LOGARITHMS

A logarithm of a number to base x is the power to which x must be raised to give a number e.g.

$$x = a^n \Rightarrow a = \log_x m$$

The following 3 statements are equivalent

(i) $16 = 2^4$ and $\log_2 16 = 4$
(ii) $27 = 3^3$ and $\log_3 27 = 3$
(iii) $64 = 8^2$ and $log_8\ 64 = 2$

Logarithms appear in all sorts of calculations in engineering and science, business and economics. Before days of calculators, they were used to assist in the process of multiplication by replacing the operation of multiplication by addition similarly they enabled the operation of division to be replaced by subtraction.

Laws of Logarithms

(1) $log_a x + log_a y = log_a xy$
(2) $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
(3) $log_a x^m = m log_a x$
(4) $log_a 1 = 0$
(5) $log_a b = \frac{log_c b}{log_c a}$ (Change of base law)

Example I

Prove the following laws of logarithms

(1) $log_a xy = log_a x + log_a y$
(2) $log_a \frac{x}{y} = log_a x - log_a y$
(3) $log_a x^m = m log_a x$
(4) $log_a b = \frac{log_c b}{log_c a}$
(5) $log_a b = \frac{1}{log_a b}$

Solution:

(1) Suppose $x = a^n$ and $y = a^m$ then equivalent logarithmic forms are

$$log_a x = n \text{ and } log_a y = m$$

Using the first rule of indices,

$$xy = a^n x\ a^m$$

$$xy = a^{n+m}$$

Now the logarithmic form of the statement

$xy = a^{n+m}$ is $log_a xy = n + m$,

but $n = log_a x$ and $m = log_a y$

Putting this result together we have:

$$log_a xy = log_a x + log_a y$$

Therefore, $\mathbf{log_a xy = log_a x + log_a y}$ (as required.)

(2) For $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

Suppose $x = a^n$ and $y = a^m$ with equivalent respective logarithmic forms
$log_a x = n$ and $log_a y = m$.

$$\frac{x}{y} = a^n \div a^m$$

$$\frac{x}{y} = a^{n-m}$$

$$log_a \left(\frac{x}{y}\right) = log_a a^{n-m}$$

$$log_a \left(\frac{x}{y}\right) = n - m$$

$$log_a \left(\frac{x}{y}\right) = log_a x - log_a y$$

$$log_a x - log_a y = log_a \frac{x}{y}$$

$\therefore log_a \left(\frac{x}{y}\right) = log_a x - log_a y$ as required

(3) For $\mathbf{log_a x^m = m log_a x}$

Suppose $x = a^n$ or $log_a x = n$.
Suppose we raise both sides of $x = a^n$ to the power m

$$x^m = (a^n)^m$$

Using the rules of indices this can be written as;

$$x^m = a^{nm}$$

Thinking of the quantity x^m as a single term, the logarithmic form is

$$\log_a x^m = \log_a a^{nm}$$

$$\log_a x^m = nm \log_a a$$

$$\log_a x^m = nm$$
$$m\log_a x^m = m(n)$$
$$\log_a x^m = m\log_x a$$

(4) For $\boldsymbol{log_a b = \frac{log_c b}{log_c a}}$

Let $log_a b = y$

$\therefore\ a^y = b$…………………… (1)

Introducing log_c on both sides of equation (1)

$$\log_c a^y = \log_c b$$
$$y\log_c a = \log_c b$$
$$y = \frac{\log_c b}{\log_c a}$$
$$log_a b = \frac{log_c b}{log_c a}$$

(5) For $log_a b = \frac{1}{log_a b}$

Let $y = log_a b$

$$b = a^y$$
$$log_b b = log_b a^y$$
$$log_b b = y log_b a$$
$$y = \frac{log_b b}{log_b a}$$
$$y = \frac{1}{log_b a}$$
$$\therefore\ log_a b = \frac{1}{log_b a}$$

Example II

Solve $\log_5 x + 2\log_x 5 = 3$

Solution

$\log_5 x + 2\log_x 5 = 3$

But $log_x 5 = \frac{1}{log_5 x}$

$$\therefore\ log_5 x + \frac{2}{log_5 x} = 3$$

Let $log_5 x = m$

$$m + \frac{2}{m} = 3$$

$m^2 + 2 = 3m$

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1\ and\ m = 2$

Since $log_5 x = 1,$

$\therefore\ 5^1 = x$

$\log_5 x = 2$

$5^2 = x$

$x = 25$

$\therefore x = 5\ or\ x = 25$

Example III

Solve $log_2 x - log_x 8 = 2$

Solution:

$$log_2 x - log_x 8 = 2$$
$$log_2 x - \frac{log_2 8}{log_2 x} = 2$$
$$log_2 x - \frac{log_2 2^3}{log_2 x} = 2$$
$$log_2 x - \frac{3log_2 2}{log_2 x} = 2$$
$$log_2 x - \frac{3}{log_2 x} = 2$$

Let $m = log_2 x$

$$m - \frac{3}{m} = 2$$

$m^2 - 3 = 2m$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3, m = -1$

$log_2 x = 3$

$2^3 = x$

$x = 8$

$\log_2 x = -1$

$2^{-1} = x$

$x = \frac{1}{2}$

Example IV

Solve for x: $3\log_2 x - \log_x 2 = 2$

Solution

$$3\log_2 x - \log_x 2 = 2$$

But $\log_x 2 = \frac{1}{log_2 x}$

$$\Rightarrow 3log_2 x - \frac{1}{log_2 x} = 2$$

Let $log_2 x = y$

$$3y - \frac{1}{y} = 2$$

$3y^2 - 1 = 2y$

$3y^2 - 2y - 1 = 0$

$3y^2 - 3y + y - 1 = 0$

$3y(y-1) + 1(y-1) = 0$

$$(3y+1)(y-1)=0$$
$$y=-\frac{1}{3}, y=1$$

Since $log_2 x = y$

$$\therefore log_2 x = -\frac{1}{3}$$
$$2^{-\frac{1}{3}} = x$$
$$log_2 x = 1$$
$$2^1 = x$$

Example IV

Solve $\ln(6x-5)=3$

Solution

$$\ln(6x-5)=3$$
$$log_e(6x-5)=3$$
$$\therefore\ e^3 = 6x-5$$
$$6x = e^3+5$$
$$x=\frac{e^3+5}{6}$$
$$x=\frac{e^3+5}{6}$$

Example V

Prove that $log_6 x = \frac{log_3 x}{1+log_3 2}$

Solution

$$log_6 x = \frac{log_3 x}{log_3 6}$$
$$=\frac{log_3 x}{log_3 3 + log_3 2}$$
$$=\frac{log_3 x}{1+log_3 2}$$
$$\therefore\ log_6 x = \frac{log_3 x}{1+log_3 2}$$

Example VI (UNEB Question)

Given that $log_3 x = p\ and\ log_{18} x = q$. Prove that $log_6 3 = \frac{q}{p-q}$

Solution

$log_{18} x = q$

$\therefore\ 18^q = x$……………………. (1)

$log_3 x = p$

$\therefore\ 3^p = x$……………………… (2)

Equating equation (1) and (2)

$18^q = 3^p$

$3^p = 18^q$

$log_6 3^p = log_6 18^q$

$plog_6 3 = qlog_6 18$

$plog_6 3 = q(log_6 6 + log_6 3)$

$plog_6 3 = qlog_6 6 + qlog_6 3$

$plog_6 3 - qlog_6 3 = qlog_6 6$

$(p-q)log_6 3 = q$

$log_6 3 = \frac{q}{p-q}$

Example VII

If $log_4 m = a, log_{12} m = b$. Prove that

$$log_3 48 = \frac{a+b}{a-b}$$

Solution

$log_4 m = a$

$4^a = m$

$4 = m^{\frac{1}{a}}$

$log_{12} m = b$

$12^b = m$

$12 = m^{\frac{1}{b}}$

$4\ \times 12 = m^{\frac{1}{a}} \times\ m^{\frac{1}{b}}$

$48 = m^{\frac{1}{a}+\frac{1}{b}}$

$48 = m^{\frac{b+a}{ab}}$

$log_3 48 = log_3 m^{\frac{b+a}{ab}}$

$log_3 48 = \frac{b+a}{ab} log_3 m$ …………. (1)

But $m = 4^a$

Substituting for $m = 4^a$ in Eqn (1);

$$log_3 48 = \frac{b+a}{ab} log_3 4^a$$
$$log_3 48 = \frac{b+a}{ab} \times a log_3 4$$

$log_3 48 = \frac{b+a}{b} log_3 4$……………….. (2)

Since $m = 4^a$, $12^b = m$

$4^a = 12^b$

$log_3 4^a = log_3 12^b$

$alog_3 4 = blog_3 12$

$alog_3 4 = b[log_3 3 + log_3 4]$

$$alog_3 4 = b + blog_3 4$$
$$alog_3 4 - blog_3 4 = b$$
$$log_3 4 = \frac{b}{a-b} \text{……………………….. (3)}$$

Substituting eqn. (3) in eqn. (2)

$$log_3 48 = \left(\frac{b+a}{b}\right) \times \left(\frac{b}{a-b}\right)$$

$$log_3 48 = \frac{b+a}{a-b}$$

Example VIII (UNEB Question)

Prove that $\log_8 x = \frac{2}{3}\log_4 x$. Hence find $\log_8 6$ if $\log_4 3$ = 0.7925.

Solution

$$\log_8 x = \frac{\log_4 x}{\log_4 8} \text{ ………………… (1)}$$

But $\log_4 8 = \frac{\log_2 8}{\log_2 4}$

$$= \frac{\log_2 2^3}{\log_2 2^2}$$

$$= \frac{3\log_2 2}{2\log_2 2}$$

$$\log_4 8 = \frac{3}{2} \text{ ……………………….. (2)}$$

substituting Eqn (2) in Eqn (1);

$$log_8 x = \frac{log_4 x}{3/2}$$

$$log_8 x = 2/3 \, log_4 6$$

$$= 2/3 \, [log_4 2 + log_4 3]$$

But, $log_4 2 = \frac{log_2 2}{log_2 4}$

$$= \frac{log_2 2}{log_2 2^2} = \frac{1}{2}$$

$$log_8 6 = 2/3 \, [0.5 + 0.7925]$$
$$= 0.867$$

Example (UNEB Question)

Solve $\log_x 5 + 4\log_5 x = 4$

Solution

$\log_x 5 + 4\log_5 x = 4$

$$\log_x 5 + 4\log_5 x = 4$$

$$\log_5 x = \frac{1}{\log_x 5}$$

$$\Rightarrow \log_x 5 + \frac{4}{\log_x 5} = 4$$

Let $m = \log_x 5$

$$m + \frac{4}{m} = 4$$

$$\Rightarrow m^2 + 4 = 4m$$
$$m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$m = 2$$
$$\log_x 5 = 2$$
$$x^2 = 5$$
$$x = \sqrt{5}, \; x = -\sqrt{5}$$

Example IX

Solve $6\log_3 x + 6\log_{27} y = 7$

$4\log_9 x + 4\log_3 y = 9$

Solution

$$6log_3 x + 6log_{27} y = 7 \text{……………… (1)}$$
$$4log_9 x + 4log_3 y = 9 \text{……………….. (2)}$$

From equation (1)

$$6log_3 x + \frac{6log_3 y}{log_3 27} = 7$$

$$6log_3 x + \frac{6log_3 y}{3log_3 3} = 7$$

$$6log_3 x + 2log_3 y = 7 \text{……………….. (3)}$$

From equation (2)

$$\frac{4log_3 x}{log_3 9} + 4log_3 y = 9$$

$$\frac{4log_3 x}{2log_3 3} + 4log_3 y = 9$$

$$2log_3 x + 4log_3 y = 9 \text{……………… (4)}$$

Let $A = \log_3 x$, $B = log_3 y$

$$6A + 2B = 7 \text{…………………… (5)}$$
$$2A + 4B = 9 \text{ …………………….. (6)}$$

Solving equation 5 and 6 simultaneously

$$A = \frac{1}{2} \; and \; B = 2$$

But, $A = log_3 x$

$$\therefore \frac{1}{2} = log_3 x$$

$$x = 3^{\frac{1}{2}}$$

$B = \log_3 y$

$2 = \log_3 y$

$y = 3^2$

$y = 9$

$x = 3^{1/2}, \quad y = 9$

Example X

Given that $log_2 x + 2log_4 y = 4$, show that $xy = 16$. Hence solve for x and y in the equations.

$log_2 x + 2log_4 y = 4$

$log_{10} x + y = 1$

Solution

$log_2 x + 2log_4 y = 4$………………….. (1)

From eqn. (1)

$$log_2 x + \frac{2log_2 y}{log_2 4} = 4$$

But $\log_2 4 = 2\log_2 2$

$= 2$

$$\log_2 x + \frac{2\log_2 y}{2} = 4$$

$\log_2 x + \log_2 y = 4$

$\log_2 xy = 4$

$2^4 = xy$

$16 = xy$…………………………. (2)

From

$log_{10} x + y = 1$

$\therefore\ 10^1 = x + y$

$y = 10 - x$ …………………………. (3)

Substituting Eqn (3) in Eqn (2);

Substituting Eqn (3) in Eqn (1)

$16 = x(10 - x)$

$16 = 10x - x^2$

$x^2 - 10x + 16 = 0$

$(x - 2)(x - 8) = 0$

$x = 2, x = 8$

$10 = x + y$

If $x = 2$, $\quad 10 = 2 + y$

$y = 8$

If $x = 8$, $\quad y = 2$

Example XI

Solve for x and y;

$log_2(x - 3y + 2) = 0$

$(\log_2 x + 1) - 1 = 2\log_2 y$

Solution

$log_2(x - 3y + 2) = 0$

$2^0 = x - 3y + 2$

$1 = x - 3y + 2$

$x - 3y = -1$ ……………….……… (1)

$log_2(x + 1) - 1 = 2log_2 y$

$log_2(x + 1) - log_2 2 = 2log_2 y$

$\log_2 2(x + 1) = log_2 y^2$

$y^2 = 2(x + 1)$…………………………. (2)

From eqn. (1)

$\frac{x+1}{3} = y$ ……………… (3)

Substituting Eqn (3) in Eqn (2)

$$\frac{(x + 1)^2}{9} = 2(x + 1)$$

$$\frac{(x + 1)^2}{9} + -2(x + 1) = 0$$

$$(x + 1)\left[\frac{x + 1}{9} - 2\right] = 0$$

$x + 1 = 0 \quad \therefore x = -1$

$$\frac{x + 1}{9} = 2$$

$x + 1 = 18$

$x = 17$

$x = -1, x = 17$

Example

Solve the equation $\log_4 x^2 - 6\log_x 4 - 1 = 0$

Solution

$\log_4 x^2 - 6\log_x 4 - 1 = 0$

But $\log_x 4 = \frac{1}{\log_4 x}$

$$\Rightarrow \log_4 x^2 - \frac{6}{\log_4 x} - 1 = 0$$

$$\Rightarrow 2\log_4 x - \frac{6}{\log_4 x} - 1 = 0$$

$$2y - \frac{6}{y} - 1 = 0$$

$2y^2 - 6 - y = 0$

$(2y + 3)(y - 2) = 0$

$y = \frac{-3}{2}, \ y = 2$

If $y = 2 \Rightarrow \log_4 x = 2$

$4^2 = x$

$x = 16$

If $y = \frac{-3}{2}, \ \Rightarrow \log_4 x = \frac{-3}{2}$

$$4^{-3/2} = x$$

$$x = \frac{1}{8}$$

Application of Indices

Example XIII

Solve the equation $2^{2x+8} - 32(2^x) + 1 = 0$

Solution

$$2^{2x+8} - 32(2^x) + 1 = 0$$

$$(2^x)^2 . 2^8 - 32 . 2^x + 1 = 0$$

$$256(2^x)^2 - 32(2^x) + 1 = 0$$

Let $2^x = y$

$$256y^2 - 32y + 1 = 0$$

$$256y^2 - 16y - 16y + 1 = 0$$

$$16y(16y - 1) - 1(16y - 1) = 0$$

$$(16y - 1)(16y - 1) = 0$$

$$y = \frac{1}{16}$$

$$2^x = \frac{1}{16}$$

$$2^x = \frac{1}{2^4}$$

$$2^x = 2^{-4}$$

$$x = -4$$

Example IV

Solve the following equations

(i) $5^{2x} - 5^{x+1} + 4 = 0$

(ii) $9^x - 12(3^x) + 27 = 0$

(iii) $4^x + 2 = 3 \times 2^x$

(iv)

Solutions

(i) $5^{2x} - 5^{x+1} + 4 = 0$

$(5^x)^2 - 5^x . 5^1 + 4 = 0$

$y^2 - 5y + 4 = 0$

$y^2 - y - 4y + 4 = 0$

$y(y - 1) - 4(y - 1) = 0$

$(y - 4)(y - 1) = 0$

$y = 4 \; and \; y = 1$

$5^x = 4$

$5^x = 1$

For $5^x = 4$,

$$\Rightarrow \log_{10} 5^x = \log_{10} 4$$

$$x = \frac{\log_{10} 4}{\log_{10} 5} = 0.86135$$

For $5^x = 1$, $\Rightarrow$ $5^x = 5^0$

$x = 0$

(ii) $9^x - 12(3^x) + 27 = 0$

$\Rightarrow (3^x)^2 - 12(3^x) + 27 = 0$

[Since $9^x = (3^2)^x = (3^x)^2$]

Let $y = 3^x$

$y^2 - 12y + 27 = 0$

$(y - 3)(y - 9) = 0$

$y = 3, y = 9$

But $y = 3^x$

$3^x = 3$

$\therefore x = 1$

$3^x = 9$

$\therefore x = 2$

(iii) $4^x + 2 = 3 \times 2^x$

$(2^2)^x + 2 = 3 \times 2^x$

$(2^x)^2 + 2 = 3 \; X \; 2^x$

Let $y = 2^x$

$y^2 + 2 = 3y$

$y^2 - 3y + 2 = 0$

$(y - 1)(y - 2) = 0$

$y = 1, y = 2$

$2^x = 1$

$x = 0$

$2^x = 2$

$x = 1$

$\therefore x = 0 \; and \; x = 1$

(iv) $9^x - 3^{x+1} = 10$

$(3^2)^x - 3^x . 3^1 = 10$

$(3^x)^2 - 3^x . 3^1 = 10$

$y^2 - 3y - 10 = 0$

$(y - 5)(y + 2) = 0$

$y = 5 , y = -2$

$3^x = 5$

$x log_{10} 3 = log_{10} 5$

$$x = \frac{\log_{10} 5}{\log_{10} 3}$$

$x = 1.465$

Example (UNEB Question)

Solve the equations:

$9^x - 3^{x+1} = 10$

Solution

$9^x - 3^{x+1} = 10$

$(3^2)^x - 3^x \times 3 = 10$

$(3^x)^2 - 3 \cdot 3^x - 10 = 0$

Let $3^x = y$

$\Rightarrow y^2 - 3y - 10 = 0$

$(y - 5)(y + 2) = 0$

$y = 5, \quad y = -2$

$3^x = 5$

$\Rightarrow \log_{10} 3^x = \log_{10} 5$

$x \log_{10} 3 = \log_{10} 5$

$$x = \frac{\log_{10} 5}{\log_{10} 3}$$

$x = 1.465$

ROOTS OF QUADRATIC EQUATIONS

Any equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$ is called a quadratic equation.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the formula used to solve quadratic equations.

$b^2 - 4ac$ is a discriminant and determines the nature of the roots.

Nature of roots of quadratic Equations

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the formula used to solve the equation $ax^2 + bx + c = 0$, we see that

Either

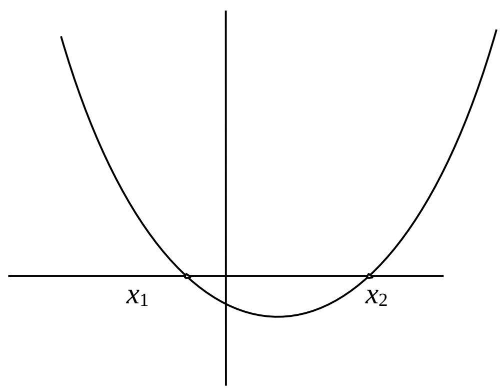
Or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Therefore in general, a quadratic equation has two solutions (called roots).

1. If $b^2 - 4ac$ is positive

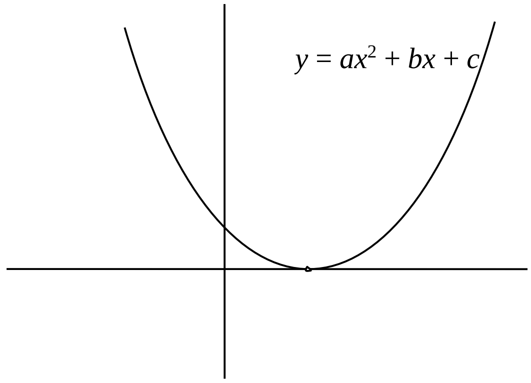
$\sqrt{b^2 - 4ac}$ can be evaluated and the equation has two real and distinct (different) roots.

Illustration



2. If $b^2 - 4ac$ is zero

$\sqrt{b^2 - 4ac}$ is zero; the equation is satisfied by only one value of x; $x = \dfrac{-b}{2a}$ and we say that it has repeated roots or equal roots.



3. If $b^2 - 4ac$ is negative

$\sqrt{b^2 - 4ac}$ has no real values; so the equation has no real roots.

To summarise the equation $ax^2 + bx + c = 0$;

Has two distinct roots if $b^2 - 4ac > 0$
Has equal roots if $b^2 - 4ac = 0$
Has no real roots if $b^2 - 4ac < 0$; and $b^2 - 4ac$ is called a discriminant.

Example I

Determine the nature of roots of the following equations:

(a) $4x^2 - 7x + 3 = 0$
(b) $x^2 + ax + a^2 = 0$
(c) $x^2 - px - q = 0$

Solutions

(a) $4x^2 - 7x + 3 = 0$

Compare $4x^2 - 7x + 3 = 0$ with $ax^2 + bx + c = 0$, it follows that $a = 4$, $b = -7$ and $c = 3$

The discriminant = $b^2 - 4ac$

$b^2 - 4ac$

$\Rightarrow (-7)^2 - 4 \times 4 \times 3$

$= 49 - 48$

$= 1 > 0$

Since $b^2 - 4ac > 0$,

$\Rightarrow$ The equation $4x^2 - 7x + 3 = 0$ has two real distinct roots

(b) $x^2 + ax + a^2 = 0$.

The discriminant is $b^2 - 4ac$

$b^2 - 4ac = (a)^2 - 4 \times 1(a^2)$

$= a^2 - 4a^2$

$= -3a^2$

Since a^2 is positive irrespective of the value of a,

$b^2 - 4ac < 0$. So the equation $x^2 + ax + a^2 = 0$ has no real roots.

(c) $\mathbf{x^2 - px - q^2 = 0}$

Comparing $x^2 - px - q^2 = 0$ with $ax^2 + bx + c = 0$, it follows that $a = 1$, $b = -p$, and $c = -q^2$.

The discriminant is $b^2 - 4ac$

$b^2 - 4ac = (-p)^2 - 4(1)(-q^2)$

$= p^2 + 4q^2$

$p^2 + 4q^2 > 0$ irrespective of the values of p and q.

Since $b^2 - 4ac > 0$,

$\Rightarrow x^2 - px - q^2 = 0$ has two real distinct roots.

Example II

Determine the nature of the roots of the following equations:

(a) $x^2 - 6x + 9 = 0$
(b) $x^2 - 2x + 1 = 0$
(c) $x^2 - 6x + 10 = 0$
(d) $4x^2 - 12x - 9 = 0$

Solution

(a) $\mathbf{x^2 - 6x + 9 = 0}$

Comparing $x^2 - 6x + 9 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -6$, $c = 9$

The discriminant is $b^2 - 4ac$

$= (-6)^2 - 4 \times 1 \times 9$

$= 36 - 36$

$= 0$

Since $b^2 - 4ac = 0$,

$\Rightarrow$ The equation $x^2 - 6x + 9 = 0$ has repeated roots.

(b) $\mathbf{x^2 - 2x + 1 = 0}$

Comparing $x^2 - 2x + 1 = 0$ with $ax^2 + bx + c = 0$, it follows that $a = 1$, $b = -2$, $c = 1$

The discriminant is $b^2 - 4ac$

$= (-2)^2 - 4 \times 1 \times 1$

$= 4 - 4$

$= 0$

Since $b^2 - 4ac = 0$,

$\Rightarrow$ The equation $x^2 - 2x + 1 = 0$ has repeated roots.

(c) $\mathbf{x^2 - 6x + 10 = 0}$

Comparing $x^2 - 6x + 10 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -6$, $c = 10$

The discriminant is $b^2 - 4ac = 0$

$= (-6)^2 - 4 \times 1 \times 10$

$= 36 - 40$

$= -4$

Since $b^2 - 4ac < 0$,

$\Rightarrow$ The equation $x^2 - 6x + 10 = 0$ has no real roots.

(d) $4x^2 - 12x - 9 = 0$

Comparing $4x^2 - 12x - 9 = 0$ with $ax^2 + bx + c = 0$ gives a = 4, b = -12, c = -9.

The discriminant is $b^2 - 4ac$

$= (4)^2 - 4 \times 4 \times (-9)$

$= 16 - 16 \times -9$

$= 16 + 144$

$= 160$

$b^2 - 4ac > 0$

$\Rightarrow$ The equation $4x^2 - 12x - 9 = 0$ has two real distinct roots.

Example III

Find the values of k for which the following equations have equal roots.

(i) $3x^2 + kx + 12 = 0$

(ii) $x^2 - 5x + k = 0$

Solution

(i) $3x^2 + kx + 12 = 0$

For a quadratic equation to have equal roots,

$$b^2 = 4ac$$

Comparing $3x^2 + kx + 12 = 0$ with $ax^2 + bx + c = 0$ gives $a = 3$, $b = k$ and $c = 12$

$b^2 = 4ac$

$\Rightarrow (k)^2 = 4 \times 3 \times 12$

$k^2 = 144$

$k = \pm 12$

$\Rightarrow k = 12, k = -12$

(ii) $x^2 - 5x + k = 0$

For the equation $x^2 - 5x + k = 0$ to have real roots,

$b^2 = 4ac$

$\Rightarrow (-5)^2 = 4 \times 1 \times k$

$25 = 4k$

$k = \frac{25}{4}$

Example IV

Prove that $kx^2 + 2x - (k - 2) = 0$ has real roots for any values of k.

Solution

$k^2 + 2x - (k - 2) = 0$

Comparing $kx^2 + 2x - (k - 2) = 0$ with $ax^2 + bx + c = 0$ gives $a = k$, $b = 2$, $c = -(k - 2)$

The discriminant is $b^2 - 4ac$

$= (2)^2 - 4 \times k[-(k - 2)]$

$= 4 + 4k(k - 2)$

$= 4 + 4k^2 - 8k$

$= 4k^2 - 8k + 4$

$= 4(k^2 - 2k + 1)$
$= 4(k - 1)^2$

Since $4(k - 1)^2 > 0$, $\Rightarrow kx^2 + 2x - (k - 2)$ has two real distinct roots for any values of k.

Example V

Find the range of values k can take for $9x^2 + kx + 4 = 0$ to have two real distinct roots.

Solution

Comparing $9x^2 + kx + 4 = 0$ with $ax^2 + bx + c = 0$ gives $a = 9$, $b = k$, $c = 4$.

The discriminant is $b^2 - 4ac$

$= (k)^2 - 4 \times 9 \times 4$

$= k^2 - 144$

For two distinct real roots, $b^2 - 4ac > 0$

$k^2 - 144 > 0$

$(k + 12)(k - 12) > 0$

For the boundary conditions, $k = -12$, $k = 12$

	$k < -12$	$-12 < k < 12$	$k > 12$
$k + 12$	–ve	+ve	+ve
$k - 12$	–ve	–ve	+ve
$(k + 12)(k - 12)$	+ve	–ve	+ve

$\Rightarrow$ For $9x^2 + kx + 4 = 0$ to have real roots, the product $(k + 12)(k - 12)$ must be positive.

$\Rightarrow k < -12$ and $k > 12$ are ranges of values for which $9x^2 + kx + 4 = 0$ has real distinct roots.

Example VI (UNEB Question)

Find the value of k for which the equation $\dfrac{x^2 - x + 1}{x - 1} = k$ has repeated roots. What are the repeated roots?

Solution

$$\frac{x^2 - x + 1}{x - 1} = k$$

$x^2 - x + 1 = kx - k$

$x^2 - x - kx + 1 + k = 0$

$x^2 - (k + 1)x + k + 1 = 0$

For repeated roots, $b^2 = 4ac$

Comparing $x^2 - (k + 1)x + k + 1 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -(k + 1)$, $c = k + 1)$

$b^2 = 4ac$

$[-(k + 1)]^2 = 4 \times 1(k + 1)$

$(k + 1)^2 = 4k + 4$

$k^2 + 2k + 1 = 4k + 4$

$k^2 - 2k - 3 = 0$

$(k - 3)(k + 1) = 0$

$\Rightarrow k = 3$ **OR** $k = -1$

If $k = 3$,

$x^2 - (k + 1)x + k + 1 = 0$

$x^2 - 4x + 4 = 0$

$(x - 2)^2 = 0$

$x = 2, x = 2$

When $x = 3$, the repeated roots are x=2 and $x = 2$

If $x = -1$;

$x^2 - (k + 1)x + k + 1 = 0$

$x^2 = 0$

$x = 0, x = 0$

When k= -1, the repeated roots are x=0 and x =0.

Maximum and Minimum values of a quadratic function

Consider $y = ax^2 + bx + c$

Using the method of completing squares, the quadratic equation can be reduced to:

(i) $a(x - p)^2 + q$

(ii) $q - a(x - p)^2$

(i) Let $y = a(x - p)^2 + \text{q}$

Since $(x - p)^2$ is never negative, the least value of y occurs when $(x - p)^2 = 0$

(ii) For $y = q - a(x - p)^2$;

Since $(x - p)^2$ is never negative

$\Rightarrow$ The maximum value of y is q.

Examples

Find the greatest or least values of the following functions:

(a) $x^2 - 2x + 5$

(b) $5 - 4x - x^2$

(c) $x^2 - 3x + 5$

(d) $2x^2 - 4x + 5$

(e) $7 + x - x^2$

(f) $x^2 - 2$

(g) $2x - x^2$

Solution

(a) $\mathbf{x^2 - 2x + 5}$

By completing squares,

$x^2 - 2x + 5 = x^2 - 2x + \left[\tfrac{1}{2}(-2)\right]^2 - \left[\tfrac{1}{2}(-2)\right]^2 + 5$

$= x^2 - 2x + 1 - 1 + 5$

$= (x - 1)^2 + 4$

$y = x^2 - 2x + 5$

$y = 4 + (x - 1)^2$

The least value of y is 4 and it occurs when $(x–1)^2$ =0

(b) $\mathbf{5 - 4x - x^2}$

By completing squares;

$5 - 5x - x^2 = 5 - (x^2 + 4x)$

$= 5 - (x^2 + 4x + 4) - {}^-4$

$= 5 - (x + 2)^2 + 4$

$= 9 - (x + 2)^2$

$y = 9 - (x + 2)^2$

The greatest value is $y = 9$ and it occurs when

$(x + 2)^2 = 0$

(c) $\mathbf{x^2 - 3x + 5}$

By completing squares;

$$x^2 - 3x + 5 = x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 5$$

$$= \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}$$

$$y = \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}$$

The least value of y is $\frac{11}{4}$ and it occurs when $\left(x - \frac{3}{2}\right)^2 = 0$

(d) $\mathbf{2x^2 - 4x + 5}$

$2(x^2 - 2x) + 5$

$2(x^2 - 2x + 1) - 2 + 5$

$3 + 2(x - 1)^2$

$y = 3 + 2(x - 1)^2$

The least value of y is 3 and it occurs when

$2(x - 1)^2 = 0$

(e) $\mathbf{7 + x - x^2}$

$7 - (x^2 - x)$

$$7 - (x^2 - x + \frac{1}{4}) - \frac{-1}{4}$$

$$7 - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$$

$$y = \frac{29}{4} - \left(x - \frac{1}{2}\right)^2$$

The greatest value of y is $\frac{29}{4}$ and it occurs when $\left(x - \frac{1}{2}\right)^2 = 0$

(f) $\mathbf{x^2 - 2}$

The least value of y is -2 and it occurs when $x^2 = 0$

(g) $\mathbf{2x - x^2}$

$y = 2x - x^2$

$y = -(x^2 - 2x)$

By completing squares;

$y = -(x^2 - 2x + 1) - {}^{-}1$

$y = -(x - 1)^2 + 1$

$y = 1 - (x - 1)^2$

The greatest value of $y = 1$ and it occurs when $x = 1$

Sum & Product of the roots of Quadratic Equations

Consider the equation $ax^2 + bx + c = 0$

$\Rightarrow x^2 + \frac{bx}{a} + \frac{c}{a} = 0$ …………………… (i)

Suppose α and β are the roots of the equation

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

We can use α and β to form an algebraic equation in which the unknown quantity x is satisfied by putting $x = \alpha$ or $x = \beta$.

$x = \alpha$ or $x = \beta$

$x - \alpha = 0$ or $x - \beta = 0$

$(x - \alpha)(x - \beta) = 0$

$x^2 - \beta x - \alpha x + \alpha\beta = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ ……………………… (ii)

Eqn (i) and Eqn (ii) have the same roots, must be precisely the same equation written in two different ways.

Equating coefficients of the same monomials in Eqn (i) and Eqn (ii);

$\Rightarrow -(\alpha + \beta) = \frac{b}{a}$

$(\alpha + \beta) = \frac{-b}{a}$

Similarly, $\alpha\beta = \frac{c}{a}$

$\Rightarrow$ For a quadratic function with roots α and β,

Sum of roots $= \alpha + \beta = \frac{-b}{a}$

Product of roots $\alpha\beta = \frac{c}{a}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$

$\alpha^3 + \beta^3 = (\alpha\beta)^3$

$(\alpha - \beta) = \sqrt{(\alpha-\beta)^2}$

$$\Rightarrow \alpha - \beta = \sqrt{\alpha^2 - 2\alpha\beta + \beta^2}$$

$$\alpha - \beta = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta}$$

$$\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 2\alpha\beta - 2\alpha\beta}$$

$$(\alpha - \beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}.$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$

$$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta}$$

⇒ **The following are important formulae used under roots of quadratic equations.**

$$(\alpha^2 + \beta^2) = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$

Example I

If α and β are roots of the equation $x^2 + 8x + 1 = 0$, find the values of

(a) $\alpha\beta$

(b) $\alpha + \beta$

(c) $\alpha^2\beta + \alpha\beta^2$

(d) $\alpha^2 + \beta^2$

Solution

(a) $x^2 + 8x + 1 = 0$

Comparing $x^2 + 8x + 1 = 0$ with $ax^2 + bx + c = 0$ gives

$a = 1, b = 8, c = 1$

$$\alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{1}{1} = 1$$

(b) $\alpha + \beta = \frac{-b}{a}$.

$$\alpha + \beta = \frac{-8}{1}$$

$$\alpha + \beta = -8$$

(c) $\alpha^2\beta + \alpha\beta^2$

$$= \alpha\beta(\alpha + \beta)$$

$= 1(-8)$

$= -8$

$\therefore \alpha^2\beta + \alpha\beta^2 = -8$

(d) $\boldsymbol{\alpha^2 + \beta^2 = (\alpha + \beta)}^2 - 2\alpha\beta$

$\alpha + \beta = -8$

$\alpha\beta = 1$

$\alpha^2 + \beta^2 = (-8)^2 - 2 \times 1$

$= 64 - 2$

$= 62$

Example II

If α and β are roots of the equation $x^2 - x - 3 = 0$, state the values of $\alpha + \beta$ and $\alpha\beta$ and find the values of:

(a) $\alpha^2 + \beta^2$

(b) $(\alpha - \beta)^2$

(c) $\alpha^3 + \beta^3$

Solution

Comparing $x^2 - x - 3 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -1$, $c = -3$

$\alpha + \beta = \frac{-b}{a}$

$\Rightarrow \alpha + \beta = \frac{-^-1}{1} = 1$

$\alpha\beta = \frac{c}{a}$

$\alpha\beta = \frac{-3}{1} = -3$

(a) $\alpha^2 + \beta^2$

$(\alpha + \beta)^2 - 2\alpha\beta$

$= (1)^2 - 2(-3)$

$= 1 - {}^-6$

$= 7$

(b) $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

$= \alpha^2 + \beta^2 - 2\alpha\beta$

But $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (\alpha + \beta)^2 - 4\alpha\beta$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= 1^2 - 4 \times (-3)$

$= 13$

(c) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha^3 + \beta^3 = (1)^3 - 3(-3)(1)$

$= 1 + 9$

$= 10$

Example III

If α and β are roots of the equation $2x^2 - 5x + 1 = 0$, find the values of:

(a) $\boldsymbol{\alpha + \beta}$

(b) $\alpha\,\beta$

(c) $\alpha^2 + 3\alpha\beta + \beta^2$

(d) $\alpha^2 - 3\alpha\beta + \beta^2$.

(e) $\alpha^3\beta + \alpha\beta^3$

(f) $\frac{1}{\beta} + \frac{1}{\alpha}$

Solution

Comparing $2x^2 - 5x + 1 = 0$ with $ax^2 + bx + c = 0$ gives

$a = 2$, $b = -5$ and $c = 1$

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = -\frac{-5}{2}$$

$$\Rightarrow \alpha + \beta = \frac{5}{2}$$

$$\alpha\,\beta = \frac{c}{a}$$

$$\Rightarrow \alpha\,\beta = \frac{1}{2}$$

(c) $\boldsymbol{\alpha^2 + 3\alpha\beta + \beta^2}$

$$= (\alpha^2 + \beta^2) + 3\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta + 3\alpha\beta$$

$$= (\alpha + \beta)^2 + \alpha\beta$$

$$= \left(\frac{5}{2}\right)^2 + \frac{1}{2}$$

$$= \frac{25}{4} + \frac{1}{2}$$

$$= \frac{27}{4}$$

(d) $\boldsymbol{\alpha^2} - 3\alpha\beta + \beta^2$

$$\alpha^2 + \beta^2 - 3\alpha\beta$$

$$(\alpha + \beta)^2 - 2\,\alpha\beta - 3\alpha\beta$$

$$(\alpha + \beta)^2 - 5\alpha\beta$$

$$= \left(\frac{5}{2}\right)^2 - 5\left(\frac{1}{2}\right)$$

$$= \frac{25}{4} - \frac{5}{2}$$

$= \dfrac{15}{4}$

(d) $\alpha^3 \beta + \alpha\beta^3$

$= \alpha \beta (\alpha^2 + \beta^2)$

$= \alpha \beta[(\alpha + \beta)^2 - 2\alpha\beta]$

$= \dfrac{1}{2}\left[\left(\dfrac{5}{2}\right)^2 - 2\times\dfrac{1}{2}\right]$

$= \dfrac{1}{2}\left(\dfrac{5}{4} - 1\right) = \dfrac{1}{8}$

(f) $\dfrac{1}{\beta} + \dfrac{1}{\alpha}$

$= \dfrac{\alpha + \beta}{\alpha\beta}$

$= \dfrac{5/2}{1/2}$

$= 5$

Example IV

If α and β are roots of the equation $6x^2 + 2x - 3 = 0$, find the values of:

(a) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ (b) $\dfrac{1}{\beta^2} + \dfrac{1}{\alpha^2}$

(c) $\dfrac{2\alpha\beta}{1 + \frac{\alpha}{\beta}}$ (d) $\dfrac{1}{\alpha\beta} - \dfrac{1}{\alpha} - \dfrac{1}{\alpha}$

(e) $\alpha^3 + \beta^3$ (f) $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$

Solution

Comparing $6x^2 + 2x - 3 = 0$ with $ax^2 + bx + c = 0$,

gives $a = 6$, $b = 2$, $c = -3$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$\frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = \frac{-2}{6} = \frac{-1}{3}$$

$$\alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{-3}{6} = \frac{-1}{2}$$

$$\frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{(\frac{-1}{3})^2 - 2(\frac{1}{2})}{{}^{-1}\!/_2}$$

$$= \frac{{}^1\!/_9 + 1}{{}^{-1}\!/_2}$$

$$= \frac{-20}{9}$$

$$\Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{-20}{9}$$

(b) $$\frac{1}{\beta^2} + \frac{1}{\alpha^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{({}^{-1}\!/_3)^2 - 2({}^{-1}\!/_2)}{({}^{-1}\!/_2)^2}$$

$$= \frac{\frac{1}{9} + 1}{{}^1\!/_4}$$

$$= \frac{{}^{10}\!/_9}{{}^1\!/_4} = \frac{40}{9}$$

$$\frac{1}{\beta^2} + \frac{1}{\alpha^2} = \frac{40}{9}$$

(c) $$\frac{2\beta}{1 + {}^\beta\!/_\alpha} = \frac{2\beta}{\frac{\alpha+\beta}{\alpha}}$$

$$= \frac{2\alpha\beta}{\alpha+\beta}$$

$$= \frac{2(\frac{-1}{2})}{\frac{-1}{3}}$$

$$= \frac{-1}{{}^{-1}\!/_3} = 3$$

$$\Rightarrow \frac{2\beta}{1 + \frac{\beta}{\alpha}} = 3$$

(c) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\,\alpha\,\beta(\alpha + \beta)$

$$= \left(\frac{-1}{3}\right)^3 - 3\left(\frac{-1}{2}\right)\left(\frac{-1}{3}\right)$$

$$= \frac{-1}{27} - \frac{1}{2}$$

$$= \frac{-29}{54}$$

(f) $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}$

$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$$

$$= \frac{(\frac{-1}{3})^3 - 3(\frac{-1}{2})(\frac{-1}{3})}{(\frac{-1}{2})^3}$$

$$= \frac{\frac{-1}{27} - \frac{1}{2}}{-1/8} = \frac{\frac{-29}{54}}{-1/8}$$

$$= \frac{116}{27}$$

Example V

If α^2 and β^2 are roots of the equation $x^2 - 21x + 4 = 0$, and α and β are both positive, find $\alpha\,\beta$ and $\alpha + \beta$.

Solution

Comparing $x^2 - 21x + 4 = 0$ with $ax^2 + bx + c = 0$ gives

$a = 1$, $b = -21$, $c = 4$

$$\alpha^2 + \beta^2 = \frac{-b}{a}$$

$$= \frac{21}{1}$$

$$= 21$$

$\alpha^2\,\beta^2 = 4$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\,\alpha\,\beta$

$(\alpha + \beta)^2 - 2\alpha\beta = 21$

$(\alpha + \beta)^2 - 2 \times 2 = 21$

$(\alpha + \beta)^2 = 25$

$(\alpha + \beta) = 5$

Example VI

Write down the equation whose roots are:

(a) 3, 4 (b) -2, ½

(c) $\frac{1}{3}, \frac{-2}{5}$ (d) $\frac{-1}{4}, 0$

(e) a^2, a^2 (f) $-(k + 1), k^2 - 3$

(g) $\frac{b}{a}$, $\frac{c^2}{b}$

Solution

Any quadratic equation is given by
x^2 – (sum of roots)x + product of roots = 0

(a) $x = 3, 4$

Sum of roots = 3 + 4
= 7

Product of roots = 3 × 4
= 12

x^2 – (sum of roots)x + product = 0
$x^2 - (7)x + 12 = 0$
$x^2 - 7x + 12 = 0$

(b) $x = -2, x = \frac{1}{2}$

Sum of roots = $-2 + \frac{1}{2}$

Sum of roots = $\frac{-3}{2}$

Product of roots = -1

x^2 – (sum of roots)x + product = 0

$x^2 - \left(\frac{-3}{2}\right)x + -1 = 0$

$x^2 + \frac{3}{2}x - 1 = 0$

$2x^2 + 3x - 2 = 0$

(c) $x = \frac{1}{3}, x = \quad x = \frac{-2}{5}$

Sum of the roots = $\frac{1}{3} + \frac{-2}{5}$

$= \frac{-1}{15}$

Product of the roots = $\frac{1}{3} \times \frac{-2}{5}$

$= \frac{-2}{15}$

x^2 – (sum of roots)x + product of roots = 0

$x^2 - \left(\frac{-1}{5}\right)x + \frac{-2}{15} = 0$

$15x + x - 2 = 0$

(d) $x = \frac{-1}{4}$, $x = 0$

Sum of roots = $\frac{-1}{4} + 0 = \frac{-1}{4}$

Product of roots = 0

x^2 – (sum of roots)x + product of roots = 0

$x^2 - \frac{-1}{4}x = 0$

$4x^2 + x = 0$

(e) $x = a^2$, $x = a^2$

Sum of the roots $= a^2 + a^2$

$= 2a^2$

Product of the roots $= a^2 \times a^2$

$= a^4$

x^2 – (sum of roots)x + product of roots = 0

$x^2 - (2a^2)x + a^4 = 0$

$x^2 - 2a^2x + a^4 = 0$

(f) $-(k + 1)$, $k^2 - 3$

Sum of roots $= -k + 1 + k^2 - 3$

$= k^2 - k - 2$

Product of roots $= -(k + 1)(k^2 - 3)$

$= -(k^3 - 3k + k^2 - 3)$

Product of roots $= -k^3 + 3k - k^2 + 3$

x^2 – (sum of roots)x + product of roots = 0

$x^2 - (k^2 - k - 2)x + 3k + 3 - k^2 - k^3 = 0$

(g) $x = \frac{b}{a}$, $x = \frac{c^2}{b}$

Sum of roots $= \frac{b}{a} + \frac{c^2}{b}$

$= \frac{b^2 + ac^2}{ab}$

Product of the roots $= \frac{b}{a} \times \frac{c^2}{b} = \frac{c^2}{a^2}$

$$x^2 - \left(\frac{b^2 + ac^2}{ab}\right)x + \frac{c^2}{a^2} = 0$$

$a^2bx^2 - a(b^2 + a^2c)x + c^2 = 0$

Example VII

The roots of the equation $x^2 - 2x + 3 = 0$ are α and β. Find the equation whose roots are:

(a) $\alpha + 2,\ \beta + 2$

(b) $\frac{1}{\beta},\ \frac{1}{\alpha}$

(c) $\alpha^2,\ \beta^2$

(d) $\frac{\alpha}{\beta},\ \frac{\beta}{\alpha}$

Solution

$x^2 - 2x + 3 = 0$

Comparing $x^2 - 2x + 3 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -2$, and $c = 3$.

$\alpha + \beta = \frac{-b}{a} = \frac{-(-2)}{1} = 2$

$\alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$

New sum of roots $= \alpha + 2 + \beta + 2$

$= \alpha + \beta + 4$

$= 2 + 4$

$= 6$

New product of the roots $= (\alpha + 2)(\beta + 2)$

$= \alpha\beta + 2\alpha + 2\beta + 4$

$= \alpha\beta + 2(\alpha + \beta) + 4$

$= 3 + 2(2) + 4$

$= 3 + 4 + 4$

$= 11$

Any quadratic equation is given by:

$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

$x^2 - (2)x + 11 = 0$

$x^2 - 2x + 11 = 0$

(b) $\frac{1}{\alpha},\ \frac{1}{\beta}$

$\alpha + \beta = 2$

$\alpha\beta = 3$

New sum of roots $= \frac{1}{\alpha} + \frac{1}{\beta}$

$= \frac{\alpha + \beta}{\alpha\beta} = \frac{2}{3}$

New product of roots $= \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta}$

$= \frac{1}{3}$

$x^2 - (\text{sum of the roots})x + \text{product of roots} = 0.$

$$x^2 - \left(\frac{2}{3}\right)x + \frac{1}{3} = 0$$

$3x^2 - 2x + 1 = 0$

(c) α^2, β^2

$\alpha + \beta = 2$

$\alpha\beta = 3$

New sum of the roots $= \alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= 2^2 - 2(3)$

$= 4 - 6$

$= -2$

New product of roots $= \alpha^2\beta^2$

$= (\alpha\beta)^2$

$= 3^2$

$= 9$

x^2 – (sum of the roots)x +product of the roots = 0

$x^2 - (-2)x + 9 = 0$

$x^2 + 2x + 9 = 0$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

$\alpha + \beta = 2$; $\alpha\beta = 3$

New sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2 - \alpha\beta}{\alpha\beta}$$

$$= \frac{2^2 - 2\times 3}{3} = \frac{4-6}{3}$$

$$= \frac{-2}{3}$$

New product of roots $= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha}$

$= 1$

x^2 – (sum of roots)x + product of roots = 0

$$x^2 - \left(\frac{-2}{3}\right)x + 1 = 0$$

$3x^2 + 2x + 3 = 0$

(e) $\alpha - \beta, \ \beta - \alpha$

$\alpha + \beta = 2, \quad \alpha\beta = 3$

New sum of the roots = $\alpha - \beta + \beta - \alpha$

$= \alpha + \beta - (\alpha + \beta)$

$= 0$

New product of the roots = $(\alpha - \beta)(\beta - \alpha)$

$= \alpha\beta - \alpha^2 - \beta^2 + \alpha\beta$

$= 2\alpha\beta - [(\alpha^2 + \beta^2)]$

$= 2\alpha\beta - [(\alpha + \beta)^2 - 2\alpha\beta]$

$= 4\alpha\beta - (\alpha + \beta)^2$

$= 4 \times 3 - (2^2)$

$= 8$

x^2 – (sum of the roots)x + product of roots = 0

$x^2 - (0)x + 8 = 0$

$x^2 + 8 = 0$

Example VIII

The roots of the equation $2x^2 + 7x - 3 = 0$ are α and β. Find the equation whose roots are $\left(\alpha + \frac{5}{\beta}\right)$ and $\left(\beta + \frac{5}{\alpha}\right)$.

Solution

Sum of the roots = $\alpha + \beta = \dfrac{-b}{a}$

$= \dfrac{-7}{2}$

Product of the roots = $\alpha\,\beta = \dfrac{c}{a} = \dfrac{-3}{2}$

New sum of the roots = $\alpha + \dfrac{5}{\beta} + \beta + \dfrac{5}{\alpha}$

$$= \alpha + \beta + 5\left(\frac{1}{\beta} + \frac{1}{\alpha}\right)$$

$$= \alpha + \beta + 5\left(\frac{\alpha + \beta}{\alpha\beta}\right)$$

$$= \frac{-7}{2} + 5\left(\frac{-7/2}{-3/2}\right)$$

$$= \frac{-7}{2} + \frac{-35}{3}$$

$$= \frac{-91}{6}$$

New product of roots = $\left(\alpha + \dfrac{5}{\beta}\right)\left(\beta + \dfrac{5}{\alpha}\right)$

$= \alpha\beta + 5 + 5 + \frac{25}{\alpha\beta}$

$= \alpha\beta + \frac{25}{\alpha\beta} + 10$

$= \frac{-3}{2} + \frac{25}{-3/2} + 10$

$= \frac{-49}{10}$

x^2 – (sum of roots)x + product of roots = 0

$x^2 - \left(\frac{-91}{6}\right)x + \frac{-49}{6} = 0$

$6x^2 + 91x - 49 = 0$

Example IX

Given that α and β are roots of the equation
$4x^2 + 7x - 5 = 0$. Find the equation whose roots are $2\alpha - 1$ and $2\beta - 1$

Solution

Comparing $4x^2 + 7x - 5 = 0$ with $ax^2 + bx + c = 0$ gives
$a = 4$, $b = 7$, and $c = -5$

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$

$\Rightarrow \alpha + \beta = \frac{-7}{4}$

Product of the roots = $\alpha\beta = \frac{c}{a}$

$= \frac{-5}{4}$

New sum of roots $= 2\alpha - 1 + 2\beta - 1$

$= 2(\alpha + \beta) - 2$

$= 2\left(\frac{-7}{4}\right) - 2$

$= \frac{-14}{4} - 2$

$= \frac{-22}{4} = \frac{-11}{2}$

New product of roots = $(2\alpha - 1)(2\beta - 1)$

$= 4\alpha\beta - 2\alpha - 2\beta + 1$

$= 4\alpha\beta - 2(\alpha + \beta) + 1$

$= 4\left(\frac{-5}{4}\right) - 2\left(\frac{-7}{4}\right) + 1$

$= -5 + \frac{14}{4} + 1$

$$= \frac{-20 + 14 + 4}{4}$$

$$= \frac{-1}{2}$$

x^2 – (sum of roots)x + product of roots = 0

$$x^2 - \left(\frac{-11}{2}\right)x + \frac{1}{2} = 0$$

$x^2 + 11x - 1 = 0$

Example (UNEB Question)

If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$.

Solution

Comparing $x^2 + 2x + 3 = 0$ with $ax^2 + bx + c = 0$ gives

$a = 1$, $b = 2$ and $c = 3$

Sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of the roots = $\frac{c}{a}$

$$\Rightarrow \alpha + \beta = \frac{-2}{1} = -2$$

$$\Rightarrow \alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$$

New sum of the roots = $\alpha^2 - \beta + \beta^2 - \alpha$

$= \alpha^2 + \beta^2 - (\alpha + \beta)$

$= (\alpha + \beta)^2 - 2\alpha\beta - (\alpha + \beta)$

$= (-2)^2 - 2(3) - (-2)$

$= 4 - 6 + 2$

$= 0$

New product of roots = $(\alpha^2 - \beta)(\beta^2 - \alpha)$

$= \alpha^2\beta^2 - \alpha^3 - \beta^3 + \alpha\beta$

$= (\alpha\beta)^2 - (\alpha^3 + \beta^3) + \alpha\beta$

$= (\alpha\beta)^2 - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)] + \alpha\beta$

$= 3^2 - [(-2)^3 - 3(3)(-2)] + 3$

$= 9 - [-8 + 18] + 3$

$= 9 - [10] + 3$

$= 2$

x^2 – (sum of the roots)x + product of roots = 0

$x^2 - (0)x + 2 = 0$

$x^2 + 2 = 0$

Example (UNEB Question)

If α and β are roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3 - 1}{\alpha}$ and $\frac{\beta^3 - 1}{\beta}$

Solution

Comparing $x^2 – px + q = 0$ with $ax^2 + bx + c = 0$ gives

$a = 1$, $b = -p$, $c = q$

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$

$$= \frac{-(-p)}{1} = p$$

Product of the roots = $\alpha\beta = \frac{c}{a} = \frac{q}{1} = 1$

New sum of the roots = $\frac{\alpha^3 - 1}{\alpha} + \frac{\beta^3 - 1}{\beta}$

$$= \frac{\alpha^3\beta - \beta + \alpha\beta^3 - \alpha}{\alpha\beta}$$

$$= \frac{-(\alpha+\beta) + \alpha\beta(\alpha^2 + \beta^2)}{\alpha\beta}$$

$$= \frac{-(\alpha+\beta) + \alpha\beta[(\alpha+\beta)^2 - 2\alpha\beta]}{\alpha\beta}$$

$$= \frac{-(p) + q[p^2 - 2q]}{q}$$

$$= \frac{-p + p^2q - 2q^2}{q}$$

New product of the roots = $\left(\frac{\alpha^3 - 1}{\alpha}\right)\left(\frac{\beta^3 - 1}{\beta}\right)$

$$= \frac{\alpha^3\beta^3 - \alpha^3 - \beta^3 + 1}{\alpha\beta}$$

$$= \frac{(\alpha\beta)^3 - (\alpha^3 + \beta^3) + 1}{\alpha\beta}$$

$$= \frac{(\alpha\beta)^3 - [(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)] + 1}{\alpha\beta}$$

$$= \frac{q^3 - [p^3 - 3q(p)] + 1}{q}$$

$$= \frac{q^3 - p^3 + 3pq + 1}{q}$$

x^2 – (sum of the roots)x + product of the roots = 0

$$x^2 - \frac{(p^2q - p - 2q^2)}{q}x + \frac{q^3 - p^3 + 3pq + 1}{q} = 0$$

$$qx^2 – (p^2q – p – 2q^2)x + q^3 – p^3 + 3pq + 1 = 0$$

Example **(UNEB Question)**

If α and β are roots of the equation $ax^2 + bx + c = 0$, express $(\alpha - \beta)(\beta - 2\alpha)$ in terms of a, b and c. Hence deduce the condition for the root to be twice the other.

Solution

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

$$(\alpha - 2\beta)(\beta - 2\alpha)$$

$$\alpha\beta - 2\alpha^2 - 2\beta^2 + 4\alpha\beta$$

$$\alpha\beta - 2(\alpha^2 + \beta^2) + 4\alpha\beta$$

$$= \alpha\beta - \alpha[(\alpha+\beta)^2 - 2\alpha\beta] + 4\alpha\beta$$

$$= 5\alpha\beta - 2[(\alpha+\beta)^2 - 2\alpha\beta]$$

$$= \frac{5c}{a} - 2\left[\left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right]$$

$$= \frac{5c}{a} - 2\left[\frac{b^2}{a^2} - \frac{2c}{a}\right]$$

$$= \frac{5c}{a} - 2\left(\frac{b^2 - 2ac}{a^2}\right)$$

$$= \frac{5ac - 2b^2 + 4ac}{a^2}$$

$$= \frac{9ac - 2b^2}{a^2}$$

$$\Rightarrow (\alpha - 2\beta)(\beta - 2\alpha)$$

For one root to be twice the other, $(\alpha - 2\beta)(\beta - 2\alpha) = 0$

$$\Rightarrow = \frac{9ac - 2b^2}{a^2} = 0$$

$$9ac = 2b^2$$

Example (UNEB Question)

Given that α and β are roots of the equation $x^2 + px + q = 0$, express $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q. Deduce that for one root to be a square of another root, $p^3 - 3pq + q^2 + q = 0$

Solution

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$

$$\alpha + \beta = -p$$

$$\alpha\beta = q$$

$$(\alpha + \beta^2)(\beta - \alpha^2)$$

$$= \alpha\beta - \alpha^3 - \beta^3 + (\alpha\beta)^2$$

$$= \alpha\beta - \left[(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\right] + (\alpha\beta)^2$$

$$= q - [(-p^3 - 3q(-p)] + q^2$$

$$= q + p^3 - 3pq + q^2$$

For one root to be a square of the other,

$(\alpha - \beta^2)(\beta - \alpha^2) = 0$

$\alpha = \beta^2, \quad \beta = \alpha^2$

$(\alpha - \beta^2)(\beta - \alpha^2) = q + p^3 - 3pq + q^2$

$\Rightarrow p^3 - 3pq + q + q^2 = 0$

Example (UNEB Question)

Given that α and β are roots of the equation $ax^2 + bx + c = 0$, determine the equation whose roots are $\alpha + \beta$ and $\alpha^3 + \beta^3$.

Solution

Sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of roots = $\alpha\beta = \frac{c}{a}$

New sum of roots = $(\alpha + \beta) + \alpha^3 + \beta^3$

$$= (\alpha+\beta) + \left[(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\right]$$

$$= \frac{-b}{a} + \left[\frac{-b^3}{a^3} - \frac{3c}{a}\left(\frac{-b}{a}\right)\right]$$

$$= \frac{-b}{a} + \left[\frac{-b^3}{a^3} + \frac{3bc}{a^2}\right]$$

$$= \frac{-b}{a} + \left[\frac{3abc - b^3}{a^3}\right]$$

$$= \frac{-a^2b + 3abc - b^3}{a^3}$$

$$= -\left(\frac{a^2b + b^3 - 3abc}{a^3}\right)$$

New product of roots = $(\alpha + \beta)(\alpha^3 + \beta^3$

$= (\alpha + \beta)[(\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)]$

$$= \frac{-b}{a}\left[\frac{-b^3}{a^3} - \frac{3c}{a}\left(\frac{-b}{a}\right)\right]$$

$$= \frac{-b}{a}\left[\frac{-b^3}{a^3} + \frac{3cb}{a^2}\right]$$

$$= \frac{-b}{a}\left[\frac{-b^3 + 3abc}{a^3}\right]$$

$$= \frac{-3ab^2c + b^4}{a^4}$$

$$= \frac{b^4 - 3ab^2c}{a^4}$$

$x^2 -$ (sum of roots)x + product of roots = 0

$$x^2 + \left(\frac{a^2b + b^3 - 3abc}{a^3}\right)x + \frac{b^4 - 3ab^2c}{a^4} = 0$$

$$a^4x^2 + (a^2b + b^3 - 3abc)x + b^4 + 3ab^2c = 0$$

Example (UNEB Question)

Given that equations $y^2 + py + q = 0$ and $y^2 + my + k = 0$ have a common root, show that $(q - k)^2 = (m - p)(pk - mq)$

Solution

Let the common root be α.

$\Rightarrow \alpha^2 + p\alpha + q = 0$ ……………………. (i)

$\alpha^2 + m\alpha + k = 0$ …………………… (ii)

Eqn (i) – Eqn (ii);

$\Rightarrow (p - m)\alpha + q - k = 0$

$-(m - p)\alpha + q - k = 0$

$\alpha = \frac{q-k}{m-p}$ ……………………….. (iii)

Substituting Eqn (iii) in Eqn (i);

$$\frac{(q-k)^2}{(m-p)^2} + p\left(\frac{q-k}{m-p}\right) + q = 0$$

$(q - k)^2 + p(m - p)(q - k) + q(m - p)^2 = 0$

$(q - k)^2 + (m - p)(pq - pk + qm - pq) = 0$

$(q - k)^2 + (m - p)(qm - pk) = 0$

$(q - k)^2 - (m - p)(pk - qm) = 0$

$(q - k)^2 = (m - p)(pk - qm)$

Example

If α and β are roots of $px^2 + qx + r = 0$, form an equation with algebraic integral coefficients whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$

Solution

$\alpha + \beta = \frac{-b}{a}$

$\Rightarrow \alpha + \beta = \frac{-q}{p}$

$\alpha\beta = \frac{c}{a}$

$\Rightarrow \alpha\beta = \frac{r}{p}$

New sum of the roots = $\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta}$

$$= \frac{(1-\alpha)(1+\beta)+(1-\beta)(1+\alpha)}{(1+\alpha)(1+\beta)}$$

$$= \frac{1+\beta-\alpha-\alpha\beta+1+\alpha-\beta-\alpha\beta}{1+\beta+\alpha+\alpha\beta}$$

$$= \frac{2-2\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{2-\frac{2r}{p}}{1+\frac{-q}{p}+\frac{r}{p}}$$

$$= \frac{2p-2r}{p-q+r}$$

$$= \frac{2(p-r)}{p-q+r}$$

New product of the roots = $\frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)}$

$$= \frac{1-\beta-\alpha+\alpha\beta}{1+\beta+\alpha+\alpha\beta}$$

$$= \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{1-(\frac{-q}{p})+\frac{r}{p}}{\frac{p-q+r}{p}}$$

$$= \frac{p+q+r}{p-q+r}$$

$$x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0$$

$$x^2 - \frac{2(p-r)x}{p-q+r} + \frac{p+q+r}{p-q+r} = 0$$

$$(p-q+r)x^2 - 2(p-r)x + p + q + r = 0$$

Revision Exercise

1. The roots of the equation $4x^2 + 4x - 1 = 0$ are α and β. Find the values of: (a) $\frac{1}{\alpha} + \frac{1}{\beta}$

   (b) $\alpha^2 + \beta^2$
2. If α and β are roots of the equation $3x^2 - 6x + 2 = 0$. Find
   (a) $\alpha^2 - 3\alpha\beta + \beta^2$
   (b) $\alpha^3\beta + \alpha\beta^3$
   (c) $\frac{1}{\alpha} + \frac{1}{\beta}$
3. If α and β are roots of the quadratic equation $x^2 - 2x - 5 = 0$. Find the quadratic equation whose roots are:
   (a) $\alpha - 4,\ \beta - 4$
   (b) $\frac{1}{\alpha + 2}, \frac{1}{\beta + 2}$
   (c) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$
4. The roots of the equation $3x^2 - 8x + 2 = 0$ are α and β. Find an equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
5. If the roots of the equation $ax^2 + bx + c = 0$ differ by 4, show that $\frac{b^2}{4a} = 4a + c$.
6. Prove that if the roots of the equation $ax^2 + bx + c = 0$ is three times the other, then $3b^2 = 16ac$.
7. The roots of the equation $x^2 + 2px + q = 0$ differ by 8. Show that $p^2 - 16 = q$.
8. The roots of the equation $x^2 + 2x + k$ are β and $\beta - 1$. Find the value of k.
9. The roots of the equation $ax^2 + bx + c = 0$ is a square of the other. Prove that $c(a - b)^3 = a(c - b)^3$.
10. If α and β are roots of the equation $px^2 + qx + r = 0$, form an equation with integral coefficients whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$.
11. Given that α and β are roots of the equation $2x^2 - 8x + 2 = 0$, show that $\alpha^3 + \beta^3 = 52$. Hence that $\alpha^6 + \beta^3 = 27$.
12. Find the relationship between p, q and r if the roots of the equation $px^2 + qx + r = 0$ double each other. Show that $\frac{\log_3 243 + \log_3(\frac{1}{3})^8 + \log_3(27^{\frac{1}{2}})^{8/3} + \log_3 a^3}{\log_3 a^2 + 2} = \frac{3}{2}$
13. If the roots of the equation $2x^2 - 3x - 1 = 0$ are α and β, find the value of $\alpha^2 + \beta^2$ and hence form the equation whose roots are α^2 and β^2.
14. Given that α is a common root of the equations $x^2 - 2x - k = 0$ and $x^2 - 5x + 2k = 0$, where $k \neq 0$. Find the numerical values of k and α.
15. In the equation $m^2x^2 + 2mnx + n^2 + 1 = 0$, m and n are constants which are real numbers. Show that the equation has no real roots for any values of m and n.
16. The roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β. Write down the expression for $(\alpha + \beta)$ and $\alpha\beta$. Express in terms of α and β

(i) $\frac{-c}{b}$ (ii) $\frac{a-b+c}{a}$

(b) The roots of the equation $2x^2 - 3x + 4 = 0$ are α and β. Prove that $\frac{2}{\alpha}$ and $\frac{2}{\beta}$ are also roots of the equation.

17. Solve the equation $2^{2(x+1)} - 5 \times 2^x + 1 = 0$
18. Solve the simultaneous equations
$2x + y + 3 = x + y + 2 = 2x^2 - 11y^2 + 3$
19. The roots of the equation $x^2 + ax + b = 0$ is the square of the other. Find the roots in terms of a and b.
20. The roots of the equations $2x^2 - 3x + 5 = 0$ are α and β. And the roots of the equation $px^2 + x = q = 0$ are
$\alpha - 1$ and $\beta - 1$. Find the value of p and q.
21. If α and β are the roots of the equation $2x^2 - x = 5$, find the equation whose roots are $\alpha + 2\beta$ and $\beta + 2\alpha$.
22. If α and β are roots of the equation $2x^2 - 3x - 4 = 0$, find the equation whose roots are $\frac{1}{\alpha} + \beta$ and $\frac{1}{\beta} + \alpha$.
23. If the roots of the equation $x^2 - 5x + 1 = 0$ are α and β, form an equation with roots $\alpha + 3\beta$, $3\alpha + \beta$.
24. If α and β are roots of the equation $3x^2 - 3x - 1 = 0$, form an equation whose roots are $\alpha - \frac{1}{\alpha}$ and $\beta - \frac{1}{\beta}$.
25. If α and β are roots of the equation $3x^2 + x + 2 = 0$,

    (a) Evaluate $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

    (b) Find the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

    (c) Show that $27\alpha^4 = 11\alpha + 10$
26. The roots of the equation $x^2 + 6x + c = 0$ differ by $2n$, where n is real and non-zero. Show that $n^2 = 9 - c$. Given that the roots have opposite signs, find the set of all possible values of n.
27. Prove that the equation $x(x - 2p) = q(x - p)$ has real roots for all values of p and q. If $p = 3$, find the non-zero value for q.
28. If the roots of the equation $x^2 + bx + c$ are α and β and the roots of the equation $x^2 + \lambda bx + \lambda^2 c = 0$ are γ and δ. Show that the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ is $x^2 - \lambda b^2 x + 2\lambda^2 c(b^2 - 2c) = 0$
29. The roots of the quadratic equation $x^2 - px + q = 0$ are α and β. Determine the equation having the roots
$\alpha^2 + \beta^{-2}$ and $\beta^2 + \alpha^{-2}$.
30. Prove that the roots of the equation
$(\gamma + 3)x^2 + (6 - 2\gamma)x + \gamma - 1 = 0$ are real if and only if γ is not greater than $\frac{3}{2}$. Find the values of γ if one root is six times the other.

31. Form the equation whose roots are the cubes of the roots of the equation $x^2 - 3x + 4 = 0$.

32. Show that if the equations $x^2 + bx + c = 0$ and $x^2 + px + q = 0$ have a common root, then

$$(c - q)^2 = (b - p)(cp - bq)$$

33. (i) Write $x^2 + 6x + 16$ in the form $(x + a)^2 + b$, where a and b are integrals to be found.

(ii) Find the minimum values of $x^2 + 6x + 16$ and state the value of x for which this minimum value occurs.

34. The roots of the equation $2x^2 + 3x - 4 = 0$ are $\alpha\beta$. Find the values of: (a) $\alpha^2 + \beta^2$

(b) $\frac{1}{\alpha} + \frac{1}{\beta}$

(c) $(\alpha + 1)(\beta + 1)$

(d) $\frac{\beta}{\alpha} + \frac{\alpha}{\beta}$

35. If the roots of the equation $3x^2 - 5x + 1 = 0$ are α and β, find the values of:

(a) $\alpha\beta^2 + \alpha^2\beta$ (b) $\alpha^2 - \alpha\beta + \beta^2$

(c) $\alpha^3 + \beta^3$ (d) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

36. The equation $4x^2 + 8x - 1 = 0$ has roots α and β. Find the values of:

(a) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (b) $(\alpha - \beta)^2$

(c) $\alpha^3\beta + \alpha\beta^3$ (d) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

37. If the roots of the equation $x^2 - 5x - 7 = 0$ are α and β, find the equations whose roots are:

(a) α^2, β^2 (b) $\alpha + 1, \beta + 1$

(c) $\alpha^2\beta,\ \alpha\beta^2$

38. The roots of the equation $2x^2 - 4x + 1$ are α and β. Find the equations with integral coefficients whose roots are:

(a) $\alpha - 2, \beta - 2$ (b) $\frac{1}{\alpha}, \frac{1}{\beta}$ (c) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

39. Find the equation with integral coefficients whose roots are the squares of the roots of the equation $2x^2 + 5x - 6 = 0$

40. The roots of the equation $x^2 + 6x + q = 0$ are α and $\alpha - 1$. Find the values of q.

41. The roots of the equation $x^2 - px + 8 = 0$ are α and $\alpha + 2$. Find the two possible values of p.

42. The roots of the equation $x^2 + 2px + q = 0$ differ by 2. Show that $p^2 = 1 + q$

43. If the roots of the equation $ax^2 + bx + c = 0$ are α and β, find expressions in terms of a, b, and c for:

(a) $\alpha^2\beta + \alpha\beta^2$ (b) $\alpha^2 + \beta^2$

(c) $\alpha^3 + \beta^3$ (d) $\frac{1}{\alpha} + \frac{1}{\beta}$

(e) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (f) $\alpha^4 + \beta^4$

44. The equation $ax^2 + bx + c = 0$ has roots α and β. Find equations whose roots are:
(a) $-\alpha, -\beta$ (b) $\alpha + 1, \beta + 1$ (c) α^2, β^2
(d) $\frac{-1}{\alpha}, \frac{-1}{\beta}$ (e) $\alpha - \beta, \beta - \alpha$ (f) $2\alpha + \beta, \alpha + 2\beta$

45. Prove that, if the difference between the roots of the equation $ax^2 + bx + c = 0$ is 1, then $a^2 = b^2 - 4ac$

46. Prove that if one root of the equation $ax^2 + bx + c = 0$ is twice the other, then $2b^2 = 9ac$

47. Prove that if the sum of the squares of the roots of the equation $ax^2 + bx + c = 0$ is 1, then $b^2 = 2ac + a^2$.

48. Prove that if the sum of the reciprocals of the roots of the equation $ax^2 + bx + c = 0$ is 1, then $b + c = 0$.

Answers

1. (a) 4 (b) $\frac{3}{4}$
2. (a) $\frac{2}{3}$ (b) $\frac{16}{9}$ (c) 3
3. (a) $x^2 + 6x + 3$ (b) $3x^2 - 6x + 1 = 0$
(c) $25x^2 - 14x + 1 = 0$
4. $3x^2 - 26x + 3 = 0$
8. $k = \frac{3}{4}$
10. $(p - q + r)x^2 + 2(r - p)x + p + q + r = 0$
13. $\frac{13}{4}$, $4x^2 - 13x + 1 = 0$
14. $k = 3, \alpha = 3$ 17. $x = 0$, $x = -2$
18. $x = -1$, $y = \frac{-1 \pm \sqrt{177}}{22}$
20. $p = -2, q = -4$ 21. $2x^2 - 3x - 4 = 0$
22. $4x^2 - 3x - 1$. 23. $x^2 - 20x + 79$.
24. $x^2 - 4x - 1 = 0$
25. (i) $\frac{-11}{4}$ (ii) $4x^2 + 11x + 9 = 0$
29. $q^2x^2 - (p^2 - 2q)(q^2 + 1)x + (q^2 + 1)^2 = 0$
30. -11, $\frac{33}{25}$
31. $x^2 + 9x + 64 = 0$
33. (i) $(x + 3)^2 + 7$ (ii) 7 at $x = -3$
34. (a) $\frac{25}{4}$, (b) $\frac{3}{4}$ (c) $\frac{-5}{2}$ (d) $\frac{-25}{8}$
35. (a) $\frac{5}{9}$, (b) $\frac{16}{9}$ (c) $\frac{80}{27}$ (d) $\frac{80}{9}$
36. (a) 72, (b) $5\frac{16}{9}$ (c) $\frac{-9}{8}$ (d) -32

37. (a) $x^2 - 39x + 49 = 0$
(b) $x^2 - 7x - 1 = 0$
(c) $x^2 + 35x - 343 = 0$

38. (a) $2x^2 + 4x + 1 = 0$
(b) $x^2 - 4x + 2 = 0$
(c) $x^2 - 6x + 1 = 0$

39. $4x^2 - 49x + 36 = 0$

40. $\frac{35}{4}$ 41. ± 6

43. (a) $\frac{-bc}{a^2}$ (b) $\frac{b^2 - 2ac}{a^2}$ (c) $\frac{b(3ac - b^2)}{a^3}$
(d) $\frac{-b}{c}$ (e) $\frac{b^2 - 2ac}{ac}$ (f) $\frac{b^4 - 4ab^2c + 2a^2c^2}{a^4}$

44 (a) $ax^2 - bx + c = 0$
(b) $ax^2 + (b - 2a)x + a - b + c = 0$
(c) $a^2x^2 + (2ac - b^2)x + c^2 = 0$
(d) $cx^2 - bx + a = 0$
(e) $a^2x^2 - (b^2 - 4ac) = 0$
(f) $a^2x^2 + 3abx + (2b^2 + ac) = 0$

POLYNOMIALS

A polynomial is an expression consisting of variables and co-efficient which only employs the operations of addition, multiplication and non-negative integer exponent.

Consider the expression

$P(x) = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \cdots k$

Where $C_0 \neq 0$ $then$ $P(x)$ is said to be a polynomial of degree n. When we solve $P(x) = 0$ we get n unequal roots. When x is equal to each of the unequal values

$$x = a_{1,}\ x = a_{2,}\ x = a_{3,} \dots x = a_n \text{ then}$$

$x - a_1, x - a_2, x - a_3, \dots x - a_n$ are factors of $P(x)$

Polynomials must have whole numbers as exponents for example $x^2 - 4x + 7$ is a polynomial but $9x^{-1} + 12x^{\frac{1}{2}}$ is not a polynomial.

Polynomials appear in a wide variety of areas of mathematics and science. For example they are used to form polynomial equations which encode a wide range of problems from elementary word problems to complicated problems in science. They are used in calculus to approximate other functions.

Remainder Theorem

If when a polynomial $P(x)$ is divided by $x - a$, the quotient is $Q(x)$ and remainder is R then;

$$P(x) = (x - a)Q(x) + R$$

If $P(a) = 0$ then $x - a$ is a factor of $P(x)$
This approach can be extended to the division of the polynomial $f(x)$ by polynomial $g(x)$ of the degree less or equal to the degree of $f(x)$.
If the division gives the quotient $Q(x)$ and remainder $R(x)$ then $f(x) = g(x)Q(x) + R(x)$
Where $R(x)$ is of lower degree than $g(x)$

The Remainder Theorem

When $P(x)$ is divided by $x - a$, the remainder is $P(a)$
Proof $P(x) = (x - a)Q(x) + R$
Equating the divisor to zero

$$x - a = 0$$
$$x = a$$
$$P(a) = (a - a)Q(x) + R$$
$$P(a) = R$$

Example I
Find the remainders when

(i) $3x^2 - 4x^2 + 5x - 8$ is divided by $x - 2$

(ii) $2x^3 - 3x^2 - 5x + 6$ is divided by $x + 2$

(iii) $2x^3 - 7x + 6$ is divided by $x - 3$

(iv) $x^5 + x - 9$ is divided by $x + 1$

Solutions

$$P(x) = Q(x)D(x) + R$$

Where $Q(x)$ is the quotient, $D(x)$ is the divisor and R is the remainder

(i) $P(x) = 3x^3 - 4x^2 + 5x - 8$

$$D(x) = x - 2$$
$$x - 2 = 0$$
$$x = 2$$
$$P(2) = 3(2^3) - 4(2^2) + 5(2) - 8$$
$$= 24 - 16 + 10 - 8$$
$$= 34 - 24$$
$$= 10$$
$$R = 10$$

The remainder when $3x^3 - 4x^2 + 5x - 8$ is divided by $x - 2\ is\ 10$

(ii) $P(x) = Q(x)D(x) + R$

$P(x) = 2x^3 - 3x^2 - 5x + 6$

$D(x) = x + 2$

Equating the divisor to 0

$\Rightarrow x + 2 = 0$

$x = -2$

$P(-2) = 2(-2)^3 - 3(-2)^2 - 5(-2) + 6$

$$P(-2) = -12$$

The remainder when $2x^3 - 3x^2 - 5x + 6$ is divided by $x + 2$ is -12

(iii) $P(x) = Q(x)D(x) + R$

$$D(x) = x - 3$$

Equating the divisor to 0

$x - 3 = 0$

$x = 3$

$P(x) = 2x^3 - 7x + 6$

$P(3) = 2(3)^3 - 7(3) + 6$

$P(3) = 54 - 21 + 6$

$P(3) = 39$

The remainder when $2x^3 - 2x + 6$ is divided by $x - 3$ is 39

(iv) $x^5 + x - 9 = P(x)$

$$D(x) = x + 1$$

Equating the divisor to 0

$$x + 1 = 0$$

$$x = -1$$
$$P(-1) = (-1)^5 + (-1) - 9$$
$$= -1 - 10$$
$$= -11$$

The remainder when $x^5 + x - 9$ is divided by $x + 1$ is -11

Suppose a polynomial $f(x)$ has a repeated factor $x - a$. So that $f(x) = (x - a)^2 \, . \, g(x)$
So by differentiating

$$f^1(x) = (x - a)^2 \, g^1(x) + 2(x - a)g(x)$$

(differentiation by product rule)

Hence if $f(x)$ has a repeated factor of $x - a$ then $(x - a)$is also a factor of $f^1(x)$

⇒ If $(x - a)^2$ is a factor of a polynomial $f(x)$ if and only if $f(a) = f^1(a) = 0$

Example II

Given that the polynomial $f(x) = x^3 + 3x^2 - 9x + k$ has a repeated linear factor, find the possible values of k.

