X = x)$ = 0 x0.1861 + 1 x 0.5628 + 2 x 0.2511 = 1.065

Revision exercise 2

1. A discrete random variable X has a probability distribution.

x	0	1	2	3	4
P(X= x)	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$

Find E(X) = 2.25

2. A discrete random variable X ha probability distribution

x	5	6	7	8	9
P(X=x)	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{3}{11}$

Find the mean = 7

3. A discrete random variable X has a probability distribution

4. A discrete random variable has a probability distribution

x	0	1	2	3
P(X=x)	c	c^2	c^2+c	$3c^2$ + 2c

Find (i) the value of c= 0.2 (ii) expectation of c=2.08

5. Find the expected number of heads when two fair coins are tossed (E(x) 1)
6. A family plans to have 3 children. Given that x is the number of boys in the family. Find the expected number of boys (=1.5)
7. If X is a random variable for the product of the scores on two tetrahedral dice, where the score is the number on which the die lands, find the expected score for the throw (=6.25)
8. A bag contains 5 black counters and 6 red counters. Two counters are drawn at random, one at a time without replacement. Find the expected number of red counters. (= $\frac{12}{11}$)
9. An unbiased tetrahedral die is tossed once. If it lands on a face marked 1, the player has to pay 10,000/=. If it lands on marked with 2 or 4 the player wins 5000/= and if it lands on a 3, the player wins 3000/=. Find the expected gain in one throw.
10. A discrete random variable X can take on values 10 and 20 only. If E(X) = 16. Write out the probability distribution for X (P(X=x) = 0.4 and P(X=x) = 0.6)
11. A discrete random variable X can take on values 0, 1, 2, and 3 only. If E(X) = 1.4, P(X$\leq$ 2) = 0.9 and P(X$\leq$1) = 0.5. Find (i) P(X=1) = 0.3 (ii) P(X =0) = 0.2
12. The discrete random variable Y has a probability distribution is given by
    P(Y=y) = cy, y = 1, 2, 3, 4
    Find (i) value of c = 0.1 (ii) E(X) = 3
13. A discrete random variable has p.d.f

$$P(X = x) = \begin{cases} k2^x, & x = 0, 1, 2, 3, 4, 5, 6 \\ 0, & otherwise \end{cases},$$

Find (i) value of k = $\frac{1}{127}$, (ii) mean = 5

14. A discrete random variable X has a probability distribution

x	0	1	2	3	4	5
P(X= x)	0.11	0.17	0.2	0.13	p	0.09

Find (i) the value of p = 0.3 (ii) Expected value of X (= 2.6)

Solutions to revision exercise 2

10. A discrete random variable X can take on values 10 and 20 only. If E(X) = 16. Write out the probability distribution for X
    Let P(X=10) = a and P(X=20) = b
    a+b = 1
    a = (1-b)………………. (i)
    10a + 20b = 16 ……. (ii)
    Eqn. (i) and eqn. (ii)
    10(1-b) + 20b = 16
    10 + 20b = 16
    b = 0.6
    a = 1- 0.6 = 0.4

Probability distribution:(P(X=x) = 0.4 and P(X=x) = 0.6)

11. A discrete random variable X can take on values 0, 1, 2, and 3 only. If E(X) = 1.4, P(X$\leq$ 2) = 0.9 and P(X$\leq$1) = 0.5. Find (i) P(X=1) (ii) P(X =0)

Let P(X = 0) = a, P(X=1)=b, P(X=2)= c p(X = 3) = d

a + b + c + d = 1 ……………. (i)

P(X$\leq$ 2) = a + b+ c = 0.9 … (ii)

Eqn. (i) and eqn. (ii)

d = 0.1

P(X$\leq$1) = a + b = 0.5 ……. (iii)

Eqn. (i) and eqn. (iii)

