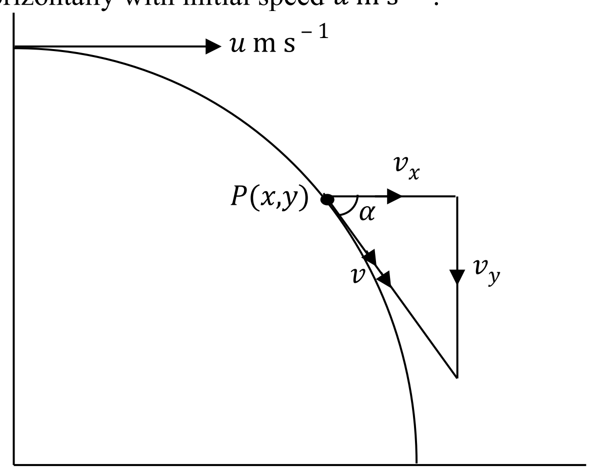
11. PROJECTILES

In projectile motion, the horizontal and vertical motion of a projectile can be considered separately. When this is done each kind of motion is linear and equations of linear motion can be applied.

11.1 Horizontal projection

This is when the initial direction of motion of the projectile is horizontal. Consider a particle projected horizontally with initial speed u m s^{-1}.



If after time t, particle passes through a point $P(x,y)$:

From $v = u + at$

$$v_x = u_x + a_x t, a_x = 0$$

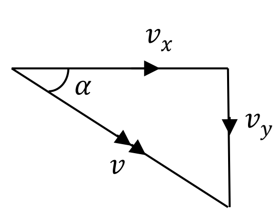
$$v_x = u_x = u$$

$$v_y = u_y + a_y t, u_y = 0, a_y = g$$

$$v_y = gt$$

The speed of the particle at time t is $v = \sqrt{v_x^2 + v_y^2}$.

The direction α of the particle at time t, is obtained from the velocity of the particle.



$$\tan\alpha = \frac{v_y}{v_x}$$

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

From $s = ut + \frac{1}{2}at^2$

$$s_x = u_x t + \tfrac{1}{2}a_x t^2, a_x = 0, u_x = u$$

$$x = ut$$

$$s_y = u_y t + \tfrac{1}{2}a_y t^2, u_y = 0, a_y = g$$

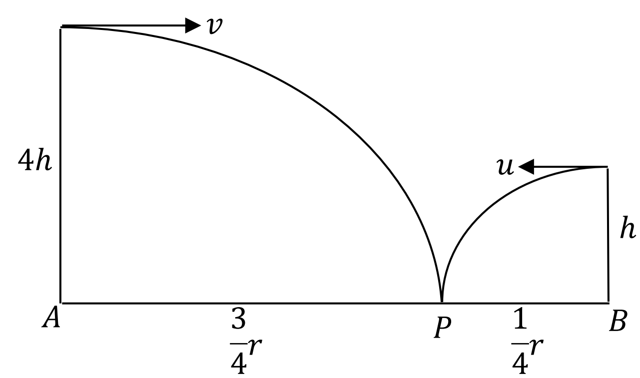
$$y = \tfrac{1}{2}gt^2$$

Example 1

A and B are two points on level ground. A vertical tower of height $4h$ has its base at A and a vertical tower of height h has its base at B. When a stone is thrown horizontally with speed v from the top of the taller tower towards the smaller tower, it lands at a point P where $AP = \frac{3}{4}AB$. When a stone is thrown horizontally with speed u from the top of the smaller tower towards the taller tower, it also lands at the point P. Show that $3u = 2v$.

Solution

Let $AB = r$



From taller tower:

From $x = u_x t$

$$\tfrac{3}{4}r = vt_1 \Rightarrow t_1 = \tfrac{3r}{4v}$$

From $y = \frac{1}{2}gt^2$

$$4h = \tfrac{1}{2}gt_1^2 \Rightarrow 4h = \tfrac{1}{2}g\left(\tfrac{3r}{4v}\right)^2$$

$$128hv^2 = 9gr^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

From smaller tower:

From $x = u_x t$

$$\frac{1}{4}r = ut_2 \Rightarrow t_2 = \frac{r}{4u}$$

From $y = \frac{1}{2}gt^2$

$$h = \frac{1}{2}gt_2^2 \Rightarrow h = \frac{gr^2}{32u^2}$$

$$32hu^2 = gr^2 \text{ .... } (ii)$$

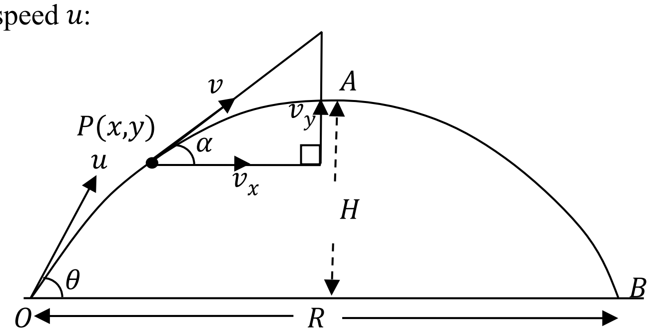
Dividing equation (ii) by equation (i)

$$\frac{u^2}{4v^2} = \frac{1}{9} \Rightarrow 9u^2 = 4v^2$$

$$3u = 2v$$

11.2 Projection from level ground

Consider a particle projected at an angle θ with initial speed u:



If the particle passes through a point $P(x,y)$ after time t.

From $v = u + at$

$$v_x = u_x = u\cos\theta$$

$$v_y = u_y + a_y t,\ a_y = -g$$

$$v_y = u\sin\theta - gt$$

The vertical component of velocity reduces with time and becomes zero at point A (maximum height) and then increases in the downward direction. Since the vertical component of velocity is zero at A, the motion of the particle is said to be horizontal.

Hence at A, $v_y = 0$

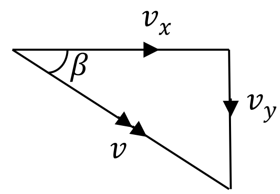
$$0 = u\sin\theta - gt$$

$$t = \frac{u\sin\theta}{g}$$

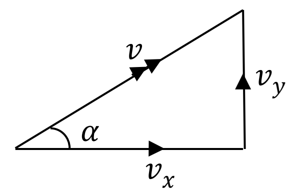
This is the time taken to reach the maximum height. The speed of the particle after time, t is $v = \sqrt{v_x^2 + v_y^2}$.

The direction of the particle is the angle between the horizontal and velocity of the particle. The angle is above the horizontal when the particle is ascending and below the horizontal when the particle is descending.

Ascent



Descent



$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) \qquad \beta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

From $s = ut + \frac{1}{2}at^2$

$$s_x = u_x t + \frac{1}{2}a_x t^2,\ a_x = 0$$

$$x = (u\cos\theta)t \text{ .... } (i)$$

$$s_y = u_y t + \frac{1}{2}a_y t^2,\ u_y = u\sin\theta,\ a_y = -g$$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2 \text{ .... } (ii)$$

11.2.1 Equation of trajectory

This is the equation of the path taken by a projectile.

From equation (i): $x = (u\cos\theta)t \Rightarrow t = \frac{x}{u\cos\theta}$

Substituting in equation (ii)

$$y = (u\sin\theta) \times \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$$

$$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$$

11.2.2 Maximum height (H)

At maximum height $v_y = 0 \Rightarrow u\sin\theta - gt = 0$

$$\Rightarrow t = \frac{u\sin\theta}{g}$$

From $y = (u\sin\theta)t - \frac{1}{2}gt^2$

$$H = u\sin\theta \times \frac{u\sin\theta}{g} - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$H = \frac{u^2 \sin^2\theta}{2g}$$

Alternatively; from $v^2 = u^2 + 2as$

$v_y^2 = u_y^2 - 2gy$, when $y = H$, $v_y = 0$

$$O = (u \sin\theta)^2 - 2gH$$

$$H = \frac{u^2 \sin^2\theta}{2g}$$

11.2.3 Time of Flight ($\boldsymbol{T}$)

This is the time taken by a particle to return to the level of projection.

From $y = (u \sin\theta)t - \frac{1}{2}gt^2$

When the particle returns to the level of projection, $y = 0$

Therefore $O = (u \sin\theta)t - \frac{1}{2}gt^2$

Either $t = 0$ or $t = \frac{2u \sin\theta}{g}$

Hence time of flight, $T = \frac{2u \sin\theta}{g}$

11.2.4 Range ($\boldsymbol{R}$)

This is the horizontal displacement covered by a particle to return to the level of projection.

From $x = (u \sin\theta)t$

When $x = R, t = T = \frac{2u \sin\theta}{g}$

$$R = (u \cos\theta) \times \frac{2u \sin\theta}{g}$$

$$R = \frac{u^2 \sin 2\theta}{g}$$

For any speed of projection, the range R, varies with the angle of projection. The maximum range, $R_{\max} = \frac{(u^2 \sin 2\theta)_{\max}}{g} = \frac{u^2}{g} \times (\sin 2\theta)_{\max}$

but $(\sin 2\theta)_{\max} = 1$ when $2\theta = 90° \Rightarrow \theta = 45°$

hence $R_{\max} = \frac{u^2}{g}$.

Note: If a particle is projected with a certain initial speed, there are always two possible angles of projection for which it hits the plane at the same horizontal displacement from the point of projection. In particular if the projection is on level ground, then the two angles of projection for a given range are complimentary. That is, if the angles are α and β then $\alpha + \beta = 90°$.

Verification:

From $R = \frac{u^2 \sin 2\theta}{g} \Rightarrow 2\theta = \sin^{-1}\left(\frac{Rg}{u^2}\right)$

Let $\sin^{-1}\left(\frac{Rg}{u^2}\right) = \phi$, then $2\theta = \phi, 180 - \phi$

$$\therefore\ \theta = \frac{1}{2}\phi, 90 - \frac{1}{2}\phi \Rightarrow \alpha = \frac{1}{2}\phi,\ \beta = 90 - \frac{1}{2}\phi$$

Hence $\alpha + \beta = 90°$.

Example 2

A particle is projected with velocity of 40 m s^{-1} at an angle of $60°$ to the horizontal. Find the maximum height and range of the particle.

Solution

$u = 40\text{ m s}^{-1}, \theta = 60°$

$$H = \frac{u^2 \sin^2\theta}{2g}$$

$$H = \frac{40^2 (\sin 60)^2}{2 \times 9 \cdot 8}$$

$$H = 61 \cdot 224\text{ m}$$

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$R = \frac{40^2 \sin 120}{9 \cdot 8}$$

$$R = 141 \cdot 392\text{ m}$$

Example 3

(a) Derive an equation of path of a particle projected from origin O at an angle α to the horizontal with initial speed $u\text{ m s}^{-1}$.

(b) A particle projected from a point O on a horizontal ground moves freely under gravity and hits the ground at A. Taking O as the origin, the equation of trajectory of the particle is $60y = 20\sqrt{3}x - x^2$, where x and y are measured in metres. Determine the:

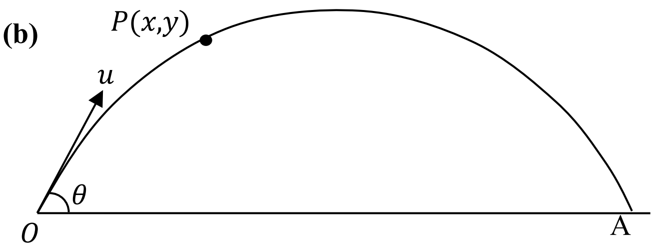
(i) initial speed and angle of projection.

(ii) distance OA. (Take g as 10 m s^{-2})

Solution

(a) See introduction.

(b)



(i) $60y = 20\sqrt{3}x - x^2 \Rightarrow y = \frac{\sqrt{3}}{3}x - \frac{x^2}{60}$

From $y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

Comparing coefficients:

$$\tan\theta = \frac{\sqrt{3}}{3} \Rightarrow \theta = 30°$$

$$\frac{g(1+\tan^2\theta)}{2u^2} = \frac{1}{60}$$

$$\frac{10\left(1+\frac{1}{3}\right)}{2u^2} = \frac{1}{60}$$

$$u = 20 \text{ m s}^{-1}$$

(ii) Along OA, $y = 0$

$$60 \times 0 = x(20\sqrt{3} - x)$$

Either $x = 0$ or $x = 20\sqrt{3}$

Hence $AB = 20\sqrt{3}$ m

Example 4

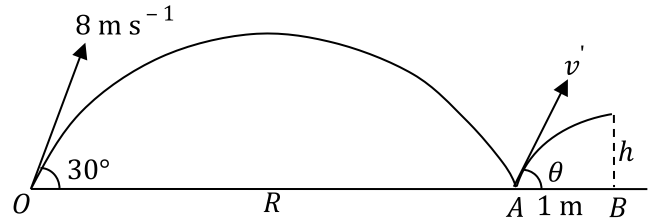
A football player projects a ball at a speed of 8 m s^{-1} at an angle of 30° with the ground. The ball strikes the ground at a point which is level with the point of projection. After impact with the ground, the ball bounces and the horizontal component of velocity remains the same but the vertical component is reversed in direction and halved in magnitude. The player running after the ball kicks it again at a point which is a horizontal distance of 1 m from the point where it bounced, so that the ball continues in the same direction. Find the:

(a) horizontal distance between the points of projection and the point at which the ball strikes the ground.

(b) (i) time interval between the ball striking the ground and the player kicking it again.

(ii) height of the ball above the ground when it is kicked again. (Take $g = 10$ m s^{-2})

Solution



(a) From $y = (u\sin\theta)t - \frac{1}{2}gt^2$

At A, $y = 0$

$O = (u\sin\theta)t - \frac{1}{2}gt^2$

Either $t = 0$ or $t = \frac{2u\sin\theta}{g}$

$\therefore\ t = T = \frac{2u\sin\theta}{g}$

From $x = (u\cos\theta)t$

When $x = R$, $t = T$

$$R = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{u^2\sin 2\theta}{g}$$

$$R = \frac{8^2\sin 60}{10}$$

$$R = 5 \cdot 5426 \text{ m}$$

(b) (i) $T = \frac{2u\sin\theta}{g} = \frac{2\times 8\sin 30}{10}$

$$T = 0 \cdot 8 \text{ s}$$

$$v_x = u\cos\theta$$

$$= 8\cos 30$$

$$v_x = 4\sqrt{3} \text{ m s}^{-1}$$

$$v_y = u\sin\theta - gt$$

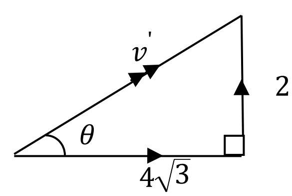
$$= 8\sin 30 - 10 \times 0 \cdot 8$$

$$v_y = -4 \text{ m s}^{-1}$$

On bouncing from ground:

$$v_x' = v_x = 4\sqrt{3} \text{ m s}^{-1}$$

$$v_y' = 2 \text{ m s}^{-1}$$



$$\tan\theta = \frac{2}{4\sqrt{3}}$$

$$\theta = 16 \cdot 1°$$

$$v' = \sqrt{2^2 + (4\sqrt{3})^2}$$

$$= 7 \cdot 211 \text{ m s}^{-1}$$

From $x = u_x t$

$$= (8\cos 30)t$$

$$1 = 4\sqrt{3}t$$

$$t = 0 \cdot 1443 \text{ s}$$

(ii) From $y = v_y' t - \frac{1}{2}gt^2$

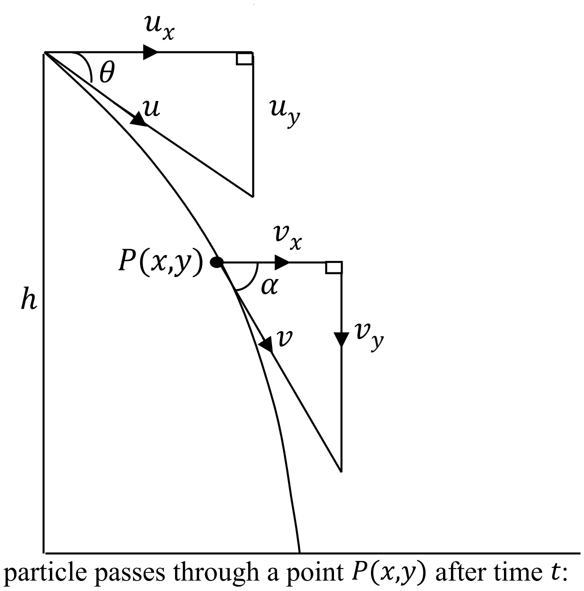
$$h = 2 \times 0 \cdot 1443 - \frac{1}{2} \times 10 \times 0 \cdot 1443^2$$

$$h = 0 \cdot 1845 \text{ m}$$

11.3 Projection from a height above level ground

11.3.1 Projection at an angle below the horizontal

Consider a particle projected from a height h, above level ground at an angle θ below the horizontal.



If the particle passes through a point $P(x,y)$ after time t:

From $v = u + at$

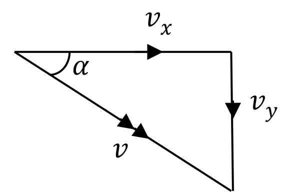
$$v_x = u_x = u\cos\theta$$

$$v_y = u_y + a_y t,\ a_y = g$$

$$v_y = u\sin\theta + gt$$

The speed of the particle after time t is $v = \sqrt{v_x^2 + v_y^2}$.