Solution

$f(x) = x^3 + 3x^2 - 9x + k$
$f^1(x) = 3x^2 + 6x - 9$
$f^1(x) = 3(x^2 + 2x - 3)$
$f^1(x) = 3(x - 1)(x + 3)$

The repeated factor of $f(x)$ is either $(x - 1)$ or $(x + 3)$

If $x - 1$ is a factor of $f(x)$, then $f(1) = 0$

$\Rightarrow 1 + 3 - 9 + k = 0$

$$k = 5$$

If $x + 3$ is a factor $f(x)$, then $f(-3) = 0$

$$-27 + 27 + 27 + k = 0$$
$$k = -27$$

The possible values of k are $k = 5 \; and \; k = -27$

Obtaining the remainder by long division

Here are the steps required for dividing by a polynomial containing more than one term

Step I: Make sure the polynomial is written in descending order. If any terms are missing, use a zero to fill in the missing term (this will help with the spacing)

Step II: Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol

Step III: Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of the division symbol

Step IV: Subtract and bring down the next term

Step V: Repeat step (II), (III) and (IV) until there are no more terms to bring down.

Step VI: Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction in the final answer.

Example

Find the remainder when $x^3 - 4x^2 + 2x - 3$ is divided by $x + 2$

Solution

STEP I Make sure the polynomial is written in descending order. If any term is missing, use zero to fill in the missing terms (this will help with the spacing). In this case, the problem is ready as it is.	$x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$
STEP II Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. In this case, we have x^3 divided by x which is x^2	x^2 $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$
STEP III Multiply (or distribute) the answer obtained in the previous step by polynomial in front of division symbol. In this case, we need x^2 and $x + 2$	x^2 $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $x^3 + 2x^2$
STEP IV Subtract and bring down the next term	x^2 $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 + 2x^2}$ $-6x^2 + 2x - 3$
STEP V Divide the term with the highest power inside the division symbol by the term with the highest power	$x^2 - 6x$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x - 3$

outside the division symbol. In this case, we have $-6x^2$ divided by x which is $-6x$.	
STEP VI Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of division symbol. In this case, we need to multiply ($-6x$) by $x+2$	$x^2 - 6x$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $-6x^2 - 12x$
STEP VII Subtract and bring down the next term	$x^2 - 6x$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $\underline{-6x^2 - 12x}$ $14x - 3$
STEP VIII Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. in this case, we have $14x$ divided by x which is $+14$	$x^2 - 6x + 14$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $\underline{-6x^2 - 12x}$ $14x - 3$
STEP IX Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of the division symbol. In this case, we need to multiply 14 by $x + 2$	$x^2 - 6x + 14$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $\underline{-6x^2 - 12x}$ $14x - 3$ $14x + 28$
STEP X Subtract and notice there are no more terms to bring down	$x^2 - 6x + 14$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $\underline{-6x^2 - 12x}$ $14x - 3$ $\underline{14x + 28}$ -31
STEP XI Write the final answer. The term remaining after the last subtract step is the	$x^2 - 6x + 14 + \frac{-31}{x+2}$

remainder and must be written as a fraction.	

$\Rightarrow$ The remainder when $x^3 - 4x^2 + 2x - 3$ is divided by $x + 2$ is -31.

Example II

By using long division, obtain remainders and quotients when

(i) $x^3 + 3x^2 - 4x - 12$ is divided by $x^2 + x - 6$
(ii) $2x^4 - 8x^3 + 5x^2 + 4$ is divided by $x - 3$
(iii) $5x^3 - 6x^2 + 3x + 14$ is divided by $x + 1$
(iv) $2x^4 + 6x^3 - 7x^2 + 9x + 11$ is divided by $x + 4$
(v) $x^4 - 16$ is divided by $x - 2$

Solution

(i) $x^3 + 3x^2 - 4x - 12$ is divided by $x^2 + x - 6$

$$\begin{array}{r|l} & x + 2 \\ \hline x^2+x-6 & x^3 + 3x^2 - 4x - 12 \\ & x^3 + x^2 - 6x \\ \hline & 2x^2 + 2x - 12 \\ & 2x^2 + 2x - 12 \\ \hline & 0 \end{array}$$

$$\frac{x^3 + 3x^2 - 4x - 12}{x^2 + x - 6} = (x + 2) - \frac{0}{x^2 + x - 6}$$

$R = 0$

$Q(x) = x + 2$ (the quotient)

(ii) $\mathbf{2x^4 - 8x^3 + 5x^2 + 4}$ is divided by $x - 3$

$2x^4 - 8x^3 + 5x^2 + 0x + 4$

$$\begin{array}{r|l} & 2x^3 - 2x^2 - x - 3 \\ \hline x - 3 & 2x^4 - 8x^3 + 5x^2 + 0x + 4 \\ & 2x^4 - 6x^3 \\ \hline & -2x^3 + 5x^2 + 0x + 4 \\ & -2x^3 + 6x^2 \\ \hline & -x^2 + 0x + 4 \\ & -x^2 + 3x \\ \hline & -3x + 4 \\ & -3x + 9 \\ \hline & -5 \end{array}$$

$$2x^3 - 2x^2 - x - 3 + \frac{-5}{x - 3}$$

$$Q(x) = 2x^3 - 2x^2 - x - 3$$

$$R = -5$$

Where $Q(x)$ = Quotient and remainder = R

(iii) $5x^3 - 6x^2 + 3x + 14$

$$
\begin{array}{r|l}
 & 5x^2 - 11x + 14 \\
\hline
x + 1 & 5x^3 - 6x^2 + 3x + 14 \\
 & \underline{5x^3 + 5x^2} \\
 & \quad -11x^2 + 3x + 14 \\
 & \quad \underline{-11x^2 - 11x} \\
 & \qquad 14x + 14 \\
 & \qquad \underline{14x + 14} \\
 & \qquad\qquad 0
\end{array}
$$

$$\frac{5x^3 - 6x^2 + 3x + 14}{x + 1} = 5x^2 - 11x + 14$$

$$R(x) = 0, Q(x) = 5x^2 - 11x + 14$$

(iv) $2x^4 + 6x^3 - 7x^2 + 9x + 11$

$$
\begin{array}{r|l}
 & 2x^3 - 2x^2 + x + 5 \\
\hline
x + 4 & 2x^4 + 6x^3 - 7x^2 + 9x + 11 \\
 & \underline{2x^4 + 8x^3} \\
 & \quad -2x^3 - 7x^2 + 9x + 11 \\
 & \quad \underline{-2x^3 - 8x^2} \\
 & \qquad x^2 + 9x + 11 \\
 & \qquad \underline{x^2 + 4x} \\
 & \qquad\quad 5x + 11 \\
 & \qquad\quad \underline{5x + 20} \\
 & \qquad\qquad -9
\end{array}
$$

$$\frac{2x^4 + 6x^3 - 7x^2 + 9x + 11}{x + 4} = 2x^3 - 2x^2 + x + 5 - \frac{9}{x + 4}$$

$$Q(x) = 2x^3 - 2x^2 + x + 5$$

$$R = -9$$

(v) $x^4 - 16$ is divided by $x - 2$

$$
\begin{array}{r|l}
 & x^3 + 2x^2 + 4x + 8 \\
\hline
x - 2 & x^4 + 0x^3 + 0x^2 + 0x - 16 \\
 & \underline{x^4 - 2x^3} \\
 & \quad 2x^3 + 0x^2 + 0x - 16 \\
 & \quad \underline{2x^3 - 4x^2} \\
 & \qquad 4x^2 + 0x - 16 \\
 & \qquad \underline{4x^2 - 8x} \\
 & \qquad\quad 8x - 16 \\
 & \qquad\quad \underline{8x - 16} \\
 & \qquad\qquad 0
\end{array}
$$

$$r = 0$$

$$(x) = x^3 + 2x^2 + 4x + 8$$

Were $r = remainder$

And $Q(x) = x^3 + 2x^2 + 14x + 8$

Obtaining the remainder by synthetic approach

Definitions :

Dividend: The number or expression you are dividing into

Divisor: The number or expression you are dividing by

Synthetic division: is a quick method of dividing a polynomial when the divisor is of the form $ax + b$ or $x - c$

Steps involved when obtaining the remainder by synthetic approach

(1) Write the value obtained after equating the divisor to 0 and the coefficients of the dividend in descending order in the first row. If any x terms are missing, place a zero in its place

(2) Bring the leading coefficient in the top row down to bottom (third) row

(3) Next multiply the number in the bottom row by c and place this product in the second row under the next coefficient and add these two terms together

(4) Continue with this process until you reach the last column

(5) The numbers in the bottom are coefficients of the quotient and the remainder. The quotient will have one degree less than the dividend

Example I

Use synthetic approach to obtain the remainder when $2x^4 - 8x^3 + 5x^2 + 4$ is divided by $x - 3$

Solution

First note that the x term is missing so we must record zero in its place

	x^4	x^3	x^2	x	x^0
$x = 3$	2	-8	5	0	4
		6	-6	-3	-9
	2	-2	-1	-3	-5

Therefore, the quotient is $2x^3 - 2x^2 - x - 3$ and the remainder is -5.

Example II

Use synthetic approach to obtain the remainders when

(i) $x^4 - 16$ is divided by $x + 1$

(ii) $5x^3 - 6x^2 + 3x + 14$ is divided by $x + 1$

(iii) $2x^4 + 6x^3 - 7x^2 + 9x + 11$ is divided by $x + 4$

Solution

(i) Equating the divisor to zero

$$x + 1 = 0$$
$$x = -1$$

	x^4	x^3	x^2	x	x^0
$x = -1$	1	0	0	0	16
		-1	1	-1	1
	1	-1	1	-1	-15

$$R = -15$$
$$Q(x) = x^3 - x^2 + x - 1$$

Where R = remainder
And $Q(x)$ = Quotient

> **Note: Synthetic method only work where the divisor is of the form**
> $$\boldsymbol{x - c \text{ or } ax + b}$$

(ii) $5x^3 - 6x^2 + 3x + 14$

Equating the divisor to zero

$x + 1 = 0$, $x = -1$

	x^3	x^2	x	x^0
$x = -1$	5	-6	3	14
		-5	11	-14
	5	-11	14	0

$$Q(x) = 5x^2 - 11x + 14$$
$$R = 0$$

Where $Q(x) = Quotient$
R is the remainder

(iii) $2x^4 + 6x^3 - 7x^2 + 9x + 11 = P(x)$

$$D(x) = x + 4$$

Equating the divisor to zero

$$x + 4 = 0$$
$$x = -4$$

	x^4	x^3	x^2	x	x^0
$x = -4$	2	6	-7	9	11
		-8	8	-4	-20
	2	-2	1	5	-9

$Q(x) = 2x^3 - 2x^2 + x + 5$
$R = -9$

Where $Q(x)$ = quotient, R = remainder

More examples on polynomials

Find the values a in the expression below if the following conditions are satisfied

(i) $x^3 + ax^2 + 3x - 5$ has a remainder -3 when divided by $x - 2$

(ii) $x^3 + x^2 - 2ax + a^2$ has a remainder 8 when divided by $x - 2$

Solution

(i) $P(x) = Q(x)D(x) + R$

$x^3 + ax^2 + 3x - 5 = Q(x)(x-2) + -3$ …… (1)

Equating the divisor to zero

$$x - 2 = 0$$
$$x = 2$$

Substituting x = 2 in Eqn (1);

$$2^3 + a(2^2) + 3 \times 2 - 5 = 0 + -3$$
$$8 + 4a + 6 - 5 = -3$$
$$9 + 4a = -3$$
$$4a = -12$$
$$a = -3$$

(ii) $x^3 + x^2 - 2ax + a^2 = Q(x)(x-2) + 8$ …. (1)

$$x - 2 = 0$$
$$x = 2$$

Substituting x = 2 in Eqn (1);

$$8 + 4 - 4a + a^2 = 0 + 8$$
$$a^2 - 4a + 4 = 0$$
$$(a - 2)^2 = 0$$
$$a = 2$$

Example II

Show that $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$ and find other factors

Solution

$$\begin{array}{r|l} & 6x^2 + 11x + 3 \\ \hline 2x - 1 & 12x^3 + 16x^2 - 5x - 3 \\ & 12x^3 - 6x^2 \\ \hline & \quad 22x^2 - 5x - 3 \\ & \quad 22x^2 - 11x \\ \hline & \qquad 6x - 3 \\ & \qquad 6x - 3 \\ \hline \end{array}$$

Since the remainder is zero

$12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$

$12x^3 + 16x^2 - 5x - 3 = (2x - 1)(6x^2 + 11x + 3)$

For $6x^2 + 11x + 3$

The other Factors are 2, 9 and product of factors is 18

$6x^2 + 2x + 9x + 3$
$2x(3x + 1) + 3(3x + 1)$
$\Rightarrow (2x + 3)(3x + 1)$
$6x^2 + 11x + 3 = (2x + 3)(3x + 1)$
$12x^3 + 16x^2 - 5x - 3 = (2x - 1)(2x + 3)(3x + 1)$
The other factors of $12x^3 + 16x^2 - 5x - 3$ are $(2x + 3)$and $(3x + 1)$

Example III

$x + 2$ is a factor of $2x^3 + 6x^2 + bx - 5$. Find the remainder when the expression is divided by $2x - 1$

Solution

$$P(x) = 2x^3 + 6x^2 + bx - 5$$

Since $x + 2$ is a factor,
$x + 2 = 0 \quad \Rightarrow x = -2$
$P(-2) = 0$ (Since $x + 2$ is a factor of $P(x)$

(Remainder = 0)
$P(-2) = 2(-2^3) + 6(-2^2) + b(-2) - 5$
$P(-2) = -16 + 24 - 2b - 5$
$P(-2) = -21 + 24 - 2b$
$P(-2) = 3 - 2b$
$3 - 2b = 0$
$b = 1.5$

Example IV

The remainder obtained when $2x^3 + ax^2 - 6x + 1$ is divided by $x + 2$ is twice the remainder obtained when the same expression is divided by $x - 3$. Find a

Solution

$$P(x) = 2x^3 + ax^2 - 6x + 1$$
$$P(-2) = 2(-2)^3 + a(-2^2) - 6(-2) + 1$$
$$= -16 + 4a + 12 + 1$$
$$P(2) = -3 + 4a$$
$$P(3) = 2(3^3) + a(3^2) - 6(3) + 1$$
$$= 54 + 9a - 18 + 1$$
$$\mathrm{P}(3) = 37 + 9a$$
$$P(-2) = 2P(3)$$
$$3 + 4a = (37 + 9a) \times 2$$
$$-3 + 4a = 74 + 18a$$
$$-77 = 14a$$
$$a = -5.5$$

Example V

A cubic polynomial $6x^3 + 7x^2 + ax + b$ has a remainder 72 when divided by $x - 2$ and exactly divisible by $x + 1$. Calculate the values of $a \; and \; b$. Show that $2x - 1$ is also a factor. Obtain the other factor factor

Solution:

$P(x) = 6x^3 + 7x^2 + ax + b$

$P(2) = 72$

$P(-1) = 0$

$P(2) = 6(2^3) + 7(2^2) + a(2) + b$

$\quad = 48 + 28 + 2a + b$

$P(2) = 76 + 2a + b$

$72 = 76 + 2a + b$

$-4 = 2a + b$………………………….. (1)

$P(-1) = 6(-1)^3 + 7(-1)^2 + a(-1) + b$

$P(-1) = -6 + 7 - a + b$

$0 = 1 - a + b$

$-1 = -a + b$………………………… (2)

Eqn. (1) – eqn. (2)

$-3 = 3a$

$a = -1$

Substituting $a = -1$ in Eqn (2);

$\Rightarrow -1 = 1 + b$

$b = -2$

$P(x) = 6x^3 + 7x^2 - x - 2$

We can apply long division to obtain other factors

$$
\begin{array}{r|l}
 & 6x^2 + x - 2 \\
\hline
x + 1 & 6x^3 + 7x^2 - x - 2 \\
 & 6x^3 + 6x^2 \\
\hline
 & \quad x^2 - x - 2 \\
 & \quad x^2 + x \\
\hline
 & \qquad -2x - 2 \\
 & \qquad -2x - 2 \\
\hline
\end{array}
$$

$\Rightarrow (6x^2 + x - 2)(x + 1)$

But $6x^2 + x - 2 = 6x^2 + 4x - 3x - 2$

$= 2x(3x + 2) - 1(3x + 2)$

$\Rightarrow 6x^3 + 7x^2 - x - 2 = (2x - 1)(3x + 2)(x + 1)$

$\Rightarrow$ The other factors are $(3x + 2)$ and $(x + 1)$

Example VI

$x - 1$and $x + 1$ are factors of $x^3 + ax^2 + bx + c$ and it leaves a remainder of 12 when divided by $x - 2$.find the values of a, b, and c.

Solution

$P(x) = x^3 + ax^2 + bx + c$

$P(1) = 0$

$P(-1) = 0$

$P(2) = 12$

$P(1) = 1^3 + a(1^2) + b(1) + c$

$P(1) = 1 + a + b + c$

$0 = 1 + a + b + c$

$\Rightarrow a + b + c = -1$…………………… (1)

$P(-1) = (-1)^3 + a(-1)^2 + b(-1) + c$

$P(-1) = -1 + a - b + c$

$0 = -1 + a - b + c$

$a - b + c = 1$……………………… (2)

$P(2) = 2^3 + a(2^2) + b(2) + c$

$12 = 8 + 4a + 2b + c$

$4a + 2b + c = 4$………………….. (3)

Eqn. (1) – eqn.(2)

$2b = -2$

$b = -1$

Eqn.(1) – eqn.(3)

$-3a - b = -5$

But b = -1

$-3a + 1 = -5$

$-3a = -6$

$a = 2$

Substituting $a = 2, b = -1$ in eqn. (1)

$2 - 1 + c = -1$

$c = -2$

$\therefore a = 2, b = -1 \; and \; c = -2$

Example IX

When a polynomial$P(x)$ is divided by $x - 2$, the remainder is 4 and when $P(x)$ is divided by $x - 3$, the remainder is 7. Find the remainder when $P(x)$ is divided by $(x - 2)(x - 3)$

Solution

P(2) = 4 and P(3) = 7

The remainder is of the form

$R(x) = ax + b$

$P(x) = Q(x)D(x) + R(x)$

$P(x) = Q(x)(x-2)(x-3) + ax + b$

Equating the divisor to zero

$(x-2)(x-3) = 0$

$x = 2, x = 3$

$P(2) = 2a + b$

$P(3) = 3a + b$

But $P(2) = 4 \; and \; P(3) = 7$

$2a + b = 4$.............................. (1)

$3a + b = 7$.............................. (2)

Eqn (2) – Eqn (1)

$$a = 3$$

Substituting $a = 3$ in eqn. (1)

$2 \times 3 + b = 4$

$6 + b = 4$

$b = 4 - 6$

$b = -2$

$R(x) = ax + b$

$R(x) = 3x - 2$

Example VIII

When a polynomial $p(x)$is divided by $x - 1$, the remainder is 5 and when $p(x)$ is divided by $x - 2$, the remainder is 7. Find the remainder when the same expression is divided by $(x-1)(x-2)$.

Solution

The remainder takes the form $R(x) = ax + b$.

$$p(x) = (x)(x-1)(x-2) + ax + b$$

Equating the divisor to zero

$(x-1)(x-2) = 0$

$$x = 1 \; and \; x = 2$$

$$p(1) = a + b$$

$$p(2) = 2a + b$$

$5 = a + b$.............................. (1)

$7 = 2a + b$............................. (2)

Eqn. (2) – eqn. (1)

$$2 = a$$

$$a = 2$$

Substituting $a = 2$ in eqn. (1)

$$5 = 2 + b$$

$$b = 3$$

The remainder is $2x + 3$

Example IX

Given that the polynomial $f(x) = Q(x)g(x) + R(x)$ where $Q(x)$ is a quotient, $g(x) = (x-\alpha)(x-\beta)$and $R(x)$ is a remainder. Show that

$$R(x) = \frac{(x-\beta)f(\alpha) + (\alpha - x)f(\beta)}{\alpha - \beta}$$

Solution

$$f(x) = Q(x)D(x) + R(x)$$

$$f(x) = Q(x)(x-\alpha)(x-\beta) + ax + b$$

Where $R(x) = ax + b$

Equating the divisor to zero

$(x-\alpha)(x-\beta) = 0$

$$x = \alpha, x = \beta$$

$f(\alpha) = a\alpha + b$.......................... (1)

$f(\beta) = a\beta + b$.......................... (2)

Eqn. (1) – eqn. (2)

$$a(\alpha - \beta) = f(\alpha) - f(\beta)$$

$$a = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$$

Substituting $a = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}$ in equation (1)

$$f(\alpha) = \alpha\left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right) + b$$

$$b = f(\alpha) - \left(\frac{\alpha f(\alpha) - \alpha f(\beta)}{\alpha - \beta}\right)$$

$$b = \frac{\alpha f(\alpha) - \beta f(\alpha) - \alpha f(\alpha) + \alpha f(\beta)}{\alpha - \beta}$$

$$b = \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}$$

But since $R(x) = ax + b$

$$R(x) = \left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right)x + b$$

$$R(x) = \frac{xf(\alpha) - f(\beta)x}{\alpha - \beta} + \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}$$

$$R(x) = \frac{xf(\alpha) - \beta f(\alpha) + \alpha f(\beta) - xf(\beta)}{\alpha - \beta}$$

$$R(x) = \frac{(x-\beta)f(\alpha) + (\alpha - x)f(\beta)}{\alpha - \beta}$$

Example

If $x^2 + 1$ a factor of $3x^4 + x^3 - 4x^2 + px + q$. Find the values of p and q

Solution

$$3x^4 + x^3 - 4x^2 + px + q = (x^2+1)\left(ax^2 + bx + c\right)$$

But

$$(x^2+1)\left(ax^2+bx+c\right) = ax^4 + bx^3 + cx^2 + ax^2 + bx + c$$

$$= ax^4 + bx^3 + \left(a+c\right)x^2 + bx + c$$

$$3x^4 + x^3 - 4x^2 + px + q = ax^4 + bx^3 + (a+c)x^2 + bx + c$$

Equating co-efficients of the same monomial;

$\Rightarrow a = 3, b = 1$

$$a + c = -4$$
$$3 + c = -4$$
$$c = -7$$
$$b = p$$
$$1 = p$$
$$c = q$$
$$q = -7$$

$\Rightarrow p = 1$ and $q = -7$

Example XII

If $f(x)$ and $g(x)$ are polynomials.

$f(x) = (x-a)^2 g(x) + Ax + B$. Find $f^1(x)$and hence find A and B in terms of $f(a)\ and\ f^1(a)$ and deduce that $x - a$ is a repeated factor of $f(x)$ if and only if $f(a) = f^1(a)$

Solution

$$f(x) = (x-a)^2 g(x) + Ax + B$$
$$f^1\left(x\right) = (x-a)^2 g^1\left(x\right) + g\left(x\right)2\left(x-a\right) + A$$
$$f^1(a) = 0 + A$$
$$A = f^1(a)$$
$$f(a) = 0 + Aa + B$$
$$f(a) = f^1(a) + B$$
$$B = f(a) - af^1(a)$$

If $(x-a)^2$ is a repeated factor of $f(x)$

$$f(x) = (x-a)^2 Q_1(x)$$
$$f(a) = (a-a)^2 Q_1(x) = 0$$
$$f(a) = 0$$
$$f^1(x) = (x-a)^2 Q_1^1(x) + Q_1(x)2(x-a)$$
$$f^1(a) = 0$$

$\Rightarrow$ If $(x - a)^2$ is a repeated factor of $f(x)$ if and only if

$$f(a) = f^1(a) = 0$$

Example (UNEB 2015)

(a) Given that $f(x) = (x - \alpha)^2 g(x)$, show that $f'(x)$ is divisible by $(x - \alpha)$

(b) A polynomial $P(x) = x^3 + 4ax^2 + bx + 3$ is divisible by $(x - 1)^2$. Use your results above to find the values of a and b. Hence solve the equation $p(x) = 0$

Solution

$f(x) = (x - \alpha)^2 g(x)$

$f'(x) = (x - \alpha)^2 g'(x) + g(x)2(x - \alpha)$

$\Rightarrow$ $f'(x)$ is divisible by $(x - \alpha)$

$p(x) = x^3 + 4ax^2 + bx + 3$

$p'(x) = 3x^2 + 8ax + b$

Since $x - 1$ is a factor of $p(x)$ and $p'(x)$,

$\Rightarrow p(1) = 0$ and $p'(1) = 0$

$1 + 4a + b + 3 = 0$

$4a + b = -4$ …………………. (i)

$p'(1) = 0$

$3 + 8a + b = 0$

$8a + b = -3$ …………………. (ii)

Eqn (ii) – Eqn (i);

$4a = 1$

$a = \frac{1}{4}$

Substituting $a = \frac{1}{4}$ in Eqn (ii)

$\Rightarrow 8\left(\frac{1}{4}\right) + b = -3$

$2 + b = -3$

$b = -5$

$p(x) = x^3 + x^2 + -5x + 3$

$p(1) = 1 + 1 + 3 - 5 = 0$

$p'(x) = 3x^2 + 2x - 5$

$p'(1) = 3 + 2 - 5$

$p'(1) = 0$

$P(x) = x^3 + x^2 - 5x + 3$

$(x^3 + x^2 - 5x + 3) = (x - 1)^2 g(x)$

$(x^3 + x^2 - 5x + 3) = (x^2 - 2x + 1)g(x)$

$$
\begin{array}{r|l}
 & x + 3 \\
\hline
x^2 - 2x + 1 & x^3 + x^2 - 5x + 3 \\
 & x^3 - 2x^2 + x \\
\hline
 & \quad 3x^2 - 6x + 3 \\
 & \quad 3x^2 - 6x + 3
\end{array}
$$

$\Rightarrow (x^3 + x^2 - 5x + 3) = (x^2 - 2x + 1)(x + 3)$

$(x^2 - 2x + 1)(x + 3) = 0$

$(x - 1)^2(x + 3) = 0$

$x - 1 = 0$ OR $x + 3 = 0$

$x = 1$ $\quad$ $x = -3$

Example **(UNEB Question)**

When the quadratic expression $ap^2 + bp + c$ is divided by $p - 1$, $p - 2$ and $p + 1$, the remainders are 1, 1 and 25 respectively. Determine the factors of the expression.

b) Express $2x^3 + 5x^2 - 4x - 3$ in the form $(x^2 + x - 2)\ Q(x) + Ax + B$; where $Q(x)$ is a polynomial in x and A and B are constants. Determine the values of A and B and the expression $Q(x)$.

Solution

Let $f(p) = ap^2 + bp + c$

Now $f(1) = a + b + c$

But $f(1) = 1$

$\Longrightarrow \quad a + b + c = 1$ ……………… (i)

$f(2) = 4a + 2b + c$

But $f(2) = 1$

$\Longrightarrow 4a + 2b + c = 1$…………….. (ii)

$f(-1) = a - b + c$

But $f(-1) = 25$

=> $a - b + c = 25$ …………….. (iii)

Eqn (i) – Eqn (ii)

$-3a - 3b = 0$

$-3a = b$…………………….(iv)

Eqn (ii) – Eqn (iii)

$3a + 3b = -24$

$a + b = -8$ ………………….(v)

$a - 3a = -8$

$-2a = -8$

$a = 4$

Substituting for a into Eqn (iv)

$b = -12$

Substituting for and b into Eqn (i)

$4 - 12 + c = 1$

$-8 + c = 1$

$c = 9$

Hence $f(p) = 4p^2 - 12p + 9$

By factorization,

$4p^2 - 12p + 9 = 4p^2 - 6p - 6p + 9$

$= 2p(2p - 3) - 3(2p - 3)$

$= (2p - 3)(2p - 3)$

Hence the factors of $4p^2 - 12p + 9$ are

$(2p - 3)$ and $(2p - 3)$

b) Let $2x^3 + 5x^2 - 4x - 3$

$\equiv (x^2 + x - 2)(2x + \mathrm{D}) + Ax + B$

By opening brackets on the L.H.S

$2x^3 + 5x^2 - 4x - 3 \equiv 2x^3 + Dx^2 + 2x^2 + Dx - 4x - 2D + Ax + B$

$2x^3 + 5x^2 - 4x - 3 \equiv 2x^3 +(D + 2)x^2 + (D - 4)x - 2D + Ax + B$

$2x^3 + 5x^2 - 4x - 3 \equiv 2x^3 + (D +2)^2 + (D + A - 4)x - 2D + B$

Equating corresponding coefficients,

For x^2,

$D + 2 = 5$

$D = 3$

For x

$-4 = D + A - 4$

$D = -A$

$3 = -A$

$A = -3$

For constant

$-3 = -6 + B$

$B = 6 - 3$

$B = 3$

Hence $2x^3 + 5x^2 - 4x - 3 \equiv (x^2 + x - 2)(2x + 3) - 3x + 3$

Alternatively

By using long division,

$$\begin{array}{r|l} & 2x+3 \\ (x^2+x-2) & 2x^3+5x^2-4x-3 \\ & \underline{2x^3+2x^2-4x} \\ & \quad 3x^2-3 \\ & \quad \underline{3x^2+3x-6} \\ & \qquad -3x+3 \end{array}$$

Hence $2x^3 + 5x^2 - 4x - 3 \equiv (x^2 + x - 2)(2x + 3) - 3x + 3$

Revision Exercise

1. Find the constants p, q and r such that
$$2y^2 - 9y + 14 = p(y-1)(y-2) + q(y-1) + r$$
2. Find the relationship between p and r so that
$A^2 + 3qA^2 + PA + R$ shall be a perfect cube for all values of A.
3. When the expression $p^6 + 4p^2 + ap + b$ is divided by $p^2 - 1$, the remainder is $2p + 3$. Find the values of a and b.
4. Find the remainder when:
(a) $4x^3 - 5x + 4$ is divided by $-(1 - 2x)$
(b) $y^5 + y - 9$ is divided by $y + 1$
5. Find the values of β in the expressions below when the following conditions are satisfied:
(a) $y^3 + \beta y^2 + 3y - 5$ has remainder -3 when divided by $y - 2$.
(b) $x^5 + 4x^4 - 6x^2 - \beta x + 2$ has a remainder 6 when divided by $\dfrac{1}{(x+2)^{-1}}$.
6. $(p - 1)$ and $(p + 1)$ are factors of the expression
$p^3 + ap^2 + bp + c$ and it leaves a remainder of 12 when divided by $p - 2$. Find the values of a, b, c.
7. The expression $ax^4 + bx^3 + 3x^2 - 2x + 3$ has a remainder $x + 1$ when divided by $x^2 - 3x + 2$. Find the values of a and b.
8. What is the value of a if $2x^2 - x - 6$, $3x^2 - 8x + 4$ and $ax^3 - 10x - 4$ have a common factor?

9. Factorise the expression $3k^3 – 11k^2 – 19k – 5$.
10. Find the values of a and b which make
 $y^4 + 6y^3 + 13y^2 + ay + b$ a perfect square.
11. If $x^2 + nx + q$ and $x^2 + dx + m$ have a common factor $(x – p)$. show that $p = \frac{m-q}{n-d}$.
12. The remainder obtained when $2x^3 + ax^2 – 6x + 1$ is divided by $(x + 2)$ is twice the remainder obtained when the same expression is divided by $(x – 1)$. Find the values of a and b.
13. Given that $(x + 2)$ is a factor of $2x^3 + 6x^2 + bx – 5$, find the remainder when the expression is divided by $(2x – 1)$
14. Find the values of p and q if the expression $2y^3 – 15y^2 + py + q$ is divisible both by $y – 4$ and $2y – 1$.
15. Use the remainder theorem to find the factors of
 $x^4 + 3x^2 – 4$.
16. Find p and q so that
 $y^4 – 7y^3 + 17y^2 – 17y + 6 = (y – 1)2(y^2 + py + q)$ Hence find all the factors of the quadratic equation.
17. Factorise (a) $2y^3 – y^2 + 2y – 1$
 (b) $2y^3 + 5y^2 + y – 2$
18. Use the synthetic approach to find the remainder when:
 (a) $8y^3 – 10y^2 + 7y + 3$ is divided by $2y – 1$
 (b) $5 + 6x + 7x^2 – x^3$ is divided by $x + 2$
19. Find the range of values of q for which
 $(2 – 3q)x^2 + (4 – q)x + 2 = 0$ has no real roots.
20. Find the value of k for which the line $y = mx + c$ is a tangent to the curve $x^2 + xy + 2 = 0$.
21. Express the polynomial $f(x) = 2y^4 + y^3 – y^2 + 8y – 4$ as a product of two linear factors and a quadratic factor $q(y)$. Prove that there are no real values of y for which $q(y) = 0$.
22. The polynomial $ax^3 + bx^2 – 5x + 1$ has $2x – 1$ and $x – 1$ as its two factors. Find a and b.
23. $f(x) = 2x^3 + px^2 + qx + 6$ where p and q are constants. When f(x) is divided by $x – 1$, the remainder is -6, when divided by $(x + 1)$ the remainder is 12. Show that $f(½) = 0$ hence write $f(x)$ as a product of linear factors.
24. Find the remainder when
 (a) $3x^5 – x^2 + 1$ is divided by $x + 2$
 (b) $x^4 – 2x^2 + 3x – 6$ is divided by $x^2 + 4x + 3$
25. Use long division to find the missing factors:
 (a) $x^5 + x^4 + 3x^3 + 5x^2 + 2x + 8 = (x^2 – x + 2)(\ldots)$
 (b) $6x^5 + x^4 – x^3 – 15x + 5 = (3x – \ldots)(\ldots.)$
26. The expression $2x^3 + ax^2 + bx + 6$ is exactly divisible by $(x – 2)$ and on division by $(x + 2)$ gives a remainder of -12. Calculate the values of a and b and factorise the expression completely.
27. $f(x) = x^2 + ax + b$ when $f(x)$ is divided by $x – 2$ the remainder is 8 and when $f(x)$ is divided by $x + 3$ the remainder is 18. Find the values of constants a and b.
28. If $f(x)$ denotes the polynomial $2x^3 – 3x^2 – 8x – 3$, find the remainders when $f(x)$ is divided by:
 (i) $x – 1$ (ii) $x + 3$ (iii) $2x + 1$
29. State the remainder when the cubic polynomial $x^3 + ax^2 – 3x + 4$ is divided by $(x – 3)$ the remainder obtained is twice the remainder obtained when the polynomial is divided by $(x – 2)$. Calculate a.
30. When $f(x) = x^4 – 2x^3 + ax^2 – bx + c$ is divided by $x – 2$, the remainder is -24 and when divided by $x + 4$, the remainder is 240. Given that $x + 1$ is a factor of $f(x)$, show that $x – 1$ is also a factor.

31. Given that $f(x) = x^3 + kx^2 - 2x + 1$, When $f(x)$ is divided by $(x - k)$, the remainder is k. Find the possible values of k.
32. When the polynomial $p(x)$ is divided by $(x - 1)$, the remainder is 5 and when p(x) is divided by $(x - 2)$, the remainder is 7. Find the remainder when $p(x)$ is divided by $(x - 1)(x - 2)$.
33. When the polynomial $p(x)$ is divided by $(x - 2)$, the remainder is 4 and when $p(x)$ is divided by $(x - 3)$ the remainder is 7. Find by writing $p(x) = (x - 2)(x - 3)q(x) + ax$, the remainder when $p(x)$ is divided by $(x - 2)(x - 3)$. If $p(x)$ is cubic when the coefficient of $x3$ is unity and $p(1) = 1$ determine $q(x)$.
34. Find the quotient and remainder when:
    (a) $6x^2 - x + 2$ is divided by $2x + 1$
    (b) $6x^2 - 7x + 5$ is divided by $2x - 3$
    (c) $x^3 + 3x^2 - 2x + 1$ is divided by $x - 2$
    (d) $2x^3 - 3x^2 - 4x + 1$ is divided by $x - 4$
    (e) $4x^2 - 3x^2 + x + 2$ is divided by $2x + 3$
35. Use the remainder theorem to find the remainder when
    (a) $3x^2 + 2x - 4$ is divided by $x - 2$
    (b) $2x^3 + 4x^2 - 6x + 5$ is divided by $x - 1$
    (c) $8x^3 + 4x + 3$ is divided by $2x - 1$
    (d) $6x^3 - 2x^2 + 5x - 4$ is divided by x
    (e) $3x^3 + 6x - 8$ is divided by $x + 3$
36. The expression $2x^3 - 3x^2 + ax - 5$ gives a remainder of 7 when divided by $x - 2$. Find the value of the constant a.
37. The remainder when $x^3 - 2x^2 + ax + 5$ is divided by $(x - 3)$ is twice the remainder when the same expression is divided by $x + 1$. Find the value of the constant a.
38. The remainder when $cx^3 + 2x^2 - 5x + 7$ is divided by $x - 2$ is equal to the remainder when the same expression is divided by $x + 1$. Find the value of the constant c.
39. Given that $x - 4$ is a factor of $2x^3 - 3x^2 - 7x + b$, where b is a constant. Find the remainder when the same expression is divided by $2x - 1$.
40. The expression $cx^3 + dx^2 + 3x + 8$ leaves a remainder of -6 when divided by $x - 2$ and a remainder of -34 when divided by $x + 2$. Find the value of the constants c and d.
41. The expression $x^3 - x^2 + ax + \mathrm{b}$ has a factor of $x + 3$, and leaves a remainder of 6 when divided by $x - 3$. Find the values of the constants a and b and hence factorise the expression.
42. The remainder when the expression $x^3 - 2x^2 + ax + b$ is divided by $x - 2$ is five times the remainder when the same expression is divided by $x - 1$, and 12 less than the remainder when the same expression is divided by $x - 3$. Find the values of constants a and b.
43. Show that $(x - 2)$ is a factor of $x^3 - 9x^2 + 26x - 24$. Find the set of values of x for which $x^3 - 9x^2 + 26x - 24 < 0$
44. The expression $6x^2 + x + 7$ leaves the same remainder when divided by $x - a$ and by $x + 2a$, where $a \neq 0$. Calculate the value of a.
45. Given that $x^2 + px + q$ and $3x^2 + q$ have a common factor $x - \mathrm{b}$, where p, q and b are non-zero. Show that $3p^2 + 4q = 0$.
46. Express the polynomial $f(x) = 2x^4 + x^3 - x^2 + 8x - 4$ as a product of two linear factors and a quadratic factor $q(x)$. Prove that there are no real values of x for which $q(x) = 0$.
47. Find the remainder when:
    (a) $x^3 + 3x^2 - 4x + 2$ is divided by $x - 1$

(b) $x^3 - 2x^2 + 5x + 8$ is divided by $x - 2$
(c) $x^5 + x - 9$ is divided by $x + 1$
(d) $x^3 + 3x^2 + 3x + 1$ is divided by $x + 2$

48. Find the values of a in the expressions below when the following conditions are satisfied.
(a) $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$.
(b) $x^3 + x^2 + \mathrm{a}x + 8$ is divisible by $x - 1$
(c) $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$
(d) $x^4 - 3x^2 + 2x + \mathrm{a}$ is divisible by $x + 1$

49. Show that $2x^3 + x^2 - 13x + 6$ is divisible by $x - 2$, and hence find the other factors of the expression.

50. Show that $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$ and find the factors of the expression.

51. Factorise:
$x^3 - 2x^2 - 5x + 6$
$x^3 - 4x^2 + x + 6$
$2x^3 + x^2 - 8x - 4$
$2x^3 + 5x^2 + x - 2$

52. Find the values of a and b if $ax^4 + bx^3 - 8x^2 + 6$ has remainder $2x + 1$ when divided by $x^2 - 1$

53. The expression $px^4 + qx^3 + 3x^2 - 2x + 3$ has remainder $x + 1$ when divided by $x^2 - 3x + 2$. Find the values of p and q.

54. The expression $ax^2 + bx + c$ is divisible by $x - 1$, has remainder 2 when divided by $x + 1$ and has remainder 8 when divided by $x - 2$. Find the values of a, b and c.

55. $(x - 1)$ and $(x + 1)$ are factors of the expression
$x^3 + ax^2 + bx + c$ and leaves a remainder of 12 when divided by $x - 2$. Find the values of a, b, and c.

56. What are the values of a and b if $x - 3$ and $x + 7$ are factors of the quadratic equation $ax^2 + 12x + b$?

57. Show that $3x^3 + x^2 - 8x + 4$ is zero when $x = 2/3$, hence factorise the expression.

Answers

1. $p = 2, q = -3, r = 7$
2. $P^3 = 27R^2$
3. $a = 1, b = -1$
4. (a) 2 (b) -11
5. (a) -3 (b) -2
6. $a = 2, b = -1, c = -2$
7. $a = 1, b = -3$
8. 3
9. $(k + 1)(x - 3)(3k + 1)$
10. $a = 12, b = 4$
12. 1½
13. -2.5
14. $p = 31, q = -12$
15. $(x + 1)(x - 1)(x - 2)$
16. $p = -5, q = 6$; $(y - 1)(y - 2)(y - 3)$
17. (a) $(y^2 + 1)(2y - 1)$ (b) $(y + 1)(y + 2)(2y - 1)$
18. (a) 5 (b) 29
19. $-16 < q < 0$
20. ± 4
21. $(2y - 1)(y - 2)(y2 - y - 2)$
22. $a = 2, b = 1$
23. (a) $p = 3, q = -11$ b) $(2x - 1)(x + 2)(x - 1)$
24. (a) -99 (b) $35x - 39$
25. (a) $x^3 + 2x^2 + 3x + 4 = 0$ (b) $2x^4 + x^3 - 5$
26. $a = -9, b = 7$, $[(x - 1)(x - 3)(2x + 1)]$
27. $a = -1, b = 6$
28. (i) -12 (ii) -60 (iii) 0
29. $a = -10$
30. $a = -9, b = 2, c = 8$

31. $k = 1, \frac{1}{2} \pm \sqrt{\frac{3}{4}}$ 32. $(2x + 3)$

33. $[3x - 2, (x - 1)]$

34.(a) $3x - 2, 4$ (b) $3x + 1, 8$

(c) $x^2 + 5x + 8, 17$ (d) $2x^2 + 5x + 16, 65$

(e) $2x^2 - 3x^2 + 3x - 4, 14$

35. (a) 24 (b) 5 (c) 6 (d) -4 (e) 1

36. $a = 4$ 37. $a = -2$

38. $c = 1$ 39. $b = 52$, remainder = -56

40. $c = 1, d = -7$ 41. -8, 12, $(x - 1)2(x + 3)$

42. 3, -1 43. $x < 2$ or $3 < x < 4$

44. $\frac{1}{6}$ 46. $(2x - 1)(x + 2)(x^2 - x + 2)$

47. (a) 2 (b) 18 (c) -11 (d) -1

48. (a) -3 (b) -10 (c) 2 (d) 4

49. $(x + 3)(2x - 1)$ 50. $(2x - 1)(2x + 3)(3x + 1)$

51. (a) $(x - 1)(x + 2)(x - 3)$ (b) $(x + 1)(x - 2)(x - 3)$

(c) $(2x + 1)(x - 2)(x + 2)$ (d) $x + 1)(x + 2)(2x - 1)$

52. $a = 3, b = 2$ 53. $p = 1, q = -3$

54. $a = 3, b = -1, c = -2$ 55. $a = 2, b = -1, c = -2$

56. $a = 3, b = -63$ 57. $(x - 1)(x + 2)(3x - 2)$

BINOMIAL THEOREM

Objectives of the topic:

- Create rows of Pascal's triangle
- Compute factorial values
- Compute binomial co-efficient by the formula
- Expand powers of binomial by Pascal's triangle and by binomial theorem
- Approximate numbers using binomial expansion

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

We can use Pascal triangle to expand expressions of the form $(a + b)^n$

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

Pascal's triangle helps us to calculate the powers of a binomial $(a + b)^n$ without actually multiplying it

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Note

The literal factors are all combination of a and b where the sum of the components of the power is 4 $a^4, a^3b,\ a^2b^2, b^4$
The degree of each term is 4. The first term is actually a^4b^0 which is $a^4(1)$.

Thus, to expand $(a+b)^5$ we would anticipate the following terms in which the sum of all the components of the powers is 5.

$$?\ a^5 +\ ?a^4b +?\ \ a^3b^2 +?a^2b^3 +\ ?ab^4 +?b^5$$

The question is what are the co-efficients. We can obtain the co-efficients from the Pascal's triangle above (line five above)

Example

Use Pascal's triangle to expand the following.

a) $(x+2)^5$
b) $(2x-3)^3$
c) $\left(\frac{1}{x}+2x^2\right)^4$
d) $(x^2-1)^4$
e) $(2x-3y)^5$
f) $(2-3x)^6$

Solution

Consider the Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

We can use Pascal's triangle to find $(x + 2)^5$

$$=?x^5+?(x)^4(2)+?x^3(2)^2+?x^2(2)^3+?x(2)^4+?x^5$$

We can obtain the coefficients from the Pascal's triangle above (line 6).

$$(\mathbf{a})(x+2)^5 = 1(x^5)+5(x^4)(2)+10(x^3)(2)^2+10(x^2)(2)^3+5(x)(2)^4+1(2)^5$$

$$(x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

(b) $(2x-3)^3 = (2x)^3 + 3(2x)^2(-3) + 3(2x)(-3)^2 + (-3)^3$

$$= 8x^3 + -36x^2 + 54x - 27$$

$$= 8x^3 - 36x^2 + 54x - 27$$

(c) $\left(\frac{1}{x} + 2x^2\right)^4$

$$= \left(\frac{1}{x}\right)^4 + 4\left(\frac{1}{x}\right)^3(2x^2) + 6\left(\frac{1}{x}\right)^2(2x^2)^2 + 4\left(\frac{1}{x}\right)(2x^2)^3 + (2x^2)^4$$

$$= \frac{1}{x^4} + \frac{8}{x} + 24x^2 + 32x^5 + 16x^8$$

$$(x^2-1)^4 = (x^2)^4 + 4(x^2)^3(-1) + 6(x^2)^2(-1)^2 + 4(x^2)(-1)^3 + (-1)^4$$

$$= x^8 + -4x^6 + 6x^4 - 4x^2 + 1$$

$$= x^8 - 4x^6 + 6x^4 - 4x^2 + 1$$

$$(2x-3y)^5 = (2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)(-3y)^4 + (-3y)^5$$

$$= 32x^5 + -240x^4y + 720x^3y - 1080x^2y^3 + 810xy^4 - 243y^5$$

$$= 32x^5 - 240x^4y + 720x^3y - 1080x^2y^3 + 810xy^4 - 243y^5$$

$$(2-3x)^6 = (2)^6 + 6(2)^5(-3x) + 15(2^4)(-3x)^2 + 20(2^3)(-3x)^3 + 15(2^2)(-3x)^4 + 6(2)(-3x)^5 + (-3x)^6$$

$= 64 - 576x + 2160x^2 - 4320x^3 + 4860x^4 + -2916x^5 + 729x^6$

We have seen that Pascal's triangle can be used to expand $(a+b)^n$ for the known value of n where n is a positive integer.

However, as n becomes large it becomes difficult to determine the co-efficient of a triangle. Imagine a task to expand $(a + b)^{10000}$.

This is so tedious yet indeed, we may not require all the terms of the expansion but just few. in the above case, we can use binomial theorem.

Binomial Theorem

It states that if n is a positive integer then

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)a^{(n-2)}b^2}{2!} + \cdots + b^n$$

It is also stated as

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

An important particular case is when $(a = 1)$ and $(b = x)$ giving

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots x^n$$

The binomial expansion discussed up to now is for the case when the exponent is positive

For the case when the number n is not positive, the binomial expansion $(1 + x)^n$ is valid when $-1 < x < 1$ OR $|x| < 1$.

Example

Expand $\left(2+\frac{x}{3}\right)^4$

Solution:

Using the binomial theorem;

$$(a+b)^n = a^n + n a^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots b^n$$

$$\left(2+\tfrac{x}{3}\right)^4 = (2)^4 + 4(2)^3\left(\tfrac{x}{3}\right) + \frac{4(3)(2^2)(\tfrac{x}{3})^2}{2!}$$

$$+\frac{4(3)(2)(2)^1(\tfrac{x}{3})^3}{3!} + \frac{4(3)(2)(1)(2^0)(\tfrac{x}{3})^4}{4!}$$

$$= 16 + \frac{32}{3}x + \frac{48x^2}{18} + \frac{48x^3}{27\times 6} + \frac{1}{81}x^4$$

$$= 16 + \frac{32}{3}x + \frac{8}{3}x^2 + \frac{8}{27}x^3 + \frac{1}{81}x^4$$

Example II

Expand $(2x - 3y)^4$

Solution

Using binomial expansion,

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \cdots b^n$$

$$(2x-3y)^4 = (2x)^4 + (4)(2x)^3(-3y) + \frac{(4)(3)(2x)^2(-3y)^2}{2!} + \frac{(4)(3)(2)(2x)(-3y)^3}{3!} + (-3y)^4$$

$$= 16x^4 + -96x^3y + \frac{432}{2}x^2y^2 + -216xy^3 + 81y^4$$

$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

Example III

Expand $\left(\frac{x}{2}+\frac{2}{x}\right)^3$

Solution

$$(a+b)^n = a^n + n a^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots b^n$$

$$\left(\frac{x}{2}+\frac{2}{x}\right)^3 = \left(\frac{x}{2}\right)^3 + 3\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right) + \frac{3(2)\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^2}{2!} + \left(\frac{2}{x}\right)^3$$

$$= \frac{x^3}{8} + 3\left(\frac{x^2}{4}\right)\left(\frac{2}{x}\right) + \frac{6\left(\frac{x}{2}\right)\left(\frac{4}{x^2}\right)}{2} + \frac{8}{x^3}$$

$$= \frac{x^3}{8} + \frac{3}{2}x + \frac{6}{x^2} + \frac{8}{x^3}$$

The r^{th} term in a binomial expansion

(co-efficient of a term in a binomial expansion)

The r^{th} term of a binomial expansion is given by

$$U_{r+1} = {}^nC_r a^{(n-r)} b^r$$

$\mathbf{U_{r+1} = {}^nC_r a^{(n-r)} b^r}$

Example I

Write down the terms indicated in the expansion of the following and simplify your answers.

a) $(x+2)^8$, term in x^5

b) $(3x-2)^5$ term in x^3

c) $\left(2x-\frac{1}{2}\right)^{12}$ term in x^7

d) $(2x+y)^{11}$ term in x^3

Solution

$(x+2)^8 = (a+b)^n$

$n = 8,\ a = x,\ b = 2$

(a) $U_{r+1} = n\mathrm{C}_r\ (x)^{8-r}(2)^r$

$= 8\ \mathrm{C}_r\ 2^r x^{8-r}$

$\Rightarrow 8-r = 5$

$r = 3$

$8\mathrm{C}_3\ x^5(2)^3 = 448x^5$

(b) $(3x-2)^5$ term in x^3

$(3x-2)^5 = (a+b)^n$

$a = 3x,\ b = -2$

$U_{r+1} = n\mathrm{C}_r\ a^{n-r}b^r$

$5C_r(3x)^{5-r}(-2)^r$

$5C_r 3^{5-r}x^{5-r}(-2)^r$

$5-r = 3$

$r = 2$

$U_{(r+1)} = 5C_2(3x)^{(5-2)}(-2)^2$

$= 5C_2(4)(27x^3)$

$= 1080x^3$

(c) $\left(2x-\frac{1}{2}\right)^{12}$

$(2x-\tfrac{1}{2})^{12} = (a+b)^n$

$a = 2x,\ b = \dfrac{-1}{2}$

$= 12\mathrm{C}_r\ (2x)^{12-r}\left(-\dfrac{1}{2}\right)^r$

$= 12\mathrm{C}_r\ 2^{12-r}x^{12-r}\left(-\dfrac{1}{2}\right)^r$

$12-r = 7$

$r = 5$

$U_6 = 12\mathrm{C}_5\ (2)^7x^7\left(-\dfrac{1}{2}\right)^5$

$= -3168x^7$

(d) $(2x+y)^{11}$ term in x^3

$U_{r+1} = n\mathrm{C}_r\ a^{n-r}b^r$

$= 11\mathrm{C}_r\ (2x)^{11-r}y^r$

$= 11C_r 2^{11-r}x^{11-r}y^r$

$11-r = 3$

$r = 8$

$$11C_8\,(2)^3x^3y^8$$
$$= 1320\,x^3y^8$$

Example III

Find the term independent of x in expansion of $\left(x^2 - \frac{1}{3x}\right)^9$

Solution

$\left(x^2 - \frac{1}{3x}\right)^9 = (a+b)^n$

$a = x^2,\ b = \frac{-1}{3x}$

$$= 9C_r\,(x^2)^{9-r}\left(-\frac{1}{3}x\right)^r$$

$= 9C_r\,x^{18-2r}\left(-\frac{1}{3}\right)^r x^{-r}$

$= 9C_r\,x^{18-3r}\left(-\frac{1}{3}\right)^r$

$18 - 3r = 0$

$r = 6$

$={}^9C_6x^0\left(-\frac{1}{3}\right)^6$

$= 84\left(\frac{1}{729}\right)$

$= \frac{28}{243}$

Example IV

Find the co-efficient of *x* in the expansion of $\left(x + \frac{2}{x^2}\right)^{10}$

Solution

$$\left(x + \frac{2}{x^2}\right)^{10} = (a+b)^n$$
$$a = x,\ b = \frac{2}{x^2},\ n = 10$$
$$U_{r+1} = nC_r\,a^{n-r}b^r$$
$$\Rightarrow 10C_rx^{10-r}\left(\frac{2}{x^2}\right)^r$$
$$= 10C_r\,x^{10-r}(2)^r(x^{-2})^r$$
$$= 10C_rx^{10-r}(2)^rx^{-2r}$$
$$= 10C_r\,2^rx^{10-3r}$$

$\Rightarrow\ 10 - 3r = 1$

$r = 3$

$= 10C_32^3x$

$= 960x$

The coefficient is 960

Example V

Find the co-efficient of the term in x^6 in the expansion $\left(x - \frac{2}{x}\right)^8$

Solution

$$\left(x-\frac{2}{x}\right)^8=(a+b)^n$$
$$a=x,\ b=\frac{-2}{x}$$
$$U_{r+1}=n\mathrm{C}_r\ a^{n-r}b^r$$
$$=8\mathrm{C}_r\ x^{8-r}\ \left(-\frac{2}{x}\right)^r$$
$$=8\mathrm{C}_r\ x^{8-r}\left(\frac{(-2)^r}{x^r}\right)$$
$$=8\mathrm{C}_r\ x^{8-r}(-2)^r x^{-r}$$
$$=8\mathrm{C}_r\ (-2)^r\ x^{8-2r}$$
$$8-2r=6$$
$$2=2r$$
$$r=1$$
$$8\mathrm{C}_1(-2)^1x^6$$
$$=-16x^6$$

The coefficient is -16

Validity of a Binomial Expansion

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\ldots$$

When n is not positive the binomial theorem is valid for $-1<x<1$ or when $|x|<1$

Example I

State what values of *x* for which the following expansions are valid:

a) $\left(1-\frac{x}{2}\right)^{-5}$

b) $(1+2x)^{\frac{1}{2}}$

c) $\left(\frac{1}{(2+x)^2}\right)$

d) $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$

Solution

$(1+x)^n$ is valid when $|x|<1$

So $\left(1-\frac{x}{2}\right)^{-5}$ is valid $\left|-\frac{x}{2}\right|<1$

$\Rightarrow \pm\left(-\frac{x}{2}\right)<1$

$-\frac{x}{2}<1$

$-x<2$

$-x<2$

$x>-2$

$-\left(-\frac{x}{2}\right)<1$

$\frac{x}{2}<1$

$x < 2$
$\Rightarrow -2 < x < 2$
$\left(1-\frac{x}{2}\right)^{-5}$ is valid when $-2 < x < 2$

(b) $(1+2x)^{\frac{1}{2}}$
$(1+x)^n$ is valid when $|x| < 1$
$|2x| < 1$
$\pm(2x) < 1$
$x < \frac{1}{2}$
$-2x < 1$
$x > -\frac{1}{2}$
$\frac{-1}{2} < x < \frac{1}{2}$
$\Rightarrow (1+2x)^{\frac{1}{2}}$ is valid when $-\frac{1}{2} < x < \frac{1}{2}$

(c) $\frac{1}{(2+x)^2} = (2+x)^{-2}$
$= 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$
it is valid when $\left|\frac{x}{2}\right| < 1$
$\pm\left(\frac{x}{2}\right) < 1$
$x < 2$
$-\frac{x}{2} < 1$
$x > -2$
$-2 < x < 2$

$\Rightarrow \frac{1}{(2+x)^2}$ is valid $-2 < x < 2$

(d) $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$, for validity $\left|\frac{1}{x}\right| < 1$
$\frac{1}{|x|} < 1, \;\; 1 < |x|$

$|x| > 1$ for expansion $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$ to be valid

More examples on Binomial Expansion

Example I (UNEB Question)

Expand $(1-x)^{\frac{1}{3}}$ as far as the term in x^3. Hence evaluate $\sqrt[3]{24}$

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

$$(1-x)^{1/3} = 1 + \frac{1}{3}(-x) + \frac{1/3(1/3-1)(-x)^2}{2!} + \frac{1/3(1/3-1)(1/3-2)(-x)^3}{3!} + \dots$$

$$(1-x)^{\frac{1}{3}} = 1 + \frac{1}{3}(-x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(-x)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!} \ldots$$

$$(1-x)^{\frac{1}{3}} = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \cdots$$

$$\sqrt[3]{24} = \sqrt[3]{27-3} = (27-3)^{\frac{1}{3}}$$

$$(27-3)^{\frac{1}{3}} = 27^{\frac{1}{3}}\left(1-\frac{3}{27}\right)^{\frac{1}{3}} = 3\left(1-\frac{1}{9}\right)^{\frac{1}{3}}$$

Comparing $\left(1-\frac{1}{9}\right)^{\frac{1}{3}}$ with $(1-x)^{\frac{1}{3}} \Rightarrow x = \frac{1}{9}$

$$\left(1-\frac{1}{9}\right)^{\frac{1}{3}} = 1 - \frac{1}{3}\left(\frac{1}{9}\right) - \frac{1}{9}\left(\frac{1}{9}\right)^2 - \frac{5}{81}\left(\frac{1}{9}\right)^3 + \cdots$$

$$\left(1-\frac{1}{9}\right)^{\frac{1}{3}} \approx 0.961506545$$

$$(27)^{\frac{1}{3}} \approx 3\left(1-\frac{1}{9}\right)^{\frac{1}{3}}$$

= 3(0.961506545)
= 2.8845

$$\Rightarrow \sqrt[3]{24} = 2.8845$$

Example II

Determine the expansion of $\left(1-\frac{x}{3}\right)^{\frac{1}{2}}$ as far as the term in x^3 Hence evaluate $\sqrt{8}$

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

$$\left(1-\frac{x}{3}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\left(-\frac{x}{3}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{x}{3}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{x}{3}\right)^3}{3!} + \cdots$$

$$\left(1-\frac{x}{3}\right) = 1 - \frac{x}{6} - \frac{x^2}{72} - \frac{1}{432}x^3 + \cdots$$

$$\sqrt{8} = (8)^{\frac{1}{2}} = (9-1)^{\frac{1}{2}}$$

$$= 9^{\frac{1}{2}}\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$$

$$= 3\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$$

Comparing $\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$ with $\left(1-\frac{x}{3}\right)^{\frac{1}{2}}$

$$\Rightarrow \frac{x}{3} = \frac{1}{9},\ x = \frac{1}{3}$$

$$\Rightarrow \left(1-\frac{1}{9}\right)^{\frac{1}{2}} = 1 - \frac{\frac{1}{3}}{6} - \frac{\left(\frac{1}{3}\right)^2}{72} - \frac{1}{432}\left(\frac{1}{3}\right)^3 + \cdots$$

$$\sqrt{8} = 3\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$$

$$\sqrt{8} = 3\left(1 - \frac{1}{18} - \frac{1}{648} - \frac{1}{11664} + \cdots \ldots \ldots \ldots\right)$$

$$\sqrt{8} = 2.8284$$

Example III

Expand $\left(1+\frac{x}{3}\right)^{15}$ up to and including the term in x^3.

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

$$\left(1+\frac{x}{3}\right)^{15} = 1 + 15\left(\frac{x}{3}\right) + \frac{15(14)\left(\frac{x}{3}\right)^2}{2!} + \frac{(15)(14)(13)\left(\frac{x}{3}\right)}{3!} + \cdots$$

$$\left(1+\frac{x}{3}\right)^{15} = 1 + \frac{15x}{3} + \frac{35x^2}{3} + \frac{455x^3}{27} + \cdots$$

Example IV

Expand $(1+2x)^{\frac{1}{2}}$ as far as the term inx^4.

Solution

$$(1+2x)^{\frac{1}{2}}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1+2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(2x)^3}{3!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(2x)^4}{4!} + \cdots$$

$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \cdots$$

$$(1+2x)^{\frac{1}{2}} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \cdots$$

Example V

Expand $\left(1-\frac{x}{2}\right)^{-5}$ as far as the fourth term

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

$$\left(1-\frac{x}{2}\right)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

$$\left(1-\frac{x}{2}\right)^{-5} = 1 + (-5)\left(-\frac{x}{2}\right) + \frac{(-5)(-6)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(-5)(-6)(-7)\left(-\frac{x}{2}\right)^3}{3!} + \cdots$$

$$\left(1-\frac{x}{2}\right)^{-5} = 1 + \frac{5}{2}x + \frac{15}{4}x^2 + \frac{35}{8}x^3 + \cdots$$

Example VI

Expand $(3+x)^{-1/2}$ as far as the third term

Solution

$$(3+x)^{-1/2} = \left[3\left(1+\frac{x}{3}\right)\right]^{-1/2}$$

$$= 3^{-1/2}\left(1+\frac{x}{3}\right)^{-1/2}$$

$$(3+x)^{-\frac{1}{2}} = 3^{(-\frac{1}{2})}\left(1+\frac{x}{3}\right)^{\frac{-1}{2}}$$

$$= \frac{1}{3^{\frac{1}{2}}}\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}$$

But $\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}$

$$= 1 + \left(-\frac{1}{2}\right)\left(\frac{x}{3}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{3}\right)^2}{2!} + \cdots$$

$$= 1 - \frac{x}{6} + \frac{x^2}{24} + \cdots$$

$$(3+x)^{-\frac{1}{2}} = 3^{-\frac{1}{2}}\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{3}}\left(1 - \frac{x}{6} + \frac{x^2}{24} + \cdots\right)$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{6\sqrt{3}}x + \frac{1}{24\sqrt{3}}x^2 + \cdots$$

Example VII

Find the expansion of $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$ as far as the term x^{-3}

Solution:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

$$\left(1+\frac{1}{x}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\left(\frac{1}{x}\right) + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{1}{x}\right)^2}{2!} + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{1}{x}\right)^3}{3!} + \ldots$$

$$\left(1+\frac{1}{x}\right)^{\frac{1}{2}} = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3} + \cdots$$

Example VIII

Expand $(1-x)^3(2+x)^6$ up to the term in x^2.

Solution

$(1-x)^3(2+x)^6$

$$= \left(1 + 3(^-x) + \frac{3(2)(-x)^2}{2!} + \ldots\right)\left(2^6 + 6(2)^5x + \frac{6(5)(2^4)x^2}{2!} + \ldots\right)$$

$= (1 - 3x + 3x^2 \ldots)(64 + 192x + 240x^2 + \cdots)$

$= 64 + 192x + 240x^2 - 192x - 576x^2 + 192x^2 + \ldots$

$= 64 - 144x^2$

Example IX

Expand $\dfrac{(1+x)^2}{(1-\frac{x}{2})^3}$ up to and including the term in x^2.

Solution

$$\frac{(1+x)^2}{\left(1-\frac{x}{2}\right)^3} = (1+x)^2\left(1-\frac{x}{2}\right)^{-3}$$

$$= (1+2x+x^2)\left(1+(-3)(\tfrac{-x}{2})+\frac{(-3)(-4)(\tfrac{-x}{2})^2}{2!}+\dots\right)$$

$$\left(1\ +\ 2x\ +\ x^2\right)\left(1+\tfrac{3}{2}x+\tfrac{3}{2}x^2...\right)$$

$$1+\frac{3x}{2}+\frac{3x^2}{2}+2x+3x^2+x^2....$$

$$1+\frac{7x}{2}+\frac{11x^2}{2}+...$$

$$=1+\frac{7}{2}x+\frac{11}{2}x^2+\dots$$

Example X

Show that, if x is small enough for its cube and higher powers to be neglected,

$$\sqrt{\frac{1-x}{1+x}}=1-x+\frac{1}{2}x^2+\dots$$

By putting $x=\frac{1}{8}$, show that $\sqrt{7}=2\frac{83}{128}$

Solution

$$\sqrt{\frac{1-x}{1+x}}=\sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}$$

$$=\frac{1-x}{\sqrt{1-x^2}}$$

$$=(1-x)(1-x^2)^{-\frac{1}{2}}$$

From

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\cdots$$

$$(1-x^2)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)(-x^2)+\cdots$$

$$(1-x^2)^{-\frac{1}{2}}=1+\frac{1}{2}x^2+\cdots$$

$$(1-x)(1-x^2)^{-\frac{1}{2}}=(1-x)\left(1+\frac{1}{2}x^2+\cdots\right)$$

$$=1+\frac{1}{2}x^2-x-\frac{1}{2}x^3+\cdots$$

$$=1-x+\frac{1}{2}x^2+\cdots$$

$$\Rightarrow\sqrt{\frac{1-x}{1+x}}=1-x+\frac{x^2}{2}+\cdots$$

By putting $x=\frac{1}{8}$

$$\sqrt{\frac{1-\frac{1}{8}}{1+\frac{1}{8}}}=1-\frac{1}{8}+\frac{\left(\frac{1}{8}\right)^2}{2}+\cdots$$

$$\sqrt{\frac{7/8}{9/8}}=1-\frac{1}{8}+\frac{1}{128}+\dots$$

$$\Rightarrow \frac{\sqrt{7}}{3} = \frac{128-16+1}{128} + \dots$$

$$\sqrt{7} = \frac{113}{128} \times 3$$

$$\sqrt{7} = \frac{339}{128}$$

$$\sqrt{7} = 2\frac{83}{128}$$

Example XI

Use Binomial expansion to expand $\sqrt{\frac{1+2x}{1-2x}}$ up to and including the term in x^3. Hence find $\sqrt{\frac{1.02}{0.98}}$ to 4 decimal places.

Hence deduce the square root of 51

Solution

$$\sqrt{\frac{(1+2x)(1+2x)}{(1-2x)(1+2x)}} = \frac{1+2x}{\sqrt{1-4x^2}}$$

$$= (1+2x)(1-4x^2)^{-\frac{1}{2}}$$

$$\text{From } (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$(1-4x^2)^{-\frac{1}{2}} = 1 + \frac{-1}{2}(-4x^2) + \cdots$$

$$\Rightarrow (1+2x)(1-4x^2)^{-\frac{1}{2}} = (1+2x)(1+2x^2+\dots)$$

$$= 1 + 2x^2 + 2x + 4x^3 + \dots$$

$$\Rightarrow \sqrt{\frac{1+2x}{1-2x}} = 1 + 2x + 2x^2 + 4x^3 + \cdots$$

$1 + 2x = 1.02$

$x = 0.01$

$$\sqrt{\frac{1.02}{0.98}} = 1 + 2(0.01) + 2(0.01)^2 + 4(0.01)^3 + \cdots$$

$$\sqrt{\frac{102}{98}} = 1.020204$$

$$\frac{\sqrt{2} \times \sqrt{51}}{\sqrt{2} \times \sqrt{49}} = 1.020204$$

$$\frac{\sqrt{51}}{7} = 1.020204$$

$$\sqrt{51} = (7 \times 1.020204)$$

$$\sqrt{51} = 7.1414$$

Example XII

Given that, the first three terms of the expansion in ascending powers of x of $(1 + x + x^2)^n$ are the same as the first three terms in the expansion of $\left(\frac{1+ax}{1-3ax}\right)^3$, find the values of a and n.

Solution

$(1+x+x^2)^n$ from

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$(1+x+x^2)^n = 1 + n(x+x^2) + \frac{n(n-1)(x+x^2)^2}{2!} + \ldots$$

$$= 1 + nx + nx^2 + \frac{n(n-1)x^2}{2} + \cdots$$

$$= 1 + nx + \left(n + \frac{n(n-1)}{2}\right)x^2 + \cdots \ldots\ldots (1)$$

$$\left(\frac{1+ax}{1-3ax}\right)^3 = (1+ax)^3(1-3ax)^{-3}$$

$$(1+ax)^3 = 1 + 3(ax) + \frac{3(2)(ax)^2}{2!} + \cdots$$

$$= 1 + 3ax + \frac{6a^2x^2}{2} + \cdots$$

$$(1+ax)^3 = 1 + 3ax + 3a^2x^2 + \cdots$$

$$(1-3ax)^{-3} = 1 + (-3)(-3ax) + \frac{(-3)(-4)(-3ax)^2}{2} + \cdots$$

$$= 1 + 9ax + 54a^2x^2 + \cdots$$

$$(1+ax)^3(1-3ax)^{-3} = (1 + 3ax + 3a^2x^2 + \cdots)(1 + 9ax + 54a^2x^2 + \cdots)$$

$$= 1 + 9ax + 54a^2x^2 + 3ax + 27a^2x^2 + 3a^2x^2 + \cdots$$

$$= 1 + 12ax + 84a^2x^2 + \cdots (2)$$

Comparing Eqn (1) and Eqn (2);

$\Rightarrow n = 12a$

$$n + \frac{n(n-1)}{2} = 84a^2$$

$$12a + \frac{12a(12a-1)}{2} = 84a^2$$

$$24a + 144a^2 - 12a = 168a^2$$

$$12a - 24a^2 = 0$$

$$12a(1-2a) = 0$$

$$a = 0 \quad a = \frac{1}{2}$$

$$n = 12a$$

$$\Rightarrow \quad n = 12 \times \frac{1}{2} = 6$$

Example XIII

Find the first three terms of the expansion

$(4+x)^{-\frac{1}{2}}$ in ascending powers of x. Deduce the approximate value of $\frac{1}{\sqrt{4.16}}$

Solution

$$(4+x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1 + \frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left[1 + (\tfrac{-1}{2})(\tfrac{x}{4}) + \frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{x}{4})^2}{2} + \ldots\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{8}x - \frac{3}{128}x^2 + \cdots\right)$$

$$(4+x)^{-\frac{1}{2}} = (\frac{1}{2} - \frac{1}{16}x - \frac{3}{256}x^2 + \cdots)$$

$$\frac{1}{\sqrt{4.16}} = \frac{1}{\sqrt{4+0.16}} = \frac{1}{2} - \frac{1}{16}\times 0.16 + \frac{3}{256}\times 0.16^2$$

$$\frac{1}{\sqrt{4.16}} \approx 0.4903$$

Example XIV

Write down the expansions in ascending powers of x up to the term in x^2.

(a) $(1+x)^{\frac{1}{2}}$

(b) $(1-x)^{\frac{-1}{2}}$

Hence or otherwise Expand $\sqrt{\frac{1+x}{1-x}}$ in ascending power of x up to the term in x^2. By substituting $x = \frac{1}{10}$, use your expansion to find the $\sqrt{11}$.

Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}(x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2}{2} + \cdots$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(\frac{-1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2} + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

$$\sqrt{\frac{1+x}{1-x}} = \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$= (1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots)(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots)$$

$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{8}x^2\ldots$$

$$= 1 + x - \tfrac{1}{2}x^2 + \ldots$$

$$\Rightarrow \sqrt{\frac{1+x}{1-x}} = 1 + x - \frac{1}{2}x^2$$

When $x = \frac{1}{10}$

$$\sqrt{\frac{1+\frac{1}{10}}{1-\frac{1}{10}}} = 1 + \frac{1}{10} - \frac{1}{2}\left(\frac{1}{10}\right)^2 + \cdots$$

$$\frac{\sqrt{11}}{\sqrt{9}} = 1 + \frac{1}{10} - \frac{1}{200} + \cdots$$

$$\sqrt{11} = 3\left(1 + \frac{1}{10} - \frac{1}{200} + \cdots\right)$$

Example (UNEB Question)

(a) By using the binomial theorem, expand $(8-24x)^{2/3}$ as far as the 4th term. Hence evaluate $4^{2/3}$ to one decimal place.

b) Find the coefficient of x in the expansion of $\left(x+\frac{2}{x^2}\right)^{10}$

Solution

a) By using binomial distribution theorem,

$$(1+y)^n = 1 + n(y) + \frac{n(n-1)}{2!}y^2 + \frac{n(n-2)}{3!}y^3 + \ldots$$

Now $(8-24x)^{2/3} = 8^{2/3}(1-3x)^{2/3} = 4(1-3x)^{2/3}$

Here $n = 2/3$ and $y = -3x$

By substitution,

$$4(1-3x)^{2/3} = 4\left[1 + \frac{2}{3}(-3x) + \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\frac{(-3x)^2}{2} + \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\frac{(-3x)^3}{6}\right]$$

$$= 4\left[1 - 2x - x^2 - \frac{4x^3}{3}\right]$$

$$= 4 - 8x - 4x^2 - \frac{16}{3}x^3$$

Evaluating $4^{2/3}$:

$$4^{2/3} = (8-4)^{2/3}$$

$$= 8^{2/3}\left(1-\frac{1}{2}\right)^{2/3} = 4\left(1-\frac{1}{2}\right)^{2/3}$$

Comparing $4(1-3x)^{2/3}$ with $4\left(1-\frac{1}{2}\right)^{2/3}$

$$\Rightarrow 3x = \frac{1}{2}$$

$$x = \frac{1}{6}$$

$$\Rightarrow 4\left(1-\frac{1}{2}\right)^{2/3} = 4 - 8\left(\frac{1}{6}\right) - 4\left(\frac{1}{36}\right) - \frac{16}{3}\left(\frac{1}{216}\right)$$

$$= 2.5 \quad (1dp)$$

b) When expanding $\left(x+\frac{2}{x^2}\right)^{10}$, we could use Pascal's triangle to get the coefficient of *x*, but this may seem tedious, so by using binomial expansion

$$\left(x+\frac{2}{x^3}\right)^{10} = x^{10}\left(1+\frac{2}{x^3}\right)^{10}$$

Here, $y = \frac{2}{x^3}$ $\quad n = 10$

$$\Rightarrow x^{10}\left(1+\frac{2}{x^3}\right)^{10} = x^{10}\left[1+\frac{20}{x^3}+\frac{90}{2}\left(\frac{4}{x^6}\right)+\frac{10\times9\times8}{6}\times\frac{8}{x^9}+\ldots\right]$$

$$x^{10}\left(1+\frac{2}{x^3}\right)^{10} = x^{10}+20x^7+18x^4+960x+\ldots$$

The coefficient of x is 960

OR By using the expansion of

$$(a+b)^n = a^n + n^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + ..+b^n$$

$$\left(x+\frac{2}{x^2}\right)^{10} = x^{10}+10x^9\left(\frac{2}{x^2}\right)+\frac{10\times9\times x^8}{2}\times\left(\frac{2}{x^2}\right)+\frac{10\times9\times8\times x^7}{6}\times\left(\frac{2}{x^2}\right)^3+..$$

$$= x^{10}+20x^2+180x^4+960x$$

The coefficient of x is 960

OR We could handle it by using direct approach

The term in $10_{C_3}x^7\left(\frac{2}{x^2}\right)^3$ is the one needed which is expanded as,

$$10_{C_3}x^7\left(\frac{2}{x^2}\right)^3 = 10_{C_3}\times x^7\times\frac{8}{x^6} = 10_{C_3}\times 8x$$

The coefficient = $10_{C_3}\times 8$

$= 120 \times 8$

$= 960$

Revision Exercise

1. Write down and simplify the terms indicated in the expression of the following in ascending powers of x.
   (a) $(1+x)^9$, 4th term
   (b) $\left(2-\frac{x}{2}\right)^{12}$, 4th term
   (c) $(3+x)^7$, 5th term
   (d) $(x+1)^{20}$, 3rd term.
2. Expand $(1+\frac{3}{2}x-x^2)^5$ in ascending powers of x as far as the term in x^4.
3. Use the Binomial theorem to expand:
   (a) $(x+y)^4$
   (b) $(a-b)^7$
   (c) $(2+p^2)^6$
   (d) (2h – k)5
   (e) $\left(x+\frac{1}{x}\right)^3$
   (f) $\left(z+\frac{1}{2z}\right)^8$
4. Expand the following using the Binomial theorem
   (a) $(1+3x)^4$
   (b) $(2x+y)^4$
   (c) $(2-3x)^6$
5. Write down and simplify the coefficients of the terms indicated in the expansions of the following
   (a) $\left(\frac{1}{2}t+\frac{1}{2}\right)^{10}$, term in t^4
   (b) $\left(4+\frac{3}{4}x\right)^6$, term in x^3

(c) $(2x - 3)^7$, term in x^5

6. Expand $\dfrac{7+x}{(1+x)(1+x^2)}$ in ascending powers of x as far as the term in x^4.
7. Expand and simplify $\left(2x+\dfrac{1}{x^2}\right)^5+\left(2x-\dfrac{1}{x^2}\right)^5$
8. Use the Binomial theorem to expand $(1 + x)^{12}$ in ascending powers of x up to and including the term in x^3.
9. Write down the coefficients of the terms indicated in the expansions of the following in ascending powers of x
   (a) $(1 + x)^{16}$, 3rd term
   (b) $(2 - x)^{20}$, 18th term
   (c) $(3 + 2x)^6$, 4th term
   (d) $\left(2+\frac{3}{2}x\right)^8$, 5th term
10. If x is so small that x^3 and higher powers can be neglected, show that $\left(1-\frac{3}{2}x\right)^5\left(2+3x\right)^6 = 64 + 96x - 720x^2$
11. The coefficient of x^3 in the expansion of $(1 + x)^2$ is four times the coefficient of x^2. Find the value of n.
12. In the Binomial expansion of $\left(1-\frac{1}{3}x\right)^n$, the fourth and fifth terms are equal. find the value of n
13. The coefficient of x^5 in the expansion of $(1 + 5x)^8$ is equal to the coefficient of x^4 in the expansion of $(a + 5x)^7$. Find the value of a.
14. If the first three terms of the expansion of $(1 + ax)^n$ in ascending powers of x are $1 - 4x + 7x^2$, find n and a.
15. Use the expansion of $(a + b)^4$ to evaluate $(1.03)^4$ correct to 2 decimal places.
16. If x is so small to allow any term in x^5 or higher powers of x to be neglected, show that $(1 + x)^6(1 - 2x^3)^{10} \approx 1 + 6x + 15x^2 - 105x^4$.
17. When $(1 + ax)^n$ is expanded in ascending powers of x, the expansion is $1 + 2x + \dfrac{15x^2}{8} + \ldots$. Find the values of n and a.
18. When $(1 + ax)^{10}$ is expanded in ascending powers of x, the series expansion is $A + Bx + Cx^2 + 15x^3 + \ldots$. Find the values of a, A, B and C.
19. Find the ratio of the term in x^7 to the term in x^8 in the expansion of $\left(1-\frac{1}{3}x\right)^n\left(3x+\frac{2}{3}\right)^7$
20. Expand the following in ascending powers of x as far as the terms in x^3 and state the values of x for which the expansions are valid.
   (a) $(1 + x)^{-2}$
   (b) $(1+x)^{1/3}$
   (c) $(1+x)^{3/2}$
   (d) $(1+\frac{x}{2})^{-3}$
21. Write down and simplify the term independent of x in the expansion of $\left(3x^2-\frac{1}{2x}\right)^9$ which is the numerically greatest term in this expansion when $x = \dfrac{1}{2}$.
22. In the binomial expansion of $(1 + x)^{n+1}$, n being an integer greater than 2, the coefficient of x^4 is six times the coefficient of x^2 in the expansion of $(1 + x)^{n-1}$. Determine the value of n.
23. Find the value of n for which the coefficients of x, x^2 and x^3 in the expansion of $(1 + x)^n$ are in arithmetical progression.
24. Express $\dfrac{2x^3}{(1+x^2)(1-x)^2}$ as a sum of three partial fractions and obtain an expansion in ascending powers of x of this expression as far as the term involving x^7.
25. If ${}^nC_r$ denotes the coefficient of x^r in the expansion of $(1 + x)^n$, prove that ${}^nC_r + 2({}^nC_{r+1}) + {}^nC_{r+2} = {}^{n+2}C_{r+2}$
26. If the coefficients of x^{r-1}, x^r, x^{r+1} in the binomial expansion of $(1 + x)^n$ are in arithmetical progression. Prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.
27. Show that if x is so small that x^4 and higher powers of x can be neglected, then

$$\left[(1+x/2)^3-(1+3x)^{1/2}\right]\div(1-\tfrac{5x}{6})=\frac{15x^2}{8}$$

28. (a) If $x-\frac{1}{4}=u$, express $x^3-\frac{1}{x^3}$ and $x^5-\frac{1}{x^5}$ in terms of u.