0.5 + c + 0.1 = 1

c = 0.4

E(X) = 0 x a + 1 x b + 2 x c + 3 x 0.1 = 1.4

= b + 2c + 0.3 = 1.4

b + 2c = 1.1

b + 2 x 0.4 = 1.1

b = 0.3

a = 0.5 – 0.3 = 0.2

Hence, (i) P(X=1) = 0.3 (ii) P(X =0) = 0.2

12. The discrete random variable Y has a probability distribution is given by

P(Y=y) = cy, y = 1, 2, 3, 4

Find (i) value of c = 0.1 (ii) E(X) = $\frac{11}{3}$

(i) $\sum P(X=x)=1$

c + 2c + 3c + 4c = 1

10c = 1; c= 0.1

(ii) E(X) =$\sum xP(X=x)$= 1 x 0.1 + 2 x 0.2 + 3 x 0.3 + 4 x 0.4 =3

13. A discrete random variable has p.d.f

$$P(X=x)=\begin{cases} k2^x, & x=0,1,2,3,4,5,6 \\ 0, & otherwise \end{cases},$$

Find

(i) (i) value of k = $\frac{1}{127}$,

$\sum P(X=x)=1$

k($2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$) = 1

k = $\frac{1}{127}$

(ii) Mean = $\sum Px(X=x)=\frac{1}{127}$ (0 x 1 + 2 x 2 + 3 x 8 + 4 x 16 + 4 x 32 + 6 x 64) = 5.01

14. A discrete random variable X has a probability distribution

x	0	1	2	3	4	5
P(X= x)	0.11	0.17	0.2	0.13	p	0.09

Find

(i) the value of p

$\sum P(X=x)=1$

0.11 + 0.17 + 0.2 + 0.13 + p + 0.09 = 1

p = 0.3

(ii) Expected value of X

E(X) =$\sum Px(X=x)$ = 0.11x0 + 0.17x1 + 0.2x2 + 0.13x3 + 0.3 x 4 + 0.09x5= 2.61

Properties of the mean

(i) E(a) = a
(ii) E(ax) = aE(x)
(iii) E(ax + b) = aE(x) + b
(iv) E(ax - b) = aE(x) - b

Example 13

A random variable X of discrete probability distribution is given by

x	1	2	3	4
P(X=x)	0.1	0.2	0.3	0.4

Find

(i) E(x) = $\sum Px(X = x)$ = 1 x 0.1 + 2 x 0.2 + 3 x 0.3 + 4 x 0.4 = 3
(ii) E(3x) = 3E(x) = 3 x 3 = 9
(iii) E(4x + 6) =4E(x) + 6 = 4 x 3 + 6 = 18

Example 14

A random variable X of discrete probability distribution is given by

x	-1	0	1	2
P(X= x)	0.25	0.10	0.45	0.20

Find

(i) P(-1$\leq$ X <1) = P(X= -1) + p(X = 0) = 0.25 + 0.10 = 0.35
(ii) E(X) =$\sum Px(X = x)$ = -1 x 0.25 + 0 x 0.10 + 1 x 0.45 + 2 x 0.20 = 0.6
(iii) E(6x – 2) = 6E(X) – 2 = 0.6 x 6 – 2 = 1.6

Variance, Var(x)

Var(x) = E(X^2) = [E(x)]^2 where E(X^2) = $\sum x^2 P(X = x)$

Example 15

A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	0.2	0.3	0.1

Find

(i) The mean = $\sum Px(X = x)$ = 1 x 0.1 + 2 x 0.3 + 3 x 0.2 + 4 x 0.3 + 5 x 0.1= 3
(ii) Var(x)
E(X^2) =$\sum Px^2(X = x)$ = 1^2 x 0.1 + 2^2 x 0.3 + 3^2 x 0.3 + 4^2 x 0.3 + 5^2 x 0.1 = 10.4
Var(X) = 10.4 – $(3)^2$ = 1.4

Example 16

The discrete random variable Y has a probability distribution is given by P(Y =y), y= -3, -2, -1, 0, 1, 2,3

Find: (i) value of c (ii) mean (iii) standard deviation

Solution

y	-3	-2	-1	0	1	2	3
P(Y= y)	3c	2c	c	0	c	2c	3c

(i) $\sum P(X = x) = 1$

3c +2c + c+ c + 2c + 3c = 1; c = $\frac{1}{12}$

(ii) Mean = $\sum Px(X = x)$ = -3 x 3c + -2 x 2c + -1 x c + 0 x 0 + 1 x c + 2 x 2c + 3 x 3c = 0

(iii) $E(X^2) = (-3)^2 \times 3c + (-2)^2 \times 2c + (-1)^2 \times c + (0)^2 \times 0 + (1)^2 \times c + (2)^2 \times 2c + (3)^2 \times 3c = 72 \times \frac{1}{12} = 6$