The direction of the particle at this time:



$$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

From $s = ut + \frac{1}{2}at^2$

$$x = u_x t + \frac{1}{2}a_x t^2,\ a_x = 0$$

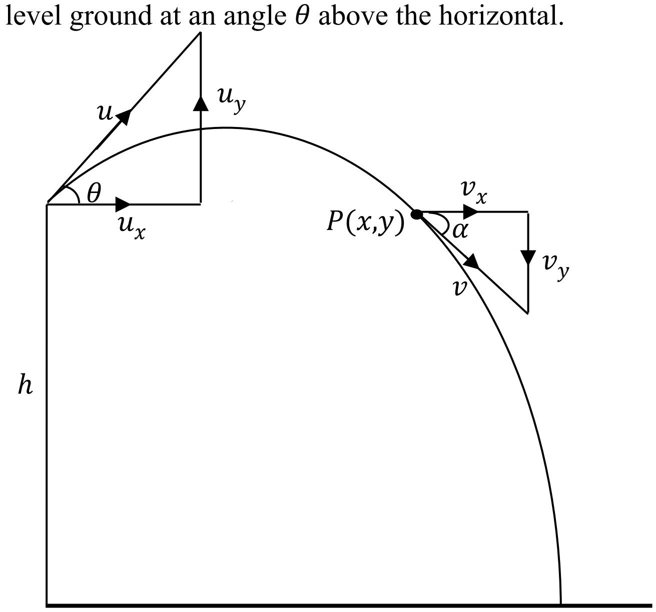
$$x = (u\cos\theta)t$$

$$y = u_y t + \frac{1}{2}a_y t^2,\ a_y = g$$

$$y = (u\sin\theta)t + \frac{1}{2}gt^2$$

11.3.2 Projection at an angle above the horizontal

Consider a particle projected from a height h, above level ground at an angle θ above the horizontal.



Vertical motion goes above the level of projection and then below the level of projection. The sign of g depends on the initial direction of motion. If the initial direction is upwards, we use $a = -g$ and if the initial direction is downwards, we use $a = g$. Positive vertical displacements are those above the level of projection, zero the at level of projection and negative below the level of projection.

If the particle passes through a point $P(x, y)$ after time t:

From $v = u + at$

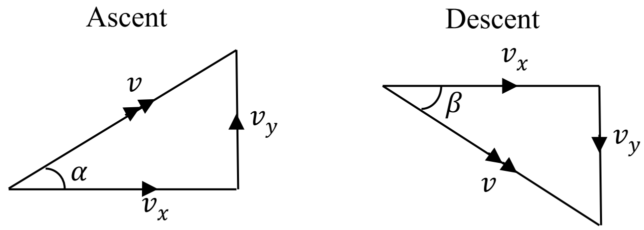
$$v_x = u_x = u\cos\theta$$

$$v_y = u_y - gt$$

$$v_y = u\sin\theta - gt$$

Speed of particle; $v = \sqrt{v_x^2 + v_y^2}$.

Direction of the particle:



$$\tan \alpha = \frac{v_y}{v_x} \qquad \tan \beta = \frac{v_y}{v_x}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) \qquad \Rightarrow \beta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

From $s = ut + \frac{1}{2}at^2$

$$x = u_x t + \frac{1}{2}a_x t^2,\ a_x = 0$$

$$x = (u\cos\theta)t$$

$$y = u_y t + \frac{1}{2}a_y t^2,\ a_y = -g$$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

From equation (i): $x = (u\cos\theta)t \Rightarrow t = \frac{x}{u\cos\theta}$

Substituting in equation (ii)

$$y = (u\sin\theta) \times \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$$

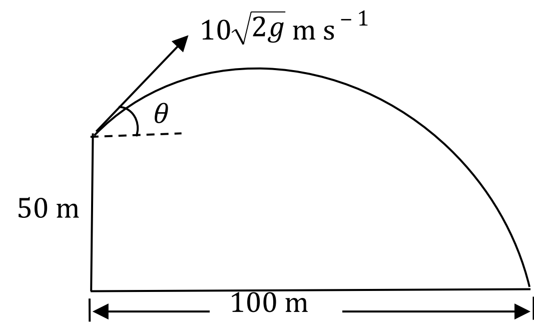
$$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$$

Note: $-h = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

Example 5

A particle is projected with a speed of $10\sqrt{2g}$ m s^{-1} from the top of a cliff 50 m high. The particle hits the sea at a distance of 100 m from the vertical through the point of projection. Show that there are two possible directions of projection which are perpendicular. Determine the time taken from the point of projection in each case.

Solution



From $y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

$$-50 = 100\tan\theta - \frac{g \times 100^2(1+\tan^2\theta)}{2(10\sqrt{2g})^2}$$

$$-50 = 100\tan\theta - \frac{10000g(1+\tan^2\theta)}{2 \times 200g}$$

$$-2 = 4\tan\theta - (1+\tan^2\theta)$$

$$\tan^2\theta - 4\tan\theta - 1 = 0$$

$$(\tan\theta - 2)^2 = 5$$

$$\tan\theta = 2 \pm \sqrt{5}$$

If the angles are θ_1 and θ_2 then; $\tan\theta_1 = 2 + \sqrt{5}$ and $\tan\theta_2 = 2 - \sqrt{5}$

The directions of projection are perpendicular if $\tan\theta_1 \times \tan\theta_2 = -1$

From $\tan\theta_1 \times \tan\theta_2 = (2+\sqrt{5})(2-\sqrt{5})$

$$= 4 - 5 = -1$$

Hence the two directions are perpendicular.

When $\tan\theta_1 = 2 + \sqrt{5}$

$$\theta_1 = 76 \cdot 7°$$

From $x = u_x t$

$$100 = (10\sqrt{2 \times 9 \cdot 8}\cos 76 \cdot 7)t_1$$

$$t_1 = 9 \cdot 8313 \text{ s}$$

When $\tan\theta_2 = 2 - \sqrt{5}$

$$\theta_2 = -13 \cdot 3°$$

$= 13 \cdot 3°$ below the horizontal

From $x = u_x t$

$$100 = (10\sqrt{2 \times 9 \cdot 8}\cos 13 \cdot 3)t_2$$

$$t_2 = 2 \cdot 3209 \text{ s}$$

Note: Angle between directions of projection can also be obtained from,

$$\theta_1 - \theta_2 = 76\cdot 7° -- 13\cdot 3° = 90°$$

Example 6

A point O is directly above point A of a horizontal plane. A particle P is projected from O with a speed of $5v$ at an angle $\cos^{-1}\left(\frac{3}{5}\right)$ above the horizontal and hits the plane at point B at a distance $\frac{48v^2}{g}$ from A.

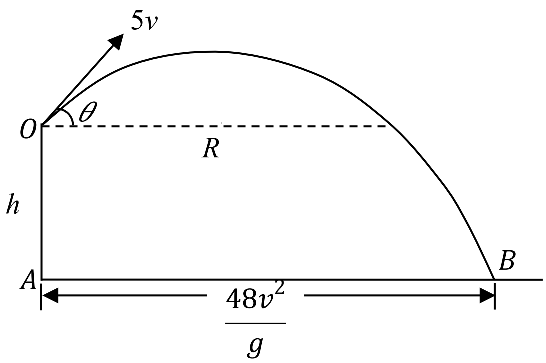
(i) Show that the height of O above A is $\frac{64v^2}{g}$.

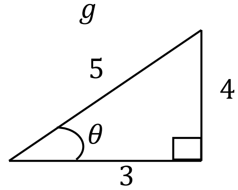
(ii) Find the distance of P from O when it is directly on level with it.

- A second particle is now projected with a speed of $24w$ from O at an angle α above the horizontal and also hits the plane at B. Find the equation involving v, w and α.
- Given that the value of α is 45°, find w in terms of v and show that the other value occurs such that $7\tan^2\alpha - 6\tan\alpha - 1 = 0$.

Solution

(i)





From $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$

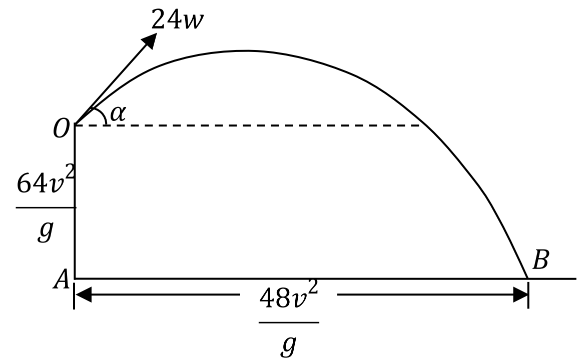
$$-h = \frac{48v^2}{g} \times \frac{4}{3} - \frac{g \times \left(\frac{48v^2}{g}\right)^2}{2 \times (5v)^2 \times \left(\frac{3}{5}\right)^2}$$

$$-h = \frac{64v^2}{g} - \frac{128v^2}{g} \Rightarrow h = \frac{64v^2}{g}$$

(ii) $R = \frac{u^2\sin 2\theta}{g}$

$$R = \frac{2 \times (5v)^2 \times \sin\theta\cos\theta}{g}$$

$$R = \frac{50v^2}{g} \times \frac{4}{5} \times \frac{3}{5} = \frac{24v^2}{g}$$



From $y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$

$$\frac{-64v^2}{g} = \frac{48v^2}{g}\tan\alpha - g\left(\frac{48v^2}{g}\right)^2 \times \frac{\sec^2\alpha}{2 \times (24w)^2}$$

$$\frac{2v^2\sec^2\alpha}{w^2} = 48\tan\alpha + 64$$

$$v^2\sec^2\alpha = 4w^2(6\tan\alpha + 8)$$

When $\alpha = 45°$

$$2v^2 = 4w^2(6 + 8)$$

$$w^2 = \frac{v^2}{28}$$

Substituting for w^2;

$$v^2\sec^2\alpha = 4 \times \frac{v^2}{28}(6\tan\alpha + 8)$$

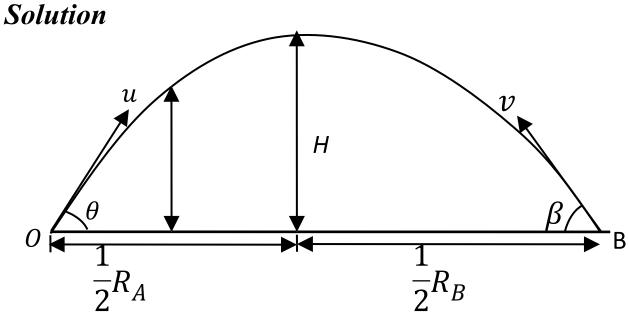
$$7(1 + \tan^2\alpha) = 6\tan\alpha + 8$$

$$7\tan^2\alpha - 6\tan\alpha - 1 = 0$$

Example 7

Two equal particles are projected at the same instant from points A and B on horizontal ground, the first from A with speed u at an angle of elevation α and the second from B with speed v at an angle of elevation β. They collide directly when they are moving horizontally in opposite directions. Find v in terms of u, α and β, and show that $AB = \frac{u^2\sin\alpha\sin(\alpha+\beta)}{g\sin\beta}$.

Solution



$$H = \frac{u^2\sin^2\alpha}{2g} = \frac{v^2\sin^2\beta}{2g}$$

$$v = \frac{u\sin\alpha}{\sin\beta}$$

$$R_A = \frac{u^2 \sin 2\alpha}{g}$$

$$R_B = \frac{v^2 \sin 2\beta}{g} = \left(\frac{u \sin\alpha}{\sin\beta}\right)^2 \times \frac{\sin 2\beta}{g} = \frac{2u^2 \sin^2\alpha \cos\beta}{g \sin\beta}$$

$$AB = \frac{1}{2}(R_A + R_B)$$

$$= \frac{1}{2}\left(\frac{u^2 \sin 2\alpha}{g} + \frac{2u^2 \sin^2\alpha \cos\beta}{g \sin\beta}\right)$$

$$= \frac{u^2 \sin\alpha}{g}\left(\frac{\sin\beta\cos\alpha + \cos\beta\sin\alpha}{\sin\beta}\right)$$

$$AB = \frac{u^2 \sin\alpha \sin(\alpha+\beta)}{g \sin\beta}$$

Example 8

(a) A particle is projected vertically upwards from a point O with speed $\frac{4}{3}v$. After it has travelled a distance $\frac{2}{5}x$ above O, on its upward motion, a second particle is projected vertically upwards from the same point with the same initial speed. Given that the particles collide at a height $\frac{2}{5}x$ above O, x and v being constant, show that:

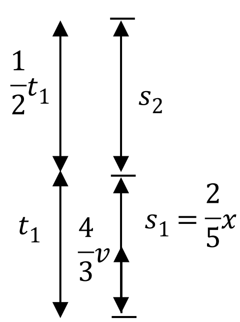
(i) at maximum height H, $8v^2 = 9gH$.

(ii) when the particles collide $9x = 20H$.

(b) A stone projected at an angle α to the horizontal with speed, u m s^{-1} just clears a vertical wall 4 m high and 10 m from the point of projection when travelling horizontally. Find the angle of projection.

Solution

(a)



(ii) (i) At maximum height, $v_y = 0$

From $v = u - gt$

$$0 = \frac{4}{3}v - g\left(\frac{3}{2}t_1\right)$$

$$t_1 = \frac{8v}{9g}$$

From $s = ut - \frac{1}{2}gt^2$

$$H = \frac{4}{3}v \times \left(\frac{3}{2} \times \frac{8v}{9g}\right) - \frac{1}{2}g \times \left(\frac{3}{2} \times \frac{8v}{9g}\right)^2$$

$$H = \frac{8v^2}{9g}$$

$$8v^2 = 9gH$$

(iii) For 1st particle:

$t = 2t_1 = \frac{16v}{9g}$ and $s = \frac{2}{5}x$ when the particles collide

$$s = \frac{4}{3}v \times \frac{16v}{9g} - \frac{1}{2}g\left(\frac{16v}{9g}\right)^2$$

$$\frac{2}{5}x = \frac{64v^2}{27g} - \frac{128v^2}{81g}$$

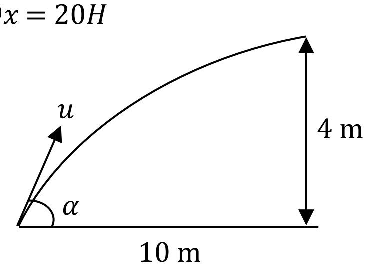
$$81gx = 160v^2$$

From $8v^2 = 9gH \Rightarrow v^2 = \frac{9gH}{8}$

Hence $81gx = 160 \times \frac{9gH}{8}$

$$9x = 20H$$

(b)



A particle travels horizontally when $v_y = 0$, that is, at maximum height.

$H = 4$ m and $\frac{1}{2}R = 10$ m$\Rightarrow R = 20$ m

From $H = \frac{u^2 \sin^2\alpha}{2g}$

$$4 = \frac{u^2 \sin^2\alpha}{2g}$$

$u^2 \sin^2\alpha = 8g$ .... (i)

From $R = \frac{u^2 \sin 2\alpha}{g}$

$$20 = \frac{u^2 \sin 2\alpha}{g}$$

$10g = u^2 \sin\alpha\cos\alpha$ .... (ii)

Dividing equation (i) by equation (ii)

$$\tan\alpha = \frac{8}{10}$$

$$\alpha = 38 \cdot 7°$$

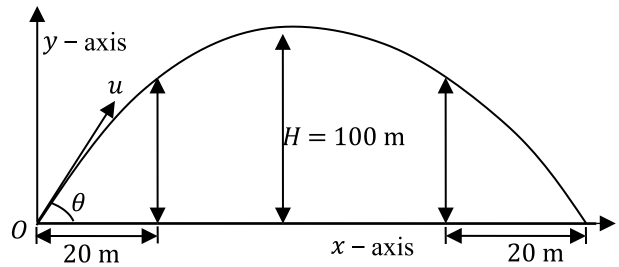
Example 9

The horizontal and vertical components of the initial velocity of a particle projected from point O on a horizontal plane are p and q respectively.

(a) Express the vertical distance y, travelled in terms of horizontal distance x, and the components p and q.

(b) Find the greatest height H, attained and range R, on the horizontal plane through O. Hence show that $y = \frac{4Hx(R-x)}{R^2}$. Given that the particle passes through the point $(20,80)$ and $H = 100$ m, find the velocity of projection.

Solution



$u_x = p, u_y = q$

(a) From $y = u_y t - \frac{1}{2}gt^2$

$$y = qt - \frac{1}{2}gt^2 \quad \ldots\ldots\ldots\ldots (i)$$

From $x = u_x t$

$$x = pt \quad \ldots\ldots\ldots\ldots (ii)$$

(b) At maximum height, $v_y = 0$

From $v_y = u_y - gt$

$$v_y = q - gt \Rightarrow 0 = q - gt \Rightarrow t = \frac{q}{g}$$

From equation (i)

$H = q \times \frac{q}{g} - \frac{1}{2}g\left(\frac{q}{g}\right)^2$

$$H = \frac{q^2}{2g} \quad \ldots\ldots\ldots\ldots (iii)$$

When $x = R, y = 0$

From equation (i)

$$O = qt - \frac{1}{2}gt^2 \Rightarrow t = \frac{2q}{g}$$

From equation (ii)

$R = p \times \frac{2q}{g}$

$R = \frac{2pq}{g} \quad \ldots\ldots\ldots\ldots (iv)$

From equation (ii) $t = \frac{x}{p}$

Substituting for t in equation (i)

$y = \frac{q}{p}x - \frac{gx^2}{2p^2} \ldots\ldots\ldots\ldots (v)$

Dividing Eqn (iii) by Eqn (iv)

$\frac{H}{R} = \frac{q}{4p} \Rightarrow \frac{q}{p} = \frac{4H}{R}$

Squaring Eqn (iv) and dividing by Eqn (iii)

$\frac{R^2}{H} = \frac{8p^2}{g} \Rightarrow p^2 = \frac{R^2 g}{8H}$

Hence from equation (v)

$y = \frac{4Hx}{R} - \frac{gx^2}{2} \times \frac{8H}{R^2 g}$

$y = \frac{4Hx(R-x)}{R^2}$

The particle passes through $(20,80)$ and $H = 100$ m

$80 = \frac{4 \times 100 \times 20(R-20)}{R^2}$

$R^2 - 100R + 2000 = 0$

$(R-50)^2 = 50^2 - 2000$

$R = 50 \pm 10\sqrt{5}$

Either $R = 72.36$ m or $R = 27.64$ m

Since $R > 40$ m $\Rightarrow R = 72.36$ m

From equation (iii) : $100 = \frac{q^2}{2 \times 9.8}$

$\Rightarrow q = 44.272$

From equation (iv) : $72.36 = \frac{2p \times 44 \cdot 272}{9 \cdot 8}$

$\Rightarrow p = 8.0088$

Speed of projection:

$$u = \sqrt{8.0088^2 + 44.272^2} = 44.9 \text{ m s}^{-1}$$

Angle of projection:

From $\tan\theta = \frac{q}{p} \Rightarrow \tan\theta = \frac{44 \cdot 272}{8 \cdot 0088} \Rightarrow \theta = 79.7°$

Example 10

(a) A particle is projected from a point O with initial velocity $3\boldsymbol{i} + 4\boldsymbol{j}$. Find in vector form the position vector of the particle at any time t.