(b) Assuming that $(1-2kx+x^2)^{-1/2}$ may be expanded in a series of ascending powers of x, obtain the expansion as far as the term in x^3. Simplify the coefficients.

29. Write down the expansion in ascending powers of x. up to the term in x^2 of (i) $(1+x)^{1/2}$

(ii) $(1+x)^{-1/2}$ and simplify the coefficients.

Hence or otherwise, expand $\sqrt{\frac{1+x}{1-x}}$ in ascending powers of x up to the term in x^2. By using $x=\frac{1}{10}$, obtain an estimate, to three decimal places for π.

30. Find the term independent of x in the expansion of

(a) $\left(x^2-\frac{1}{3x}\right)^9$ (b) $\left(x-\frac{1}{x}\right)^8\left(x+\frac{1}{x^2}\right)^4$

Answers

1. (a) $84x^3$ (b) $-14080x^3$
   (c) $945x^4$ (d) $190x^2$.
2. $1-\frac{15}{2}x+\frac{35}{2}x^2-\frac{15}{4}x^3-\frac{515}{16}x^4 \ldots$
3. (a) $x^4+4x^3y+6x^2y^2+4xy^3+y^4$
   (b) $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a2b^5+7ab^6-b^7$
   (c) $64+192p^2+240p^4+160p^6+60p^8+12p^{10}+p^{12}$
   (d) $32h^5-80h^4c+80h^3k^2-40h^2k^3+10hk^4-k^5$
   (e) $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$
   (f) $z^8-4z^6+7z^4-7z^2+\frac{7}{4z^2}+\frac{7}{6z^4}-\frac{1}{16z^6}+\frac{1}{256z^8}$
4. (a) $1+12x+54x^2+108x^3+81x^4$
   (b) $16x^4+32x^3y+24x^2y^2+8xy^3+y^4$
   (c) $64-576x+2160x^2-4320x^3+4860x^4-2916x^5+729x^6$
5. (a) $\frac{105}{512}$ (b) 540 (c) 6048
6. $7-6x-x^2+7x^4+\ldots$.
7. $64x^5+\frac{160}{x}+\frac{20}{x^7}$.
8. $1+12x+66x^2+220x^3$
9. (a) 120 (b) -9120 (c) 4320 (d) 5670.

11. 14 **12.** 15 **13.** 2 **14.** 18, -½

15. 1.13 **17**. 16, ⅛ **18.** ½, 1, 5, $11\frac{1}{4}$

19. $\frac{8}{45x}$

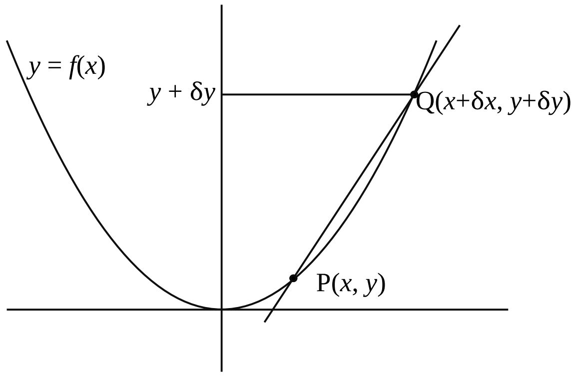
20. (a) $1-2x+3x^2-4x^3$, $-1 < x < 1$

DIFFERENTIATION I

Suppose we have a smooth function *f(x)* which is represented graphically by the curve $y = f(x)$. Then we can draw a tangent to the curve at point *P*. It is important to be able to calculate the slope of the tangent of the curve. A graphical method can be used but this is rather imprecise, so we use the following analytical method.

We chose a second point Q on the curve which is near P and join the two points with a tangent line *PQ* called secant and calculate the slope of the line.

Then we can allow Q to approach P so that the secant swings around until it just touches the curve and become a tangent. The limit of the slope of the secant is required to find the slope of the tangent.



The gradient of the secant PQ =

$$\frac{\Delta y}{\Delta x} = \frac{\partial y}{\partial x} = \frac{f(x+\partial x) - f(x)}{x + \partial x - x}$$

$$= \frac{f(x+\partial x) - f(x)}{\partial x}$$

$\Rightarrow$ The gradient of the tangent at P($f'(x)$)

$$f'(x) = \lim \frac{f(x+\partial x) - f(x)}{\partial x}$$

$$\text{as } \partial x \longrightarrow 0$$

Example

Find the gradient of the tangent to the curve $y = x^2$.

Solution

The gradient of the tangent to the curve $y = f(x)$

$$\frac{dy}{dx}(f'x) = \lim \frac{f(x+\partial x) - f(x)}{\partial x}$$

$$\partial x \longrightarrow 0$$

$f(x) = x^2$

$f(x + \partial x) = (x + \partial x)^2$

$= x^2 + 2x\partial x + \partial x^2$

$$\Rightarrow f'(x) = \left[\lim \frac{x^2 + 2x\partial x + (\partial x)^2 - x^2}{\partial x} \right]$$

$$\partial x \longrightarrow 0$$

$$f'(x) = \lim_{\partial x \longrightarrow 0}(2x + \partial x)$$

$$= 2x$$

$$\Rightarrow \frac{dy}{dx} = 2x$$

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1} = f'(x)$

For example: If $y = x^4$

$$\frac{dy}{dx} = 4x^{4-1}$$

$$= 4x^3$$

Example I

Differentiate the following functions:

(a) $x^3 + 2x^2 + 3x$

(b) $4x^4 - 3x^2 + 5$

(c) $ax^2 + bx + c$

Solution

(a) $y = x^3 + 2x^2 + 3x$

$$\frac{dy}{dx} = 3x^{3-1} + 2 \times 2x^{2-1} + 3 \times 1(x^{1-1})$$

$$= 3x^2 + 4x + 3x^0$$

$$= 3x^2 + 4x + 3$$

(b) $y = 4x^4 - 3x^2 + 5$

$$\frac{dy}{dx} = 4x^{4-1} - 2 \times 3x^{2-1} + 0$$

$$= 4x^3 - 6x$$

(c) $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

Example III

Find the gradient of the curve $y = x(2 - x)$ at $x = 2$

Solution

$y = x(2 - x)$

$y = 2x - x^2$

$$\frac{dy}{dx} = 2 - 2x$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 2 - 2 \times 2$$

$$= -2$$

Example IV

Find the gradient of the curves at the given points:

(a) $y = (4x - 5)^2$ $(\frac{1}{2}, 9)$

(b) $y = 3x^3 - 2x^2$ (-2, -24)
(c) $y = (x + 2)(x - 4)$ (3, -5)

Solution

(a) $\mathbf{y = (4x - 5)^2}$

$y = 16x^2 - 40x + 25$

$\frac{dy}{dx} = 32x - 40$

$\left.\frac{dy}{dx}\right|_{(½, 9)} = 32 \times \frac{1}{2} - 40$

$= 16 - 40$

$= -24$

(b) $\mathbf{y = 3x^3 - 2x^2}$

$\frac{dy}{dx} = 9x^2 - 4x$

$\left.\frac{dy}{dx}\right|_{(-2, -24)} = 9 \times -2^2 - 4(-2)$

$= 36 + 8$

$= 44$

(c) $\mathbf{(x + 2)(x - 4)}$

$y = x^2 - 2x - 8$

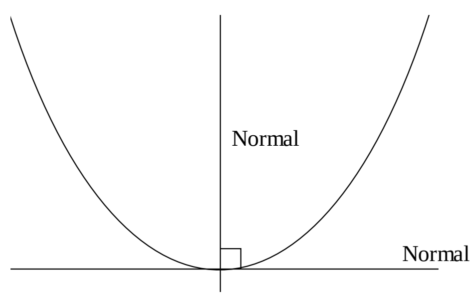
$\frac{dy}{dx} = 2x - 2$

$\left.\frac{dy}{dx}\right|_{(3, -5)} = 2 \times 3 - 2$

$= 4$

Tangents and Normals to curves

A tangent is a line which touches a curve at only one point. A normal is a line which is perpendicular to the tangent.



Example I

Find the equations of the tangents and normal to the curve at the given points:

(a) $y = x^2$ (2, 4)
(b) $y = 3x^2 + 2$ (4, 50)
(c) $y = 3x^2 - x + 1$ (0, 1)
(d) $3 - 4x - 2x^2$ (0, 1)

Solution

(a) $\boldsymbol{y = x^2}$

$\frac{dy}{dx} = 2x$

$\left.\frac{dy}{dx}\right|_{(2,\,4)} = 2 \times 2$

$= 4$

$\Rightarrow$ The gradient of the tangent = 4

Let n be the gradient of the normal

$n \times 4 = -1$

$n = \frac{-1}{4}$

Equation of the tangent:

$\Rightarrow \frac{y-4}{x-2} = 4$

$y - 4 = 4(x - 2)$

$y - 4 = 4x - 8$

$y = 4x - 4$

Equation of the normal:

$\Rightarrow \frac{y-4}{x-2} = \frac{-1}{4}$

$4(y - 4) = 1(x - 2)$

$4y - 16 = x - 2$

$4y = x - 14$

(b) $\boldsymbol{y = (3x^2 + 2)}$

$\frac{dy}{dx} = 6x$

$\left.\frac{dy}{dx}\right|_{(4,50)} = 6 \times 4$

$= 24$

Gradient of tangent = 24

Let the gradient of the normal be n

$n \times 24 = -1$

$n = \frac{-1}{24}$

Equation of the tangent:

$\Rightarrow \frac{y-50}{x-4} = 24$

$y - 50 = 24(x - 4)$

$y - 50 = 24x - 96$

$y = 24x - 96 + 50$

$y = 24x - 46$

Equation of the normal:

$$\Rightarrow \frac{y-50}{x-4} = \frac{-1}{24}$$

$24(y - 50) = -1(x - 4)$

$24y - 1200 = -x + 4$

$24y + x = 1204$

(c) $y = 3x^2 - x + 1$ (0, 1)

$$\frac{dy}{dx} = 6x - 1$$

$$\left.\frac{dy}{dx}\right|_{(0,\,1)} = 6 \times 0 - 1 = -1$$

$$\left.\frac{dy}{dx}\right|_{(0,\,1)} = -1$$

Let the gradient of the normal be n

$n \times -1 = -1$

$n = 1$

Equation of the tangent:

$$\Rightarrow \frac{y-1}{x-0} = -1$$

$y - 1 = -x$

$y = -x + 1$

Equation of the normal:

$$\Rightarrow \frac{y-1}{x-0} = 1$$

$y - 1 = x$

$y = x + 1$

(d) $y = 3 - 4x - 2x^2$ (1, -3)

$$\frac{dy}{dx} = -4 - 4x$$

$$\left.\frac{dy}{dx}\right|_{(1,\,-3)} = -4 - 4 \times 1$$

$= -8$

Let the gradient of the normal be n

$n \times -8 = -1$

$$n = \frac{1}{8}$$

Equation of the tangent:

$$\Rightarrow \frac{y - {}^{-}3}{x-1} = -8$$

$y + 3 = -8(x - 1)$

$y + 3 = -8x + 8$

$y = -8x + 5$

Equation of the normal:

$$\Rightarrow \frac{y - {}^{-}3}{x - 1} = \frac{1}{8}$$

$8(y + 3) = x – 1$

$8y + 24 = x – 1$

$8y + 25 = x$

Example II

Find the coordinates of a point on $y = x^2$ at which the gradient is 2. Hence find the equation of the tangent to the curve $y = x^2$ whose gradient is 2.

Solution

$y = x^2$

$\frac{dy}{dx} = 2x$

$\Rightarrow 2x = 2$

$x = 1$

If $x = 1$, from $y = x^2$;

$y = 1^2$

$y = 1$

$\Rightarrow$ The point is (1, 1)

Equation of the tangent:

$\frac{y - 1}{x - 1} = 2$

$y – 1 = 2(x – 1)$

$y = 2x – 1$

Example III

Find the equation of the normal to the curve $y = x^2 + 3x – 2$ at the point where it cuts the x-*axis*.

Solution

$y = x^2 + 3x – 2$

$\frac{dy}{dx} = 2x + 3$

At the *y*-axis, $x = 0$

From $y = x^2 + 3x – 2$,

$\Rightarrow y = 0^2 + 3 \times 0 – 2$

$y = -2$

(0, -2)

$\left.\frac{dy}{dx}\right|_{(0, -2)} = 2 \times 0 + 3$

$= 3$

The gradient of the tangent = 3

Let the gradient of the normal be *n*

$n \times 3 = -1$

$n = \frac{-1}{3}$

$$\frac{y - {}^{-}2}{x - 0} = \frac{-1}{3}$$

$3(y - -2) = -1(x)$

$3(y + 2) = -x$

$3y + 6 = -x$

$3y + x + 6 = 0$

Example IV

Find the value of k for which $y = 2x + k$ is a normal to the curve $y = 2x^2 - 3$.

Solution

$y = 2x + k$

Comparing y = $2x + k$ with $y = mx + c$;

$\Rightarrow m = 2$

$\therefore$ Gradient of the normal = 2

$y = 2x^2 - 3$

$$\frac{dy}{dx} = 4x$$

Let the gradient of the normal be n.

$4x \times n = -1$

$$n = \frac{-1}{4x}$$

Since the gradient of the normal = 2,

$$\Rightarrow \frac{-1}{4x} = 2$$

$$x = \frac{-1}{8}$$

$y = 2x^2 - 3$

$$y = 2\left(\frac{1}{64}\right) - 3$$

$$y = \frac{2}{64} - 3$$

$$y = \frac{-190}{64} = \frac{95}{32}$$

$\left({}^{-1}\!/_{8},\ {}^{-95}\!/_{32}\right)$

From $y = 2x + k$

$$\frac{-95}{32} = 2 \times \frac{-1}{8} + k$$

$$\frac{-95}{32} = \frac{-1}{4} + k$$

$$k = \frac{-95}{32} + \frac{1}{4} = \frac{-87}{32}$$

Example V

Find the equations of the tangents to the curve

$y = (2x - 1)(x + 1)$ at the points where the curve cuts the x-axis. Find the point of intersection of these tangents.

Solution

$y = (2x - 1)(x + 1)$

$y = 2x^2 + x - 1$

At the x-axis, $y = 0$

$\Rightarrow 0 = (2x - 1)(x + 1)$

$x = \frac{1}{2}$, $x = -1$

(½, 0) and (-1, 0)

$\Rightarrow$ The curve cuts the x-axis at (½, 0) and (-1, 0)

$y = 2x^2 + x - 1$

$$\frac{dy}{dx} = 4x + 1$$

$$\left.\frac{dy}{dx}\right|_{(\frac{1}{2},\ 0)} = 4 \times \frac{1}{2} + 1$$

$$= 3$$

$$\frac{y - 0}{x - \frac{1}{2}} = 3$$

$y = 3x - \frac{3}{2}$ …………………….. (i)

$$\left.\frac{dy}{dx}\right|_{(-1,\ 0)} = 4 \times -1 + 1$$

$$= -3$$

$$\Rightarrow \frac{y - 0}{x - {}^{-}1} = -3$$

$y = -3(x + 1)$

$y = -3x - 3$ ……………………..(ii)

Equating Eqn (i) and Eqn (ii);

$$\Rightarrow 3x - \frac{3}{2} = -3x - 3$$

$$6x = -3 + \frac{3}{2}$$

$$6x = \frac{-3}{2}$$

$$x = \frac{-1}{4}$$

Substituting $x = \frac{-1}{4}$ in Eqn (i);

$$y = 3 \times \frac{-1}{4} - \frac{3}{2}$$

$$y = \frac{-9}{4}$$

$\Rightarrow$ The two tangents intersect at $\left(\frac{-1}{4}, \frac{-9}{4}\right)$

Example VI

Find the coordinates of the point on $y = x^2 - 5$ at which the gradient is 3. Hence find the value of c for which the line

$y = 3x + c$ is a tangent to $y = x^2 - 5$

Solution

$y = x^2 - 5$

$\frac{dy}{dx} = 2x$

$2x = 3 \Rightarrow x = \frac{3}{2}$

When $x = \frac{3}{2}$,

$$y = \left(\frac{3}{2}\right)^2 - 5$$

$$y = \frac{9}{4} - 5$$

$$y = \frac{-11}{4}$$

$\left(\frac{3}{2}, \frac{-11}{4}\right)$

$y = 3x + c$

$\left(\frac{3}{2}, \frac{-11}{4}\right)$ satisfies $y = 3x + c$

$$\frac{-11}{4} = 3 \times \frac{3}{2} + c$$

$$\frac{-11}{4} = \frac{9}{2} + c$$

$$c = \frac{-29}{4}$$

Example VII

A tangent to the parabola $x^2 = 16y$ is perpendicular to the line $x - 2y - 3 = 0$. Find the equation of this tangent and the coordinates of its point of contact.

Solution

$x^2 = 16y$

$2x\, dx = 16\, dy$

$$\frac{dy}{dx} = \frac{2x}{16} = \frac{x}{8}$$

$x - 2y - 3 = 0$

$x - 3 = 2y$

$$y = \frac{1}{2}x - \frac{3}{2}$$

Since the tangent is perpendicular to the line $y = \frac{1}{2}x - \frac{3}{2}$,

Let the gradient of the tangent be t.

$t \times \frac{1}{2} = -1$

t = -2

$\frac{x}{8}$ = -2

x = -16

When x = -16,

$-16^2 = 16y$

y = 16

(-16, 16)

$\Rightarrow \frac{y-16}{x-{}^{-}16}$ = -2

y – 16 = -2(x + 16)

y – 16 = -2x – 32

y + 2x + 16 = 0

$\Rightarrow$ The equation of the tangent is $y + 2x + 16 = 0$ and the point of contact is (-16, 16)

Example VIII

Find the equation of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

Solution

$y = x^3 - 6x^2 + 12x + 2$

Comparing $y = 3x$ with $y = mx + c$ gives $m = 3$

$\frac{dy}{dx} = 3x^2 - 12x + 12$

$\Rightarrow 3x^2 - 12x + 12 = 3$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1$ and $x = 3$

If x = 1,

$y = 1^3 - 6 \times 1^2 + 12 \times 1 + 2$

y = 1 – 6 + 12 + 2

y = 9

If $x = 3$, $y = 3^3 - 6 \times 3^2 + 36 + 2$

y = 27 – 54 + 38

y = 11

$\Rightarrow$ The points are (1, 9) and (3, 11)

$\Rightarrow \frac{y-9}{x-1}$ = 3

y – 9 = 3(x – 1)

y – 9 = 3x – 3

y = 3x + 6

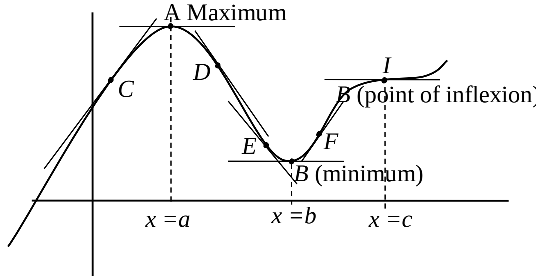
$\frac{y-11}{x-3}$ = 3

y – 11 = 3(x – 3)

y – 11 = 3x – 9

y = 3x + 2

Maximum, Minimum and Inflexion points of a curve



Points A, B, and I are stationary (turning points) of the curve. We say that $f(x)$ has a maximum value at $x = a$, if $f(a)$ is greater than any value immediately preceding or following, we say that a function $f(x)$ has a minimum value at $x = b$, if $f(b)$ is less than any value immediately preceding or following.

The tangent to the curve at points A, B and C are horizontal (parallel to the x-axis).

$\Rightarrow$ The gradient of each tangent to the curve is zero;

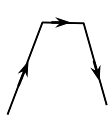
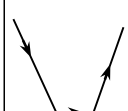
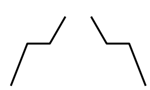
$$f(x) = 0$$

At points immediately to the left of the maximum point, C the slope of the tangent is positive. i.e. $f'(x) > 0$ while points immediately to the right at point D, the slope is negative i.e. $f'(x) < 0$.

In other words, at the maximum $f'(x)$ changes sign from + to (−).

At the minimum point, $f'(x)$ changes sign from − to +. We can see this at E and F.

Recall $f'(x) = \frac{dy}{dx}$.

	Maximum	Minimum	Inflexion
Sign of $\frac{dy}{dx}$ changes when moving through stationary values.	+ 0 − 	− + 0 	+ 0 +, − 0 − 

To locate maximum, minimum, and inflexion points of a curve without necessarily drawing the curve, we proceed as follows:

(a) Find the gradient $\frac{dy}{dx}$ of the curve

(b) Equate to zero the expression for $\frac{dy}{dx}$.

(c) Find the values of x which satisfy this equation.

(d) Consider the sign of $\frac{dy}{dx}$ on either sides of these points.

(e) Find the value(s) of y which correspond(s) to the values of x.

Distinguishing stationary points using the second derivative method

In order to distinguish the turning points, we find the second derivative.

If $\frac{d^2y}{dx^2} < 0$ at (x_1, y_2), $\Rightarrow (x_1, y_1)$ is a point of maximum

If $\frac{d^2y}{dx^2} > 0$ at (x_1, y_1), $\Rightarrow (x_1, y_1)$ is a minimum point;

If $\frac{d^2y}{dx^2} = 0$ at (x_1, y_1), $\Rightarrow (x_1, y_1)$ is a point of inflexion.

Example I

Find the coordinates of the stationary points of the curve $y = 2x^3 - 24x$ and distinguish between them.

Solution

$y = 2x^3 - 24x$

$\frac{dy}{dx} = 6x^2 - 24$

At stationary points, $\frac{dy}{dx} = 0$

$\Rightarrow 6x^2 - 24 = 0$

$x^2 - 4 = 0$

$(x + 2)(x - 2) = 0$

$x = -2$ and $x = 2$

If $x = -2$, $y = 2(-2)^3 - 24(-2)$

$y = -16 + 48$

$y = 32$

$\Rightarrow$ (-2, 32) is a stationary point.

If $x = 2$, $y = 2(2)^3 - 24(2)$

$y = 16 - 48$

$y = -32$

$\Rightarrow$ (2, -32) is a stationary point

$\frac{dy}{dx} = 6x^2 - 24$

$\frac{d^2y}{dx^2} = 12x$

$\left.\frac{d^2y}{dx^2}\right|_{(-2,\ 32)} = 12 \times -2$

$= -24 < 0$

Since $\frac{d^2y}{dx^2} < 0$, $\Rightarrow$ (-2, 32) is a point of maxima.

$\frac{d^2y}{dx^2} = 12x$

$\left.\frac{d^2y}{dx^2}\right|_{(2,\ -32)} = 12 \times 2$

$= 24 > 0$

Since $\frac{d^2y}{dx^2} > 0$, $\Rightarrow$ (2, -32) is a point of minima.

Example II

Investigate the nature of stationary points of the following curves.

(a) $y = x(x^2 - 12)$
(b) $y = x^2(3 - x)$
(c) $y = x(x - 8)(x - 15)$
(d) $y = x^3(2 - x)$
(e) $y = 3x^4 + 16x^3 + 24x + 3$

Solution

(a) $\boldsymbol{y = x(x^2 - 12)}$

$y = x^3 - 12x$

$\frac{dy}{dx} = 3x^2 - 12$

At a stationary point, $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 12 = 0$

$x^2 - 4 = 0$

$x = \pm 2$

If $x = 2$, $y = x(x^2 - 12)$

$y = 2(4 - 12)$

$y = 2(-8)$

$\Rightarrow$ (2, -16) is a stationary point.

If $x = -2$, $y = -2(-2^2 - 12)$

$y = -2(4 - 12)$

$y = -2(-8)$

$y = 16$

(-2, 16) is a turning point.

$\frac{d^2y}{dx^2} = 6x$

$\left.\frac{d^2y}{dx^2}\right|_{(2,\,-16)} = 6 \times 2 = 12$

$\Rightarrow$ (2, -16) is a point of minima

$\left.\frac{d^2y}{dx^2}\right|_{(-2,\,16)} = 6 \times -2 = -12 < 0$

$\Rightarrow$ (-2, 16) is a point of maxima.

(b) $\boldsymbol{y = x^2(3 - x)}$

$y = 3x^2 - x^3$

$\frac{dy}{dx} = 6x - 3x^2$

At a turning point, $\frac{dy}{dx} = 0$

$6x - 3x^2 = 0$

$3x(2 - x) = 0$

$x = 0$ and $x = 2$

If $x = 0$, $y = x^2(3 - x)$

$y = 0$

$\Rightarrow (0, 0)$ is a stationary point.

If $x = 2$, $y = 2^2(3 - 2)$

$y = 4$

$\Rightarrow$ (2, 4) is a stationary point

Turning points:

$\frac{dy}{dx} = 6x - 3x^2$

$\frac{d^2y}{dx^2} = 6 - 6x$

$\left.\frac{d^2y}{dx^2}\right|_{(0,\,0)} = 6$

$\Rightarrow$ (0, 0) is a point of minima

$\left.\frac{d^2y}{dx^2}\right|_{(2,\,4)} = 6 - 6 \times 2$

$= -6 < 0$

$\Rightarrow$ (2, 4) is a point of maxima.

(c) $\boldsymbol{y = x(x - 8)(x - 15)}$

$y = x^3 - 23x^2 + 120x$

$\frac{dy}{dx} = 3x^2 - 46x + 120$

At stationary points, $\frac{dy}{dx} = 0$

$3x^2 - 46x + 120 = 0$

$x = 12, \; x = \frac{10}{3}$

If $x = 12$, $y = x(x - 8)(x - 15)$

$y = 12(12 - 8)(12 - 15)$

$y = 12(4)(-3)$

$y = -144$

$\Rightarrow (12, -144)$ is a stationary point

When $x = \frac{10}{3}$, $y = \frac{10}{3}\left(\frac{10}{3} - 8\right)\left(\frac{10}{3} - 15\right)$

$y = \frac{10}{3}\left(\frac{-14}{3}\right)\left(\frac{-35}{3}\right) = \frac{4900}{27}$

$\Rightarrow \left(10/3, \; 4900/27\right)$ is a stationary point.

$\frac{d^2y}{dx^2} = 6x - 46$

$\left.\frac{d^2y}{dx^2}\right|_{(12,\,-144)} = 6 \times 12 - 46$

$= 26 > 0$

$\Rightarrow$ (12, -144) is a point of minima.

$$\left.\frac{d^2y}{dx^2}\right|_{(10/3,\ 4900/9)} = 6 \times \frac{10}{3} - 46$$

$= -26 < 0$

$\Rightarrow \left(10/3,\ 4900/9\right)$ is a point of maxima.

(d) $\boldsymbol{y = x^3(2 - x)}$

$y = 2x^3 - x^4$

$$\frac{dy}{dx} = 6x^2 - 4x^3$$

At stationary points, $\frac{dy}{dx} = 0$

$6x^2 - 4x^3 = 0$

$2x^2(3 - 2x) = 0$

$x = 0, \quad x = \frac{3}{2}$

If $x = 0$, $y = x^3(2 - x)$

$y = 0^3(2 - 0)$

$y = 0$

(0, 0) is a stationary point.

If $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^3\left(2 - \frac{3}{2}\right)$

$y = \frac{27}{8}\left(\frac{1}{2}\right) = \frac{27}{16}$

$\Rightarrow \left(3/2,\ 27/16\right)$ is a stationary point

$$\frac{d^2y}{dx^2} = 12x - 12x^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,\ 0)} = 0$$

$\Rightarrow$ (0, 0) is a point of inflexion.

$$\left.\frac{d^2y}{dx^2}\right|_{(3/2,\ 27/16)} = 12 \times \frac{3}{2} - 12\left(\frac{3}{2}\right)^2 = -9$$

$\Rightarrow \left(3/2,\ 27/16\right)$ is a point of maxima

(e) $\boldsymbol{y = 3x^4 + 16x^3 + 24x^2 + 3}$

$$\frac{dy}{dx} = 12x^3 + 48x^2 + 48x$$

At stationary points, $\frac{dy}{dx} = 0$

$12x^3 + 48x^2 + 48x = 0$

$12x(x + 4x + 4) = 0$

$x = 0, \qquad x = -2$

If $x = 0$, $y = 3$

$\Rightarrow$ (0, 3) is a stationary point.

If $x = -2$, $y = 3(-2)^4 + 16(-2)^3 + 24(-2)^2 + 3$

$y = 48 - 128 + 96 + 3$

$y = 19$

$\Rightarrow$ (-2, 19) is a stationary point.

$$\frac{d^2y}{dx^2} = 36x^2 + 96x + 48$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,\,3)} = 48 > 0$$

$\Rightarrow$ (0, 3) is a point of minima.

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,\,19)} = 36(-2)^2 + 96(-2) + 48 = 0$$

$\Rightarrow$ (-2, 19) is a point of inflexion.

Example II

If $p = 4s^2 - 10s + 7$, find the minimum value of p and the values of s which gives the minimum value of p.

Solution

$p = 4s^2 - 10s + 7$

$$\frac{dp}{ds} = 8s - 10$$

For minimum value of p, $\frac{dp}{ds} = 0$

$8s - 10 = 0$

$$s = \frac{10}{8} = \frac{5}{4}$$

$p = 4s^2 - 10s + 7$

$$p_{\min} = 4\left(\frac{5}{4}\right)^2 - 10\left(\frac{5}{4}\right) + 7$$

$$p_{\min} = \frac{100}{16} - \frac{50}{4} + 7$$

$$p_{\min} = \frac{3}{4}$$

$$\frac{dp}{ds} = 8s - 10$$

$$\frac{d^2p}{ds^2} = 8 > 0$$

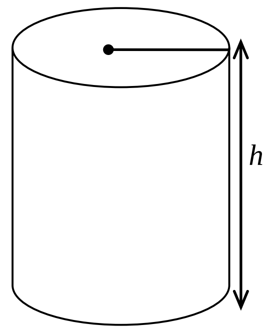
$\Rightarrow p$ is minimum when $S = \frac{5}{4}$ and the minimum value of p is $\frac{3}{4}$.

Example IV

A cylindrical can is made so that the sum of the height and the circumference of its base is 45π cm. Find the radius of the base of the cylinder if the volume of the can is maximum.

Solution

Let the radius of the base be r and the height h cm.



(Height + circumference) = 45π.

$h + 2\pi r = 45\pi$

$h = 45\pi - 2\pi r$ .... (i)

$V = \pi r^2 h$ …………………………….. (ii)

Substituting Eqn (i) in Eqn (ii);

$V = \pi r^2(45\pi - 2\pi r)$

$V = 45\pi^2 r^2 - 2\pi^2 r^3$

$\frac{dV}{dr} = 90\pi^2 r - 6\pi^2 r^2$

For the maximum volume, $\frac{dV}{dr} = 0$.

$90\pi^2 r - 6\pi^2 r^2 = 0$

$6\pi^2 r(15 - r) = 0$

$r = 0$ or $r = 15$

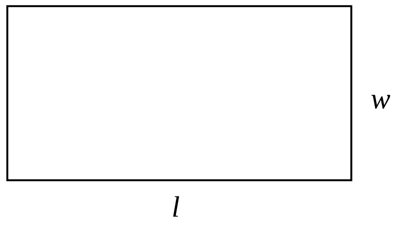
But $r \neq 0$

$\Rightarrow r = 15$ cm

Example V

Onyango wishes to fence a rectangular farm. He wants the sum of the length and the width of the farm to be 42 cm. Calculate the length and width of the farm for the area of the farm to be as maximum as possible.

Solution



Let the length and width of the rectangular farm be l and w respectively.

$l \times w = 42$

$l = 42 - w$

$A = l \times w$

$A = (42 - w)w$

$A = 42w - w^2$

$$\frac{dA}{dw} = 42 - 2w$$

For the maximum area, $\frac{dA}{dw} = 0$

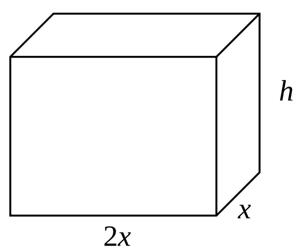
$$\Rightarrow 42 - 2w = 0$$
$$w = 21$$
$$l = 42 - w$$
$$= 42 - 21$$
$$= 21$$

Example VI

The length of a rectangular block is twice its width, and the total surface area is 108 cm^2. Show that if the width of the block is x cm, the volume is $\frac{4}{3}x(27 - x^2)$. Find the dimensions of the block if the volume is maximum.

Solution

Let the width be x cm



$$V = l \times w \times h$$
$$V = 2x \times x \times h$$
$$V = 2x^2h \text{ ....(i)}$$

Total surface area $A = 2(lw + wh + hl)$

$$108 = 2(2x^2 + xh + 2xh)$$
$$54 = 2x^2 + 3xh$$

$$\frac{54 - 2x^2}{3x} = h \text{ .... (ii)}$$

Substituting Eqn (ii) in Eqn (i);

$$\Rightarrow V = 2x^2\left(\frac{54 - 2x^2}{3x}\right)$$

$$V = 2x\left(\frac{54 - 2x^2}{3}\right)$$

$$V = \frac{4x}{3}(27 - x^2)$$

For the maximum volume, $\frac{dV}{dx} = 0$

$$V = \frac{4x}{3}(27 - x^2)$$

$$V = \frac{4}{3}(27x - x^3)$$

$$\frac{dV}{dx} = \frac{4}{3}(27 - 3x^2)$$

For V_{max}, $\frac{dV}{dx} = 0$

$$\frac{4}{3}(27 - 3x^2) = 0$$

$$\Rightarrow 27 - 3x^2 = 0$$

$$x^2 = 9$$

$$\Rightarrow x = 3$$

$$l = 2x$$

$$l = 6$$

$$h = \frac{54 - 2x^2}{3x}$$

$$= \frac{54 - 2(3^2)}{3 \times 3}$$

$$= \frac{54 - 18}{9}$$

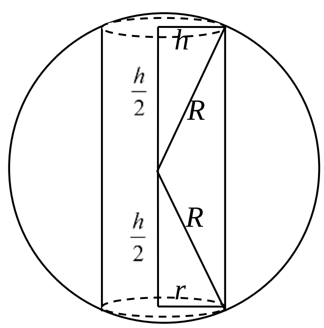
$$h = 4$$

Example VII

A cylindrical volume V is to be cut from a solid sphere of radius R. Prove that the maximum volume of the cylindar, V is $V = \frac{4\pi R^3}{3\sqrt{3}}$

Solution

Let the height of the cylinder be h



$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

$$r^2 + \frac{h^2}{4} = R^2$$

$$r^2 = R^2 - \frac{h^2}{4}$$

$$V = \pi r^2 h$$

$$V = \pi\left(R^2 - \frac{h^2}{4}\right)h$$

$$V = \pi R^2 h - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2$$

For the maximum volume, $\frac{dV}{dh} = 0$

$$\pi R^2 - \frac{3}{4}\pi h^2 = 0$$

$$h^2 = \frac{4R^2}{3}$$

$$h = \frac{2R}{\sqrt{3}}$$

$$V = \pi r^2 h$$

$$V = \pi r^2\left(\frac{2R}{\sqrt{3}}\right)$$

But $r^2 = R^2 - \frac{h^2}{4}$

$$\Rightarrow r^2 = R^2 - \frac{1}{4}\left(\frac{4R^2}{3}\right)$$

$$r^2 = \mathrm{R}^2 - \frac{1}{3}R^2$$

$$r^2 = \frac{2}{3}R^2$$

$$V = \pi r^2 h$$

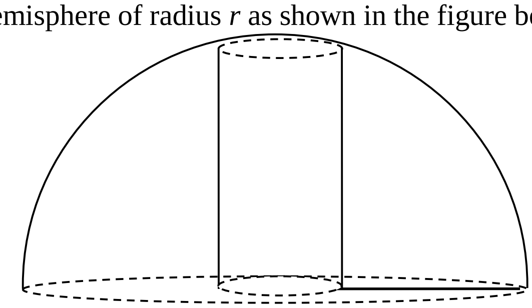
$$h = \frac{2R}{\sqrt{3}},\ r^2 = \frac{2}{3}R^2$$

$$V_{\max} = \pi\left(\frac{2R^2}{3}\right)\left(\frac{2R}{\sqrt{3}}\right)$$

$$V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}$$

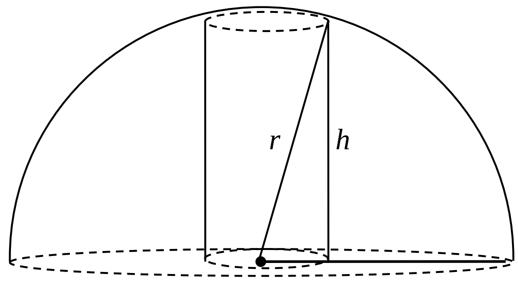
Example VIII

A cylinder is inscribed in a hemisphere of radius r as shown in the figure below.



Find the maximum volume of the cylinder in terms of r.

Solution



$x^2 + h^2 = r^2$

$x^2 = r^2 - h^2$

Volume of the cylinder, $V = \pi x^2 h$

$V = \pi(r^2 - h^2)h$

$V = \pi r^2 h - \pi h^3$

$\frac{dV}{dh} = \pi r^2 - 3\pi h^2$

For maximum volume, $\frac{dV}{dh} = 0$

$\pi r^2 - 3\pi h^2 = 0$

$\pi(r^2 - 3h^2) = 0$

$\frac{r^2}{3} = h^2$

$h = \frac{r}{\sqrt{3}}$

$x^2 = r^2 - h^2$

$x^2 = r^2 - \frac{r^2}{3}$

$x^2 = \frac{2r^2}{3}$

$V = \pi x^2 h$

$x^2 = \frac{2r}{3}$, $h = \frac{r}{\sqrt{3}}$

$V_{\max} = \pi \cdot \frac{2r^2}{3} \cdot \left(\frac{r}{\sqrt{3}}\right) = \frac{2\pi r^3}{3\sqrt{3}}$

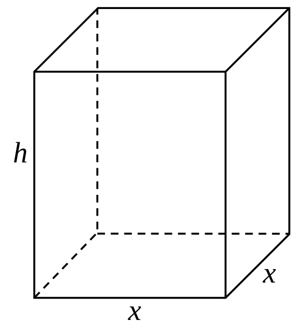
Example IX

A rectangular block has a base x cm square. Its surface area is 150 cm^2. Prove that the volume of the block is $\frac{1}{2}(75x - x^3)$.

(a) Calculate the dimensions of the block when the volume is maximum.

(b) The maximum volume.

Solution



S.A = 2(lw + wh + hl)

$150 = 2(x^2 + xh + xh)$

$75 = (x^2 + 2xh)$

$\frac{75 - x^2}{2x} = h$

$V = l \times w \times h$

$V = x^2h$

$V = x^2\left(\frac{75 - x^2}{2x}\right)$

$V = \frac{x}{2}(75 - x^2)$

$V = \frac{1}{2}(75x - x^3)$

$\frac{dV}{dx} = \frac{1}{2}(75 - 3x^2)$

For maximum volume, $\frac{dV}{dx} = 0$.

$\frac{1}{2}(75 - 3x^2) = 0$

$75 - 3x^2 = 0$

$x^2 = 25$

$x = 5$

$h = \frac{75 - x^2}{2x}$

$h = \frac{75 - 25}{10}$

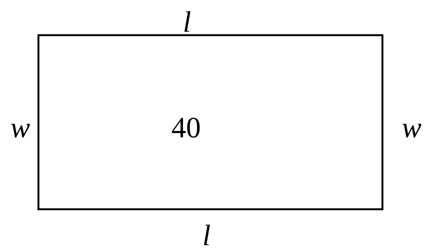
$h = 5$

Example X

(a) A variable rectangular flower garden has a constant perimeter of 40. Find the length of the side when the area is maximum.

(b) A variable rectangle has a constant area of 36 cm^2. Find the length of the sides when the perimeter is maximum.

Solution



Perimeter of the flower garden $P = 2(l + w)$

$40 = 2(l + w)$

$20 = l + w$

$l = 20 - w$

$A = lw$

$A = (20 - w)w$

$A = 20w - w^2$

$\frac{dA}{dw} = 20 - 2w$

For the maximum area, $\frac{dA}{dw} = 0$

$20 - 2w = 0$

$w = 10$

$l = 20 - w$

$l = 10$

(b) $\boldsymbol{P = 2(l + w)}$

$lw = 36$

$l = \frac{36}{w}$

$P = 2\left(\frac{36}{w} + w\right)$

$P = \frac{72}{w} + 2w$

$P = 72w^{-1} + 2w$

$\frac{dP}{dw} = -72w^{-2} + 2$

$= \frac{-72}{w^2} + 2$

For the maximum perimeter, $\frac{dP}{dw} = 0$

$\frac{-72}{w^2} + 2 = 0$

$\frac{72}{w^2} = 2$

$w^2 = 36$

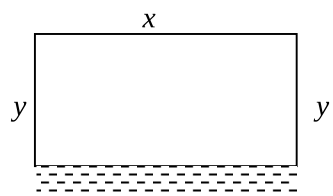
$w = 6$

$l = 6$

Example XI

Mukasa wishes to enclose a rectangular piece of land of area 1250 cm^2 whose one side is bound by a straight bank of a river. Find the least possible length of barbed wire required.

Solution



$xy = 1250$

$y = \frac{1250}{x}$

$P = x + y + y$

$P = x + 2y$

$P = x + \left(2 \times \frac{1250}{x}\right)$

$P = x + \frac{2500}{x}$

$\frac{dP}{dx} = 1 - \frac{2500}{x^2}$

For the least possible length, $\frac{dP}{dx} = 0$

$\Rightarrow 1 - \frac{2500}{x^2} = 0$

$1 = \frac{2500}{x^2}$

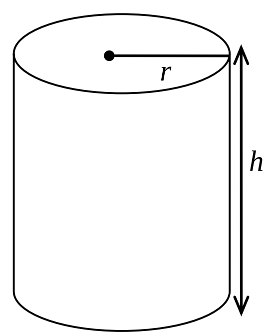
$x = 50$

$y = \frac{1250}{50} = 25$

Example XII

A closed right circular cylinder of base radius r cm and height h cm has volume of 54π cm^3. Show that S, the total surface area of the cylinder, is given by $S = \frac{108\pi}{r} + 2\pi r^2$ hence find the radius and height which makes the surface area minimum.

Solution



$V = \pi r^2 h$

$54\pi = \pi r^2 h$

$\frac{54}{r^2} = h$

Surface area of a cylinder $A = 2\pi r^2 + 2\pi rh$

$$A = 2\pi r^2 + 2\pi r\left(\frac{54}{r^2}\right)$$

$$A = 2\pi r^2 + \frac{108\pi}{r^3}$$

$$\frac{dA}{dr} = 4\pi r - \frac{108\pi}{r^2}$$

For the minimum surface area, $\frac{dA}{dr} = 0$

$$4\pi r - \frac{108\pi}{r^2} = 0$$

$$4\pi r^3 - 108\pi = 0$$

$$r^3 = \frac{108}{4}$$

$$r^3 = 27$$

$$r = 3$$

$$h = \frac{54}{r^2} = \frac{54}{9}$$

$$h = 6$$

Example XIII

A company that manufactures dog food wishes to pack the feed in closed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 250π cm^3 and the minimum possible surface area?

Solution

$$A = 2\pi r^2 + 2\pi rh$$

$$V = \pi r^2 h$$

$$250\pi = \pi r^2 h$$

$$h = \frac{250}{r^2}$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right)$$

$$A = 2\pi r^2 + \frac{500\pi}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{500\pi}{r^2}$$

For minimum surface area, $\frac{dA}{dr} = 0$

$$4\pi r - \frac{500\pi}{r^2} = 0$$

$$\pi(4r^3 - 500) = 0$$

$$r^3 = 125$$

$$r = 5 \text{ cm}$$

$$h = \frac{250}{r^2}$$

$$h = 10 \text{ cm}$$

Example **(UNEB Question)**

Write down the expression of the volume V and surface area S of a cylinder of radius r and height h. If the surface area S of the cylinder is kept constant, show that the volume of the cylinder will be maximum when $h = 2r$

Solution

$S = 2\pi r^2 + 2\pi rh$

$$h = \frac{S - 2\pi r^2}{2\pi r}$$

$$V = \pi r^2 h$$

$$V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right)$$

$$V = \frac{1}{2}(Sr - 2\pi r^3)$$

$$\frac{dV}{dr} = \frac{1}{2}(S - 6\pi r^2)$$

For maximum volume, $\frac{dV}{dr} = 0$

$\Rightarrow S - 6\pi r^2 = 0$

$S = 6\pi r^2$

$$h = \frac{S - 2\pi r^2}{2\pi r}$$

$$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r}$$

$h = 2r$

For maximum volume, $h = 2r$

Example **(UNEB Question)**

A right circular cone of radius r cm has a maximum volume. The sum of its vertical height h and circumference of its base is 15 cm. If the radius varies, show that the maximum volume of the cone is $\frac{125}{3\pi}$ cm^3.

Solution

The base is circular

$\Rightarrow$ The circumference of the base = $2\pi r$

$2\pi r + h = 15$

$h = 15 - 2\pi r$

Volume of the cone = $\frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi r^2(15 - 2\pi r)$$

$$= \frac{1}{3}\pi(15r^2 - 2\pi r^3)$$

$$\frac{dV}{dr} = \frac{\pi(30r - 6\pi r^2)}{3}$$

For maximum volume, $\frac{dV}{dr} = 0$

$$\frac{\pi}{3}(30r - 6\pi r^2) = 0$$
$$30r - 6\pi r^2 = 0$$
$$6r(5 - \pi r) = 0$$
$$5 = \pi r$$
$$r = \frac{5}{\pi}$$
$$V = \frac{1}{3}\pi r^2 h$$
$$h = 15 - 2\pi r; \text{ But } r = \frac{5}{\pi}$$
$$h = 15 - 2\pi\left(\frac{5}{\pi}\right)$$
$$h = 5$$
$$V = \frac{\pi}{3}\left(\frac{5}{\pi}\right)^2 \times 5$$
$$V = \frac{125}{3\pi} \text{ cm}^3$$

Example

A match box consists of an outer cover open at both ends into which a rectangular box without a top. The length of the box is one and a half times the width. The thickness of the material is negligible and the volume of the match box is 25 cm^3. If the width is x cm, find in terms of x the area of the material used. Hence show that if the least area of the material is to be used to make the box, the length should be 3.7 approximately.

Solution

Area of the inner surface $= 2(lw) + 2(lh)$

$$= 2\left(\frac{3x}{2} \times x\right) + 2\left(\frac{3x}{2} \times h\right)$$
$$= 3x^2 + 3xh$$

Area of the water surface = $(lw + 2lh + 2wh)$

$$= \frac{3x}{2} \cdot x + \frac{3x}{2} \times 2 \times h + 2xh$$
$$= \frac{3x^2}{2} + 5xh$$

The total surface of the match box

$$= \frac{3x^2}{2} + 5xh + 3xh + 3x^2$$
$$A = \frac{9x^2}{2} + 8xh \text{ ………………… (i)}$$

From volume = $l \times w \times h$,

$$V = \frac{3x^2 h}{2}$$

$$\Rightarrow 25 = \frac{3x^2h}{2}$$

$h = \frac{50}{3x^2}$ ………………………….. (ii)

Substituting Eqn (2) in Eqn (i);

$$A = \frac{9x^2}{2} + 8x\left(\frac{50}{3x^2}\right)$$

$$A = \frac{9x^2}{2} + \frac{400}{3x}$$

For the least area, $\frac{dA}{dx} = 0$

$$\frac{dA}{dx} = 9x - \frac{400}{3x^2}$$

$$\frac{dA}{dx} = 9x - \frac{400}{3x^2} = 0$$

$27x^3 - 400 = 0$

$$x^3 = \frac{400}{27}$$

$$x = \sqrt[3]{\frac{400}{27}}$$

$$l = \frac{3x}{2}$$

$$l = \frac{3}{2} \times \sqrt[3]{\frac{400}{27}}$$

$l = 3.68403$ cm

$l \approx 3.7$ cm

Techniques of Differentiation

Chain, Product, and Quotient rules

We can now move to some more properties involved in differentiation. To summarise, so far we have found that:

1. The derivative of a sum is a sum of its derivatives.
2. The derivative of a difference is the difference of the derivatives.

However, it turns out that:

1. The derivative of a product of derivative $f(x)g(x)$ is not a product of the derivative.

$$\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$$

2. The derivative of a quotient is not the quotient of the derivative

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$$

3. The derivative of the composition $f(x)$ is not the composition of the derivatives.

The chain, product and quotient rules tell us how to differentiate in these three situations.

Chain Rule

The chain rule states that:

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

Example I

Given that $y = (x^2 + 7)^{100}$, find $\frac{dy}{dx}$

Solution

$$y = (x^2 + 7)^{100}$$

Let $t = x^2 + 7$

$$\frac{dt}{dx} = 2x$$

$$\Rightarrow y = t^{100}$$

$$\frac{dy}{dt} = 100t^{99}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 100t^{99} \times 2x$$

$$= 200xt^{99}$$

$$= 200x(x^2 + 7)^{99}.$$

Example II

Given that $y = (x^7 – x^2)^{42}$, find $\frac{dy}{dx}$.

Solution

$$y = (x^7 – x^2)^{42}$$

Let $t = x^7 – x^2 \Rightarrow y = t^{42}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$y = t^{42}$

$$\frac{dy}{dt} = 42t^{41}$$

$t = x^7 - x^2$

$$\frac{dt}{dx} = 7x^6 - 2x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$= 42(t^{41}) \times (7x^6 - 2x)$

$= 42(7x^6 - 2x)t^{41}$

$= 42(7x^6 - 2x)(x^7 - x^2)^{41}$

Example III

Find $\frac{dy}{dx}$ in terms of t in the following expressions:

(a) $x = t^2$, $y = 4t - 1$

(b) $y = 3t^2 + 2t$, $x = 1 - 2t$

(c) $x = 2\sqrt{2}$, $y = 5t - 4$

(d) $x = \frac{1}{t}$, $y = t^2 + 4t - 3$

(e) $x = \frac{2}{3+\sqrt{t}}$, $y = \sqrt{t}$

Solution

(a) $x = t^2$, $y = 4t - 1$

$y = 4t - 1$

$$\frac{dy}{dt} = 4$$

$x = t^2$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 4 \times \frac{1}{2t} = \frac{2}{t}$$

(b) $y = 3t^2 + 2t$, $x = 1 - 2t$

$$\frac{dy}{dt} = 6t + 2$$

$$\frac{dx}{dt} = -2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = (6t + 2) \times \frac{-1}{2}$$

$$\frac{dy}{dx} = -3t - 1$$

(c) $x = 2\sqrt{t}$, $y = 5t - 4$

$$\frac{dx}{dt} = 2 \times \frac{1}{2} t^{-1/2}$$

$$= \frac{1}{\sqrt{t}}$$

$y = 5t - 4$

$$\frac{dy}{dt} = 5$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 5 \times \sqrt{t}$$

$$= 5\sqrt{t}$$

(d) $x = \frac{1}{t}$, $y = t^2 + 4t - 3.$

$$\frac{dx}{dt} = -1t^{-1-1} = \frac{1}{t^2}$$

$y = t^2 + 4t - 3$

$$\frac{dy}{dt} = (2t + 4)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$= (2t + 4) \times -t^2$

$= -2t^3 - 4t^2$

(e) $x = \frac{2}{3+\sqrt{t}}$, $y = \sqrt{t}$

$$x = 2(3+\sqrt{t})^{-1}$$

$$\frac{dx}{dt} = -2(3 + \sqrt{t})^{-2} \cdot \frac{t^{-1/2}}{2}$$

$$= \frac{-1}{(3+\sqrt{t})^2\sqrt{t}}$$

$y = \sqrt{t}$

$$\frac{dy}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{t}} \times {}^{-}(3+\sqrt{t})^2\sqrt{t}$$

$$= \frac{{}^{-}(3+\sqrt{t})^2}{2}$$

Example IV

Find $\frac{dy}{dx}$ in terms of t if $x = at^2$ and $y = 2at$

Solution

$x = at^2$

$\frac{dx}{dt} = 2at$

$y = 2at$

$\frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$

Example V (UNEB Question)

A curve is defined by the parametric equations

$x = t^2 - t$

$y = 3t + 4$

Find the equation of the tangent to the curve at (2, 10)

Solution

$x = t^2 - t$ and $y = 3t + 4$.

$\frac{dx}{dt} = 2t - 1$

$\frac{dy}{dt} = 3$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= 3 \times \frac{1}{2t-1}$

$= \frac{3}{2t-1}$

At point (2, 10), $x = 2$ and $y = 10$.

$x = t^2 - t$

$y = 3t + 10$

Substituting, for $x = 2$,

$2 = t^2 - t$

$t^2 - t - 2 = 0$

$t^2 - 2t + t - 2 = 0$

$t(t - 2) + 1(t - 2) = 0$

$(t - 2)(t + 1) = 0$

Either $t - 2 = 0$,

$t = 2$

Or $t + 1 = 0$

$t = -1$

Substituting for $y = 10$,

$10 = 3t + 4$

$3t = 6$

$t = 2$

For $\frac{dy}{dx} = \frac{3}{2t - 1}$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3}{2(2)-1} = \frac{3}{4-1} = 1$$

$$\Rightarrow \frac{y-10}{x-2} = 1$$

$y - 10 = x - 2$

$y = x - 2 + 10$

$y = x + 8$

Example VI

If $x = at^2$, $y = 2at$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

Solution

$x = at^2$, $y = 2at$

$\frac{dx}{dt} = 2at$; $\qquad \frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2a \times \frac{1}{2at}$$

$$= \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{-1}{t^2} \times \frac{1}{2at} = \frac{1}{2at^3}$$

Example VII

A curve is represented parametrically by

$$x = (t^2 - 1)^2; \qquad y = t^3$$

Find $\frac{dy}{dx}$

Solution

$x = (t^2 - 1)^2$, $y = t^3$

$\frac{dx}{dt} = 2(t^2 - 1)2t$

$= 4t(t^2 - 1)$

$y = t^3$

$$\frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 3t^2 \times \frac{1}{4t(t^2 - 1)}$$

$$= \frac{3t}{4(t^2 - 1)}$$

Product Rule

Consider $y = uv$, where v and u are functions of x.

$$y + \partial y = (u + \partial u)(v + \partial v)$$

$$y + \partial y = uv + u\partial v + v\partial u + \partial u\partial v$$

As $\partial u \longrightarrow 0$, $\partial v \longrightarrow 0$

$$\partial u\partial v \approx 0$$

$$\Rightarrow \partial y + y = uv + u\partial v + v\partial u$$

$$\partial y = uv + u\partial v + v\partial u - y$$

$$\partial y = uv + u\partial v + v\partial u - uv$$

$$\partial y = u\partial v + v\partial u$$

$$\frac{\partial y}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

As $\partial x \longrightarrow 0$

$$\frac{\partial y}{\partial x} \approx \frac{dy}{dx}, \quad \frac{\partial v}{\partial x} \approx \frac{dv}{dx} \text{ and } \frac{\partial u}{\partial x} \approx \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Example I

Differentiate the following

(a) $(x^2 + 1)(x^3 + 2)$

(b) $x^2(x + 1)^3$

(c) $(1+x)^{3/2}(x-1)^{5/4}$

(d) $(x-1)\sqrt{x^2+1}$

(e) $\sqrt{(x+1)(x-2)^3}$

(f) $(x-1)^2\sqrt[3]{1-2x}$

Solution

(a) $\boldsymbol{y = (x^2 + 1)(x^3 + 2)}$

Let $u = x^2 + 1$, $v = x^3 + 2$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^2 + 1)(3x^2) + (x^3 + 2)2x$$
$$= 3x^4 + 3x^2 + 2x^4 + 4x$$
$$= 5x^4 + 3x^2 + 4x$$
$$= 5x^4 + 3x^2 + 4x$$
$$\Rightarrow \frac{dy}{dx} = 5x^4 + 3x^2 + 4x.$$

(b) $\boldsymbol{y = x^2(x + 1)^3}$

Let $u = x^2$, $v = (x + 1)^3$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{dy}{dx} = x^2 3(x + 1)^2 \cdot 1 + (x + 1)^3 2x$$
$$= x(x + 1)^2[3x + 2(x + 1)]$$
$$= x(x + 1)^2(5x + 2)$$
$$= x(x+ 1)^2(5x + 2)$$

(c) $\boldsymbol{y =} (1+x)^{3/2}(x-1)^{5/4}$

$u = (1+x)^{3/2}$, $v = (x-1)^{5/4}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\Rightarrow \frac{dy}{dx} = (1+x)^{3/2} \cdot \frac{5}{4}(x-1)^{1/4} \cdot 1 + (x-1)^{5/4} \cdot \frac{3}{2}(x+1)^{1/2} \cdot 1$$
$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{1/4}(x+1)^{1/2}\left[\frac{5}{2}(x+1) + \frac{3(x-1)}{1}\right]$$
$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{1/4}(x+1)^{1/2}\left(\frac{5+5x+6(x-1)}{2}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{1/4}(x+1)^{1/2}\left(\frac{11x-6}{2}\right)$$
$$\frac{dy}{dx} = \frac{(x-1)^{1/4}(x+1)^{1/2}(11x-6)}{4}$$

(d) $y = (x-1)\sqrt{x^2+1}$

Let $u = x - 1$, $v = \sqrt{x^2+1}$

$$\frac{dy}{dx} = (x-1) \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x + \left(\sqrt{x^2+1}\right) \cdot 1$$
$$\frac{dy}{dx} = x(x-1)(x^2+1)^{-1/2} + (x^2+1)^{1/2}$$
$$\frac{dy}{dx} = (x^2+1)^{-1/2}\left[x(x-1) + x^2 + 1\right]$$
$$\frac{dy}{dx} = \frac{1}{(x^2+1)^{1/2}}\left[x^2 - x + x^2 + 1\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}\left[2x^2 - x + 1\right]$$

(e) $y = \sqrt{(x+1)(x-2)^3}$

$$y = \left[(x+1)(x-2)^3\right]^{1/2}$$

$$y = (x+1)^{1/2}(x-2)^{3/2}$$

$$\frac{dy}{dx} = (x+1)^{1/2} \cdot \frac{3}{2}(x-2)^{1/2} \cdot 1 + (x-2)^{3/2}\frac{1}{2}(x+1)^{-1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-1/2}(x-2)^{1/2}\left[3(x+1) + (x-2)\right]$$

$$\frac{dy}{dx} = \frac{(x-2)^{1/2}}{2(x+1)^{1/2}}\left[3x + 3 + x - 2\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x-2}{x+1}}\left[4x+1\right]$$

(f) $y = (1 - x)^2\sqrt[3]{1-2x}$

Let $u = (1 - x)^2$, $v = (1 - 2x)^{1/3}$

$y = uv$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (1-x)^2 \cdot \frac{1}{3}(1-2x)^{-2/3} \cdot -2 + (1-2x)^{1/3} \cdot 2(1-x)(^-1)$$

$$\frac{dy}{dx} = 2(1-x)(1-2x)^{-2/3}\left[\frac{-1}{3}(1-x) + {}^-1(1-2x)\right]$$

$$\frac{dy}{dx} = \frac{2(1-x)}{(1-2x)^{2/3}}\left[\frac{-1+x-3(1-2x)}{3}\right]$$

$$\frac{dy}{dx} = \frac{2(1-x)}{(1-2x)^{2/3}}\left[\frac{-1-3+x+6x}{3}\right]$$

$$\frac{dy}{dx} = \frac{2(1-x)}{\sqrt[3]{(1-2x)^2}}\left[\frac{7x-4}{3}\right]$$

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{1-x}{\sqrt[3]{(1-2x)^2}}\right)(4x-8)$$

Example (UNEB Question)

Given that $R = q\sqrt{(1000 - q^2)}$, find:

(a) $\frac{dR}{dq}$

(b) The value of q when R is maximum.

Solution

(a) $R = q\sqrt{(1000 - q^2)}$

Let $u = q$, $v = \sqrt{(1000 - q^2)}$;

$$\frac{dR}{dq} = u\frac{dv}{dq} + v\frac{du}{dq}$$

$$\frac{dR}{dq} = q \cdot \frac{1}{2}(1000 - q^2)^{-1/2} \times -2q + \sqrt{1000 - q^2} \times 1$$

$$\frac{dR}{dq} = -q^2(1000 - q^2)^{-1/2} + \sqrt{1000 - q^2}$$

$$= (1000 - q^2)^{-1/2}\left[(-q^2 + 1000 - q^2)\right]$$

$$= \frac{1000 - 2q^2}{\sqrt{1000 - q^2}}$$

(b) For R_{max}, $\frac{dR}{dq} = 0$

$$\Rightarrow \frac{1000 - 2q^2}{\sqrt{1000 - q^2}} = 0$$

$1000 - 2q^2 = 0$

$q^2 = 500$

$q^2 = 100 \times 5$

$q = \sqrt{100 \times 5}$

$q = \pm 10\sqrt{5}$

$q = 10\sqrt{5}$ or $q = -10\sqrt{5}$

Quotient Rule

Consider $y = \frac{u}{v}$, where u and v are functions of x.

$$y = \frac{u}{v}$$

$$y + \partial y = \frac{u + \partial u}{v + \partial v}$$

$$\partial y = \frac{u + \partial u}{v + \partial v} - y$$

$$\partial y = \frac{u + \partial u}{v + \partial v} - \frac{u}{v}$$

$$\partial y = \frac{(u + \partial u)(v - \partial v)}{(v + \partial v)(v - \partial v)} - \frac{u}{v}$$

$$= \frac{uv + u\partial v + v\partial u - \partial u\partial v}{v^2 - (\partial v)^2} - \frac{u}{v}$$

As $\partial v \longrightarrow 0$, $\partial u \longrightarrow 0$ and $\partial v\partial u \approx 0$, $(\partial v)^2 \approx 0$

$$\partial y = \frac{uv + v\partial u - u\partial v}{v^2} - \frac{u}{v}$$

$$\partial y = \frac{u}{u} + \frac{v\partial u - u\partial v}{v^2} - \frac{u}{v}$$

$$= \frac{v\partial u - u\partial v}{v^2}$$

$$\frac{\partial y}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

As $\partial x \longrightarrow 0,\ \frac{\partial y}{\partial x} \longrightarrow \frac{dy}{dx}$

$$\frac{\partial u}{\partial x} \longrightarrow \frac{du}{dx}$$

$$\frac{\partial v}{\partial x} \longrightarrow \frac{dv}{dx}$$

$$\boxed{\mathbf{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}}$$

Example

Differentiate the following:

(a) $\frac{x^2+1}{x^2-1}$ **(b)** $\frac{x}{\sqrt{x^2+1}}$

(c) $\sqrt{\frac{(x+2)^3}{x-1}}$ **(d)** $\sqrt{\frac{(x+1)^3}{x+2}}$

(e) $\frac{(1-\sqrt{x})^2}{\sqrt{x^2-1}}$ **(f)** $\frac{2x^2-x^3}{\sqrt{x^2-1}}$

Solutions

(a) $\frac{x^2+1}{x^2-1}$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = x^2 + 1;\ v = x^2 - 1$

$$\frac{dy}{dx} = \frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x[(x^2-1)-(x^2+1)]}{(x^2-1)^2}$$

$$\frac{dy}{dx} = 2x\left(\frac{-2}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

(b) $\dfrac{x}{\sqrt{x^2+1}}$

$u = x, \quad v = \sqrt{x^2+1}$

$y = \dfrac{u}{v}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1} \times (1) - x \cdot \frac{1}{2}(x^2+1)^{-1/2} \times 2x}{\left(\sqrt{x^2+1}\right)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)^{-1/2}[(x^2+1) - x^2]}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)^{-1/2}[1]}{x^2+1}$$

$$\frac{dy}{dx} = \frac{1}{(x^2+1)^{3/2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2+1)^3}}$$

(c) $y = \sqrt{\dfrac{(x+2)^3}{x-1}}$

$$y = \frac{(x+2)^{3/2}}{(x-1)^{1/2}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x-1)^{1/2} \times \frac{3}{2}(x+2)^{1/2} - (x+2)^{3/2} \times \frac{1}{2}(x-1)^{-1/2}}{\left[(x-1)^{1/2}\right]^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(x+2)^{1/2}}{(x-1)^{3/2}}(3x-3-x-2)$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x+2}{(x-1)^3}}\ (2x-5)$$

(d) $\boldsymbol{y} =$ $\sqrt{\dfrac{(x+1)^3}{x+2}}$

$$y = \frac{(x+1)^{3/2}}{(x+2)^{1/2}}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}=\frac{(x+2)^{1/2}\cdot\frac{3}{2}(x+1)^{1/2}-(x+1)^{3/2}\cdot\frac{1}{2}(x+2)^{-1/2}\cdot 1}{[(x+2)^{1/2}]^2}$$

$$\frac{dy}{dx}=\frac{\frac{1}{2}(x+2)^{-1/2}(x+1)^{1/2}[3(x+2)-(x+1)]}{x+2}$$

$$\frac{dy}{dx}=\frac{1}{2}\frac{(x+1)^{1/2}}{(x+2)^{3/2}}(3x+6-x-1)$$

$$\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{x+1}{(x+2)^3}}(2x+5)$$

$$\frac{dy}{dx}=\frac{(2x+5)}{2}\sqrt{\frac{x+1}{(x+2)^3}}$$

(e) $y=\frac{(1-\sqrt{x})^2}{\sqrt{x^2-1}}$

$u=(1-\sqrt{x})^2, \quad v=\sqrt{x^2-1}$

$y=\frac{u}{v}$

$$\frac{dy}{dx}=\frac{(\sqrt{x^2-1})2(1-\sqrt{x})\cdot\frac{-1}{2}x^{-1/2}-(1-\sqrt{x})^2\frac{1}{2}(x^2-1)^{-1/2}\cdot 2x}{(\sqrt{x^2-1})^2}$$

$$\frac{dy}{dx}=\frac{(x^2-1)^{-1/2}(1-\sqrt{x})x^{-1/2}\left[2(x^2-1)\cdot\frac{-1}{2\sqrt{x}}-(1-\sqrt{x})x\right]}{x^2-1}$$

$$\frac{dy}{dx}=\frac{1-\sqrt{x}}{\sqrt{x}(x^2-1)^{3/2}}\left[\frac{x^2+1-(\sqrt{x})x(1-\sqrt{x})}{\sqrt{x}}\right]$$

$$\frac{dy}{dx}=\frac{1-\sqrt{x}}{\sqrt{x}(x^2-1)^{3/2}}\left[\frac{-x^2+1-x\sqrt{x}+x^2}{\sqrt{x}}\right]$$

$$\frac{dy}{dx}=\frac{1-\sqrt{x}}{\sqrt{(x^2-1)^3}}\left[\frac{1-x\sqrt{x}}{x}\right]$$

Example (UNEB Question)

Differentiate:

(a) $(x+1)^{1/2}(x+2)^2$

(b) $\frac{2x^2+3x}{(x-4)^2}$

Solution

(a) $(x+1)^{1/2}(x+2)^2$

$y=uv$

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x+1)^{1/2}2(x+2) + (x+2)^2 \cdot \frac{1}{2}(x+1)^{-1/2}$$

$$\frac{dy}{dx} = (x+1)^{-1/2}(x+2)\left[2(x+1) + \frac{1}{2}(x+2)\right]$$

$$\frac{dy}{dx} = \frac{x+2}{\sqrt{x+1}}\left[\frac{4(x+1)+x+2}{2}\right]$$

$$\frac{dy}{dx} = \frac{x+2}{\sqrt{x+1}}\left[4x+4+x+2\right]$$

$$\frac{dy}{dx} = \frac{x+2}{\sqrt{x+1}}\left[5x+6\right]$$

(b) $y = \frac{2x^2+3x}{(x-4)^2}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x-4)^2 \cdot (4x+3) - (2x^2+3x) \cdot 2(x-4)}{[(x-4)^2]^2}$$

$$\frac{dy}{dx} = \frac{(x-4)\left[(x-4)(4x+3) - 2(2x^2+3x)\right]}{(x-4)^4}$$

$$\frac{dy}{dx} = \frac{-19x-12}{(x-4)^3}$$

Differentiation of Implicit Functions

Example I

Find $\frac{dy}{dx}$ when $x^2 + 2xy + y^2 = 8$

Solution

$$\frac{d}{dx}(x^2 + 2xy + y^2) = \frac{d}{dx}(8)$$

$$2xdx + 2(xdy + ydx) + 2ydy = 0$$

$$2x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x + 2y) = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2(x+y)}{2(x+y)}$$

$$\frac{dy}{dx} = -1$$

Example II

If $x^2 - 3xy + y^2 - 2y + 4x = 0$, find $\frac{dy}{dx}$

Solution

$$x^2 - 3xy + y^2 - 2y + 4x = 0$$

$$\frac{d}{dx}(x^2 - 3xy + y^2 - 2y + 4x) = \frac{d}{dx}(0)$$

$$2x\,dx - 3(xdy + ydx) + 2ydy - 2dy + 4dx = 0$$

$$2x - 3x\frac{dy}{dx} - 3y + 2y\frac{dy}{dx} - 2\frac{dy}{dx} + 4 = 0$$

$$\frac{dy}{dx}(2y - 3x - 2) = -4 - 2x$$

$$\frac{dy}{dx} = \frac{-4-2x}{2y-3x-2}$$

Example III

Find $\frac{dy}{dx}$ when $3x^2 - 4xy = 7$

Solution

$$3x^2 - 4xy = 7$$

$$\frac{d}{dx}(3x^2 - 4xy) = \frac{d}{dx}(7)$$

$$6x\,dx - 4(x\,dy + y\,dx) = 0$$

$$6x - 4x\frac{dy}{dx} - 4y = 0$$

$$6x - 4y = 4x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x-4y}{4x}$$

Example IV

If $x^2 + 3xy - y^2 = 0$, find $\frac{dy}{dx}$ at (1, 1).

Find the equation of the tangent and normal at (1, 1)

Solution

$$x^2 + 3xy - y^2 = 0$$

$$\frac{d}{dx}(x^2 + 3xy - y^2) = \frac{d}{dx}(3)$$

$$2x\,dx + 3(x\,dy + y\,dx) - 2y\,dy = 0$$

$$2x + 3x\frac{dy}{dx} + 3y - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3x - 2y) = -2x - 3y$$

$$\frac{dy}{dx} = \frac{-2x-3y}{3x-2y}$$

$$\left.\frac{dy}{dx}\right|_{(1,\,1)} = \frac{-2-3}{3-2} = -5$$

$$\Rightarrow \frac{y-1}{x-1} = -5$$

$y - 1 = -5(x - 1)$
$y - 1 = -5x + 5$
$y = -5x + 6$ is the equation of the tangent

Let the gradient of the normal be n

$n \times -5 = -1$

$n = \frac{1}{5}$

$\Rightarrow \frac{y-1}{x-1} = \frac{1}{5}$

$5(y - 1) = x - 1$
$5y - 5 = x - 1$
$5y - 4 = x$ is the equation of the normal.

Example V

Find the x-stationary points of the curve
$x^3 - y^3 - 4x^2 + 3y = 11x + 4$

Solution

$x^3 - y^3 - 4x^2 + 3y = 11x + 4$

$\frac{d}{dx}(x^3 - y^3 - 4x^2 + 3y) = \frac{d}{dx}(11x + 4)$

$3x^2\, dx - 3y^2\, dy - 8x\, dx + 3dy = 11\ dx$

$(3 - 3y^2)\, dy = (11 - 3x^2 - 8x)\ dx$

$\frac{dy}{dx} = \frac{11-3x^2-8x}{3-3y^2}$

At stationary points, $\frac{dy}{dx} = 0$

$\Rightarrow \frac{11-3x^2-8x}{3-3y^2} = 0$

$11 - 3x^2 - 8x = 0$
$3x^2 + 8x - 11 = 0$

$x = 1, \quad x = \frac{-11}{3}$

Application of Differentiation

Small Changes

If $A(x, y)$ is a general point in the curve with equation $y = f(x)$ and $B(x+\delta x, y+\delta y)$ is a point in the curve close to A, then δx is a small increase in x and δy is a small increase in y

We know from differentiation that

$$\Rightarrow \lim_{\partial x \longrightarrow 0}\left(\frac{\partial y}{\partial x}\right) = \frac{dy}{dx}$$

So when ∂x is small, we can say that $\frac{\partial y}{\partial x} \approx \frac{dy}{dx}$

$\partial y \approx \frac{dy}{dx} \cdot \partial x$

The approximation can be used to estimate the value of a function close to a known value $y + \delta y$ can be estimated if y is known.

Example I

Given that $y = 3x^2 + 2x - 4$. Use small changes to find the small change in y when x increases from 2 to 2.02.

Solution

$$y = 3x^2 + 2x - 4$$

$$\frac{dy}{dx} = 6x + 2$$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

$$\partial x = (2.02 - 2) = 0.02$$

$$x = 2; \qquad \partial x = 0.02$$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

$$\partial y = (6x + 2)\ \partial x$$

$$\partial y = [(6 \times 2) + 2] \times 0.02$$

$$\partial y = 0.28$$

Example II

Use small changes to estimate $\sqrt{101}$

Solution

$$y = \sqrt{101}$$

$$y + \partial y = \sqrt{x + \partial x}$$

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$x = 100, \quad \partial x = 1$$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

$$\partial y = \frac{1}{2\sqrt{x}} \cdot (\partial x)$$

$$\partial y = \frac{1}{2\sqrt{100}} \cdot (1)$$

$$\partial y = \frac{1}{20}$$

$$\partial y = 0.05$$

$$y + \partial y = \sqrt{x + \partial x}$$

$$x = 100, \quad y = \sqrt{x}$$

$$y = \sqrt{100} = 10$$

$$10 + 0.05 = \sqrt{100 + 1}$$

$$10.05 = \sqrt{101}$$

Example III
In an experiment, the diameter x of a metal is measured and the volume V cm^3 is calculated using the formula $V = \frac{1}{6}\pi x^3$. If the diameter is found to be 10 cm with a possible error of 0.1cm, estimate the possible error in the volume calculated.
Solution

$$V = \frac{1}{6}\pi x^3$$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

$$\partial x = 0.1,\ x = 10$$

$$\partial y = \frac{1}{2}\pi x^2 \cdot (0.1)$$

$$\partial y = \frac{1}{2}\pi (10)^2 \times (0.1)$$

$$= 5\pi$$

Hence the possible error in the volume is 5π cm^3

Example IV
Find the approximate value of $\sqrt[3]{1003}$
Solution

$$y = \sqrt[3]{x}$$

$$x = 1000,\ \partial x = 3$$

$$y + \partial y = \sqrt[3]{1000 + 3}$$

$$y + \partial y = \sqrt[3]{1003}$$

$$y = \sqrt[3]{x}$$

$$y = x^{1/3}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

$$\partial y = \frac{dy}{3x^{2/3}} \cdot 3$$

$$= \frac{1}{3(1000)^{2/3}} \cdot 3$$

$$= \frac{3}{300} = 0.01$$

$$y + \partial y = \sqrt[3]{1003}$$

$$y = \sqrt[3]{x}$$

$$y = \sqrt[3]{1000} = 10$$

$$10 + 0.01 = \sqrt[3]{1003}$$

$$10.01 = \sqrt[3]{1003}$$

Example I
Use small changes to find the cube root of 1005
Solution

$$y = \sqrt[3]{1005}$$
$$y + \partial y = \sqrt[3]{x + \partial x}$$
$$y = x^{1/3}$$
$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$
$x = 100,\ \partial x = 5$
$$y + \partial y = \sqrt[3]{x + \partial x}$$
$$y = x^{1/3}$$
$$y = 1000^{1/3}$$
$$y = 10$$
$$y + \partial y = \sqrt[3]{1005}$$
$$10 + \partial y = \sqrt[3]{1005}$$
$$\partial y = \frac{dy}{dx} \cdot \partial x$$
$$\partial y = \frac{1}{3x^{2/3}} \cdot 5$$
$$\partial y = \frac{1}{3(1000)^{2/3}} \times 5$$
$$\partial y = \frac{5}{300}$$
$$\partial y = 0.016667$$
$$10 + 0.016667 = \sqrt[3]{1005}$$
$$10.016667 = \sqrt[3]{1005}$$
$$\sqrt[3]{1005} = 10.01667$$

Example
Use small changes to find $\sqrt{627}$.
Solution

$$y = \sqrt{x}$$
$$y + \partial y = \sqrt{x + \partial x}$$
$$y = \sqrt{x}$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$$
$$\partial y = \frac{dy}{dx} \cdot \partial x$$
$x = 625,\ \partial x = 2$

$$\partial y = \frac{1}{2 \times \sqrt{625}} \times 2$$

$$\partial y = \frac{1}{25}$$

$$y = \sqrt{x}$$

$$y = \sqrt{625}$$

$$y = 25$$

$$25 + \frac{1}{25} = \sqrt{625 + 2}$$

$$\sqrt{625} = 25 + 0.04$$

$$\sqrt{625} = 25.04$$

Percentage Small Changes

An error of 3% is made in measuring the radius of the sphere. Find the percentage error in the volume.

Solution

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\partial r = \frac{3}{100} r$$

$$\partial V = \frac{\partial V}{\partial r} \cdot \partial r$$

$$\partial V = 4\pi r^2 \cdot \frac{3}{100} r$$

$$\partial V = \cdot \frac{12\pi r^3}{100}$$

$$\frac{\partial V}{V} \times 100 = \frac{\frac{12\pi r^3}{100}}{V} \times 100 = \frac{\frac{12\pi r^3}{100}}{\frac{4}{3}\pi r^3} \times 100$$

$$= 9\%$$

Example II

The height of a cylinder is 10 cm and the radius is 4 cm. Find the approximate percentage increase in the volume when the radius increases from 4 to 4.02 cm.