Var (x) = $E(X^2) - (E(x))^2 = 6 - (0)^2 = 6$

S.D = $\sqrt{Var\ (X)} = \sqrt{6} = 2.45$

Example 17

Two marbles are drawn without replacement from a box containing 3 red marbles and 4 white marbles. The marbles are randomly drawn. If X is the random variable for the number of red marble drawn find

(i) Expected number of red marbles

P(X= 0) = P(W∩W) = $\frac{4}{7} x \frac{3}{6} = \frac{2}{7}$

P(X= 1) = P(W∩R) + P(R∩W) = $\frac{4}{7} x \frac{3}{6} + \frac{3}{7} x \frac{4}{6} = \frac{4}{7}$

P(X= 2) = P(R∩R) = $\frac{3}{7} x \frac{2}{6} = \frac{1}{7}$

The probability distribution table

x	0	1	2
P(X=x)	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

E(x) = $\sum Px(X = x) = \frac{2}{7} x\ 0 + \frac{4}{7} x\ 1 + \frac{1}{7} x\ 2 = \frac{6}{7}$

(ii) Standard deviation of X

$E(x^2) = \sum Px^2(X = x) == \frac{2}{7} x\ 0 + \frac{4}{7} x\ 1^2 + \frac{1}{7} x\ 2^2 = \frac{8}{7}$

Var (x) = $E(x^2) - (E(X))^2 = \frac{8}{7} - \left(\frac{6}{7}\right)^2 = \frac{20}{49}$

S.D = $\sqrt{Var\ (X)} = \sqrt{\frac{20}{49}} = 0.6389$

Example 19

A vendor stocks 12 copies of a magazine each week and the probability for each possible total number of copies sold is shown below

Number of copies	9	10	11	12
probability	0.2	0.35	0.30	0.15

(a) Estimate the mean and variance of the number of copies

Mean = $\sum Px(X = x)$ = 9 x 0.2 + 10 x 0.35 + 11 x 0.3 + 12 x 0.15 = 10.4

$E(X^2) = 9^2 \times 0.2 + 10^2 \times 0.35 + 11^2 \times 0.3 + 12^2 \times 0.15 = 109.1$

Var(x) = $109.1 - (10.4)^2 = 0.94$

(b) The vendor buys the magazine at 8,500/= and sells at 14,500/=. Any copies not sold are destroyed. Construct a probability distribution table for vendor's weekly profit and hence find the expected weekly profit

Profit = S.P – C.P

Profit for 9 copies = 9 x 14,500 – 12 x 8500 = 28500
Profit for 10 copies= 10 x 14,500 – 12 x 8500 =43000
Profit for 11 copies= 11 x 14,500 – 12 x 8500 = 57500
Profit for 12 copies= 12 x 14,500 – 12 x 8500 = 72000

y	28500	43000	57500	72000
P(Y=y)	0.2	0.35	0.30	0.15

E(Y) = 0.2 x 28500 + 0.35 x 43000 + 0.30 x 57500 + 0.15 x 72000 = 48000/=

Example 20

The table below shows the number of red and green balls put in three identical boxes A, B and C.

Boxes	A	B	C
Red balls	4	6	3
Green balls	2	7	5

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn".

(a) Draw a probability distribution table for X (06marks)

Using combination

$$P(X = 0) = \frac{1}{3}\left[\frac{^4C_2}{^6C_2} + \frac{^6C_2}{^{13}C_2} + \frac{^3C_2}{^8C_2}\right]$$

$$= \frac{1}{3}\left[\frac{2}{5} + \frac{5}{26} + \frac{2}{28}\right] = \frac{1273}{5460}$$

$$P(X = 1) = \frac{1}{3}\left[\frac{^2C_1\ x\ ^4C_1}{^6C_2} + \frac{^7C_1\ x\ ^6C_1}{^{13}C_2} + \frac{^3C_1\ x\ ^3C_1}{^8C_2}\right]$$

$$= \frac{1}{3}\left[\frac{8}{15} + \frac{7}{13} + \frac{15}{28}\right] = \frac{8777}{16380}$$

$$P(X = 2) = \frac{1}{3}\left[\frac{^2C_2}{^6C_2} + \frac{^7C_2}{^{13}C_2} + \frac{^5C_2}{^8C_2}\right]$$

$$= \frac{1}{3}\left[\frac{1}{15} + \frac{7}{26} + \frac{5}{14}\right] = \frac{946}{4095}$$

x	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$

(b) Calculate the mean and variance of X (06marks)