(b) A particle P is projected from a point A with an initial velocity of 60 m s^{-1} at an angle of $30°$ to the horizontal. At the same instant a particle Q is projected in opposite direction with an initial speed of 50 m s^{-1} from a point at the same level with A and 100 m from A. Given that the particles collide, find the:
   (i) angle of projection of Q.
   (ii) time when the collision occurs.

Solution

(a) $u = (3\boldsymbol{i} + 4\boldsymbol{j}) \text{ m s}^{-1}$

$x = u_x t$

$x = 3t$

$y = u_y t - \frac{1}{2}gt^2$

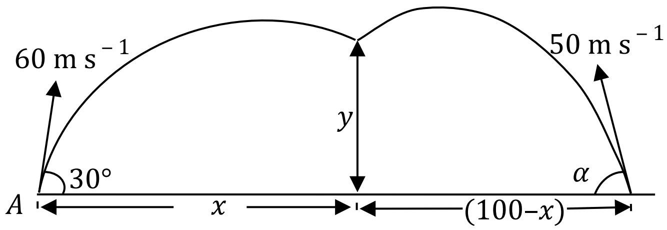
$= 4t - \frac{1}{2} \times 9 \cdot 8t^2$

$= 4t - 4 \cdot 9t^2$

$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$$\boldsymbol{r}(t) = 3t\boldsymbol{i} + (4t - 4\cdot 9t^2)\boldsymbol{j}$$

(b) $u_P = 60 \text{ m s}^{-1}, \theta = 30°$ and $u_Q = 50 \text{ m s}^{-1}$



For P:

$$x = (60\cos 30)t$$

$$x = (30\sqrt{3})t \text{ .... } (i)$$

$$y = (60\sin 30)t - \tfrac{1}{2}gt^2$$

$$y = 30t - \tfrac{1}{2}gt^2 \text{ .... } (ii)$$

For Q:

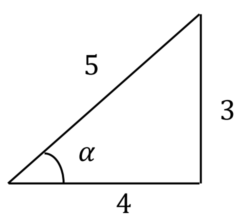
$$(100 - x) = (50\cos\alpha)t \text{ .... } (iii)$$

$$y = (50\sin\alpha)t - \tfrac{1}{2}gt^2 \text{ .... } (iv)$$

(i) From equation (ii) and equation (iv):

$$(50\sin\alpha)t - \tfrac{1}{2}gt^2 = 30t - \tfrac{1}{2}gt^2$$

$$50\sin\alpha = 30 \Rightarrow \sin\alpha = \tfrac{3}{5}$$



$$\alpha = 36.9°$$

(ii) From equation (i) and equation (iii):

$$100 - 30\sqrt{3}t = (50\cos\alpha)t$$

$$100 = \left(30\sqrt{3} + 50 \times \tfrac{4}{5}\right)t$$

$$t = 1\cdot 0874 \text{ s}$$

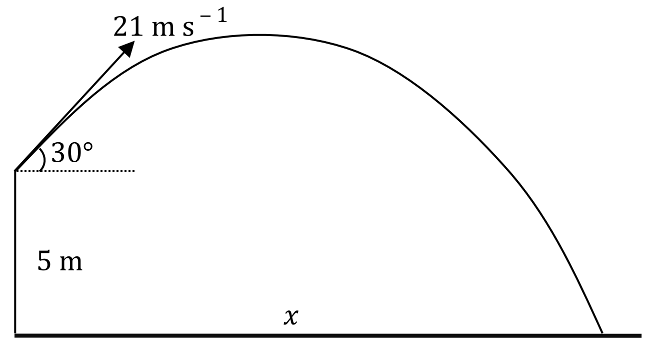
Example 11

(a) A particle is projected at an angle of elevation 30° with a speed of 21 m s^{-1}. If the point of projection is 5 m above the horizontal ground find the horizontal distance that the particle travels before striking the ground.

(b) A boy throws a ball at an initial speed of 40 m s^{-1} at an angle of elevation α. Show that the times of flight corresponding to a horizontal range of 80 m are positive roots of the equation $T^4 - 64T^2 + 256 = 0$. (Take $g = 10$ m s^{-2})

Solution

(a)



From $y = u_y t - \tfrac{1}{2}gt^2$

$$-5 = (21\sin 30)t - \tfrac{1}{2} \times 10t^2$$

$$-5 = \tfrac{21}{2}t - 5t^2 \Rightarrow t^2 - \tfrac{21}{10}t - 1 = 0$$

$$\left(t - \tfrac{21}{20}\right)^2 - \left(\tfrac{21}{20}\right)^2 - 1 = 0 \Rightarrow t = \tfrac{21}{20} \pm \tfrac{29}{20}$$

Either $= \tfrac{21}{20} - \tfrac{29}{20} = -\tfrac{2}{5}$ or $t = \tfrac{21}{20} + \tfrac{29}{20} = \tfrac{5}{2}$

Hence $t = \tfrac{5}{2}$ s

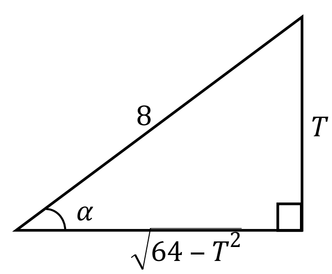
$$x = u_x t$$

$$x = (21\cos 30) \times \tfrac{5}{2} = 45\cdot 466 \text{ m}$$

(b) $u = 40 \text{ m s}^{-1}, g = 10 \text{ m s}^{-2}, \ R = 80 \text{ m}$

$$T = \frac{2u\sin\alpha}{g}$$

$$T = \frac{2 \times 40\sin\alpha}{10} \Rightarrow \sin\alpha = \frac{T}{8}$$



$$R = \frac{u^2\sin 2\alpha}{g}$$

$$80 = \frac{40^2 \times 2\sin\alpha\cos\alpha}{10} \Rightarrow \frac{1}{4} = \sin\alpha\cos\alpha$$

$$\frac{1}{4} = \frac{T}{8} \times \frac{\sqrt{64 - T^2}}{8}$$

$$16^2 = \left(T\sqrt{64 - T^2}\right)^2 \Rightarrow 256 = T^2\left(64 - T^2\right)$$

$$T^4 - 64T^2 + 256 = 0$$

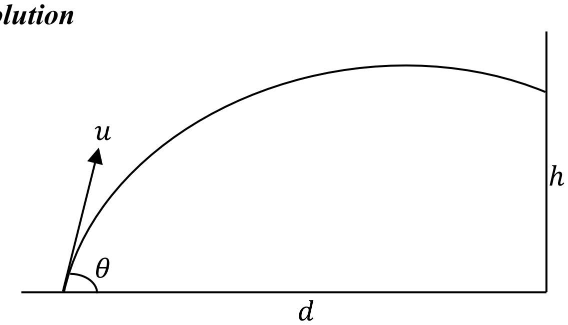
Example 12

A boy throws a stone at a vertical wall a distance d away. Given that R is the maximum range on the horizontal through the point of projection that can be attained by the speed of projection, show that:

(a) the height above the point of projection of highest point on that wall that he can hit is $\left(\frac{R^2 - d^2}{2R}\right)$.

(b) in this case the angle of projection is $\tan^{-1}\left(\frac{R}{d}\right)$.

Solution



$$R = \frac{u^2}{g} \Rightarrow u = \sqrt{Rg}$$

From $x = u_x t$

$$d = (u\cos\theta)t \text{ .... } (i)$$

From $y = u_y t - \frac{1}{2}gt^2$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2 \text{ ....} (ii)$$

From equation (i) $t = \frac{d}{u\cos\theta}$

Substituting in equation (ii)

$$y = u\sin\theta \times \frac{d}{u\cos\theta} - \frac{1}{2}g\left(\frac{d}{u\cos\theta}\right)^2$$

$$y = d\tan\theta - \frac{gd^2(1 + \tan^2\theta)}{2u^2}$$

But $u = \sqrt{Rg}$, $y = h$

$$\therefore h = d\tan\theta - \frac{d^2(1 + \tan^2\theta)}{2R}$$

For $h_{\max}$; $\frac{dh}{d\theta} = 0$

$$\frac{dh}{d\theta} = d\sec^2\theta - \frac{d^2}{2R}\left(2\sec^2\theta\tan\theta\right)$$

$$\therefore d\sec^2\theta\left(1 - \frac{d}{R}\tan\theta\right) = 0$$

$$1 - \frac{d}{R}\tan\theta = 0 \Rightarrow \tan\theta = \frac{R}{d}$$

(b) $\theta = \tan^{-1}\left(\frac{R}{d}\right)$

(a) $h_{\max} = d \times \frac{R}{d} - \frac{d^2}{2R}\left(1 + \frac{R^2}{d^2}\right)$

$$h_{\max} = R - \frac{d^2 + R^2}{2R}$$

$$h_{\max} = \left(\frac{R^2 - d^2}{2R}\right)$$

Example 13

A stone is thrown from a height of $1 \cdot 5$ m above a level ground with a speed of 10 m s^{-1} and hits a bottle standing on a wall 4 m high and 5 m away.

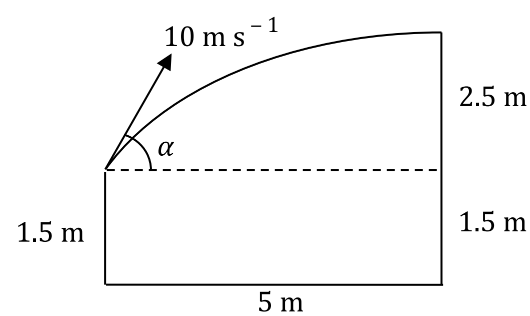
(i) Show that if α is the angle of projection of the stone then $\tan^2\alpha - 4\tan\alpha + 3 = 0$.

(ii) The horizontal component of the stones velocity has to be at least 6 m s^{-1} for the bottle to be knocked off. By solving the above equation, or otherwise, show that α has to be 45° for the bottle to be knocked off.

(iii) If α is 45°, find the direction in which the stone is moving when it hits the bottle.

(iv) If the bottle has a velocity of 3 m s^{-1} after being struck find where it hits the ground. (Use $g = 10$ m s^{-2})

Solution



(i) After time t:

$$x = (10\cos\alpha)t \text{ .... } (i)$$

$$y = (10\sin\alpha)t - \frac{1}{2}gt^2 \text{ .... } (ii)$$

From equation (i): $t = \frac{x}{10\cos\alpha}$

Substituting in equation (ii)

$$y = 10\sin\alpha \times \frac{x}{10\cos\alpha} - \frac{1}{2}g\left(\frac{x}{10\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2(1 + \tan^2\alpha)}{200}$$

When $x = 5$ m, $y = 2.5$ m

$$2.5 = 5\tan\alpha - \frac{10 \times 5^2(1+\tan^2\alpha)}{200}$$

$$2.5 = 5\tan\alpha - \tfrac{5}{4}(1+\tan^2\alpha)$$

$$5\tan^2\alpha - 20\tan\alpha + 15 = 0$$

$$\tan^2\alpha - 4\tan\alpha + 3 = 0$$

(ii) $\tan^2\alpha - 4\tan\alpha + 3 = 0$

$$(\tan\alpha - 2)^2 - 4 + 3 = 0$$

$$\tan\alpha = 2 \pm 1$$

Either $\tan\alpha = 2 \Rightarrow \alpha = 63 \cdot 4°$ or $\tan\alpha = 1$

$$\Rightarrow \alpha = 45°$$

When $\alpha = 63 \cdot 4°$;

$$v_x = u_x = 10\cos 63 \cdot 4 = 4 \cdot 472 \text{ m s}^{-1}$$

When $\alpha = 45°$;

$$v_x = u_x = 10\cos 45 = 5\sqrt{2} \text{ m s}^{-1}$$

$$= 7.071 \text{ m s}^{-1}$$

Hence for the bottle to be knocked off $\alpha = 45°$ since in this case $u_x \geq 6 \text{ m s}^{-1}$

(iii) $v_x = u_x = 5\sqrt{2} \text{ m s}^{-1}$

From $x = u_x t$

When $x = 5$ m

$$5 = (10\cos 45)t \Rightarrow t = \tfrac{\sqrt{2}}{2}s$$

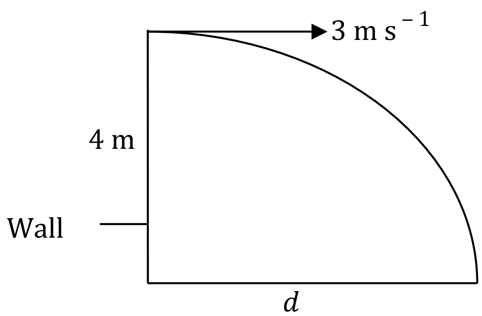
$$v_y = u_y - gt$$

$$v_y = 10\sin 45 - 10t$$

$$\Rightarrow v_y = 5\sqrt{2} - 10 \times \frac{\sqrt{2}}{2} = 0$$

Hence it will be moving horizontally

(iv)



From $y = \frac{1}{2}gt^2$

$$4 = \tfrac{1}{2} \times 10t^2 \Rightarrow t = 0 \cdot 8944 \text{ s}$$

From $x = ut$

$$d = 3 \times 0 \cdot 8944 = 2 \cdot 68 \text{ m}$$

Exercises

Exercise: 11A

1. A stone is thrown horizontally with speed u from the edge of a vertical cliff of height h. The stone hits the ground at a point which is a distance d horizontally from the base of the cliff. Show that 2 $hu^2 = gd^2$.
2. (a) A particle is projected with speed u at an angle of elevation θ from O on level ground. Show that the equation of its trajectory is
$$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}.$$
   (b) A particle is projected with speed $10\sqrt{g} \text{ m s}^{-1}$ from a point O on the ground at an elevation θ. If the particle must clear a vertical tower of height 40 m and at a horizontal distance 40 m from O, prove that $2 \leq \tan\theta \leq 3$.
3. A particle is projected with a speed of 28 m s^{-1} at an angle θ to the horizontal where $\tan\theta = \frac{4}{3}$. Find the:
   (i) speed and direction of motion after 2 s.
   (ii) horizontal and vertical distance travelled in this time.
4. A particle is projected from a level ground towards a vertical pole 4 m high and 30 m away from the point of projection. It just clears the pole in one second, find:
   (a) its initial speed and angle of projection.
   (b) the distance beyond the pole where the particle will fall.
5. A particle is projected from a point O, $24 \cdot 5$ m above a horizontal plane. After 5 seconds it hits the plane at a point whose horizontal distance from O is 100 m. Find the horizontal and vertical components of the initial velocity of the particle and the greatest height reached above the plane.
6. If the horizontal range of a particle projected with velocity u is R, show that the maximum height H attained is given by the equation $16gH^2 - 8u^2H + gR^2 = 0$.

7. (a) A stone thrown at an angle θ to the horizontal takes T seconds in its flight and moves R metres on the horizontal range, show that $2R\tan\theta = gT^2$.
   (b) If the stone in (a) above is now thrown from a point O with an initial speed of 30 m s^{-1} so as to pass through a point 40 m from O horizontally and 10 m above O. Show that there are two possible angles of projection for which this is possible. If these angles are θ_1 and θ_2, show that $\tan(\theta_1 + \theta_2) + 4 = 0$.
   (Take $g = 10 \text{ m s}^{-2}$).

8. A particle P, projected from a point A on horizontal ground, moves freely under gravity and hits the ground again at B. Taking A as the origin, AB as the $x-$axis and the upward vertical at A as the $y-$axis, the equation of path of P is $y = x - \frac{x^2}{40}$, where x and y are measured in metres. Calculate the:
   (i) distance AB.
   (ii) greatest height above AB attained by P.
   (iii) magnitude and direction of the velocity of P at A.
   (iv) time taken by P to reach B from A.

9. A ball is projected from a point A and falls at a point B which is in level with A and at a distance of 160 m from A. The greatest height attained by the ball is 40 m. Find the:
   (a) angle and speed at which the ball is projected.
   (b) time taken for the ball to attain its greatest height. [Use $g = 10 \text{ m s}^{-2}$]

10. A particle is projected from a point O on horizontal ground with an initial velocity whose horizontal and vertical components are $3u$ and $5u \text{ m s}^{-1}$ respectively. Find the equation of the trajectory of the particle. Given that it just clears an obstacle 5 m high and 9 m from O, find the value of u and the distance from O of the point at which the particle strikes the ground.

11. A ball is projected from a horizontal ground and has an initial velocity of 20 m s^{-1} at an angle of elevation $\tan^{-1}\left(\frac{7}{24}\right)$. When the ball is travelling horizontally, it strikes a vertical wall. How high above the ground does the impact occur?

12. A stone is thrown from the top of a vertical cliff, 100 m above sea level. The initial velocity of the stone is 13 m s^{-1} at an angle of elevation of $\tan^{-1}\left(\frac{5}{12}\right)$. Find the time taken for the stone to reach the sea and its horizontal distance from the cliff at that time. (Take $g = 10 \text{ m s}^{-2}$)

Exercise: 11B

1. A particle projected from point A with speed u at an angle α to the horizontal hits the horizontal plane through A at B. Show that if the particle is to be projected from A with the same angle of elevation to the horizontal so as to hit a target at a height h above B, the speed of projection must be

$$\frac{u^2\sin\alpha}{\left(u^2\sin^2\alpha - \frac{1}{2}gh\right)^{\frac{1}{2}}}$$

2. A particle that is projected from a point on level ground and attains a maximum height H, just clears two vertical walls each of height h. Prove that the time taken by the particle to fly between the walls is $\sqrt{\frac{8(H-h)}{g}}$.