Solution

$$V = \pi r^2 h$$

$$\frac{dV}{dr} = 2\pi r h$$

$$\partial V = \frac{\partial V}{\partial r} \cdot \partial r$$

$$\partial r = 4.02 - 4 = 0.02$$

$\partial V = 2\pi rh(0.02)$

$\partial V = 2\pi \times 10 \times 4(0.02)$

$\partial V = 1.6\pi$

$V = \pi r^2 h$

$V = \pi (4)^2 \times 10$

$V = 160\pi$

Percentage increase in the volume is $\frac{\partial V}{V} \times 100$

$= \frac{1.6\pi}{160\pi} \times 100 = 1\%$

Example III

The period T of a simple pendulum is calculated from the formula $T = 2\pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum and g is the acceleration due to gravity constant. find the percentage change in the period caused by lengthening the pendulum by 2%.

Solution

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$T = \frac{2\pi l^{1/2}}{g^{1/2}}$$

$$\frac{dT}{dl} = \frac{\pi l^{-1/2}}{g^{1/2}} = \frac{\pi}{g^{1/2} l^{1/2}}$$

$$\partial T = \frac{dT}{dl} \cdot \partial l$$

$$\partial T = \frac{\pi}{g^{1/2} l^{1/2}} \cdot \frac{2}{100} l$$

$$\partial T = \frac{2\pi}{100 g^{1/2}} l^{1/2}$$

$$\partial T = \frac{2\pi}{100}\sqrt{\frac{l}{g}}$$

Percentage change in period = $\frac{\partial T}{T} \times 100$

$$= \frac{\frac{2\pi}{100}\sqrt{l/g}}{2\pi\sqrt{l/g}} \times 100$$

$$= 1\%$$

Example

An error of 2.5% is made in measuring the area of a circle. What is the percentage error in the circumference?

Solution

$$\partial A = \frac{\partial A}{\partial r} \cdot \partial r$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\partial A = \frac{2.5A}{100}$$

$$\partial A = 2\pi r \cdot \partial r$$

$$\frac{2.5}{100} A = 2\pi r \ \partial r$$

$$\frac{2.5}{100} \pi r^2 = 2\pi r \ \partial r$$

$$\partial r = \frac{1.25r}{100}$$

$$C = 2\pi r$$

$$\frac{\partial C}{\partial r} = 2\pi$$

$$\partial C = \frac{\partial C}{\partial r} \cdot \partial r$$

$$\partial C = 2\pi \times \frac{1.25}{100} r$$

$$\partial C = \frac{2.5\pi r}{100}$$

Percentage error in circumference = $= \frac{\partial C}{C} \times 100$

$$= \frac{\frac{2.5\pi r}{100}}{2\pi r} \times 100 = 1.25\%$$

Example

If l is the length of a pendulum and t is the time of a complete swing, it is known that $l = kt^2$. The length of the pendulum is increased by x%. x is so small. Find the corresponding increase in the time of the string.

Solution

$$l = kt^2$$

$$\frac{dl}{dt} = 2kt$$

$$\partial l = \frac{dl}{dt} \cdot \partial t$$

$$\partial l = 2kt \cdot \partial t$$

$$\frac{x}{100} l = 2kt \cdot \partial t$$

$$\partial t = \frac{\frac{xl}{100}}{2kt} = \frac{xl}{200kt}$$

$$\partial t = \frac{x(kt^2)}{200kt} = \frac{xt}{200}$$

Percentage increase in time = $\frac{\partial t}{t} \times 100$

$$= \frac{\frac{xt}{200}}{t} \times 100 = \frac{x}{2}\%$$

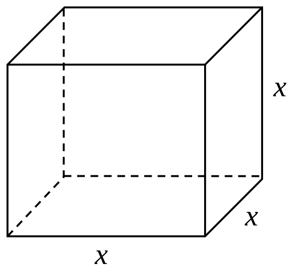
Rates of Change

Application of derivatives

Example I

A side of a cube is increasing at a rate of 6cm/s. Find the rate of increase in the volume of the cube when the length of the side is 8cm.

Solution



$$V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$\frac{dx}{dt} = 6 \text{ cm/s}$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\frac{dV}{dt} = 3x^2 \times 6$$

$$\frac{dV}{dt} = 18x^2$$

$$\left.\frac{dV}{dt}\right|_{x=8} = 18 \times 8^2$$

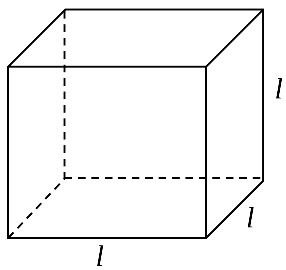
$$= 1152$$

$$\frac{dV}{dt} = 1152 \text{ cm}^3\text{/s}$$

Example II

The volume of a cube is increasing at a rate of 2 cm^3/s. Find the rate of change of the side of the base when the length is 3 cm.

Solution



$$V = l^3$$

$$\frac{dV}{dt} = 2\text{cm}^3\text{/s}$$

$$\frac{dV}{dl} = 3l^2$$

$$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$$

$$\frac{dV}{dt} = 3l^2 \times \frac{dl}{dt}$$

$$2 = 3l^2 \frac{dl}{dt}$$

$$\frac{dl}{dt} = \frac{2}{3l^2}$$

$$\left.\frac{dl}{dt}\right|_{l=3} = \frac{2}{3 \times 3^2} = \frac{2}{3 \times 9}$$

$$\frac{dt}{dl} = \frac{2}{27} \text{ cm/s}$$

Example III

The area of the circle is increasing at a rate of 3cm^2/s. Find the rate of change of the circumference when its radius is 2cm.

Solution

$$\frac{dA}{dt} = 3$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$3 = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{3}{2\pi r}$$

$$\left.\frac{dr}{dt}\right|_{r=2} = \frac{3}{2\pi \times 2}$$

$$\frac{dr}{dt} = \frac{3}{4\pi}$$

$$C = 2\pi r$$

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dC}{dt} = 2\pi \times \frac{3}{4\pi}$$

$$\frac{dC}{dt} = \frac{6}{4}$$

$$\frac{dC}{dt} = 1.5\text{cm/s}$$

Example III (UNEB Question)
A spherical balloon is inflated such that the rate at which its radius is increasing is 0.5cm/s. Find the rate at which:
(a) the volume is increasing at the instant when r = 5.0cm
(b) the surface area is increasing when r = 8.5 cm
Solution

$$V = \frac{4}{3}\pi r^3, \quad \frac{dr}{dt} = 0.5 \text{ m/s}$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 2\pi r^2$$

$$\left.\frac{dV}{dt}\right|_{r=5} = 2\pi(5)^2 = 50\pi \text{ cm}^2\text{/s}$$

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

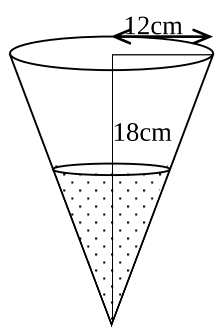
$$\frac{dA}{dt} = 8\pi r \times 0.5$$

$$\frac{dA}{dt} = 4\pi r$$

$$\left.\frac{dA}{dt}\right|_{r=8.5} = 4\pi(8.5)$$

$$\left.\frac{dA}{dt}\right|_{r=8.5} = 34\pi \text{ cm}^2\text{/s}$$

Example IV
A hollow circular cone is held vertex downwards beneath a tap leaking at a rate of 2cm^3/s. Find the rise of water level when the level is 6 cm. Given that the height of the cone is 18 cm and its radius is 12 cm.
Solution



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = 2 \text{ cm}^3\text{/s}$$

$$\frac{r}{h} = \frac{12}{18} = \frac{2}{3}$$

$$r = \frac{2}{3}h$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\tfrac{2}{3}h\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{4h^2}{9}\right)h$$

$$V = \frac{4}{27}\pi h^3$$

$$\frac{dV}{dh} = \frac{12}{27}\pi h^2$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{12\pi h^2}{27}\times\frac{dh}{dt}$$

$$2 = \frac{12\pi h^2}{27}\times\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{54}{12\pi h^2}$$

$$h = 6$$

$$\frac{dh}{dt} = \frac{54}{12\pi(6)^2}$$

$$\frac{dh}{dt} = \frac{54}{432\pi} = \frac{1}{8\pi} \text{ cm/s}$$

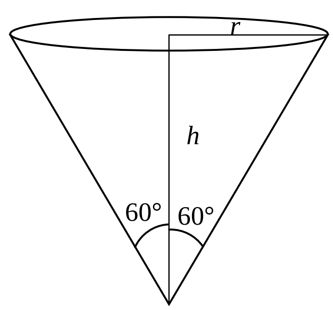
Example V

An inverted right circular cone of vertical angle 120° is collecting water from a tap at a steady rate of 18π cm^3/min. Find:

(a) the depth of the water after 12 minutes

(b) the rate of increase of the depth at this instant.

Solution



Volume of the cone $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = 18\pi \text{ cm}^3\text{/min}$$

$1 \text{ min} \longrightarrow 18\pi \text{ cm}^3$

$12 \text{ min} \longrightarrow x \text{ cm}^3$

$x = 12 \times 18\pi$

$= 216\pi \text{ cm}^3$

$$\tan 60 = \frac{r}{h} \Rightarrow \sqrt{3} = \frac{r}{h}$$

$$r = \sqrt{3}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(\sqrt{3}\ h)^2 h$$

$V = \pi h^3$

$V = \pi h^3$

$216\ \pi = \pi h^3$

$216 = h^3$

$h = 6$ cm

$V = \pi h^3$

$$\frac{dV}{dh} = 3\pi h^2$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$18\pi = 3\pi h^2 \times \frac{dh}{dt}$$

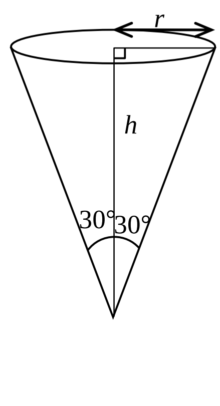
$$\frac{dh}{dt} = \frac{18}{3h^2}$$

$$\left.\frac{dh}{dt}\right|_{h=6} = \frac{18}{3 \times 6^2} = \frac{1}{6} \text{ cm/min}$$

Example VI

An inverted cone with vertical angle of 60° is collecting water leaking from a tap at a rate of 2cm^3/s. If the height of water collected is 10cm, find the rate at which the depth is decreasing at that instant.

Solution



$$\tan 30 = \frac{r}{h}$$

$$\frac{1}{\sqrt{3}} = \frac{r}{h} \quad \Rightarrow h = \sqrt{3}\ h$$

$$r = \frac{h}{\sqrt{3}}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{1}{3}\pi\left(\frac{h^2}{3}\right)h$$

$$= \frac{1}{9}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{3}\pi h^2$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$0.2 = \frac{1}{3}\pi h^2 \cdot \frac{dh}{dt}$$

$$\frac{0.6}{\pi h^2} = \frac{dh}{dt}$$

When $h = 10$,

$$\frac{dh}{dt} = \frac{0.6}{\pi h^2}$$

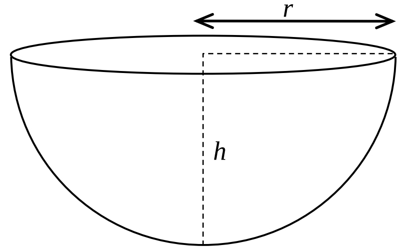
$$\frac{dh}{dt} = \frac{0.6}{\pi (10)^2}$$

$$\frac{dh}{dt} = \frac{0.6}{100\pi}$$

$$\frac{dh}{dt} = \frac{6}{1000\pi}\ \text{cm/s}$$

Example

A hemispherical bowl is being filled with water at a uniform rate when the height of water is h cm. The volume is $\pi(rh^2 - \frac{1}{3}h^3)\ \text{cm}^3$, r being the radius of the sphere. Find the rate at which the water level is rising when it is half-way to the top, given that $r = 6$ and the bowl fills in 1 minute.



$$V = \pi(rh^2 - \tfrac{1}{3}h^3)$$

When it is full, $r = h$

$$V = \pi(h^3 - \tfrac{1}{3}h^3)$$

$$V = \frac{2\pi h^3}{3}$$

$$r = h = 6$$

$$V = \frac{2}{3}\pi \times 6^3$$

$$= 144\pi \text{ cm}^3$$

$$\frac{dV}{dt} = 144\pi \text{ cm}$$

(*Because the bowl fills in a minute*)

When the bowl is not full, $r \neq h$

r = 6 cm

$$V = \pi(rh^2 - \tfrac{1}{3}h^2)$$

$$V = \pi(6h^2 - \tfrac{1}{3}h^2)$$

$$\frac{dV}{dh} = \pi(12h - h^2)$$

When h = 3, $\frac{dV}{dh} = \pi(36 - 9)$

$$\frac{dV}{dh} = 27\pi$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = 27\pi \times \frac{dh}{dt}$$

$$144\pi = 27\pi \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{144\pi}{27\pi} = \frac{48}{9}$$

$$\frac{dh}{dt} = \frac{16}{3} \text{ cm/min} = \frac{4}{45} \text{ cm/s}$$

Example

A horse trough has a triangular cross-section area of height 50 cm and base 60cm and height 2m long. A horse is drinking steadily and when the water level is 5cm below the top, it is being lowered at a rate of 1cm/min. Find the rate of consumption in litres per minute.

Solution

h = 50

$$V = \left(\frac{1}{2} \times b \times h\right) \times l$$

l = 200 cm

$$V = \frac{1}{2} \times b \times h \times 200$$

$$V = 100bh$$

$$\frac{h}{b/2} = \frac{50}{30}$$

$$\frac{2h}{b} = \frac{5}{3}$$

$$2h = \frac{5}{3}b$$

$$b = \frac{6h}{5}$$

$$V = 100\left(\frac{6h}{5}\right)h = 120h^2$$

$$\frac{dV}{dh} = 240h$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = 240h \times 1$$

$$\frac{dV}{dt} = 240h$$

$$\frac{dV}{dt} = (240 \times 20) = 4800 \text{ cm}^3\text{/min}$$

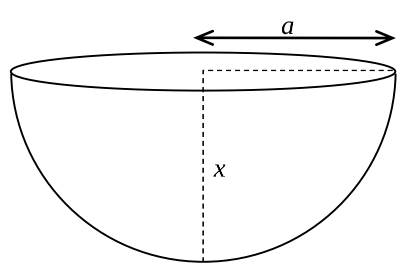
$$\frac{dV}{dt} = 4.8 \text{ litres/minute}$$

Example (UNEB Question)

A hemispherical bowl of radius a cm is initially full of water. The water runs out through a small hole at the bottom of the bowl at a constant rate such that it empties the bucket in 24 s. Given that when the depth of water is x cm and the volume of water is $\frac{1}{3}\pi x^2(3a - x)$ cm^3, show that the depth of water at that instant is decreasing at a rate of $a^3(36(2a - x))^{-1}$. Find how long it will take for the depth of water to be $\frac{1}{3}a$ cm and the rate at which the depth is increasing at that instant.

Solution

$$V = \frac{1}{3}\pi x^2(3a - x)$$



When it is full of water, $x = a$

$$V = \frac{1}{3}\pi a^2(3a - a)$$

$$V = \frac{1}{3}\pi a^2(2a)$$

$$V = \frac{2}{3}\pi a^3$$

Because it empties in 24s

$$24\text{s} \longrightarrow \frac{2}{3}\pi a^3 \text{ cm}^3$$

$$1 \text{ s} \longrightarrow x$$

$$x = \frac{2}{72}\pi a^3 \text{ cm}^3\text{/s}$$

$$\frac{dV}{dt} = \frac{\pi a^3}{36} \text{ cm}^3\text{/s}$$

When $x \neq a$

$$V = \frac{1}{3}\pi x^2(3a - x)$$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$\frac{\pi a^3}{36} = (2\pi ax - \pi x^2)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{\pi a^3}{36\pi x(2a - x)}$$

$$\frac{dx}{dt} = a^3[36x(2a - x)]^{-1}$$

$$h = \frac{1}{3}a$$

$$V = \frac{1}{3}\pi x^2(3a - x)$$

$$V = \frac{1}{3}\pi\left(\tfrac{1}{3}a\right)^2(3a - \tfrac{1}{3}a)$$

$$V = \frac{1}{3}\pi\left(\frac{a^2}{9}\right)\left(\frac{8a}{3}\right)$$

$$V = \frac{8\pi a^3}{81}$$

Volume of water in the bowl = $\frac{8\pi a^3}{81}$

Volume of the water emptied

$$= \frac{2\pi a^3}{3} - \frac{8\pi a^3}{81} = \frac{46\pi a^3}{81}$$

$$\frac{dV}{dt} = \frac{\pi a^3}{36}$$

$$1\text{ s} \longrightarrow \frac{\pi a^3}{36}$$

$$x\text{ s} \longrightarrow \frac{46\pi a^3}{81}$$

$$x = \frac{\frac{46\pi a^3}{81}}{\frac{\pi a^3}{36}} = \frac{46\pi a^3}{81} \times \frac{36}{\pi a^3}$$

x = 20.4445 cm

(b) $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$, $-1 < x < 1$

(c) $1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$, $-1 < x < 1$

(d) $1 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{5}{15}x^3$, $-2 < x < 2$

21. $\frac{567}{16}$, 6th **22**. 8 **23**. 7

24. $\frac{1}{1+x^2} + \frac{1}{(1-x)^2} - \frac{2}{1-x}$, $2x^3 + 4x^4 + 4x^5 + 4x^6 + 6x^2$

23 (a) $u^3 + 3u$, $u^5 + 5u^3 + 5u$

(b) $1 + kx + \frac{1}{2}(3k^2 - 1)x^2 + \frac{k}{2}(5k^2 - 3)x^3$

29(i) $1 + \frac{1}{2}x - \frac{1}{8}x^2$

(ii) $1 + \frac{1}{2}x + \frac{3}{8}x^2$, $1 + x + \frac{1}{2}x^2$; 3.315

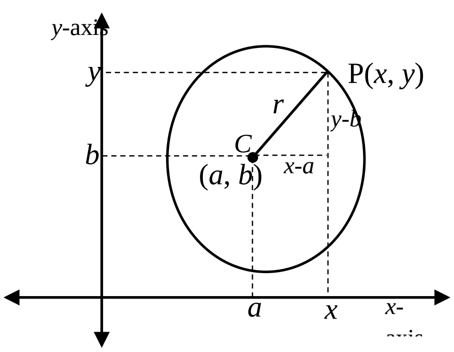
30(a) $\frac{258}{243}$ **(b)** -307.

CIRCLES

A circle is a 2-dimensional shape in Euclidean geometry made by drawing a curve that is always the same distance from the center

A circle can also be defined as a locus of all points P(x, y) which are equidistant from the same given point fixed point C(a, b) [center]

Suppose that the distance of the points P from the given point C (a, b) is r



$$(x - a)^2 + (y - b)^2 = r^2$$

$(x-a)^2+(y-b)^2=r^2$ is the equation of the circle wit center (a, b) and radius r
If the center C is (0, 0) then the equation of the circle is $x^2+y^2=r^2$
For $(x-a)^2+(y-b)^2=r^2$

$$x^2-2ax+a^2+y^2-2by+b^2=r^2$$
$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0$$

Suppose $-a=g, -b=f, C=a^2+b^2-r^2$

$\Rightarrow C=g^2+f^2-r^2$

The equation of the circle becomes

$$x^2+y^2+2gx+2fy+C=0$$

$x^2+y^2+2gx+2fy+C=0$ is the standard equation of a circle with center (-g, -f) and radius $r=\sqrt{g^2+f^2-C}$

Example I

Find the center and the radius of the circles below

(a) $(x-1)^2+(y-2)^2=9$
(b) $(x+1)^2+(y-3)^2=25$
(c) $x^2+y^2-4x-2y=4$
(d) $2x^2+2y^2-2x+2y=1$

Solution

(a) $(x-1)^2+(y-2)^2=9$

Comparing $(x-1)^2+(y-2)^2=9$ with $(x-a)^2+(y-b)^2=r^2$

$a=1, b=2, r^2=9$

$\Rightarrow$ The center is C (1, 2) and $r=3$

$(x-1)^2+(y-2)^2=9$ is a circle with radius 3 units and center (1, 2)

(b) $(x+1)^2+(y-3)^2=25$

Compare $(x+1)^2+(y-3)^2=25$ with $(x-a)^2+(y-b)^2=r^2$

$$a=-1, b=3, r^2=25$$

$r=5$

The center is (-1, 3)

$\therefore (x+1)^2+(y-3)^2=25$ is the equation of the circle with center (-1, 3) and radius 5.

(c) $\boldsymbol{x^2+y^2-4x-2y=4}$

$x^2+y^2-4x-2y-4=0$

Comparing $x^2+y^2-4x-2y-4$ with

$x^2+y^2+2gx+2fy+C=0$

$$2g=-4, 2f=-2$$
$$C=-4$$
$$g=-2, f=-1$$
$$C=-4$$

Since the center is (-g, -f),

The center is (2, 1)

Radius $=\sqrt{g^2+f^2-C}$

$$r = \sqrt{(-2)^2 + (-1)^2 - (-4)}$$
$$r = \sqrt{4 + 1 + 4}$$
$$r = 3$$

$x^2 + y^2 - 4x - 2y = 4$ is a circle with radius 3 units and center (2, 1)

(d) $\mathbf{2x^2 + 2y^2 - 2x + 2y = 1}$

$$\frac{2x^2}{2} + \frac{2y^2}{2} - \frac{2x}{2} + \frac{2y}{2} = \frac{1}{2}$$
$$x^2 + y^2 - x + y - \frac{1}{2} = 0$$

Comparing $x^2 + y^2 - x + y - \frac{1}{2}$ with
$x^2 + y^2 + 2gx + 2fy + C = 0$

$2gx = -x$, $2fy = y$, $c = \frac{-1}{2}$

$g = \frac{-1}{2}$, $f = \frac{1}{2}$

Center $(-g, -f)$
Centre $(\frac{1}{2}, -\frac{1}{2})$

Radius $= \sqrt{g^2 + f^2 - C}$

Radius $= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 - \frac{-1}{2}}$

Radius $= \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}}$

$= \sqrt{1}$
$= 1$

$2x^2 + 2y^2 - 2x + 2y = 1$ is the equation of the circle with center $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius 1.

Example III

Find the equation of the circle with the following centers and radii

(a) Center (2, 3) radius 1
(b) Center (3, -4) radius 5
(c) Center $\left(\frac{-3}{2}, 2\right)$ and radius $\frac{1}{2}$
(d) Center $\left(\frac{-1}{4}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$
(e) Center (0, -5) and radius 5

Solution

(a) Center (2, 3) radius 1

Given a circle of centre (a, b) and radius r. The equation of the circle is $(x - a)^2 + (y - b)^2 = r^2$.
Consider the equation of the circle
$(x - a)^2 + (y - b)^2 = r^2$ with center (a, b) and radius r

$$(x - 2)^2 + (y - 3)^2 = 1^2$$
$$(x - 2)^2 + (y - 3)^2 = 1$$

$x^2 - 4x + 4 + y^2 - 6y + 9 = 1$
$x^2 + y^2 - 4x - 6y + 13 - 1 = 0$
$x^2 + y^2 - 4x - 6y + 12 = 0$

The equation of the circle with center (2, 3) and radius 1 is $x^2 + y^2 - 4x - 6y + 12 = 0$

(b) Center (3, -4) radius 5
$(x-a)^2 + (y-b)^2 = r^2$
$(x-3)^2 + (y--4)^2 = 5^2$
$(x-3)^2 + (y+4)^2 = 5^2$
$x^2 - 6x + 9 + y^2 + 8y + 16 = 25$
$x^2 + y^2 - 6x + 8y = 0$

The equation of the circle with center (3, -4) and radius 5 is $x^2 + y^2 - 6x + 8y = 0$

(c) Center $\left(\frac{-3}{2}, 2\right)$ and radius $\frac{1}{2}$

$$\left(x - \frac{-3}{2}\right)^2 + (y-2)^2 = \left(\frac{1}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + (y-2)^2 = \frac{1}{4}$$

$$x^2 + 3x + \frac{9}{4} + y^2 - 4y + 4 = \frac{1}{4}$$

$$x^2 + y^2 + 3x - 4y + 6 = 0$$

Equation of the circle with center $\left(\frac{-3}{2}, 2\right)$ and radius $r = \frac{1}{2}$ is $x^2 + y^2 + 3x - 4y + 6 = 0$

(d) Center $\left(\frac{-1}{4}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$

$$\left(x - \frac{-1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\sqrt{2}\right)^2$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}(2)$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

$$x^2 + \frac{1}{2}x + \frac{1}{16} + y^2 - y + \frac{1}{4} = \frac{1}{2}$$

$$x^2 + y^2 + \frac{1}{2}x - y + \frac{1}{16} + \frac{1}{4} - \frac{1}{2} = 0$$

$$x^2 + y^2 + \frac{1}{2}x - y - \frac{3}{16} = 0$$

$$16x^2 + 16y^2 + 8x - 16y - 3 = 0$$

(e) Center (0, -5) and radius 5

$$(x-0)^2 + (y--5)^2 = 5^2$$

$$x^2 + (y+5)^2 = 5^2$$
$$x^2 + y^2 + 10y + 25 = 25$$
$$x^2 + y^2 + 10y = 0$$

Example III

State which of the following are equations of the circles

(a) $x^2 + y^2 - 5 = 0$
(b) $x^2 + y^2 + 10 = 0$
(c) $x^2 + y^2 + c = 0$
(d) $x^2 + y^2 + bxy = 1$
(e) $9x^2 + 9y^2 = 1$
(f) $7x^2 + 3x - y^2 + 2y = 16$
(g) $x^2 + 3x - y^2 = 7$
(h) $x^2 + y^2 + 2x - 8y = 1$
(i) $x^2 + 2xy + y^2 = 4$

Solution

(a) $x^2 + y^2 - 5 = 0$
$x^2 + y^2 = 5$
$(x-0)^2 + (y-0)^2 = \left(\sqrt{5}\right)^2$
$x^2 + y^2 - 5 = 0$ is an equation of a circle

(b) $x^2 + y^2 + 10 = 0$
$x^2 + y^2 = -10$
$(x-0)^2 + (y-0)^2 = \left(\sqrt{-10}\right)^2$
$x^2 + y^2 + 10 = 0$ is not an equation of a circle, since $r = \sqrt{-10}$ is not real.

(c) $x^2 + y^2 + c = 0$
$x^2 + y^2 = -c$
$(x-0)^2 + (y-0)^2 = \left(\sqrt{-c}\right)^2$
$x^2 + y^2 + c = 0$ is an equation of the circle when $c < 0$.

(d) $x^2 + y^2 + bxy = 1$

$$x^2 + y^2 + bxy = 1$$

Comparing $x^2 + y^2 + bxy = 1$ with $x^2 + y^2 + 2gx + 2fy + C = 0$
$\Rightarrow x^2 + y^2 + bxy = 1$ is not an equation of a circle because of the component of bxy

(e) $9x^2 + 9y^2 = 1$
$\frac{9x^2}{9} + \frac{9y^2}{9} = \frac{1}{9}$
$x^2 + y^2 = \frac{1}{9}$
$(x-0)^2 + (y-0)^2 = \left(\frac{1}{3}\right)^2$

$9x^2 + 9y^2 = 1$ is a circle

(f) $7x^2 + 3x - y^2 + 2y = 16$

Is not a circle because the co-efficient of $x^2 and\ y^2$ are not the same

(g) $x^2 + 3x - y^2 = 7$

Is not a circle because the co-efficient of x^2 and y^2 are not the same.

(h) $x^2 + y^2 + 2x - 8y = 1$

Is a circle

(i) $x^2 + 2xy + y^2 = 4$

Is not a circle

Example IV (UNEB Question)

The equation of the circle with center O is given by $x^2 + y^2 + Ax + By + C = 0$ where A, B and C are constants. Given that 4A = 3B, 3A = 2C and C = 9

Determine

(a) The coordinates of the center of the circle

(b) The radius of the circle

Solution

4A = 3B …………………… (1)

3A = 2C …………………… (2)

C = 9 ………………………. (3)

Substituting eqn. (3) in eqn. (2)

$$3A = 2(9)$$
$$3A = 18$$
$$A = 6$$

Substituting A = 6 in Eqn (1)

4 × 6 = $3B$

$B = 8$

$x^2 + y^2 + Ax + By + C = 0$

$x^2 + y^2 + 6x + 8y + 9 = 0$

Comparing $x^2 + y^2 + 6x + 8y + 9 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0$

$2g = 6 \Rightarrow g = 3$

$2f = 8, \Rightarrow f = 4$

$C = 9$

Centre (-3, -4)

Radius = $\sqrt{g^2 + f^2 - C}$

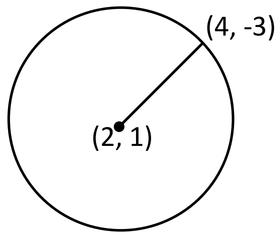
Radius = $\sqrt{(-3)^2 + (-4)^2 - 9}$

= $\sqrt{9 + 16 - 9}$

= 4

Example V
Find the equation of a circle whose center is (2, 1) and passes through (4, -3)

Solution

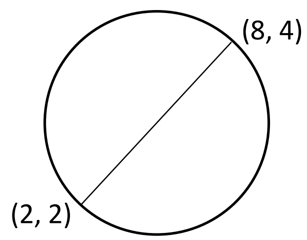


$$r = \sqrt{(2-4)^2 + (1 - -3)^2}$$

$r = \sqrt{4+16}$
$r = \sqrt{20}$
$(x-a)^2 + (y-b)^2 = r^2$
$(x-2)^2 + (y-1)^2 = (\sqrt{20})^2$
$x^2 - 4x + 4 + y^2 - 2y + 1 = 20$

$x^2 + y^2 - 4x - 2y - 15 = 0$

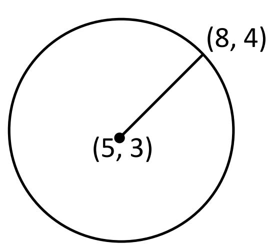
Example VI
The points (8, 4) and (2, 2) are end points of the diameter of the circle. Find the center, the radius and the equation of the circle
Solution



$$C\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$C\left(\frac{2+8}{2}, \frac{2+4}{2}\right)$$

$$C(5, 3)$$



$$r = \sqrt{(5-8)^2 + (3-4)^2}$$

$$r = \sqrt{9+1}$$

$$r = \sqrt{10}$$

$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-5)^2 + (y-3)^2 = 10$$
$$x^2 - 10x + 25 + y^2 - 6y + 9 = 10$$
$$x^2 + y^2 - 10x - 6y + 34 - 10 = 0$$
$$x^2 + y^2 - 10x - 6y + 24 = 0$$

Example VI

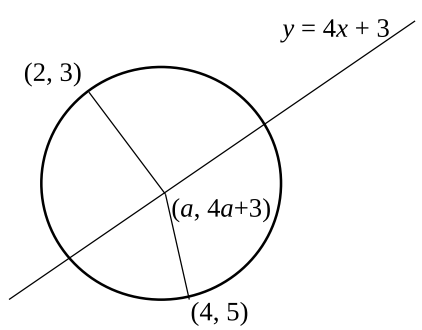
Find the equation of a circle passing through points (2, 3) and (4, 5) having its center on the line $y = 4x + 3$

Solution

$$y = 4x + 3$$

Let the center be (x, y). Since it lies on the line
$y = 4x + 3 = 0$, let the *x*-co-ordinate of the center be ***a***.
Then the *y*-co-ordinate

$$y = 4a + 3$$



$$r_1 = \sqrt{(a-2)^2 + (4a+3-3)^2}$$
$$r_1 = \sqrt{(a-2)^2 + (4a)^2}$$
$$r_1 = \sqrt{a^2 - 4a + 4 + 16a^2}$$
$$r_1 = \sqrt{17a^2 - 4a + 4}$$

$$r_2 = \sqrt{(a-4)^2 + (4a+3-5)^2}$$
$$r_2 = \sqrt{(a-4)^2 + (4a-2)^2}$$
$$r_2 = \sqrt{a^2 - 8a + 16 + 16a^2 - 16a + 4}$$
$$r_2 = \sqrt{17a^2 - 24a + 20}$$

$$r_1 = r_2 = r$$
$$\sqrt{17a^2 - 4a + 4} = \sqrt{17a^2 - 24a + 20}$$

$17a^2 - 4a + 4 = 17a^2 - 24a + 20$

$20a = 16$

$$a = \frac{16}{20} = \frac{4}{5}$$

$$r_1 = r = \sqrt{17 \times (\tfrac{4}{5})^2 - 4(\tfrac{4}{5}) + 4}$$

$$r = \sqrt{\tfrac{17\times16}{25} - \tfrac{16}{5} + 4}$$

$$r = \sqrt{\frac{292}{25}}$$

Centre$(a, 4a+3)$

centre $\left(\tfrac{4}{5},\ \tfrac{4\times4}{5} + 3\right)$

centre $\left(\tfrac{4}{5},\ \tfrac{31}{5}\right)$

$$(x - \tfrac{4}{5})^2 + (y - \tfrac{31}{5})^2 = \left(\sqrt{\tfrac{292}{25}}\right)^2$$

$$(x - \tfrac{4}{5})^2 + (y - \tfrac{31}{5})^2 = \frac{292}{25}$$

$$x^2 - \frac{8x}{5} + \frac{16}{25} + y^2 - \frac{62y}{5} + \frac{961}{25} = \frac{292}{25}$$

$$x^2 + y^2 - \frac{8x}{5} - \frac{62y}{5} + \frac{977}{25} - \frac{292}{25} = 0$$

$$x^2 + y^2 - \frac{8x}{5} - \frac{62y}{5} + \frac{685}{25} = 0$$

$\Rightarrow 5x^2 + 5y^2 - 8x - 62y + 137 = 0$

Example

What is the equation of the circle whose center lies on the $x - 2y + 2 = 0$ which touches the positive axes.

Solution

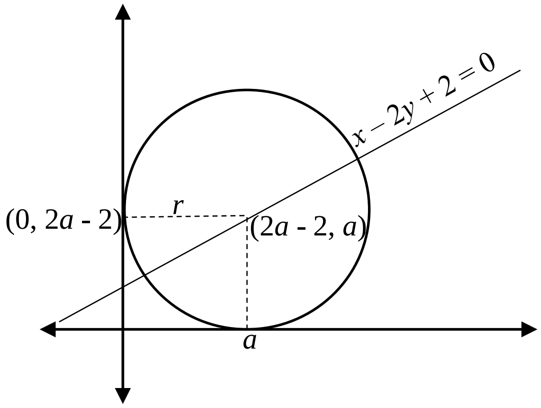
Let the y-coordinate of the centre be a

$x - 2y + 2 = 0$

$x - 2a + 2 = 0$

$x = 2a - 2$

$(2a - 2, a)$



$2a - 2 = a$

$a = 2$

The center is (2, 2); radius $r = 2$

$$(x-2)^2 + (y-2)^2 = 2^2$$
$$x^2 - 4x + 4 + y^2 - 4y + 4 = 4$$
$$x^2 + y^2 - 4x - 4y + 4 = 0$$

Equation of circle passing through three points

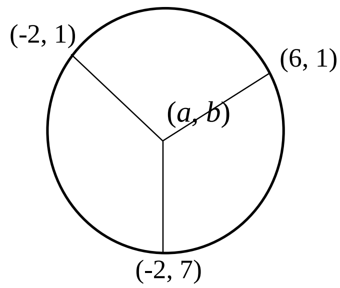
Example I

Find the equation of the circle passing through the points

(a) A(-2, 1) B(6, 1) and C(-2, 7)
(b) A(-1, 4) B(2, 5) and C(0, 1)
(c) A(3, 1) B(8, 2) and C(2, 6)
(d) A(5, 7) B(1, 6) and C(2, 2)

Solution

(a) A(-2, 1) B(6, 1) and C(-2, 7)



$$r_1 = \sqrt{(a - {}^{-}2)^2 + (b-1)^2}$$
$$r_2 = \sqrt{(a-6)^2 + (b-1)^2}$$
$$r_3 = \sqrt{(a - {}^{-}2)^2 + (b-7)^2}$$

Equating the radii; $r_1 = r_2 = r$

$$\sqrt{(a - {}^{-}2)^2 + (b-1)^2} = \sqrt{(a-6)^2 + (b-1)^2}$$

$$(a+2)^2 + (b-1)^2 = (a-6)^2 + (b-1)^2$$

$$a^2 + 4a + 4 + b^2 - 2b + 1 = a^2 - 12a + 36 + b^2 - 2b + 1$$

$$a^2 + b^2 + 4a - 2b + 5 = a^2 + b^2 - 12a - 2b + 37$$

$$4a - 2b + 5 = -12a - 2b + 37$$

$$16a = 32$$

$$a = 2$$

Also $r_1 = r_3 = r$

$$\sqrt{(a - -2)^2 + (b-1)^2} = \sqrt{(a - -2)^2 + (b-7)^2}$$

$$(a+2)^2 + (b-1)^2 = (a+2)^2 + (b-7)^2$$

$$a^2 + 4a + 4 + b^2 - 2b + 1 = a^2 + 4a + 4 + b^2 - 14b + 49$$

$$b^2 - 2b + 1 = b^2 - 14b + 49$$

$$12b = 48$$

$$b = 4$$

Center $(a, b) = (2, 4)$

$$radius = \sqrt{(a - -2)^2 + (b - 1)^2}$$

$$= \sqrt{(2 - -2)^2 + (4 - 1)^2}$$

$$= \sqrt{16 + 9}$$

$$= 5$$

$$(x - a)^2 + (y - b) = r^2$$

$$(x - 2)^2 + (y - 4)^2 = 5^2$$

$$x^2 - 4x + 4 + y^2 - 8y + 16 = 25$$

$$x^2 + y^2 - 4x - 8y + 20 = 25$$

$$x^2 + y^2 - 4x - 8y - 5 = 0$$

Alternatively; Consider the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

At (-2, 1)

$$-2^2 + 1^2 + 2g(-2) + 2f(1) + c = 0$$

$-4g + 2f + c = -5$…………………. (1)

At (6, 1), $6^2 + 1^2 + 2g(6) + 2f(1) + c = 0$

$$36 + 1 + 12g + 2f + c = 0$$

$12g + 2f + c = -37$……………….. (2)

At (-2, 7), $-2^2 + 7^2 + 2g(-2) + 2f(7) + c = 0$

$$4 + 49 - 4g + 14f + c = 0$$

$-4g + 14f + c = -53$……………… (3)

Solving equation (1), 2 and 3 simultaneously

$$g = -2, f = -4, c = -5$$

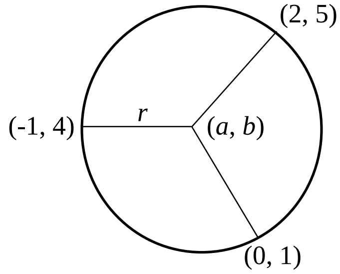
Substituting $g = -2, f = -4, c = -5$ in the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

$$x^2 + y^2 + 2x(-2) + 2y(-4) + -5 = 0$$

$$x^2 + y^2 - 4x - 8y - 5 = 0$$

(As before)

(b) A(-1, 4) B(2, 5) and C(0, 1)



$$r_1 = \sqrt{(a - -1)^2 + (b - 4)^2}$$

$$r_2 = \sqrt{(a-2)^2 + (b-5)^2}$$

$$r_3 = \sqrt{(a-0)^2 + (b-1)^2}$$

$$r_1 = r_2 = r$$

$$\sqrt{(a+1)^2 + (b-4)^2} = \sqrt{(a-2)^2 + (b-5)^2}$$

$$(a+1)^2 + (b-4)^2 = (a-2)^2 + (b-5)^2$$

$$a^2 + 2a + 1 + b^2 - 8b + 16 = a^2 - 4a + 4 + b^2 - 10b + 25$$

$$2a - 8b + 17 = -4a - 10b + 29$$

$$6a + 2b = 12$$

$$\Rightarrow 3a + b = 6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly, $r_1 = r_3 = r$

$$\sqrt{(a+1)^2 + (b-4)^2} = \sqrt{(a-0)^2 + (b-1)^2}$$

$$(a+1)^2 + (b-4)^2 = (a-0)^2 + (b-1)^2$$

$$a^2 + 2a + 1 + b^2 - 8b + 16 = a^2 + b^2 - 2b + 1$$

$$2a - 8b + 17 = -2b + 1$$

$$2a - 6b = -16$$

$$a - 3b = -8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From eqn. (1)

$$b = 6 - 3a$$

Substituting $b = 6 - 3a$ in eqn. (2)

$$a - 3(6 - 3a) = -8$$

$$a - 18 + 9a = -8$$

$$10a = 10$$

$$a = 1$$

$$\Rightarrow b = 6 - 3 \times 1$$

$$b = 3$$

Center (1, 3)

$$r = \sqrt{(a+1)^2 + (b-4)^2}$$

$$r = \sqrt{(1 - -1)^2 + (3-4)^2}$$

$$r = \sqrt{4+1}$$

$$r = \sqrt{5}$$

$$(x-a)^2 + (y-b)^2 = r^2$$

$$(x-1)^2 + (y-3)^2 = (\sqrt{5})^2$$

$$x^2 - 2x + 1 + y^2 - 6y + 9 = 5$$

$$x^2 + y^2 - 2x - 6y + 5 = 0$$

Alternatively

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

At (-1, 4): $(-1)^2 + 4^2 + 2g(-1) + 2f(4) + c = 0$

$$1 + 16 - 2g + 8f + c = 0$$

$$-2g + 8f + c = -17$$

At (2, 5): $(2)^2 + 5^2 + 2g(2) + 2f(5) + c = 0$

$$4 + 25 + 4g + 10f + c = 0$$
$$4g + 10f + c = -29 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

At (0, 1): $0^2 + 1^2 + 2g(0) + 2f(1) + c = 0$

$$2f + c = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (3)$$

Solving eqn. 1, 2 and 3 simultaneously

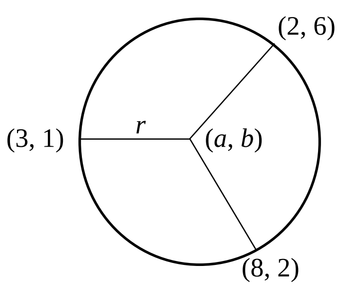
$$g = 1, f = -3, c = 5$$

Substituting $g = -1, f = -3, c = 5$ in the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

$$x^2 + y^2 + 2x(1) + 2y(-3) + 5 = 0$$

$$x^2 + y^2 - 2x - 6y + 5 = 0 \text{ (as before)}$$

(c) A(3, 1) B(8, 2) and C(2, 6)



$$r_1 = \sqrt{(a-3)^2 + (b-1)^2}$$
$$r_2 = \sqrt{(a-2)^2 + (b-6)^2}$$
$$r_2 = \sqrt{(a-8)^2 + (b-2)^2}$$
$$r_1 = r_2 = r$$
$$\sqrt{(a-3)^2 + (b-1)^2} = \sqrt{(a-2)^2 + (b-6)^2}$$

$$(a-3)^2 + (b-1)^2 = (a-2)^2 + (b-6)^2$$

$$a^2 - 6a + 9 + b^2 - 2b + 1 = a^2 - 4a + 4 + b^2 - 12b + 36$$
$$-6a - 2b + 10 = -4a - 12b + 40$$
$$-2a + 10b = 30$$
$$-a + 5b = 15$$
$$a = 5b - 15 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (1)$$

Similarly; $r_1 = r_3 = r$

$$\sqrt{(a-3)^2 + (b-1)^2} = \sqrt{(a-8)^2 + (b-2)^2}$$

$$(a-3)^2 + (b-1)^2 = (a-8)^2 + (b-2)^2$$
$$a^2 - 6a + 9 + b^2 - 2b + 1 = a^2 - 16a + 64 + b^2 - 4b + 4$$
$$-6a - 2b + 10 = -16a - 4b + 68$$

$$10a + 2b = 58$$
$$\Rightarrow 5a + b = 29 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Substituting equation (1) in (2)

$$5(5b - 15) + b = 29$$

$$25b - 75 + b = 29$$
$$26b = 104$$
$$b = 4$$
$$a = 5b - 15$$
$$a = 5 \times 4 - 15$$
$$a = 5$$

Center (5, 4)

$$r = \sqrt{(5-3)^2 + (4-1)^2}$$
$$r = \sqrt{4+9}$$
$$r = \sqrt{13}$$
$$(x-5)^2 + (y-4)^2 = (\sqrt{13})^2$$
$$x^2 - 10x + 25 + y^2 - 8y + 16 = 13$$
$$x^2 + y^2 - 10x - 8y + 41 = 13$$
$$x^2 + y^2 - 10x - 8y + 28 = 0$$

Alternatively

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

At (3, 1): $3^2 + 1^2 + 2g(3) + 2f(1) + c = 0$

$$9 + 1 + 6g + 2f + c = 0$$

$$6g + 2f + c = -10 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

At (8, 2): $8^2 + 2^2 + 2g(8) + 2f(2) + c = 0$

$$64 + 4 + 16g + 4f + c = 0$$

$$16g + 4f + c = -68 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

At (2, 2): $2^2 + 2^2 + 2g(2) + 2f(2) + c = 0$

$$4 + 4 + 4g + 4f + c = 0$$
$$4g + 4f + c = -8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Solving eqn. 1, 2 and 3 simultaneously

$$g = -5, f = -4, c = 28$$

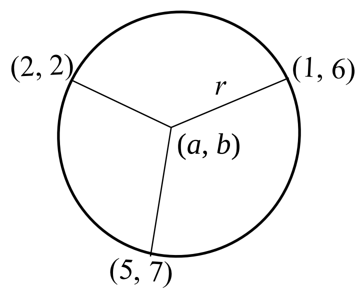
Substituting the values of g, f and c in the general equation

$$x^2 + y^2 + 2gx + 2fy + C = 0$$
$$x^2 + y^2 + 2(-5)x + 2y(-4) + 28 = 0$$

$$x^2 + y^2 - 10x - 8y + 28 = 0$$

(As before)

(d) A(5, 7) B(1, 6) and C(2, 2)



$$r_1 = \sqrt{(a-2)^2 + (b-2)^2}$$

$$r_2 = \sqrt{(a-1)^2 + (b-6)^2}$$

$$r_3 = \sqrt{(a-5)^2 + (b-7)^2}$$

Equating the radii

$$r_1 = r_2 = r$$

$$\sqrt{(a-2)^2 + (b-2)^2} = \sqrt{(a-1)^2 + (b-6)^2} \text{ .... (1)}$$

$$\sqrt{(a-2)^2 + (b-2)^2} = \sqrt{(a-5)^2 + (b-7)^2} \text{ .... (2)}$$

From equation (1)

$$(a-2)^2 + (b-2)^2 = (a-1)^2 + (b-6)^2$$

$$a^2 - 4a + 4 + b^2 - 4b + 4 = a^2 - 2a + 1 + b^2 - 12b + 36$$

$$-4a - 4b + 8 = -2a - 12b + 37$$

$$12b - 4b - 4a + 2a = 37 - 8$$

$$8b - 2a = 29 \text{.............................. (3)}$$

From eqn. (2)

$(a-2)^2 + (b-2)^2 = (a-5)^2 + (b-7)^2$

$$a^2 - 4a + 4 + b^2 - 4b + 4 = a^2 - 10a + 25 + b^2 - 14b + 49$$

$$-4a - 4b + 8 = -10a - 14b + 74$$

$$6a + 10b = 74 - 8$$

$$6a + 10b = 66$$

$$3a + 5b = 33 \text{............................ (4)}$$

Solving Eqn (3) and (4) simultaneously

$$\Rightarrow a = \frac{7}{2}, \text{ and } \; b = \frac{9}{2}$$

Center $\left(\frac{7}{2}, \frac{9}{2}\right)$

$$r = \sqrt{\left(\frac{7}{2} - 2\right)^2 + \left(\frac{9}{2} - 2\right)^2}$$

$$r = \sqrt{\frac{9}{4} + \frac{25}{4}}$$

$$r = \frac{\sqrt{34}}{2}$$

$$(x - a)^2 + (y - b)^2 = r^2$$

$$\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{34}{4}$$

$$x^2 - 7x + \frac{49}{4} + y^2 - 9y + \frac{81}{4} = \frac{34}{4}$$

$$x^2 + y^2 - 7x - 9y + \frac{96}{4} = 0$$

$$x^2 + y^2 - 7x - 9y + 24 = 0$$

Alternatively

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

At (5, 7): $5^2 + 7^2 + 2g(5) + 2f(7) + c = 0$

$$25 + 49 + 10g + 14f + c = 0$$

$$10g + 14f + c = -74 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

At (1, 6): $1^2 + 6^2 + 2g(1) + 2f(6) + c = 0$

$$1 + 36 + 2g + 12f + c = 0$$

$2g + 12f + c = -37 \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2)

At (2, 2): $2^2 + 2^2 + 2g(2) + 2f(2) + c = 0$

$$4g + 4f + c = -8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Solving eqn. 1, 2 and 3 simultaneously

$$g = \frac{-7}{2}, f = \frac{-9}{2}, c = 24$$

Substituting g, f and c in the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + C = 0.$$

$$x^2 + y^2 + 2\left(\frac{-7}{2}\right)x + 2\left(\frac{-9}{2}\right)y + C = 0$$

$$x^2 + y^2 - 7x - 9y + 24 = 0$$

Parametric Equations of circle

Consider a circle $(x - a)^2 + (y - b)^2 = r^2$ the parametric equations of the above circles are $x - a = r\cos\theta$ and $y - b = r\sin\theta$

$$\therefore x = a + \cos\theta \;\; and \; y = b + r\sin\theta$$

Example I

Find the parametric equation of the circle $(x-4)^2 + (y-3)^2 = 4$

Solution

$$(x-4)^2 + (y-3)^2 = 2^2$$
$$x - 4 = r\cos\theta$$
$$y - 3 = r\sin\theta$$
$$r = 2$$
$$x - 4 = 2\cos\theta$$
$$y - 3 = 2\sin\theta$$
$$x = 4 + 2\cos\theta$$
$$y = 3 + 2\sin\theta$$

The parametric equations of the circle $(x-4)^2 + (y-3)^2 = 4$ are

$$x = 4 + 2\cos\theta$$

$$y = 3 + 2\sin\theta$$

Example II

Find the parametric equations of the circle

$$(x+1)^2 + (y-2)^2 = 9$$

Solution

Comparing $(x+1)^2 + (y-2)^2 = 9$ with the equation of the circle $(x-a)^2 + (y-b)^2 = r^2$

$$a = -1, b = 2, r = 3$$
$$x + 1 = r\cos\theta$$
$$y - 2 = r\sin\theta$$
$$x + 1 = 3\cos\theta$$
$$y - 2 = 3\sin\theta$$
$$x = 3\cos\theta - 1$$
$$y = 2 + 3\sin\theta$$

Example III

Find the parametric equations of the circle

$$x^2 + y^2 - 4x - 2y + 1 = 0$$

Solution

$$x^2 + y^2 - 4x - 2y + 1 = 0$$

By completing squares;s

$$(x^2 - 4x + 4) - 4 + y^2 - 2y + 1 = 0$$
$$(x-2)^2 + (y-1)^2 = 4$$
$$x - 2 = r\cos\theta$$
$$y - 1 = r\sin\theta$$
$$x - 2 = 2\cos\theta$$
$$y - 1 = 2\sin\theta$$
$$x = 2 + 2\cos\theta$$
$$y = 1 + 2\sin\theta$$

Example IV
Find the parametric equation of a circle
$x^2 + y^2 - 6x + 4y - 12 = 0$
Solution

$$x^2 + y^2 - 6x + 4y - 12 = 0$$
$$x^2 - 6x + y^2 + 4y - 12 = 0$$

By completing squares;

$$x^2 - 6x + 9 - 9 + y^2 + 4y^2 + 4 - 4 - 12 = 0$$
$$(x-3)^2 + (y+2)^2 = 25$$
$$(x-a)^2 + (y-b)^2 = 5^2$$
$$a = 3, b = -2, r = 5$$
$$x - 3 = r\cos\theta$$
$$y + 2 = r\sin\theta$$
$$x - 3 = 5\cos\theta$$
$$y + 2 = 5\sin\theta$$
$$x = 3 + 5\cos\theta$$
$$y = -2 + 5\sin\theta$$

Example V
Find the Cartesian equation of the circle with parametric equations
$x = -2 + 3\cos\theta$
$y = 3 + 3\sin\theta$
Solution

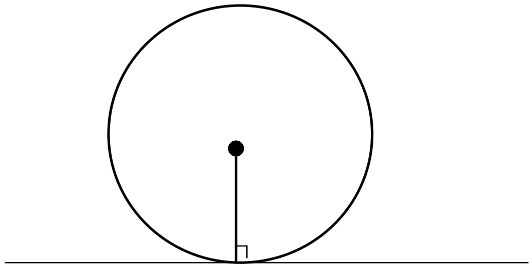
$$\frac{x+2}{3} = \cos\theta$$
$$\frac{y-3}{3} = \sin\theta$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$\frac{(x+2)^2}{9} + \frac{(y-3)^2}{9} = 1$$

But $\frac{(x+2)^2}{9} + \frac{(y-3)^2}{9} = 1$

$$(x+2)^2 + (y-3)^2 = 9$$
$$x^2 + 4x + 4 + y^2 - 6y + 9 = 9$$
$$x^2 + 4x + y^2 - 6y + 4 = 0$$
$$x^2 + y^2 + 4x - 6y + 4 = 0$$

Tangents to the Circle

A tangent to the circle is a line which touches the circle at only one point and makes 90^0 with the radius of the circle.



Length of the tangent to a circle

Example

Find the length of the tangent from (5, 7) to the circle

$$x^2 + y^2 - 4x - 6y + 9 = 0$$

Solution

Comparing $x^2 + y^2 + 4x - 6y - 9 = 0$ with

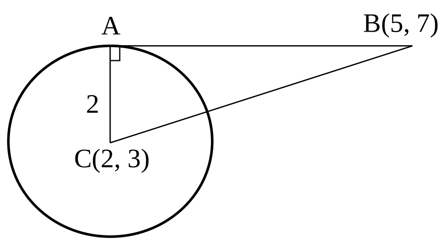
$x^2 + y^2 + 2gx + 2fy + C = 0.$

$$g = -2, f = -3, c = 9$$

Center (-g, -f)

Center (2, 3)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{4 + 9 - 9}$$
$$r = 2$$



$$CB = \sqrt{(2-5)^2 + (3-7)^2}$$
$$CB = \sqrt{9 + 16}$$
$$CB = 5$$
$$AB^2 + AC^2 = CB^2$$
$$AB^2 + 2^2 = 5^2$$
$$AB^2 = 5^2 - 2^2$$
$$AB^2 = 21$$
$$AB = \sqrt{21}\, units$$

The length of the tangent is $\sqrt{21}$ units

Example II

Find the lengths of the tangents from the given points to the following circles

(a) $x^2 + y^2 + 4x - 6y + 10 = 0, (0, 0)$

(b) $x^2 + y^2 + 6x + 10y - 2 = 0, (-2, 3)$

Solution

(a) $x^2+y^2+4x-6y+10=0,(0,0)$

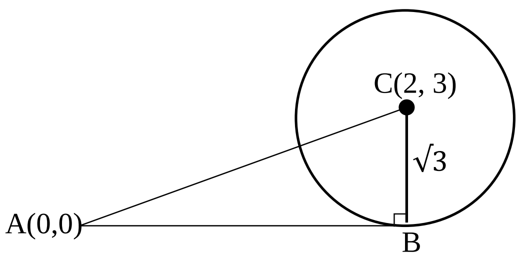
Comparing $x^2+y^2+4x-6y+10=0$ with $x^2+y^2+2gx+2fy+C=0.$

$$2gx=-4x$$
$$g=-2$$
$$2fy=-6y$$
$$f=-3$$

Center (-*g*, -*f*)

Center (2, 3)

$$r=\sqrt{g^2+f^2-c}$$
$$r=\sqrt{4+9-10}$$
$$r=\sqrt{3}$$



$$AC=\sqrt{(0-2)^2+(0-3)^2}$$
$$AC=\sqrt{4+9}$$
$$AC=\sqrt{13}$$
$$AB^2+CB^2=AC^2$$
$$AB^2+(\sqrt{3})^2=(\sqrt{13})^2$$
$$AB^2+3=13$$
$$AB^2=10$$
$$AB=\sqrt{10}$$

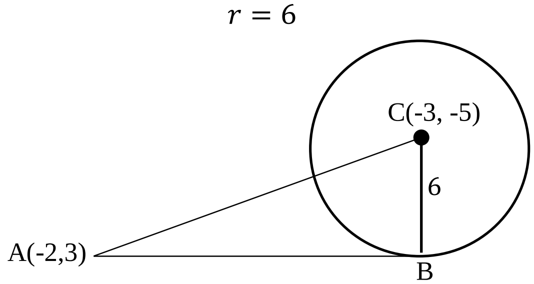
(b) $\boldsymbol{x^2+y^2+6x+10y-2=0},(\boldsymbol{-2},\boldsymbol{3})$

Comparing $x^2+y^2+6x+10y-2=0$ with $x^2+y^2+2gx+2fy+C=0.$

$$g=3, f=5, c=-2$$

Center (-3, -5)

$$r=\sqrt{g^2+f^2-c}$$
$$r=\sqrt{9+25--2}$$
$$r=6$$



$$AC = \sqrt{(-2 - -3)^2 + (3 - -5)^2}$$
$$AC = \sqrt{1 + 64}$$
$$AC = \sqrt{65}$$
$$AB^2 + 6^2 = (\sqrt{65})^2$$
$$AB^2 + 36 = 65$$
$$AB^2 = 65 - 36$$
$$AB^2 = 29$$
$$AB = \sqrt{29}$$

Alternative method of finding length of the tangent to a circle

The length of a tangent drawn from a point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + C = 0$ is given by

$$L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$= \sqrt{S_1}$ where L = length of the tangent

The square of the length of the tangent from the point P is called a power point with respect to the circle.

Example I
Find the length of the tangent drawn from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$

Solution
Comparing $x^2 + y^2 + 6x - 4y - 3 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0$

$$g = 3, f = -2, c = 3$$

$$(x_1, y_1) = (5, 1)$$

$$L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$$L = \sqrt{5^2 + 1^2 + 2g(5) + 2f(1) + c}$$

$$L = \sqrt{5^2 + 1^2 + 2 \times 3(5) + 2(-2)(1) - 3}$$

$$L = \sqrt{25 + 1 + 30 - 4 - 3}$$

$$L = 7\ Units$$

Example II
If the length of the tangent from the point (f, g) to the circle $x^2 + y^2 = 4$ is four times the length of the tangent from (f_1, g_1) it to the circle $x^2 + y^2 = 4x$, show that $15f_1^2 + 15g_1^2 - 64f_1 + 4 = 0$

Solution

$$x^2 + y^2 - 4 = 0$$
$$g = 0, f = 0, c = -4$$
$$L_1 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$$L_1 = \sqrt{g_1^2 + f_1^2 + 0 + 0 + -4}$$

$$L_1 = \sqrt{g_1{}^2 + f_1{}^2 - 4}$$

For $x^2 + y^2 - 4x = 0$, $g = -2$ and $f = 0$

$$L_2 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$$L_2 = \sqrt{g_1{}^2 + f_1{}^2 - 2(-2)g_1 + 0 + 0}$$

$$L_2 = \sqrt{g_1{}^2 + f_1{}^2 + 4g_1}$$

But $L_1 = 4L_2$

$$\sqrt{g_1{}^2 + f_1{}^2 - 4} = 4\sqrt{g_1{}^2 + f_1{}^2 + 4g_1}$$

$$g_1{}^2 + f_1{}^2 - 4 = 16\left(g_1{}^2 + f_1{}^2 + 4g_1\right)$$

$$g_1{}^2 + f_1{}^2 - 4 = 16g_1{}^2 + 16f_1{}^2 + 64g_1$$

$$15g_1 + 15f_1 + 64g_1 + 4 = 0 \text{ (as required)}$$

Equation of a Tangent

Example I

Find the equation of the tangent to the circle $x^2 + y^2 + 2x - 2y - 8 = 0$ at $(2, 2)$

Solution

$$\frac{d}{dx}(x^2 + y^2 + 2x - 2y - 8) = \frac{d}{dx}(0)$$

$$2xdx + 2ydy + 2dx - 2dy = 0$$

$$2x + 2y\frac{dy}{dx} + 2 - 2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - 2) = -2 - 2x$$

$$\frac{dy}{dx} = \frac{-2 - 2x}{2y - 2}$$

$$\frac{dy}{dx} = \frac{-2(1 + x)}{2(y - 1)}$$

$$\frac{dy}{dx} = \frac{-1(1 + x)}{y - 1}$$

$$\frac{dy}{dx} = \frac{-1 - x}{y - 1}$$

$$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{-1 - 2}{2 - 1}$$

$$\frac{dy}{dx} = -3$$

$$\frac{y - 2}{x - 2} = -3$$

$$y - 2 = -3(x - 2)$$

$$y - 2 = -3x + 6$$

$$y = -3x + 8$$

Alternatively

Note: The equation of the tangent to the circle $x^2 + y^2 = a^2\ at\ (x_1, y_1)$ is $xx_1 + yy_1 = a^2$

The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + C = 0$ at x_1, y_1 is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

We can now find the equation of the tangent to the $x^2 + y^2 + 2x - 2y - 8 = 0\ at\ (2, 2)$

Comparing $x^2 + y^2 + 2x - 2y - 8 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0$

$$g = 1, f = -1, c = -8$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$x(2) + y(2) + g(x + 2) + f(y + 2) - 8 = 0$$

$$2x + 2y + 1(x + 2) - 1(y + 2) - 8 = 0$$

$$2x + 2y + x + 2 - y - 2 - 8 = 0$$

$$3x + y = 8$$

$$y = -3x + 8\ (as\ before\ )$$

Example II

Find the equation of the tangent to the circle $2x^2 + 2y^2 - 8x - 5y - 1 = 0$ at C (1, -1)

Solution

$$2x^2 + 2y^2 - 8x - 5y - 1 = 0$$
$$4xdx + 4ydy - 8dx - 5dy = 0$$
$$4x + 4y\frac{dy}{dx} - 8 - 5\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(4y - 5) = 8 - 4x$$
$$\frac{dy}{dx} = \frac{8 - 4x}{4y - 5}$$
$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{8 - 4(1)}{4 \times {}^{-}1 - 5} = \frac{4}{-9}$$
$$\frac{y - -1}{x - 1} = \frac{-4}{9}$$
$$9(y + 1) = -4(x - 1)$$
$$9y + 9 = -4x + 4$$
$$9y = -4x - 5$$

Alternative method

From $2x^2 + 2y^2 - 8x - 5y - 1 = 0$,

$x^2 + y^2 - 4x - \frac{5y}{2} - \frac{1}{2} = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow g = -2,\ f = \frac{-5}{4},\ c = \frac{-1}{2}$

$x_1 = 1, y_1 = -1$

The equation of the tangent is given by

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$x(1) + y(-1) + -2(x + 1) + \frac{-5}{4}(y - 1) + \frac{-1}{2} = 0$$

$x - y - 2x - 2 + \frac{-5y}{4} + \frac{5}{4} - \frac{1}{2} = 0$

$-x - \frac{9y}{4} - \frac{5}{4} = 0$

$-4x - 9y - 5 = 0$

$4x + 9y + 5 = 0$

Example III

The tangent to the circle $x^2 + y^2 - 4x + 6y - 77 = 0$ at the point (5, 6) meets the axes at A and B. find A and B

Solution

$$x^2 + y^2 - 4x + 6y - 77 = 0$$

$$2xdx + 2ydy - 4dx + 6dy = 0$$

$$2x + 2y\frac{dy}{dx} - 4 + 6\frac{dy}{dx} = 0$$

$$(2y + 6)\frac{dy}{dx} = 4 - 2x$$

$$\frac{dy}{dx} = \frac{4 - 2x}{2y + 6}$$

$$\left.\frac{dy}{dx}\right|_{(5,6)} = \frac{4-2(5)}{2(6)+6}$$

$$\frac{dy}{dx} = \frac{-6}{18}$$

$$= \frac{-1}{3}$$

$$\frac{y - 6}{x - 5} = \frac{-1}{3}$$

$$3(y - 6) = -1(x - 5)$$

$$3y - 18 = -x + 5$$

$$3y = -x + 23$$

$$x + 3y = 23$$

Alternative method

Comparing $x^2 + y^2 - 4x + 6y - 77 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -2, f = 3, c = -77$
$x_1 = 5, \; y_1 = 6$
The equation of the tangent is

$$x_1x + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$\Rightarrow 5x + 6y + -2(x + 5) + 3(y + 6) – 77 = 0$
$5x + 6y – 2x – 10 + 3y + 18 – 77 = 0$
$3x + 9y = 69$
$\Rightarrow x + 3y = 23$, as before.
At the x- axis (A), $y = 0$

$$0 = -x + 23$$
$$x = 23$$

The tangent meets the x- axis at (23, 0)
At the y- axis (B), $x = 0$

$$3y = 23$$
$$y = \frac{23}{3}$$

The curve cuts the y- axis at $(0, \frac{23}{3})$

Example VII

Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0 \; at \; (4, -1)$

Solution

$$x^2 + y^2 - 30x + 6y + 109 = 0$$
$$\frac{d}{dx}(\, x^2 + y^2 - 30x + 6y + 109) = \frac{d}{dx}(0)$$
$$2xdx + 2ydy - 30dx + 6dy = 0$$
$$2x + 2y\frac{dy}{dx} - 30 + 6\frac{dy}{dx} = 0$$
$$(2y + 6)\frac{dy}{dx} = 30 - 2x$$
$$\frac{dy}{dx} = \frac{30 - 2x}{2y + 6}$$
$$\frac{dy}{dx} = \frac{15 - x}{y + 3}$$
$$\left.\frac{dy}{dx}\right|_{(4,\,-1)} = \frac{15-4}{-1+3} = \frac{11}{2}$$
$$\frac{y - -1}{x - 4} = \frac{11}{2}$$
$$2y + 2 = 11x - 44$$
$$2y = 11x - 46$$
$$0 = 11x - 2y - 46$$

Alternatively

Given a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ the equation of the tangent at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Comparing $x^2 + y^2 - 30x + 6y + 109 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -15, f = 3, c = 109$, $x_1 = 4$, $y_1 = -1$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$\Rightarrow x(4) + y(-1) + -15(x+4) + 3(y + -1) + 109 = 0$

$$4x - y - 15x - 60 + 3y - 3 + 109 = 0$$
$$-11x + 2y + 46 = 0$$
$$11x - 2y - 46 = 0\ (as\ before)$$

Example IV

Show that $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$

Solution

$$y = mx + c$$
$$x^2 + y^2 = a^2$$
$$c + (mx + c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$
$$(1 + m^2)x^2 + (2mc)x + c^2 - a^2 = 0$$
$$B^2 = 4AC\ (for\ tangency)$$
$$(2mc)^2 = 4(1 + m^2)[c^2 - a^2]$$
$$4m^2c^2 = 4(1 + m^2)[c^2 - a^2]$$
$$m^2c^2 = (1 + m^2)[c^2 - a^2]$$
$$m^2c^2 = c^2 - a^2 + m^2c^2 - m^2a^2$$
$$c^2 = a^2 + m^2a^2$$
$$c^2 = a^2(1 + m^2)$$

Example V

Show that the line $y = x + 1$ touches the circle
$x^2 + y^2 - 8x - 2y + 9 = 0$.

Solution

$$x^2 + y^2 - 8x - 2y + 9 = 0$$

$y = x + 1$

$$x^2 + (x + 1)^2 - 8x - 2(x + 1) + 9 = 0$$
$$x^2 + x^2 + 2x + 1 - 8x - 2x - 2 + 9 = 0$$
$$2x^2 - 8x + 8 = 0$$
$$x^2 - 4x + 4 = 0$$

For the line to touch the circle

$$B^2 = 4AC$$
$$(-4)^2 = 4(4)(1)$$
$$16 = 16$$

The line y = x + 1 touches the circle

$$x^2 + y^2 - 8x - 2y + 9 = 0$$

Note:

> If $y = mx + c$ is a line and $x^2 + y^2 = a^2$ is a circle then
> (i) $C^2 > a^2(1 + m^2)$ the line is a secant to the circle
> (ii) If $C^2 = a^2(1 + m^2)$ the line touches the circle
> (iii) If $C^2 < a^2(1 + m^2)$ the line doesn't meet the circle

Example VI

For what values of *c* will the line *y* = 2*x* + *c* be tangent to the circle $x^2 + y^2 = 5^2$

Solution

$$y = 2x + c$$
$$x^2 + y^2 = 5^2$$
$$x^2 + (2x + c)^2 = 5$$
$$x^2 + 4x^2 + 4xc + c^2 = 5$$
$$5x^2 + 4xc + c^2 - 5 = 0$$

For tangency $B^2 = 4AC$

$$(4c)^2 = 4(5)(c^2 - 5)$$
$$16c^2 = 20c^2 - 100$$
$$100 = 4c^2$$
$$25 = c^2$$
$$c = 5$$

Example VII

For what values of α, does the line $3x + 4y = \alpha$ touch the circle $x^2 + y^2 - 10x = 0$?

Solution

$$3x + 4y = \alpha \text{ .... (i)}$$
$$x^2 + y^2 - 10x = 10 \text{ .... (ii)}$$

Substituting $y = \frac{\alpha - 3x}{4}$ in Eqn (ii)

$$\Rightarrow x^2 + \left(\frac{\alpha - 3x}{4}\right)^2 - 10x = 0$$
$$x^2 + \frac{\alpha^2 - 6\alpha x + 9x^2}{16} - 10x = 0$$
$$16x^2 + \alpha^2 - 6x\alpha + 9x^2 - 160x = 0$$
$$25x^2 + (-6\alpha - 160)x + \alpha^2 = 0$$

For tangency $B^2 = 4AC$

$$(-6\alpha - 160)^2 = 4 \times 25 (\alpha^2)$$
$$36\alpha^2 + 1920\alpha + 25600 = 100\alpha^2$$

$$64\alpha^2 - 1920\alpha - 25600 = 0$$
$$\alpha^2 - 30\alpha - 400 = 0$$
$$(\alpha - 40)(\alpha + 10) = 0$$
$$\alpha = 40, \alpha = -10$$

Example VIII

Find the equation of the tangents to the circle

$x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the line $4x + 3y + 5 = 0$

Solution

Let the tangent be $y = mx + c$

Since the tangent is parallel to $4x + 3y + 5 = 0$ $(y = -\frac{4x}{3} - \frac{5}{3})$

$$m = \frac{-4}{3}$$
$$y = -\frac{4x}{3} + c$$

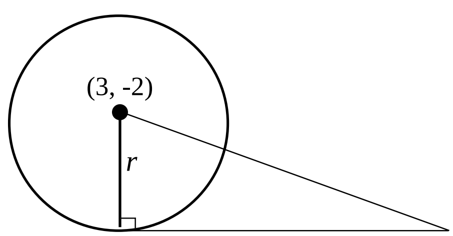
$3y + 4x = 3c$ is equation of the tangent

Comparing $x^2 + y^2 - 6x + 4y - 12 = 0$ with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -3, f = 2, c = -12$$

Center (+3, -2)



$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{9 + 4 - -12}$$
$$r = 5$$

But we can obtain *r* using the formula for perpendicular distance of a point from a line

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$
$$r = \left|\frac{4(3) + 3(-2) + -3c}{\sqrt{4^2 + 3^2}}\right|$$
$$5 = \left|\frac{12 - 6 - 3c}{5}\right|$$
$$5 = \pm\left(\frac{6 - 3c}{5}\right)$$

$$25 = 6 - 3c$$
$$3c = 6 - 25$$
$$3c = -19$$
$$5 = -\left(\frac{6 - 3c}{5}\right)$$
$$25 = -6 + 3c$$
$$31 = 3c$$

Since the tangents to the circle are given by

$$4x + 3y = 3c$$

$\Rightarrow$ The equations of the tangents are $4x + 3y = -19$ and $4x + 3y = +31$

Example ix

(i) Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are parallel to line $3x - 4y - 1 = 0$

(ii) Which are perpendicular to the line $3x – 4y – 1 = 0$

Solution

Comparing $x^2 + y^2 + 2gx + 2fy + c = 0$ with $x^2 + y^2 - 2x - 4y - 4 = 0$

$$g = -1, f = -2, c = -4$$

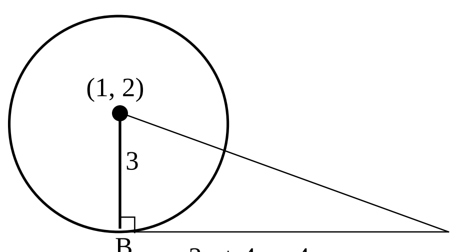
Center (1, 2)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{1 + 4 - -4}$$
$$r = 3$$
$$3x - 4y - 1 = 0$$
$$\frac{3x}{4} - \frac{1}{4} = y$$

Since the tangents are parallel to the line

$\Rightarrow$ $m = \frac{3}{4}$ for the tangent $y = mx + c$

$y = \frac{3x}{4} + C,\ (4y - 3x) = 4C$ are the equations of the tangents



$$r = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$3 = \left|\frac{-3(1) + 4(2) + 4c}{\sqrt{-3^2 + 4^2}}\right|$$

$$3 = \left|\frac{5 + 4c}{5}\right|$$

$$3 = \pm\left(\frac{5 + 4c}{5}\right)$$

$$3 = \frac{5 + 4c}{5}$$

$$15 = 5 + 4c$$

$$4c = 10$$

$$3 = -\left(\frac{5 + 4c}{5}\right)$$

$$15 = -5 - 4c$$

$$4c = -20$$

Since the equations of the tangent that are parallel to the line $3x - 4y - 1 = 0$ are $-3x + 4y = 4c$

$\Rightarrow$ The required tangents are:

$$-3x + 4y = 10$$

$$-3x + 4y = -20$$

(ii) Let the tangents that are perpendicular to the line $3x - 4y - 1 = 0$ be $y = mx + c$

$$3x - 4y - 1 = 0$$

$$4y = 3x - 1$$

$$y = \frac{3x}{4} - \frac{1}{4}$$

$$m = \frac{3}{4}$$

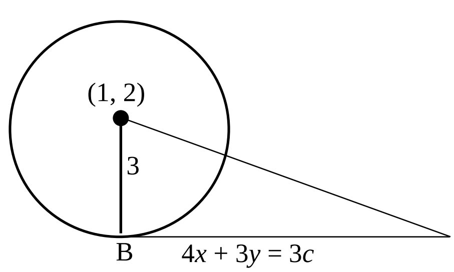
$$\Rightarrow m_1 = \frac{-4}{3}$$

$$y = \frac{-4x}{3} + c$$

$$3y + 4x = 3c$$

Center (1, 2)

$$r = 3$$



$$r = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$3 = \left|\frac{4(1) + 3(2) - 3c}{\sqrt{4^2 + 3^2}}\right|$$

$$3 = \left|\frac{4 + 6 - 3c}{5}\right|$$

$$3 = \frac{10 - 3c}{5}$$

$$15 = 10 - 3c$$

$$3c = -5$$

$$3 = \frac{-(10 - 3c)}{5}$$

$$15 = -10 + 3c$$

$$3c = 25$$

Since the tangent are;

$$3x + 4y = 3c$$

$$3x + 4y = 25$$

$$3x + 4y = -5$$

Director Circle

The locus of the point of intersection of two perpendicular tangents is called the Director circle of a given circle. The Director circle of a circle is a concentric circle having radius equal to $\sqrt{2}$ times the original radius.

Example

Find the equation of the director circle of the circle $(x - 2)^2 + (y + 1)^2 = 2$

Solution

$$(x - 2)^2 + (y + 1)^2 = 2$$

Center (2, -1)

Radius $r = \sqrt{2}$

The center of the director circle is (2, -1) and the radius of the director circle is

$$\sqrt{2} \times r$$

$$= \sqrt{2}\, X\, \sqrt{2}$$

$$= 2$$

The equation of the director circle is

$$(x - 2)^2 + (y + 1)^2 = 2^2$$

$$x^2 - 4x + 4 + y^2 + 2y + 1 = 4$$

$$x^2 + y^2 - 4x + 2y + 1 = 0$$

Example II

Find the equation of a director circle of the circle whose diameters are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and has an area of 154.

Solution

$2x - 3y + 12 = 0$…………………… (1)

$x + 4y - 5 = 0$……………………… (2)

Solving eqn. (1) and (2) simultaneously

$$x = -3, y = 2$$

The center of a circle is (-3, 2)

$$\pi r^2 = 154$$

$$\frac{22}{7} X\, r^2 = 154$$

$$r = 7$$

Radius of the director circle is $7\sqrt{2}$

The equation of the director circle is

$$(x - 3)^2 + (y - 2)^2 = (7\sqrt{2})^2$$

$$(x - 3)^2 + (y - 2)^2 = 98$$

$$x^2 + 6x + 9 + y^2 - 4y + 4 = 98$$

Therefore, $x^2 + y^2 + 6x - 4y - 85 = 0$ is the equation of the director circle.

Equation of a common chord of two circles

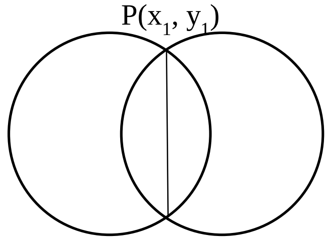
Let the equations of two intersecting circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \text{…… (1)}$$

And

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \text{……. (2)}$$

Intersect at $P(x_1, y_1)$ and $Q(x_2, y_2)$



Now we observe from the figure that $P(x_1, y_1)$ lies on both given equations therefore, we get

$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = 0$…… (3)

$x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2 = 0$ ….. (4)

Eqn. (3) − Eqn. (4)

$2(g_1 - g_2)x_1 + 2(f_1 - f_2)y_1 + c_1 - c_2 = 0$ … (5)

Again we observe from the above figure that point $Q(x_2, y_2)$ lies on both circles

$$x_2^2 + y_2^2 + 2g_1x_2 + 2f_1y_2 + c_1 = 0 \text{ …... (6)}$$
$$x_2^2 + y_2^2 + 2g_2x_2 + 2f_2y_2 + c_2 = 0 \text{ …... (7)}$$

Eqn. 6 – eqn. 7

$$2(g_1 - g_2)x_2 + 2(f_1 - f_2)y_2 + c_1 - c_2 = 0 \text{ …… (8)}$$

From eqn. 5 and 8, it's evident that the points $P(x_1, y_1)$and $Q(x_2, y_2)$ lie on $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ which is a linear equation in x and y.

Note: While finding the equation of the common chord of two given intersecting circle, we fast express each equation in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Example

Determine the equation of the chord of the two intersecting circles $x^2 + y^2 - 4x - 2y - 31 = 0$ and $2\,x^2 + 2y^2 - 6x + 8y - 35 = 0$ and prove that the common chord is perpendicular to the line joining the two centres of the circles.