1	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$
xP(X =x)	0	$\frac{8777}{16380}$	$\frac{1892}{4095}$
x^2P(X =x)	0	$\frac{8777}{16380}$	$\frac{3784}{4095}$

$E(X) = \frac{8777}{16380} + \frac{1892}{4095} = 0.9979$

$E(X^2) = \frac{8777}{16380} + \frac{3784}{4095} = 1.4599$

Var(X) = 1.4599 – 0.9979

= 0.4642

Properties of the variance

(i) Var(a) = 0
(ii) Var(aX) = a^2Var(X)
(iii) Var(aX + b) = a^2Var(X)
(iv) Var(aX – b) = a^2Var(X)

Example 21

A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X = x)	0.2	0.25	0.4	0.1	0.05

Find

(i) Mean = $\sum Px(X = x)$ = 1 x 0.2 + 2 x 0.25 + 3 x 0.4 + 4 x 0.1 + 5 x 0.05 = 2.55
(ii) The variance
E(X^2) = 1^2 x 0.2 + 2^2 x 0.25 + 3^2 x 0.4 + 4^2 x 0.1 + 5^2 x 0.05 = 7.65
Var(x) = E(X^2) – $(E(X))^2$ = 7.65 – $(2.55)^2$ = 1.148
(iii) Var (3x -2) = 3^2Var(x) = 9 x 1.148 = 10.332

Example 22

A random variable X of a discrete probability distribution given by

x	10	20	30
P(X = x)	0.2	0.3	0.5

Find

(i) E(X) = 10 x 0.2 + 20 x 0.3 + 30 x 0.5 =22
(ii) Var (X) = E(X^2) – $(E(X))^2$
E(X^2) = 10^2 x 0.2 + 20^2 x 0.3 + 30^2 x 0.5 = 520
Var(x) = 520 – 22^2 = 36
(iii) Var(4X + 3) = 4^2Var(x) = 16 x 36 = 576

Revision exercise 3

1. A random variable X of discrete probability distribution is given by

x	1	2	3
P(X = x)	0.2	0.3	0.5

Find (i) E(X) = 2.3 (ii) E(X^2) = 5.9 (iii) Var (X) = 0.61

2. A random variable X of discrete probability distribution is given by

x	-1	0	1	2
P(X = x)	0.25	0.1	0.45	0.2

Find: (i) P(-1≤ X < 2) = 0.8 (ii) E(X) = 0.6 (iii) E(2x + 3) = 4.2

3. A random variable X of a discrete probability distribution
P(X = 0) = 0.05, P(X = 1) = 0.45 P(X = 2) = 0.5

Find: (i) E(X) = 1.45, (i) E(X^2) = 2.45 (iii) Var (X) = 0.348

4. A random variable X of discrete probability distribution is given by
   P(X = 1) 0.1, P(X= 2) = 0.2, P(X = 3) = 0.3, P(X =4)= 0.4
   Find (i) E(X) = 3 (ii) Var (X) = 1 (iii) P(X = 2/X≥2) = $\frac{2}{9}$
5. The discrete random variable Y has a probability distribution P(Y = y) = k =1, 2, 3, 4, 5,6
   Find (i) mean, μ = 3.5 (ii) E(3X +4) = $15\frac{1}{6}$ (iii) E(X^2)=14.5 (iv) standard deviation = 1.708
6. The discrete random variable R has a probability distribution is given by
   P(R = r) = $\frac{3r+1}{22}$; r = 0, 1, 2, 3
   Find (i) mean, μ= $\frac{24}{11}$, E(R2) = $\frac{61}{11}$ (iii) E(3R-2) = $\frac{50}{11}$
7. The discrete random variable R has a probability distribution given by
   $$P(R = r) = \begin{cases} \frac{2r+1}{20}; r = 0, 1, 2,3 \\ \frac{11-r}{20}, \quad r = 4, 5 \end{cases}$$
   Find (i) E(R) = 2.55, (ii) Var(R) = 1.45
8. The discrete random variable X has a probability distribution given by
   $$P(X = x) = \begin{cases} kx, & x = 1, 2, 3, 4, 5 \\ k(10 - x), & x = 6, 7,8, 9 \end{cases}$$
   Find (i) constant, k = 0.04, (i) E(X) = 5 (iii) Var(X) = 4
9. The discrete random variable X has a probability distribution is given by
   P(X =x) = kx, x = 1, 2, 3,…..n ; where k is a constant
   Show that k = $\frac{2}{n(n+1)}$, hence find in terms of n the mean X = $\frac{1}{3}(2n + 1)$
10. A random variable X of a discrete probability distribution given by
    P(X = 0) = P(X=1) = 0.1, P(X = 2) = 0.2, P(X= 3) = P(X = 4) = 0.3. Find Var (X) = 1.64
11. A random variable X of a discrete probability distribution given by
    P(X = 2) = 0.1; P(X= 4) = 0.3; P(X = 6) = 0.5; P(X = 8) = 0.1. Find Var (X) 2.56