3. A particle is projected upwards with a speed of 20 m s^{-1} at an angle θ to the horizontal from a point h metres above a horizontal plane. The particle takes t seconds to hit the plane.
   (i) Show that t is a positive root of the equation $5t^2 - 20t\sin\theta - h = 0$.
   (ii) Given that the particle is moving horizontally after 1 second and the total time of flight is 3 seconds, calculate the value of θ and show that $h = 15 \text{ m}$.
   (iii) Determine the horizontal distance travelled and the greatest height attained by the particle above the horizontal plane.
   (iv) Show that the angle at which the particle strikes the plane is $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$.
   (Take $g = 10 \text{ m s}^{-2}$)

4. (a) A particle is projected with a speed of $2\sqrt{70} \text{ m s}^{-1}$ at an angle of elevation θ. The particle just clears a wall 5 m high and 20 m away from the point of projection. Find the possible values of θ.
   (b) A particle is projected from the top of a cliff H metres above the ground at an angle α above the horizontal. If the particle hits the horizontal plane through the bottom of the cliff, at a distance D from the base of the

cliff. Show that the maximum height attained by the particle above the ground is given by

$$\frac{4H(H + D\tan\alpha) + D^2\tan^2\alpha}{4(H + D\tan\alpha)}$$

5. A particle is projected with a speed of $2\sqrt{gh}\ \text{m s}^{-1}$ at an angle α to the horizontal. If it clears a vertical pole of height h which is at a horizontal distance $2h$ from the point of projection prove that $1 < \tan\alpha < 3$.

6. A particle is projected from a point O on a level ground at an elevation α, and while still rising it passes through a point P with speed v in a direction β to the horizontal. Prove that the time it takes to reach point P is $\frac{v\sin(\alpha - \beta)}{g\cos\alpha}$.

7. (a) A particle is fired with a velocity of $35\ \text{m s}^{-1}$ from the edge of a vertical cliff of height $20\ \text{m}$ and hits the sea at a distance $50\ \text{m}$ from the foot of the cliff. Show that the two possible angles of projection are perpendicular and find the angles.

(b) Two points P and Q on a horizontal ground are $30\ \text{m}$ apart. A ball is thrown from P with a velocity of $20\ \text{m s}^{-1}$ at $30°$ to PQ. A second ball is thrown from Q at $60°$ to QP at the same time. Find the speed of projection of the second ball if they collide. Also find the time taken for the collision to occur.

8. A boy of height $1 \cdot 5\ \text{m}$ throws a ball at $20\ \text{m s}^{-1}$ from the level of his head to land at a point which is $25\ \text{m}$ from his feet, on a level ground. Find the:

(i) two possible angles of projection.

(ii) velocity of the ball as it passes through a point $1\ \text{m}$ above the ground if he used the smaller angle of projection.

9. Two particles P and Q are projected simultaneously from a point O with the same speed but at different angles of elevation and they both pass through a point C which is at a horizontal distance $2h$ from O and at a height h above the level of O. The particle P is projected at an angle $\tan^{-1}2$ above the horizontal. Show that:

(i) the speed of projection is $\sqrt{\frac{10gh}{3}}$.

(ii) Q is projected at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ to the horizontal.

(iii) the time interval between the times of arrival of the two particles at C is $(3 - \sqrt{5})\sqrt{\left(\frac{2h}{3g}\right)}$.

10. Two particles are projected at the same instant from points A and B at the same level, the first from A towards B with a velocity u at $45°$ above AB towards B and a second from B towards A at $60°$ above BA with speed v. If the particles collide when each reaches its maximum height find the ratio $v^2 : u^2$ and given that $AB = a$, prove that $u^2 = ga(3 - \sqrt{3})$.

11. Two particles A and B are projected simultaneously, A from the top of a vertical cliff and B from the base. Particle A is projected horizontally with speed $3u\ \text{m s}^{-1}$ and B is projected at an angle θ above the horizontal with speed $5u\ \text{m s}^{-1}$. The height of the cliff is $56\ \text{m}$ and the particles collide after 2 seconds. Find the vertical and horizontal distances from the point of collision to the base of the cliff and the values of u and θ.

Exercise: 11C

1. Initially a particle is at an origin O and is projected with a velocity of $a\boldsymbol{i}\ \text{m s}^{-1}$. After t seconds, the particle is at a point with position vector $(30\boldsymbol{i} – 10\boldsymbol{j})$ m. Find the values of t and a.

2. A stone is thrown with an initial velocity of $(24\boldsymbol{i} + 10\boldsymbol{j})\ \text{m s}^{-1}$ from the edge of a vertical cliff. The stone hits the sea at a point level with the base of the cliff and at a distance of $72\ \text{m}$ from it. Find the:

(i) time for which the stone is in air.

(ii) height of the cliff.

(iii) maximum height reached by the stone.

(iv) velocity with which the stone hits the sea.

3. (a) A particle is projected from the top of a cliff $87 \cdot 5\ \text{m}$ high. It takes 5 s to hit the ground. If it covers a horizontal distance of $120\ \text{m}$, calculate the speed and direction of projection.

(b) A particle is projected vertically upwards from ground level with a speed of $u\ \text{m s}^{-1}$ and clears the top of a pole H metres high in t s and returns to the top of the pole after $\frac{1}{2}t$ s. Show that:

(i) $12u^2 = 25gH$.

(ii) the speed at the top of the pole is $\frac{1}{5}u$.

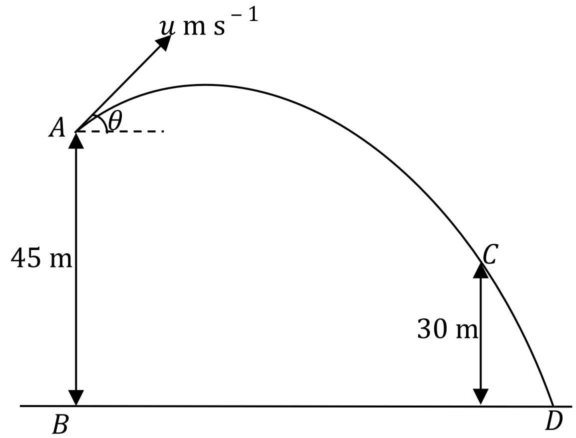
4. A particle is projected from the origin and has an initial velocity of $(7\boldsymbol{i} + 5\boldsymbol{j})\ \text{m s}^{-1}$. Given that the particle passes through the point P, position vector $(x\boldsymbol{i}–30\boldsymbol{j})$ m, find the time taken for this to occur and the value of x. (Take $g = 10 \text{ m s}^{-2}$)

5. A particle projected from origin O with initial velocity $u\boldsymbol{i} + v\boldsymbol{j}$ passes through points A and B with position vectors $20(\boldsymbol{i} + \boldsymbol{j})$ and $25\boldsymbol{i}$ respectively. Find the values of u and v. Show that a particle projected from O with velocity $v\boldsymbol{i} + u\boldsymbol{j}$ also passes through B. (Use $g = 10 \text{ m s}^{-2}$)

6. (a) Initially a particle is projected with a velocity $\begin{pmatrix}20\\0\end{pmatrix}$ m s^{-1} from a point with position vector $\begin{pmatrix}10\\90\end{pmatrix}$ m. Find the distance of the particle from the origin after 4 seconds.
   (b) A footballer kicks a ball with a velocity of 52 m s^{-1} at an angle $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. Determine the:
   (i) time for which the ball is at least 12 m above the ground level.
   (ii) maximum height and time taken to reach it.

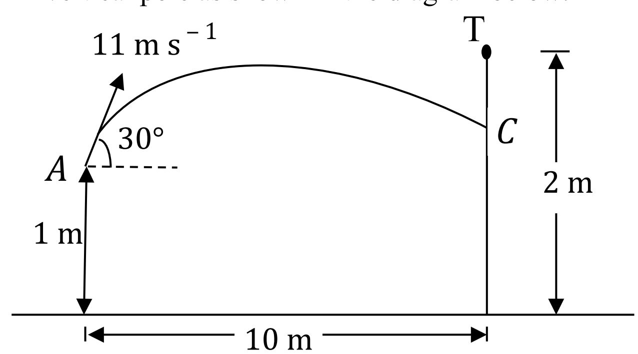
7. Two particles A and B are projected simultaneously under gravity, A from O on horizontal ground and B from a point 40 m vertically above O. B is projected horizontally with speed 28 m s^{-1}. If the particles hit the ground simultaneously at the same time; determine the:
   (a) time taken for B to reach the ground and horizontal distance travelled.
   (b) magnitude and direction of the velocity with which A is projected. Show that just before hitting the ground, the direction of the motion of A and B differ by approximately $18 \cdot 4°$.

8. A particle is projected from a point A with speed $u \text{ m s}^{-1}$ at an angle θ, where $\cos\theta = \frac{4}{5}$. The point B, on horizontal ground, is vertically below A and $AB = 45$ m. After projection, the particle moves freely under gravity passing through point C, 30 m above the ground and lands at point D, as shown in the diagram below.



Given that the particle passes through C with speed $24 \cdot 5 \text{ m s}^{-1}$, find the:
   (a) value of speed u.
   (b) direction of the particle at C.
   (c) distance BD.

9. The aim of a game is to throw a ball B from a point A to hit a target T which is placed at the top of a vertical pole as shown in the diagram below.



The point A is 1 m above the horizontal ground and the height of pole is 2 m. If the ball hits the pole at C, where the pole is at a horizontal distance of 10 m from A and the ball is projected from A with a speed of 11 m s^{-1} at an angle of 30°.
   (a) (i) Calculate the time taken by B to move from A to C.
   (ii) Find the distance CT.
   (b) The ball is thrown again from A with the speed of projection of B increasing to $v \text{ m s}^{-1}$ and the angle of elevation remaining 30°. Given that B hits T, calculate the value of v.

10. A particle is fired with speed u at an angle θ to the horizontal at a height h, above the horizontal ground. It takes a time T, to reach the horizontal ground. Prove that

$$T = \frac{u\sin\theta}{g}\left\{1 \pm \left(1 + \frac{2gh}{u^2\sin^2\theta}\right)^{\frac{1}{2}}\right\}$$

Answers to exercises

Exercise: 11A

1. 2. (a) (b) 3. (i) $17 \cdot 032$ m s^{-1} ; $9 \cdot 5°$ above horizontal (ii) $33 \cdot 6$ m ; $25 \cdot 2$ m

4. (a) $31 \cdot 292$ m s^{-1} ; $16 \cdot 5°$ to the horizontal (b) $24 \cdot 42$ m

5. 20 m s^{-1}; $19 \cdot 6$ m s^{-1}; $44 \cdot 1$ m

8. (i) 40 m (ii) 10 m (iii) 20 m s^{-1} at 45° to horizontal (iv) $2\sqrt{2}$ s

9. (a) 45° ; 40 m s^{-1} (b) $2\sqrt{2}$ seconds

10. $y = \frac{5}{3}x - \frac{49x^2}{90u^2}$; $2 \cdot 1$ m s^{-1} ; $13 \cdot 5$ m 11. $1 \cdot 6$ m 12. 5 s ; 60 m

Exercise: 11B

1. 2. 3. (i) (ii) 30° (iii) $30\sqrt{3}$ m; 20 m (iv)

4. (a) 40.6° ; $63 \cdot 4°$ 7.(a) 10.9° below horizontal; 79.1° above horizontal (b) $11 \cdot 547$ m s^{-1}; 1.299 s 8. (i) 15.0° ; $71 \cdot 5°$ (ii) 20.24 m s^{-1} at $17 \cdot 4°$ below horizontal 9. (i) (ii) (iii)

10. 2 : 3 11. 36.4 m ; 42 m ; 7 m s^{-1}; 53.1°

Exercise: 11C

1. $\frac{10}{7}$; 21 2. (i) 3 s (ii) $14 \cdot 1$ m (iii) $19 \cdot 2$ m (above the sea level) ; (iv) $(24\boldsymbol{i} - 19 \cdot 4\boldsymbol{j})$ m s^{-1}

3. (a) 25 m s^{-1} ; $16 \cdot 3°$ above horizontal (b) (i) (ii)

4. 3 s ; 21 m 5. 5 ; 25

6. (a) $90 \cdot 744$ m (b) (i) $2 \cdot 62$ s (ii) $20 \cdot 4$ m ; $2 \cdot 04$ s

7. (a) $\frac{20}{7}$ s ; 80 m (b) $14\sqrt{5}$ m s^{-1} at $26 \cdot 6°$ to the horizontal

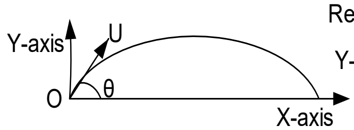
8. (a) $17 \cdot 5$ m s^{-1} (b) $55 \cdot 2°$ below horizontal (c) 60 m

9. (a) (i) $1 \cdot 05$ s (ii) $0 \cdot 626$ m (b) $11 \cdot 699$ m s^{-1} 10.

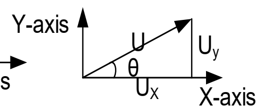
2. PROJECTILE MOTION

This is the motion of a body which after being given an initial velocity moves under the influence

Consider a ball projected at O with an initial velocity um/s at an angle θ to the horizontal.



Resolving initial velocity



$u_y = u\sin\theta$ -------------------(1)

Also: $\cos\theta = \frac{u_x}{u}$

$u_x = u\cos\theta$ -------------------(2)

From the figure: $\sin\theta = \frac{u_y}{u}$

Equation (1) is the initial vertical component of velocity

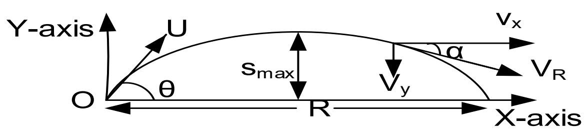
Equation (2) is the initial horizontal component of velocity

Note

The horizontal component of velocity [$u_x = u\cos\theta$] is constant through the motion and therefore the acceleration is zero.

MATHEMATICAL FORMULAR IN PROJECTILES

All formulas in projectiles are derived from equations of linear motion



Finding velocity at any time t.

Horizontally: $v = u_x + at$

$u_x = u\cos\theta$,

$a = 0$ (constant velocity)

$$\boxed{v_x = u\cos\theta}$$

Vertically: $v = u_y + at$

$u_y = u\sin\theta$

a= $-$g

$$\boxed{v_y = u\sin\theta - gt}$$

Velocity at any time t

$$\boxed{v = \sqrt{V_x^2 + V_y^2}}$$

Direction of motion

$\boxed{\alpha = \tan^{-1}\left(\frac{V_y}{V_x}\right)}$ to the horizontal

Finding distances at any time t

horizontally : $s_x = u_x t + \frac{1}{2} at^2$

$u_x = u\cos\theta$, a= 0

$$\boxed{x = u\cos\theta\, t}$$

Vertically: $s_y = u_y t + \frac{1}{2} at^2$

$u_y = u\sin\theta$, $a = g$

$$\boxed{y = u\sin\theta t - \frac{1}{2}gt^2}$$

TERMS USED IN PROJECTILES

1. **MAXIMUM HEIGHT [GREATEST HEIGHT] [S_{max}]**

For vertical motion : at max height v=0,

$u_y = u\sin\theta$, $a = -g$, s= S_{max}

$v_y{}^2 = u_y{}^2 + 2gs$

$0 = (u\sin\theta)^2 - 2gS_{max}$

$2gS_{max} = u^2\sin^2\theta$

$$\boxed{Smax = \frac{u^2\sin^2\theta}{2g}}$$

Note : $\sin^2\theta = (\sin\theta)^2$ but $\sin^2\theta \neq \sin\theta^2$

2. **TIME TO REACH MAX HEIGHT [t]**

Vertically $v = u_y + at$ at max height v=0

$u_y = u\sin\theta$, $a = g$

$0 = u\sin\theta - gt$

$$\boxed{t = \frac{u\sin\theta}{g}}$$

3. TIME OF FLIGHT [T]

It refers to the total time taken by the projectile to move from the point of projection to the point where it lands on the horizontal plane through the point of projection.

Vertically: $S_y = u_y t + ½ at^2$

at point A when the projectile return to the plane $S_y=0$,

t=T(time of flight), $a = -g \quad u_y = u \sin\theta$

$$0 = usin\theta T - \frac{gT^2}{2}$$

$$T\left(usin\theta - \frac{gT}{2}\right) = 0$$

Either $T = 0$ or $\left(usin\theta - \frac{gT}{2}\right) = 0$

$$\left(usin\theta - \frac{gT}{2}\right) = 0$$

$$usin\theta = \frac{gT}{2}$$

$$\boxed{T = \frac{2\,u\,sin\theta}{g}}$$

Note: The time of flight is twice the time to maximum height

4. RANGE [R]

It refers to the horizontal distance from the point of projection to where the projectile lands along the horizontal plane through the point of projection.

Neglecting air resistance the horizontal component of velocity $\boldsymbol{ucos\theta}$ remains constant during the flight

Horizontally : $S_x = u_x t + ½ at^2$

$u_x = ucos\theta$, a=0 (constant velocity), t=T

$$R = ucos\theta\, T + ½\, x 0 x T^2$$

$$R = ucos\theta T$$

But $T = \frac{2\,u\,sin\theta}{g}$

$$R = \frac{\boldsymbol{u^2\, 2sin\theta\, cos\theta}}{\boldsymbol{g}}$$

But from trigonometry $2sin\theta cos\theta = sin2\theta$

$$\boxed{R = \frac{u^2 \text{ Sin } 2\theta}{g}}$$

5. MAXIMUM RANGE [R_{max}]

For maximum range sin2θ =1, R=R_{max}

2 θ= $sin^{-1}(1)$

$$2\,\theta = 90°$$

$$R_{max} = \frac{u^2 sin90}{g}$$

$$\boxed{R_{max} = \frac{u^2}{g}}$$

6. EQUATION OF A TRAJECTORY

A trajectory is a path described by a projectile.

A trajectory is expressed in terms of horizontal distance x and vertical distance y.