Solution

$x^2 + y^2 - 4x - 2y - 31 = 0$……….. (1)

$$2\,x^2 + 2y^2 - 6x + 8y - 35 = 0$$

$x^2 + y^2 - 3x + 4y - \frac{35}{2} = 0$……….. (2)

Eqn. (1) – Eqn. (2)

$$-x - 6y + \frac{27}{2} = 0$$
$$-2x - 12y + 27 = 0$$
$$y = \frac{-2x}{12} + \frac{27}{12}$$

The equation of the chord:

The gradient of the chord is $\frac{-1}{6}$

Comparing $x^2 + y^2 - 4x - 2y - 31 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -2, f = -1$$

Center (2, 1) = C_1

Comparing $x^2 + y^2 - 3x + 4y - \frac{35}{2} = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -\frac{3}{2}, f = 2$$

Center $(\frac{3}{2}, -2)$ = C_2

The gradient joining the two centers

$$= \frac{-2 - 1}{\frac{3}{2} - 2}$$
$$= \frac{-3}{-\frac{1}{2}} = 6$$

Gradient of chord × gradient of line joining the two centres

$$6 \times \frac{-1}{6} = -1$$

The chord is perpendicular to the line joining the two centers

Example

Show that the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 2y - 4 = 0$ passes through the origin

Solution

$x^2 + y^2 = 4$

$x^2 + y^2 - 4 = 0$ …………………… (1)

$x^2 + y^2 - 4x - 2y - 4 = 0$ ……….. (2)

Eqn. (2) – eqn. (1)

$$4x + 2y = 0$$

$y = -2x$ is the equation of the common chord

At (0, 0), $x = 0, y = 0$

$$0 = -2 \times 0$$

$$0 = 0$$

The common chord passes through the origin.

Example

Find the equation of the common chord of the circles

$x^2 + y^2 - 4x - 2y + 1 = 0$

$x^2 + y^2 + 4x - 16y - 10 = 0$

Solution

$x^2 + y^2 - 4x - 2y + 1 = 0$ ………… (1)

$x^2 + y^2 + 4x - 16y - 10 = 0$ ………. (2)

Eqn. (2) – eqn. (1)

$+8x - 14y - 11 = 0$

$$14y = -11 + 8x$$

Example

Find the point of intersection of the two circles

$x^2 + y^2 - 2x - 6y + 6 = 0$ and

$x^2 + y^2 - 6x - 6y + 14 = 0$

Solution

When we are finding the point of intersection, we first find the equation of the common chord and then we solve it simultaneously with one of the equations of the circles

$x^2 + y^2 - 2x - 6y + 6 = 0$ ……….. (1)

$x^2 + y^2 - 6x - 6y + 14 = 0$ ……… (2)

Eqn. (1) – eqn. (2)

$$4x - 8 = 0$$
$$x = 2$$

$x = 2$ is the equation of the common chord

Substituting $x = 2$, in eqn. (1)

$$2^2 + y^2 - 2 \times 2 - 6y + 6 = 0$$
$$y^2 - 6y + 6 = 0$$
$$y = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 6}}{2 X 1}$$
$$y = \frac{6 \pm \sqrt{12}}{2}$$
$$y = 3 \pm \sqrt{3}$$
$$\left(2, 3 - \sqrt{3}\right) and \left(2, 3 + \sqrt{3}\right)$$

The point of intersection of both circles is $\left(2, 3 - \sqrt{3}\right) and \left(2, 3 + \sqrt{3}\right)$

Example

Find the point of intersection of the circles

$x^2 + y^2 + 2x + 2y - 23 = 0$ and $x^2 + y^2 - 10x - 7y + 31 = 0$

Solution

$x^2 + y^2 + 2x + 2y - 23 = 0$………… (1)

$x^2 + y^2 - 10x - 7y + 31 = 0$ ………. (2)

Eqn (1) – Eqn (2)

$$12x + 9y - 54 = 0$$
$$4x + 3y = 18$$
$$y = \frac{18 - 4x}{3}$$
$$x^2 + \left(\frac{18 - 4x}{3}\right)^2 + 2x + 2\left(\frac{18 - 4x}{3}\right) - 23 = 0$$
$$x^2 + \frac{324 - 144x + 16x^2}{9} + 2x + \frac{36 - 8x}{3} - 23 = 0$$
$$9\,x^2 + 16x^2 - 144x + 18x + 108 - 24x + 324 + 108 - 207 = 0$$
$$25x^2 - 150x + 225 = 0$$
$$x^2 - 6x + 9 = 0$$
$$(x - 3)^2 = 0$$
$$x = 3$$
$$y = \frac{18 - 4 \times 3}{3}$$
$$y = 2$$

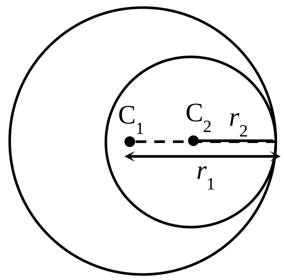
The point of intersection is (3, 2)

Types of intersecting circles

(1) Touching each other internally

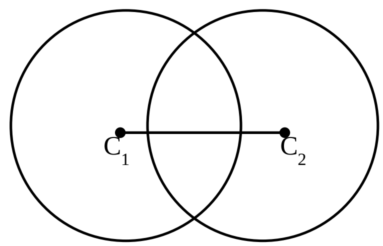
Two circles touch each other internally if the distance between their centers is equal to the distance between their radii

$$C_1C_2 = r_1 - r_2$$

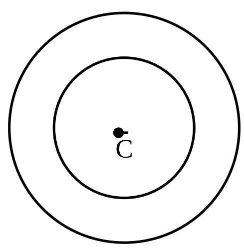


$$C_1C_2 = r_1 - r_2$$

(2) Circle intersect at two distinct points when $C_1C_2 < r_1 - r_2$

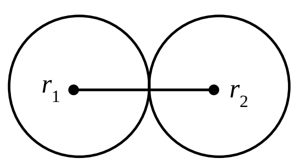


(3) Concentric circles



These are circles with the same center.

(4) Circle which touches each other externally if the distance between their centers is equal to the sum of their radii.



Example

Prove that the circles $x^2 + y^2 - 10x - 7y + 31 = 0$ and $x^2 + y^2 + 2x + 2y - 23 = 0$ touch each other externally.

Solution

$$x^2 + y^2 - 10x - 7y + 31 = 0$$
$$x^2 + y^2 + 2x + 2y - 23 = 0$$

Comparing $x^2 + y^2 - 10x - 7y + 31 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -5, = -\frac{7}{2}, c = 31$$

Center $\left(5, \frac{7}{2}\right)$

Radius $= \sqrt{g^2 + f^2 - c}$

$$r = \sqrt{(-5)^2 + \left(\frac{-7}{2}\right)^2 - 31}$$

$$r = \frac{5}{2}$$

Comparing $x^2 + y^2 + 2x + 2y - 23 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = 1, f = 1, c = -23$$

Center (-1, -1)

$$radius = \sqrt{(-1)^2 + (1)^2 - -23}$$

$$r = 5$$

$$C_1\left(5, \frac{7}{2}\right) \; and \; C_2(-1, -1)$$

$$C_1C_2 = \sqrt{(5 - {}^{-}1)^2 + \left(\tfrac{7}{2} - -1\right)^2}$$

$$= \sqrt{36 + \frac{81}{4}}$$

$$C_1C_2 = \frac{15}{2}$$

$$C_1C_2 = 7.5$$

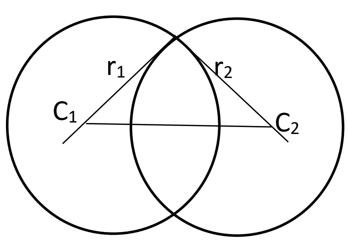
$$r_1 + r_2 = 7.5$$

Since $C_1C_2 = r_1 + r_2$

The two circles touch each other externally

Orthogonal Circle

Two circles are said to be orthogonal if the tangents at their point of intersection cut at right angles as illustrated below.



$$r_1^2 + r_2^2 = (C_1C_2)^2$$

Example

Prove that the circles $x^2 + y^2 + 4x - 2y - 11 = 0$ and $x^2 + y^2 - 4x - 8y + 11 = 0$ are orthogonal

Solution

Comparing $x^2 + y^2 + 2gx + 2fy + c = 0$ with $x^2 + y^2 + 4x - 2y - 11 = 0$

$$g = 2, f = -1, c = -11$$

Center $C_1(-2, 1)$

$$r_1 = \sqrt{2^2 + -1^2 - -11}$$

$$r_1 = 4$$

Similarly

Comparing $x^2 + y^2 - 4x - 8y + 11 = 0$ with

$x^2 + y^2 + 2gx + 2fy + c = 0$

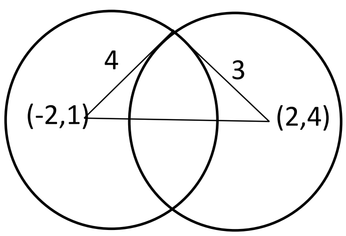
$$g = -2, f = -4, c = 11$$

Center $C_2(2, 4)$

$$r_2 = \sqrt{(-2)^2 + (-4)^2 - 11}$$

$$r_2 = \sqrt{4 + 16 - 11}$$

$$r_2 = 3$$



$$C_1C_2 = \sqrt{(-2-2)^2 + (1-4)^2}$$

$$\overline{C_1C_2} = 5$$

Since $r_1^2 + r_2^2 = \overline{C_1C_2}^2$

The two circles are orthogonal

Example (UNEB Question)

13. a) Form the equation of a circle that passes through the points *A* (-1, 4), *B* (2, 5) and *C* (0, 1)

b) The line $x + y = c$ is a tangent to the circle

$x^2 + y^2 - 4y + 2 = 0$. Find the coordinates of the point of contact of the tangent for each value of *c*.

Solution

General equation of the circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At *A*(-1, 4),

$-2g + 8f + c = -17$............................. (i)

At *B*(2, 5);

$4g + 10f + c = -29$............................. (ii)

At *C*(0, 1):

$2f + c = -1$....................................... (iii)

2 Eqn (i) + Eqn (ii)

$10c = 50$

$c = 5$

From Eqn (iii);

$2f + 5 = -1$

$2f = -6$

$f = -6$

From Eqn (i)

$-2g + 8(-3) + 5 = -17$

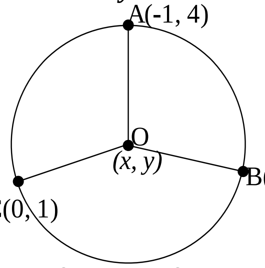
$-2g = 24 - 17 - 5$

$g = -1$

Hence the equation of the circle is

$\mathbf{x^2 + y^2 - 2x - 6y + 5 = 0}$

Alternatively



$(0 - x)^2 + (1 - y)^2 = (-1 - x)^2 + (4 - y)^2$

$3y - x = 8$.................................. (i)

Also

$(0 - x)^2 + (1 - y)^2 = (2 - x)^2 + (5 - y)^2$

$2y + x = 7$...................................... (ii)

Eqn (i) + Eqn (ii)

$5y = 15$

$y = 3$

$3(3) - x = 8$

$x = 1$

Centre of the circle = (1, 3) and the radius is

$\sqrt{(0-1)^2 + (1-3)^2} = \sqrt{5}$

Equation of the circle is $\mathbf{x^2 + y^2 - 2x - 6y + 5 = 0}$

b) $x^2 + y^2 - 4y + 2 = 0$,

And $y = c - x$

At the point of contact,

$x^2 + (c - x)^2 - 4(c - x) + 2 = 0$

$2x^2 + (4 - 2c)x + (c^2 - 4c + 2) = 0$

For tangency, $b^2 = 4ac$

$(4 - 2c)^2 = 4 \times 2 \times (c^2 - 4c + 2)$

$4(2 - c)^2 = 8(c^2 - 4c + 2)$

$(2 - c)^2 = 2(c^2 - 4c + 2)$

$4 - 4c + c^2 = 2c^2 - 8c + 4$

$c^2 - 4c = c(c - 4) = 0$

Either $c = 0$ or $c = 4$

If $c = 0$, $y = -x$

$\Longrightarrow \quad x^2 + x^2 + 4x + 2 = 0$

$2x^2 + 4x + 2 = 0$

$x^2 + 2x + 1 = (x + 1)^2 = 0$

$\Longrightarrow \quad x = -1$

Therefore $y = 1$
The point is (-1, 1)
If $c = 4$, $y = 4 - x$

$$(4 - x)^2 + x^2 - 4(4 - x) + 2 = 0$$
$$16 - 8x + x^2 + x^2 - 16 + 4x + 2 = 0$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = (x - 1)^2 = 0$$
$$x = 1, y = 3$$

The point is (1, 3)

Example **(UNEB Question)**

a) Find the equation of a circle which passes through the points (5, 7), (1, 3) and (2, 2).

b) i) If $x = 0$ and $y = 0$ are tangents to the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, show that $c = g^2 = f^2$.

ii) Given that the line $3x – 4y + 6 = 0$ is also a tangent to the circle in (b) (i) above, determine the equation of the circle lying in the first quadrant. (06 *marks*)

Solution

(a) The equation of the circle is given by;

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Substituting for (5, 7),

$$25 + 49 + 10g + 14f + c = 0$$
$$74 + 10g + 14f + c = 0$$
$$10g + 14f + c = -74 \text{ …………..(i)}$$

Substituting for (1, 3)

$$1 + 9 + 2g + 4f + c = 0$$
$$2g + 6f + c = -10 \text{ ……………(ii)}$$

Substituting for (2, 2)

$$4 + 4 + 4g + 4f + c = 0$$
$$4g + 4f + c = -8 \text{.......................... (iii)}$$

Eqn (i) − Eqn (ii)

$$6g + 10f = -64$$
$$g + 6f = -8 \text{.................................. (iv)}$$

Eqn (i) − Eqn (iii)

$$6g + 10f = -66$$
$$3g + 5f = -33 \text{................................ (v)}$$

3 Eqn (iv) − Eqn (v)

$$3g + 3f = -24$$
$$3g + 5f = -33$$
$$-2f = 9$$
$$f = \frac{-9}{2}$$

Substituting for f in Eqn (iv)

$$g = \frac{9}{2} - 8 = -7/2$$

Substituting for f and g in Eqn (iii)

$$4\left(-7/2\right) + 4\left(\frac{-9}{2}\right) + c = -8$$

$$-28 - 36 + 2c = -16$$
$$2c = 64 - 16$$
$$c = \frac{48}{2} = 24$$

The equation of the circle is $x^2 + y^2 - 7x - 9y + 24 = 0$

b) Given $x^2 + y^2 + 2gx + 2fy + c = 0$
When $y = 0$, $x^2 + 2gx + c = 0$
For tangency, $b^2 = 4ac$

$(2g)^2 = 4c$

$4g^2 = 4c$

$g^2 = c$

When $x = 0$, $y^2 + (2f)^2 + c = 0$
For tangency, $b^2 = 4ac$

$(2f)^2 = 4c$

$4f^2 = 4c$

$f^2 = c$

Hence $c = g^2 = f^2$

ii) From the line $3x - 4y + 6 = 0$

$4y = 3x + 6$

$$y = \frac{3x+6}{4}$$

$$y^2 = \frac{(3x+6)^2}{16}$$

Substituting for y and y^2 in the equation

$x^2 + y^2 + 2gx + 2fy + c = 0$

$$x^2 + \frac{(3x+6)^2}{16} + 2fx + 2f\left(\frac{3x+6}{4}\right) + f^2 = 0$$

$16x^2 + (3x + 6)^2 + 32fx + 8f(3x + 6) + 16f^2 = 0$
$16x^2 + 9x^2 + 36x + 36 + 32fx + 24fx + 48f + 32f^2 = 0$
$25x^2 + (36 + 54f)x + (36 + 48f + 16f^2) = 0$
For tangency, $b^2 = 4ac$

$(36 + 54f)^2 = 4 \times 24(36 + 48f + 16f^2)$

$(36 + 54f)^2 = 100(36 + 48f + 16f^2)$

By opening brackets and simplifying we obtain

$2f^2 - f - 3 = 0$

$2f^2 - 3f + 2f - 3 = 0$

$f(2f - 3) + 1(2f - 3) = 0$

$(2f - 3)(f + 1) = 0$

Either $2f - 3 = 0$

$2f = 3$

$f = 3/2$

Or $f + 1 = 0$

$f = -1$

Now $f = g$

$\Longrightarrow$ $g = 3/2$ or -1

Centre of the circle is $(-g, -f)$. Since it is in the first quadrant, then the centre is (1, 1)
But $c = g^2 = f^2 = 1$
The equation of the circle is $x^2 + y^2 - 2x - 2y + 1 = 0$

LOCI

When a point moves in the plane according to some given conditions, the path along which it moves is called a locus.

A locus is a set of points which satisfy certain geometric conditions. Many geometric shapes are most naturally and easily described as a loci. For example a circle is a set of points in the plane which are fixed at distance r from a given point *P* (center).

Problems involving describing a certain locus is often solved by explicitly finding equations for the coordinates of the points in the locus. Here is a step by step procedure for finding plane loci

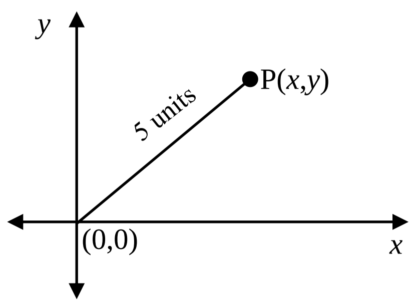
Step I: If possible, choose a coordinate system that will make computations and equations as simple as possible

Step II: Write the given conditions in mathematics from involving the coordinates (x, y).

Step III: Simplify the equations.

Step IV: Identify the shape out by the equations.

Example I
Find the locus of a circle with center at the origin and radius 5 units.
Solution

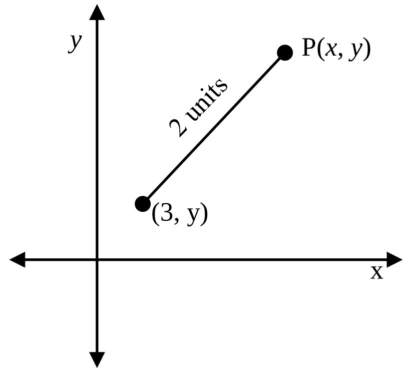


$$\sqrt{(x-0)^2+(y-0)^2}=5$$

$$x^2+y^2=5$$

The locus is $x^2 + y^2 = 5$

Example II
What is the locus of a point which moves so that its distance from the point (3, 1) is 2 units?
Solution



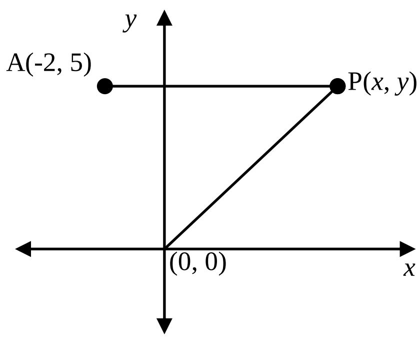
$$\sqrt{(x-3)^2+(y-1)^2}=2$$
$$(x-3)^2+(y-1)^2=2^2$$
$$x^2-6x+9+y^2-2y+1=4$$
$$x^2+y^2-6x-2y+6=0$$

The locus is a circle with center (3, 1) and radius 2

Example III

What is the locus of point which is equidistant from the origin (0, 0) and the point (-2, 5)

Solution



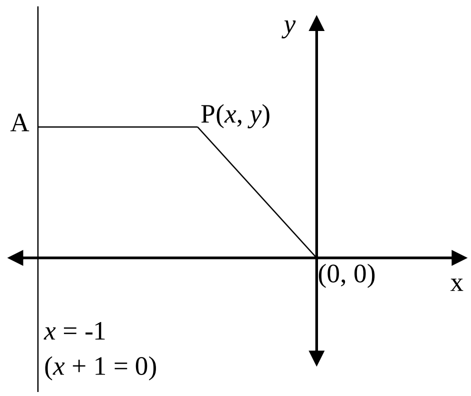
$$A\,p=p\,B$$
$$\sqrt{(x+2)^2+(y-5)^2}=\sqrt{x^2+y^2}$$
$$(x+2)^2+(y-5)^2=x^2+y^2$$
$$x^2+4x+4+y^2-10y+25=x^2+y^2$$
$$4x-10y+29=0$$

The locus is a straight line with a positive gradient.

Example IV

Find the locus of a point which is equidistant from the line $x=-1$ and the origin.

Solution



The perpendicular distance of the line $ax + by + c=0$ from (x_1, y_1) is $\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Comparing $x + 1 = 0$ with $ax + by + c = 0$

$$a = 1, b = 0, c = 1$$

The perpendicular distance of the point (x, y) from the line $x + 1 = 0$ is

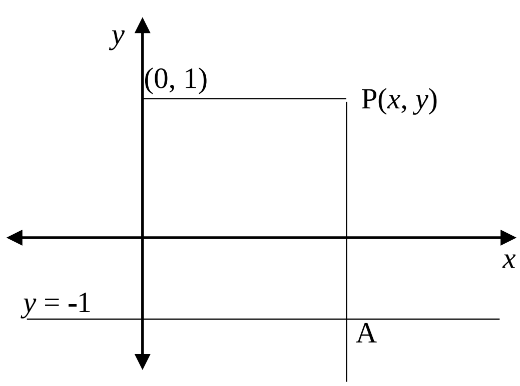
$$AP = \left|\frac{1(x) + 0(y) + 1}{\sqrt{(1)^2 + 0^2}}\right| = x + 1$$

$$Ap = x + 1$$
$$pB = \sqrt{x^2 + y^2}$$
$$Ap = pB$$
$$x + 1 = \sqrt{x^2 + y^2}$$
$$(x + 1)^2 = x^2 + y^2$$
$$x^2 + 2x + 1 = x^2 + y^2$$
$$y^2 = 2x + 1$$

The locus is a parabola

Example

Find the locus of a point which is equidistant from the point (0, 1) and the line $y = -1$



Comparing $y + 1 = 0$ $(y = -1)$ with general equation of the line $ax + by + c = 0$

$$a = 0, b = 1, c = 1$$

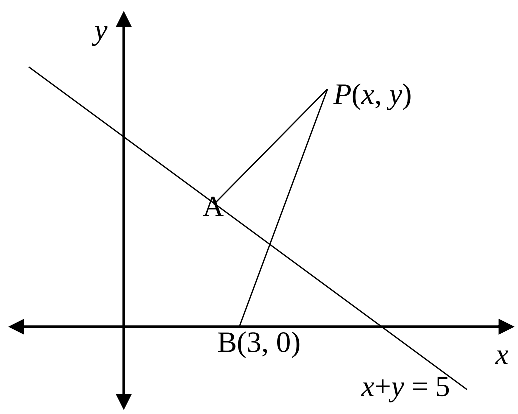
The perpendicular distance of the point $P(x, y)$ from the line is $y = -1$ $\quad (y + 1 = 0)$

$$\Rightarrow \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$\left|\frac{0(x)+1(y)+1}{\sqrt{0^2+1^2}}\right| = y+1$$
$$AP = y+1$$
$$PB = \sqrt{(x-0)^2+(y-1)^2}$$
$$PB = \sqrt{(x^2+(y-1)^2}$$
$$AP = PB$$
$$y+1 = \sqrt{x^2+(y-1)^2}$$
$$(y+1)^2 = x^2+(y-1)^2$$
$$y^2+2y+1 = x^2+y^2-2y+1$$
$$x^2 = 4y$$

The locus is a parabola

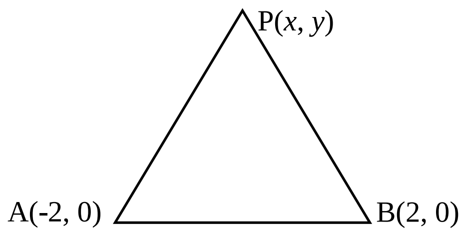
Example VI (UNEB Question)
A point p is twice as far from the line $x + y = 5$ as from the point (3, 0). Find the locus of P.
Solution



$AP = PB$
The perpendicular distance of point (x, y) from the line $x + y - 5 = 0$ is

$$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$
$$\Rightarrow \left|\frac{1(x)+1(y)-5}{\sqrt{1^2+1^2}}\right|$$
$$AP = \left|\frac{x+y-5}{\sqrt{2}}\right|$$
$$PB = \sqrt{(x-3)^2+y^2}$$
$$AP = 2PB$$
$$\frac{x+y-5}{\sqrt{2}} = 2\sqrt{(x-3)^2+y^2}$$
$$x+y-5 = 2\sqrt{2}\left[\sqrt{(x-3)^2+y^2}\right]$$
$$(x+y-5)^2 = 8[(x-3)^2+y^2]$$
$$7x^2+7y^2-2xy-58x+10y+47=0$$

Example VII
Find the locus of a point which moves so that the sum of squares of its distances from $(-2, 0)$ and (2, 0) is 26



Solution

$$(\overline{AP})^2 + (\overline{PB})^2 = 26$$

$$\left(\sqrt{(x+2)^2 + (y-0)^2}\right)^2 + \left(\sqrt{(x-2)^2 + y^2}\right)^2 = 26$$

$$(x+2)^2 + y^2 + (x-2)^2 + y^2 = 26$$

$$x^2 + 4x + 4 + y^2 + x^2 - 4x + 4 + y^2 = 26$$

$$2x^2 + 2y^2 = 18$$

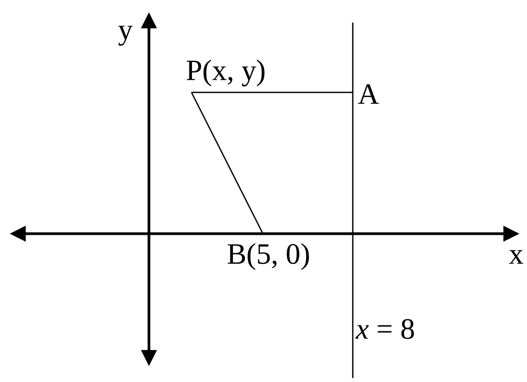
$$x^2 + y^2 = 9$$

The locus is a circle with center $(0, 0)$ and radius 3 units

Example VIII

Find the locus of the point P which moves so that its distance from the point $(5, 0)$ is a half its distance from the line $x - 8 = 0$

Solution



$$PB = \frac{1}{2}PA$$

$$2PB = PA$$

$$PA = 8 - x$$

$$PB = \sqrt{(x-5)^2 + y^2}$$

$$2PB = PA$$

$$2\sqrt{(x-5)^2 + y^2} = (8 - x)$$

$$4[(x-5)^2 + y^2] = (8 - x)^2$$

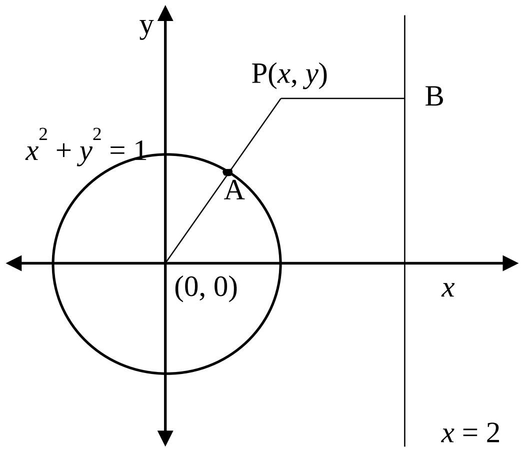
$$4[x^2 - 10x + 25 + y^2] = 64 - 16x + x^2$$

$$3x^2 + 4y^2 - 24x + 36 = 0$$

Example IX

Find the locus of a point which is equidistant from the line $x = 2$ and the circle $x^2 + y^2 = 1$

Solution



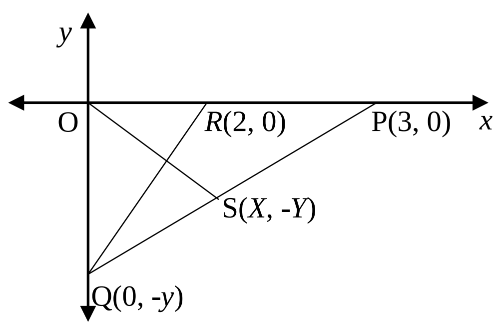
$$AP = B$$
$$\sqrt{x^2 + y^2} - 1 = 2 - x$$
$$\sqrt{x^2 + y^2} = 3 - x$$
$$x^2 + y^2 = (3 - x)^2$$
$$x^2 + y^2 = 9 - 6x + x^2$$
$$y^2 + 6x - 9 = 0$$

The locus is a parabola

Example X

The points $R(2, 0)$ and $P(3, 0)$ lie on the x-axis and $Q(0, -y)$ on the y- axis. The perpendicular from the origin to QR meets PQ at point $S(X, -Y)$. Find the locus of S.

Solution



Since S is in terms of X and Y

Then the locus of S must be in terms of X and Y

From the figure above,

(The gradient of PQ) = (Gradient of SQ)

$$\frac{0 - -y}{3 - 0} = \frac{-Y - -y}{X - 0}$$
$$\frac{y}{3} = \frac{-Y + y}{X}$$
$$yX = -3Y + 3y \text{ …………….. (1)}$$

(Gradient of RQ) X (Gradient of OS) = -1

$$\left(\frac{-Y}{X}\right) X \left(\frac{y}{2}\right) = -1$$

$$\frac{-Yy}{2X} = -1$$

$$y = \frac{2X}{Y} \text{……………………..(2)}$$

Substituting eqn. 2 in (1)

$$\frac{2X^2}{Y} = -3Y + 3\left(\frac{2X}{Y}\right)$$

$$2X^2 = -3Y^2 + 6X$$

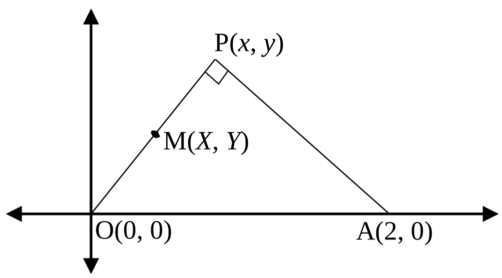
$$3Y^2 = -2X^2 + 6X$$

$$3Y^2 = 2X(3 - X) \text{ is the locus of S}$$

Example XI

Variable lines through the point O(0, 0) and A(2, 0) intersect at right angles at the point *P*. Show that the locus of the midpoint of OP is $y^2 + x(x - 1) = 0$

Solution



(The gradient of *OP*) × (Gradient *AP*) = -1

$$\left(\frac{y}{x}\right) \times \frac{-y}{2 - x} = -1$$

$$y^2 = x(2 - x) \text{………………………… (1)}$$

Let the midpoint Op be M(*X*, *Y*)

$$X = \frac{0 + x}{2}$$

$$x = 2X$$

$$Y = \frac{0 + y}{2}$$

$$2Y = y$$

$$y = 2Y$$

But *x* and *y* satisfy the above equation.

Substituting *x* = 2*X* and *y* = 2*Y* in Eqn (1);

$$(2Y)^2 = 2X[(2 - (2X)]$$

$$4Y^2 = 2X[2 - 2X]$$

$$Y^2 = +X[1 - X]$$

$$Y^2 = -X[X - 1]$$

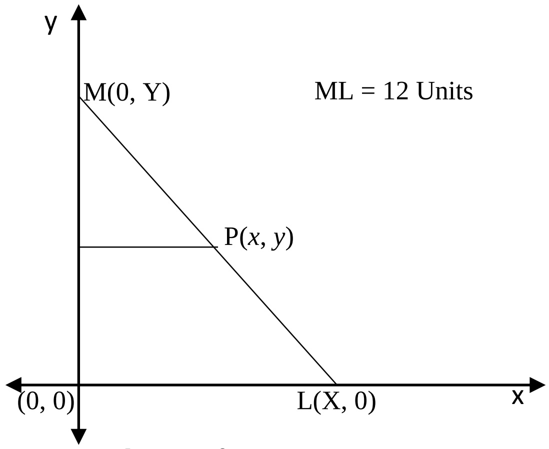
$$Y^2 + X[X - 1] = 0$$

Example XII

P is a point on a line of length 12 units, which moves so that it's length lie on the axes. Find the locus of P when its

(a) The midpoint of line,
(b) The point of trisection near the *y*-axis.

Solution



Since P is a midpoint of LM

$$x = \frac{0 + X}{2}$$

$$2x = X$$

Similarly, $y = \frac{0+Y}{2}$

$2y = Y$

Applying Pythagoras theorem on triangle OLM

$$X^2 + Y^2 = 12^2$$

$$X^2 + Y^2 = 144$$

$$X^2 + Y^2 = 144 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

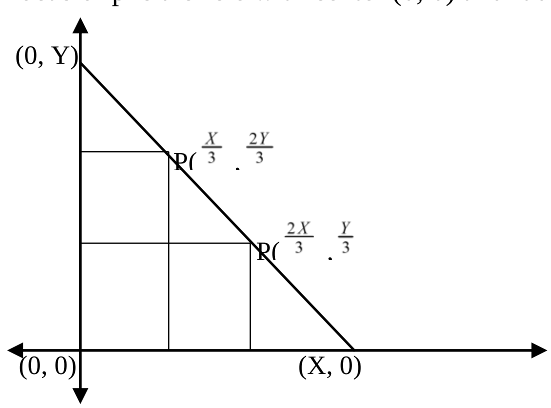
Substituting $X = 2x$ and $Y = 2y$ in equation (1)

$$(2x)^2 + (2y)^2 = 144$$

$$4x^2 + 4y^2 = 144$$

$$x^2 + y^2 = 36$$

The locus of p is a circle with center (0, 0) and radius 6



Since P(x, y) is a point of intersection near the y-axis

$$x = \frac{1}{3}X, y = \frac{2}{3}Y$$

$$3x = X, \frac{3y}{2} = Y$$

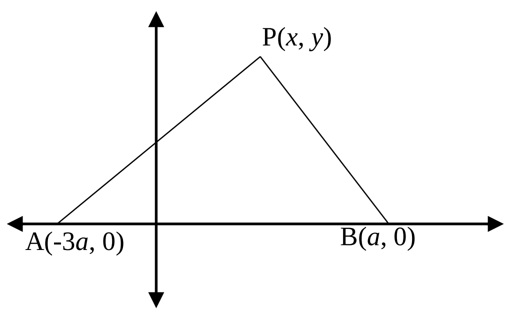
Substituting $X = 3x$ and $Y = \frac{3y}{2}$ in Eqn (1)

$$(3x)^2 + \left(\frac{3y}{2}\right)^2 = 144$$
$$9x^2 + \frac{9y^2}{4} = 144$$
$$x^2 + \frac{y^2}{4} = 16$$
$$4x^2 + y^2 = 64$$

Example XIII

The fixed points *A* and *B* have coordinates (-3*a*, 0) and (*a*, 0) respectively. Find the locus of P which moves in the coordinate plane so that *AP* = 3*pB*. Show that the locus is a circle, S which touches the axis of *y* and has a center at the point $\left(\frac{3a}{2}, 0\right)$. A point Q moves in such a way that the perpendicular distance of Q from the y-axis is equal to the length of the tangent from Q to the circle S. find the equation of the locus of Q. show that this locus is also a locus of points which are equidistant from the line 4*x* + 3*a* = 0 and the point $\left(\frac{3a}{4}, 0\right)$.

Solution



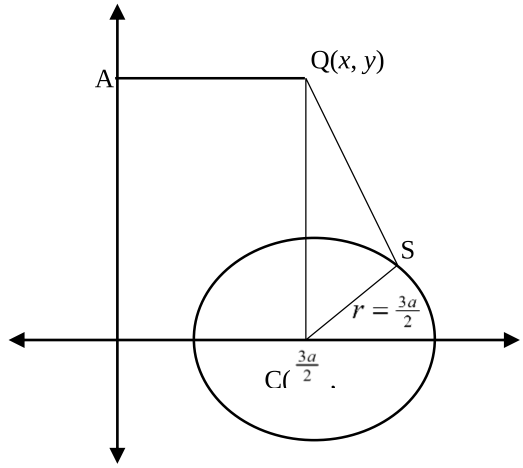
$$AP = 3PB$$
$$\Rightarrow \sqrt{(x+3a)^2 + y^2} = 3\sqrt{(x-a)^2 + y^2}$$
$$(x+3a)^2 + y^2 = 3[(x-a)^2 + y^2]$$
$$x^2 + 6ax + 9a^2 + y^2 = 9[x^2 + y^2 - 2ax + a^2]$$
$$x^2 + 6ax + 9a^2 + y^2 = 9x^2 + 9y^2 - 18ax + 9a^2$$
$$8x^2 + 8y^2 - 24ax = 0$$
$$x^2 + y^2 - 3ax = 0$$

Comparing $x^2 + y^2 + 2gx + 2fy + c = 0$

$$2g = -3a$$
$$g = -\frac{3a}{2}$$
$$f = 0$$

Center $\left(\frac{3a}{2}, 0\right)$

$$r = \sqrt{\left(\frac{3a}{2}\right)^2 + 0^2 - 0}$$
$$r = \frac{3a}{2}$$



$$CS^2 + QS^2 = CQ^2$$

$$\frac{9a^2}{4} + SQ^2 = \left(x - \frac{3a}{2}\right)^2 + y^2$$

$$SQ^2 = \left(x - \frac{3a}{2}\right)^2 + y^2 - \frac{9a^2}{4}$$

$$SQ^2 = x^2 - 3ax + \frac{9a^2}{4} + y^2 - \frac{9a^2}{4}$$

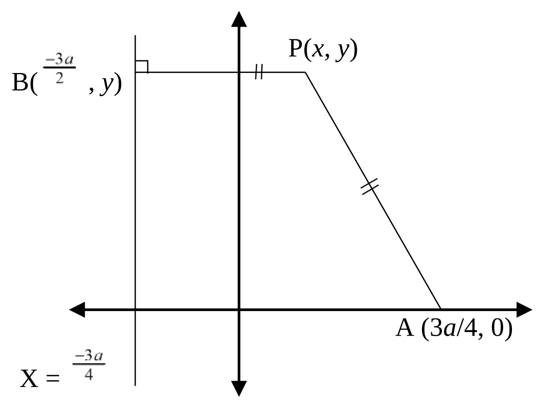
$$SQ^2 = x^2 + y^2 - 3ax$$

$$AQ = SQ$$

$$AQ^2 = SQ^2$$

$$x^2 = x^2 + y^2 - 3ax$$

$$y^2 = 3ax$$



$$x = \frac{-3a}{4}$$

$$4x + 3ax = 0$$

$$PB = d = \left|\frac{4(x) + 3a}{\sqrt{4^2}}\right|$$

$$PB = \frac{4x + 3a}{4}$$

$$\frac{4x + 3a}{4} = \sqrt{\left(x - \frac{3a}{4}\right)^2 + y^2}$$

$$\frac{16x^2 + 24ax + 9a^2}{16} = x^2 - \frac{3}{2}ax + \frac{9}{16}a^2 + y^2$$
$$x^2 + \frac{3}{2}ax + \frac{9}{16}a^2 = x^2 - \frac{3}{2}ax + \frac{9}{16}a^2 + y^2$$
$$y^2 = 3ax$$

Revision Exercise

1. Find the equation of the circle which passes through the origin and the points (2, 0), (3, -1).

2. Fid the radii and coordinates of the centres of the following circles.
(a) $x^2 + y^2 + 4x - 6y + 12 = 0$
(b) $x^2 + y^2 - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 - 3x = 0$
(d) $x^2 + y^2 + 3x - 4y - 6 = 0$

3. Find the equations of the circle with the following centres and radii:
(a) (3, 2), 4
(b) (-1, -2), 1
(c) (0, 0), 5
(d) (½, 0), $\frac{3}{2}$
(e) (4, -1), $\sqrt{3}$

4. Find the equation of the circle which has the points (0, -1) and (2, 3) as ends of its diameter.

5. What is the equation of the circle with centre (2, -3) and touches the x-axis?

6. Find the equation of the curve having AB as diameter where A is the point (1, 8) and B(3, 14).

7. Find the range of the values of k for which each of the following represents a circle with non-zero radius.
(a) $x^2 + y^2 = k$
(b) $x^2 + ky^2 - 2x - 8 = 0$
(c) $kx^2 + y^2 + 4y + 9 = 0$
(d) $2x^2 + 2y^2 + kxy - 9 = 0$

8. Find the equation of the diameter of the circle $x^2 + y^2 - 6x + 2y = 15$, which when produced passes through the point (8, -2).

9. Find the radii of the two circles with centres at the origin which touch the circle $x^2 + y^2 - 8x - 6y + 24 = 0$

10. Find the equation of the tangent to the circle $(x - 2)^2 + (y - 3)^2 = 16$ at the general point $((2 + 4\cos\theta), (3 + 4\sin\theta))$. Hence find the equation of the tangent at the point $(4, 3+2\sqrt{3})$.

11. Find the equation of the tangents to the following circles at the given points:
(a) $x^2 + y^2 = 5$, (-2, 1)
(b) $x^2 + y^2 - 4x + 2y = 3$. (0, -3)
(c) $x^2 + y^2 + 6y - 1 = 0$, (3, -4)
(d) $2x^2 + 2y^2 + 9x - 4y + 4 = 0$, (-2, 3)

12. Find the equation of the circle whose centres lies on the line $y = 3x - 7$ and which passes through the points (1, 1) and (2, -1).

13. Show that the distance of the centre of the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ from the y-axis is equal to the radius. What does this prove about the y-axis and the centre?

14. Prove that the circles $x^2 + y^2 - 4x - 6y = 0$ and $x^2 + y^2 - 4x - 6y = 3$ are concentric. Find the radius of the common centre.

15. Find the lengths of the tangents drawn from the following points to the given circles:

(a) (6, -1), $x^2 + y^2 = 12$

(b) (-1, 3), $x^2 + y^2 - 8x + 4y + 19 = 0$

(c) (4, -2), $x^2 + y^2 - 10y - 4 = 0$

(d) (3, -4), $x^2 + y^2 + x - 3y = 0$.

16. If O is the origin and P, Q are the intersections of the circle $x^2 + y^2 + 4x + 2y - 20 = 0$ and the straight line

$x - 7y + 20 = 0$. Show that OP and OQ are perpendicular. Find the equation to the circle through O, P and Q.

17. The circle x2 + y2 + 2gx + 2fy + c = 0 passes through the points A(-1, -2), B(1, 2), C(2, 3). Write down three equations which must be satisfied by g, f, c. Solve these equations and write down the equations of the circle *ABC*.

18. Prove that the line $y = 3x - 1$ neither cuts nor touches the circle $(x - 1)^2 + (y - 1)^2 = 9$

19. Find the greatest and least distance of a point P from the origin as it moves round the circle

(a) $x^2 + y^2 - 24x - 10y + 48 = 0$

(b) $x^2 + y^2 + 6x - 8y - 24 = 0$

20. A circle which passes through the origin cuts off intercepts of lengths 4 and 6 units on the positive x and y-axes respectively. Find the equation to the circle and the equations to the tangents to the circle at the points other than the origin where it cuts the axes.

21. A is the point (3, -1) and B is the point (5, 3). Show that the locus of the point P, which moves so that $PA^2 + PB^2 = 28$ is a circle Find its centre and radius.

22. Prove that the line $y = 2x - 3$ is a tangent to the circle $(x - 5)^2 + (y - 2)^2 = 5$

23. Find the equation of the circle which has the points

(-7, 3) and (1, 9) at the end of a diameter. Find also the equation of the tangents to the circle which are parallel

(a) to the x-axis

(b) to the y-axis

24. The point (a, b) is the midpoint of a chord of the circle $x^2 + y^2 = R2$. Show that the equation to the chord is $ax + by = a^2 + b^2$.

25. A circle touches the x-axis and cuts off a constant length $2a$ from the y-axis. Show that the equation to the locus of its centre is a curve $y^2 - x^2 = a^2$.

26. Find the length of the chord joining the points in which the straight line $(x/a) + (y/b) = 1$ meets the circle $x^2 + y^2 = R^2$.

27. Show that the line $2x - 3y + 26 = 0$ is a tangent to the circle $x^2 + y^2 - 4x + 6y - 104 = 0$ and find the equation to the diameter through the point of contact.

28. Find the length of the tangent to the circle $x^2 + y^2 - 4 = 0$ from the point (x, y) and deduce the equation of the locus of P, when it moves so that the length of the tangents to the circle is always equal to the distance of P from the point (1, 0).

29. Prove that the line $x - y - 3 = 0$ is a common tangent to the circles $x^2 + y^2 - 2x - 4y - 3 = 0$ and $x^2 + y^2 + 4x - 7y - 13 = 0$. What are the coordinates of the point in which it meets the other common tangent?

30. Show that the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 7y - 4 = 0$ passes through the origin.

31. Show that the following pair of circle are orthogonal:

(a) $x^2 + y^2 - 6x - 8y + 9 = 0$, $x^2 + y^2 = 9$
(b) $x^2 + y^2 - 4x + 2 = 0$, $x^2 + y^2 + 6y - 2 = 0$
(c) $x^2 + y^2 - 6y + 8 = 0$, $x^2 + y^2 - 4x + 2y - 14 = 0$
(d) $x^2 + y^2 + 10x - 4y - 3=0$, $x^2 + y^2 - 2x - 6y + 5=0$

32. Prove that the line $y = 2x$ is a tangent to the curve $x^2 + y^2 - 8x - y + 5 = 0$ and find the coordinates of the point of contact.

33. A and B have coordinates (-3, 0) and (3, 0). Show that the locus of a point P which moves such that $PB = 2PA$ is a circle with centre (-5, 0) and radius 4.

34. A triangle has vertices (0, 6), (4, 0), (6, 0). Find the equation of the circle through the midpoint of the sides and show that it passes through the origin.

35. Prove that the following pairs of circles touch each other and state whether the contact is external or internal.
(a) $x^2 + y^2 - 2x = 0$, $x^2 + y^2 - 8x + 12 = 0$
(b) $x^2 + y^2 - 2x - 2y = 18$, $x^2 + y^2 - 14x - 8y + 60 = 0$
(c) $x^2 + y^2 - 12x - 2y = 12$, $x^2 + y^2 - 4x + 4y + 4 = 0$
(d) $x^2 + y^2 - 4x + 2y = 8$, $x^2 + y^2 + 6x - 13y + 22 = 0$

36. Prove that the circle $x^2 + y^2 - 2x - 6y + 1 = 0$ cuts the circle $x^2 + y^2 - 8x - 8y + 31 = 0$ in two distinct places and find the equation of the common chord.

37. Points A(0, 2) and B(4, -2) lie on the circumference of a given circle. Points C(-3, -3) and D(7, 2) lie outside the circle but the centres of the circle lie on the CD. Find the equation of the circle.

38. Show that the $x^2 + y^2 + 4x - 2y - 11 = 0$ and $x^2 + y^2 - 4x - 8y + 11 = 0$ intersect at right angles.

39. Show that the line $x + 3y - 1 = 0$ touches the circle $x^2 + y^2 - 3x - 3y + 2 = 0$

40. Show that the locus of a point which moves such that the square of its distance from the point (3, 4) is proportional to its distance from the line $x + y = 0$, one of the locus being the point (1, 2), is a circle and find its centre and radius.

Answers

1. $x^2 + y^2 - 2x + 4y = 0$
2. (a) 1, (-2, 3) (b) 2, (1, 2)
   (c) $\frac{3}{2}$, $(\frac{3}{2}, 0)$ (d) $\frac{7}{2}$, $(\frac{-7}{2}, 2)$
3. (a) $x^2 + y^2 - 6x - 4y - 3 = 0$
   (b) $x^2 + y^2 + 2x + 4y + 4 = 0$
   (c) $x^2 + y^2 = 25$
   (d) $x^2 + y^2 - x - 2 = 0$
4. $x^2 + y^2 - 2x - 2y - 3 = 0$
5. $x^2 + y^2 - 4x + 6y + 4 = 0$
6. $x^2 + y^2 - 4x - 22y + 115 = 0$
7. (a) $k > 0$ (b) $k = 1$
   (b) No value of k (c) $k = 0$
8. $x + 5y + 2 = 0$
9. (4, 6)
10. $(y - 3)\sin\theta + (x - 2)\cos\theta = 4$, $y\sqrt{3} + x = 10 + 3\sqrt{3}$.
11. (a) $y = 2x + 5$ (b) $x + y + 3 = 0$
    (c) $x + y + 3 = 0$ (d) $x + 8y = 32$.

12. $x^2 + y^2 - 5x - y + 4 = 0$
13.
14. (2, 3)
15. (a) 5 (b) 7 (c) 6 (d) $2\sqrt{10}$
16. $x^2 + y^2 + 5x - 5y = 0$
17. $x^2 + y^2 - 16x + 8y - 5 = 0$
18.
19. (a) 24, 2 (b) 12, 2
20.
21. (4, 1), 3
22.
23. $x^2 + y^2 - 4x - 6y = 0$
$2x - 3y = 8$
$y = 1, y = 11$
$x = 2, x = -8$
24.
25.
26. $2\sqrt{\left(R^2 - \frac{a^2b^2}{a^2+b^2}\right)}$
27. $3x + 2y = 0$
28. $\sqrt{(x^2 + y^2 - 4}$, $2x - 5 = 0$
29. (7, 4)
30. (1, 2)
31.
32. $x^2 + y^2 - 5x - y = 0$
33.
34. $3x + y = 15$
35.
36.
37. $x^2 + y^2 - 2x + 2y = 8$

Locus

1. L and M are the feet of perpendiculars from a point P onto the axes. Find the locus of P when it moves so that LM is length 4 units.
2. A variable line through the point (3, 4) cuts the axes at Q and R. and the perpendiculars to the axes at Q and R intersect at P. What is the locus of the point P?
3. A variable joint P lies on the curve $xy = 12$. Q is the midpoint of the line joining P to the origin. Find the locus of Q.
4. P is a variable line on the curve $y = 2x^2 + 3$ and O is the origin. Q is the point of intersection of OP nearer the origin. Find the locus of Q.
5. A line parallel to the x-axis cuts the curve $y^2 = 4x$ at P and the line $x = -1$ at Q. Find the locus of the midpoint of PQ.
6. Find the locus of a point which moves so that the sum of the squares of its distance from the lines $x + y = 0$ and $x - y = 0$ is 4.
7. A is the point (1, 0), B is the point (2, o) and O is the origin. A point P moves so that the angle BPO is a right angle and Q is the midpoint of AP. What is the locus of Q?
8. A line parallel to the y-axis meets the curve $y = x^2$ at P and the line $y = x + 2$ at Q. Find the locus of the midpoint of PQ.

Answers to Locus Questions

1. $x^2 + y^2 = 16$
2. $xy = 3y + 4x$
3. $xy = 3$
4. $y = 6x^2 + 1$
5. $y^2 = 8x + 4$
6. $x^2 + y^2 = 4$
7. $4x^2 + 4y^2 - 8x + 3 = 0$
8. $2y = x^2 + x + 2$

SERIES AND SEQUENCES

Objectives:

After reading this book, you should be able to

- Recognize the difference between a sequence and series
- Recognize an arithmetic progression
- Find the n^th term of an arithmetic progression
- Find the sum of an arithmetic progression
- Recognize a geometric progression
- Find the n^th term of a geometric progression A. P
- Find the sum to infinity of an a G.P with common ratio when $|r| < 1$
- Understand $\sum notation$ for sum of series
- Be familiar with standard formula for $\sum r, \sum r^2, \sum r^3$.
- Prove expressions by mathematical induction.
- Apply geometric and arithmetic progression to solve word problems.

Introduction

A **sequence** is a set of number stated in a definite order such that each number forming a set can be obtained from the previous one according to some rule.

For example

(a) $\{2, 4, 6, 8, 10 \ldots \ldots\}$

(b) $\{1, -1, 1, -1, 1 \ldots \ldots \ldots \ldots.\}$

(c) $\{10, 11, 9, 12, 8, 13 \ldots \ldots \ldots.\}$

(d) $\{1, 4, 9, 16, 25 \ldots \ldots \ldots \ldots \ldots.\}$

(e) $\left\{8, -4, 2, -1, \frac{1}{2} \ldots \ldots \ldots \ldots\right\}$

(f) $\{1, 1, 2, 3, 5, 8 \ldots \ldots \ldots \ldots \ldots.\}$

A **series** is obtained by adding up all the terms of the sequence for example. Suppose we have a sequence

$u_1, u_2, u_3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots u_n$

The serie can be obtained from the above sequence. The serie is $u_1 + u_2 + u_3 + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots u_n$

From the examples of sequences above. It is possible to give a formular for the general or n^th term of each of the above series**.**

Thus for a, $u_n = 2n$, where $n = 1, 2, 3, \ldots.$

For *b*, $u_n = (-1)^{n+1}$ where $n = 1, 2, 3, \ldots.$

For *c* , u_n is not easy to find but we can give a separate formulae for odd and even terms of the sequence.

$$u_{2n} = 10 + n$$
$$u_{2n+1} = 10 - n$$

For d $u_n = n^2$ where n = 1, 2, 3, ….

For e, the formula u_n is not obvious

To obtain each successive term we divide by 2 then change sign. Thus at each stage we multiply by $\left(-\frac{1}{2}\right)$ which means that

$$u_n = 8\left(-\frac{1}{2}\right)^{n-1}$$

For f, each term is a sum of two previous terms. This can be expressed formally as a relation between u_n *and* u_{n-1}, u_{n-2} ($u_n = u_{n-1} + u_{n-2}$). In this case more advanced technique are needed to find the formula.

If the sequence ends after a certain number of terms it's said to be **finite**. A sequence which continues indefinitely is said to be **infinite**.

ARITHMETIC PROGRESSION (A.P)

Arithmetic progression is a serie in which one term is obtained from the previous one by adding a constant number.
For example:
$(a)\ 1 + 2 + 3 + 4 + \cdots 98 + 99$
$(b)\ 6 + 10 + 14 + 18 + \cdots 46 + 50$
$(c) 10 + 7 + 4 + 1 + \cdots . -47 - 50$

The constant number is called a common difference. In the above examples the common differences are 1 for *a*, 4 for *b* and -3 for *c* respectively.

The arithmetic progression is completely defined when the first term (***a***) and common difference, (***d***) are given
Generally, the arithmetic progression series is given by

$$a + (a + d) + (a + 2d) + \cdots a + (n - 1)d$$
$$a = first\ term$$
$$d = common\ difference$$
$$a + (n - 1)d = n^{th}\ term$$

Example 1
Which of the following series are arithmetic progression. Write down the common difference of those that are

$(a)\ 7 + 11 + 15 + \cdots$
$(b) - 7 - 5 - 3 + \cdots$
$(c) - 17 - 12 - 7 + \cdots + 13$
$(d)\ 1 + 1\frac{1}{4} + 1\frac{1}{2} + \cdots 9\frac{3}{4}$
$(e)\ 1^2 + 2^2 + 3^2 + 4^2 \ldots$

Solution
$(a)\ 7 + 11 + 15 + \cdots$

$a = 7$
$d_1 = 11 - 7 = 4$
$d_2 = 15 - 11 = 4$
$\Rightarrow d_1 = d_2 = d = 4 (common\ difference)$
$\Rightarrow 7 + 11 + 15 + \cdots$

Is an arithmetic progression with first term *a* = 7 and common difference *d* = 4

$(b) - 7 - 5 - 3 + \cdots$
$a = -7, \quad d_1 = -5 - -7 = 2$
$d_2 = -3 - -5 = 2$
$d_1 = d_2 = d = 2$ (*common difference*)

It is an arithmetic progression with first term *a* = -7 and common difference 2

$(c) - 17 - 12 - 7 + \cdots 13$
$a = -17$
$d_1 = -12 - -17 = 5$
$d_2 = -7 - -12 = 5$

It is an arithmetic progression with the first term
a = -17 and common difference (*d*) = 5

$(d)\ 1 + \frac{5}{4} + \frac{3}{2} + \cdots \frac{39}{4}$
$a = 1$
$d_1 = \frac{5}{4} - 1 = \frac{1}{4}$
$d_2 = \frac{3}{2} - \frac{5}{4} = \frac{1}{4}$
$a = 1, \ d_1 = d_2 = d = \frac{1}{4}$

$\Rightarrow 1 + \frac{5}{4} + \frac{3}{2} + \ldots \frac{39}{4}$ is an arithmetic progression

$(e)\ 1^2 + 2^2 + 3^2 + 4^2 \ldots$
$a = 1^2 = 1$
$d_1 = 2^2 - 1^2 = 3$

$d_2 = 3^2 - 2^2 = 5$

$d_3 = 4^2 - 3^2 = 7$

$d_1 \neq d_2 \neq d_3.$

$1^2 + 2^2 + 3^2 + 4^2 + \ldots$ is not an arithmetic progression.

Example II

Write down the terms indicated in each of the following arithmetic progression
$(i)\ 2 + 6 + 10 + 14 + \cdots 12^{th}$
$(ii)\ 10 + 8 + 6 + 4 \ldots 15^{th}$
$(iii)\ 10 + 8 + 6 + 4 + \cdots$
$(iv) - 6\frac{1}{2} - 5 - 3\frac{1}{2} - 2 + \cdots 12^{th}$

Solution

$2 + 6 + 10 + 14 + \ldots 12^{th}$

The n^{th} term of an A.P (u_n) is given by:

$u_n = a + (n-1)d$

$a = 2, d = 4$

$\Longrightarrow u_n = 2 + (n-1)4$

$u_n = 2 + 4n - 4$

$u_n = 4n - 2$

12^{th} term, n=12

$\Longrightarrow\ u_{12} = 4 \times 12 - 2$

$= 48 - 2$

$= 46$

$(ii) 10 + 8 + 6 + 4 \ldots 15^{th}$

n^{th} $term,\ u_n = a + (n-1)d$

$u_{15} = 10 + 14 \times -2$

$= 10 - 28$

$= -18$

$(ii)\ 7 + 3 - 1 - 5 + \cdots 19^{th}$

$a = 7$

$d = 3 - 7 = -4$

$u_n = a + (n-1)d$

$u_{19} = 7 + (19-1)(-4)$

$= 7 + -72$

$u_{19} = -65$

(iv) $-6\frac{1}{2} + \ -5 - 3\frac{1}{2} - 2 + \cdots$

$a = -6.5$

$d = 1.5$

$n^{th}\ term\, u_n = a + (n-1)d$

$u_n = -6.5 + (n-1)(1.5)$

$u_{12} = -6.5 + (12-1) \times 1.5$

$= -6.5 + 16.5$

$= 10$

Example III

Find the number of terms in each of the following progression

$(a)\ 5 + 8 + 11 + 14 + \cdots 59 + 62$

$(b)\ 1 + 6 + 11 + 16 + \cdots 501 + 506$

$(c) - 193 - 189 - 185 + \cdots - 21 - 17$

$(d) 2\frac{1}{4} + 2\frac{17}{20} + 3\frac{9}{20} + \cdots 20\frac{1}{4} + 20\frac{17}{20}$

Solution

$5 + 8 + 11 + 14 + \cdots 59 + 62$

$a = 5 \qquad d = 8 - 5 = 3$

$n^{th} term = a + (n-1)d$

$62 = a + (n-1)d$

$62 = 5 + (n-1)3$

$62 - 5 = 3(n-1)$

$$\frac{57}{3} = -1$$
$$20 = n$$

$(b)\ 1 + 6 + 11 + 16 + \cdots 501 + 506$

$a = 1 \qquad d = 6 - 1 = 5$

$n^{th}\ term\ u_n = a + (n-1)d$

$u_n = 1 + (n-1)5$

$506 = 1 + (n-1)5$

$505 = 5(n-1)$

$101 = n - 1$

$n = 102$

$(c) -193 + -189 + \ -185 + \cdots - 21 + \ -17$

$a = -193$

$d = -189 - \ -193$

$d = 4$

The n^{th} term

$u_n = a + (n-1)d$

$u_n = -193 + (n-1)4$

$-17 = -193 + (n-1)4$

$-17 + 193 = 4(n-1)$

$\frac{176}{4} = n - 1$

$n = 45$

Example IV

An AP is given by $k, \frac{2k}{3}, \frac{k}{3}, 0$ ...

(i) Find the sixth term

(ii) Find the nth term

(iii) If the 20th term is 15. Find K

Solution

$$a = k, d = \frac{2k}{3} - k = \frac{-k}{3}$$

$$nth\ term\ (u_n) = k + (n-1)\left(\frac{-k}{3}\right)$$

$$6^{th}\ term\ (u_6) = k + 5\left(-\frac{k}{3}\right)$$

$$= k - \frac{5k}{3}$$

$$= -\frac{2k}{3}$$

$$(ii)\ n^{th}\ term = k + (n-1)\left(-\left(\frac{k}{3}\right)\right)$$

$$= k - \frac{nk}{3} + \frac{k}{3}$$

$$= \frac{4k}{3} - \frac{nk}{3}$$
$$= \frac{k}{3}(4 - n)$$

But $u_{20} = 15$

$$\Rightarrow 15 = \frac{k}{3}(4 - 20)$$
$$45 = k(-16)$$
$$k = \frac{-45}{16}$$

Sum of Arithmetic Progression

Generally, the arithmetic progression is given by
$a + (a + d) + (a + 2d) + \ldots a + (n - 2)d + a + (n-1)d$
Let the sum of the arithmetic progression be S_n
$S_n = a + (a + d) + (a + 2d) + \cdots a + (n - 2)d + a + (n - 1)d$……….. (1)

Suppose the terms are added in opposite order
$S_n = a + (n - 1)d + a + (n - 2)d + \cdots (a + 2d) + (a + d) + a$………….. (2)

$adding\ eqn\ (1) and\ (2)$
$\Rightarrow 2S_n = 2a + (n - 1)d + 2a + (n - 1)d$
$+2a + (n - 1)d + \cdots 2a + (n - 1)d$
$+2a + (n - 1)d + 2a + (n - 1)d$ ………….. (3)
Since there are *n* terms of equation (3) that have been added together. The total is $2S_n = n(2a + (n - 1)d)$

$$\Rightarrow 2S_n = n(2a + (n - 1)d$$
$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$\boxed{S_n = \frac{n}{2}(2a + (n - 1)d)}$$

This is the formula for the sum of arithmetic progression

Example I
Find the sum of the first 50 terms of an A.P

$$1 + 3 + 5 + 7 + 9 + \cdots$$

Solution

$$a = 1, d = 2$$
$$S_n = \frac{n}{2}(2a + (n - 1)d)$$
$$S_{50} = \frac{50}{2}(2 \times 1 + 49 \times 2)$$
$$S_{50} = 25(2 + 98)$$
$$S_{50} = 25(100)$$
$$S_{50} = 2500$$

Example II
Find the sum of $5 + 9 + 13 + \cdots$ to 20 terms
Solution

$$a = 5, d = 4$$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$S_{20} = \frac{20}{2}(2 \times 5 + (20-1) \times 4)$$
$$S_{20} = 10(10 + 19 \times 4)$$
$$S_{20} = 10(86)$$
$$S_{20} = 860$$

Example III
Find the sum 27 + 22 + 17 + … to 10 terms
Solution

$$a = 27, d = 22 - 27$$
$$d = -5$$
$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$S_{10} = \frac{10}{2}(2 \times 27 + (10-1) \times -5)$$
$$S_{10} = \frac{10}{2}(54 + \ -45)$$
$$S_{10} = 45$$

Example IV
Find the sum of the following A.P

- **(i)** 6+10+14+…50
- **(ii)** 10+7+4+…-50
- **(iii)** 5+9+13+…101
- **(iv)** $1 + 1\frac{1}{4} + 1\frac{1}{2} + \cdots 9\frac{3}{4}$
- **(v)** 83+80+77+…5

Solution

$(i)\ a = 6, d = 4$

$U_n = a + (n-1)d$

$50 = 6 + (n-1)4$

$44 = 4(n-1)$

$11 = n - 1$

$n = 12$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{12} = \frac{12}{2}(2 \times 6 + (12-1) \times 4)$

$= 6(12 + 44)$

$S_{12} = 6(56)$

$= 336$

$(ii)\ 10 + 7 + 4 + \cdots - 50$

$a = 10, \qquad d = 7 - 10 = -3$

$n^{th}\ term = a + (n-1)d$

$-50 = 10 + (n-1)(\text{-}3)$

$-60 = -3(n-1)$

$20 = n - 1$
$n = 21$
$S_n = \frac{n}{2}(2a + (n-1)d)$

$$S_{21} = \frac{21}{2}[2 \times 10 + (21 - 1) \times -3]$$

$= \frac{21}{2}(20 + \ -60)$
$= \frac{21}{2}(-40)$
$= {}^{-}420$

(iii) $5 + 9 + 13 \ldots 101$
$a = 5, \quad d = 4,$
$n^{th}\ term = a + (n-1)d$
$101 = 5 + (n-1)4$
$96 = 4(n-1)$
$24 = n - 1$
$n = 25$
$S_n = \frac{n}{2}(2a + (n-1)d)$
$S_{25} = \frac{25}{2}(2 \times 5 + (25 - 1) \times 4)$
$= \frac{25}{2}(10 + 96)$
$= \frac{25}{2}(106)$
$S_{25} = 1325$

(iv) $1 + 1\frac{1}{4} + 1\frac{1}{2} \ldots 9\frac{3}{4}$

$1 + \frac{5}{4} + \frac{3}{2} \ldots \frac{39}{4}$

$a = 1, \ \ d = \frac{1}{4}$

$\frac{39}{4} = a + (n-1)d$
$\frac{39}{4} = 1 + (n-1)\frac{1}{4}$
$\frac{39}{4} = 1 + \frac{1}{4}n - \frac{1}{4}$
$\frac{39}{4} = \frac{3}{4} + \frac{1}{4}n$
$39 = 3 + n$
$n = 36$
$S_{36} = \frac{36}{2}\left(2 \times 1 + (36 - 1) \times \frac{1}{4}\right)$

$S_{36} = 18(2 + 8.75)$
$= 193.5$

(v) $83 + 80 + 77 + \cdots 5$
$a = 83,\ \ d = 80 - 83$
$d = -3$
$u_n = a + (n-1)d$
$5 = 83 + (n-1)(-3)$
$5 = 83 + \ -3n + 3$
$5 = 86 - 3n$
$3n = 81, \qquad n = 27$
$S_{27} = \frac{27}{2}(2 \times 83 + (27-1) \times -3)$

$= \frac{27}{2}(166 + \ -78)$
$= \frac{27}{2}(88)$
$= 1188$

Example V
In an A.P the sum of the first 10 terms is 520 and the 7th terms doubles the 3rd term. Find the first term *a* and common difference (*d*).
Solution
$S_n = \frac{n}{2}(2a + (n-1)d)$
$S_{10} = 520$
$S_{10} = \frac{10}{2}(2a + (10-1)d)$
$S_{10} = 5(2a + 9d)$
$520 = 5(2a + 9d)$
$104 = 2a + 9d \ldots\ldots\ldots\ldots(1)$
7th term
$u_n = a + (n-1)d$
$u_7 = a + 6d$
3rd term
$u_3 = a + 2d$
$u_7 = 2u_3$
$a + 6d = 2(a + 2d)$
$a + 6d = 2a + 4d$
$0 = a - 2d$
$a = 2d$
Substituting eqn (2) in eqn (1)
$104 = 2(2d) + 9d$
$104 = 4d + 9d$
$13d = 104$
$d = \frac{104}{13}$
$d = 8$
$a = 2d$

$a = 16$

Example V

The first and last terms of an A.P with 25 terms are 29 and 179 respectively. Find the sum of the series and it's common difference

Solution

$n = 25$, $a = 29$, n^{th} term

$u_n = a + (n-1)d$

$179 = 29 + (25-1)d$

$150 = 24d$

$d = 6.25$

$S_{25} = \frac{25}{2}(2 \times 29 + (25-1) \times 6.25)$

$S_n = 12.5(58 + 150)$

$= 2600$

Example VI

The nth term of the series is 10-3n. Find the first three terms of the series and the sum of the first 15 terms.

Solution

$n^{th}\ term = 10 - 3n$

$first\ term\ n = 1$

$\Rightarrow a = 10 - 3 \times 1$

$= 7$

$2^{nd}\ term\ n = 2$

$= 10 - 3 \times 2$

$= 4$

$3^{rd}\ term = 10 - 3 \times 3$

$= 1$

$7 + 4 + 1 + \cdots$

$a = 7, \quad d = -3$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{15} = \frac{15}{2}(2 \times 7 + (15-1) \times -3)$

$= \frac{15}{2}(14 + \ -42)$

$= -210$

Example VIII

Given that the first and third terms of an A.P are 13 and 25 respectively. Find the 100^{th} term and the sum of the first 15 terms

Solution

$a = 13$

$a + 2d = 25$

$13 + 2d = 25$

$2d = 12$

$d = 6$

$n^{th}\ term = a + (n-1)d$
$n^{th}\ term = 13 + (n-1)6$
$100^{th}\ term = 13 + (100-1) \times 6$
$u_{100} = 13 + 99 \times 6$
$= 607$
$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{15} = \frac{15}{2}(2 \times 13 + (100-1) \times 6)$

$S_n = 4650$

Example VII

The second and seventh terms of an A.P are -5 and 10 respectively. Find the fifth term and the least number of terms that must be taken for their sum to exceed 200

Solution

$a + d = -5$………………………. (1)
$a + 6d = 10$………………………. (2)
$eqn\ (2) - eqn\ (1)$
$5d = 15$
$d = 3$
$a + d = -5$
$a = -5 - d$

$a = -5 - 3$
$a = -8$
$u_n = a + (n-1)d$
$u_5 = -8 + (5-1) \times 3$
$u_5 = -8 + 12$
$u_5 = 4$

$S_n = \frac{n}{2}(2a + (n-1)d)$
$S_n = \frac{n}{2}(2 \times -8 + (n-1)3)$
$S_n = \frac{n}{2}(-16 + 3n - 3)$
$S_n = \frac{n}{2}(-19 + 3n)$
$S_n > 200$
$S_n = \frac{n}{2}(-19 + 3n) > 200$
$-19n + 3n^2 > 400$
$3n^2 - 19n - 400 > 0$

$$3\left(n^2 - \frac{19}{3}n\right) > 400$$

$$n^2 - \frac{19}{3}n > \frac{400}{3}$$

By completing squares;

$$n^2 - \frac{19n}{3} + \frac{361}{36} - \frac{361}{36} > \frac{400}{3}$$

$$\left(n - \frac{19}{6}\right)^2 > \frac{400}{3} + \frac{361}{36}$$

$$\left(n - \frac{19}{6}\right)^2 > \frac{5161}{36}$$

$$n - \frac{19}{16} > \sqrt{\frac{5161}{36}}$$

$$n > \frac{19}{6} + \sqrt{\frac{5161}{36}}$$

$$n > 15.1400167$$

$$n = 16$$

Example VIII

In an A.P the sum of the first 15 terms is 615 and the 13th term is six times the second term. Find the first three terms

Solution

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{15} = \frac{15}{2}(2a + (15-1)d) = 615$

$15(a + 7d) = 615$

$15a + 105d = 615$

$a + 7d = \frac{615}{15}$

$a + 7d = 41$

$\Rightarrow a = 41 - 7d$ ………………… (1)

$13^{th}\ term = a + (13-1)d$

$u_{13} = a + 12d$

$2^{nd}\ term = a + d$

$u_{13} = 6u_2$

$a + 12d = 6a + 6d$

$0 = 5a - 6d$ ………………… (2)

Substituting Eqn (1) in Eqn (2)

$\Rightarrow 0 = 5(41 - 7d) - 6d$

$0 = 205 - 35d - 6d$

$0 = 205 - 41d$

$d = 5$

$a = 41 - 7d$

$\Rightarrow a = 41 - 35$

$a = 6$

The first three terms are: 6, 11, 17 ...

Example IX

In an A.P the sum of the first 2n terms is equal to the sum of the next n terms. If the first term is 12 and common difference is 3. Find the non-zero value of n.

Solution

In an A.P the sum of the first 2*n* terms

$$S_{2n} = \frac{2n}{2}(2 \times 12 + (2n-1)3)$$
$$= n(24 + 6n - 3)$$
$$= n(21 + 6n)$$
$$= 21n + 6n^2$$

The total of 2*n* terms and the next n terms is 3n terms.

$$S_{3n} = \frac{3n}{2}[(2 \times 12 + (3n-1)3]$$

$$S_{3n} = \frac{3n}{2}[24 + 9n - 3]$$

$$= \frac{3n}{2}(21 + 9n)$$
$$= \frac{63n}{2} + \frac{27n^2}{2}$$

The sum of the next *n* terms after 2*n* terms is

$$\left(\frac{63n}{2} + \frac{27n^2}{2}\right) - (21n + 6n^2)$$
$$\frac{63n + 27n^2 - 42n - 12n^2}{2}$$
$$\frac{21n + 15n^2}{2}$$

Sum of first 2*n* terms = sum of the next *n* terms.

$$\Rightarrow 21n + 6n^2 = \frac{21n + 15n^2}{2}$$
$$42n + 12n^2 = 21n + 15n^2$$
$$0 = 3n^2 - 21n$$
$$0 = 3n(n-7)$$
$$n = 0 \qquad n = 7$$
$$n \neq 0 \quad \Rightarrow n = 7$$

Example X

The sum of the first n terms of a certain series is $3n^2 + n$. Show that the series is an A.P find the first term and common difference.

Solution

$S_n = 3n^2 + n$

$S_{n-1} = 3(n-1)^2 + n - 1$

$S_{n-1} = 3n^2 - 5n + 2$

$$n^{th}\ term = S_n - S_{n-1}$$

$= (3n^2 + n) - (3n^2 - 5n + 2)$

$= 6n - 2$

$if\ n = 1 \quad (1^{st}\ term)$

$a = 4$

$if \;\; n = 2\ (second\ term)$
$6 \times 2 - 2 = 10$
$If\ n = 3\ \ (third\ term)$
$6 \times 3 - 2 = 16$
$first\ term = 4$
Common difference=6
4, 10, 16, …
⇒ The series is an A.P

Example XI
The sum to n terms of a particular series is given by

$$S_n = 17n - 3n^2$$

(a) Find an expression for the sum of (n-1) terms
(b) Find the expression for the n^{th} term of the series.
Show that the series is arithmetic progression. Find the first term and common difference.
Solution:
$S_n = 17n - 3n^2$
$S_{(n-1)} = 17(n-1) - 3(n-1)^2$
$= 17n - 17 - 3(n^2 - 2n + 1)$
$= 17n - 17 - 3n^2 + 6n - 3$
$= 23n - 3n^2 - 20$
$n^{th}\ term = S_n - S_{n-1}$
$= (17n - 3n^2) - (23n - 3n^2 - 20)$
$= -6n + 20$
$= 20 - 6n$
$For\ n = 1.(first\ term)$
$20 - 6 = 14$
$For\ n = 2.(second\ term)$
$20 - 6 \times 2 = 8$
$For\ n = 3\ (third\ term)$
$20 - 6 \times 3 = 2$
$First\ term\ a = 14$
Common difference
$d = -6$

Example XII
The sum of the three terms in A.P is 30 and the sum of their squares is 398. Find the numbers.
Solution
$a + (a + d) + a + 2d = 30$
$3a + 3d = 30$
$a + d = 10$ ………………………. (1)
$a^2 + (a + d)^2 + (a + 2d)^2 = 398$
$3a^2 + 6ad + 5d^2 = 398$ ……….. (2)
$from\ eqn\ (1)$
$a = (10 - d)$
Substituting $a = 10 - d$ in Eqn (2);
$\Rightarrow 3(10 - d)^2 + 6d(10 - d) + 5d^2 = 398$
$3(100 - 20d + d^2) + 60d - 6d^2 + 5d^2 = 398$

$300 - 60d + 3d^2 + 60d - d^2 = 398$
$2d^2 = 98$
$d^2 = 49$
$d = \pm 7$
$If\ d = 7$
$a = 3$
3, 10, 17 ...
$If\ \ d = -7$
$a + d = 10$
$a - 7 = 10$
$a = 17$
17, 10, 3
The numbers are 3, 10 and 17.

Example XIII
Show that the sum to 20 terms of the series

$$\log a + \log ab + \log ab^2 + \log ab^3 + \cdots$$

can be written in the form $\log a^x b^y$ and find x and y.
Solution

$$d = \log ab - \log a$$
$$= \log b$$
$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$S_{20} = \frac{20}{2}[(2\log a + (20-1)\log b]$$
$$= 10[(\log a^2 + 19\log b]$$
$$= 10[(\log a^2 + \log b^{19}]$$
$$= 10[\log a^2 + \log b^{19}]$$
$$10\left[\log a^2 b^{19}\right]$$
$$\log(a^2 b^{19})^{10}$$
$$= \log a^{20} b^{190}$$
$$= \log a^x b^y$$
$$x = 20, \quad y = 190$$

GEOMETRIC PROGRESSION

A geometric progression G.P is a sequence where each new terms after the first is obtained by multiplying the preceding term by a constant **r**, called a common ratio. If the first term of the sequence is 'a' then the geometric progression is

$$a + ar + ar^2 + ar^3 + \cdots ar^{n-1}$$

$a = first\ term, r = common\ ratio$
$ar^{n-1} = n^{th}\ term.$
Examples of geometric progression are:
$(i)\ 2 + 6 + 18 + 54 + \cdots$
$(ii)\ 1 + 2 + 4 + 8 + \cdots 128 + 256$
$(iii) 27 - 9 + 3 - 1 + \cdots \frac{1}{27} - \frac{1}{81}$
From the above examples

$$2 + 6 + 18 + 54 + \cdots$$

First term $(a = 2)$

common ratio $= \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$

For $1 + 2 + 4 + 8 + \cdots 128 + 256$

a = 1, common ratio (r) $= \frac{2}{1} = \frac{4}{2} = \frac{8}{4} = 2$

For $27 - 9 + 3 + \ -1 + \cdots \frac{1}{27} - \frac{1}{81}$

First term a = 27

common ratio $r = \frac{-9}{27} = \frac{3}{-9} = -\frac{-1}{3}$

Example I

Which of the following series are geometric progression write down the common ratios of those that are:

(i) $3 + 9 + 27 + 81 \ldots$
(ii) $-1 + 2 - 4 + 8 \ldots$
(iii) 1+1.1+1.21 + 1.331…
(iv) 1+1/4+1/16+1/64+ …
(v) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} \ldots$

Solution

(i) 3 + 9 + 27 + 81 …

$$r_1 = \frac{9}{3} = 3$$
$$r_2 = \frac{27}{9} = 3$$
$$r_3 = \frac{81}{27} = 3$$

$r_1 = r_2 = r_3 = common\ ration = r$=3

$\Rightarrow 3 + 9 + 27 + 81$ …is a geometric progression

$(ii) - 1 + 2 + \ -4 + 8$

$$r_1 = \frac{2}{-1} = -2$$
$$r_2 = \frac{-4}{2} = -2$$
$$r_3 = \frac{8}{-4} = -2$$
$$r_1 = r_2 = r_3 = r = -2$$

Common ratio

$\Rightarrow -1 + 2 + \ -4 + 8$ is a geometric progression with a common ratio $r = -2$

$(iii)\ 1 + 1.1 + 1.21 + 1.331 \ldots$

$$r_1 = \frac{1.1}{1} = 1.1$$
$$r_2 = \frac{1.21}{1.1} = 1.1$$
$$r_3 = \frac{1.331}{1.21} = 1.1$$

It is a geometric progression with a common ratio r=1.1

(iv) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} \ldots$

$$r_1 = \left(\frac{\frac{1}{4}}{1}\right) = \frac{1}{4}$$

$$r_2 = \left(\frac{\frac{1}{16}}{\frac{1}{4}}\right) = \frac{1}{16} \times 4 = \frac{1}{4}$$

$$r_3 = \left(\frac{\frac{1}{64}}{\frac{1}{16}}\right) = \frac{1}{64} \times 16 = \frac{1}{4}$$

It is a geometric progression with a common ratio $r = \frac{1}{4}$

$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} + \ldots$

$$r_1 = \frac{1/6}{1/2} = \frac{1}{6} \times 2 = \frac{1}{3}$$

$$r_2 = \frac{1/12}{1/6} = \frac{6}{12} = \frac{1}{2}$$

$$r_3 = \frac{1/36}{1/12} = \frac{12}{36} = \frac{1}{3}$$

$r_1 \neq r_2$

$\Rightarrow \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36}$ … is not a geometric progression because the common ratio is not the same.