Cumulative distribution function F(X)

F(X) is given by FX = $\sum P(X = x)$

Note F(+∞) = 1 where +∞ is the upper limit.

Example23

A discrete random variable has a probability distribution

x	1	2	3	4	5
P(X = x)	0.2	0.25	0.4	0.1	0.05

Find the cumulative distribution function

Solution

x	1	2	3	4	5
F(X))	0.2	0.45	0.85	0.95	1

Example 24

The random variable X has a cumulative function below

X	-1	0	1	2
F(X)	0.25	0.35	0.80	1

Find the probability distribution function

X	-1	0	1	2
F(X)	0.25	0.1	0.45	0.2

Example 25

A discrete random variable has a cumulative distribution

x	1	2	3	4	5
F(X)	0.2	0.32	0.67	0.91	1

Find (i) probability distribution function

x	1	2	3	4	5
F(X)	0.2	0.12	0.35	0.24	0.09

(ii) P(X = 3) =0.35

(iii) P(X>2) = P(X= 3) + P(X =4) + P(X=5) = 1- 0.12 = 0.68

Example 26

The random variable X has a cumulative function

X	1	2	3	4
F(X)	0.1	0.5	0.8	1

Find (i) mean (ii) Var(X) (iii) mode

Solution

X	1	2	3	4
P(X = x)	0.1	0.4	0.3	0.2

(i) Mean = $\sum xP(X = x)$ = 1 x 0.1 + 2 x 0.4 + 3 x 0.3 + 4 x 0.2 = 2.6

(ii) Var(X)

$E(X^2) = 1^2 \times 0.1 + 2^2 \times 0.4 + 3^2 \times 0.3 + 4^2 \times 0.2 = 5.92$

$Var(X) = E(X^2) - (E(X))^2 = 5.92 - (2.6)^2 = 0.84$

(iii) Mode 2

Revision Exercise 4

1. A discrete random variable has a cumulative distribution

x	0	1	2	3	4
F(X)	01	0.3	0.6	0.8	1

Find (i) E(X) = 15.2 (ii) Var(X)=1.56 (iii) Var(6X + 2) = 56.16

2. The random variable X has a cumulative function below

x	1	2	3	4
F(X)	0.13	0.54	0.75	1

Find (i) P(X=2) = 0.41 (ii) P(X>1) = 0.87 (iii) P(X ≥ 3) = 0.46 (iv) P(X<2) = 0.13 (V)E(X) = 2.58

3. A discrete random variable X has a cumulative distribution

x	3	4	5	6	7
F(X)	0.01	0.23	0.64	0.85	1

Find (i) probability distribution function (ii) Var (X) = 0.9724)

4. A discrete random variable has a cumulative probability function F(X) = $\frac{x^2}{9}$, x = 1, 2, 3.

   Find (1) F(2) = $\frac{4}{9}$ (ii) P(X = 2)= $\frac{1}{3}$ (iii) E(2X -3) = $\frac{17}{9}$

5. A discrete random variable has a cumulative probability function. F(X) = k, x = 1, 2,3

   Find the

   (i) constant k = $\frac{1}{3}$

   (ii) P(X<3) = $\frac{2}{3}$

   (iii) Standard deviation, σ = 0.816

6. A discrete random variable has a cumulative probability function

   F(X) = $1-\left(1-\frac{x}{4}\right)^x$ x = 1, 2, 3, 4

   Find the

   (i) F(3) = $\frac{63}{64}$

   (ii) F(2) =$\frac{3}{4}$

   (iii) Var (X) = 0.547

Thank you

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