For horizontal motion at any time t

$$x = ucos\,\theta t$$

$t = \frac{x}{ucos\theta}$ ----------------[1]

For vertical motion at any time t

$y = usin\theta t - ½\, gt^2$ -------------[2]

Putting t into equation [2]

$$y = usin\theta \frac{x}{ucos\theta} - \frac{1}{2} g \left(\frac{x}{ucos\theta}\right)^2$$

$$\boxed{\boldsymbol{y} = \boldsymbol{xtan\theta} - \frac{gx^2}{2u^2\cos^2\theta}}$$

since $y = a\,x - b\,x^2$

the motion is parabolic

Either $\boxed{y = xtan\theta - \frac{g\,x^2\, Sec^2\theta}{2\,u^2}}$

Or $\boxed{y = xtan\theta - \frac{g\,x^2(1+tan^2\theta)}{2\,u^2}}$

A. Objects projected upwards from the ground at an angle to the horizontal

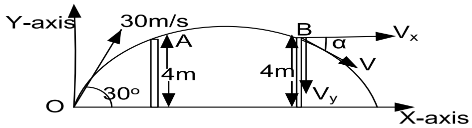
1. A Particle is projected with a velocity of 30ms⁻¹ at an angle of elevation of 30°. Find
   i) The greatest height reached
   ii) The time of flight
   iii) Horizontal range
   iv) The velocity and direction of motion at a height of 4m on its way downwards

Solution

(i) $Smax = \frac{u^2\, sin^2\theta}{2g} = \frac{30^2 sin^2 30}{2x9.81} = 11.47m$

(ii) $\boldsymbol{T} = \frac{\boldsymbol{2\,u\,sin\theta}}{\boldsymbol{g}} = \frac{2\,x\,30\,sin30}{9.81} = 3.06s$

(iii) $R = \frac{u^2 \sin 2\theta}{g} = \frac{30^2 \sin 2x30}{9.81} = 79.45m$



For vertical motion

$y = usin\theta t - ½ gt^2$

$4 = 30sin30t - ½ 9.81x\, t^2$

$4.905t^2 - 15t + 4 = 0$

$t = 2.76s$ or $t = 0.30s$

The value of $t = 0.30s$ is the correct time since it's the smaller value for which the body moves upwards.

$v_x = ucos\theta$

$v_x = 30cos\, 30 = 25.98m/s$

$v_y = usin\theta - gt$

$v_y = 305x30 - 9.81x0.30 = 12.06m/s$

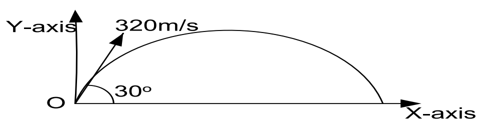
$v = \sqrt{V_x^2 + V_y^2} = \sqrt{25.98^2 + 12.06^2} = 28.64m/s$

Direction : $\alpha = tan^{-1}\frac{V_y}{V_x} = tan^{-1}\left(\frac{12.06}{25.98}\right)$ =**24.9°**

Velocity is 28.64m/s at 24.9° to horizontal

2. A projectile is fired with a velocity of 320m/s at an angle of 30° to the horizontal. Find
   (i) time to reach the greatest height
   (ii) its horizontal range
   (iii) maximum range

Solution



i) At max height $v = 0$,

$v = usin\theta - gt$

$0 = 320sin30 - 9.81t$

$t = \frac{320sin30}{9.81} = 16.31s$

ii) range $R = ucos\theta x$time of flight

Time of flight =twice time to max height

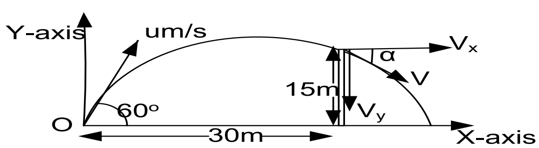
R= $320cos30x\,[2x16.31] = 9039.92m$

iii) max range

$R_{max} = \frac{u^2}{g} = \frac{320^2}{9.81} = 10438.33\text{m}$

3. A projectile fired at an angle of 60° above the horizontal strikes a building 30m away at a point 15m above the point of projection. Find
   (i) Speed of projection
   (ii) Velocity when it strikes a building

Solution



(i) Horizontal distance at time t : $x = ucos\theta t$

$30 = utcos60$

$t = \frac{60}{u}$

Also vertical distance at any time t

$y = usin\theta - \frac{1}{2}gt^2$

$15 = usin60x\frac{60}{u} - \frac{1}{2}x9.81\left(\frac{60}{u}\right)^2$

$15 = 51.96152423 - \frac{4.905\, x\, 3600}{u^2}$

$u = \sqrt{477.7400383} = 21.86m/s$

ii)but since $t = \frac{60}{u}$

$t = \frac{60}{21.86} = 2.75s$

$v_x = ucos\theta$

$v_x = 21.86cos60 = 10.93ms^{-1}$

$v_y = usin\theta - gt$

$v_y = 21.81sin60 - 9.81x2.75 = -8.09ms^{-1}$

velocity at any time

$v = \sqrt{V_x^2 + V_y^2} = \sqrt{10.93^2 + (-8.09)^2}$

$= 13.60ms^{-1}$

$\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{8.09}{10.9}\right) = 36.6°$

The velocity is $13.60ms^{-1}$at $36.6°$to the horizontal

Alternatively

$y = xtan\theta - \frac{gx^2}{2u^2COS^2\theta}$

$y = 15m, x = 30m, \theta = 60°, u = ?$

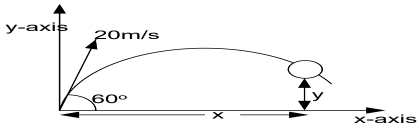
$15 = 30tan60 - \frac{9.81x30^2}{2u^2Cos^2 60}$

$15 = 51.96152423 - \frac{17658}{u^2}$

$u = \sqrt{477.7400383} = 21.86m/s$

4. A body is projected at an angle of 60° above horizontal and passes through a net after 10s. Find the horizontal and vertical distance moved by the body after it, was projected at a speed of 20m/s

Solution



Horizontal motion : $x = ucos\theta xt$

$x = 20cos60x10$

$x = 100m$

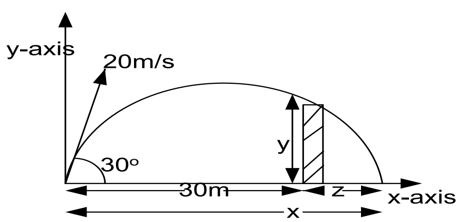
Vertical motion; $y = usin\theta t - ½\, gt^2$

$y = 20(sin60)x10 - ½\, x9.81x10^2$

$y = -317.29m$

5. A ball is kicked from the spot 30m from the goal post with a velocity of 20m/s at 30° to the horizontal. The ball just clears the horizontal bar of a goal post. Find;
   (i) Height of the goal post
   (ii) How far behind the goal post does the ball land

Solution



horizontal motion : $x = ucos\theta xt$

$30 = 20cos30xt$

$t = 1.732s$

For vertical motion: $y = usin\theta t - ½\, gt^2$

$y = (20sin30)x1.732 - \frac{1}{2}x9.81x(1.732)^2$

$y = 2.61m$

Height of the goal post = 2.61m

ii) Time of flight

$T = \frac{2\text{ u sin}\theta}{\text{g}} = \frac{2x20xsin30}{9.81} = 2.04s$

iii) Horizontal distance: $x = ucos\theta xt$

$x = 20cos30x2.04$ =35.33m

but $x = 20 + z$

$35.33 = 30 + z$

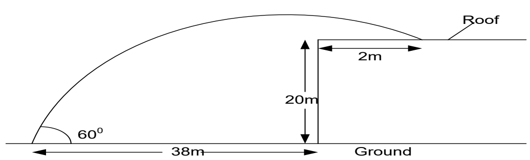
$z = 5.33m$ The ball 5.33m behind the goal

EXERCISE : 3

1. A particle is projected at an angle of 60° to the horizontal with a velocity of 20m/s. calculate the greatest height the particle attains **An[15.29m]**
2. A stone is projected at an angle of 60° to the horizontal with a velocity of 30m/s. calculate;
   (a) the highest point reached
   (b) Range
   (c) Time taken for flight
   (d) Height of the stone at the instant that the path makes an angle of 30° with the horizontal
   An[33.75m, 78m, 5.2s, 33.3m]
3. A particle is projected from level ground towards a vertical pole, 4m high and 30m away from the point of projection. It just passes the pole in one second. Find
   i) Its initial speed and angle of projection **An [39.29m/s, 16.5°]**
   ii) The distance beyond the pole where the particle will fall **An [24.42m]**
4. A particle is projected with a velocity of 30m/s at an angle of 40° above the horizontal plane. find ;
   a) The time for which the particle is in the air.
   b) The horizontal distance it travels **An [3.9s, 22.9m/s]**
5. A body is projected with a velocity of 200ms^{-1} at an angle of 30° above the horizontal. Calculate
   (a) Time taken to reach the maximum height
   (b) Its velocity after 16s **An [10.2s, 183m/s at 19.1°]**
6. A particle is projected from a level ground in such a way that its horizontal and vertical components of velocity are 20ms^{-1} and 10ms^{-1} respectively. Find
   (a) Maximum height of the particle
   (b) Its horizontal distance from the point of projection when it returns to the ground
   (c) The magnitude and direction of the velocity on landing **An [5.0m, 40m, 22.4m/s at 26.6° below horizontal]**
7. A particle is projected with a speed of 25ms^{-1} at 30° above the horizontal. Find;

(a) Time taken to reach the height point of trajectory
(b) The magnitude and direction of the velocity after 2.0s **An [1.25s, 22.9m/s at 19.1° below horizontal]**

8. A projectile is launched with a velocity of 1800m/s at an angle 60° with the horizontal. Determine the speed of the projectile at a height of 32km when falling downwards **An[1616.23m/s]**
9. A hammer thrown in athletics consists of a metal sphere of mass 7.26kg with a wire handle attached, the mass of which can be neglected. In a certain attempt it is thrown with an initial velocity which makes an angle of 45° with the horizontal and its flight takes 4.00s. stating any assumptions find;
   (i) The horizontal distance travelled
   (ii) Kinetic energy of the sphere just before it strikes the ground **An [80.0m, $2.90x10^3$J]**
10. A soft ball is thrown at an angle of 60 above the horizontal. It lands a distance 2m from the edge of a flat roof of height 20m. the edge of the roof is 38m horizontally from the thrower.



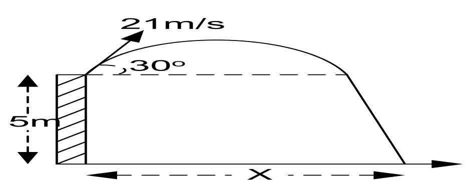
   (i) The speed at which the ball was thrown **An (25.4 ms^{-1})**
   (ii) The velocity with which the ball strikes the roof **An (15.64 ms^{-1} at $36.2°$ below the horizontal)**

   Calculate
11. A stone thrown upwards at an angle θ to the horizontal with speed ums^{-1} just clears a vertical wall 4m high and 10m from the point of projection when travelling horizontally. Find the angle of projection **An[38.66°]**

B. **Objects projected upwards from a point above the ground at an angle to the horizontal**

1. A particle is projected at an angle of elevation of 30° with a speed of 21m/s. If the point of projection is 5m above the horizontal ground, find the horizontal distance that the particle travels before striking the ground

   Solution



$u = -5m$ since it's below the point of projection

For vertical motion: $y = usin\theta t - ½\, gt^2$

$$-5 = 21sin30t - \frac{9.81t^2}{2}$$

$$4.905t^2 - 10.5t - 5 = 0$$

t= $2.54s$ or t = $-0.40s$

Time of flight $t = 2.54s$

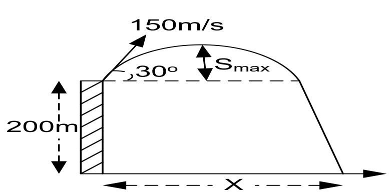
For horizontal motion

$x = ucos\theta xt = 21(cos30)\, x2.54 = 46.19m$

The horizontal distance = $46.19m$

2. A bullet is fired from a gun placed at a height of 200m with a velocity of $150ms^{-1}$ at an angle of 30° to the horizontal find
   i) Maximum height attained
   ii) Time taken for the bullet to hit the ground

   Solution



i) $Smax = \frac{u^2\, sin^2\theta}{2g} = \frac{150^2\, sin^2 30}{2x9.81} = 286.70m$

The max height attained is 286.70m from the point of projection

ii) Time taken for the bullet to hit the ground

Vertical motion : $y = usin\theta t - ½\, gt^2$

$y = -200m$ since its below the point of projection

$$-200 = 150sin30t - ½\, x9.81t^2$$

$$-200 = 75t - 4.905t^2$$

$$t = 17.61s \ or \ t = -2.32s$$

Time taken is 17.61s

Trial :1

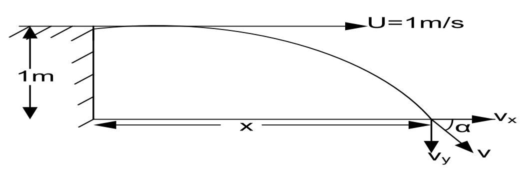
1. A particle is projected with a velocity of $10ms^{-1}$ at an angle of 45^0 to the horizontal, it hits the ground at a point which is 3m below its point of projection. Find the time fro which it is in the air and the horizontal distance covered by the particle in this time **An[1.76s, 12.42m]**
2. A pebble is thrown from the top of a cliff at a speed of 10m/s and at 30^0 above the horizontal. it hits the sea below the cliff 6.0s later , find;
   a) The height of the cliff . **An[150m, 52m]**
   b) The distance from the base of the cliff at which the pebble falls into the sea.

C. An object projected horizontally from a height above the ground

Examples

1. A ball rolls off the edge of a table top 1m high above the floor with a horizontal velocity $1ms^{-1}$. Find;
   i) The time it takes to hit the floor
   ii) The horizontal distance it covered
   iii) The velocity when it hits the floor

Solution



u=$1ms^{-1}$ θ=0^0 y=-1m below the point of projection

vertical motion: $y = usin\theta t - ½\, gt^2$

$-1 = 1xsin0xt - ½\, x9.81t^2$

$-1 = -4.905t^2$

$t = 0.45s$

ii) $x = ucos\theta t = 1xcos0x0.45 = 0.45m$

iii) velocity when it hits the ground

$v_x = ucos\theta = 1cos0 = 1m/s$

$v_y = usin\theta - gt$

$v_y = 1sin0 - 9.81x0.45 = -4.4m/s$

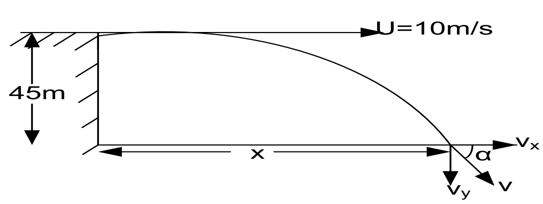
$\boldsymbol{v} = \sqrt{V_x^2 + V_y^2} = \sqrt{(1)^2 + (-4.4)^2} = 4.5ms^{-1}$

Direction: $\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{4.4}{1}\right) = 77.2°$

The velocity is $4.5ms^{-1}$ at $77.2°$to the horizontal

2. A ball is thrown forward horizontally from the top of a cliff with a velocity of 10m/s. the height of a cliff above the ground is 45m. calculate
   i) Time to reach the ground
   ii) Distance from the cliff where the ball hits the ground
   iii) Direction of the ball just before it hits the ground

Solution



u=$10ms^{-1}$ θ=0^0 y=-45m below the point of projection

For vertical motion

$y = usin\theta t - ½\, gt^2$

$-45 = 10xsin0xt - ½\, x9.81t^2$

$t = 3.03s$

ii) $x = ucos\theta t = 10xcos0x3.03 = 30.3m$

iv) velocity when it hits the ground

$v_x = ucos\theta = 10cos0 = 10m/s$

$v_y = usin\theta - gt$

$v_y = 10sin0 - 9.81x3.03 = 29.72m/s$

$v = \sqrt{V_x^2 + V_y^2} = \sqrt{(10)^2 + (29.72)^2} = 31.36ms^{-1}$

$\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{29.72}{10}\right) = 71.4°$

The velocity is $31.36ms^{-1}$ at $71.4°$to the horizontal

Trial:2

1. A pencil is accidentally knocked off the edge of a horizontal desktop. The height of the desk is 64.8cm and the pencil hits the floor a horizontal distance of 32.4cm from the edge of the desk, What was the speed of the pencil as it left the desk. **An[$0.9ms^{-1}$]**

2. An aero plane moving horizontally at 150ms^{-1} releases a bomb at a height of 500m. the hits the intended target. What was the horizontal distance of aero plane from the target when the bomb was released. **An(1500m)**

UNEB 2016 No1 (b)

A particle is projected from a point on a horizontal plane with a velocity, u, at an angle, θ, above the horizontal. Shwo that the maxmum horizontal range R_{max} is given by $R_{max} = \frac{u^2}{g}$ where g is acceleration due to gravity. (04marks)

UNEB 2014 No1 (a)

(i) What is a **projectile motion** (01marks)

(ii) A bomb is dropped from an aero plane when it is directly above a target at a height of 1402.5m. the aero plane is moving horizontally with a speed of 500kmh^{-1}. Determine whether the bomb will hit the target. **An (misses target by 2347.2m)** (05marks)

UNEB 2012 No 3 (d)

(i) Derive an expression for maximum horizontal distance travelled by a projectile in terms of the initial speed u and the angle of projection θ to the horizontal [02 marks]

(ii) Sketch a graph to show the relationship between kinetic energy and height above the ground in a projectile.