Example II

State the common ratio and the next two term of the following geometric progression (G.P)

$(i)\ 1 + \ -2 + 4 + \ -8 + \cdots$

$(ii) \frac{1}{4} + \frac{1}{2} + 1 + \cdots$

$(iii) 200 + 50 + 12\frac{1}{2} + \cdots$

$(iv)\ 162 + 54 + 18 + \cdots$

Solution

Let the next two terms be x and y

$1 + \ -2 + 4 + \ -8 + x + y + \cdots$

$r = -\frac{2}{1} = -2 \ = \ common\, ratio$

$\Rightarrow \frac{x}{-8} = -2$

$x = 16$

And $\frac{y}{x} = -2$

$\frac{y}{16} = -2$

$y = -32$

$\Rightarrow The\ ext\ two\ numbers\ of\ the\ sequence$
$1 + \ -2 + 4 + \ -8 + \cdots are\ 16\ and - 32$

$(ii)\frac{1}{4} + \frac{1}{2} + 1 + \cdots$

$r = \frac{1/2}{1/4} = \frac{1}{2} \times \frac{4}{1} = 2$

Let the next numbers of the sequence be m and n.

$\frac{1}{4} + \frac{1}{2} + 1 + m + n + \cdots$

$\frac{m}{1} = 2 \Rightarrow \quad m = 2$

$\frac{n}{m} = 2, \qquad n = 2m$

$n = 4$

$\Rightarrow The\ next\ two\ numbers\ of\ the\ sequence$

$\frac{1}{4} + \frac{1}{2} + 1 + \cdots are\ 2\ and\ 4$

$(iii)\ 200 + \ -50 + 12\frac{1}{2} + \cdots$

$r = -\frac{50}{200} = -\frac{1}{4}$

Let the next two numbers of the series be m and n

$\Rightarrow 200 + \ -50 + 12\frac{1}{2} + m + n + \cdots$

$\frac{m}{12.5} = -\frac{1}{4}$

$4m = -12.5$

$m = -3.125, \ \ m = \frac{-25}{8}$

$m = -3\frac{1}{8}$

$\frac{n}{m} = -\frac{1}{4}$

$n = \frac{-1}{4}m$

$n = \frac{-1}{4}\left(\frac{-25}{8}\right)$

$\Rightarrow \ \ The\ next\ two\ terms\ of\ the\ series$

$200 + \ -50 + 12\frac{1}{2} + \cdots$ are $-\frac{25}{8}$ and $\frac{25}{32}$

(iv) $162 + 54 + 18 + \cdots$

Let the next terms of the sequence be m and n.

$162 + 54 + 18 + m + n + \ldots$

$\frac{m}{18} = \frac{1}{3}$

$3m = 18$

$m = 6$

$$\frac{n}{m} = \frac{1}{3}$$

$$\frac{n}{6} = \frac{1}{3}$$

$3n = 6$

$n = 2$

$\Rightarrow$ The next two terms of the sequence

$162 + 54 + 18 + m + n + \ldots$ are 6 and 2

Example III

Write down the terms indicated in each of the following progression

$(i) \frac{1}{2} + 1 + 2 + \cdots 8^{th}$

$(ii)\ 200 - 50 + 2\frac{1}{2} + \cdots 5^{th}$

$(iii)\ 162 + 54 + 18 + \cdots 6^{th}$

$(iv) -\frac{4}{9} - \frac{2}{3} - 1 + \cdots 7^{th}$

Solution

The n^{th} of a geometric progression is given by;

$U_n = ar^{n-1}$

$For \quad \frac{1}{2} + 1 + 2 + \cdots 8^{th}$

$a = \frac{1}{2} \quad r = \frac{1}{½} = 2$

$\Rightarrow U_n = \frac{1}{2}(2^{n-1})$

For the 8^{th} term, $\quad n = 8$

$\Rightarrow U_8 = \frac{1}{2}(2^7) = 64$

$(ii)\ 200 - 50 + 12.5 + \cdots$

The n^{th} term $U_n = ar^{n-1}$

$a = 200 \;\; r = -\frac{1}{4}$

$U_n = 200\left(-\frac{1}{4}\right)^{n-1}$

For the 5^{th} term

$U_5 = 200\left(-\frac{1}{4}\right)^{5-1}$

$U_5 = \frac{25}{32}$

$\Rightarrow The\ 5^{th}\ term\ is \frac{25}{32}$

Example IV

$162 + 54 + 18 + \cdots 6^{th}\ term$

$$U_n = ar^{n-1}$$
$$r = \frac{54}{162} = \frac{1}{3}$$
$$a = 162$$
$$U_n = 162\left(\frac{1}{3}\right)^{n-1}$$
$$6^{th}\ term = 162\left(\frac{1}{3}\right)^{6-1}$$
$$= \frac{2}{3}$$

$\Rightarrow$ The 6th term is $\frac{2}{3}$

$$-\frac{4}{9} - \frac{2}{3} - 1 + \cdots 7^{th}$$

$$a = -\frac{4}{9}, \qquad r = -\left(\frac{\frac{2}{3}}{\frac{-4}{9}}\right)$$

$$r = \frac{-2}{3} \times \frac{9}{-4} = \frac{3}{2}$$
$$U_n = ar^{n-1}$$
$$= \frac{-4}{9}\left(\frac{3}{2}\right)^{n-1}$$

For the 7th term, $n = 7$

$$U_7 = \frac{-4}{9}\left(\frac{3}{2}\right)^{7-1}$$
$$= \frac{-4}{9}\left(\frac{729}{64}\right) = \frac{-81}{16}$$
$$\Rightarrow The\ 7^{th}\ term\ is\ \mathbf{\frac{-81}{16}}$$

Example VI

How many term are in a geometric progression

$2 + 4 + 8 + \cdots + 128$

Solution:

The nth term is given by $U_n = ar^{n-1}$

$$a = 2, \qquad r = \frac{4}{2} = 2$$
$$\Rightarrow 128 = 2 \times (2)^{n-1}$$
$$128 = 2^n$$
$$2^7 = 2^n$$
$$n = 7$$

Example VII

The first term of a G.P with positive common ratio is 80. If the sum of the first three term is 185, find the common ratio.

Solution

$a = 80 \quad (first\ term).$

$a + ar + ar^2 = 185$

$$a(1 + r + r^2) = 185$$
$$80(1 + r + r^2) = 185$$
$$1 + r + r^2 = \frac{185}{80}$$
$$1 + r + r^2 = \frac{37}{16}$$
$$16 + 16r + 16r^2 = 37$$
$$16r^2 + 16r - 21 = 0$$
$$(4r + 7)(4r - 3) = 0$$
$$r = \frac{3}{4}, \qquad r = -\frac{7}{4}$$

Example VIII

In a G.P the second term exceeds the first by 20 and the fourth exceeds the second by 15. Find the two possible values of the first term

Solution

$2^{nd}\ term = ar$

$1^{st}\ term = a$

$a + 20 = ar$

$20 = ar - a$……………………. (1)

$4^{th} term = ar^3$

$second\ term = ar$

$ar + 15 = ar^3$

$15 = ar^3 - ar$ ……………………. (2)

$Eqn\ (1) \div Eqn(2)$

$$\frac{20}{15} = \frac{a(r-1)}{ar(r^2-1)}$$
$$\frac{4}{3} = \frac{r-1}{r(r^2-1)}$$
$$\frac{4}{3} = \frac{r-1}{r^3-r}$$
$$4r^3 - 4r = 3r - 3$$
$$4r^3 - 4r - 3r + 3 = 0$$
$$4r^3 - 7r + 3 = 0$$
$$r = 1, \qquad r = -\frac{3}{2}, \qquad r = \frac{1}{2}$$

If $r = \frac{1}{2}$, $20 = ar - a$

$$\Rightarrow 20 = \frac{1}{2}a - a$$
$$20 = -\frac{a}{2}$$
$$a = -40$$

When $r = \frac{-3}{2}$,

$$20 = -\frac{3}{2}a - a$$
$$20 = -\frac{5}{2}a$$
$$-\frac{40}{5} = a$$
$$a = -8$$
When r = 1, a is not defined.

Example IX
The second and fifth term in a G.P are 405 and -120 respectively. Find the seventh term.

Solution
$ar = 405$ ………………………… (1)
$ar^4 = -120$……………………… (2)
Eqn (2) ÷ Eqn (1)
$$r^3 = \frac{-120}{405}$$
$$r^3 = \frac{-8}{27}$$
$$r = -\frac{2}{3}$$
$$a\left(-\frac{2}{3}\right) = 405$$
$$a = -607.5$$
$$ar^6 = (-607.5)\left(-\frac{2}{3}\right)^6$$
$$= \frac{-160}{3}$$

Example IX
The second, fourth and eighth terms of an A.P are in geometric progression, the sum of the third and fifth terms is 20. Find the first four terms of the progression.
Solution
$second\ term = a + d$
$4^{th}\ term = a + 3d$
$8^{th}\ term = a + 7d$
$(a + d),\ \ a + 3d,\ \ a + 7d$
$$\frac{a + 3d}{a + d} = \frac{a + 7d}{a + 3d}$$
$(a + 3d)^2 = (a + d)(a + 7d)$
$a^2 + 6ad + 9d^2 = a^2 + 7ad + ad + 7d^2$
$2d^2 - 2ad = 0$
$2d(d - a) = 0$
$d = 0\ \ but\ \ d \neq 0$
$d = a$………………………… (1)
$a + 2d + a + 4d = 20$
$2a + 6d$=20
$a + 3d = 10$ ………………………… (2)

Substitute Eqn (1) in Eqn (2)

$a + 3a = 10$

$a = 2.5$

$2.5, 5, 7.5, 10 \ldots$

Sum of a Geometric Progression

Let the sum of a geometric progression be S_n

$S_n = a + ar + ar^2 + \cdots ar^{n-2} + ar^{n-1}$ ……… (1)

$multiplying\ equation\ (1) by\ r$

$rS_n = ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n$…… …(2)

$Eqn\ (1) - Eqn(2)$

$\Rightarrow S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad for\ r < 1$$

$$S_n = \frac{a(r^n - 1)}{r - 1} for\ r > 1$$

Example I

Find the sum of six terms of the progression

$$2 + 6 + 18 + 54 + \cdots$$

Solution

$$a = 2, \qquad r = \frac{6}{2} = 3$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_6 = \frac{2(3^6 - 1)}{3 - 1}$$

$$= 728$$

Example II

Find the sum of the first 20 terms of a G.P with first term 3 and common ratio 1.5

Solution

$$a = 3, \quad r = 1.5$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{20} = \frac{3(1.5^{20} - 1)}{1.5 - 1}$$

$$S_{20} = 19945.54038$$

Example III

Find the sum of the first six terms of a G.P

$$100 + 10 + 1 + \cdots$$

Solution

$$a = 100, \qquad r = \frac{10}{100} = \frac{1}{10}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{100\left(1-\left(\frac{1}{10}\right)^6\right)}{1-\frac{1}{10}}$$

$$S_6 = 111.111$$

Example IV

Find the sum of the following G.P Progression

a) $\frac{1}{4}+\frac{1}{2}+\cdots+64$

b) $1000+200+\cdots+0.32$

c) $\frac{1}{4}-\frac{1}{2}+\cdots+64$

d) $2-3+\cdots+22\frac{25}{32}$

Solution

$$\frac{1}{4}+\frac{1}{2}+\cdots+64$$

$$a=\frac{1}{4}, \qquad r=\left(\frac{\frac{1}{2}}{\frac{1}{4}}\right)=\frac{1}{2}\times 4=2$$

$$ar^{n-1}=u_n$$

$$\Rightarrow \frac{1}{4}(2)^{n-1}=64$$

$$4\times\frac{1}{4}(2)^{n-1}=64\times 4$$

$$2^{n-1}=256$$

$$2^{n-1}=2^8$$

$$n-1=8$$

$$n=9$$

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$S_9=\frac{\frac{1}{4}(2^9-1)}{2-1}$$

$$S_9=\frac{1}{4}(511)$$

$$S_9=127.75$$

(ii) $1000+200+\cdots+0.32$

$$a=1000,\ \ r=\frac{200}{1000}=0.2$$

$$ar^{n-1}=u_n$$

$$1000\,(0.2)^{n-1}=0.32$$

$$(0.2)^{n-1}=0.00032$$

$$(0.2)^{-1} = (0.2)^5$$
$$n - 1 = 5$$
$$n = 6$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_6 = \frac{\frac{1}{4}(1-0.2^6)}{1-0.2}$$
$$S_6 = \frac{1}{4}\left(\frac{0.999936}{0.8}\right)$$
$$= 0.31248$$

(c) $\frac{1}{4} - \frac{1}{2} + \cdots 64$

$$a = \frac{1}{4} \quad r = -\frac{\frac{1}{2}}{\frac{1}{4}} = -\frac{1}{2} \times 4 = -2$$
$$ar^{(n-1)} = 64$$
$$\frac{1}{4}(-2)^{n-1} = 64$$
$$(-2)^{n-1} = 64 \times 4$$
$$-2^{n-1} = 256$$
$$(-2)^{n-1} = (-2)^8$$
$$n - 1 = 8$$
$$n = 9$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_9 = \frac{\frac{1}{4}(1-(-2)^9)}{1--2}$$
$$S_9 = \frac{1}{4}\left(\frac{1--512}{3}\right)$$
$$S_9 = 42.75$$

(d) $2 + \ -3 + \cdots 22\frac{25}{32}$

$$a = 2, \qquad r = \frac{-3}{2}$$
$$ar^{n-1} = 22\frac{25}{32}$$
$$2\left(-\frac{3}{2}\right)^{n-1} = \frac{729}{32}$$
$$\left(\frac{-3}{2}\right)^{n-1} = \frac{729}{32} \div 2$$
$$\left(\frac{-3}{2}\right)^{n-1} = \frac{729}{64}$$

$$\left(-\frac{3}{2}\right)^{n-1} = \left(-\frac{3}{2}\right)^{6}$$
$$n - 1 = 6$$
$$n = 7$$
$$S_n = \frac{a(1 - r^n)}{1 - r}$$
$$S_7 = \frac{2\left(1 - \left(-\frac{3}{2}\right)^7\right)}{1 - \frac{-3}{2}}$$
$$S_7 = 14.46875$$

Example V

Find the sum of the first n terms of the G.P

$$\frac{1}{12} + \frac{1}{4} + \frac{3}{4} + \cdots$$

how many terms of the series are needed to reach a sum greater than 100

Solution

$$\frac{1}{12} + \frac{1}{4} + \frac{3}{4} + \cdots$$
$$a = \frac{1}{12}, \quad r = \frac{1/4}{1/12} = \frac{1}{4} \times 12$$
$$r = 3$$
$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_n = \frac{\frac{1}{12}(3^n - 1)}{3 - 1}$$
$$S_n = \frac{1}{12}\left(\frac{3^n - 1}{2}\right)$$
$$S_n = \frac{3^n - 1}{24}$$
$$S_n > 100$$
$$\frac{3^n - 1}{24} > 100$$
$$3^n - 1 > 2400$$
$$3^n > 2401$$
$$\log_{10} 3^n > \log_{10} 2401$$
$$n > \frac{\log_{10} 2401}{\log_{10} 3}$$
$$n > 7.08497$$
$$n = 8$$

Example VI

The sum of the first seven term of a G.P is 7 and the sum of next seven terms is 896. Find the common ratio of the progression. If the k^{th} term is the first term of a G.P which is greater than 1, find k.

Solution

$sum\ of\ the\ first\ seven\ terms = 7$

$sum\ of\ the\ next\ seven\ term = 896$

$sum\ of\ the\ 14\ terms = 7 + 896$

$= 903$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{14} = \frac{a(r^{14} - 1)}{r - 1}$$

$$\frac{a(r^{14} - 1)}{r - 1} = 903$$

$a(r^{14} - 1) = 903(r - 1) \ldots\ldots\ldots (1)$

$S_7 = 7$

$$\frac{a(r^n - 1)}{r - 1} = 7$$

$a(r^7 - 1) = 7(r - 1) \ldots\ldots\ldots\ldots (2)$

Eqn (1) ÷ Eqn (2)

$$\frac{r^{14} - 1}{r^7 - 1} = 129$$

$let\ r^7 = P$

$$\frac{P^2 - 1}{P - 1} = 129$$

$$\frac{(p+1)(p-1)}{p-1} = 129$$

$p + 1 = 129$

$p = 128$

But $p = r^7$

$r^7 = 128$

$\Rightarrow r = 2$

Example VII

A G.P has first term 16 and common ratio ¾. If the sum of the first n terms is greater than 60. Find the least possible values of n.

Solution

$a = 16, \qquad r = \frac{3}{4}$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\frac{16\left(1 - \left(\frac{3}{4}\right)^n\right)}{1 - \frac{3}{4}} > 60$$

$$1 - \left(\frac{3}{4}\right)^n > \frac{15}{16}$$

$$\frac{1}{16} > \left(\frac{3}{4}\right)^n$$
$$\log_{10}\left(\frac{1}{16}\right) > \log_{10}\left(\frac{3}{4}\right)^n$$
$$\log_{10}\left(\frac{1}{16}\right) > n\log_{10}\frac{3}{4}$$
$$\frac{\log_{10}(\frac{1}{16})}{\log_{10}(\frac{3}{4})} < n$$
$$9.638 < n$$
$$\Rightarrow \quad n = 10$$

Sum to Infinity of A G.P

Consider the general geometric progression (G.P)
$a + ar + ar^2 + \cdots + ar^{n-1}$
The sum of the *n* terms is denoted by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ , for } |r| < 1$$

Since *n* cannot be negative it implies that as *n* increases r^n decreases for $|r| < 1$

$$Sn = \frac{a(1-r^n)}{1-r}$$
$$Sn = \frac{a}{1-r} - \frac{a\,r^n}{1-r}$$

But as *n* tends to a big positive value ($n \to \infty$),

$$\frac{a\,r^n}{1-r} \approx 0$$

$$\Rightarrow S_\infty = \frac{a}{1-r} \text{ (sum to infinity of a G.P)}$$

We say that the G.P with a common ratio, *r* and first term (*a*) converges when $-1 < r < 1$ and the limit of the sum is $\frac{a}{1-r}$. There is no limit of a geometric progression whose common ratio lies outside the range. $|r| \leq 1$

Example I

Find the sum to infinity of a G.P with first term 3 and common ratio $\frac{1}{2}$

Solution

$$S_\infty = \frac{a}{(1-r)}$$
$$S_\infty = \frac{3}{1-\frac{1}{2}}$$
$$S_\infty = \frac{3}{\frac{1}{2}}$$
$$S_\infty = 6$$

Example II

Find the sum to infinity of

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$$

Solution

$$a = 1,\ \ r = \frac{1}{3}$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{1}{1-\frac{1}{3}}$$
$$= \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

Example III (UNEB Question)
Find how many terms of the series

$$1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$$

must be taken so that the sum will differ from the sum to infinity by less than 10^{-6}
Solution

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_n = \frac{1\left(1-\left(\frac{1}{5}\right)^n\right)}{1-\frac{1}{5}}$$
$$S_n = \frac{1-\left(\frac{1}{5}\right)^n}{\frac{4}{5}}$$
$$S_n = \frac{5}{4}\left(1-\left(\frac{1}{5}\right)^n\right)$$
$$S_\infty = \frac{1}{1-r}$$
$$S_\infty = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$$
$$S_\infty - S_n < 10^{-6}$$
$$\frac{5}{4} - \left(\frac{5}{4}\left(1-\left(\frac{1}{5}\right)^n\right)\right) < 10^{-6}$$
$$\frac{5}{4}\left(\frac{1}{5}\right)^n < 10^{-6}$$
$$\left(\frac{1}{5}\right)^n < \frac{10^{-6} \times 4}{5}$$
$$\left(\frac{1}{5}\right)^n < 8 \times 10^{-7}$$
$$(0.2)^n < 8 \times 10^{-7}$$
$$\mathrm{n}(\log_{10} 0.2) < \log_{10} 8 \times 10^{-7}$$
$$n > \frac{lo\,g_{10}\, 8 \times 10^{-7}}{log_{10} 0.2}$$

(*Sign changes because the denominator is negative*)

$$n > 8.7227$$

$$n > 8.7227$$

$$n = 9$$

Example IV

The sum to infinity of a G.P is twice the sum of the first two terms. Find the possible values of the common ratio.

Solution

$$S_\infty = \frac{a}{1-r}$$

$$S = a + ar$$

$$2(a + ar) = \frac{a}{1-r}$$

$$2(1 + r) = \frac{1}{1-r}$$

$$2(1 - r^2) = 1$$

$$1 - r^2 = \frac{1}{2}$$

$$1 - \frac{1}{2} = r^2$$

$$r^2 = \frac{1}{2}$$

$$r = \pm\frac{1}{\sqrt{2}}$$

Example V

The first and fourth terms of a geometric series are 135 and ⁻40 respectively. Find its common ratio and the sum to infinity.

Solution

$$a = 135$$

$$ar^3 = -40$$

$$135r^3 = -40$$

$$r^3 = -\frac{40}{135}$$

$$r = \frac{-2}{3}$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{135}{1 - \frac{-2}{3}}$$

$$S_\infty = 81$$

Example VI

The sum of the first n terms of a geometric progression is $8 - 2^{3-2n}$. Find the first term, its common ratio and its sum to infinity.

Solution

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$8 - 2^{3-2n} = \frac{a(1-r^n)}{1-r}$$
$$2^3 - 2^{3-2n} = \frac{a(1-r^n)}{1-r}$$
$$2^3(1-(2^{-2})^n = \frac{a(1-r^n)}{1-r}$$
$$2^3\left(1-\left(\frac{1}{4}\right)^n\right) = \frac{a(1-r^n)}{1-r}$$

$$\tfrac{1}{4} = r \quad by \text{ Comparison}$$

$$\frac{a}{1-r} = 2^3,$$

The sum to infinity = 8

$$\frac{a}{3/4} = 2^3$$
$$\Longrightarrow a = 8 \times \frac{3}{4}$$
$$a = 6$$
$$S_\infty = \frac{a}{1-r}$$
$$= \frac{6}{1-\frac{1}{4}}$$
$$= \frac{6}{3/4} = 8$$

Example (UNEB Question)

In an arithmetic progression $u_1 + u_2 + u_3 + ...$, $u_4 = 15$ and $u_{16} = -3$. Find the greatest integer N such that $u_N \geq 0$. Determine the sum of the first N terms of the progression.

Solution

Let a = 1st term of the A.P. and
d = common difference of the A.P.

The n^{th} term of the A.; $u_n = a + (n-1)d$

=> 4th term, $u_4 = a + 3d$

But $u_4 = 15$

=> $a + 3d = 15$.............................. (i)

The 16th term $u_{16} = a + 15d$

But $u_{16} = -3$

=> $a + 15d = -3$............................. (ii)

Eqn (i) − Eqn (ii)

$-12d = 18$

$d = -1.5$

From eqn (i)

$a + 3(-1.5) = 15$

$a = 15 + 4.5$

$a = 19.5$

Substituting for a and d in $u_n = a + (n-1)d$

=> $u_n = 19.5 + (n - 1) \times -1.5$

$= 19.5 - 1.5(n - 1)$

Now for $u_n \geq 0$

=> $19.5 - 1.5(n - 1) \geq 0$

$19.5 - 1.5n + 1.5 \geq 0$

$21 - 1.5n \geq 0$

$21 \geq 1.5n$

$14 \geq n$

Hence $n \leq 14$

The greatest integer, N is 14

Sum of n terms of the A.P. is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_n = \frac{14}{2}\left[2\times 19.5 + (14-1)\times -1.5\right]$$

$$= 7\left[39 + (13\times -1.5)\right]$$

$$= 7(39 - 19.5)$$

$$= 136.5$$

Example (UNEB Question)

Show that $\ln 2^r$, $r = 1, 2, 3$ is an arithmetic progression.

ii) Find the sum of the first 10 terms of the progression.

iii) Determine the least value of m for which the sum of the first $2m$ terms exceeds 883.7.

Solution

a) i) For $r = 1$,

$\ln 2^1 = \ln 2$

For $r = 2$

$$\ln 2^2 = 2\ln 2$$

Common difference = 2ln2 – ln2 = ln2

For $r = 3$

$\ln 2^3 = 3\ln 2$

Common difference = 3 ln 2 – 2 ln 2 = ln2

Since the difference between any two consecutive terms is the same i.e. ln 2, Therefore the progression is an arithmetic progression.

ii) For an *A.P.*, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Where $n = 10$, $a = \ln 2$ and $d = \ln 2$

$$\Rightarrow S_{10} = \frac{10}{2}\left[2\ln 2 + (10-1)\ln 2\right]$$

$$= 5\left[2\ln 2 + (10-1)\ln 2\right]$$

$$= 55\ln 2$$

$= 38.1231$ (4 dp)

iii) Give $a = \ln 2$

$d = \ln 2$

$n = 2m$

$$\Rightarrow S_{2m} = \frac{2m}{2}\left[2\ln 2 + (2m-1)\ln 2\right]$$

$$= m\left[(2 + 2m - 1)\ln 2\right]$$

$$= m\left[\left(1 + 2m\right)\ln 2\right]$$

For $S_{2m} > 883.7$,

$$\Rightarrow m(1+2m)\ln 2 > 883.7$$

$$m + 2m^2 > \frac{883.7}{\ln 2}$$

$$2m^2 + m > 1274.9$$

$$2m^2 + m - 1274.9 > 0$$

$$m > \frac{-1 \pm \sqrt{1 + 8 \times 1274.9}}{4}$$

$$m > \frac{-1 \pm \sqrt{10200.2}}{4}$$

$$m > \frac{-1 \pm 100.996}{4}$$

$$m > 24.999$$

Hence the least value of m is 25.

Example (UNEB Question)

The first term of an arithmetic progression (A.P) is 73 and the 9th is 25. Determine

i) The common difference of the A.P.

ii) The number of terms that must be added to give a sum of 96.

b) A geometric progression (G.P.) and an arithmetic progression (A.P) have the same first term. The sums of their first, second and third terms are 6, 10.5 and 18 respectively. Calculate the sum of their fifth terms.

Solution

Let a = 1st term and

d = common difference of an A.P

Now 1st term = 73 and

The 9th term = 25

But 9th term = $a + 8d$

=> $a + 8d = 25$

$73 + 8d = 25$

$8d = 25 - 73$

$8d = 48$

$d = -6$

Hence the common difference of the A.P is -6

ii) Let n = number of terms required

$$S_n = \frac{n}{2}\left[2a + \left(n-1\right)d\right]$$

But $S_n = 96$

$$\Rightarrow \frac{n}{2}\left[2a + \left(n-1\right)d\right] = 96$$

Substituting for a = 73 and d = -6

$$\frac{n}{2}\left[146+(n-1)\times -6\right]=96$$

$$\frac{n}{2}\left[146-6(n-1)\right]=96$$

$$n\left[73-3(n-1)\right]=96$$

$73n - 3n^2 + 3n = 96$

$3n^2 - 76n + 96 = 0$

$$n=\frac{76\pm\sqrt{76^2-4\times 3\times 96}}{2\times 3}$$

$$=\frac{76\pm\sqrt{5776-1152}}{6}$$

$$=\frac{76\pm 68}{6}$$

$$n=\frac{76+68}{6}$$

$$=24$$

Hence the terms that must be added to give a sum of 96 are 24

b) Let a = 1st term of both A.P. and G.P.

d = common difference of A.P. and

r = common difference of a G.P.

Now for an A.P:

1st term = a

2nd tern = $a + d$

3rd term = $a+2d$

4th term = $a + 3d$

5th term = $a + 4d$

Also for a G.P.:

1st term = a

2nd term = ar

3rd term = ar^2

4th term = ar^3

5th term = ar^4

Sum of their 1st terms = $2a$

$\Longrightarrow \quad 2a = 6$

$a = 3$

Sum of their second terms = $a + d + ar$

$\Longrightarrow \quad a + d + ar = 10.5$

$3 + d + 3r = 10.5$

$d + 3r = 7.5$................................ (i)

Sum of their third terms = $a + 2d + ar^2$

$\Longrightarrow \quad a + 2d + ar^2 = 18$

$3 + 2d + 3r^2 = 18$

$2d + 3r^2 = 15$

Making d the subject from Eqn (ii)

$d = 7.5 - 3r$

Substitute for d into Eqn (ii)

$2(7.5 - 3r) + 3r^2 = 15$

$15 - 6r + 3r^2 = 15$

$3r^2 - 6r = 0$
$r = 2$
$d = 7.5 - 3 \times 2 = 1.5$
The sum of their 5th terms = $a + 4d + ar^4$
$= 3 + 4 \times 1.5 + 3 \times 2^4$
$= 3 + 6 + 48$
$= 57$

Example (UNEB Question)

a) The n^{th} term of a series is $U_n = a3^n + bn + c$. given that $U_1 = 4$, $U_2 = 13$ and $U_3 = 46$, find the values of a, b and c.

Solution

a) Given $U_n = a3^n + bn + c$
Substituting for $n = 1, 2, 3...$
For $n = 1$
⇨ $3a + b + c = 4$ ....(i)
For $n = 2$,
⇨ $9a + 2b + c = 13$ ....(ii)
For $n = 3$
⇨ $27a + 3b + c = 46$...............................(iii)
Eqn (ii) − Eqn (i)
$6a + b = 9$... (iv)
Eqn (iii) − Eqn (ii)
$18a + b = 33$..................................... (v)
Eqn (v) − Eqn (iv)
$12a = 24$
$a = 2$
Substitute for a in Eqn (iv)
$12 + b = 9$
$b = -3$
Substituting for b in Eqn (i),
$6 - 3 + c = 4$
$c = 1$
$\therefore a = 2, b = -3$ and $c = 1$

Example (UNEB Question)

9.**(a)** The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the common difference and the first term. Hence find the sum of the first 60 terms.

b) A cable 10 m long is divided into ten pieces whose lengths are in a geometric progression. The length of the longest piece is 8 times the length of the shortest piece. Calculate to the nearest centimeter the length of the third piece.

Solution:

(a) The nth term of A.P. is given by
$U_n = a + (n - 1)d$
Given $U_{10} = 29$,
$\Longrightarrow a + 9d = 29$ ………………….…………(i)
Given $U_{15} = a + 14d$
$\Longrightarrow a + 14d = 44$ ……………………..…...(ii)
Eqn (i) – Eqn (ii);

$5d = 15$
$d = 3$
Substituting for d in Eqn (i)
$a + 27 = 29$
$a = 2$
The sum of the first n terms of A.P. is given by

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

$$S_{60} = \frac{60}{2}[2(2) + 59(3)]$$
$$= 30[4 + 177]$$
$$= 5430$$

b) Given length of cable = 10m
Number of divisions = 10
Here, length of divisions makes a G.P.
By considering the first to be the shortest and the 10^{th} to be the longest,
Let 1^{st} term = a
10^{th} term = ar^9
3^{rd} term = ar^2
$ar^9 = 8a$
$r^9 = 8$

$$r = 8^{1/9} \qquad = 2^{1/3}$$

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

$$10 = a\left(\frac{8^{10/9} - 1}{8^{1/9} - 1}\right)$$

$$10 = a\left(\frac{2^{10/3} - 1}{2^{1/3} - 1}\right)$$

$$10 = a\left(\frac{9.0794}{0.2599}\right)$$

$$a = 0.2863 \qquad (4d.p.)$$

Length of the 3^{rd} piece, $U_3 = ar^2$
$U_3 = 0.2862 \times 2^{2/3}$
$= 0.4544$m
$\underset{\_}{\sim}$ 45 cm

Application of Geometric Progression

UNEB Question
A credit society gives out a compound interest of 4.5% per annum. Mugagga deposited 300,000 at the beginning of each year. How much money will he have at the end of the 4^{th} year in no withdrawal is made between this period.
Solution

$$A = A_1 + A_2 + \cdots A_n$$

$$A_1 = 300{,}000\left(1 + \frac{4.5}{100}\right)$$
$$A_2 = 300{,}000\left(1 + \frac{4.5}{100}\right)^2$$
$$A_3 = 300{,}000\left(1 + \frac{4.5}{100}\right)^3$$
$$A_3 = 300{,}000\ \left(1 + \frac{4.5}{100}\right)^4$$

$$A = 300{,}000(1 + 0.045) + 300{,}000\ (1 + 0.045)^2 + \cdots 300{,}000\ (1 + 0.045)^4$$

$$A = 300{,}000(1.045 + 1.045)^2 + \cdots 1.045^4)$$

$$A = 300{,}000a\frac{(r^n - 1)}{r - 1}$$

$$a = 1.045$$

$$r = \frac{(1.045)^2}{1.045} = 1.045$$

$$A = \frac{300{,}000 \times 1.045(1.045^4 - 1)}{1.045 - 1}$$

$$A = 1341212.918$$

Example II (UNEB Question)

A Finance society gives out a compound interest of 8% per annum. Moses deposited £100 into a saving account at the beginning of each year. How much will his saving be worth after ten years?

Solution:

$$A = A_1 + A_2 + A_3 + \cdots A_{10}$$
$$A = 100\left(1 + \frac{8}{100}\right) + 100\left(1 + \frac{8}{100}\right)^2 + \cdots 100\left(1 + \frac{8}{100}\right)^{10}$$
$$A = 100(1.08 + 1.08^2 + \cdots 1.08^{10})$$
$$A = 100\frac{a(r^n - 1)}{r - 1}$$
$$A = \frac{100 \times 1.08(1.08^{10} - 1)}{1.08 - 1}$$
$$A = 1564.55$$

€1564.5548746

Example III

5 millions are invested each year at a rate of 15% interest. In how many years will it accumulate to more than 50 millions.

$$A = A_1 + A_2 + A_3 + \cdots A_n$$
$$A = 5(1.15 + 1.15^2 + \cdots 1.15^n)$$
$$a = 1.15$$
$$r = \frac{1.15^2}{1.15} = 1.15$$

$$A = \frac{5a(r^n - 1)}{r - 1}$$

$$A = \frac{5 \times 1.15(1.15^n - 1)}{1.15 - 1}$$

$$\Rightarrow \frac{5 \times 1.15(1.15^n - 1)}{1.15 - 1} > 50.$$

$$1.15^n > \frac{53}{23}$$

$$\log_{10} 1.15^n > \log_{10}\left(\frac{53}{23}\right).$$

$$n \log_{10} 1.15 > \left(\log_{10} \frac{53}{23}\right).$$

$$n \log_{10} 1.15 > \left(\log_{10} \frac{53}{23}\right)$$

$$n > \frac{\log_{10} \frac{53}{23}}{\log_{10} 1.15}$$

$$n > 5.97$$

$$n = 6.$$

Example IV

A man pays a premium of £100 at the beginning of every year to insurance company on understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? Give your answer correct to three significant figure.

Solution

$$A = 100\left(1 + \frac{5}{100}\right)^1 + 100\left(1 + \frac{5}{100}\right)^2 + \cdots 100\left(1 + \frac{5}{100}\right)^{15}$$

$$A = 100[(1.05 + 1.05^2 + \cdots 1.05^{15})]$$

$$A = \frac{100\ a\ (r^n - 1)}{r - 1}$$

$$A = \frac{100 \times 1.05(1.05^{15} - 1)}{1.05 - 1}$$

$$A = 2265.74917.$$

$$A = £2270\ (to\ 3\ significant\ figure)$$

Proof by induction of summation of other series

Mathematical induction is a method of proof in which a statement is proved for one step in a process and it’s shown that if the statement holds for that step, it holds for the next. Proof by induction is not a direct proof because the method doesn’t produce the expression it’s self but it can only be used to prove that a given expression is a required sum.

Mathematic proof by induction involves 3 steps

Step 1: (initial step) is to prove the given statement for all natural numbers.

Step 2: (induction step) prove the given statement for any natural number implies the given statement for the next natural numbers.

Step 3: (conclusion step).

If you have just covered induction with your teacher and you are feeling uneasy about the whole thing. If you are anything like me, you are having the same thought that I had. Can't you prove anything is true. If you assume it to be true in the first place, what is left is to prove.

If you have already proved isn't that cheating any how

Well………

First step don't get mad at your teacher.

Induction makes a perfect sense to him and he honestly thinks he has explained it clearly. When I was still at school like you I had a good math teacher so I had good lessons of induction from my teacher.

But still I didn't trust induction so I tried to pick up where my teacher had left me and fill in some gaps.

Before the lesson ended. I asked my teacher, '*excuse me teacher*'

Excuse me teacher, '*excuse me teacher, excuse me teacher. Mathematical proof by induction is confusing me because I cannot seem to reason with myself as to how to go about getting to the solution and also asked why is it called mathematical proof by induction not substitution induction*'.

Example 1

Prove by induction that

$1 + 2 + 3 + \cdots n = \frac{1}{2}n(n+1)$

Solution

$$1 + 2 + 3 + \cdots n = \frac{1}{2}n(n+1)$$

$$For\ n = 1$$

$$1 = \frac{1}{2}(1)(2)$$

$$1 = 1.$$

It is true for $n = 1$

Assume the result holds for some general value of n say $n = k$.

$$1 + 2 + 3 + \cdots K = \frac{1}{2}k(k+1)$$

It must also be true for the next value of n i.e

$$n = k + 1$$

Adding the next term, $n = k + 1$

$$\Rightarrow 1 + 2 + 3 + \cdots + k + k + 1 = \frac{1}{2}k(k+1) + k + 1$$

$$= (k + 1)[\tfrac{1}{2}k + 1]$$

$$= \frac{1}{2}(k+1)[k+2]$$

$$= \frac{1}{2}(k+1)(k+2)$$

$\Rightarrow 1 + 2 + 3 + \ldots + k + k + 1 = \frac{1}{2}(k + 1)(k + 2)$

For the next integer

$$n = k + 1 = 2.$$
$$k + 1 = 2.$$
$$k = 1.$$
$$\Longrightarrow 1 + 2 = \frac{1}{2}(1 + 1)(1 + 2)$$
$$3 = \frac{1}{2}(2)(3)$$
$$3 = 3.$$

Since it's true for $n = 1$, $n = 2$ and so on hence it's true for all positive integral values of n.

Example II

Prove by induction that

$1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{1}{6}n(n + 1)(2n + 1).$

Solution

$1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

$For\ n = 1$

$1 = \frac{1}{6}(1)(2)(3).$

$1 = 1$

It's true for $n = 1$.

Assume the result holds for the general value of n say n=k, it must be true for the next value of n i.e $n = k + 1$

$1^2 + 2^2 + 3^2 + \cdots k^2 = \frac{1}{6}k(k + 1)(2k + 1)$

Adding the next term $n = k + 1$

$$1^2 + 2^2 + 3^2 + \cdots k^2 + (k + 1)^2 = \frac{1}{6}(k)(k + 1)(2k + 1) + (k + 1)^2$$
$$= \frac{1}{6}(k + 1)[2k^2 + k + 6(k + 1)]$$
$$= \frac{1}{6}(k + 1)[2k^2 + 7k + 6]$$
$$= \frac{1}{6}(k + 1)[(k + 2)(2k + 3)]$$

$1^2 + 2^2 + 3^2 + \cdots k^2 + (k + 1)^2 = \frac{1}{6}(k + 1)(2k + 3)$

For the next integer

$n = k + 1 = 2$

$k = 1.$

$1^2 + 2^2 = \frac{1}{6}(2)(3)(5)$

$5 = 5$

Since it is true for $n = 1$ and $n = 2$ and so on, and so on hence it's true for all positive integral values of n.

Example III

Prove by induction that

$$a + ar + ar^2 + \cdots ar^{n-1} = \frac{a(1 - r^k)}{1 - r}$$

Solution

For $n = 1$.

$$a = a\frac{(1 - r)}{1 - r}$$

$$a = a$$

It's true for $n = 1$

Assume the result holds for the general value of n say $n = k$

$$a + ar + ar^2 + \cdots ar^{k-1} = a\frac{(1 - r^k)}{1 - r}$$

It must be true for the next integer

$n = k + 1$.

Adding the next term

$n = k + 1$

$$a + ar + ar^2 + \cdots ar^{k-1} + ar^k = a\frac{(1 - r^k)}{1 - r} + ar^k$$

$$= \frac{a(1 - r^k) + ar^k(1 - r)}{1 - r}$$

$$= \frac{a - ar^k + ar^k - ar^{k+1}}{1 - r}$$

$$= \frac{a - ar^{k+1}}{1 - r}$$

$$= \frac{a(1 - r^{k+1})}{1 - r}$$

$$a + ar + ar^2 + \cdots ar^{k-1} + ar^k = \frac{a(1 - r^{k+1})}{1 - r}$$

For the next integer

$$n = k + 1 = 2.$$

$$k = 1.$$

$$a + ar = \frac{a(1 - r^2)}{1 - r}$$

$$a + ar = \frac{a(1 + r)(1 - r)}{1 - r}$$

$$a + ar = a(1 + r)$$

$$a + ar = a + ar$$

Since it's true for $n = 1,\ n = 2,\ n = k$ so on hence it's true.

Example IV

Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$.

Solution

$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots \frac{1}{n(n+1)}$

$$\Rightarrow \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots \frac{1}{n(n+1)} = \frac{n}{n + 1}$$

For $n = 1$

$$\frac{1}{2}=\frac{1}{2.}$$

It's true for $n=1$.

Assume the result holds for general value of n, say $n=k$.

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots\frac{1}{k(k+1)}=\frac{k}{k+1}$$

It must be true for the next integer $n=k+1$

adding the next term $n=k+1$

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+2)}$$

$$\Rightarrow\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

For the next integer, $n=k+1=2, k=1$.

$$\frac{1}{2}+\frac{1}{6}=\frac{2}{3}.$$

$$\frac{3+1}{6}=\frac{2}{3}.$$

$$\frac{4}{6}=\frac{2}{3}.$$

Since it is true for $n=1, n=2$ and so on hence it's true for all positive integral value of n.

Example V

Prove by induction that $\sum_{m=1}^{n} m^3=\frac{1}{4}n^2(n+1)^2$

Solution

$$\sum_{m=1}^{n} m^3=1^3+2^3+3^3+\cdots n^3$$

$$\Rightarrow 1^3+2^3+3^3+\cdots n^3=\frac{1}{4}n^2(n+1)^2$$

$for\ n=1$

$$1^3=\frac{1}{4}(1^2)(2)^2$$

$$1=1.$$

It's true for $n=1$.

Assume the result holds for the general value of

$n=k$

$$1^3 + 2^3 + 3^3 + \cdots k^3 = \frac{1}{4}k^2(k+1)^2$$

It must be true for next integer $n = k + 1$

Adding the next term $n = k + 1$

$$1^3 + 2^3 + 3^3 + \cdots k^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$1^3 + 2^3 + 3^3 + \ldots k^3 + (k+1)^3 = = \frac{1}{4}(k+1)^2(k+2^2$$

$$1^3 + 2^3 = \frac{1}{4}(2)^2(3^2)$$

$$9 = 9$$

Since it true for $n = 1$, $n = 2$ and so on, it is true for all positive integral value of n.

Example VI

Prove by induction that

$$1 \times 2 + 2 \times 3 + \cdots n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

Solution

$$(1 \times 2) + (2 \times 3) + \cdots n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

For $n = 1$

$$1 \times 2 = \frac{1}{3}(1)(2)(3)$$

$$2 = 2.$$

Its true result holds for general value of n say
$n = k$.

$$(1 \times 2) + (2 \times 3) + \cdots k(k+1) = \frac{1}{3}k(k+1)(k+2)$$

It must be true for the next integer $n = k + 1$

$$(1 \times 2) + (2 \times 3) + \cdots k(k+1) + (k+1)(k+2)$$

$$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$

$$= \frac{1}{3}(k+1)(k+2)[k+3].$$

$$= \frac{1}{3}(k+1)(k+2)(k+3)$$

$$1 \times 2 + 2 \times 3 + \cdots k(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$$

For the next integer, $n = k + 1 = 2$, $k = 1$

$$1 \times 2 + 2 \times 3 = \frac{1}{3}(2)(3)(4).$$

$$8 = 8$$

Since it is true for $n = 1, n = 2$ and so on hence it's true for all positive integral values of n.

Example VII

Prove by induction that

$\frac{d}{dx}(x^n) = nx^{n-1}$.

Solution

$\frac{d}{dx}(x^n) = nx^{n-1}$.

For $n = 1$.

$$\frac{d}{dx}(x) = 1.$$

Assume the result holds for the general value of n say

$n = k$.

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}$$

It must be true for the next integer $n = k + 1$

$\frac{d}{dx}\left(x^{k+1}\right) = (k + 1)x^k$.

For the next integer, $n = k + 1 = 2, k = 1$

$$\frac{d}{dx}(x^2) = 2x$$

Since it is true for $n = 1, n = 2$, and so on, it is true for positive integral values of n.

Example VIII

Prove by induction.

$$1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(4n^2 - 1)$$

Solution

$$1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(4n^2 - 1)$$

For $n = 1$

$$1^2 = \frac{1}{3}(1)(3)$$

$$1 = 1$$

It is true for $n = 1$.

Assume the results holds for the general value of $n = k$.

$1^2 + 3^2 + 5^2 + \cdots \ldots .(2k - 1)^2 = \frac{1}{3}k(4k^2 - 1)$

It must be true for the next integer $n = k + 1$

Adding the next term $n = k + 1$

$1^2 + 3^2 + 5^2 + \cdots (2k - 1)^2 + (2k + 1)^2 = \frac{1}{3}k(4k^2 - 1) + (2k + 1)^2$

$$= \frac{1}{3}k(2k + 1)(2k - 1) + (2k + 1)^2$$

$$= \frac{1}{3}(2k + 1)[2k^2 - k + 3(2k + 1)]$$

$$= \frac{1}{3}(2k + 1)[(2k^2 + 5k + 3)]$$

$$= \frac{1}{3}(2k+3)(2k+1)(k+1)$$

$1^2 + 2^2 + \cdots (2k-1)^2 + (2k+1)^2$

$$= \frac{1}{3}(k+1)(2k+1)(2k+3)$$

For the next integer

$n = k + 1 = 2.$

$k = 1$

$1^2 + 3^2 = \frac{1}{3}(2)(3)(5)$

$10 = 10$

Since it is true for $n = 1, n = 2$ and so on hence it is true for all positive integral values of n.

Example IX

Prove by induction that $6^n - 1$ is divisible by 5 for all positive integral value of n

Solution

$let\ a_n = \frac{6^n - 1}{5}$

$for\ n = 1$

$a_1 = \frac{6-1}{5} = 1.$

It's true for $n = 1$

Assume it holds for the general value of n say $n = k$.

$a_k = \frac{6^k - 1}{5}$

$5a_k = 6^k - 1$ ………………… (i)

It must be true for the next integer $n = k + 1$

$a_{k+1} = \frac{6^{k+1} - 1}{5}$

$5a_{k+1} = 6^{k+1} - 1$ ……………..…. (ii)

$eqn(ii) - eqn(i)$

$$5a_{k+1} - 5_{ak} = \left(6^{k+1} - 1\right) - \left(6^k - 1\right)$$

$$= 6^{k+1} - 6^k$$

$$= 6^k(6^1 - 1)$$

$$= 5(6^k)$$

For the next integer

$n = k + 1 = 2$

$k = 1.$

$5a_2 - 5a_1 = 5 \times 6$

$5a_2 - 5 \times 1 = 30.$

$5a_2 = 35.$

$a_2 = 7$

Since it is true for $n = 1$, n = 2 and so on, it is true for all the positive integral values of n.

Example X

Prove by induction that $4^n - 1$ is divisible by 3 for all positive integral value of n.

Solution

$$a_n = \frac{4^n - 1}{3}$$
$for\ n = 1$
$$a_1 = \frac{4^1 - 1}{3} = 1$$
It is true for $n = 1$

Assume the result holds for general value of n say $n = k$
$$a_k = \frac{4^k - 1}{3}$$
$3a_k = 4^k - 1$ …………………… (i)

It must be true for the next integer $n = k + 1$
$$a_{k+1} = \frac{4^{k+1} - 1}{3}$$
$3a_{k+1} = 4^{k+1} - 1$ ……………….. (ii)

$eqn\ (ii) - eqn(i)$
$$3a_{k+1} - 3a_k = \left(4^{k+1} - 1\right) - \left(4^k - 1\right)$$
$= 4^{k+1} - 4^k$
$= 4^k(4 - 1)$
$= 3\left(4^k\right).$

For the next integer, $n = k + 1 = 2, k = 1.$

$3a_2 - 3a_1 = 3(4^1).$
$3a_2 - 3 = 12$
$3a_2 = 15$
$a_2 = 5.$

Since it's true for $n = 1$ $n = 2, n = k$ and so on it's true for all positive integral values of n.

Example

Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all positive values of n.

Solution
$$a_n = \frac{8^n - 7n + 6}{7}$$
For $n = 1$
$$a_1 = \frac{8 - 7 + 6}{7} = 1$$

It's true for $n = 1$ for the general value of n say $n = k.$
$$a_k = \frac{8^k - 7k + 6}{7}$$
$7a_k = 8^k - 7k + 6$ ………………….. (1)

It must be true for the next integer $n = k + 1$
$$a_{k+1} = \frac{8^{k+1} - 7(k + 1) + 6}{7}$$
$7a_{k+1} = 8^{k+1} - 7k - 7 + 6.$
$$7a_{k+1} = 8^{k+1} - 7k - 1 \text{………………….. (2)}$$
$eqn\ (2) - eqn\ (1).$

$7a_{k+1-}7a_k = \left(8^{k+1} - 7k - 1\right) - \left(8^k - 7k + 6\right)$

$= 8^{k+1} - 8^k - 7$

$= 8^k.8 - 8^k - 7.$

$= 8^k(8 - 1) - 7$

$= 7\,8^k - 7$

$= 7\left(8^k - 1\right)$

$7(a_{k+1} - a_k) = 7\left(8^k - 1\right)$

For the next integer

$n = k + 1 = 2, k = 1$

$7(a_2 - a_1) = 7(8^1 - 1).$

$a_2 - a_1 = 7$

$a_2 = 7 + a_1$

$a_2 = 7 + 1$

$a_2 = 8$

Since it is true for $n = 1, n = 2$ and so on it's true for all positive integral values of n.

Example **(UNEB Question)**

Prove by induction that $2^n + 3^{2n-3}$is always divisible by 7 for $n \geq 2$

Solution

$$a_n = \frac{2^n + 3^{2n-3}}{7}$$

$for\ n = 2$

$$a_2 = \frac{2^2 + 3}{7} = 1$$

$it's\ true\ for\ n = 1.$

Assume the result holds for the general value of n say $n = k$

$$a_k = \frac{2^k + 3^{2k-3}}{7}$$

$7a_k = 2^k + 3^{2k-3}$ … ………………….. (1)

For the next integer, $n = k + 1$

$$a_{k+1} = \frac{2^{k+1} + 3^{2k-1}}{7}.$$

$7a_{k+1} = 2^{k+1} + 3^{2k-1}$ … ………………. (2)

Eqn (2) – Eqn (1)

$7a_{k+1} - 7a_k = \left(2^{k+1} + 3^{2k-1}\right) - \left(2^k + 3^{2k-3}\right)$

$= 2^k.2 + 3^{2k-3}.3^2 - 2^k - 3^{2k-3}$

$= 2\left(2^k\right) - 2^k + 3^2\left(3^{2k-3}\right) - 3^{2k-3}$

$= 2^k(2 - 1) + 3^{2k-3}(9 - 1)$

$= 2^k + 8(3^{2k-3})$

$= 2^k + 3^{2k-3} + 7\,3^{2k-3}$

$= 7a_k + 7 3^{2k-3}$

$= 7\left(a_k + 3^{2k-3}\right)$

$7(a_{k+1} - a_k) = 7\left(a_k + 3^{2k-3}\right)$

For the next integer $n = k + 1 = 3$, $k = 2$

$7(a_3 - a_2) = 7(a_2 + 3)$

$a_3 - a_2 = a_2 + 3$
$a_3 = 2a_2 + 3$
$a_3 = 2 + 3$
$a_3 = 5$

Since it is true for $n = 2,\ n = 3\ n = k$ and so on it's true for all positive of n.

Example XI

Prove by induction that $8^n - 6^n$ is always divisible by 7 for all even integers of n.

Solution

$$a_n = \frac{8^n - 6^n}{7}$$

For $n = 2$, $a_2 = \frac{8^2 - 6^2}{7} = \frac{64 - 36}{7} = 4$

Assume the result holds for the general even integer $n = k$

$$a_k = \frac{8^k - 6^k}{7}$$

$7a_k = 8^k - 6^k$. ……………… (1)

For the next even integer, $a_{k+2} = \dfrac{8^{k+2} - 6^{k+2}}{7}$

$7a_{k+1} = 8^{k+1} - 6^{k+1}$ ………... (2)

Eqn (2) – Eqn (1)

$7(a_{k+2} - a_k) = 8^{k+2} - 6^{k+2} - 8^k + 6^k$
$7(a_{k+2} - a_k) = 8(8^2 - 1) - 6^k(6^2 - 1)$
$= 8^k(63) - 35(6^k)$
$a_{k+2} - a_k = 9(8^k) - 5(6^k)$
$a_{k+2} = a_k + 9(8^k) - 5(6^k)$

For the next integer $n = k + 2 = 4$, $k = 2$

$a_4 = a_2 + 9(8^2) - 5(6^2)$
$a_4 = 4 + 396$
$a_4 = 400$

From $a_n = \dfrac{8^n - 6^n}{7}$, $a_4 = \dfrac{8^4 - 6^4}{7} = 400$

Since it is true for $n = 2$, $n = 4$, and so on, hence it is true for all positive even integers.

Revision Exercise

1. Which of the following series are arithmetical progressions? Write down the common difference of those that are.
   (i) $7 + 8\frac{1}{2} + 10 + 11\frac{1}{2} + \ldots$
   (ii) $-2 - 5 - 8 - 11 + \ldots$
   (iii) $1 + 1.1 + 1.2 + 1.3 + \ldots$
   (iv) $1 + 1.1 + 1.1 + 1.11 + \ldots$
   (v) $\frac{1}{2}, + \frac{5}{6} + \frac{7}{6} + \frac{3}{2} + \ldots$
   (vi) $1^2 + 2^2 + 3^2 + 4^2 + \ldots$
   (vii) $n + 2n + 3n + 4n + \ldots$
   (viii) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

2. Write down the first term and common difference of each of the following arithmetic progressions.
   (a) 8 + 11 + 14 + 17 + …
   (b) 23 + 25 + 27 + 291 + …
   (c) 19 + 16 + 13 + 10 + …
   (d) $13\frac{1}{2}$ + 15 + $16\frac{1}{2}$ + 18 + …
3. Find the sums of the following arithmetic progressions
   (a) 4 + 11 + … to 16^{th} term
   (b) 3 + $8\frac{1}{2}$ + … to 20^{th} term
   (c) 19 + 13 + … to 10^{th} term.
   (d) -9 – 1 + … to 8^{th} term.
4. Write down the terms indicated in each of the following Arithmetic progressions.
   (a) 3 + 11 + …, 10^{th} and 19^{th}
   (b) 8 + 5 + …, 15^{th}, 31^{st} term.
   (c) $\frac{1}{4}+\frac{7}{8}+...$ 12^{th}, n^{th}
   (d) 50 + 48 + … , 100^{th}, n^{th}
5. Find the 18^{th} term of a series that has an n^{th} term given by $(2 + 3n)$
6. Find the sum of the arithmetic progression -7 – 3 + 1 + … from the 7^{th} to the 13^{th} term inclusive.
7. Find the number of terms in the following arithmetic progressions:
   (a) 2 + 4 + 6 + … + 46
   (b) 50 + 47 + 44 + … + 14
   (c) 2 + 4 + … + $4n$
   (d) $a + (a + d) + … + l$
8. Find the 31^{st} term of a series that has an n^{th} term given by $\frac{1}{3}(10+2n)$
9. Find the sum of all odd numbers between 0 and 500 which are divisible by 7.
10. The 2^{nd} term of an arithmetic progression is 15 and the 15^{th} is 21. Find the common difference, the first term and the sum of the first 10 terms.
11. Find the 5^{th} and 7^{th} terms of a series that has an n^{th} term given by $(-1)^n(2n + 1)$
12. Show that the sum $1 + 3 + 5 + … + (2n – 1)$ is always a perfect square.
13. The 4^{th} term of an arithmetic progression is 18, and the common difference is -5. Find the 1^{st} term and the sum of the 1^{st} 16 terms.
14. Find an expression for the n^{th} term of each of the following arithmetic progression and use your answer to write down the 100^{th} term of each series.
    (a) 5 + 8 + 11 + 14 + …
    (b) 5 + 2 + -1 – 4 + …
    (c) 12½ + 16 + 19½ + 23 + …
15. A piece of string of length 5m is cut into n pieces in such way that the lengths of the piece are in an arithmetic progression. If the lengths of the longest and shortest pieces are 1m and 25 cm respectively, calculate n.
16. Find the difference between the sums of the first 10 terms of the A.P whose first terms are 12 and 8 and whose common differences are respectively 2 and 3.
17. Find the sum of each of the following arithmetic progressions:
    (a) 2 + 4 + 6 + 8 + 10 + … 146
    (b) 100 + 95 + 90 + 85 + 80 + … + 20
    (c) 4 + 10 + 16 + 22 + 28 + … +334
    (d) 5¼ + 4½ + 3¾ + … + -3
18. The 10^{th} term of an arithmetic progression is 10 and the sum of the first 10 terms is -35. Find the first term and the common difference of the progression.
19. The first term of an arithmetic progression is -12, and the last term is 40. If the sum is 196, find the number of terms and the common difference.

20. In an arithmetic progression, u_5 = -0.5 and S_7 = 21. Find a, d, and u_6.
21. The sum of the first four terms of an arithmetic progression is twice the 5th term. Show that the common difference is equal to the first term.
22. Find the sum of the even number divisible by 3 lying between 400 and 500.
23. The sum of the first ten terms of an arithmetic progression is 120 and the sum of the first 20 terms is 840. Find the sum of the first 30 terms.
24. Find the arithmetic mean of:
    (a) 3 and 27
    (b) 3 and -27
    (c) $\frac{1}{3}$ and $\frac{1}{27}$
    (d) log 3 and log 27
25. Show that the sum of the integers from 1 to n is $\frac{1}{2}n(n+1)$.
26. An arithmetic progression has a common difference, d. If the sum to 20 terms is 25 times the first term, find in terms of d, the sum to 30 terms.
27. Three numbers in an arithmetic progression have sum 33 and product 1232. Find the numbers.
28. Show that the sum of the first n terms of the arithmetic progression with first term a and common difference d is $\frac{1}{2}n(2a+(n-1)d)$.
29. In an arithmetic progression, a = -61 and d = 4. Find the least value of n such that $S_n > 0$.
30. An arithmetic progression has first term -5 and common difference 1.5. Find the greatest number of terms the arithmetic progression can have given that the sum of the terms does not exceed 450.

Answers

1. (a) 1½ (b) -3 (c) 0.1 (d) $\frac{1}{3}$ (g) n
2. (a) 8.3 (b) 23, 2 (c) 19, -3 (d) 13½, 1½
3. (a) 904 (b) 1188 (c) 88 (d) 193½
4. (a) 75, 147 (b) -34, -82 (c) $7\frac{1}{8}$, $\frac{1}{8}(5n-3)$
   (d) -148, $(52 - 2n)$
5. 56 6. 1512
7. (a) 23 (b) 13 (c) $2n$ (d) $\frac{(l-a)}{(d+1)}$
8. 24 9. 9072 10. 2, 13, 220
11. 13, -15 13. 33, -72
14. (a) $2 + 3n$, 302
    (b) $8 - 3n$, -292
    (c) ½$(18 + 7n)$, 359
15. 8 16. 5
17. (a) 5402 (b) 1000
    (c) 9464 (d) 13½
18. -17, 3 19. 14, 4
20. 13½, -3½, -14½
22. 7650 23. 2160
24. (a) 15 (b) -15 (c) $\frac{5}{27}$ (d) log 9
26. $1575d$ 27. 8, 11, 14
29. 32 30. 28.

Revision Exercise two

1. For each of the following G.Ps, state the common ratio and the next two terms.
   (a) $4 + 20 + 100 + 500 + \ldots$
   (b) $24 + 12 + 6 + \ldots$
   (c) $45 + 15 + 5 + \ldots$
2. Write down the terms indicated in each of the following G.Ps. Do not simplify your answer.
   (a) $5 + 10 + \ldots$, 11^{th}, 20^{th}
   (b) $10 + 25 + \ldots$, 7^{th}, 19^{th}
   (c) $\frac{2}{3} + \frac{3}{4} + \ldots$, 12^{th}, n^{th}
   (d) $3 - 2 + \ldots$, 8^{th}, n^{th}
3. Find the sum of the following G.Ps
   (a) $100 + 10 + \ldots$ to 7 terms
   (b) $1 - \frac{1}{3} + \ldots$ to 6 terms
   (c) $3 - 6 + \ldots$ to n terms
   (d) $a^p + a^{p+3} + a^{p+6} + \ldots$ to k terms
4. Using the formula $S_n = \frac{a(r^n - 1)}{(r - 1)}$, find S_5 and S_6 for the G.P 18, -9, 4½, … and hence deduce the value of U_6.
5. Find the number of terms in the following geometric progressions:
   (a) $81 + 27 + 9 + \ldots + 1/27$
   (b) $0.03 + 0.06 + 0.12 + \ldots -1\frac{11}{16}$
   (c) $\frac{8}{81} - \frac{4}{27} + \frac{2}{9} - \ldots - 1\frac{11}{16}$
   (d) $5 + 10 + 20 + \ldots + 5 \times 2^n$
   (e) $a + ar + ar^2 + \ldots + ar^{n-1}$
6. Find the distinct numbers p and q such that p, q, 10 are in arithmetic progression and p, q, 10 are in geometric progression.
7. Find the value of the common ratio of the G.P that has a third term equal to 6and 8^{th} term equal to 1458.
8. The third term of a geometric progression is 10 and the 6^{th} term is 80. Find the common ratio, the first term and the sum of the first six terms.
9. Find the geometric mean of:
   (a) 3 and 27
   (b) $\frac{1}{3}$ and $\frac{1}{27}$
   (c) 10^3 and 10^{27}
10. In a geometric progression, the 7^{th} term equals 8 and the 9^{th} term equals 18. Find the possible values of the common ratio.
11. The third term of a G.P is 2 and the fifth is 18. Find two possible values of the common ratio and the second term in each case.
12. Given that the geometric mean of the numbers $4x - 3$ and $9x + 4$ is $6x - 1$, find the value of x.
13. Find the sum of the first ten terms of a G.P that has a sixth term $\frac{32}{33}$ and a seventh term of $1\frac{21}{33}$.
14. The three numbers $n - 1$, n, $n + 3$ are consecutive terms of a geometric progression. Find n and the term after $n + 3$.
15. If the sum of the first two terms of a G.P is 162 and the sum of its first four terms is 180, find the sum of the first 6 terms, Find also the possible values of the sixth term..

16. The geometric mean of two numbers a and b ($b > a$) is equal to four fifth of the arithmetic mean of the two numbers. If $a = 6$, find the value of b.
17. A man starts saving on 1st April. He saves 1p the first day, 2p the second, 4p the third and so on. If he managed to keep on saving under this system until the end of the month (30 days), how much would he have saved?
18. The sum of the first six terms of a G.P is nine times the sum of the first 3 terms. Find the common ratio.
19. Prove that the arithmetic mean of two different numbers exceeds the geometric mean of the same two numbers.
20. Show that the sum of the series $4 + 12 + 36 + 108 + \dots$ to 20 terms is greater than 3×10^9.
21. The sum of $(n + 12)$ terms of the G.P $2 + 4 + 8 + \dots$ is twice the sum of n terms of the G.P $2 + 12 + 48 + \dots$. Calculate the value of n.
22. A geometric series has first term 5 and common ratio 3. Find the least number of terms the series can have if its series exceeds 2000.
23. Find the ratio of the sum of the first 10 terms of the series $\log x + \log x^2 + \log x^4 + \log x^8 + \dots$ to the first term.
24. A geometric series has first term 35 and common ratio 2^x. State the set of values of x for which the series is convergent. Find the value of x for which the sum to infinity of the series is 40.
25. The sum of the first two terms of a G.P is 9 and the sum to infinity of the G.P is 25. If the G.P has a positive common ratio r, find r and the first term.
26. The 2nd, 4th, 8th terms of an A.P are in geometric progression and the sum of the third and fifth terms is 20. Find the first four terms of the progression.
27. For each of the following geometric series, find the range of values of x for which the sum to infinity of the series exists.
    (a) $x + x^2 + x^3 + x^4 + \dots$
    (b) $1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \dots$
    (c) $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots$
    (d) $\left(x + \frac{1}{2}\right) + \left(x + \frac{1}{2}\right)^2 + \left(x + \frac{1}{2}\right)^3 + \dots$
28. S is the sum of n terms of a geometric progression, P is the product of the n terms and R is the sum of the reciprocals of the terms. Prove that $\left(\frac{S}{R}\right)^n = P^2$.
29. Three unequal numbers a, b, c are such that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetical progression. Prove that b, a, c are in arithmetical progression.
30. Prove that the G.P $1 + \frac{2x}{3 + x^2} + \left(\frac{2x}{3 + x^2}\right)^2 + \dots$ is convergent for all values of x and find the limit of its sum.
31. Prove by induction that:
    (i) $3^{3n+2} + 5^{n+1}$ is divisible by 3
    (ii) $n^3 + 2n$ is divisible by 3.
    (iii) $4^{n+1} + 5^{2n-1}$ is a multiple of 21.
    (iv) $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Answers

1. (a) 5, 2500, 12500

(b) $\frac{1}{2}$, 3, $1\frac{1}{2}$

(c) $\frac{1}{3}$, $1\frac{2}{3}$, $\frac{5}{9}$

2. (a) 5×2^{10}, 5×2^{19}

(b) $10(\frac{5}{2})^6$, $10(\frac{5}{2})^{18}$

(c) $\frac{2}{3}(\frac{9}{8})^{11}$, $\frac{2}{3}(\frac{9}{8})^{n-1}$

3. (a) 111.1111

(b) $\frac{182}{243}$

(c) $1 - (-2)^n$

(d) $\frac{a^p(a^{3k}-1)}{a^3-1}$

4. $12\frac{3}{8}$, $11\frac{13}{16}$, $\frac{-9}{16}$

5. (a) 8 (b) 7 (c) 8 (d) $n + 1$ (e) n

6. -5, 2½

7. 3 8. 2, 2½, 167½

9. (a) 9 (b) $\frac{1}{9}$ (c) 10^5

10. ±1½ 11. ±3, $\pm\frac{2}{3}$

12. 13 13. 31

14. 6, 13½ 15. 182, ½, -1

16. 24 17. £1,070,000

18. 2 21. 12

22. 7 23. 1023

24. $x < 0$ 25. $\frac{4}{5}$, 5

26. 2½, 5, 7½, 10

27. (a) $|x| < 1$ (b) $|x| < 3$

(c) $|x| > 1$ (d) $\frac{-3}{2} < x < \frac{1}{2}$

COMPLEX NUMBERS

A complex number is represented by an expression of the form $a + b\boldsymbol{i}$ where a and b are real numbers and **i** is a symbol with a property $\boldsymbol{i^2} = -1$.

$\boldsymbol{i} = \sqrt{-1}$ was introduced by a Swiss mathematician Euler. Traditionally the letters Z and W are used to stand for complex numbers.

Given a complex numbers $z = a + b\boldsymbol{i}$.

The real part of a complex number z is $Re(z) = a$ and the imaginary part of z is $Im(z) = b$.

Both $Re(z)$ and $Im(z)$ real numbers.

Thus the real part of $\boldsymbol{Z} = 4 - 3\boldsymbol{i}$ is $Re(w) = 4$ and imaginary part of $\boldsymbol{Z}$ is $\text{Im}(Z) = -3$

By identifying the real number $\boldsymbol{a}$ with a complex number $a + oi$ we consider ℝ (real numbers) to be subset of ℂ (complex numbers).

Consider the equation $x^2 + 9 = 0$, this can be written as $x^2 = -9$ and we can see that the equation has no real roots since we cannot find the real root of a negative number, But with $\boldsymbol{i}^2 = -1$ (Euler) we are able to find the square root of complex numbers.

$$x^2 = -9$$
$$x^2 = 9\boldsymbol{i^2}$$
$$\sqrt{\boldsymbol{x^2}} = \sqrt{9\,\boldsymbol{i}^2}$$
$$x = \pm 3\boldsymbol{i}$$
$$x = 3\boldsymbol{i} \qquad \boldsymbol{x} = -3\boldsymbol{i}$$

Example

Solve the following equations

(a) $4x^2 + 49 = 0$

(b) $x^2 + 2x + 6 = 0$

Solution

$$4x^2 + 49 = 0$$
$$4x^2 = -49$$
$$x^2 = -\frac{49}{4}$$
$$x^2 = \frac{49}{4}\boldsymbol{i}^2$$
$$x = \sqrt{\frac{49}{4}\boldsymbol{i}^2}$$
$$x = \pm\frac{7}{2}\boldsymbol{i}$$

(b) $x^2 + 2x + 6 = 0$

From, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{\left(-2 \pm \sqrt{(2)^2 - 4(1)(6)}\right)}{2 \times 1}$$

$$x = \frac{-2 \pm \sqrt{-20}}{2}$$
$$x = \frac{-2 \pm \sqrt{4\boldsymbol{i}^2 \times 5}}{2}$$
$$x = \frac{-2 \pm 2\boldsymbol{i}\sqrt{5}}{2}$$
$$x = -1 + \boldsymbol{i}\sqrt{5}$$
$$x = -1 - \boldsymbol{i}\sqrt{\mathbf{5}}$$

With this new concept we are in position to find the roots of any quadratic equation.

When the imaginary part of a complex number is zero, the complex number becomes a real number. Thus, all real numbers are complex numbers.

Definition
Given a complex number $\boldsymbol{z} = x + \boldsymbol{i}y$, the complex conjugate of **Z** denoted by $\bar{z}$ *or* z^* is a complex number given by $\bar{z} = x - \mathbf{i}y$. Therefore if $z = 4 + 3\boldsymbol{i},\ w = -2 + 4\boldsymbol{i}$

Then $\bar{z} = 4 - 3\mathbf{i}$, $\bar{w}$ -2 – 4**i**

Algebra of complex numbers

1. Addition

Given that two complex numbers
$z_1 = x_1 + \boldsymbol{i}y_1,\ z_2 = x_2 + \boldsymbol{i}y_2$. Then
$z_1 + z_2 = x_1 + \boldsymbol{i}y_1 + x_2 + \boldsymbol{i}y_2$
$= x_1 + x_2 + \boldsymbol{i}(y_1 + y_2)$

Therefore if $z_1 = 3 + 5\boldsymbol{i}$ and $z_2 = 2 - 7\boldsymbol{i}$

$$z_1 + z_2 = 3 + 5\boldsymbol{i} + 2 - 7\boldsymbol{i}$$
$$= (3 + 2) + 5\boldsymbol{i} - 7\boldsymbol{i}$$
$$= 5 - 2\boldsymbol{i}$$

Example

1. Subtraction:

$$z_1 = x_1 + \mathbf{i}y_1$$
$$z_2 = x_2 + \mathbf{i}y_2$$
$$z_1 - z_2 = (x_1 + \mathbf{i}y_1) - (x_2 + \mathbf{i}y_2)$$
$$= x_1 - x_2 + \mathbf{i}y_1 - iy_2$$
$$= (x_1 - x_2) + \mathbf{i}(y_1 - y_2)$$

$$z_1 = 4 - 3\boldsymbol{i}$$
$$z_2 = 6 - 14\boldsymbol{i}$$

Find $(z_1 - z_2)$

$$(z_1 - z_2) = (4 - 3\boldsymbol{i}) - (6 - 14\boldsymbol{i})$$
$$= 4 - 6 - 3\boldsymbol{i} + 14\boldsymbol{i}$$
$$= -2 + 11\boldsymbol{i}$$

2. Multiplication

$$z_1 = x_{\mathbf{1}} + \boldsymbol{i}y_{\mathbf{1}},\quad z_2 = x_2 + \boldsymbol{i}y_2$$

$$z_1 z_2 = (x_1 + \boldsymbol{i}y_1)(x_2 + \boldsymbol{i}y_2)$$
$$= x_1 x_2 + x_1 y_2 \boldsymbol{i} + y_1 x_2 \boldsymbol{i} + i^{\mathbf{2}} y_1 y_2$$
$$= x_1 x_2 - y_1 y_2 + (y_1 x_2 + x_1 y_2)\boldsymbol{i}$$

Example

$z_1 = 3 + 5\boldsymbol{i}$, $\quad z_2 = 2 - 7\boldsymbol{i}$

Find $z_1 z_2$

Solution

Find $z_1 z_2 = (3 + 5i)(2 - 7i)$

$$= 3(2 - 7i) + 5i(2 - 7i)$$
$$= 6 - 21i + 10i - 35i^2$$
$$= 6 + 35 + (10 - 21)i$$
$$= 41 - 11i$$

3. Division

$$z_1 = x_1 + \boldsymbol{i}y_1 \; and \; z_2 = x_2 + iy_2$$
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1(x_2 - iy_2)}{x_2 + iy_2(x_2 - iy_2)}$$
$$\frac{z_1}{z_2} = \frac{x_1 x_2 - x_1 y_2 i + x_2 y_1 i - \boldsymbol{i}^2 y_1 y_2}{(x_2)^{\mathbf{2}} - \boldsymbol{i}^{\mathbf{2}} {y_2}^{\mathbf{2}}}$$
$$= \frac{x_1 x_2 + y_1 y_2 + (x_2 y_1 - x_1 y_2)i}{{x_2}^2 + {y_2}^2}$$
$$= \frac{x_1 x_2 + y_1 y_2}{{x_2}^2 + {y_2}^2} + \frac{(x_2 y_1 - x_1 y_2)i}{{x_2}^2 + {y_2}^2}$$

Example I

Simplify $z = \frac{2+6i}{3-i}$

$$z = \frac{2 + 6i}{3 - i} = \frac{2 + 6i(3 + i)}{(3 - i)(3 + i)}$$
$$= \frac{2(3 + i) + 6i(3 + i)}{3^2 - i^2}$$
$$= \frac{6 + 2i + 18i + 6i^2}{10}$$
$$= \frac{6 + 20i - 6}{10}$$
$$= \frac{0 + 20i}{10}$$
$$= 2i$$

Example II

Express $\frac{-1+2i}{1+3i}$ in the form *a* + *b***i**

Solution

$$\frac{-1 + 2i}{1 + 3i} = \frac{-1 + 2i(1 - 3i)}{1 + 3i(1 - 3i)}$$

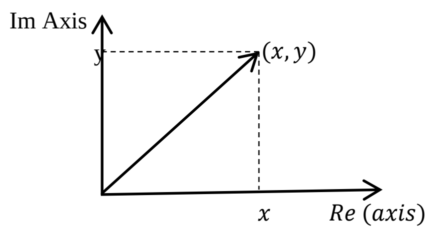
$$= \frac{-1 + 3i + 2i - 6i^2}{(1)^2 - (3i)^2}$$
$$= \frac{5i + 5}{10}$$
$$= \frac{5 + 5i}{10}$$
$$= \frac{1}{2} + \frac{1}{2}i$$

The Argand Diagram

Complex numbers can be represented graphically on a graph of Real (Re) and Imaginary (Im) axes called a **complex plane**. The complex plane is similar to the Cartesian plane where the imaginary axis corresponds to the *y*-axis and the real axis corresponds to the x-axis. The diagram representing the complex number in complex plane is called an **argand diagram** named after JR argand 1806.

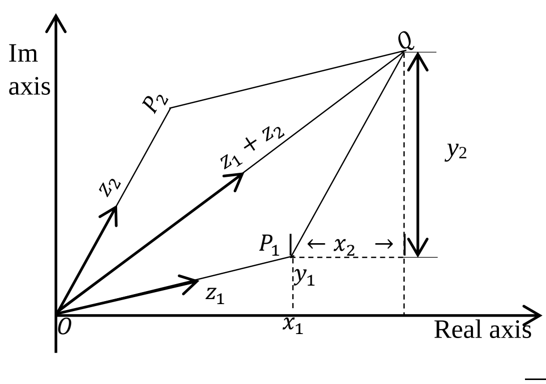
On the argand diagram a complex number is represented by a line with an arrow on the head to show direction

If $z = x + iy$ we can represent z on argand diagram as shown below.



If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$

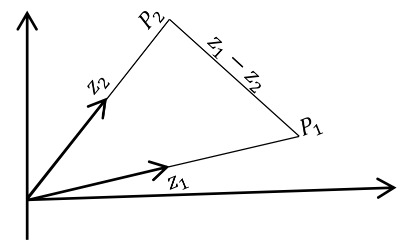


z_1, z_2 *and* $z_1 + z_2$ is represented by vectors $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$ *and* $\overrightarrow{OQ}$ respectively. The diagram shows that $\overrightarrow{P_1Q}$ is equal $\overrightarrow{OP_2}$ in magnitude and direction

$$\overrightarrow{OQ} = \overrightarrow{OP_1} + \overrightarrow{P_1Q} = \overrightarrow{OP_1} + \overrightarrow{OP_2}$$

Thus the sum of two complex numbers $z_1$and z_2 is represented in the argand diagram by the sum of the corresponding vectors $\overrightarrow{OP_1}$ *and* $\overrightarrow{OP_2}$

Representing $z_1 - z_2$ on the argand diagram.



$(z_1 - z_2) = OP_1 - OP_2$

$= \overrightarrow{P_1P_2}$

Since $\overrightarrow{OP_1} - \overrightarrow{OP_2} = \overrightarrow{P_1P_2}$

$z_1 - z_2$ can be represented by P_1P_2

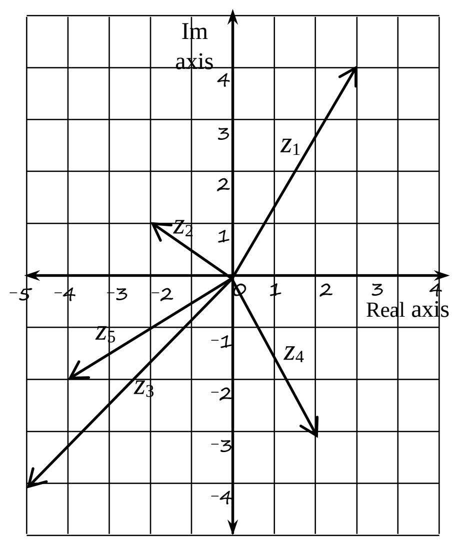
Example

Represent the following complex numbers on the argand diagram.

$z_1 = 3 + 4i,\quad z_2 = -2 + i, z_3 = -5 - 4i,$

$z_4 = 2 - 3i,\qquad z_5 = -4 - 2i,$

Solution



Modulus of a complex number

Given a complex number $z = x + iy$, the magnitude or length of z is denoted be $|z|$ is defined by

$$|z| = \sqrt{x^2 + y^2}$$

Example I

Given $z = 1 + \sqrt{3}i$ find $|z|$

Solution

$z = 1 + (\sqrt{3})i$

$|z| = \sqrt{(1)^2 + (\sqrt{3})^2}$

$= \sqrt{4}$

$= 2$

Example II

Find $|z|$ if $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution

$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$|z| = \sqrt{(\frac{1}{2})^2 + (\frac{-\sqrt{3}}{2})^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= \sqrt{1}$

$= 1$

$\Rightarrow |z| = 1$

Example III

$z = -3 + 4i$ find $|z|$

Solution

$z = -3 + 4i$

$|z| = \sqrt{(-3)^2 + (4)^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5$

Properties of modulus

If z_1 and z_2 are complex numbers then

(i) $|z_1 z_2| = |z_1||z_2|$

(ii) $\left|\frac{z_1}{z_1}\right| = \frac{|z_1|}{|z_2|}$

Example I

$z_1 = 5 - 12i$ and $z_2 = 3 - 4i$

Find $|z_1 z_2|$ and $\left|\frac{z_1}{z_2}\right|$

Solution

$z_1 = 5 - 12i$, $z_2 = 3 - 4i$

$$|z_1 z_2| = |z_1||z_2|$$

$\Rightarrow |(5 - 12i)(3 - 4i)|$

$= |5 - 12i||3 - 4i|$

$$= \sqrt{(5)^2 + (-12)^2}\sqrt{(3)^2 + (-4)^2}$$

$= \sqrt{169} \times \sqrt{25}$

$= 13 \times 5$

$= 65$

Alternatively

$$z_1 z_2 = (5 - 12i)(3 - 4i)$$

$$= 15 - 20i - 36i + 48i^2$$
$$= 15 - 48 - 56i$$
$$= -33 - 56i$$
$$|z_1 z_2| = \sqrt{(-33)^2 + (-56)^2}$$
$$= 65\ units$$

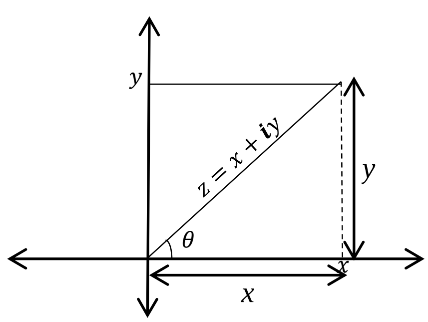
$$z_1 = 5 - 12i, \quad z_2 = 3 - 4i$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{(5)^2 + (-12)^2}}{\sqrt{(3)^2 + (-4)^2}}$$
$$= \frac{13}{5}$$

Alternatively, $\frac{z_1}{z_2} = \frac{5-12i}{3-4i}$

$$\frac{(5 - 12i)(3 + 4i)}{(3 - 4i)(3 + 4i)}$$
$$\frac{z_1}{z_2} = \frac{15 + 20i - 36i - 48i^2}{(3)^2 - (4i)^2}$$
$$= \frac{63 - 16\text{i}}{9 + 16}$$
$$= \frac{63}{25} - \frac{16}{25}i$$
$$\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{63}{25}\right)^2 + \left(-\frac{16}{25}\right)^2}$$
$$\left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(63)^2 + (16)^2}{25^2}}$$
$$= \frac{65}{25} = \frac{13}{5}$$

Argument of a complex number *Z* (arg Z)

The argument of a complex number z is defined to be the angle (θ) which the complex number *z* makes with the positive *x*-axis.



From the diagram above,

$$\tan\theta = \frac{y}{x} \Longrightarrow \theta = tan^{-1}\left(\frac{y}{x}\right)$$

Note: For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of $360°$.

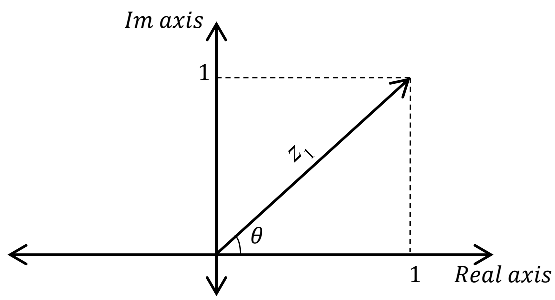
To avoid confusion we usually work with the value of θ for which $-\pi < \theta < \pi$ or $-180 < \theta < 180$. This is called the principle argument of z denoted by **arg z**.
In practice the formula $\tan\theta = \frac{y}{x}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Which is often used to find the principal argument of a complex number z, despite the fact that it tends to two possible values for θ in the permitted range. The formula is necessary but not sufficient to help us obtain the arg z. The correct value of arg z is chosen with the aid of a sketch.

Example
Find the principal argument of the following complex number
$(a)\ 1+i \quad (b) -1-i\sqrt{3} \quad (c) -5$
$(d) -\sqrt{3}+i \quad (e)\sqrt{3}-i$

Solution
Consider $z_1 = 1 + i$



$$\theta = tan^{-1}\left(\frac{1}{1}\right) = 45°$$

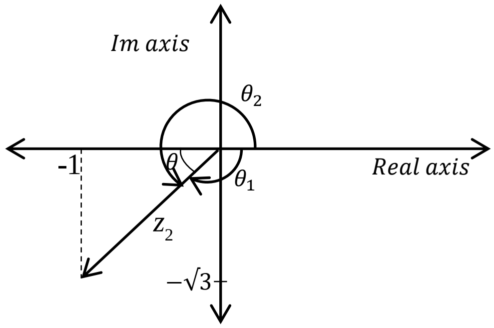
Since $180° = \pi$ radians

$$\theta = \frac{45\pi}{180} = \frac{\pi}{4}$$

$$\Longrightarrow \arg z_1 = 45°$$

$$\arg z_1 = \frac{45\pi}{180} = \frac{\pi}{4}$$

(b) Let $z_2 = -1 - i\sqrt{3}$



$$\tan \quad = \frac{\sqrt{3}}{1}$$

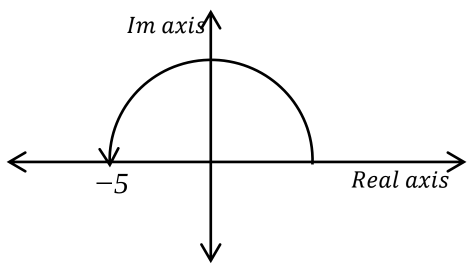
$$\theta = tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$\theta = 60°$$

$$\arg z_2 = \theta_1$$

$$\Longrightarrow \arg z_2 = -120°$$

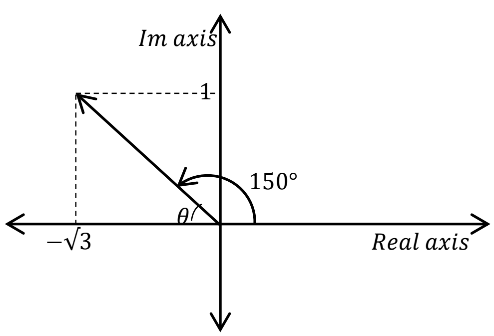
OR $\arg z_2 = -\frac{2}{3}\pi$



$$z_3 = -5 + 0i$$

$\arg z_3 = 180°$ or $\arg z_3 = \pi$

(d) Let $z_4 = -\sqrt{3} + i$



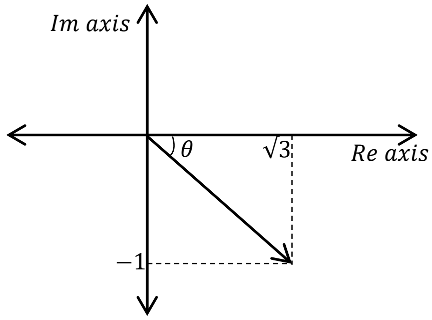
$\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$

$z_4 = -\sqrt{3} + i$

$\arg z_4 = 150°$, from the sketch above

(e) $\sqrt{3} - i$

$z_5 = \sqrt{3} - i$



$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$

$$\arg z_5 = -30° \text{ from the above diagram}$$

Properties of Arguments

Given the two complex numbers z_1 and z_2 then

$$\mathbf{arg}(\boldsymbol{z_1 z_1}) = \mathbf{arg}\,\boldsymbol{z_1} + \mathbf{arg}\,\boldsymbol{z_2}$$

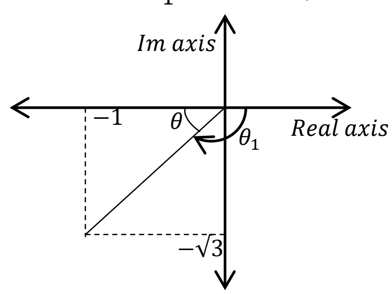
$$\mathbf{arg}\left(\frac{\boldsymbol{z_1}}{\boldsymbol{z_2}}\right) = \mathbf{arg}\,\boldsymbol{z_1} - \mathbf{arg}\,\boldsymbol{z_2}$$

Example I

Given that $z_1 = -1 - i\sqrt{3}$ and $z_2 = 1 + i$. Find the $\arg(z_1 z_2)$ *and* $\arg\left(\frac{z_1}{z_2}\right)$

Solution

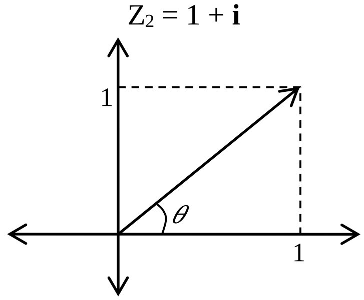
$z_1 = -1 - i\sqrt{3}$



$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$$

$$\arg z_1 = \theta_1 = -120°$$

$Z_2 = 1 + \mathbf{i}$



$$\arg z_2 = tan^{-1}\left(\frac{1}{1}\right) = 45°$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

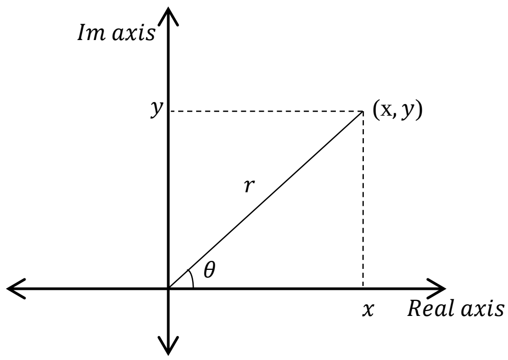
$= -120 + 45°$
$= -75°$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

$= -120 - 45$
$= -165$

Modulus–argument form of a complex number

(*Polar form of a complex number*)



Consider a complex number $z = x + iy$ making an angle θ with the positive $x -$ axis

$$\arg z = \theta$$

From the diagram above $\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$$x = r\cos\theta \quad r\sin\theta = y$$

$$z = x + iy$$

$$z = r\cos\theta + ir\sin\theta$$

$$z = r\ (\cos\theta + i\sin\theta)$$

(modulus argument form a complex number)

Where $r = |z| = \sqrt{x^2 + y^2}$

Example

Express the following complex numbers in modulus –argument

a) $5 + 5i\sqrt{3}$
b) $\sqrt{2} + i$
c) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$
d) $-3\sqrt{2} + 3\sqrt{2}i$
e) $-5i$
f) $-5 - 12i$

Solutions

$z_1 = 5 + 5i\sqrt{3}$

$r = \sqrt{(5)^2 + \left(5\sqrt{3}\right)^2}$

$= \sqrt{25 + 75}$
$= 10$

$\arg z_1 = tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) = 60°$

$z_1 = 5 + 5i\sqrt{3} = 10(\cos 60 + i \sin 60)$

(b) $z_2 = \sqrt{2} + i$

$|z_2| = r = \sqrt{\left(\sqrt{2}\right)^2 + (1)^2}$

$= \sqrt{3}$

$\arg z_2 = tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.3°$

$z_2 = r(\cos\theta + i \sin\theta$

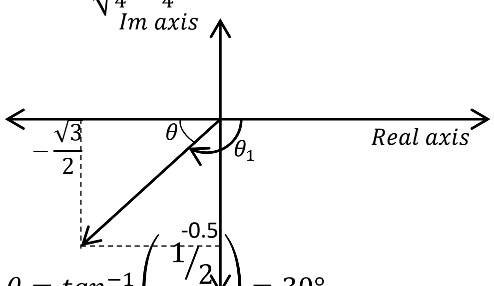
$\sqrt{3}[\cos 35.3 + i \sin 35.3]$

(c) $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$|z_3| = \sqrt{\left(\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2\right)}$

$= \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$



$\theta = tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = 30°$

$z_3 = \frac{-\sqrt{3}}{2} - \frac{1}{2}\pi$

$\arg z_3 = -150°$

$z_3 = r(\cos\theta + i \sin\theta)$

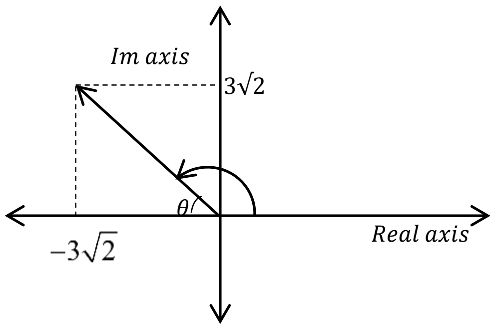
$z_3 = 1(\cos -150 + i \sin -150)$

(d) $z_4 = -3\sqrt{2} + \left(3\sqrt{2}\right)\mathrm{i}$

$|z_4| = \sqrt{\left(-3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2}$

$= \sqrt{36}$

$= 6$



$\theta = tan^{-1}\left(\frac{3\sqrt{2}}{3\sqrt{2}}\right)$

$\theta = 45°$

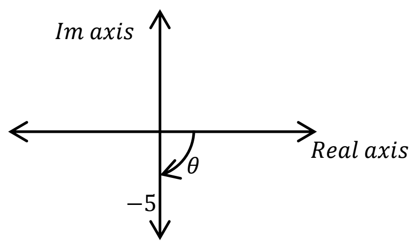
$\arg(z_4) = +135°$

$z_4 = 6(\cos 135 + i \sin 135)$

(e) $z_5 = -5i = 0 + -5i$

$|z_5| = \sqrt{0^2 + (-5)^2}$

$|z_5| = 5$



$\arg z_5 = -90$

$z_5 = 5(\cos -90 + i \sin -90)$

$z_6 = 3 + 4i$

$r = |z_6| = \sqrt{(3)^2 + (4)^2}$

$= \sqrt{25}$

$= 5$

$\theta = tan^{-1}\left(\frac{4}{5}\right) = 53.1$

$z_6 = 5(\cos 53.1° + i \sin 53.1°)$

(g) $z_6 = -5 - 12i$

$|z_6| = \sqrt{(-5)^2 + (-12)^2}$

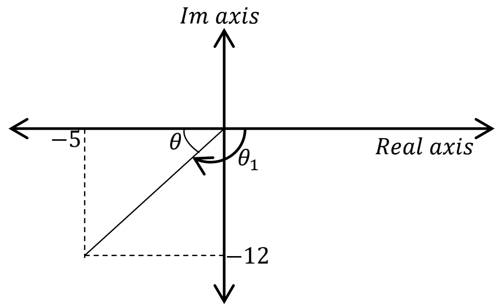
$= \sqrt{169}$

$= 13$

$\theta = tan^{-1}\left(\frac{12}{5}\right)$

$\theta = 67.4$

$z_7 = -5 - 12i$



$\arg z_7 = -112.6°$

$|z_7| = \sqrt{(-5)^2 + (-12)^2}$

$= \sqrt{25 + 144}$

$= \sqrt{169}$

$= 13$

$13(\cos -112.6 + i \sin -122.6)$

Example II

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Show that

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

And $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

Solution

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 + \cos\theta_1 (i\sin\theta_2) + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)]$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$= r_1 r_2[(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))]$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\cos\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_1(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{cos^2\theta_2 + sin^2\theta_2}\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}$$

(as required)

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$

Example III

Given that $z_1 = 1 + i$

$$z_2 = \sqrt{3} - i$$

Find in polar form $z_1 z_2$ $and \frac{z_1}{z_2}$

Solution

$$z_1 = 1 + i$$
$$|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\arg z_1 = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$
$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$z_2 = \sqrt{3} - i$$
$$|z_1| = r_2 = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$
$$= \sqrt{4}$$
$$= 2$$
$$\arg z_2 = -30°$$
$$= -\frac{\pi}{6}\ radians$$
$$\arg z_2 = -\frac{\pi}{6}$$
$$z_2 = 2\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)$$
$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$
$$= 2\sqrt{2}(\cos\left(\frac{\pi}{4} + \ -\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{-\pi}{6}\right)$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$
$$= \frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{4} - \frac{-\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{-\pi}{6}\right)\right]$$
$$\frac{\sqrt{2}}{2}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$$

Demoivre's Theorem

Demoivres theorem states that for real values of n

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

Proving Demoivre's theorem by mathematical induction

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

For *n* =1, $(\cos\theta + i\sin\theta)^1 = (\cos\theta + i\sin\theta)$

It's true for *n* =1

Assume the results holds for the general value of *n*=*k*

$$(\cos\theta + i\sin\theta)^k = (\cos k\theta + i\sin k\theta)$$

It must be true for the next integer $n = k + 1$

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k (\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin\theta\sin k\theta$$

$$= [(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)]$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$\Longrightarrow (\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

For the next integer , $n = k + 1 = 2$

$\Longrightarrow k = 1$

$$(\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta)$$

Since it's true for n=1, n = 2 and so on it's true for all positive integral values of n.

Example I

Find the value of $\left(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi\right)^{12}$

Solution

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\left(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi\right)^{12} =$$

$$\left(\cos\frac{1}{4}\pi \times 12 + i\sin\frac{1}{4}\pi \times 12\right)$$

$$= \cos 3\pi + i\sin 3\pi$$

$$= -1$$

Example II

Express $\left(1 - i\sqrt{3}\right)^4$ in the form $a + bi$

Solution

$\left(1 - i\sqrt{3}\right)^4$

Let $z = 1 - i\sqrt{3}$

$$|z| = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2} = 2$$

$$\arg z = -60$$

$$= -\frac{\pi}{3}$$

$$\arg z = -\frac{\pi}{3}$$

z = r(cos θ + $i$sinθ)

$$z = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3})\right)$$

$$z^4 = 2^4\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)^4$$
$$= 16\left(\cos \tfrac{-4\pi}{3} + i\sin \tfrac{-4\pi}{3}\right)$$
$$= 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= -8 + 8\sqrt{3i}$$

Example III

Evaluate $\frac{1}{\left(1-i\sqrt{3}\right)^3}$

Solution:

$$\frac{1}{1 - i\sqrt{3}} = \left(1 - i\sqrt{3}\right)^{-3}$$

Let $z = \left(1 - i\sqrt{3}\right)$

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2}$$
$$= 2$$

$$\arg z = -\frac{\pi}{3}$$

$$\left(1 - i\sqrt{3}\right) = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)$$

$$\left(1 - i\sqrt{3}\right)^{-3} = 2^{-3}\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)^{-3}$$
$$= \frac{1}{8}\left(\cos -\frac{\pi}{3} \times -3 + i\sin -\frac{\pi}{3} \times -3\right)$$
$$= \frac{1}{8}(\cos +\pi + i\sin +\pi)$$
$$= -\frac{1}{8}$$

Example IV

Express $\sqrt{3} + i$ in modulus –argument form. Hence find

$$(\sqrt{3} + i)^{10} \text{ and } \frac{1}{\left(\sqrt{3} + i\right)^7} \text{ in the form of } a + bi$$

Solution

Let $z = \sqrt{3} + i$

$$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1} = 2$$

$$\arg z = \frac{\pi}{6}$$

$$z = 2\left(\cos \frac{\pi}{6} + i\sin \frac{\pi}{6}\right)$$

$$\left(\sqrt{3} + i\right)^{10} = 2^{10}\left(\cos\left(\frac{10\pi}{6}\right) + i\sin\left(\frac{10\pi}{6}\right)\right)$$
$$= 2^{10}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= \frac{1024}{2} - \frac{i(1024)\sqrt{3}}{2}$$

$$= 512 - 512\sqrt{3}i$$

$$\frac{1}{(\sqrt{3}+i)^7} = (\sqrt{3}+i)^{-7}$$

$$= \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^{-7}$$

$$= 2^{-7}\left(\cos -7\times\frac{\pi}{6} + i\sin -7\times\frac{\pi}{6}\right)$$

$$= \frac{1}{128}\left(\cos\frac{-7\pi}{6} + i\sin\frac{-7\pi}{6}\right)$$

$$= \frac{1}{128}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= -\frac{\sqrt{3}}{256} + \frac{1}{256}i$$

$$= -\frac{\sqrt{3}}{256} + \frac{1}{256}i$$

Example V

Express (-1+i) in modulus – argument form. Hence show that $(-1+i)^{16}$ is real and that $\frac{1}{(-1+i)^6}$ is pure imaginary.

Solution

$z = -1 + i$

$|z| = \sqrt{(-1)^2 + 1^2}$

$= \sqrt{2}$

$\arg z = 135°$

$z = \sqrt{2}(\cos 135 + i\sin 135)$

$z^{16} = (\sqrt{2})^{16}(\cos 135\times 16 + i\sin 135\times 16)$

$= 256(\cos 2160 + i\sin 2160)$

$= 256(1)$

$= 256$

$\Longrightarrow (-1+i)^{16} = 256$ So it is purely real

As required

$$\frac{1}{(-1+i)^6} = (-1+i)^{-6}$$

$z^{-6} = (\sqrt{2})^{-6}(\cos 135\times -6 + i\sin 135\times -6)$

$$= \frac{1}{8}(0+i) = \frac{1}{8}i$$

$\Longrightarrow z^{-6}$ is purely imaginary.

Example VI

a) $(\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^3$

b) $\frac{1}{(\cos\theta+i\sin\theta)^2}$

c) $\frac{\cos\theta+i\sin\theta}{(\cos\theta+i\sin\theta)^4}$

d) $\frac{\left(\cos\frac{\pi}{17}+i\sin\frac{\pi}{17}\right)^8}{\left(\cos\frac{\pi}{17}-i\sin\frac{\pi}{17}\right)^9}$

e) $\frac{(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)}{(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2})}$

f) $\frac{\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^8}{\left(\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}\right)^3}$

Solutions

(a) $(\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^3$

$= (\cos\theta + i\sin\theta)^{2+3}$

$= (\cos\theta + i\sin\theta)^5$

$= (\cos 5\theta + i\sin 5\theta)$

(b) $\frac{1}{(\cos\theta+i\sin\theta)^2}$

$= (\cos\theta + i\sin\theta)^{-2}$

$= \cos -2\theta + i\sin -2\theta$

$= \cos 2\theta - i\sin 2\theta$

(c) $\frac{\cos\theta+i\sin\theta}{(\cos\theta+i\sin\theta)^4}$

$= (\cos\theta + i\sin\theta)^1(\cos\theta + i\sin\theta)^{-4}$

$= (\cos\theta + i\sin\theta)^{1+-4}$

$= (\cos\theta + i\sin\theta)^{-3}$

$= \cos -3\theta + i\sin -3\theta$

$= (\cos 3\theta - i\sin 3\theta)$

(d) $\frac{(\cos\frac{\pi}{17}+i\sin\frac{\pi}{17})^8}{(\cos\frac{\pi}{17}-i\sin\frac{\pi}{17})^9}$

$\frac{(\cos\pi+i\sin\pi)^{8/17}}{(\cos\pi+i\sin\pi)^{-9/17}}$

$(\cos\pi+i\sin\pi)^{\frac{8}{17}-\frac{-9}{17}}$

$(\cos\pi+i\sin\pi)^1$

(e) $\frac{(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)}{(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2})}$

$\frac{(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)^2}{(\cos\theta+i\sin\theta)^{1/2}}$

$$\frac{(\cos\theta + i\sin\theta)^3}{(\cos\theta + i\sin\theta)^{1/2}}$$

$$(\cos\theta + i\sin\theta)^{5/2}$$

$$\cos\tfrac{5}{2}\theta + i\sin\tfrac{5}{2}\theta$$

(f) $\frac{\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^8}{\left(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right)^3}$

$$\frac{\left(\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^2\right)^8}{\left(\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^{-3}\right)^3}$$

$$\frac{\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^{16}}{\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^{-9}}$$

$$\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^{16--9}$$

$$\cos\left(25 \times \frac{\pi}{5}\right) + i\sin\left(25 \times \frac{\pi}{5}\right)$$

$$= (\cos 5\pi + i\sin 5\pi)$$

Example VII

Use De-moivre's theorem to show that

$$\tan 3\,\theta = \frac{3\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta}$$

Solution

$$(\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$$

$$but\ (\cos\theta + i\sin\theta)^3 =$$

$$= cos^3\theta + 3(i\sin\theta)cos^2\theta + 3(i\sin\theta)^2\cos\theta + (i\sin\theta)^3$$

$$= (\cos^3\theta - 3\sin^2\theta\cos\theta) +$$

$$i(3\sin\theta\cos^2\theta - \sin^3\theta)$$

$$= \cos 3\theta + i\sin 3\theta$$

Equating real to real and imaginary to imaginary;

$$\Rightarrow \sin 3\theta = 3\sin\theta\, cos^2\theta - sin^3\theta \ldots .. (1)$$

$$\cos 3\theta = \cos^3\theta - 3sin^2\theta\cos\theta \ldots\ldots\ldots\ldots..\ldots (2)$$

Eqn (1) ÷ Eqn (2)

$$\Rightarrow \tan 3\theta = \frac{3\sin\theta\, cos^2\theta - sin^3\theta}{cos^3\theta - 3sin^2\theta\cos\theta}$$

$$\tan 3\theta = \frac{\frac{3\sin\theta\, cos^2\theta}{\cos^3\theta} - \frac{\sin^3\theta}{cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\sin^2\theta\cos\theta}{\cos^3\theta}}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^2\theta}{1 - 3\tan^2\theta}$$

Example VIII
Use Demovre's theorem to show that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Solution

$(\cos\theta + i\sin\theta)^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$

$= \cos^4\theta + (4\cos^3\theta\sin\theta)i - 6\,cos^2\theta\,sin^2\theta - (4\cos\theta\sin^3\theta)\,i + \sin^4\theta$

$= (\cos 4\,\theta + i\sin 4\theta)$

Equating real to real and imaginary to imaginary

$\cos 4\theta = \cos^4\theta - 6\,cos^2\theta sin^2\theta + \sin^4\theta \;\ldots.(i)$

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \;\ldots\ldots\ldots(ii)$

Eqn (ii) ÷ Eqn (i)

$$\tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\,cos^2\theta sin^2\theta + \sin^4\theta}$$

$$\tan 4\theta = \frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta} - \frac{6cos^2\theta\,sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6tan^2\theta + \tan^4\theta}$$

Example IX
Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$
$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Hence show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Solution

$$z = \cos\theta + i\sin\theta$$
$$z^n = (\cos\theta + i\sin\theta)^n$$
$$= (\cos n\theta + i\sin n\theta)$$
$$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos -n\theta + i\sin -n\theta$$
$$= \cos n\theta - i\sin n\theta$$
$$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$
$$= 2\cos n\theta$$
$$z^n - \frac{1}{z^n}$$

$$= (\cos\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$$
$$= 2i\sin n\theta$$
$$\text{from } z^n + \frac{1}{z^n} = 2\cos n\theta$$
$$z + \frac{1}{z} = 2\cos\theta$$
$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$
$$z - \frac{1}{z} = 2i\sin\theta$$
$$z + \frac{1}{z} = 2\cos\theta$$
$$\left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4$$

But $(z + \frac{1}{z})^4$

$$z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$
$$\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 = \left(z + \frac{1}{z}\right)^4$$
$$2\cos 4\,\theta + 4(2\cos 2\theta) + 6 = (2\cos\theta)^4$$

$16\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$

$$\cos^4\theta = \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6)$$
$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

Example XI

Given that $z = \cos\theta + i\sin\theta$ show that

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Hence or otherwise show that

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Solution

$$z^n - \frac{1}{z^n} = 2i\sin\theta$$
$$z - \frac{1}{z} = 2i\sin\theta$$
$$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5$$
$$\left(z - \frac{1}{z}\right)^5 = i^5(32)\sin^5\theta$$
$$\left(z - \frac{1}{z}\right)^5 = (32i \times i^4\sin^5\theta)$$
$$\left(z - \frac{1}{z}\right)^5 = 32i\sin^5\theta$$

$$\text{but } \left(z-\frac{1}{z}\right)^5 = z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2$$
$$+10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$
$$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$z^n - \frac{1}{z^n} = 2i\sin n\theta$

$z^5 - \frac{1}{z^5} = 2i\sin 5\theta$

$z - \frac{1}{z} = 2i\sin\theta$

$$\left(z-\frac{1}{z}\right)^5 = (2i\sin\theta)^5$$
$$2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta) = 32i\sin^5\theta$$
$$\sin^5\theta = \frac{1}{32}(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta)$$
$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Example XII

Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$

Solution

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$
$$z + \frac{1}{z} = 2\cos\theta$$
$$\left(z+\frac{1}{z}\right)^6 = (2\cos\theta)^6$$
$$\left(z+\frac{1}{z}\right)^6 = 64\cos^6\theta$$

But $\left(z+\frac{1}{z}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z}\right)^2 + 20z^3\left(\frac{1}{z}\right)^3 + 15z^2\left(\frac{1}{z}\right)^4 + 6z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$

$$= \left(z^6 + \frac{1}{z^6}\right) + \left(6z^4 + \frac{6}{z^4}\right) + \left(15z^2 + \frac{15}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ … … … … … … … … … . (1)

$\left(z-\frac{1}{z}\right) = 2i\sin\theta$

$\left(z-\frac{1}{z}\right)^6 = 64i^6\sin^6\theta$

$\left(z-\frac{1}{z}\right)^6 = -64\sin^6\theta$

But

$$\left(z-\frac{1}{z}\right)^6 = z^6 + 6z^5\left(-\frac{1}{z}\right) + 15z^4\left(-\frac{1}{z}\right)^2 + 20z^3\left(-\frac{1}{z}\right)^3 + 15z^2\left(-\frac{1}{z}\right)^4 + 6z\left(-\frac{1}{z}\right)^5 + \left(-\frac{1}{z}\right)^6$$

$$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$= 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$\Rightarrow 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20 = -64\sin^6\theta$$

$$-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Eqn (2) – Eqn (1)

$$\Rightarrow 64\cos^6\theta - -64\sin^6\theta = 24\cos 4\theta + 40$$

$$\cos^6\theta + \sin^6\theta = \frac{8}{64}(3\cos 4\theta + 5)$$

$$\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$$

Solving Complex Equations

Given that *x* and *y* are real numbers. Find the values of *x* and *y* which satisfy the equation.

$$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$$

Solution

$$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$$

$$\frac{2y+4i}{2x+y} = \frac{y}{x-i}$$

$$\frac{2y+4i}{2x+y} = \frac{y}{x-i}\times\frac{x+i}{x+i}$$

$$\frac{2y+4i}{2x+y} = \frac{xy+iy}{x^2+1}$$

$$\frac{2y}{2x+y} + \frac{4i}{2x+y} = \frac{xy}{x^2+1} + \frac{yi}{x^2+1}$$

Equating real to real and imaginary to imaginary

$$\Rightarrow \frac{2y}{2x+y} = \frac{xy}{x^2+1} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{4}{2x+y} = \frac{y}{x^2+1} \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

From equation (1)

$$2y(x^2+1) = xy(2x+y)$$

$$2x^2y + 2y = 2x^2y + xy^2$$

$$2y - xy^2 = 0$$

$$y(2-xy) = 0$$

$$y = 0 \ \ or\ xy = 2$$

From Eqn (2), $\frac{4}{2x+y} = \frac{y}{x^2+1}$

$4x^2 + 4 = 2xy + y^2$

For $y = 0$, $4x^2 + 4 = 0$

$x^2 + 1 = 0$

$x^2 = -1$

$x^2 = i^2$

$x = \pm i$

For $xy = 2$, $y = \dfrac{2}{x}$

$$\Rightarrow 4x^2 + 4 = 4 + \frac{4}{x^2}$$

$$4x^2 - \frac{4}{x^2} = 0$$

Let $x^2 = m$

$$4m - \frac{4}{m} = 0$$

$4m^2 - 4 = 0$

$m^2 - 1 = 0$

$(m + 1)(m - 1) = 0$

$m = 1$, $m = -1$

When $m = 1$, $x^2 = 1 \Rightarrow x = \pm 1$

When $m = -1$, $x^2 = i^2 \Rightarrow x = \pm i$

$xy = 2$

If $x = 1$, $y = 2$

If $x = -1$, $y = -2$

If $x = i$, $y = -2i$

If $x = -i$, $y = 2i$

Example II

Find the values of x and y in

$$\frac{x}{2+3i} - \frac{y}{3-2i} = \frac{6+2i}{1+8i}$$

Solution

$$\frac{x}{2+3i} - \frac{y}{3-2i} = \frac{6+2i}{1+8i}$$

$$\frac{x(2-3i)}{2+3i(2-3i)} - \frac{y(3+2i)}{(3-2i)(3+2i)} = \frac{(6+2i)(1-8i)}{(1+8i)(1-8i)}$$

$$\frac{(2x-3xi)}{13} - \frac{(3y+2yi)}{13} = \frac{(6-48i+2i+16)}{65}$$

$$\frac{2x-3y}{13} - \frac{3x+2y}{13}i = \frac{22}{65} - \frac{46}{65}i$$

$$\Rightarrow \frac{2x-3y}{13} = \frac{22}{65}$$

$$5(2x-3y) = 22$$

$$10x - 15y = 22 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly, $\frac{3x+2y}{13} = \frac{46}{65}$

$$5(3x+2y) = 46$$

$$15x + 10y = 46 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . . (2)$$

Solving eqn (1) and eqn (2) simultaneously

$$\Longrightarrow x = 2.8 \quad y = 0.4$$

Example III

Find the values of *x* and *y* if $\frac{x}{1+i} + \frac{y}{2-i} = 2 + 4i$

Solution

$$\frac{x}{1+i} + \frac{y}{2-i} = 2 + 4i$$

$$\frac{x(1-i)}{(1+i)(1-i)} + \frac{y(2+i)}{(2-i)(2+i)} = 2 + 4i$$

$$\frac{x - xi}{2} + \frac{2y + yi}{5} = 2 + 4i$$

$$5(x - xi) + 2(2y + yi) = 2 + 4i$$

$$5x - 5xi + 4y + 2yi = 20 + 40i$$

Equating real to real and imaginary to imaginary;

$5x + 4y = 20$ …………… (1)

$2y - 5x = 40$ …………… (2)

Solving Eqn (1) and Eqn (2) simultaneously;

$$y = 10$$

$$x = -4$$

Example IV

Find the values of x and y. given that

$$\frac{xi}{1 + iy} = \frac{3x + 4i}{x + 3y}$$

Solution

$$\frac{xi(1 - iy)}{(1 + iy)(1 - iy)} = \frac{(3x + 4i)}{x + 3y}$$

$$\frac{xi + xy}{1 + y^2} = \frac{3x + 4i}{x + 3y}$$

$$\frac{xy}{1 + y^2} + \frac{xi}{1 + y^2} = \frac{3x}{x + 3y} + \frac{4i}{x + 3y}$$

$$\Rightarrow \frac{xy}{1 + y^2} = \frac{3x}{x + 3y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$$

$$\frac{x}{1 + y^2} = \frac{4}{x + 3y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From equation (1)

$$x^2y + 3xy^2 = 3x + 3xy^2$$

$$\Rightarrow x^2y = 3x$$

$$\Rightarrow x^2y - 3x = 0$$

$$x(xy - 3) = 0$$

$$x = 0 \quad or\ xy = 3$$

From eqn (2)

$$x^2 + 3xy = 4 + 4y^2 \ \ldots\ldots\ldots(3)$$

When *x* = 0, $0 = 4 + 4y^2$

$-1 = y^2$

$$y = \pm i$$

When $xy = 3$

$$y = \frac{3}{x}$$

Substituting $y = \frac{3}{x}$ and $xy = 3$ in Eqn (3);

$$x^2 + 3(3) = 4 + 4\left(\frac{3}{x}\right)^2$$

$$x^2 + 9 = 4 + \frac{36}{x^2}$$

$$x^2 + 9 = 4 + \frac{36}{x^2}$$

$$x^2 - \frac{36}{x^2} + 5 = 0$$

$$let\ x^2 = P$$

$$P - \frac{36}{P} + 5 = 0$$

$$P^2 - 36 + 5P = 0$$

$$P^2 + 5P - 36 = 0$$

$$(P + 9)(P - 4) = 0$$

$$(x^2 + 9)(x^2 - 4) = 0$$

$$x^2 - 4 = 0 \quad \Rightarrow \quad x = \pm 2$$

$$x = 2, \quad x = -2$$

When $x = 2, \quad y = \frac{3}{2}$

When $x = -2, \quad y = -\frac{3}{2}$

$$x^2 + 9 = 0$$

$$x^2 = -9$$

$$x = \pm 3i$$

$$when\ x = 3i$$

$$y = \frac{3}{3i} = \frac{1}{i}$$

$$y = -i$$

$$when\ x = -3i$$

$$y = +i$$

Example V

If z is a complex number such that $z = \frac{P}{2-i} + \frac{q}{1+3i}$. Where p and q are real. If $|z| = 7$, arg P = $\frac{\pi}{2}$. Find the value of p and q.

Solution

$$z = \frac{P}{2 - i} + \frac{q}{1 + 3i}$$

$$\mathrm{z} = \frac{\mathrm{P}(2 + \mathrm{i})}{(2 - \mathrm{i})(2 + \mathrm{i})} + \frac{q(1 - 3\mathrm{i})}{(1 + 3\mathrm{i})(1 - 3\mathrm{i})}$$

$$\mathrm{z} = \frac{2\mathrm{p} + \mathrm{Pi}}{5} + \frac{\mathrm{q} - 3\mathrm{qi}}{10}$$

$$z=\frac{2(2p+pi)+q-3qi}{10}$$
$$z=\frac{4p+2pi+q-3qi}{10}$$
$$z=\frac{4p+q+(2p-3q)i}{10}$$

$$\arg \quad = \tan^{-1}\left(\frac{\frac{2P-3q}{10}}{\frac{4P+q}{10}}\right)=\frac{\pi}{2}$$

$$tan^{-1}(\frac{2p-3q}{4P+q})=\frac{\pi}{2}$$

$$\frac{2p-3q}{4P+q}=\infty$$

$$4P+q=0$$
$$q=-4P$$
$$|z|=7$$
$$\sqrt{\left(\frac{4P+q}{10}\right)^2+\left(\frac{2P-3q}{10}\right)^2}=7 \text{ .... (1)}$$

Substituting q = -4p in Eqn (1)

$$\sqrt{0^2+\left(\frac{14p}{10}\right)^2}=7$$

$$\frac{14p}{10}=7$$
$$p=5$$
$$q=-4\times 5$$
$$q=-20$$

Example VI

Given that (1 + 5i)p – 2q = 3 + 7i, find p and q

(a) When p and q are real

(b) When p and q are conjugate complex numbers

Solution

(a) $(1+5i)P-2q=3+7i$

$P+5Pi-2q=3+7i$

$P-2q+5Pi=3+7i$

$P-2q=3$ .... (1)

$5P=7$ .... (2)

From Eqn (2),

$$P=\frac{7}{5}$$

$$\Rightarrow \frac{7}{5}-2q=3$$

$$\frac{7}{5}-3=2q$$

$-\frac{8}{5} = 2q$

$-\frac{8}{10} = q$

$q = -\frac{4}{5}$

$\Rightarrow p = \frac{7}{5}, \quad q = -\frac{4}{5}$

(b) Let $p = x + iy$

$q = x - iy$

$(1 + 5i)(x + iy) - 2(x - iy) = 3 + 7i$

$x + iy + 5xi - 5y - 2x + 2yi = 3 + 7i$

$(x - 5y - 2x) + (y + 5x + 2y)i = 3 + 7i$

$(-x - 5y) + (3y + 5x)i = 3 + 7i$

$-x - 5y = 3$

$x = -3 - 5y$ .... (1)

$3y + 5x = 7$ .... (2)

Substituting Eqn (1) in Eqn (2)

$3y + 5(-3 - 5y) = 7$

$3\text{y} - 15 - 25\text{y} = 7$

$-22\text{y} = 22$

$y = -1$

$x = -3 - 5(-1)$

$x = -3 + 5$

$x = 2$

$p = x + iy$

$p = 2 - i$

$q = 2 + i$

Square root of Complex Numbers

Example I

Find the square root of $35 - 12i$

Solution

Let $\sqrt{35 - 12i} = x + iy$

$\left(\sqrt{35 - 12i}\right)^2 = (x + iy)^2$

$35 - 12i = x^2 + 2xyi + i^2y^2$

$35 - 12i = x^2 - y^2 + 2xyi$

$\Rightarrow x^2 - y^2 = 35$

$2xy = -12$

$xy = -6$

$y = -\frac{6}{x}$

$x^2 - \frac{36}{x^2} = 35$

$x^4 - 36 = 35x^2$

$x^4 - 35x^2 - 36 = 0$
Let $x^2 = m$
$m^2 - 35m - 36 = 0$
$(m - 36)(m + 1) = 0$
$(x^2 - 36)(x^2 + 1) = 0$

But x is real
$\Rightarrow x^2 - 36 = 0$
$x = \pm 6$
When $x = 6$ $\quad y = -\frac{6}{6}$
$y = -1$
$when\, x = -6, \quad y = 1$
$\Rightarrow \sqrt{35 - 12\mathrm{i}} = 6 - \mathrm{i}$
or $\sqrt{35 - 12i} = -6 + i$

Example VIII
Find the square root of $5 - 12i$
solution
Let $\sqrt{5 - 12i} = x + iy$

$$5 - 12i = (x + iy)^2$$
$$5 - 12i = x^2 + 2xyi + yi^2$$
$$5 - 12i = x^2 - y^2 + 2xyi$$

Equating real to real and imaginary to imaginary;

$$\Rightarrow x^2 - y^2 = 5 \text{ ………………………(1)}$$
$$2xy = -12$$
$$xy = -6$$
$$y = -\frac{6}{x} \text{ ………………………(2)}$$

Substituting Eqn (2) in Eqn (1)

$$x^2 - \frac{36}{x^2} = 5$$
$$(x^2)^2 - 36 = 5x^2$$
$$let\ m = x^2$$
$$m^2 - 36 = 5m$$
$$m^2 - 5m - 36 = 0$$
$$(m - 9)(m + 4) = 0$$
$$(x^2 - 9)(x^2 + 4) = 0$$
$$x^2 = 9$$
$$x = \pm 3$$
$$when\ x = 3, \ \ y = -2$$
$$when\ x = -3, \quad y = 2$$
$$\sqrt{5 - 12i} = 3 - 2i$$
$$or\ \sqrt{5 - 12i} = 3 + 2i$$

Example IX
Find the roots of $z^2 - (1 - i)z + 7i - 4 = 0$

Solution

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{(1-i) \pm \sqrt{(1-i)^2 - 4(1)(7i-4)}}{2 \times 1}$$

$$z = \frac{1 - i \pm \sqrt{1 - 2i - 1 - 28i + 16}}{2}$$

$$\frac{1 - i \pm \sqrt{16 - 30i}}{2}$$

But $\sqrt{16 - 30i} = a + bi$

$16 - 30i = a^2 + 2ab\mathrm{i} - b^2$

$a^2 - b^2 = 16$

$2ab = -30$

$ab = -15$

$a = \frac{-15}{b}$

$\left(\frac{15}{b}\right)^2 - b^2 = 16$

$\frac{225}{b^2} - b^2 = 16$

Let $m = b^2$

$\frac{225}{m} - m = 16$

$m^2 + 16m - 225 = 0$

$m = 9, \quad m = -25$

$b^2 = 9$

$b = \pm 3$

$ab = -15$

$a = 5$

When $b = -3, \quad a = 5$

When $b = 3, \quad a = -5$

$a + bi = 5 - 3i, \; -5 + 3i$

$\sqrt{16 - 30i} = \pm(5 - 3i)$

$$z = \frac{1 - i \pm (5 - 3i)}{2}$$

$$z = 3 - 2i$$

$$z = -2 + i$$

Example X

Show that $1 + 2i$ is a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$

Solution

$$z = 1 + 2i$$

$$z^2 = (1 + 2i)^2$$

$$= 1 + 4i + 4i^2$$

$$= -3 + 4i$$

$$z^3 = z \times z^2 = (1 + 2i)(-3 + 4i)$$

$$= -3 + 4i - 6i - 8$$

$$= -11 - 2i$$

$2z^3 - z^2 + 4z + 15$

$\Rightarrow 2(-11 - 2i) - (-3 + 4i) + (4(1 + 2i) + 15$

$= -22 - 4i + 3 - 4i + 4 + 8i + 15$

$= -22 + 22 - 8i + 8i$

$= 0 + 0i$

$= 0$

$\Rightarrow 1 + 2i$ is a root of the equation.

Since $z = 1 + 2i$ is a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$

The complex conjugate $\bar{z} = 1 - 2$i must also be a root of the above equation

$\Rightarrow 1 - 2i = z$ is also a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$

$$2z^3 - z^2 + 4z + 15 = 0$$

$$z = 1 + 2i$$

$$z = 1 - 2i$$

Sum of roots $= 1 + 2i + 1 - 2i$

$$= 2$$

Product of roots = $(1)^2 - (2i)^2$

$$= 1 + 4$$

$$= 5$$

$$z^2 - \binom{sum\ of}{roots} z + product = 0$$

$$z^2 - 2z + 5 = 0$$

$\Rightarrow z^2 - 2z + 5$ is a factor of $2z^3 - z^2 + 4z + 15$

$$\begin{array}{r|l} & 2z + 3 \\ z^2 - 2z + 5 & 2z^3 - z^2 + 4z + 15 \\ & 2z^3 - 4z^2 + 10z \\ \hline & 3z^2 - 6z + 15 \\ & 3z^2 - 6z + 15 \\ \hline & 0 \end{array}$$

$$(2z + 3)(z^2 - 2z + 15) = 0$$

$$z = -\frac{3}{2} \quad z = 1 + 2i\ and\ z = 1 - 2i$$

Example XI

Given that $2 + 3i$ is a root of the equation

$z^3 - 6z^2 + 21z - 26 = 0$. Find the other roots

Solution

$z = 2 + 3i$ is a root $\Rightarrow z = 2 - 3i$ is also a root of the equation $z^3 - 6z^2 + 21z - 26 = 0$

Sum of roots $= 2 + 3i + 2 - 3i$

$= 4$

Product of roots = (2 + 3i)(2 – 3i)

$= 2^2 - (3i)^2$
$= 4 + 9$
$= 13$
$\Rightarrow z^2 - 4z + 13$ is a factor of
$z^3 - 6z^2 + 21z - 26 = 0$

$$\begin{array}{r|l} & z - 2 \\ \hline z^2 - 4z + 13 & z^3 - 6z^2 + 21z - 26 \\ & z^3 - 4z^2 + 13z \\ \hline & -2z^2 + 8z - 26 \\ & -2z^2 + 8z - 26 \\ \hline & 0 \end{array}$$

$$\Rightarrow (z - 2)(z^2 - 4z + 13) = 0$$

$\Rightarrow z = 2, z = 2 + 3i \; and \; z = 2 - 3i$ are roots of equation of $z^3 - 6z^2 + 21z - 26 = 0$

Example XII
Show that $1 + i$ is a root of the equation
$z^4 + 3z^2 - 6z + 10 = 0$. Hence find other roots
Solution
$z = 1 + i$
$z^2 = 1 + 2i + i^2$
$z^2 = 1 + 2i - 1$
$z^2 = 2i$
$z^3 = z^2 . z$
$= 2i(1 + i)$
$= 2i - 2$

$$z^4 = (z^2)^2 = (2i)^2 = 4i^2$$

$= -4$
$\Rightarrow z^4 + 3z^2 - 6z + 10$
$= (-4) + 3(2i) - 6(1 + i) + 10$
$= -4 + 6i - 6 - 6i + 10$
$= -10 + 10 + 6i - 6i$
$= 0 + 0i$
$= 0$
$z = 1 + i$ is a root of the equation
$\Rightarrow 1 - i$ is also a root of the equation
Sum of the roots$= 1 + i + 1 - i$

$$= 2$$

Product of roots = $(1)^2 - i^2 = 2$

$$z^2 - (sum \; of \; roots)z + product = 0$$
$$z^2 - 2z + 2 = 0$$

$\Rightarrow z^2 - 2z + 2$ is a factor of $z^4 + 3z^2 - 6z + 10$

$$
\begin{array}{r|l}
 & z^2 + 2z + 5 \\
\hline
z^2 - 2z + 2 & z^4 + 3z^2 - 6z + 10 \\
 & \underline{z^4 - 2z^3 + 2z^2} \\
 & 2z^3 + z^2 - 6z + 10 \\
 & \underline{2z^3 - 4z^2 + 4z} \\
 & 5z^2 - 10z + 10 \\
 & \underline{5z^2 - 10z + 10} \\
 & 0
\end{array}
$$

$$(z^2 - 2z + 2)(z^2 + 2z + 5) = 0$$

$$\Rightarrow z^2 + 2z + 5 = 0$$

$$z^2 - 2z + 2 = 0$$

For $z^2 + 2z + 5 = 0$, $z = \dfrac{-2 \pm \sqrt{(2)^2 - 4\times1\times5}}{4\times1}$

$$z = \frac{-2 \pm \sqrt{16i^2}}{2}$$

$$z = \frac{-1 \pm 4i}{2}$$

$$z = -1 + 2i$$

$$z = -1 - 2i$$

$\Rightarrow$ $-1 + 2i$, $-1 - 2i$, $1 + i$, $1 - i$ are roots of the equation $z^4 + 3z^2 - 6z + 10 = 0$

Example XIII

Show that $1 - i$ is a root of the equation

$4z^4 - 8z^3 + 9z^2 - 2z + 2 = 0.$ Find the other roots.

Solution

$z = 1 - i$

$z^2 = (1 - i)^2$

$= 1 - 2i + i^2$

$= -2i$

$z^3 = z^2.z$

$= -2i(1 - i)$

$= -2i + 2i^2$

$= -2 - 2i$

$z^4 = (z^2)^2 = (-2i)^2$

$= -4$

$4z^4 - 8z^3 + 9z^2 - 2z + 2 =$

$4(-4) - 8(-2 - 2i) + 9(-2i) - 2(1 - i) + 2$

$= -16 + 16 + 16i - 18i - 2 + 2i + 2$

$= 0 + 0i = 0$

Since $z = 1 - i$ is a of the equation and it implies that $1 + i$ is also a root.

Sum of roots$= 1 - i + 1 + i$

$$= 2$$

Product of the roots = $(1 + i)(1 - i)$

$1^2 - i^2 = 2$

$$\Rightarrow z^2 - (2z) + 2 = 0$$

$\Rightarrow z^2 - 2z + 2$ is a factor of
$z^4 - 8z^3 + 9z^2 - 2z + 2 = 0.$

$$\begin{array}{r|l} & 4z^2 + 1 \\ \hline z^2 - 2z + 2 & 4z^4 - 8z^3 + 9z^2 - 2z + 2 \\ & \underline{4z^4 - 8z^3 + 8z^2} \\ & \qquad\qquad z^2 - 2z + 2 \\ & \qquad\qquad z^2 - 2z + 2 \end{array}$$

$$(z^2 - 2z + 1)(4z^2 + 1) = 0$$

$$4z^2 = -1 \qquad z^2 = -\frac{1}{4}$$

$$\Rightarrow z^2 = \frac{1}{4}i^2\,, \;\; 2 = \pm\frac{1}{2}i$$

Example XIV

Given that $z = 2 - i$ is a root of the equation
$z^3 - 3z^2 + z + k = 0,$ k is real. Find other roots.

Solution

$z = 2 - i$
$z^2 = (2 - i)^2$
$= 4 - 4i + i^2$
$= 3 - 4i$
$z^3 = (2 - i)(3 - 4i)$
$= 6 - 8i - 3i + 4i^2$
$= 2 - 11i$
$\Rightarrow (2 - 11i) - 3(3 - 4i) + 2 - i + k = 0$
$2 - 11i - 9 + 12i + 2 - i + k = 0$
$-11i + 11i + 4 - 9 + k = 0$
$0 - 5 + k = 0$
$k = 5$
$\Rightarrow z^3 - 3z^2 + z + 5 = 0$
$z = 2 - i$
$z = 2 + i$

$$z = 2 - i$$
$$z = 2 + i$$

Sum of roots = 4
Product of roots = 5

$z^2 - 4z + 5 = 0$ is a factor of
$z^3 - 3z^2 + z + 5 = 0$

$$\begin{array}{r|l} & z + 1 \\ \hline z^2 - 4z + 5 & z^3 - 3z^2 + z + 5 \\ & \underline{z^3 - 4z^2 + 5z} \\ & \qquad z^2 - 4z + 5 \end{array}$$

$$(z + 1)(z^2 - 4z + 5) = 0$$
$$(z + 1) = 0 \;\; z = -1$$

$\Rightarrow z = -1, \; z = 2 + i, \quad z = 2 - i$ are roots of the equation $z^3 - 3z^2 + z + k = 0$ where $k = 5$

Example XIV

Solve for z_1 and z_2 in the simultaneous equations below

$$z_1 + (1 - i)z_2 = 0$$

$3z_2 - 3z_1 = 2 - 5i$

Solution

$z_1 + (1 - i)z_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$3z_2 - 3z_1 = 2 - 5i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (2)$

From eqn (1)

$z_1 = -(1 - i)z_2$

$substitute \; in \; eqn \; (2)$

$3z_2 - 3[(-1(1 - i)z_2)] = 2 - 5i$

$3z_2 + 3(1 - i)z_2 = 2 - 5i$

$3z_2 - 3iz_2 + 3z_2 = 2 - 5i$

$6z_2 - 3iz_2 = 2 - 5i$

$z_2(6 - 3i) = 2 - 5i$

$z_2 = \dfrac{2 - 5i}{6 - 3i}$

$z_2 = \dfrac{(2 - 5i)(6 + 3i)}{(6 - 3i)(6 + 3i)}$

$z_2 = \dfrac{12 + 6i - 30i + 15}{36 + 9}$

$z_2 = \dfrac{27 - 24i}{45}$

$z_2 = \dfrac{9 - 8i}{15}$

$z_2 = \dfrac{9}{15} - \dfrac{8i}{15}$

$z_1 = -(1 - i)z_2$

$z_1 = -\left((1 - i)\left(\dfrac{9 - 8i}{15}\right)\right)$

$z_1 = -\left(\dfrac{9 - 8i - 9i - 8}{15}\right)$

$z_1 = -\left(\dfrac{1 - 17i}{15}\right)$

$z_1 = \dfrac{-1 + 17i}{15}$

$z_1 = -\dfrac{1}{15} + \dfrac{17i}{15}$

Example XV

Solve the equation $z^3 - 1$

Solution

$$z^3 - 1 = (z)^3 - (1)^3$$
$$= (z-1)(z^2 + z + 1)$$

Since $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$z^3 - 1 = (z-1)(z^2 + z + 1) = 0$$
$$z = 1$$
$$z^2 + z + 1 = 0$$
$$z = \frac{(-1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2 \times 1}$$
$$z = \frac{-1 \pm \sqrt{3i^2}}{2}$$
$$z = -\frac{1}{2} + \frac{(\sqrt{3})i}{2}$$
$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$
$$z = 1, z = -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \qquad z = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

Alternatively we can use Demovre's theorem

$$z^3 - 1 = 0$$
$$z^3 = 1$$
$$z^3 = 1 + 0i$$
$$z = (1 + 0i)^{\frac{1}{3}}$$
$$let\ P = 1 + 0i$$
$$|P| = \sqrt{1} = 1$$
$$\arg P = tan^{-1}\left(\frac{0}{1}\right) = 0$$
$$P = r[\cos(0) + i\sin(0)]$$
$$P = 1(\cos 0 + i \sin 0)$$

$z = P^{1/3}$

$$z = 1^{\frac{1}{3}}(\cos(0 + 360n) + i\sin(0 + 360n))$$
$$For\ n = 0, 1, 2 \ldots$$

(Depending on the number of roots you want)

For n = 0, $z = 1^{\frac{1}{3}}(\cos(0 + 360) + i\sin(0 + 360))^{\frac{1}{3}}$

$$z = 1^{\frac{1}{3}}(\cos 120 + i \sin 120)$$
$$z = 1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$z = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

For n = 1, $z = 1^{\frac{1}{3}}[\cos(0 + 360 \times 1) + i\sin(0 + 360 \times 1)]^{\frac{1}{3}}$

$$z = 1(\cos 120 + i \sin 120)$$
$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

For n = 2, $z = 1^{\frac{1}{3}}[\cos(0+360\times 2)+i\sin(0+360\times 2)]^{\frac{1}{3}}$

$$z = 1(\cos 240 + i\sin 240)$$

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\Rightarrow z = 1, \qquad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \qquad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Example XVI

Solve: $z^3 + 27 = 0$

Solution

$$z^3 + 3^3 = (z+3)(z^2 + 3z + 9)$$

From $a^3 + b^3 = (a+b)(a^2 + ab + b^2)$

$$\Rightarrow z^3 + 3^3 = (z+3)(z^2 + 3z + 9)$$

$$z = -3$$

$$z^2 + 3z + 9 = 0$$

$$z = \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2}$$

$$z = \frac{-3 \pm \sqrt{27i^2}}{2}$$

$$z = -3, \qquad z = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$$

$$z = \frac{-3}{2} - \frac{3\sqrt{3}}{2}i$$

Alternatively, we can use Demovre's theorem

$$z^3 + 27 = 0$$

$$z^3 = -27$$

$$z = (-27 + 0i)^{\frac{1}{3}}$$

$$let\ P = -27 + 0i$$

$$|P| = \sqrt{(-27)^2 + oi)}$$

$$= 27$$

$$\arg P = 180$$

$$P = 27(\cos 180 + i\sin 180)$$

$$z = P^{\frac{1}{3}} = 27^{\frac{1}{3}}(\cos 180 + i\sin 180)^{\frac{1}{3}}$$

$$z = 27^{\frac{1}{3}}(\cos(180 + 360n) + i\sin(180 + 360n))^{\frac{1}{3}}$$

When n = 0, $z = 27^{\frac{1}{3}}[(\cos 180 + i\sin 180)]^{\frac{1}{3}}$

$$z = 3(\cos 60 + i\sin 60)$$

$$= 3\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

For n = 1, $z = 27^{\frac{1}{3}}[(\cos(180 + 360\times 1) + i\sin(180 + 360\times 1)])^{\frac{1}{3}}$

$$z = 3(\cos 180 + i\sin 180)$$

$$z = -3$$

For $n = 2, z = 27^{\frac{1}{3}}[(\cos(180 + 360 \times 2) + i\sin(180 + 360 \times 2)])^{\frac{1}{3}}$

$$z = 3(\cos 300 + i\sin 300)$$

$$= \frac{3}{2} + -\frac{3\sqrt{3}}{2}i$$

$$= \frac{3}{2} - \frac{3}{2}\sqrt{3}i$$

$z = -3,\ z = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$ and $z = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$

Example XVII

Solve the equation

$z^4 + 1 = 0$

$z^4 = -1 + 0i$

$z^4 = (-1 + 0i)^{\frac{1}{4}}$

$let\ P = -1 + 0i$

$|P| = 1$

$\arg P = 180$

$P = 1(\cos 180 + i\sin 180)$

$z = P^{\frac{1}{4}} = 1^{\frac{1}{4}}[(\cos(180 + 360n) + i\sin(180 + 360n))]^{\frac{1}{4}}$

$For\ n = 0,\ z = 1^{\frac{1}{4}}(\cos 45 + i\sin 45)$

$$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

For $n = 1$

$z = 1^{\frac{1}{4}}(\cos 540 + i\sin 540)^{\frac{1}{4}}$

$z = 1(\cos 135 + i\sin 135)$

$$z = \frac{-\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}$$

For $n = 2, z = 1^{\frac{1}{4}}(\cos 900 + i\sin 900)^{\frac{1}{4}}$

$z = 1(\cos 225 + i\sin 225)$

$$z = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

For $n = 3$

$z = 1^{\frac{1}{4}}(\cos 1260 + i\sin 1260)^{\frac{1}{4}}$

$z = 1(\cos 315 + i\sin 315)$

$$z = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$For\ z^4 + 1 = 0$

$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,\quad -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,\ -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

Example XVIII

Find the fourth roots of -16

Solution

$z = (-16)^{1/4} = (-16 + 0i)^{1/4}$
Let $P = -16 + 0i$

$$z = P^{\frac{1}{4}} = (-16 + 01)^{\frac{1}{4}}$$
$$|P| = 16$$
$$\arg P = 180$$
$$z = P^{\frac{1}{4}} = 16^{\frac{1}{4}}[(\cos(180 + 360n) + i \sin(180 + 360n))]^{\frac{1}{4}}$$
$$For\ n = 0$$
$$z = 2(\cos 45 + i \sin 45)$$
$$z = \sqrt{2} + \sqrt{2}i$$
$$for\ n = 1$$
$$z = 2(\cos 540 + i \sin 540)$$
$$z = 2(\cos 135 + i \sin 135)$$
$$= -\sqrt{2} + i\sqrt{2}$$

For $n = 2$, $z = 2(\cos 225 + i \sin 225)$
$$= -\sqrt{2} - i\sqrt{2}$$
For $n = 3$, $z = 2(\cos 315 + i \sin 315)$
$$z = \sqrt{2} - \sqrt{2}i$$
$\Rightarrow For\ z = (-16 + 0i)^{\frac{1}{4}}$
$$z = \sqrt{2} - (\sqrt{2})i, -\sqrt{2} + (\sqrt{2})i$$
$$\sqrt{2} + (\sqrt{2})i, \quad -\sqrt{2} - (\sqrt{2})i$$

Example XIX
Find the cube roots of 27i

$$z = (0 + 27i)^{\frac{1}{3}}$$
$$let\ P = 0 + 27i$$
$$|P| = \sqrt{0^2 + 27^2}$$
$$= 27$$
$$\arg = tan^{-1}\left(\frac{27}{0}\right) = 90$$
$$P = 27(\cos 90 + i \sin 90)$$
$$z = 27^{\frac{1}{3}}(\cos 90 + i \sin 90)^{\frac{1}{3}}$$
$$z = 27^{\frac{1}{3}}(\cos(90 + 360n) + i \sin(90 + 360n))^{\frac{1}{3}}$$
$$For\ n = 0$$
$$z = 3(\cos 30 + i \sin 30)$$
$$z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$
$$n = 1$$
$$z = 3(\cos 150 + i \sin 150)$$
$$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -\frac{3\sqrt{3}}{2} + \frac{3i}{2}$$

$$for\ = 2$$
$$z = 3(\cos 270 + i \sin 270)$$
$$= -3i$$

Loci in the complex plane

What is a locus

A locus is a path possible position of a variable point, that obeys a given condition. It can be given as Cartesian equation or it can be described in words.

Example I

The complex number z is represented by the point P on the Argand diagram.

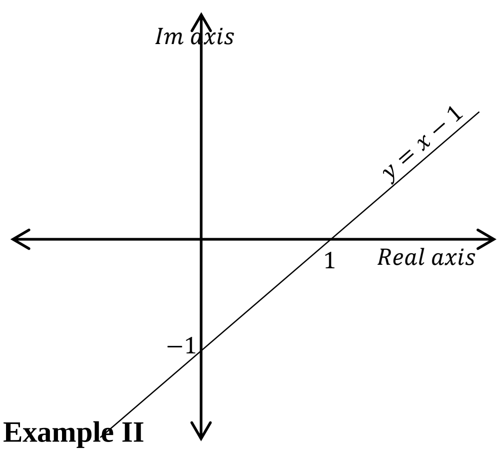
Given that $|z - 1 - i| = |z - 2|$ find in the simplest form the Cartesian equation of the locus

Solution

$$|z - 1 - i| = |z - 2|$$
$$let\ z = x + iy$$

$$|x + iy - 1 - i| = |x + iy - 2|$$
$$|x - 1 + (y - 1)i| = |x - 2 + iy|$$
$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-2)^2 + y^2}$$
$$(x-1)^2 + (y-1)^2 = (x-2)^2 + y^2$$
$$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 - 4x + 4 + y^2$$
$$-2x - 2y + 2 = -4x + 4$$
$$2x - 2 = 2y$$
$$y = x - 1$$

The locus is a straight line with a positive gradient $y = x - 1$) which can be represented on the complex plane.



Example II

Given that $|z - 2| = 2|z + i|$. Show that the locus of P is a circle.

Solution

$$|z - 2| = 2|z + i|$$

Let $z = x + iy$

$$|x + iy - 2| = 2|x + iy + i|$$

$$|(x-2)+iy| = 2|x+(y+1)i|$$

$$\sqrt{(x-2)^2+y^2} = 2\sqrt{x^2+(y+1)^2}$$

$$(x-2)^2+y^2 = 4(x^2+(y+1)^2)$$

$$x^2-4x+4+y^2 = 4x^2+4y^2+8y+4$$

$$0 = 3x^2+3y^2+4x+8y$$

$$x^2+y^2+\frac{4}{3}x+\frac{8y}{3} = 0$$

This is sufficient to justify that locus is a circle.

Comparing $x^2+y^2+\frac{4}{3}x+\frac{8y}{3} = 0$ With

$$x^2+y^2+2gx+2fy+c = 0$$

$$2g = \frac{4}{3}$$

$$g = \frac{2}{3}$$

$$2\,fy = \frac{8y}{3}$$

$$f = \frac{4}{3}$$

$$centre(-g,-f)$$

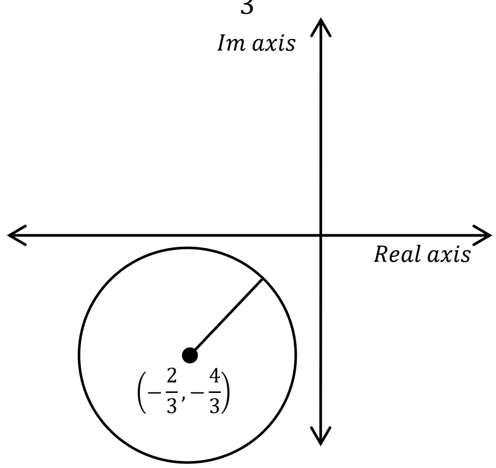
$$centre\left(-\frac{2}{3},-\frac{4}{3}\right)$$

$$r = \sqrt{g^2+f^2-c}$$

$$r = \sqrt{\frac{4}{9}+\frac{16}{9}-0}$$

$$r = \sqrt{\frac{20}{9}}$$

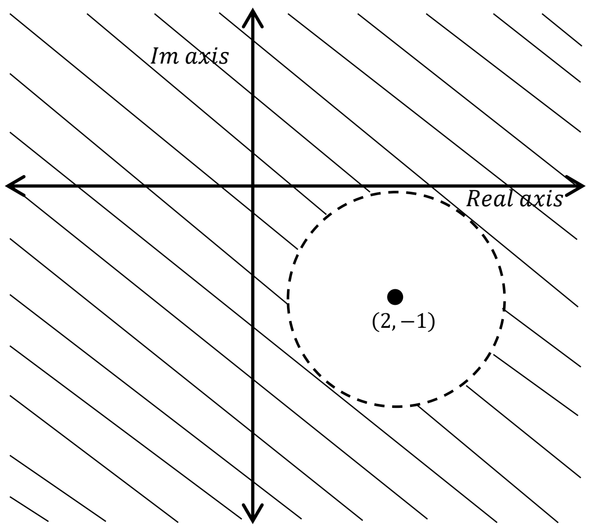
$$r = \frac{2}{3}\sqrt{5}$$



Example III
Show the region represented by $|z - 2 + i| < 1$
Solution
Let $z = x + iy$

$$|x + iy - 2 + i|$$
$$|x - 2 + (y + 1)i| < 1$$
$$\sqrt{(x - 2) + (y + 1)^2} < 1$$
$$(x - 2)^2 + (y + 1)^2 < 1$$

It's a circle with centre (2, -1) and radius less than 1.
It can be illustrated on the argand diagram



In order to represent $(x - 2)^2 + (y + 1)^2 < 1$ on the diagram, we can either take a point inside the circle or outside the circle as our test point.

Taking (2,-1) as the test point.

$$\Rightarrow (2 - 2)^2 + (-1 + 1)^2 < 1$$
$$0 + 0 < 1$$
$$0 < 1$$

$(2, -1)$(the point inside the circle satisfies our locus). It implies that (2,-1) lies in the wanted region. Therefore, we shade the region outside the circle.

Example IV
Given that

$$\left|\frac{z - 1}{z + 1}\right| = 2$$

find the Cartesian equation of the locus of z and represent the locus by the sketch on the argand diagram. Shade the region for which the inequalities.

$$\left|\frac{z - 1}{z + 1}\right| > 2$$

Solution

$$z = x + iy$$

$$\left|\frac{x + iy - 1}{x + iy + 1}\right| = 2$$

$$\left|\frac{(x - 1 + iy)}{(x + 1) + iy}\right| = 2$$

$$\frac{|x - 1 + iy|}{|(x + 1) + iy|} = 2$$

$$|(x - 1) + iy| = 2|(x + 1) + iy|$$

$$\sqrt{(x - 1)^2 + y^2} = 2\sqrt{(x + 1)^2 + y^2}$$

$$(x - 1)^2 + y^2 = 4((x + 1)^2 + y^2)$$

$$x^2 - 2x + 1 + y^2 = 4(x^2 + 2x + 1 + y^2)$$

$$3x^2 + 3y^2 + 10x + 3 = 0$$

$$x^2 + y^2 + \frac{10}{3}x + 1 = 0$$

The locus is a circle comparing

$$x^{\mathbf{2}} + y^2 + \frac{10}{3}x + 1 = 0 \text{ with}$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = \frac{10}{3}, \qquad g = \frac{5}{3}, \;\; 2f = 0 \text{ and } f = 0$$

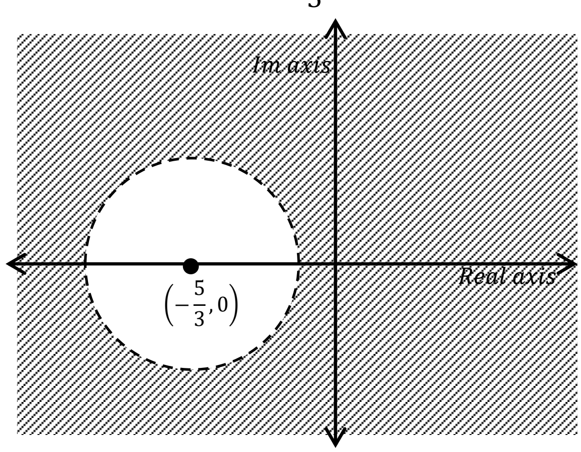
Center $(\frac{-5}{3}, 0)$

$$r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{\frac{25}{9} + 0 - 1} = \frac{4}{3}$$

For $\left|\frac{z-1}{z+1}\right| > 2$

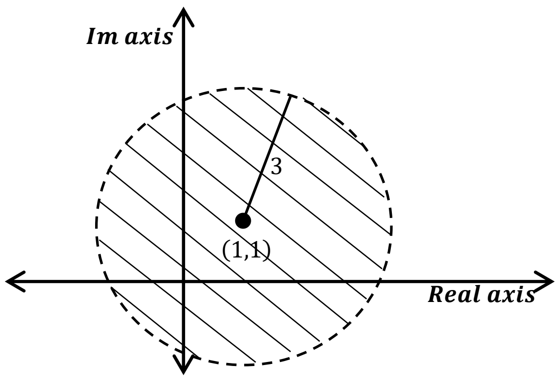
$$\Rightarrow x^2 + y^2 + \frac{10}{3}x + 1 > 0$$



Example V
Shade the region represented by $|z-1-i|<3$
Solution
Note: Shade the region represented by $|z-1-i|<3$. Implies that we shade the wanted region.
Let $x+iy$

$$|x+iy-1-i|<3$$
$$|x-1+i(y-1)|<3$$
$$\sqrt{(x-1)^2+(y-1)^2}<3$$
$$(x-1)^2+(y-1)^2<9$$

It is a circle with centre (1, 1) and radius less than 9



Taking (1, 1) as our test point

$$(1-1)^2+(1-1)^2$$
$$(0+0)<9$$

⇒The region inside the circle is the wanted region.

Example VI
Show that when

$$\text{Re}\left(\frac{z+i}{(z+2)}\right)=0,$$

the point P(x, y) lies on a circle with centre $-1,-\frac{1}{2})$ and radius$\frac{1}{2}\sqrt{5}$
Solution

$$\text{Re}\left(\frac{x+iy+i}{x+iy+2}\right)=0$$
$$Re\left(\frac{x+(y+1)i}{x+2+iy}\right)=0$$

$$\text{Re}\left[\frac{(x+(y+1)i)(x+2-iy)}{((x+2)+iy)(x+2-iy)}\right]=0$$

$$Re\left(\frac{x(x+2)-xyi+(y+1)(x+2)i+y(y+1)}{(x+2)^2+y^2}\right)$$

$$\text{Re}\left(\frac{x^2+2x+y^2+y+[(y+1)(x+2)-xy]i}{(x+2)^2+y^2}\right)$$

$$\text{Re}\left(\frac{x^2+2x+y^2+y}{(x+2)^2+y^2}+\frac{[(y+1)(x+2)-xy]i}{(x+2)^2+y^2}\right)=0$$

$$\Rightarrow \frac{x^2 + 2x + y^2 + y}{(x+2)^2 + y^2} = 0$$
$$x^2 + y^2 + 2x + y = 0$$

Comparing with
$x^2 + y^2 + 2x + y = 0$ with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$2g = 2, g = 1$$
$$2fy = y$$
$$f = \frac{1}{2}$$
$$centre\left(-1, -\frac{1}{2}\right)$$
$$radius = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{1 + \frac{1}{4} - 0}$$
$$= \frac{\sqrt{5}}{2}$$
$$= \frac{1}{2}\sqrt{5}$$

Example VII

Given that $z = x + iy$. where x and y are real. Show that $\text{Im}\left(\frac{z+i}{z+2}\right) = 0$ is equation of a straight line

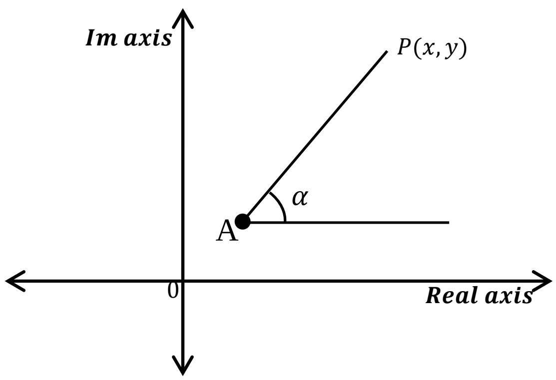
Solution

$$\text{Im}\left(\frac{x+iy+i}{x+iy+2}\right) = 0$$
$$\text{Im}\left[\frac{(x+(y+1)i)(x+2)-iy}{(x+2+iy)(x+2-iy)}\right] = 0$$
$$Im\left(\frac{x(x+2) - xyi + (y+1)(x+2)i + y(y+1)}{(x+2)^2 + y^2}\right) = 0$$
$$\Rightarrow \frac{-xy + (y+1)(x+2)}{(x+2)^2 + y^2} = 0$$
$$\frac{-xy + xy + 2y + x + 2}{(x+2)^2 + y^2} = 0$$
$$2y + x + 2 = 0$$
$$y = -\frac{x}{2} + 1$$

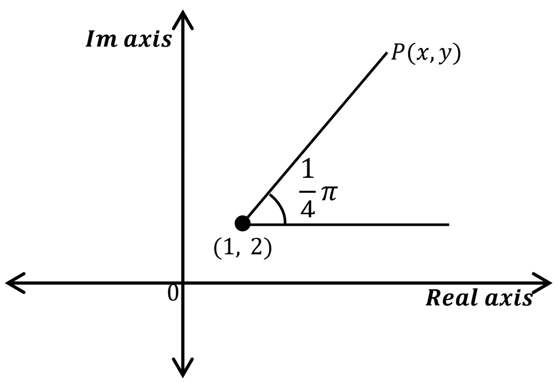
Which is a straight line with a negative gradient.

Loci in and diagram for arguments of complex numbers

If $\arg(z - A) = \alpha$ is the equation of half line with end point A inclined at an angle α to the real axis



Example I
Sketch the loci defined by the equation
$\arg(z-1-2i)=\frac{1}{4}\pi$
Solution
$z-1-2i=z-(1+2i)$
Thus if A is a point representing $1+2i$
$\arg\big(z-(1+2i)\big)$is the angle AP makes with the positive real axis. Hence the equation $\arg(z-1-2i)=\frac{1}{4}\pi$ represents the half line with end point (1, 2) inclined at angle $\frac{1}{4}\pi$ to the real axis.



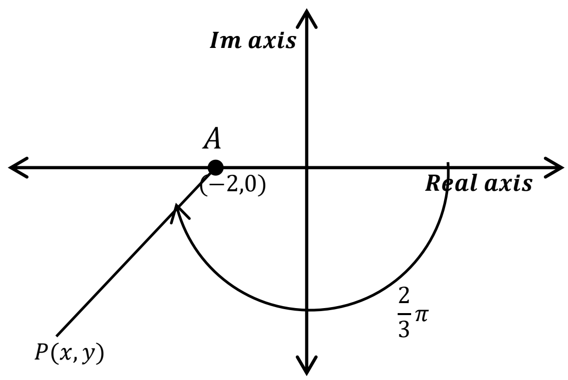
Example II
Sketch the locus of the equation.

$$\arg(z+2)=-\frac{2}{3}\pi$$

Solution

$$\arg(z+2)=-\frac{2\pi}{3}$$
$$z+2=(z-\ -2)$$

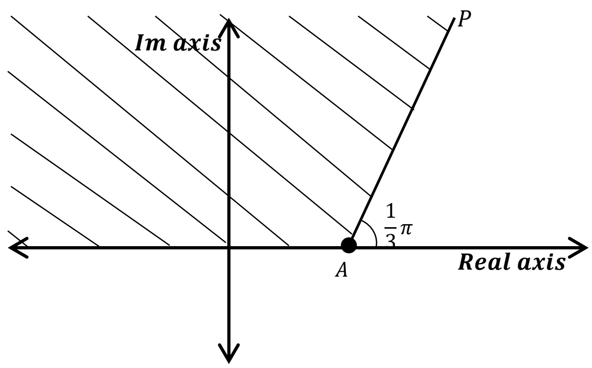
Thus *A* is a point (-2, 0). Arg(z-2) is the angle AP makes with the real axis. Hence $\arg(z--2)=-\frac{2}{3}\pi$ represents a half line with end point (-2, 0) inclined at angle $\frac{2}{3}\pi$ measured clockwise from the positive axis.