UNEB 2010 No (d)

iii) Calculate the range of a projectile which is fired at an angle of 45° to the horizontal with a speed of 20m/s **An [40.77m]**

UNEB 2009 No 1 (d)

A stone is projected at an angle of 20° to the horizontal and just clears a wall which is 10m high and 30m form the point of projection. Find the;

i) Speed of projection (04marks)

ii) Angle which the stone makes with the horizontal as it clears the wall (03marks)

An[73.78m/s, 16.9°]

UNEB 2006 No 1 (c)

A projectile is fired horizontally from the top of a cliff 250m high. The projectile lands 1.414x10^{3}m from the bottom of the cliff. Find the

i) Initial speed of the projectile (05 marks)

ii) Velocity of the projectile just before it hits the ground (05 marks)

An [198m/s, 210m/s at 19.5°]

UNEB 2000 No 3 (b)

(i) Define the terms time of flight and range as applied to projectile motion (02 marks)

(ii) A projectile is fired in air with a speed um/s at an angle θ to the horizontal. Find the time of flight of the projectile (02marks)

MARCH UNEB 1995 No 1

a) (i) write the equation of uniformly accelerated motion (03 marks)

(ii) Derive the expression for the maximum horizontal distance travelled by a projectile in terms of the initial speed u and the angle of projectile θ to horizontal (04 marks)

b) A bullet is fired from a gun placed a height of 200m with a velocity of 150m/s at an angle of 30° to the horizontal. Find

i) The maximum height attained

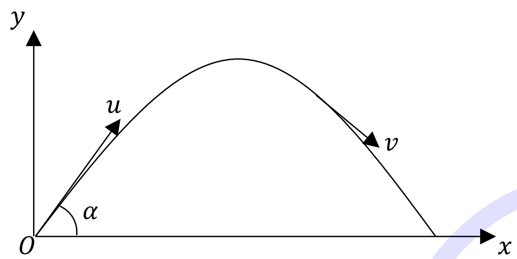
ii) The time for the bullet to hit the ground (07marks)

PROJECTILES

A projectile is a particle which is given an initial velocity and then moves freely under gravity.
It is assumed that gravity is the only force acting on the particle i.e. air resistance is negligible.
If its initial velocity is vertical, then the particle will move in a straight line under gravity.
If its initial velocity is not vertical, the particle will move in a curve (a parabola).
Examples artillery shells, shot putts, high jumpers and balls in games such as tennis, football, basketball, volley ball, cricket, golf, only to mention but a few.

Analysis of results

Consider a particle projected with initial velocity u at an angle α to the horizontal and has velocity v at time t



Its flight can be analysed by considering horizontal and vertical motion separately and using the equations for uniform acceleration in a straight line, i.e.

$$v = u + at$$
$$s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$

	Horizontal motion	Vertical motion
u	$u_x = u\cos\alpha$	$u_y = u\sin\alpha$
a	$\ddot{x} = 0$	$\ddot{y} = -\text{g}$
v	$v_x = u\cos\alpha$	$v_y = u\sin\alpha - \text{g}t$
s	$x = (u\cos\alpha)t$	$y = (u\sin\alpha)t - \frac{1}{2}\text{g}t^2$

Vector approach

If a particle is projected with initial velocity $a\text{i} + b\text{j}$, then its velocity at any time t can be expressed in the form

$$\text{v} = a\text{i} + (b - \text{g}t)\text{j}$$

and its position at any time t can be expressed in the form

$$\text{r} = at\text{i} + \left(bt - \frac{1}{2}\text{g}t^2\right)\text{j}$$

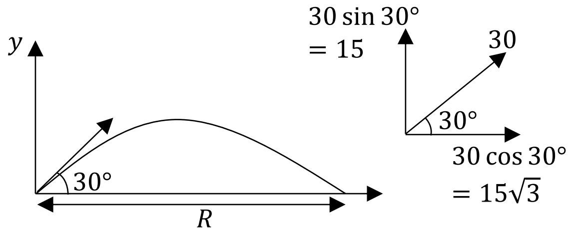
Any problem on projectiles may be solved using vector methods but in general it is unwise to do so unless the problem is phrased in vector terms.

Example 1

A particle is projected from ground level with speed 30 ms^{-1} at an angle of 30° to the horizontal. Calculate

(a) the time of flight
(b) the range

Solution



(a) Consider vertical motion
When the particle reaches the ground, $s = 0$
$u = 15$, $v = ?$, $a = -9.8$, $s = 0$, $t = ?$
Using $s = ut + \frac{1}{2}at^2$

$$0 = 15t + \frac{1}{2}(-9.8)t^2$$
$$0 = t(15 - 4.9t)$$
$$t = 0 \text{ or } t = 3.06 \text{ s}$$

$t = 0$ is the starting time, $t = 3.06$ is the time of flight

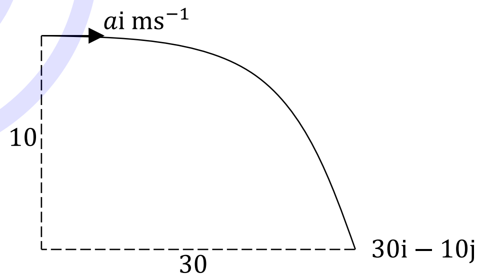
(b) The horizontal velocity is given by $15\sqrt{3}\ \text{ms}^{-1}$ and it is constant. Since the particle travels for 3.06 s at this velocity,

$$\text{Range, } R = 15\sqrt{3} \times 3.06 = 79.5 \text{ m}$$

Example 2

Initially a particle is at an origin O and is projected with a velocity $a\text{i}$ ms^{-1}. After t seconds, the particle is at the point with position vector $(30\text{i} - 10\text{j})$ m. Find the values of t and a.

Solution



Considering vertical motion;
$u = 0$, $a = \text{g}$, $s = 10$, $t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$10 = \frac{1}{2}\text{g}t^2$$
$$10 = 4.9t^2$$
$$t^2 = 2.04$$
$$t = 1.43$$

Considering horizontal motion;
$u = a$, $a = 0$, $s = 30$, $t = 1.43$

$$s = ut + \frac{1}{2}at^2$$
$$30 = at$$
$$30 = a \times 1.43$$
$$a = 20.98$$

$$t = \frac{x}{u\cos\alpha}$$

$$y = u\sin\alpha\left(\frac{x}{u\cos\alpha}\right) - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$$

$$\boldsymbol{y = x\tan\alpha - gx^2(1+\tan^2\alpha)}$$

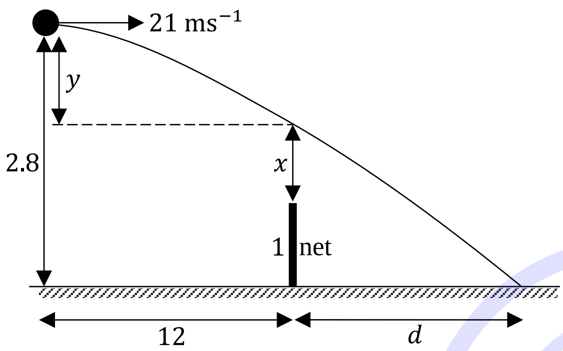
Example 3

A tennis ball is served horizontally with an initial speed of 21 ms^{-1} from a height of 2.8 m.

(a) By what distance does the ball clear a net 1 m high situated 12 m horizontally from the server?

(b) How far behind the net does the netball land?

Solution



(a) Consider vertical motion from serving point to the net

$$u = 0, a = g, s = y, t = ?$$

Horizontal motion;

Time taken $= \frac{\text{Distance}}{\text{Speed}} = \frac{12}{21} = \frac{4}{7}$ s

Considering horizontal motion;

$$s = ut + \frac{1}{2}at^2$$

$$y = \frac{1}{2}\times 9.8\left(\frac{4}{7}\right)^2$$

$$y = 1.6\text{ m}$$

Now;

$$x + 1.6 + 1 = 2.8$$

$$x + 2.6 = 2.8$$

$$x = 0.2\text{ m}$$

∴ Required distance is 0.2 m

(b) Considering vertical motion from serving point to ground

$$u = 0, a = g, s = 2.8, t = 0$$

$$2.8 = 0 + \frac{1}{2}(9.8)t^2$$

$$t^2 = 0.571$$

$$t = 0.756\text{ s}$$

Horizontally;

$$12 + d = 21\times 0.756$$

$$12 + d = 15.876$$

$$d = 3.876\text{ m}$$

The distance behind the net where the balls is 3.876 m

Example 4

Two particles A and B are projected simultaneously under gravity; A from a point O on horizontal ground and B from a point 40 m vertically above O. B is projected horizontally with speed 28 ms^{-1}. If the particles hit the ground simultaneously at the same point, calculate

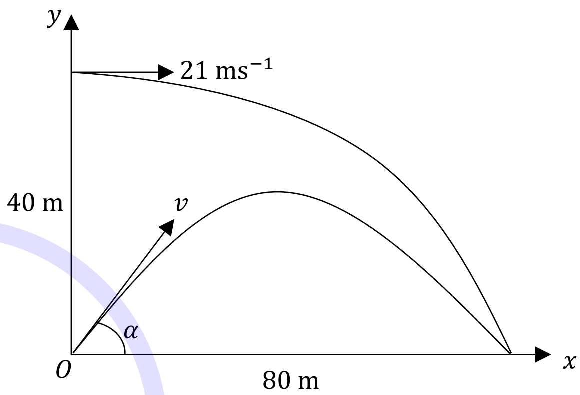
(a) the time taken for B to reach the ground and the horizontal distance it has then travelled

(b) the magnitude and direction of the velocity with which A is projected and the horizontal distance then travelled.

(c) Show that, just prior to hitting the ground, the directions of motions of A and B differ by about $18\frac{1}{2}°$

Solution

Take the horizontal and vertically upward displacements from O as x and y respectively.



(a)

For B: $\dot{x}_B = 28$, $\dot{y}_B = -gt$

$$x_B = 28t, y_B = 40 - \frac{1}{2}gt^2$$

When B strikes the ground, $y_B = 0$.

So $t^2 = \frac{40}{4.9}$

$$t = \frac{20}{7} = 2.86\text{ s}$$

At this time, $x_B = 28t = 28\left(\frac{20}{7}\right) = 80$ m

Therefore, B strikes the ground 2.86 s later having travelled a distance 80 m horizontally

(b) For A: Let the horizontal and vertically upwards components of the velocity of projection be $v\cos\alpha$ and $v\sin\alpha$.

Then $x_A = (v\cos\alpha)t$, $\quad y_A = (v\sin\alpha)t - \frac{1}{2}gt^2$.

Since A and B travel the same distance horizontally in the same time, in order to collide, they must have the same horizontal speed i.e.

$$v\cos\alpha = 28$$

When $t = \frac{20}{7}$, $y_A = 0$, so $v(\sin\alpha)t = \frac{1}{2}gt^2$

$$v\sin\alpha = \frac{9.8}{2}\left(\frac{20}{7}\right) = 14$$

and $y_B = -9.8\left(\frac{20}{7}\right) = -28$

$$\tan\alpha = \frac{1}{2}$$

$$\Rightarrow \alpha = 26.6°$$

$$t = \frac{16V}{g}$$

Considering vertical motion;

$u = 5V\sin\theta = 5V \times \frac{4}{5} = 4V,\ a = -g,\ s = -h$

$$s = ut + \frac{1}{2}at^2$$

$$-h = 4V\left(\frac{16V}{g}\right) - \frac{1}{2}g\left(\frac{16V}{g}\right)^2$$

$$-h = \frac{64V^2}{g} - \frac{1}{2}g\left(\frac{256V^2}{g^2}\right)$$

$$-h = \frac{64V^2}{g} - \frac{128V^2}{g}$$

$$h = \frac{64V^2}{g}$$

(ii) When P is directly level with O, considering vertical motion

$s = 0,\ u = 4V,\ a = -g,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = 4Vt - \frac{1}{2}gt^2$$

$$0 = t\left(4V - \frac{1}{2}gt\right)$$

$$t = 0 \text{ or } t = \frac{8V}{g}$$

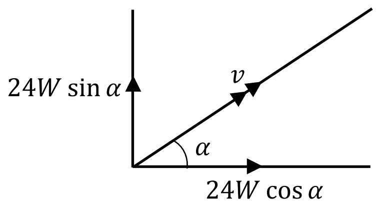
$$\therefore t = \frac{8V}{g}$$

Considering horizontal motion;

$s = OP = ?,\ t = \frac{8V}{g},\ u = 3V$

$$OP = 3V \times \frac{8V}{g} = \frac{24V^2}{g}$$

For the second particle;



Considering horizontal motion;

$u = 24W\cos\alpha,\ a = 0,\ t = T,\ s = \frac{48V^2}{g}$

$$\frac{48V^2}{g} = (24W\cos\alpha)T$$

$$T = \frac{2V^2}{gW\cos\alpha}$$

Considering vertical motion

$u = 24W\sin\alpha,\ a = -g,\ s = -\frac{64V^2}{g}$

$$-\frac{64V^2}{g} = (24W\sin\alpha)T - \frac{1}{2}gT^2$$

$$-\frac{64V^2}{g} = (24W\sin\alpha)\left(\frac{2V^2}{gW\cos\alpha}\right) - \frac{1}{2}g\left(\frac{2V^2}{gW\cos\alpha}\right)^2$$

$$-\frac{64V^2}{g} = \frac{48V^2}{g}\tan\alpha - \frac{g}{2}\left(\frac{4V^4}{g^2W^2\cos^2\alpha}\right)$$

$$-64V^2 = 48V^2\tan\alpha - \frac{2V^4}{W^2}\sec^2\alpha$$

$$-32 = 24\tan\alpha - \frac{V^2}{W^2}(1 + \tan^2\alpha)$$

$$-32 = 24W^2\tan\alpha - V^2(1 + \tan^2\alpha)$$

$$-32W^2 = 24W^2\tan\alpha - V^2 - V^2\tan^2\alpha$$

$$V^2\tan^2\alpha - 24W^2\tan\alpha + V^2 - 32W^2 = 0$$

When $\alpha = 45°$;

$$V^2\tan^2 45° - 24W^2\tan 45° + V^2 - 32W^2 = 0$$

$$V^2 - 24W^2 + V^2 - 32W^2 = 0$$

$$2V^2 - 56W^2 = 0$$

$$V^2 = 28W^2$$

$$W = \frac{V}{2\sqrt{7}}$$

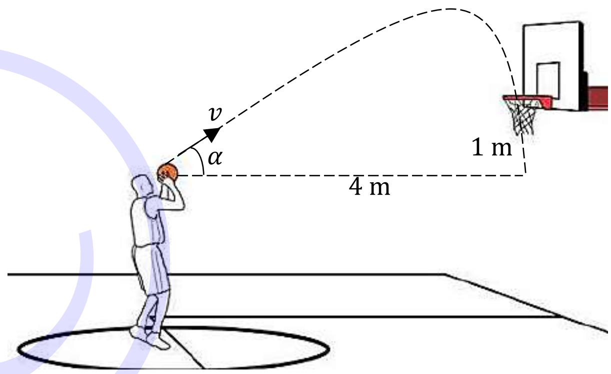
Now

$$28W^2\tan^2\alpha - 24W^2\tan\alpha + 28W^2 - 32W^2 = 0$$

$$28\tan^2\alpha - 24\tan\alpha - 4 = 0$$

$$7\tan^2\alpha - 6\tan\alpha - 1 = 0$$

Example 7



The motion of the ball in a successful free shot in basketball is illustrated above.

The ball is projected from a position, distance 4 m horizontally and 1 m vertically from the basket, with speed v ms^{-1} at an angle α to the horizontal. The ball falls into the basket.

(a) Show that v and α must satisfy

$$1 = 4\tan\alpha - \frac{78.4}{v^2}\sec^2\alpha$$

(b) Use this equation to find the required speed of projection when angle α equals 45°

(c) Also, use this equation to find the two possible trajectories when $v = 8.0$ ms^{-1}

For the ball to fall through the basket, the angle made with the vertical at the basket should be as small as possible. Which of your two solutions above would be preferred?

Solution

(a) Considering horizontal motion

$u = v\cos\alpha,\ s = 4\text{ m},\ a = 0,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

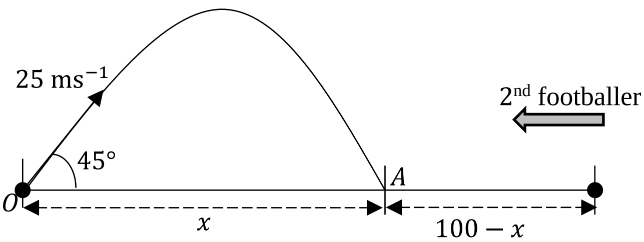
$$4 = (v\cos\alpha)t$$

$$t = \frac{4}{v\cos\alpha}$$

Example 17

Two footballers 100 m apart, stand facing each other. One of them kicks the ball from the ground such that the ball takes off at a velocity of 25 ms^{-1} at 45° to the horizontal. Find the speed at which the second footballer should run towards the first baller in order to trap the ball as it touches the ground, if he starts running at the instant the ball is kicked.

Solution



Considering the ball;

At the point A, the vertical displacement, $y = 0$

$$\therefore \text{From} \quad y = ut\sin\theta - \frac{1}{2}gt^2$$

$$0 = 25t\sin 45° - \frac{1}{2} \times 9.8 \times t^2$$

Either $t = 0$ i.e. at O or $25\sin 45° - \frac{1}{2} \times 9.8t = 0$

$$\therefore 4.9t = 25 \times \sin 45°$$

$$t = 3.61\ s$$

From $x = ut\cos\theta$

$$x = 25 \times 3.61 \times \cos 45° = 63.82 \text{ m}$$

Alternatively; as the ball touches the ground, it has travelled a distance equal to the range.

$$R = \frac{u^2\sin 2\theta}{g}$$

$$x = \frac{25^2\sin 90°}{9.8} = 63.82 \text{ m}$$

Consider the second footballer;

He is supposed to travel a distance of

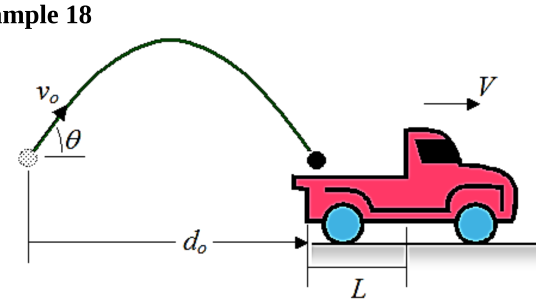
$$(100 - x) = 100 - 63.82 = 36.18 \text{ m}$$

Since the second footballer starts running at the instant the ball is kicked and is supposed to run so as to trap the ball as it falls, he should take the same time as that taken by the ball to land.

$\Rightarrow$ He should take 3.61 s

$$\therefore \text{speed} = \frac{\text{Distance}}{\text{time}} = \frac{36.18}{3.61} = 10.02 \text{ ms}^{-1}$$

Example 18



A ball is kicked at an angle $\theta = 45°$. It is intended that the ball lands in the back of a moving truck which has a trunk of length $L = 2.5$ m. If the initial horizontal distance from the back of the truck to the ball, at the instant of the kick is $d_o = 5$ m, and the truck moves directly away from the ball at a velocity of $V = 9$ ms^{-1} as shown above, what is the maximum and minimum velocity so that the ball lands in the trunk? (Assume that the initial height of the ball is equal to the height of the ball at the instant it begins to enter the trunk)

Solution

When the ball begins to enter the trunk, the horizontal distance, d_x travelled by the ball is given by $d_x = (v_o\cos\theta)t$ which is also equal to the distance $d_o + Vt$ for minimum v_o and $d_o + L + Vt$ for maximum v_o where t is the total time.

Considering vertical motion;

$u = v_o\sin 45°$, $a = -9.8$, $s = 0$, $t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = (v_o\sin 45°)t - 4.9t^2$$

$$t = 0 \text{ or } t = \frac{v_o\sin 45°}{4.9}$$

$$\therefore t = 0.1443v_o$$

The position of the two ends of the truck bed are given by

$$5 + 9t \text{ and } 7.5 + 9t$$

Considering horizontal motion;

For the ball to land on the back of the truck bed,

$$(v_o\cos 45°)t = 5 + 9t$$

$$v_o\cos 45° \times 0.1443v_o = 5 + 9(0.1443v_o)$$

$$0.102v_o^2 = 5 + 1.299v_o$$

$$0.102v_o^2 - 1.299v_o - 5 = 0$$

$$v_o = \frac{1.299 \pm \sqrt{1.299^2 - 4(0.102)(-5)}}{2(0.102)}$$

$$v_o = 15.83 \text{ or } v_o = -3.1$$

$$\therefore v_o = 15.83 \text{ ms}^{-1}$$

For the ball to fall on the front of the trunk bed,

$$(v_o\cos 45°)t = 7.5 + 9t$$

$$v_o\cos 45° \times 0.1443v_o = 7.5 + 9(0.1443v_o)$$

$$0.102v_o^2 = 7.5 + 1.299v_o$$

$$0.102v_o^2 - 1.299v_o - 7.5 = 0$$

$$v_o = \frac{1.299 \pm \sqrt{1.299^2 - 4(0.102)(-7.5)}}{2(0.102)}$$

$$v_o = 17.05 \text{ or } v_o = -4.31$$

$$\therefore v_o = 17.05 \text{ ms}^{-1}$$

The minimum velocity of the ball is 15.83 ms^{-1} and the maximum velocity is 17.05 ms^{-1}

Example 19

Two particles, A and B, are projected from the same fixed point O, with the same speed u ms^{-1}, at angles of elevation θ and 2θ respectively. It is further given that B is projected $\frac{2}{3}$ s after A and $\tan\theta = \frac{3}{4}$.

If A and B collide in the subsequent motion determine the value of u.

Solution

Suppose the collision takes place t seconds after A was projected.

For A collision, both particles must have the same x and y displacements, at time t and $t - \frac{2}{3}$

$$24T = 78.4$$
$$T = \frac{49}{15} = 3.26 \text{ s}$$

Finally, vertically for P;

$$v = u + at$$
$$v = 26\sin\theta - 9.8t$$
$$v = 26 \times \frac{12}{13} - 9.8 \times \frac{49}{15}$$
$$v = 24 - \frac{2401}{75}$$
$$v = -\frac{601}{75} = -8.01 \text{ ms}^{-1}$$

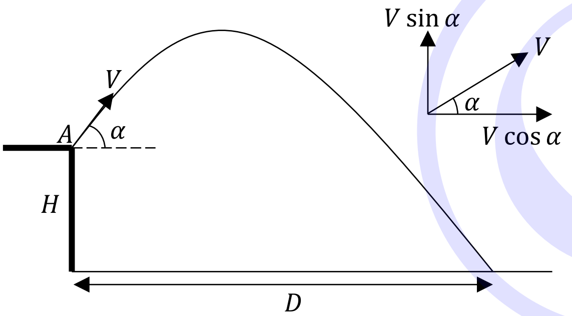
The negative sign implies that it is falling

Example 21

A particle is projected at an angle α above the horizontal, from a vertical cliff face of height H above level horizontal ground. It first hits the ground at a horizontal distance D, from the bottom of the cliff edge.
Assuming that air resistance can be ignored, show that the greatest height achieved by the particle from the level horizontal ground is

$$H + \frac{D\tan^2\alpha}{4(H + D\tan\alpha)}$$

<u>Solution</u>



Let T be the time taken to reach the maximum height.

$$v = u + at$$
$$0 = V\sin\alpha - gT$$
$$T = \frac{V\sin\alpha}{g}$$

Hence the maximum height above the ground is given by

$$H + ut + \frac{1}{2}at^2 = H + VT\sin\alpha - \frac{1}{2}gT^2$$
$$= H + V\left(\frac{V\sin\alpha}{g}\right)\sin\alpha - \frac{1}{2}g\left(\frac{V\sin\alpha}{g}\right)^2$$
$$= H + \frac{V^2\sin^2\alpha}{g} - \frac{1}{2}\frac{V^2\sin^2\alpha}{g}$$
$$= H + \frac{V^2\sin^2\alpha}{2g}$$

We need to get rid of V from the expression. Let τ be the flight time

Horizontally;

$$D = (V\cos\alpha)\tau$$
$$\tau = \frac{D}{V\cos\alpha}$$

Vertically;

$$s = ut + \frac{1}{2}at^2$$
$$-H = (V\sin\alpha)\tau - \frac{1}{2}g\tau^2$$
$$-H = V\tau\sin\alpha - \frac{1}{2}g\tau^2$$

Substituting for τ;

$$-H = V\left(\frac{D}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{D}{V\cos\alpha}\right)^2$$
$$-H = D\tan\alpha - \frac{g}{2}\frac{D^2}{V^2\cos^2\alpha}$$
$$\frac{g}{2}\frac{D^2}{V^2\cos^2\alpha} = H + D\tan\alpha$$
$$\frac{2V^2\cos^2\alpha}{gD^2} = \frac{1}{H + D\tan\alpha}$$
$$V^2\left(\frac{2\cos^2\alpha}{gD^2}\right) = \frac{1}{H + D\tan\alpha}$$
$$V^2 = \frac{gD^2}{2\cos^2\alpha} \times \frac{1}{H + D\tan\alpha}$$

Finally substitute into $H + \frac{V^2\sin^2\alpha}{2g}$

$$= H + \frac{gD^2}{2\cos^2\alpha} \times \frac{1}{H + D\tan\alpha} \times \frac{\sin^2\alpha}{2g}$$
$$= H + \frac{D\sin^2\alpha}{4\cos^2\alpha\,(H + D\tan\alpha)}$$
$$= H + \frac{D\tan^2\alpha}{4(H + D\tan\alpha)}$$

Example 22

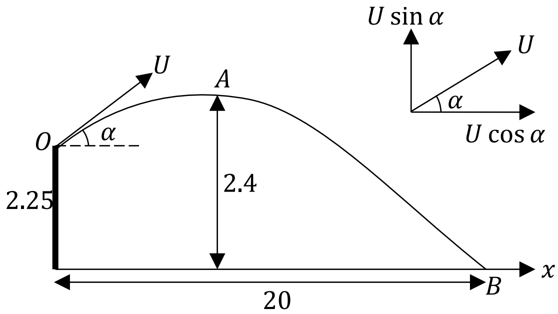
A tennis player standing on a level horizontal court serves the ball from a height of 2.25 m above the court. The ball reaches a maximum height of 2.4 m above the court and first hits the court at a horizontal distance of 20 m from the point where the player served the ball. The ball rises for T_1 s and falls for T_2 s. The ball is modelled as a particle moving through still air without any resistance.

(a) Show clearly that

$$\frac{T_2}{T_1} = 4$$

(b) Determine the magnitude and direction of the velocity of the ball
  (i) when it was first served.
  (ii) as it lands on the court.

<u>Solution</u>



Vertically, O to A;

$$v = u + at$$

$$0 = U\sin\alpha - gT_1$$
$$U\sin\alpha = gT_1$$

Vertically, O to A;

$$s = ut + \frac{1}{2}at^2$$
$$0.15 = (U\sin\alpha)T_1 - gT_1^2$$
$$\frac{3}{10} = 2(U\sin\alpha)T_1 - gT_1^2$$
$$\frac{3}{10} = 2(gT_1)T_1 - gT_1^2$$
$$\frac{3}{10} = gT_1^2$$
$$T_1^2 = \frac{3}{10g}$$

Vertically, A to B;

$$s = ut + \frac{1}{2}at^2$$
$$2.4 = \frac{1}{2}gT_2^2$$
$$4.8 = gT_2^2$$
$$gT_2^2 = \frac{24}{5}$$
$$T_2^2 = \frac{24}{5g}$$

Combining the results;

$$\frac{T_2^2}{T_1^2} = \frac{24/5g}{3/10g} = 16$$
$$\therefore \frac{T_2}{T_1} = 4$$

(b) (i)

Flight time, $T = T_1 + T_2 = \sqrt{\frac{3}{10g}} + \sqrt{\frac{24}{5g}} = \sqrt{\frac{3}{98}} + \sqrt{\frac{24}{49}}$

$$= \frac{\sqrt{6}}{14} + \frac{2}{7}\sqrt{6} = \frac{5}{14}\sqrt{6}$$

Horizontally from O to B;

$$20 = U\cos\alpha \times \frac{5}{14}\sqrt{6}$$

Vertically from O to B;

$$-2.25 = U\sin\alpha \times \frac{5}{14}\sqrt{6} - \frac{1}{2}g\left(\frac{5}{14}\sqrt{6}\right)^2$$
$$-2.25 = U\sin\alpha \times \frac{5}{14}\sqrt{6} - 3.75$$
$$1.5 = U\sin\alpha \times \frac{5}{14}\sqrt{6}$$

Dividing the equations;

$$\frac{U\sin\alpha \times \frac{5}{14}\sqrt{6}}{U\cos\alpha \times \frac{5}{14}\sqrt{6}} = \frac{1.5}{20}$$
$$\tan\alpha = \frac{3}{20}$$
$$\alpha = 4.29°$$

Hence $20 = U\cos\alpha \times \frac{5}{14}\sqrt{6}$

$$U = \frac{20 \times 14}{5\sqrt{6}\cos 4.29°}$$
$$U = 22.9\ \text{ms}^{-1}$$

(ii) Now using $v^2 = u^2 + 2as$ for OB

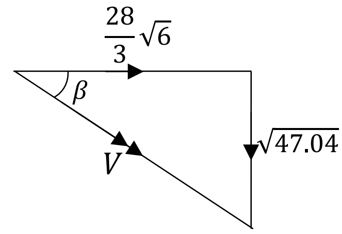
$$v^2 = (U\sin\alpha)^2 - 2 \times 9.8 \times (-2.25)$$
$$v^2 = U^2\sin^2\alpha + 44.1$$

From $1.5 = (U\sin\alpha) \times \frac{5}{14}\sqrt{6}$

$$U\sin\alpha = \frac{7}{10}\sqrt{6}$$

Thus, $v^2 = \frac{147}{50} + 44.1$

$$v^2 = 47.04$$
$$v = \sqrt{47.04}$$



Thus $V = \sqrt{47.04 + \frac{1568}{3}}$

$$V = 23.87\ \text{ms}^{-1}$$
$$\tan\beta = \frac{\sqrt{47.04}}{\frac{28}{3}\sqrt{6}}$$
$$\beta = 16.7°$$

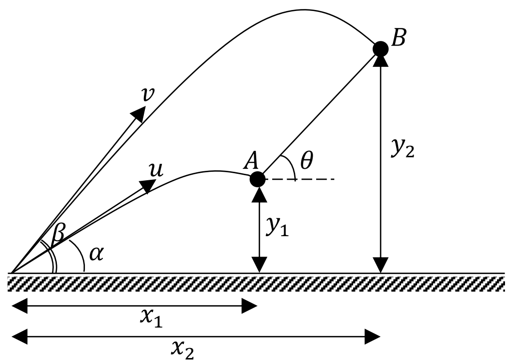
Example 23

Two particles are projected simultaneously from the same point with angles of projection α and β and initial speeds u and v. Show that at any time during their flight the line joining them is inclined to the horizontal at

$$\arctan\frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta}$$

Solution

Let the angle made by the line joining them to the horizontal be θ



Consider the motion of particle A;

Horizontal motion;

$$x_1 = uT\cos\alpha$$

Vertical motion;

$u = u\sin\alpha$, $a = -g$, $s = y_1$

$$y_1 = (u\sin\alpha)T - \frac{1}{2}gT^2$$
$$y_1 = T(u\sin\alpha - \frac{1}{2}gT)$$

Consider the motion of particle B;

Horizontal motion;

$$x_2 = vT\cos\beta$$

Vertical motion;

$$y_2 = (v\sin\beta)T - \frac{1}{2}gT^2$$

$$y_2 = T(v\sin\beta - \frac{1}{2}gT)$$

Now

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\tan\theta = \frac{T\left(v\sin\beta - \frac{1}{2}gT\right) - T\left(u\sin\alpha - \frac{1}{2}gT\right)}{vT\cos\beta - uT\cos\alpha}$$

$$\tan\theta = \frac{T\left[v\sin\beta - \frac{1}{2}gT - u\sin\alpha + \frac{1}{2}gT\right]}{T[v\cos\beta - u\cos\alpha]}$$

$$\tan\theta = \frac{v\sin\beta - u\sin\alpha}{v\cos\beta - u\cos\alpha} \times \frac{-1}{-1}$$

$$\tan\theta = \frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta}$$

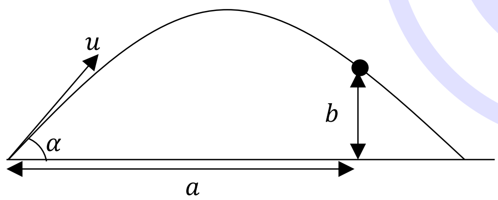
$$\theta = \tan^{-1}\frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta}$$

Example 24

A particle is projected from a point O with initial speed u to pass through a point which is at a horizontal distance a from O and a distance b vertically above the level of O. Show that there are two possible angles of projection. If these angles are α_1 and α_2, prove that $\tan(\alpha_1 + \alpha_2) = -(a/b)$

Solution

Let the angle of projection be α



$$s = ut + \frac{1}{2}at^2$$

Considering horizontal motion;

$$a = (u\cos\alpha)t$$

$$t = \frac{a}{u\cos\alpha}$$

Considering vertical motion;

$$b = (u\sin\alpha)t - \frac{1}{g}t^2$$

$$b = (u\sin\alpha)\left(\frac{a}{u\cos\alpha}\right) - \frac{1}{2}\left(\frac{a}{u\cos\alpha}\right)^2$$

$$b = a\tan\alpha - \frac{ga^2}{2u^2}\sec^2\alpha$$

$$b = a\tan\alpha - \frac{ga^2}{2u^2}(1 + \tan^2\alpha)$$

$$\frac{2u^2b}{ga^2} = \frac{2u^2}{ag}\tan\alpha - 1 - \tan^2\alpha$$

$$\tan^2\alpha - \frac{2u^2}{ag}\tan\alpha + 1 + \frac{2u^2b}{ga^2} = 0$$

$$\tan^2\alpha - \frac{2u^2}{ag}\tan\alpha + \left(\frac{ga^2 + 2u^2b}{ga^2}\right) = 0$$

This is a quadratic equation in terms $\tan\alpha$ hence two possible angles of projection.

Now;

$\tan\alpha_1 + \tan\alpha_2 = \frac{2u^2}{ag}$ and $\tan\alpha_1\tan\alpha_2 = \left(\frac{2u^2b + ga^2}{ga^2}\right)$

From compound angle formula;

$$\tan(\alpha_1 + \alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1\tan\alpha_2}$$

$$= \frac{(2u^2/ag)}{1 - \left(\frac{2u^2b + ga^2}{ga^2}\right)}$$

$$= \frac{(2u^2/ag)}{\left(\frac{ga^2 - 2u^2b - ga^2}{ga^2}\right)}$$

$$= \frac{2u^2}{ag} \div \frac{-2u^2b}{ga^2}$$

$$= \frac{2u^2}{ag} \times \frac{ga^2}{-2u^2b}$$

$$= -\frac{a}{b}$$

$$\therefore \tan(\alpha_1 + \alpha_2) = -\left(\frac{a}{b}\right)$$

Example 25

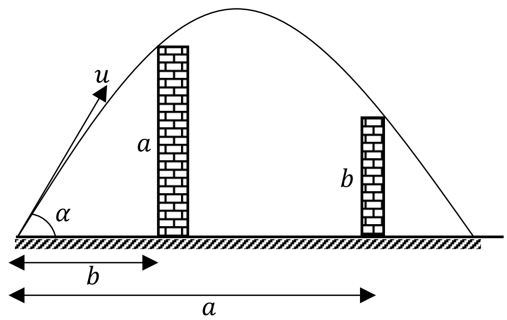
A ball is projected so as just to clear two walls, the first of height a at a distance b from the point of projection, and the second of height b at a distance a from the point of projection. Show that the range on a horizontal plane is

$$\frac{a^2 + ab + b^2}{a + b}$$

and that the angle of projection exceeds $\tan^{-1} 3$

Solution

Let the speed of projection be u and the angle of projection be α



Using the trajectory equation

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$$

For clearance at first wall,

$$a = b\tan\alpha - \frac{gb^2}{2u^2}\sec^2\alpha$$

$$b\tan\alpha - a = \frac{gb^2}{2u^2}\sec^2\alpha \quad \ldots.\text{(i)}$$

For clearance at second wall,

$$b = a\tan\alpha - \frac{ga^2}{2u^2}\sec^2\alpha$$

$$\sin 2\theta = \frac{4}{5}$$
$$2\theta = 53.13°$$
$$\theta = 26.6°$$

(b) Now for the ratio of times, we can use

Time for no bounce throws, t is $\frac{u\sqrt{2}}{g}$

Total time for one bounce through, $T = t_1 + t_2$

$$T = \frac{2u\sin\theta}{g} + \frac{u\sin\theta}{g} = \frac{3u\sin\theta}{g} = \frac{3u\sin 26.6°}{g} = \frac{1.34u}{g}$$

The ratio of the times is given by;

$$\frac{t}{T} = \frac{u\sqrt{2}}{g} \div \frac{1.34u}{g} = \frac{u\sqrt{2}}{g} \times \frac{g}{1.34u} = 1.06$$

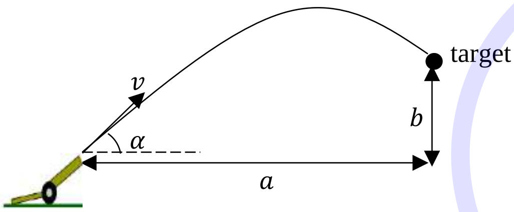
Example 27

The muzzle speed of a gun is v and it is desired to hit a small target at a horizontal distance a away and at a height b above the gun. Show that this is impossible if $v^2(v^2 - 2gb) < g^2a^2$, but that, if $v^2(v^2 - 2gb) > g^2a^2$, there are two possible elevations for the gun.

Show that if $v^2 = 2ag$ and $b = \frac{3}{4}a$, there is only one possible elevation, and find the time taken to hit the target.

Solution

Let the angle of elevation be α



The trajectory equation is $y = x\tan\alpha - \frac{gx^2(1+\tan^2\alpha)}{2v^2}$

For the shell to hit the target at $x = a$, $y = b$

$$b = a\tan\alpha - \frac{ga^2(1+\tan^2\alpha)}{2v^2}$$

$$\Rightarrow ga^2\tan^2\alpha - 2av^2\tan\alpha + 2v^2b + ga^2 = 0 \text{ .. (i)}$$

This a quadratic equation in $\tan\alpha$, which must have real solutions for the target to be hit.

(a) The target cannot be hit if the discriminant < 0

i.e. if $4a^2v^4 - 4(ga^2)(2v^2b + ga^2) < 0$

$$v^4 - 2gbv^2 - g^2a^2 < 0$$
$$v^2(v^2 - 2gb) < g^2a^2$$

(b) If the discriminant > 0, i.e. if $v^2(v^2 - 2gb) > g^2a^2$, then equation (i) has two real distinct solutions for $\tan\alpha$ and hence for the elevation

(c) If $v^2 = 2ga$ and $b = \frac{3}{4}a$, then (i) becomes

$$ga^2\tan\alpha - 4ga^2\tan\alpha + 3ga^2 + ga^2 = 0$$
$$\tan^2\alpha - 4\tan\alpha + 4 = 0$$
$$(\tan\alpha - 2)^2 = 0$$
$$\tan\alpha = 2$$

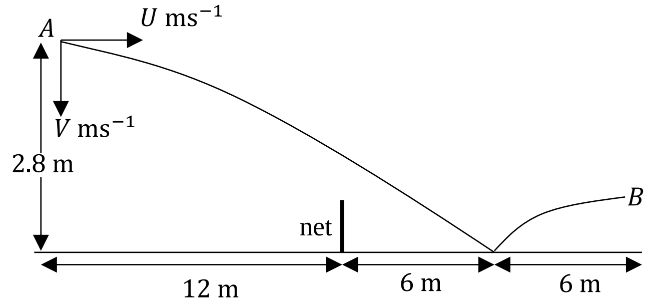
i.e. only one possible elevation, $\tan^{-1} 2$

Horizontal displacement, $a = (v\cos\alpha)t = \sqrt{2ga}\left(\frac{1}{\sqrt{5}}\right)t$

$$t = \frac{a\sqrt{5}}{\sqrt{2ga}} = \sqrt{\frac{5a}{2g}}$$

Time taken for the shell to reach the target is $\sqrt{\frac{5a}{2g}}$

Example 28



A ball is projected from a point A a height of 2.8 m above the ground with velocity components U and V vertically and horizontally respectively. It passes over a net at a horizontal distance of 12 m from A, landing 6 m behind the net. It bounces to a point B which is at a height of 0.75 m and at a distance of 6 m from the point of bounce. The ball takes 0.6 s to cover the 24 m. Assuming that there is no air resistance and the ball bouncing does not affect the horizontal velocity of the ball, find

(a) U and V

(b) the distance by which the ball just clears the net if it is 1 m high?

(c) the direction of the ball at B

Solution

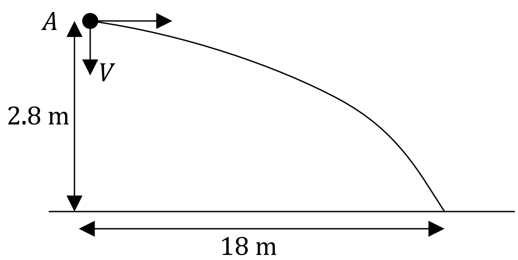
(a) Since the bouncing does not affect the horizontal velocity of the ball,

$$s_x = u_xt + \frac{1}{2}a_xt^2$$
$$24 = U \times 0.6$$
$$U = 40 \text{ ms}^{-1}$$

Considering the motion of the ball from the A to where it bounces.

Time taken is given by

$$\frac{\text{distance}}{\text{speed}} = \frac{s_x}{u_x} = \frac{18}{40} = 0.45 \text{ s}$$



Vertically;

$$s_y = u_yt + \frac{1}{2}a_yt^2$$
$$2.8 = V(0.45) + \frac{1}{2}(9.8)(0.45)^2$$
$$2.8 = 0.45V + 0.99225$$
$$0.45V = 1.80775$$
$$V = 4.02 \text{ ms}^{-1}$$

Force

A force is necessary to cause a body to accelerate. More than one force may act on a body. If the forces on a body are in **equilibrium**, i.e. balance out, then the body may be at rest or moving in a straight line at constant speed.

If there is a resultant force on the body, then the body will accelerate.

Force is a **vector**, i.e. has magnitude and direction.

S.I unit of force is the Newton (N).

1 Newton is the force needed to give a body of mass 1 kg an acceleration of 1 ms^{-2}

Mass and weight

Mass and weight are different.

The mass of a body is a measure of the matter contained in the body. A massive body will need a larger force to change its motion. The mass of a body may be considered to be constant, whatever the position of the body, provided that none of the body is destroyed or changed.

Mass is a scalar, i.e. it has magnitude only.

S.I unit of mass is the kilogram (kg)

The weight of the body is the force with which the earth attracts it. It is dependent upon the body's distance from the earth.

Weight is a vector, since it is a force.

S.I unit of weight is the Newton (N).

The weight, W, in newtons, and mass m, in kilograms are connected by the relation $W = mg$, where g is the acceleration due to gravity in ms^{-2}.

NEWTON'S LAWS OF MOTION

Newton's three laws are the fundamental basis of the study of mechanics at this level. Although there is no direct proof of these laws, predictions made using them agree very closely with observations.

1st law

Every body will remain at rest or continue to move in a straight line at constant speed unless an external force acts on it.

Consequences

- If a body has an acceleration, then there must be a force acting on it
- If a body has no acceleration, then the forces acting on it must be in equilibrium

2nd law

The rate of change of momentum of a moving body is proportional to the external force acting on it and takes place in the direction of that force.

So when an external force acts on a body of constant mass, the force produces an acceleration which is directly proportional to the force.

Consequences

- The basic equation of motion for constant mass is

  Force = **mass** × **acceleration**

  (in N) (in kg) (in ms^{-2})
- The force and acceleration of the body are both in the same direction
- A constant force on a constant mass gives a constant acceleration.

3rd law

If a body A exerts a force on B, then B exerts an equal and opposite force on A.

Consequences

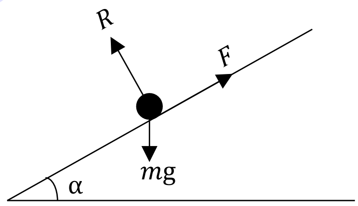
These forces between the bodies are often called reactions. In a rigid body, the internal forces occur as equal and opposite pairs and the net effect is zero. So only external forces need to be considered.

Problem solving

1. Draw a clear force diagram
2. If there is no acceleration, i.e. the body is either at rest or moving with uniform velocity, then the forces balance in each direction
3. If there is an acceleration
   (a) mark it on the diagram using $\xrightarrow{a}$
   (b) write down, if possible, an expression for the resultant force
   (c) use Newton's 2nd law i.e. write the equation of motion

   force = mass × acceleration

Body at rest on a rough inclined plane

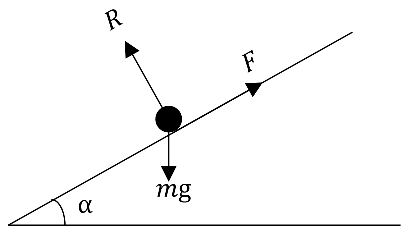


No acceleration so forces balance:

∥ to plane ⇒ $F = mg\sin\alpha$

⊥ to plane ⇒ $R = mg\cos\alpha$

Body sliding down rough plane at constant speed

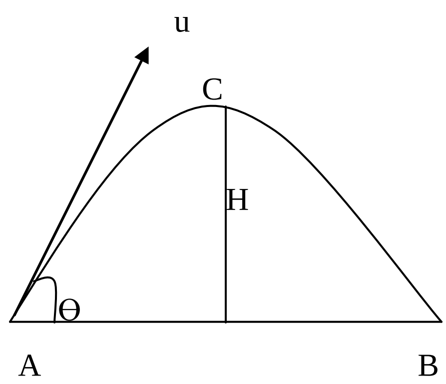


∥ to plane ⇒ $F = mg\sin\alpha$

⊥ to plane ⇒ $R = mg\cos\alpha$

PROJECTILE

Consider the motion of an object which is projected with a velocity u at an angle Q to the horizontal



Θ - angle of projection.

ACB followed by the object is called its trajectory.

Horizontal motion

Horizontal component of velocity is got by

$V_x = U_x + a_x t$. Where Vx, U_x and a_x are the velocity of a body at any time t, initial component of velocity and horizontal acceleration respectively.

But $U_x = U\cos\Theta$, $a_x = 0$

Hence $V_x = U\cos\Theta$ -------------(1)

From the above equation the horizontal velocity is constant throughout motion.

The horizontal distance travelled after time t is

$$X = u_x t + \frac{1}{2} a_x t^2$$

Where X is the horizontal distance covered by the object

But $a_x = 0$

$$\therefore\ x = Ut\cos\theta \text{..........} \text{....}(2)$$

Vertical motion

$V_y = U_y + a_y t$ where V_y, U_y and a_y are the vertical velocity of a body at any time, t, initial velocity component of velocity and vertical acceleration respectively.

$U_y = U \sin\theta, a_y = -g$

$V_y = U \sin\theta - gt$ ....(3)

The vertical displacement, y, is obtained below

$y = U_y t + ½ a_y t^2$

But $U_y = U\sin\Theta$, $a_y = -g$

Hence

$y = (U\sin\Theta)\, xt - ½gt^2$……………………………………….(4)

Speed, V, at any time t is given by

$V = \sqrt{(V_x^2 + V_y^2)}$ ……………………………………….(5)

The angle, α, the body makes with the horizontal after t is given by

$\text{Tan}\ \alpha = \frac{Vy}{Vx} = \frac{U \sin\theta - gt}{U \cos\theta}$ ……………………………………….(6)

<u>Maximum height, H</u>

At maximum height, Vy =0

$Vy^2 = U_y^2 + 2aH$

$0 = (U \sin\theta)^2 - 2ghH$

$H = \frac{U^2 \sin^2\theta}{2g}$.......... ....(7)

<u>Time to reach the maximum heights</u>

Using V = u + at

$$0 = Uy + a_y t$$

$$0 = U \sin\theta - gt$$

$$t = U\frac{\sin\theta}{g} \text{.......... ....}(8)$$

<u>Time of flight, T</u>

The time taken by the projectile to move from the point of projection to a point on the plane through the point of projection where the projection lies i.e. time taken to move from A to B.

$$atB, \; y = 0$$

$$y = ut\sin\theta - \frac{gt^2}{2}$$

$$0 = 2at\sin\theta - gt^2$$

$$0 = t(2u\sin\theta - gt)$$

$$\text{either } t = 0 \text{ or } t = \frac{2u\sin\theta}{g}$$

Hence $T = \frac{2u\sin\theta}{g}$ ……………………………………….(9)

Note: Time of flight is twice the time taken to reach height.

<u>Ranges , R:</u>

It is the distance between the point of projection and a point on the plane through the point of projection where the projectile lands i.e horizontal distance AB.

X= Ut cosΘ

When X=R, t = T= $\frac{2u\sin\theta}{g}$

$$\therefore R = u.\frac{2u\sin\theta}{g}.\cos\theta$$

$$R = \frac{2u^2\sin\theta\cos\theta}{g} \text{…………}(10)$$

$$R = \frac{u^2\sin 2\theta}{g}$$

Equation of trajectory

$$t = \frac{x}{u\cos\theta} \text{.......... ....}(1)$$

$$y = ut\sin\theta - \frac{gt^2}{2} \text{.......... ....}(2)$$

Substitute equation (1) into equation 2.

$$y = U.\frac{x}{U\cos\theta}.\sin\theta - \frac{g}{2}\frac{x^2}{U^2\cos^2\theta}$$

$$y = \frac{\sin\theta x}{\cos\theta} - \frac{gx^2}{2U^2\cos^2\theta}$$

$$y = (\tan\theta)x - \left(\frac{g}{2u^2Cos^2\theta}\right)x^2$$

$$y = x\tan\theta - \tfrac{1}{2}gx^2\frac{\sec^2\theta}{u_2}$$

The above equation is in the form $y = Ax - Bx^2$, where A and B are constants which is an equation of a parabola. Therefore the trajectory is a parabola.

Note: For any given initial speed, the range is maximum when $\sin\Theta = 1$ or $\Theta = 45^0$

$R_{max} = \frac{U^2}{g}$ *(Prove it !!!!)*

Example

1. Prove that the time of flight T and the horizontal range R, of a projectile are connected by the equation. $gT^2 = 2R\tan\alpha$

Where α is the angle of projection

From equations (9) and (10)

$Tg = 2U\sin\alpha$ ….(a) ,

$Rg = 2U^2\sin\alpha\cos\alpha$ …….(b)

Eqn $(a)^2$ ÷ eqn (b)

$$\frac{(Tg)^2}{Rg} = \frac{4U^2\sin^2\alpha}{2U^2\sin\alpha\cos\alpha}$$

$$\frac{T^2 g}{R} = \frac{2\sin\alpha}{\cos\alpha}$$

Hence $T^2 g = 2R\tan\alpha$

2. Two footballers, 120m apart, stand facing each other. One of them kicks a ball from the ground such that the ball takes off at a velocity of $30ms^{-1}$ at 38^0 to the horizontal. Find the speed at which the second footballer must run towards the first footballer in order to trap the ball as it touches the ground, if he starts running at the instant the ball is kicked.

For the first footballer, the time he ball takes to touch the ground is

c) $T = \frac{2u\sin\theta}{g}$

$$= \frac{2x30 \times \sin 38}{9.8}$$

$$= 3.78\ s$$

$$R = \frac{u^2 \sin 2\alpha}{g}$$

$$R = \frac{30^2 \times \sin 76}{9.8}$$

$$R = 89.1$$

The time taken by the second footballer to reach the ball is 3.78s.

The distance travelled by the second footballer is s = 120 – 89.1 = 30.9m

Therefore the speed of the second footballer distance / time = 30.9/3.78 = $8.2ms^{-1}$

3. A projectile is fired from ground level with a velocity of $500ms^{-1}$, 30^0 to the horizontal. Find the horizontal range, the greatest height to which it rises and time taken to reach the greatest height. What is the least speed with which it could be projected in order to achieve the same horizontal range?

$u = 500\ ms^{-1}$ $\qquad \alpha = 30^0$

(i) $\quad \text{Range} = \frac{u^2 \sin 2\alpha}{g}$

$$= \frac{500^5}{9.81} \sin(2 x 30)$$

$$= \underline{\underline{22069.96\ m}}$$

(ii) $H = \frac{u^2 \sin^2 \alpha}{2g}$

$$= \frac{500^2 (\sin 30)^2}{2 \times 9.81}$$

= 3185.5m

(iii) Time taken to reach the greatest height.

$$T = \frac{u \sin \alpha}{g}$$

T = (500sin30)/9.81 = 25.5s

(b) $U_{min} = (Rg)^{1/2}$

$(22069.96 \times 9.81)^{1/2}$

$\underline{\underline{465.3 ms^{-1}}}$

Exercise 8:

(1) A body is thrown from the top of a tower 30.4m high with a velocity of 24ms $^{-1}$ at an elevation of 3 0^0 above the horizontal. Find the horizontal distance from the roof of the tower of the point where it hits the ground.

(2) A body is projected at such an angle that the horizontal range is three times the greatest height. Given that the range of projection is 400m, find the necessary velocity of projection and angle of projection.

(3) A projectile fired at an angle of 60^0 above the horizontal strikes a building 30m away at a point 15m above the point of projection.(i) Find the speed of projection.
(ii) Find the velocity of the projectile when it strikes the building.

4. An object *P* is projected upwards from a height of 60m above the ground with a velocity of $20\,ms^{-1}$ at 30^0 to the horizontal. At the same time, an object *Q* is projected from the ground upwards towards *P* at 30^0 to the horizontal. *P* and *Q* collide at a height 60m above the ground while they are both moving downwards. Find,

(i) The speed of projection of Q.

(ii) The horizontal distance between the points of projection.

(iii) The kinetic energy of *P* just before the collision with Q if the as of *P* is 0.5 kg.