Example III
Show by shading the region represented by

$$\frac{1}{3}\pi \leq \arg(z-2) \leq \pi$$

Solution
The equations $\arg(z-2) = \frac{1}{3}\pi$ *and* $\arg(z-2) = \pi$ represent half lines with end point (2, 0). Hence the inequality $\frac{1}{3}\pi \leq \arg(z-2) \leq \pi$
Represent the two lines and region between them



Example IV
Sketch the separate argand diagram the loci defined by

$(i)\ \arg(z+1-3i) = -\dfrac{1}{6}\pi$

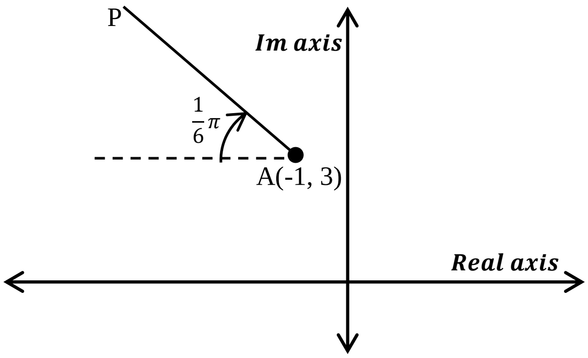
$(ii)\ \arg(z+2+i) = \dfrac{1}{2}\pi$

Solution

$(\arg(z+1-3i) = -\dfrac{1}{6}\pi$

$z-(-1+3i) = -\dfrac{1}{6}\pi$

Thus A is a point (-1, 3)
Arg(z – (-1 + 3*i*) is the angle AP makes with the real axis Hence $\arg(z+1-3i) = -\frac{1}{6}\pi$ is equation of the half line with end point (-1, 3) inclined at an angle of $\frac{1}{6}\pi$ measured clockwise from the real axis

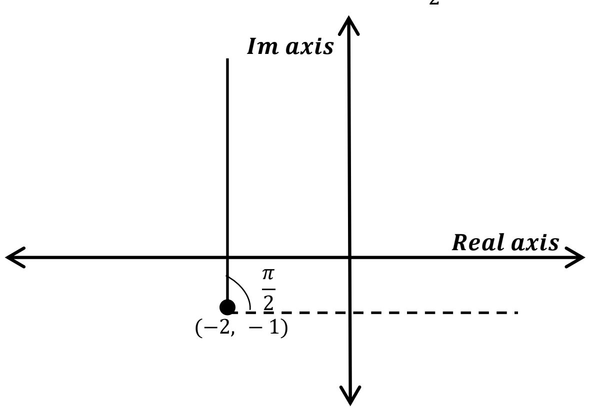


(ii) $\arg(z + 2 + i) = \frac{1}{2}\pi$

$\arg\big(z - (-2 - i)\big) = \frac{1}{2}\pi$

Thus, point A is $(-2, -1)$.

$\arg(z - (-2 - i))$ is the angle AP makes with the real axis and $\arg(z + 2 + i) = \frac{1}{2}\pi$ is the equation of the line through A inclined at and angle of $\frac{1}{2}\pi$ to the real axis



Sketching of loci involving $\boldsymbol{arg}\ \left(\frac{z-a}{z-b}\right) = \boldsymbol{\gamma}$

Equation involving arg $\left(\frac{z-a}{z-b}\right)$are more difficult to interprete. If $\arg(z - a) = \alpha$,

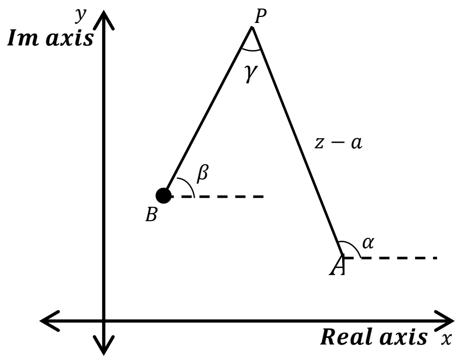
$$\arg(z - b) = \beta, \arg\left(\frac{z-a}{z-b}\right) = \gamma\ ,$$
$$\arg(z - a) - \arg(z - b) = \gamma$$

$\alpha - \beta = \gamma.$ $\quad \gamma = (\alpha - \beta) \pm 2\pi$ if necessary

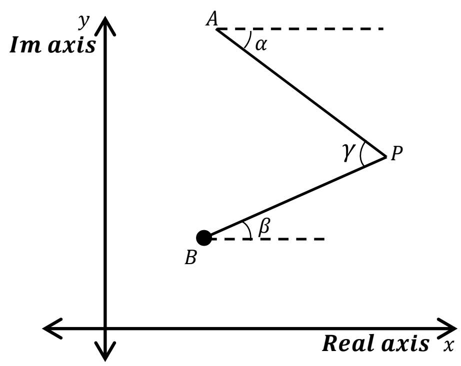
Thus γ is the angle which the vector AP makes with the vector BP.

If the turn from BP to AP is anti-clockwise the α is negative

$$\text{for} \arg\left(\frac{z-a}{z-b}\right) > 0$$



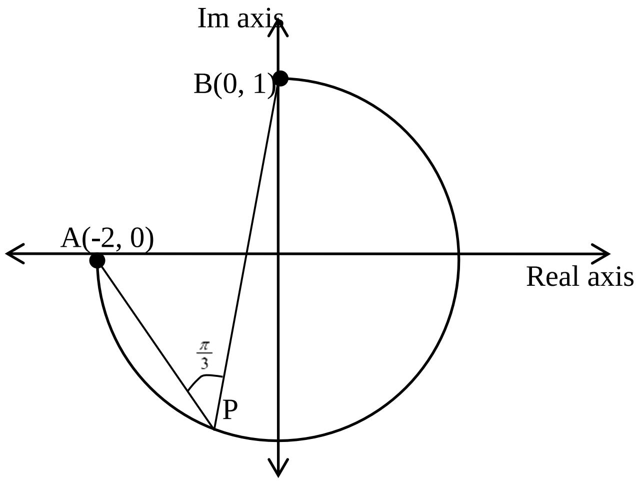
$$\arg\left(\frac{z-a}{z-b}\right) < 0$$



For instance, if $\arg\left(\frac{z-3}{z-1}\right) = \frac{1}{4}\pi,$ then the locus of P is a circular arc with end point A(3, 0) and (1, 0) such that $\angle APB = \frac{1}{4}\pi$

Similarly if $\arg\left(\frac{z+2}{z-i}\right) = \frac{1}{3}\pi$ then the locus of P is a circular arc with end points A (-2, 0) and B(0, +1) such that $\angle APB = \frac{1}{3}\pi$ since both cases the given arguments are positive, the arcs must be drawn so that the turn from BP to AP is anti-clockwise.

$$\arg\left(\frac{z+2}{z-i}\right) = \frac{1}{3}\pi$$



Example II
Sketch on different argand diagram the loci defined by the equations.

$(a)\ \arg\left(\frac{z-1}{z+1}\right)=\frac{1}{3}\pi$

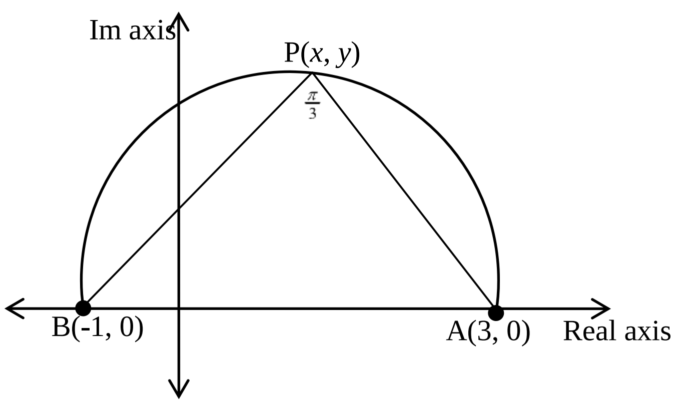
$(b)\ \arg\left(\frac{z-3}{z-2i}\right)=\frac{1}{4}\pi$

$(c)\ \arg\left(\frac{z}{z-4+2i}\right)=\frac{1}{2}\pi$

Solution

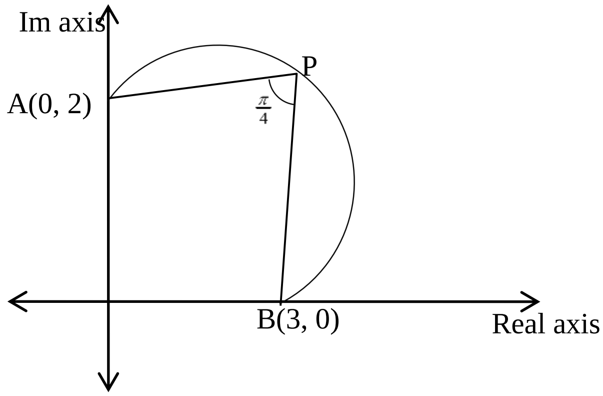
$\arg\left(\frac{z-1}{z+1}\right)=\frac{1}{3}\pi$

The locus of P is a circular arc with end point A(1, 0) and B(-1, 0) such that $\angle APB=\frac{1}{3}\pi$



$(b)\ \arg\left(\frac{z-3}{z-2i}\right)=\frac{1}{4}\pi$

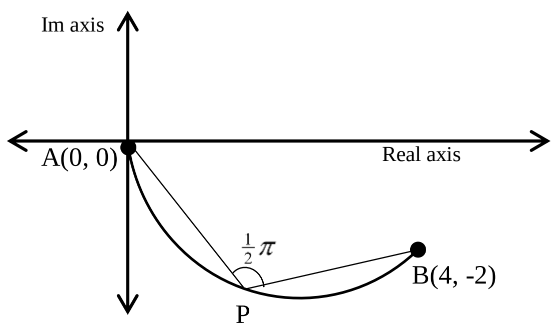
The locus of P is a circular arc with end points (3, 0) (0, 2) such that $\angle APB=\frac{1}{4}\pi$



$(c)\ \arg\left(\frac{z}{z-4+2i}\right) = \frac{1}{2}\pi$

$\arg(\frac{z}{z-(4-2i)} = \frac{1}{2}\pi$

$\arg\left(\frac{z}{z-4+2i}\right)$ is a circle with end points $A(0,0)$ and B$(4,-2)$ such that $\angle$APB $= \frac{1}{2}\pi$



Example

Find the locus of $\arg\left(\frac{z}{z-6}\right) = \frac{\pi}{2}$

Solution

$$let\ z = x + iy$$

$$\arg\left(\frac{z}{z-6}\right) = \arg z - \arg(z-6)$$

$$\Rightarrow\ \arg(z) - \arg(z-6) = \frac{\pi}{2}$$

$$\arg(x+iy) - \arg(x+iy-6) = \frac{\pi}{2}$$

$\tan^{-1}(\frac{y}{x}) - \tan^{-1}(\frac{y}{x-6}) = \frac{\pi}{2}$

$$let\ A = tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan A = \frac{y}{x}$$

$$B = \tan^{-1}\left(\frac{y}{x-6}\right)$$

$$\tan B = \frac{y}{x-6}$$

$$(A-B) = \frac{\pi}{2}$$

$$\tan(A-B) = \tan\left(\frac{\pi}{2}\right)$$

$$\frac{\tan A - \tan B}{1 + \tan A + \tan B} = \infty$$

$$\frac{\frac{y}{x} - \frac{y}{x-6}}{1 + \frac{y^2}{x(x-6)}} = \infty$$

$$\frac{\frac{y(x-6) - xy}{x(x-6)}}{\frac{x^2 - 6x + y^2}{x(x-6)}} = \infty$$

$$\frac{xy - 6y - xy}{x^2 + y^2 - 6^x} = \infty$$

$\Rightarrow x^2 + y^2 - 6x = 0$ which is a circle.

Revision Exercise 1

1. Prove that if $|Z| = \mathbf{r}$, then $ZZ^* = \mathbf{r}^2$.
2. Express $\sqrt{3}$ + i in modulus-argument form. Hence find $(\sqrt{3}+i)^{10}$ and $\frac{1}{(\sqrt{3}+i)^7}$ in the form $a + ib$.
3. Express -1 + i in modulus-argument form. Hence show that $(-1 + i)^{16}$ is real and that $\frac{1}{(-1+i)^6}$ is purely imaginary, giving the value of each.
4. Simplify the following expression:
   (a) $\frac{(\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7})^3}{(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7})^4}$ (b) $\frac{(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})^8}{(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5})^3}$
5. Find the expressions for cos 3θ in terms of cos θ,
   sin 3θ in terms of sin θ and tan 3θ in terms of tan θ.
6. Express sin 5θ and cos 5θ/cos θ in terms of sin θ.
7. Prove that $\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$. By considering the equation tan 5θ = 0, show that $\tan^2(\pi/5) = 5 - 2\sqrt{5}$
8. Find expressions for cos 6θ/sinθ in terms of cos θ and for tan 6θ in terms of tan θ.
9. Express in terms of cosines of multiples of θ:
   (a) $\cos^5\theta$ (b) $\cos^7\theta$ (c) $\cos^4\theta$
10. Express in terms of sines of multiples of θ:
    (a) $\sin^3\theta$ (b) $\sin^7\theta$ (c) $\cos^4\theta\sin^3\theta$
11. Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos^4\theta + 5)$

12. Evaluate (a) $\int_0^{\pi} \sin^4\theta \, d\theta$ (b) $\int_0^{\pi/2} \cos^4\theta \sin^2\theta \, d\theta$

13. (a) Express the following complex numbers in a form having a real denominator.

$$\frac{1}{3-2i}, \quad \frac{1}{(1+i)^2}$$

(b) Find the modulus and principal arguments of each of the complex numbers $Z = 1 + 2i$ and $W = 2 -$ I, and represent Z and W clearly by points A and B in an Argand diagram. Find also the sum and product of Z and W and mark the corresponding points C and D in your diagram.

14. If the complex number $x + iy$ is denoted by Z, then the complex conjugate number $x - iy$ is denoted by Z^*,
    (a) Express $|Z^*|$ and (Z^*) in terms of $|Z|$ and arg(Z).
    (b) If a, b, and c are real numbers, prove that if
    $aZ^2 + bZ + c = 0$, then then $a(Z^*)^2 + b(Z^*) + c = 0$
    (c) If p and q are complex numbers and $q \neq 0$, prove $\left(\frac{p}{q}\right)^* = \frac{p^*}{q^*}$

15. Find the values of a and b such that $(a + ib)^2 = i$. Hence or otherwise solve the equation $z^2 + 2z + 1 - i = 0$, giving your answer in the form $p + iq$, where p and q are real numbers.

16. If $Z = \frac{1}{2}(1+i)$, write down the modulus and argument for each of the numbers Z, Z^2, Z^3, Z^4. Hence or otherwise, show in the Argand diagram, the points representing the number $1 + Z + Z^2 + Z^3 + Z^4$.

17. If $Z = 3 - 4i$, find
    (i) Z^* (ii) ZZ^* (iii) $(ZZ)^*$

18. Simplify each of the following:
    (a) $(3 + 4i) + (2 + 3i)$ (b) $(2 - 4i) - 3(5 - 3i)$
    (b) $(2i)^2$ (c) i^4

19. Simplify each of the following:
    (a) $(2 + i)(3 - i)$ (b) $(5 - 2i)(6 + i)$
    (c) $(4 - 3i)(1 - i)$ (d) $(3 + i)(2 - 5i)$

20. Express each of the following in the form $a + ib$
    (a) $\frac{20}{3+i}$ (b) $\frac{4}{1+i}$
    (c) $\frac{2i}{1-i}$ (d) $\frac{1}{1-2i}$

21. Solve the following equations:
    (a) $x^2 + 25 = 0$
    (b) $2x^2 + 32 = 0$
    (c) $4x^2 + 9 = 0$
    (d) $x^2 + 2x + 5 = 0$

22. If $3 - 2i$ and $1 + i$ are two of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + c = 0$, find the values of a, b, c, d and e.

23. Find the square roots of the following complex numbers:
    (a) $5 + 2i$
    (b) $15 + 8i$
    (c) $7 - 24i$

24. Find the quadratic equations have the roots:
    (a) $3i, -3i$ (b) $1 + 2i, 1 - 2i$
    (c) $2 + i, 2 - i$ (d) $2 + 3i, 2 - 3i$

25. Find real and imaginary parts of the complex Z when:

(i) $\dfrac{Z}{Z+1}$ = 1 + 2i

(ii) $\dfrac{Z+i}{Z+1}=\dfrac{Z+i}{Z-3}$

26. Find the modulus and principal argument of the following complex numbers
(a) 3i (b) 15 (c) -3i (d) -1

27. Find the modulus and principle argument of:
(a) $\dfrac{1-i}{1+i}$ (b) $\dfrac{-1-7i}{4+3i}$
(c) $\dfrac{1+i}{2-i}$ (d) $\dfrac{(3+i)^2}{1-i}$

28. If Z_1 and Z_2 are complex numbers, solve the simultaneous equations
$4Z^1 + 3Z^2 = 23$
$Z^1 + iZ^2 = 6$
giving your answer in the form $x + iy$

29. Given that $2 + i$ is a root of the equation
$Z^3 - 11Z + 20 = 0$. Find the remaining roots.

30. Show that $1 + i$ is a root of the equation $x^4 + 3x^2 - 6x + 10 = 0$. Hence write down the quadratic factor of $x^4 + 3x^2 - 6x + 10$ and find all the roots of the equation.

31. The complex number satisfies the equation $\dfrac{Z}{Z+2}=2-i$. Find the real and imaginary parts of Z and the modulus and argument of Z.

32. If $Z_1 = 4\left(\cos\frac{13\pi}{24}+i\sin\frac{13\pi}{24}\right)$ and $Z_2 = 2\left(\cos\frac{5\pi}{24}+i\sin\frac{5\pi}{24}\right)$, find $\dfrac{Z_1}{Z_2}$ and Z_1Z_2 in the form $a + ib$.

33. If $Z_1 = 2\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ and $Z_2 = 6\left(\cos\frac{-3\pi}{4}+i\sin\frac{-3\pi}{4}\right)$, find:
(i) $\left|\dfrac{Z_1}{Z_2}\right|$ (ii) $\arg\left(\dfrac{Z_1}{Z_2}\right)$ (iii) $\left|\dfrac{Z_2}{Z_1}\right|$
(iv) $\arg\left(\dfrac{Z_2}{Z_1}\right)$

34. One root of the equation $Z^2 + aZ+b=0$ where a and b are real constants, is $2+3i$. Find the values of a and b.

35. If $Z1$ and $Z2$ are two complex numbers such that $|Z_1 - Z_2| = Z_1 + Z_2|$, show that the difference of their arguments is $\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$

36. (a) Find the modulus and argument of $\dfrac{(2-i)^2(3i-1)}{i+3}$
(b) If $Z_1=\dfrac{1+7i}{1-i}$ and $Z_2=\dfrac{17-7i}{2+2i}$. Find the moduli of Z_1, Z_2, $Z_1 + Z_2$ and Z_1Z_2.

37. Use Demoivre's theorem to show that:
$$\frac{(\cos 3\theta+i\sin 3\theta)^5(\cos\theta-i\sin\theta)^3}{(\cos 5\theta+i\sin 5\theta)^7(\cos 2\theta-i\sin 2\theta)^5}=\cos 13\theta-i\sin 13\theta$$

38. Use Demoivre's theorem to show that:
$\cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta$
$\sin 4\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta$

39. Show that $\left(\dfrac{1+\sin\theta+i\sin\theta}{1+\sin\theta-i\sin\theta}\right)^n=\cos n(\frac{\pi}{2}-\theta)+i\sin n(\frac{\pi}{2}-\theta)$

40. Use Demoivre's theorem to find the value of $\frac{\sqrt{-3}-1}{\sqrt{-3}+1}$

41. Find the two square roots of *I* and the four values of $(-16)^{1/4}$.

42. Find the three roots of the equation $(1 - Z)^3 = Z^3$

43. If *W* is a complex cube root of unity, show that

$$(1 + W - W^2)^3 - (1 - W + W^2)^3 = 0$$

44. Use Demoivre's theorem to find the four fourth roots of $8(-1 + i\sqrt{3})$ in the form $a + ib$, giving *a* and *b* correct to 2 decimal places.

45. Use Demoivre's theorem to show that

$$\frac{\cos 5x}{\cos x} = 1 - 12\sin^2 x + 16\sin^4 x$$

46. Prove that if $\frac{Z-6i}{Z+8}$ is real, the locus of the point representing the complex number *Z* in the Argand diagram is a straight line.

47. Prove that if $\frac{Z-2i}{2Z-1}$ is purely imaginary, the locus of the point representing *Z* in the Argand diagram is a circle and find its radius.

48. If *Z* is a complex number and $\left|\frac{Z-i}{Z+1}\right| = 2$, find the equation of the curve in the Argand diagram on which the point representing it lie.

49. The complex numbers $Z - 2$ and $Z - 2i$ have arguments which are
    (i) equal and
    (ii) differ by $\frac{1}{2}\pi$ and each argument lies between $-\pi$ and π. In each case, find the locus of the point which represents *Z* in the Argand diagram and illustrate by a sketch.

50. Show by shading on an Argand diagram the region in which both $|Z - 3 - i| \geq |Z - 3 - 5i|$

Answers

1.

2. (a) 1, (b) $-i$ (c) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (d) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

3. $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$; $512 - 512\sqrt{3}i$, $\frac{\sqrt{3}}{256} + \frac{1}{256}i$

4. $\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$; $256 - \frac{1}{8}i$

5. (a) 1, (b) -1

6. $4\cos^3\theta - 3\cos\theta - 4\sin^3\theta$, $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^3\theta}$

7. $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$, $1 - 12\sin^2\theta + 16\sin^4\theta$

8. .

9. $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$, $32\cos^5\theta - 32\cos^3\theta + 6\cos = \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta}$

10. (a) $\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$,
    (b) $\frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$
    (c) $\frac{1}{16}(2\cos\theta - \cos 3\theta - \cos 5\theta)$

11. (a) $\frac{1}{4}(3\sin\theta - \sin 3\theta)$, $\frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin\theta - \sin 7\theta)$,
    (c) $\frac{1}{64}(3\sin\theta + \sin 3\theta - \sin 5\theta - \sin 7\theta)$

12. (a) $\frac{3\pi}{8}$, (b) $\frac{\pi}{32}$

13. (a) $\frac{3+2i}{13}, \frac{1}{2}i$ (b) $\sqrt{5}$, 63.4°, $\sqrt{5}$, -26.6°, 3 + i,
4 + 3i.

15. $a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}$ or $a = \frac{-1}{\sqrt{2}}, b = \frac{-1}{\sqrt{2}}$
$Z = -1 + \frac{1}{\sqrt{2}+\frac{i}{\sqrt{2}}}$ or $Z = -1 - \frac{1}{\sqrt{2}-\frac{i}{\sqrt{2}}}$

16. $\frac{\sqrt{2}}{2}$, 45°; $\frac{1}{2}$, 90°; $\frac{\sqrt{2}}{4}$, 135°; $\frac{1}{4}$, 180°

17. (i) 3 + 4i (ii) 25 (iii) -7 + 24i

18. (a) 5 + 7i (b) -13 + 5i (c) -4 (d) 1

19 (a) 7 + i (b) 32 – 7i (c) 1 – 7i (d) 11 – 13i

20. (a) 6 – 2i (b) 2 – 2i (c) -1 + i (d) $\frac{1}{5} + \frac{2}{5}i$

21 (a) $x \pm 5i$ (b) $x = \pm 4i$ (c) $\pm \frac{3}{2} i$ (d) $x = -1 \pm 2i$

22. a = 1, b = -8, c = 27, d = -38, e = 26

23. (a) ±(3 + 2i) (b) ±(4 + i) (c) ±(4 – 3i)

24. (a) $x^2 + 9 = 0$ (b) $x^2 - 2x + 5 = 0$
(c) $x^2 - 4x + 5 = 0$ (d) $x^2 - 4x + 13 = 0$

25 (i) -1, ½ (ii) $\frac{1}{5}, \frac{-2}{5}$

26. (a) 3, π/2 (b) 15, 0 (c) 3, -π/2 (d) 1, π

27. (a) 1, -π/2 (b) $\sqrt{2}, \frac{-3\pi}{4}$, (c) $\frac{\sqrt{10}}{5}$, 1.25

28. 2 + 3i 19. 2 – i, -4

31. (i) Re(Z) = -3, Im(Z) = -1 (ii) $\sqrt{10}$, -2.82 rads

32. $1 + \sqrt{3}i$; $-4\sqrt{2} + 4\sqrt{2}\,i$

33. (i) 1/3 (ii) $\frac{-7\pi}{12}$ (iii) 3 (iv) $\frac{7\pi}{12}$

34. -4, 13

36. (a) 5, 0.6435 rad (b) 5, 6.5, 2.061, 32.5

41. $\frac{\pm(1+i)}{\sqrt{2}}$, $\pm\sqrt{2} \pm i\sqrt{2}$

42. $\frac{1}{2}, \frac{1}{2}(1 \pm i\sqrt{3})$. 44. ±(1.73 + i), ± (1 – 1.73i)

47. centre ¼ + i, radius $\frac{1}{4}\sqrt{7}$

48. $\left(x + \frac{5}{3}\right)^2 + y^2 = \frac{16}{9}$

49. (i) $x + y = 2$ (ii) $(x-1)^2 + (y-1)^2 = 2$

Exercise 2

Show on the Argand diagram the region represented by the following:

1. arg $z = \frac{1}{4}\pi$,
2. $\arg(z - i) = \frac{1}{3}\pi$
3. $\arg(z + 1 - 3i) = \frac{1}{6}\pi$
4. $\arg(z - 3 + 2i) = \pi$

5. $\arg(z + 2 + i) = \frac{1}{2}\pi$

6. $\arg(z - 1 - i) = -\frac{1}{4}\pi$

7. $|z + 1| = |z - 3|$,

8. $|z| = |z - 6i|$

9. $\left|\frac{z-i}{z-1}\right| = 1$

10. (a) $\arg\left(\frac{z-1}{z+1}\right) = \frac{1}{3}\pi$

11. (a) $\arg\left(\frac{z-3}{z-2i}\right) = \frac{1}{4}\pi$ (b) $\left(\frac{z}{z-4+2i}\right) = \frac{1}{2}\pi$

In questions 12 to 24 find the Cartesian equation of the locus of the point *P* representing the complex number *z*. Sketch the locus of *P* each case.

12. $2|z + 1| = |z - 2|$
13. $|z + 4i| = 3|z - 4|$
14. $\left|\frac{z}{z-4}\right| = 5$
15. $\left|\frac{z+i}{z-5-2i}\right| = 1$
16. $\left|\frac{z}{z+6}\right| = 5$
17. $\left|\frac{z-1}{z+1-i}\right| = \frac{2}{3}$
18. $z - 5 = \lambda i(z + 5)$, where λ is a real parameter
19. $\frac{z+2i}{z-2} = \lambda i$, where λ is a real number.
20. $z = 3i + \lambda(2 + 5i)$, where λ is a real parameter.
21. $\text{Im}(z^2) = 2$
22. $\text{Re}\ (z^2) = 1$
23. $\text{Re}\left(z - \frac{1}{z}\right) = 0$
24. $\text{Im}\left(z + \frac{9}{z}\right) = 0$

In questions 27 to 34 shade in separate Argand diagrams the regions represented by:

25. $|z - i\,| \le 3$
26. $|z - 4 + 3i| < 4$
27. $0 \le \arg z \le \frac{1}{3}\pi$
28. $\frac{1}{4}\pi < \arg z < \frac{3}{4}\pi$
29. $-\frac{1}{6}\pi < \arg(z-1) < \frac{1}{6}\pi$
30. $-\frac{1}{2}\pi \le \arg(z+i) \le \frac{2}{3}\pi$
31. $|z| > |z + 2|$
32. $|z + i| \le |z - 3i|$
33. Represent each of the following loci in an Argand diagram.
    (a) $\arg(z - 1) = \arg(z + 1)$
    (b) $\arg z = \arg(z - 1 \text{ I } i)$

(c) arg$(z - 2) = \pi + \arg z$
(d) arg$(z - 1) = \pi + \arg(z - i)$

34. Find the least value of $|z + 4|$ for which
    (a) $\text{Re}(z) = 5$ (b) $\text{Im}(z) = 3$
    (c) $|z| = 1$ (d) $\arg z = \frac{1}{4}\pi$
35. Given that the complex number z varies such that $|z - 7| = 3$, find the greatest and least values of $|z - i|$.
36. Given that the complex number w and z vary subject to the conditions $|z - 12| = 7$ and $|z - i| = 4$, find the greatest and least values of $|w - z|$.
37. In an Argand diagram, the point P represents the complex number z, where $z = x + iy$. Given that $z + 2 = \lambda i(z + 8)$, where λ is a real parameter, find the Cartesian equation of the locus of P as λ varies. If also $z = \mu(4 + 3i)$, where λ is real, prove that there is only one possible position for P.
38. (i) Represent on the same Argand diagram the loci given by the equations $|z - 3| = 3$ and $|z| = |z - 2|$. Obtain the complex numbers corresponding to the point of intersection of these loci. (ii) Find a complex number z whose argument is $\pi/4$ and which satisfies the equation $|z + 2 + i| = |z - 4 + i|$.

Answers

12. $x^2 + y^2 + 4x = 0$, 13. $x^2 + y^2 - 9x - 9x - y + 16$
15. $5x + 3y = 14$. 16. $2x^2 + 2y^2 + 25x + 75 = 0$
17. $5x^2 + 5y^2 - 26x + 8y + 1 = 0$.
18. $x^2 + y^2 = 25$, excluding (-5, 0)
19. $x^2 + y^2 - 2x + 2y = 0$, excluding (2, 0).
20. $5x - 2y + 6 = 0$ 21. $xy = 1$. 22. $x^2 - y^2 = 1$
23. $x(x^2 + y^2 - 1) = 0$, excluding (0, 0)
24. $y(x^2 + y^2 - 9) = 0$, excluding (0, 0).
34. (a) 9, (b) 3, (c) 3, (d) 4.
35. $5\sqrt{2} + 3$, $5\sqrt{2} - 3$. 38. 24, 2.
37. $x^2 + y^2 + 10x + 16 = 0$
38.(i) $1 \pm i\sqrt{5}$ (ii) $1 + i$.

Revision Exercise 3

Show on the Argand diagram the region represented by the following:

1. $\left(\dfrac{z+1}{z-1}\right) = \frac{1}{3}\pi$
2. $\left|\dfrac{z-2-3i}{z+2+i}\right| = 1$
3. Express the complex number $z_1 = \dfrac{11+2i}{3-4i}$ in the form $x + iy$ where x and y are real. Given that $z_2 = 2 - 5i$, find the distance between the points in the Argand diagram which represent z_1 and z_2. Determine the real numbers α and β such that $\alpha z_1 + \beta z_2 = -4 + i$.
4. (i) Find two complex numbers z satisfying the equation $z^2 = -8 - 6i$.
   (ii) Solve the equation $z^2 - (3 - i)z + 4 = 0$ and represent the solutions on an Argand diagram by vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where **O** is the origin. Show that triangle OAB is right-angled.
5. If z and w are complex numbers, show that:
   $$|z-w|^2 + |z+w|^2 = 2\{|z|^2 + |w|^2\}$$
   Interpret your results geometrically.

6. A regular octagon is inscribed in the circle $|z| = 1$ in the complex plane and one of its vertices represents the number $\frac{1}{\sqrt{2}}(1+i)$. Find the numbers represented by the other vertices.

7. (i) Two complex numbers z_1 and z_2 each have arguments between 0 and π. If $z_1z_2 = i - \sqrt{3}$ and $\frac{z_1}{z_2} = 2i$, find the values of z_1 and z_2 giving the modulus and argument of each.

   (ii) Obtain in the form $a + ib$ the solutions of the equation $z^2 - 2z + 5 = 0$, and represent the solutions on an Argand diagram by the points A and B.
   The equation $z^2 - 2pz + q = 0$ is such that p and q are real, and its solutions in the Argand diagram are represented by the points C and D. Find in the simplest form the algebraic relation satisfied by p and q in each of the following cases:
   (a) $p^2 < q$, $p \neq 1$ and A, B, C, D are the vertices of a triangle;
   (b) $p^2 > q$ and $\angle CAD = \frac{1}{2}\pi$

8. (a) If $-\pi < \arg z_1 + \arg z_2 \leq \pi$, show that $\arg(z_1z_2) = \arg z_1 + \arg z_2$. The complex numbers $a = 4\sqrt{3} + 2i$ and $b = \sqrt{3} + 7i$ are represented in the Argand diagram by points A and B respectively. O is the origin. Show that triangle OAB is equilateral and find the complex number c which the point C represents where $OABC$ is a rhombus. Calculate $|c|$ and $\arg c$.

   (b) z is a complex number such that $z = \frac{p}{2-q} + \frac{q}{1+3i}$ where p and q are real. If $\arg z = \pi/2$ and $|z| = 7$ find the values of p and q.

9. .

10. (a) Show that $(1 + 3i)^3 = -(26 + 18i)$.
    (b) Find the three roots z_1, z_2, z_3 of the equation $z^3 =-1$
    (c) Find in the form $a + ib$, the three roots z'_1, z'_2, z'_3 of the equation $z^3 = 26 + 18i$.
    (d) Indicate in the same Argand diagram the points represented by z_r and z'_r for $r = 1, 2, 3$, and prove that the roots of the equations may be paired so that $|z_1 - z_2| = z_2 - z'_2 = |z_3 - z'_3| = 3$.

11. Write down or obtain the non-real cube roots of unity, w_1 and w_2, in the form $a + ib$, where a and b are real. A regular hexagon is drawn in an Argand diagram such that two adjacent vertices represent w_1 and w_2, respectively and centre of the circumscribing circle of the hexagon is the point (1, 0). Determine in the form $a + ib$, the complex numbers represented by the other four vertices of the hexagon and find the product of these four complex numbers.

12. A complex number w is such that $w^3 = 1$ and $w \neq 1$. Show that:
    (i) $w^2 + w + 1 = 0$
    (ii) $(x + a + b)(x + wa + w^2b)(x + w^2a + wb)$
    is real for real x, a and b, and simplify this product. Hence or otherwise find the three roots of the equation $x^3 - 6x + 6 = 0$, giving your answers in terms of w and cube roots of integers.

13. (i) Find, without the use of tables, the two square roots of $5 - 12i$ in the form $x + iy$, where x and y are real.
    (ii) Represent on an Argand diagram the loci $|z - 2| = 2$ and $|z - 4| = 7$. Calculate the complex numbers corresponding to the points of intersection of these loci.

14. (i) Given that $(1 + 5i)p - 2q = 7i$, find p and q when (a) p and q are real (b) p and q are conjugate complex numbers.
    (ii) Shade on the Argand diagram the region for which $3\pi/4 < \arg z < \pi$ and $0 < |z| < 1$. Choose a point in the region and label it A. If A represents the complex number z, label clearly the points B, C, D and E which represent $-z$, iz, $z + 1$ and z^2 respectively.

15. (i) Show that $z = 1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Find the other roots of the equation.

(ii) Sketch the curve in the Argand diagram defined by $|z - 1| = 1$, Im $z \geq 0$. Find the value of z at the point P in which this curve is cut by the line $|z - 1| = |z - 2|$. Find also the value of arg z and arg$(z - 2)$ at P.

16. (i) If $z = 1 + i\sqrt{3}$, find $|z|$ and $|z^5|$, and also the values of arg z and arg(z^5) lying between $-\pi$ and π. Show that Re$(z^5) = 16$ and find the value of Im(z^5).

(ii) Draw the line $|z| = |z - 4|$ and the half line arg$(z - i) = \pi/4$ in the Argand diagram. Hence find the complex number that satisfies both equations.

17. (i) Without using tables, simplify $\dfrac{(\cos\frac{\pi}{9} + i\cos\frac{\pi}{9})^4}{(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9})^5}$.

(ii) Express $z_1 = \dfrac{7+4i}{3-2i}$ in the form $p + qi$, where p and q are real. Sketch in an Argand diagram the locus of the points representing complex numbers z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of z subject to this condition.

18. (i) Given that $z = 1 - i$, find the values of $r(>0)$ and θ, $-\pi < \theta < \pi$, such that $z = r(\cos\theta + i\sin\theta)$. Hence or otherwise find $1/z$ and z^6, expressing your answers in the form $p + iq$, where $q, r \in \mathbb{R}$.

(ii) Sketch on an Argand diagram the set of points corresponding to the set A, where $A = \{z{:}z \in \mathbb{C}, \arg(z - i) = \pi/4\}$. Show that the set of points corresponding to the set B, where $B = \{z{:}z \in \mathbb{C}, |z + 7i| = 2|z - 1|\}$, forms a circle in the Argand diagram. If the centre of this circle represents the numbers z_1, show that $z_1 \in A$.

19. Use De Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

20. (i) If $(1 + 3i)z_1 = 5(1 + i)$, express z_1 and z_1^2 in the form $x + iy$, where x and y are real. Sketch in an Argand diagram the circle $|z - z_1| = |z_1|$ giving the coordinates of its centre.

(ii) If $z = \cos\theta + i\sin\theta$, show that:

$$z = \frac{1}{z} = 2i\sin\theta \qquad z^n = \frac{1}{z^n} = 2i\sin n\theta$$

Hence or otherwise, show that

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

21. .

22. (i) Given that x and y are real, find the values of x and y which make satisfy the equation $\dfrac{2y+4i}{2x+y} - \dfrac{y}{x-i} = 0$

(ii) Given that $z = x + iy$, where x and y are real, (a) Show that $\text{Im}\left(\dfrac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a straight line (b) Show that, when $\text{Re}\left(\dfrac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a circle with centre (-1, -½) and radius $\frac{1}{2}\sqrt{5}$

23. (i) Find $|z|$ and arg z for which the complex numbers z given by (a) $12 - 5i$, (b) $\dfrac{1+2i}{2-i}$, giving the argument in degrees (to the nearest degree) such that $-180° < \arg z \leq 180°$.

(ii) By expressing $\sqrt{3} - i$ in modulus-argument form, or otherwise, find the least positive integer n such that $(\sqrt{3} - i)^n$ is real and positive.

(iii) The point P in the Argand diagram lies outside or on the circle of radius 4 with centre at (-1, -1). Write down in modulus form the condition satisfied by the complex number z represented by point P.

24. Sketch the circle C with Cartesian equation $x^2 + (y - 1)^2 = 1$. The point P representing the non-zero complex number z lies on C. Express $|z|$ in terms of θ, the argument of z. Given that $z' = 1/z$, find the

modulus and argument of z' in terms of θ. Show that, whatever the position of P on the circle C, the point P' representing z' lies on a certain line, the equation of which is to be determined.

25. (a) The sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ for values of z such that $|z| < 1$ is $1/(1 - z)$. By substituting $z = ½(\cos\theta + i\sin\theta)$ in this result and using De Moivre's theorem, or otherwise, prove that $\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \frac{1}{2^n}\sin n\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}$

Answers

3. $1 + 2i$; $5\sqrt{2}$; -2, -1

4. (i) $\pm(1 - 3i)$, (ii) $2 - 2i$, $1 + i$

5. **sum of squares of a parallelogram = sum of squares of sides**

TRIGONOMETRY

Trigonometry is a branch of mathematics that studies relationships involving lengths and angles of a triangle. It comes from two Greek words – *trigonom* (triangle) and *metron* (measure).

There is an enormous number of the uses of trigonometry and trigonometric functions. For instance, the technique of triangulation is used in astronomy to measure the distance between land marks. Although it was first applied in spheres, it had a greater application to planes. Surveyors have used trigonometry for many centuries.

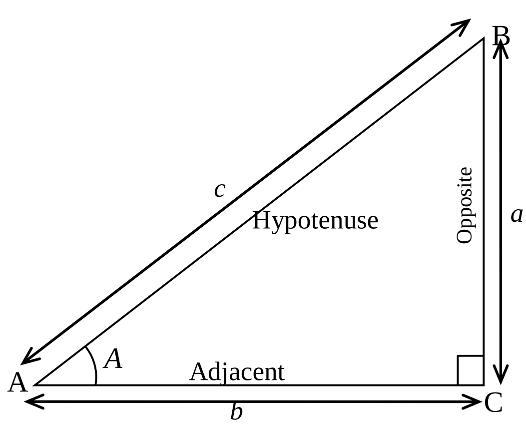
Within mathematics, it is used in calculus (perhaps its greatest application), linear algebra, and statistics.

Trigonometric tables were created over 2000 years ago for computation in astronomy.

A student is expected to be familiar with the definitions of trigonometric ratios for acute angles.

If one angle is 90° and one of the other angles is known, the third can be determined because the three angles of any triangle add up to 180°. The two acute angles therefore add up to 90° (complimentary angles).

Once the angles are known, the ratios of the sides are determined regardless of the overall size of the triangle. If the length of one side is known, the other two are determined. These ratios are given by the following trigonometric functions of known angle, A; where a, b, and c refer to the lengths of the sides accompanying the figure.



Sine function (sin)

This is the ratio of the opposite side of the triangle to its hypotenuse.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

Cosine function (cos)

This is the ratio of the adjacent side to the hypotenuse

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

Tangent function (tan)

This is the ratio of the opposite to the adjacent side.

$$\begin{aligned}\tan A &= \frac{a}{b} = \frac{a}{c} \times \frac{c}{b} \\ &= \left(\frac{a}{c}\right) \div \left(\frac{b}{c}\right) \\ &= \frac{\sin A}{\cos A} \\ \tan A &= \frac{\sin A}{\cos A}\end{aligned}$$

The hypotenuse is the side opposite to the 90° angle. It is the longest side of a triangle and one of the sides adjacent to A.

The term perpendicular and base are sometimes used for opposite and adjacent sides respectively.

Many people find it easy to remember what sides of the right angle are equal to sine, cosine, or tangent by memorising the mnemonic **SOH-CAH-TOA**.

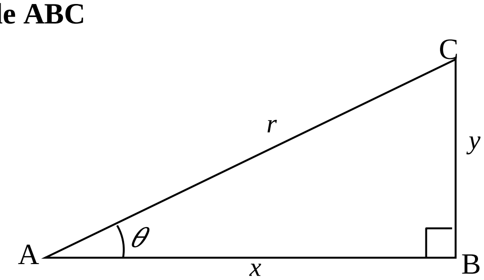
The reciprocals of the functions are named cosecant (cosec), secant (sec) and cotangent (cot)

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{c}{a}$$

$$\sec A = \frac{1}{\cos A} = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{c}{b}$$

$$\cot A = \frac{1}{\tan} = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{\cos A}{\sin A} = \frac{b}{a}$$

Consider the following triangle ABC



$$\sin\theta = \frac{y}{r}\ ,\ \cos\theta = \frac{x}{r}\text{; and } \tan\theta = \frac{y}{r}$$

$$y = r\sin\theta; \qquad x = r\cos\theta$$

Applying the Pythagoras' theorem to triangle ABC;

$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\cos^2\theta + \sin^2\theta = 1$

$\mathbf{\cos^2\theta + \sin^2\theta = 1}$ …………………………. (i)

$\mathbf{\cos^2\theta + \sin^2\theta = 1}$

Dividing equation (i) by $\cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$1 + \tan^2\theta = \sec^2\theta$

$\mathbf{1 + \tan^2\theta = \sec^2\theta}$ …………………………….. (ii)

$\mathbf{1 + \tan^2\theta = \sec^2\theta}$

Dividing Eqn (i) by $\sin^2\theta$

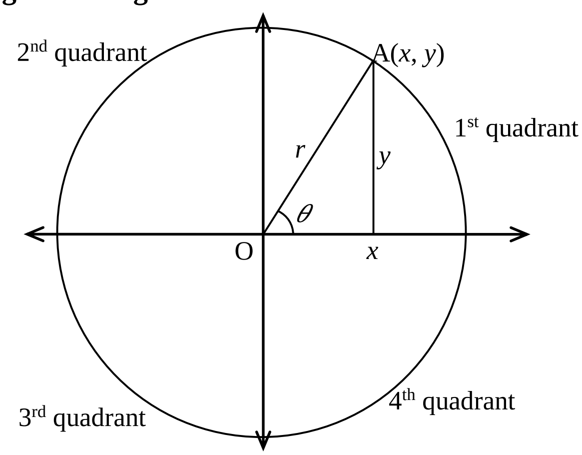
$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$\cot^2\theta + 1 = \text{cosec}^2\theta$

$\mathbf{1 + \cot^2\theta = cosec^2\theta}$ ……………..…………….. (iii)

$\mathbf{1 + \cot^2\theta = cosec^2\theta}$

Trigonometric Ratios for general angle



Angles measured from the *x*-axis in the anti-clockwise sense are termed as positive angles while those measured in the clockwise sense are negative angles.

When A is in the 1st quadrant, *x* and *y* are positive. When A is in the 2nd quadrant, *x* is negative and *y* is positive. When A is in the third quadrant, *x* and *y* are all negative. When A is in the 4th quadrant, *x* is positive and *y* is negative. *r* is taken to be positive for all positions of the line *OA*.

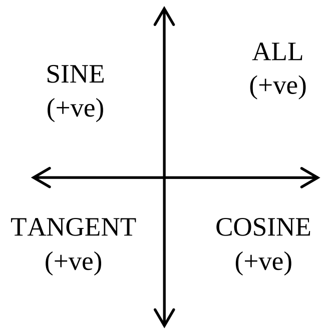
The trigonometrical ratios for angles *xOA* of any magnitude are defined precisely in the same way as for acute angles.

Thus $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$

The appropriate signs are attached to *x* and *y* according to the position of point A. hence for angles in which OA lies in the 1st quadrant; since *x* and *y* and *r* are positive, the sine, cosine, and tangent will all be positive.

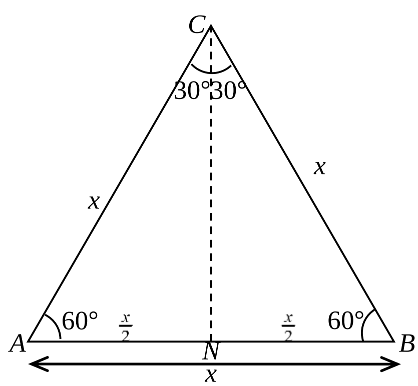
For angles in which OA lies in the 2nd quadrant, since *y* and *r* are positive and *x* negative, the sine is positive. Cosine and tangent are negative.

For angles in which OA is in the 3rd quadrant, sine and cosine are both negative but tangent is positive. In the 4th quadrant, sine and tangent are negative while cosine is positive. This is illustrated below.

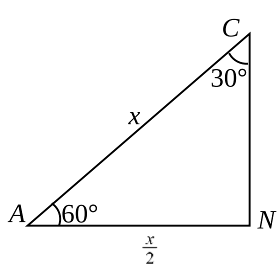


Trigonometric ratios of 30°, 45°, and 60°

Consider the equilateral triangle *ABC* of side *x*



Considering triangle CAN:



Applying the Pythagoras' theorem:

$$\left(\tfrac{x}{2}\right)^2 + (\overline{CN})^2 = x^2$$

$$\frac{x^2}{4} + (\overline{CN})^2 = x^2$$

$$(\overline{CN})^2 = x^2 - \frac{x^2}{4}$$

$$\overline{CN}^2 = \frac{3x^2}{4}$$

$$\overline{CN} = \frac{\sqrt{3}x}{2}$$

Using ***SOH-CAH-TOA***

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin\theta = \frac{\text{CN}}{\text{AC}}$$

$$\sin 60° = \frac{\frac{\sqrt{3}}{2}x}{x}$$

$$= \frac{\sqrt{3}}{2}$$

$$\sin 30° = \frac{x/2}{x} = \frac{1}{2}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\sin 60° = \frac{\frac{x\sqrt{3}}{2}}{x} = \frac{\sqrt{3}}{2}$$

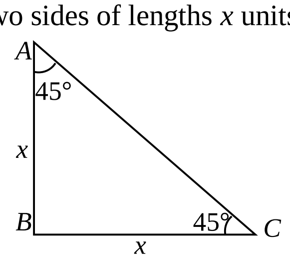
$$\cos 30° = \frac{\frac{\sqrt{3}}{2}x}{x} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{\frac{x\sqrt{3}}{2}}{x/2} = \sqrt{3}$$

$$\tan 30 = \frac{x/2}{x\sqrt{3}/2}$$

$$\tan 30 = \frac{1}{\sqrt{3}}$$

Consider a right isosceles triangle with two sides of lengths x units.



Applying the Pythagoras' theorem on ABC:

$x^2 + x^2 = AC^2$

$2x^2 = AC^2$

$AC = x\sqrt{2}$

Applying *SOH-CAH-TOA*

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin 45° = \frac{x}{AC}$$

$$\sin 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\sin 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\sin 45° = \frac{x}{x} = 1$$

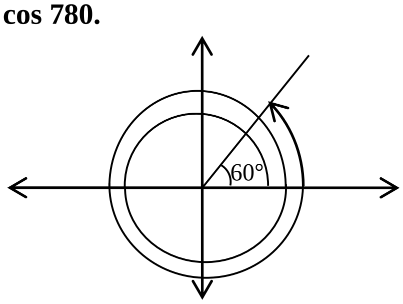
Example I

Write down the values of the following, leaving surds in your answers (*the calculator should not be used*).

(a) cos 780°
(b) sin 780°
(c) tan 780°
(d) sin 540°
(e) cos 540°
(f) cos 210°
(g) sin 150°
(h) sin(-270°)
(i) sin 225°
(j) sin 405°
(k) tan(-60°)

Solution

(a) cos 780.



cos 780° = cos 60°

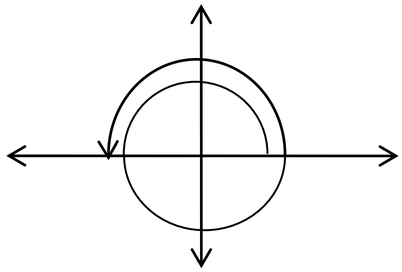
$$= \frac{1}{2}$$

sin 780° = sin 60°

$$= \frac{\sqrt{3}}{2}$$

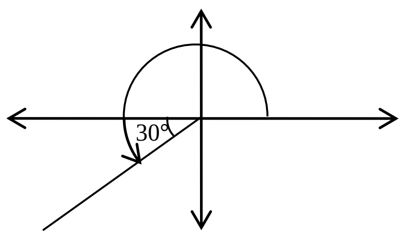
tan 780° = tan 60° = $\sqrt{3}$

sin 540°



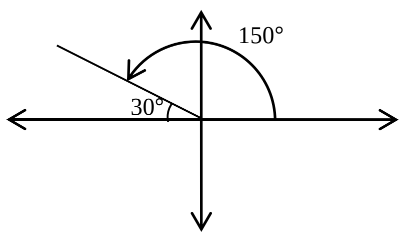
sin 540° = sin 180° = 0°
cos 540° = cos 180° = 0°

cos 210°



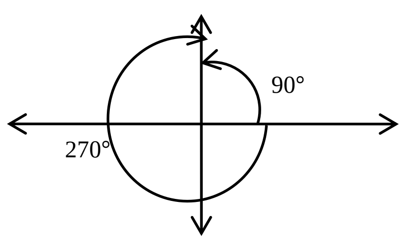
cos 210° = -cos 30° = $\frac{-\sqrt{3}}{2}$

sin 150°



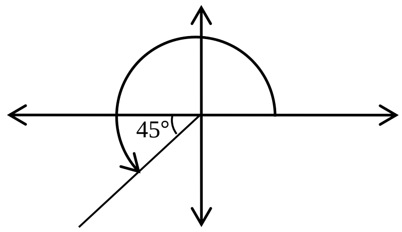
sin 150 = +sin 30 = $\frac{1}{2}$

sin -270°



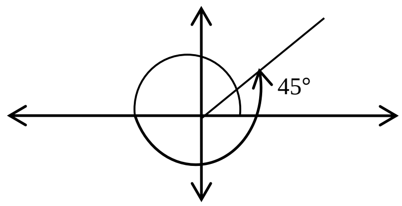
sin -270 = +sin 90° = 1

sin 225°



sin 225° = -sin 45° = $\frac{-1}{\sqrt{2}}$

sin 405°



sin 405° = sin 45° = $\frac{1}{\sqrt{2}}$

Trigonometric Curves

For any angle θ, a single value of sin θ or cos θ can be found. The same applies to tan θ unless when $\theta = \pm 90°$ and $\pm 270°$ for which the values of tan θ are not defined. Thus sin θ and cos θ are functions which are defined for all negative values of θ.

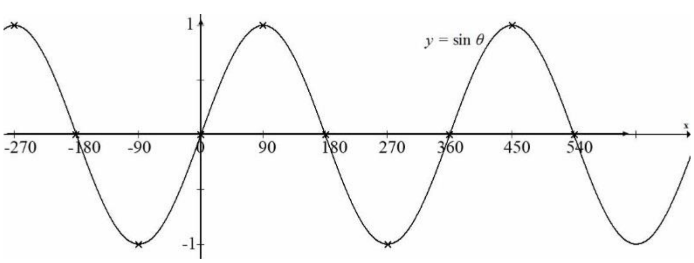
Tan θ is a function which is defined for all positive and negative values of θ except $\pm 90°$ and $\pm 270°$.

To draw the graphs of sinθ, cosθ and tanθ, we construct a table of values, giving ordered pairs of these functions and hence plot the graph.

Example

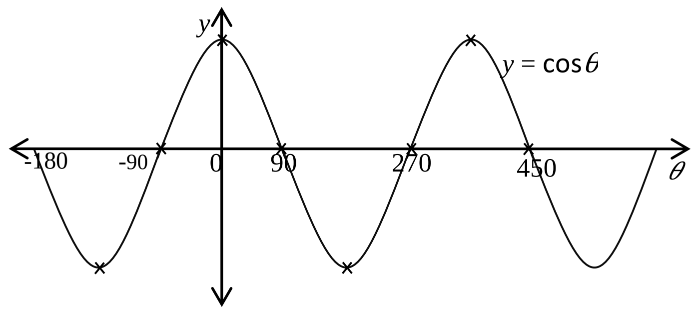
$y = \sin\theta$

θ	-270	-180	-90	0	90	180	270	360	450	540
$y = \sin\theta$	1	0	-1	0	1	0	-1	0	1	0



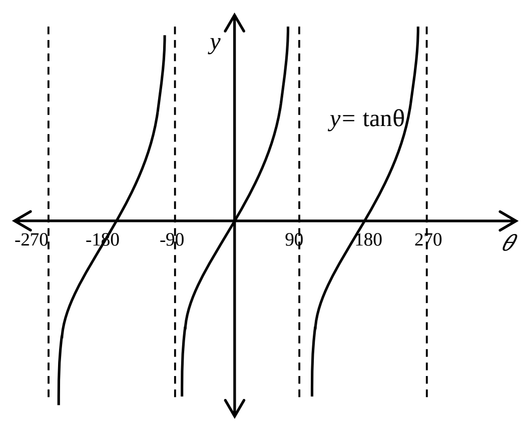
$y = \cos\theta$

θ	-180	-90	0	90	180	270	360	450
$y = \cos\theta$	1	0	1	0	-1	0	1	0



$y = \tan\theta$

θ	-270	-180	-90	0	90	180	270	360	450
$y = \tan\theta$	∞	0	∞	0	∞	0	∞	0	∞



From the graph of sin θ and cos θ, the maximum values of cos θ and sin θ are 1 and 1 respectively. The minimum value of cos θ and sin θ are -1 and -1 respectively.

The graphs for sinθ and cosθ repeat themselves at regular intervals of 360° while that of tanθ repeat itself at regular interval of 180°. These intervals are called periods. These trigonometric functions are examples of periodic functions.

Trigonometric Equations

Trigonometric equations differ from algebraic equations in that they often have unlimited number of solutions.

Example I

Solve the following equations for $0 \leq \theta \leq 360°$

(a) $\sin\theta = \frac{-1}{2}$

(b) $\sec\theta = 2$

(c) $\tan\theta = -\sqrt{3}$

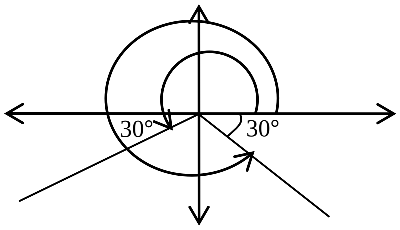
(d) $\sin^2\theta = \frac{1}{2}$

Solutions

$\sin\theta = \frac{-1}{2}$

The acute angle whose sine is $\frac{1}{2}$ is 30°. But $\sin\theta$ is negative in the 3rd and 4th quadrants.

(a)



$\Rightarrow$ For sin θ = $\frac{-1}{2}$

θ = 210°

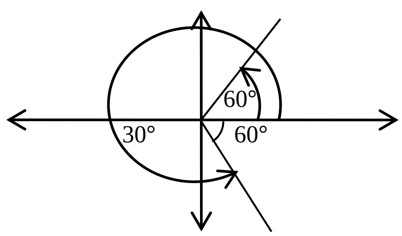
θ = 330°

(b) sec θ = 2

$$\frac{1}{\cos\theta} = 2$$

$$\Rightarrow \frac{1}{2} = \cos\theta \quad \Rightarrow \cos\theta = \frac{1}{2}$$

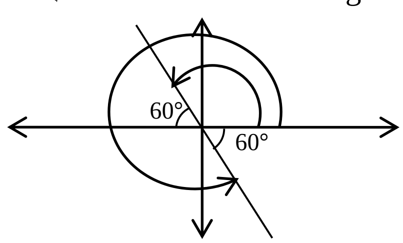
The acute angle whose cosine is $\frac{1}{2}$ is 60° but cos θ° is positive in the 1st and 4th quadrants.



For $\cos\theta = \frac{1}{2}$, θ = 60°, 300°

(c) $\tan\theta = -\sqrt{3}$

The acute angle whose tangent is $\sqrt{3}$ is 60° but $\tan\theta$ is negative in the 2nd and 4th quadrants.



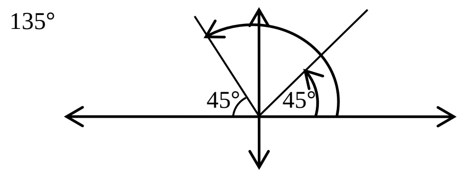
$\Rightarrow$ For $\tan\theta = -\sqrt{3}$, θ = 120°, 300°

(d) $\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

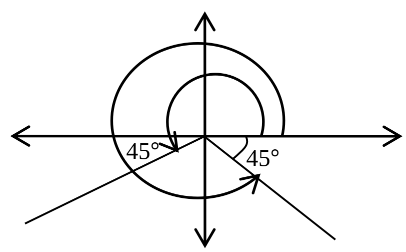
$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta\ \frac{-1}{\sqrt{2}}$

The acute angle whose sine is $\frac{1}{\sqrt{2}}$ is 45° but $\sin\theta$ is positive in the 1st and 2nd quadrants.



$\Rightarrow$ For $\sin\theta = \frac{1}{\sqrt{2}}$, θ = 45, 135

For $\sin\theta = \frac{-1}{\sqrt{2}}$



For sin θ = $\frac{-1}{\sqrt{2}}$

$\theta°$ = 225°, 315°

For $\sin^2\theta = \frac{1}{2}$, θ = 45, 135°, 225°, 315°

Example II

Solve the following equations for -180° ≤ θ ≤ 180°.

(a) Sin(2θ + 30) = 0.8

(b) $\tan^2\theta + \tan\theta = 0$

(c) $\sin^2\theta + \sin\theta = 0$

(d) $2\sin^2\theta - \sin\theta - 1 = 0$

Solution

(a) $\sin(2\theta + 30°) = 0.8$

$2\theta + 30° = \sin^{-1}(0.8)$

$2\theta + 30° = 53.1°, 126.9°$

$\Rightarrow$ 2θ = 23.1, 96.9

θ = 11.55, 48.45

For $\sin(2\theta + 30°) = 0.8$, θ = 11.55, 48.45.

(b) $\tan^2\theta + \tan\theta = 0$

$\tan\theta(\tan\theta + 1) = 0$

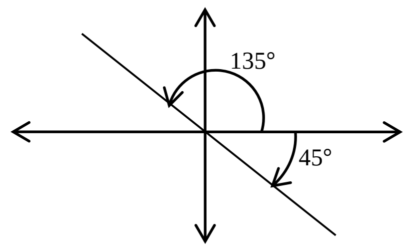
$\tan\theta = 0$ **OR** $\tan\theta = -1$

For $\tan\theta = 0$,

$\theta = \tan^{-1}0$

$\theta = 0, -180, 180$

For $\tan\theta = -1$, the acute angle whose tangent is 1 is 45°. But $\tan\theta$ is negative in the 2nd and 4th quadrants.



For $\tan\theta = -1$, $\theta = 135°, -45°$

$\Rightarrow \tan^2\theta + \tan\theta = 0$

$\theta = -180°, -45°, 0, 135°, 180°$

(c) $\sin^2\theta + \sin\theta = 0$

$\sin\theta(\sin\theta + 1) = 0$

$\sin\theta = 0, \quad \sin\theta = -1$

For $\sin\theta = 0°$, $\theta = 0, 180°, -180°$

For $\sin\theta = -1$,

The acute angle whose sine is 1 is 90°. Sine is negative in the 3rd and 4th quadrants.

For $\sin\theta = -1$, $\theta = -90$

For $\sin^2\theta + \sin\theta = 0°$, $\theta = -180°, -90°, 0°, 180°$

(d) $2\sin^2\theta - \sin\theta - 1 = 0$

$$\sin\theta = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2 \times 2}$$

$$\sin\theta = \frac{1 \pm 3}{4}$$

$$\Rightarrow \sin\theta = 1, \ \sin\theta = \frac{-1}{2}$$

For $\sin\theta = 1$,

$\theta = \sin^{-1}(1)$

$\theta = 90°$

For $\sin\theta = \frac{-1}{2}$,

$\theta = -30°, -150°$

$\Rightarrow \theta = -30°, -150°, 90°$

Example III

Solve the following equations from 0° to 360° inclusive.

(a) $\cos 3\theta = \frac{\sqrt{3}}{2}$

(b) $\tan(3\theta - 45°) = \frac{1}{2}$

(c) $\sec 2\theta = 3$

(d) $4\cos 2\theta = 1$

(e) $\tan^2\theta = \frac{1}{3}$

(f) $\sin^2 2\theta = 1$

Solutions

(a) $\mathbf{\cos 3\theta} = \frac{\sqrt{3}}{2}$

$$3\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$3\theta = 30°, 330°, 390°, 690°, 750°, 1050°$

$\Rightarrow \theta = 10°, 110°, 130°, 230°, 250°, 350°$

(b) $\tan(3\theta - 45°) = \frac{1}{2}$

$$3\theta - 45 = \tan^{-1}\left(\frac{1}{2}\right)$$

$3\theta - 45 = 26.6, 206.6, 386.6, 566.6, 746.6, 926.6$

$\Rightarrow \theta = 23.9°, 83.9°, 143.9°, 203.9°, 263.9°, 323.9°$

(c) $\mathbf{\sec 2\theta = 3}$

$$\frac{1}{\cos 2\theta} = 3$$

$$\frac{1}{3} = \cos 2\theta$$

$$2\theta = \cos^{-1}(\tfrac{1}{3})$$

$2\theta = 70.5°, 289.5°, 430.5°, 649.5°$

$\theta = 35.25°, 144.75°, 215.25°, 324.75°$

(d) $\mathbf{\tan^2}\theta = \frac{1}{3}$

$$\tan\theta = \pm\frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}} \text{ or } \tan\theta = -\frac{1}{\sqrt{3}}$$

For $\tan\theta = \frac{1}{\sqrt{3}}$, $\theta = 30°, 210°$

For $\tan\theta - \frac{1}{\sqrt{3}}$, $\theta = 150°, 330°$

$\Rightarrow$ When $\tan^2\theta = \frac{1}{3}$, $\theta = 30°, 150°, 210°, 230°$

(e) $\sin^2 2\theta = 1$

$\sin 2\theta = \pm 1$

For $\sin 2\theta = 1$,

$2\theta = 90°, 450° \Rightarrow \theta = 45°, 225°$

$\sin 2\theta = -1$,

$2\theta = 270, 630 \Rightarrow \theta = 135°, 315°$

$\Rightarrow$ When $\sin^2 2\theta = 1$,

$\theta = 45°, 135°, 225°, 315°$

Example IV

Solve the following equations for values of θ from -180° to 180°

(f) $\tan\theta = \cot\theta + 3$

(g) $\sec\theta = 2\cos\theta$

(h) $5\sin\theta + 6\text{cosec}\,\theta = 17$

(i) $3\cos\theta + 2\sec\theta + 7 = 0$

Solution

(a) $\tan\theta = 4\cot\theta + 3$

$$\tan\theta = \frac{4}{\tan\theta} + 3$$

$\tan^2\theta = 4 + 3\tan\theta.$

$\tan^2\theta - 3\tan\theta - 4 = 0$

$$\tan\theta = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$$

$$\tan\theta = \frac{3 \pm 5}{2}$$

$\tan\theta = 4,$ $\quad$ $\tan\theta = -1$

When $\tan\theta = 4$,

$\theta = \tan^{-1}(4)$

$\theta = 76°, -104°$ (for $-180° \leq \theta \leq 180°$)

When $\tan\theta = -1$,

$\theta = \tan^{-1}(-1) = -45°, 135°$ (for $-180° \leq \theta \leq 180°$)

$\Rightarrow$ For $\tan\theta = 4\cot\theta + 3°$,

$\theta = -104°, -145°, 76°, 135°$

(b) $\sec\theta = 2\cos\theta$

$$\frac{1}{\cos\theta} = 2\cos\theta$$

$1 = 2\cos^2\theta$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}}$$

For $\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = 45°, -45°.$

For $\cos\theta = \frac{-1}{\sqrt{2}}$, θ = 135°, -135°

∴ For sec θ = 2cos θ, θ = -135°, -45°, 45°, 135°.

(c) 5sin θ + 6cosec θ = 17

Solution

5sinθ + 6cosecθ = 17

$$5\sin\theta + \frac{6}{\sin\theta} = 17$$

$5\sin^2\theta + 6 = 17\sin\theta$

$5\sin^2\theta - 17\sin\theta + 6$

$$\sin\theta = \frac{17 \pm \sqrt{(-17)^2 - 4(5)\times 6}}{2\times 5}$$

$$\sin\theta = \frac{17 \pm \sqrt{289 - 120}}{10}$$

$$\sin\theta = \frac{17 \pm 13}{10}$$

sinθ = 3

sinθ = 0.4

$\theta = \sin^{-1}(0.4) \Rightarrow \theta$ = 23.6, 156.4

$\theta = \sin^{-1}(3) \Rightarrow \theta$ has no value since sinθ is maximum when it is 1

(d) 3cos θ +2sec θ +7 = 0

$$3\cos\theta + \frac{2}{\cos\theta} + 7 = 0$$

$3\cos^2\theta + 2 + 7\cos\theta = 0$

$3\cos^2\theta + 7\cos\theta + 2 = 0$

$$\cos\theta = \frac{-7 \pm \sqrt{(7)^2 - 4\times 3\times 2}}{2\times 3}$$

$$\cos\theta = \frac{-7 \pm 5}{6}$$

$$\cos\theta = \frac{-1}{3}$$

cosθ = -2

For cosθ = -2, θ has no values because the minimum of cosθ is -1

For $\cos\theta = \frac{-1}{3}$

θ = 109.5°, -109.5°.

Example IV

Solve the following equations from 0° to 360°

- **(a)** $3 - \cos\theta = 2\sin^2\theta$
- **(b)** $\cos^2\theta + \sin\theta + 1 = 0$
- **(c)** $\sec^2\theta = 3\tan\theta - 1$
- **(d)** $\text{cosec}^2\theta = 3 + \cot\theta$
- **(e)** $3\tan^2\theta + 5 = 7\sec\theta$

Solutions

(a) $3 - \cos\theta = 2\sin^2\theta$

$3 - 3\cos\theta = 2(1 - \cos^2\theta)$

$3 - 3\cos\theta = 2 - 2\cos^2\theta$

$2\cos^2\theta - 3\cos\theta + 1 = 0$

$$\cos\theta = \frac{3 \pm \sqrt{(-3)^2 - 4\times 2\times 1}}{2\times 2}$$

$$\cos\theta = \frac{3 \pm \sqrt{9-8}}{4}$$

$\cos\theta = 1$, **OR** $\cos\theta = \frac{1}{2}$

For $\cos\theta = 1$,

$\theta = \cos^{-1}(1)$

$\theta = 0°, 360°$

For $\cos\theta = \frac{1}{2}$,

$\theta = \cos^{-1}(1/2)$

$\theta = 60°, 300°$

$\Rightarrow$ For $3 - 3\cos\theta = 2\sin^2\theta$, $\theta = 0°, 60°, 300°, 360°$

(b) $\mathbf{\cos^2\theta + \sin\theta + 1 = 0}$

$1 - \sin^2\theta + \sin\theta + 1 = 0$

$\sin^2\theta - \sin\theta - 2 = 0$

$$\sin\theta = \frac{1 \pm \sqrt{(-1)^2 - 4\times 1\times -2}}{2}$$

$$\sin\theta = \frac{1\pm 3}{2}$$

$\sin\theta = 2$ **OR** $\sin\theta = -1$

For $\sin\theta = 2$, the value of θ is not defined because $\sin\theta$ is maximum at 1

For $\sin\theta = -1$, $\theta = 270°$

(c) $\mathbf{\sec^2}\theta = 3\tan\theta - 1$

$\sec^2\theta = 1 + \tan^2\theta$

$\Rightarrow 1 + \tan^2\theta = 3\tan\theta - 1$

$\tan^2\theta - 3\tan\theta + 2 = 0$

$$\tan\theta = \frac{3 \pm \sqrt{(-3)^2 - 4\times 1\times 2}}{2\times 1}$$

$$\tan\theta = \frac{3\pm 1}{2}$$

$\tan\theta = 2$ **OR** $\tan\theta = \mathbf{1}$

For $\tan\theta = 2$,

$\theta = \tan^{-1}(2) = 63.4°, 243.4°$

For $\tan\theta = 1$,

$\theta = \tan^{-1}(1) = 45°, 225°$

$\therefore$ For $\sec^2\theta = 3\tan\theta - 1$, $\theta = 45°, 63.4°, 243.4°, 225°$.

(d) **$\mathbf{cosec^2}\theta = 3 + \cot\theta$**

But $\text{cosec}^2\theta = 1 + \cot^2\theta$

$\Rightarrow 1 + \cot^2\theta = 3 + \cot\theta$

$\cot^2\theta - \cot\theta - 2 = 0$

$$\cot\theta = \frac{1 \pm \sqrt{(-1)^2 - 4\times 1\times(-2)}}{2\times 1}$$

$$\cot\theta = \frac{1\pm 3}{2}$$

$\cot\theta = 2$ **OR** $\cot\theta = -1$

$\Rightarrow \tan\theta = \frac{1}{2}$ **OR** $\tan\theta = -1$

For $\tan\theta = \frac{1}{2}$, $\theta = \tan^{-1}(\frac{1}{2})$

$\theta = 26.6°, 206.6°$

For $\tan\theta = -1$, $\theta = 135°, 315°$

$\Rightarrow$ For $\text{cosec}^2\theta = 3 + \cot\theta$,

$\theta = 26.6°, 135°, 206.6°, 315°$

(e) $3\tan^2\theta + 5 = 7\sec\theta$

$3(\sec^2\theta - 1) + 5 = 7\sec\theta$

$3\sec^2\theta - 3 + 5 = 7\sec\theta$

$3\sec^2\theta - 7\sec\theta + 2 = 0$

$$\sec\theta = \frac{7 \pm \sqrt{(-7)^2 - 4\times 3\times 2}}{3\times 2}$$

$$\sec\theta = \frac{7\pm 5}{6}$$

$\sec\theta = 2$ **OR** $\sec\theta = \frac{1}{3}$

$\Rightarrow \cos\theta = \frac{1}{2}$ **OR** $\cos\theta = 3$

For $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$

For $\cos\theta = 3$, θ is not defined because $\cos\theta$ is maximum at 1.

(f) $2\cot^2\theta + 8 = 7\text{cosec}\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$\cot^2\theta = \text{cosec}^2\theta - 1$

$2(\text{cosec}^2\theta - 1) + 8 = 7\text{cosec}\theta$

$2\text{cosec}^2\theta - 2 + 8 = 7\text{cosec}\theta$

$2\text{cosec}^2\theta - 7\text{cosec}\theta + 6 = 0$

$$\text{cosec}\,\theta = \frac{7 \pm \sqrt{(-2)^2 - 4\times 2\times 2}}{2\times 2}$$

$$\text{cosec}\,\theta = \frac{7\pm 5}{4}$$

$\text{cosec}\theta = 3$, **OR** $\text{cosec}\theta = \frac{1}{2}$

$\Rightarrow \sin\theta = \frac{1}{3}$, **OR** $\sin\theta = 2$

For $\sin\theta = \frac{1}{3}$, $\theta = 19.5, 160.5$

For $\sin\theta = 2$, $\theta = \sin^{-1}(2)$

The values of θ are not defined.

Example I (UNEB Questions)

Find all the values of θ, $0^0 \le \theta \le 360^0$, which satisfy the equation

$$\sin^2\theta - \sin 2\theta - 3\cos^2\theta = 0.$$

Solution

a) $\sin^2\theta - 2\sin\theta\cos\theta - 3\cos^2\theta = 0$

Dividing through by $\cos^2\theta$,

$\tan^2\theta - 2\tan\theta - 3 = 0$

$\tan^2\theta - 3\tan\theta + \tan\theta - 3 = 0$

$\tan\theta(\tan\theta - 3) + 1(\tan\theta - 3) = 0$

$(\tan\theta - 3)(\tan\theta + 1) = 0$

Either $\tan\theta - 3 = 0$

$\tan\theta = 3$

$\theta = \tan^{-1}(3)$

$\theta = 71.6^0, 251.6^0$

Or $\tan\theta + 1 = 0$

$\tan\theta = -1$

$\theta = \tan^{-1}(-1)$

$\theta = 135^0, 315^0$

Example II (UNEB Question)

Solve $\cos\theta + \sin 2\theta = 0$ for $0^0 \le \theta \le 360^0$.

$\cos\theta + \sin 2\theta = 0$

$\cos\theta + 2\sin\theta\cos\theta = 0$

$\cos\theta(1 + 2\sin\theta) = 0$

Either $\cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\theta = 90^0, 270^0$

Or $1 + 2\sin\theta = 0$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\theta = \sin^{-1}(-\frac{1}{2})$

$\theta = 210^0, 330^0$

For $0^0 \le \theta \le 360^0$, $\theta = 90^0, 210^0, 270^0, 330^0$

Example III (UNEB Question)

Solve $\cot^2\theta = 5(\text{cosec}\ \theta + 1)$ for $0° \leq \theta \leq 360°$

Solution

(a) $\cot^2 \theta = 5(\text{cosec}\ \theta + 1)$

But $\cot^2\theta = \text{cosec}^2\theta - 1$

$\text{cosec}^2 \theta - 1 = 5(\text{cosec}\ \theta + 1)$

$\text{cosec}^2 \theta - 1 = 5\ \text{cosec}\ \theta + 5$

$\text{cosec}^2 \theta - 5\ \text{cosec}\ \theta - 6 = 0$

$\text{cosec}^2 \theta - 6\ \text{cosec}\ \theta + \text{cosec}\ \theta - 6 = 0$

$\text{cosec}\ \theta(\text{cosec}\ \theta - 6) + 1(\text{cosec}\ \theta - 6) = 0$

$\text{cosec}\ \theta - 6)\ (\text{cosec}\ \theta + 1) = 0$

Either $\text{cosec}\ \theta = 6$

$$\frac{1}{\sin\theta} = 6$$

$$\sin\theta = \frac{1}{6}$$

$$\theta = 9.6°, 170.4°$$

Or $\text{cosec}\ \theta + 1 = 0$

$$\frac{1}{\sin\theta} = -1$$

$$\theta = 270°$$

Hence $\theta = 9.6°$, $170.4°$ and $270°$

Example IV (UNEB Question)

Solve $2\sin 2x = 3\cos x$, for $-180^0 \leq x \leq 180°$.

Solution

$2\sin 2x = 3\cos x$

$2\sin 2x - 3\cos x = 0$

But $\sin 2x = 2\sin x\cos x$

$4\sin x\cos x - 3\cos x = 0$

$\cos x\ (4\sin x - 3) = 0$

$\cos x = 0$

$x = \cos^{-1}(0)$

$x = 90^0, -90$

$4\sin x - 3 = 0$

$$\sin x = \frac{3}{4}$$

$$x = \sin^{-1}\left(\frac{3}{4}\right)$$

$x = 48.6°, 131.4°$

$\Rightarrow$ $x = (^-90°, 48.6°, 90°, 131.4°)$ are the solutions to the equation $2\sin 2x = 3\cos x$

Example V (UNEB Question)

Solve the equation $\cos x + \cos 2x = 1$ for values of x from 0^0 to 360^0 inclusive

Solution

$\cos x + \cos 2x = 1$

But $\cos 2x = 2\cos^2 x - 1$

By substitution, we have

$\cos x + 2\cos^2 x - 1 = 1$

$2\cos^2 x + \cos x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4 \times 2 \times (-2)}}{2 \times 2}$$

$$\cos x = \frac{-1 \pm \sqrt{1^2 + 16}}{4}$$

$$= \frac{-1 \pm \sqrt{17}}{4}$$

Taking $\cos x = \dfrac{-1 + \sqrt{17}}{4}$

$x = 38.7°$, $321.3°$

Taking $\cos x = \dfrac{-1 - \sqrt{17}}{4}$

$\cos x = -1.280776406$

(The values of x are not defined because x is maximum at 1)

Hence $x = 38.7^0, 321.3^0$

Example VI (UNEB Question)

Solve $7\tan\theta + \cot\theta = 5\sec\theta$ for $0° \le \theta \le 180°$.

Solution

(a) $7 \tan \theta + \cot \theta = 5 \sec \theta$

$$7\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{5}{\cos\theta}$$

Multiplying through by $\cos\theta \sin\theta$

$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7 \sin^2 \theta + 1 - \sin^2 \theta = 5 \sin \theta$

$6 \sin^2 \theta - 5 \sin \theta + 1 = 0$

$6\sin^2 \theta - 3 \sin \theta - 2 \sin \theta + 1 = 0$

$3 \sin \theta (2 \sin \theta - 1) -1 (2 \sin \theta - 1) = 0$

$(2\sin \theta - 1)(3 \sin \theta - 1) = 0$

Either $2 \sin \theta = 1$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$\theta = 30^0, 150^0$

Or $3 \sin \theta - 1 = 0$

$$\sin\theta = \frac{1}{3}$$

$$\theta = \sin^{-1}\left(\frac{1}{3}\right)$$

$\theta = 19.5^0, 160.5^0$

$\Rightarrow 19.5^0, 30^0, 150^0, 160.5^0$ are the solutions to the equation

Example VII (UNEB Question)

Solve the equation 4cos$x$ – 2cos2x = 3 for $0^0 \leq x \leq \pi$.

Solution

$4\cos x - 2(2\cos^2 x - 1) = 3$

$4\cos x - 4\cos^2 x + 2 = 3$

$4\cos x - 4\cos^2 x - 1 = 0$

$4\cos^2 x - 4\cos x + 1 = 0$

$4\cos^2 x - 2\cos x - 2\cos x + 1 = 0$

$2\cos x(2\cos x - 1) - 1(2\cos x - 1) = 0$

$(2\cos x - 1)(2\cos x - 1) = 0$

$\Longrightarrow 2\cos x - 1 = 0$

$2\cos x = 1$

$\cos x = \frac{1}{2}$

$x = 60°, 300°$

$$x = \frac{\pi}{3}, \frac{3\pi}{3}.$$

Elimination of θ from a set of equations

Example

Eliminate θ from the following equations:

(i) $x = a\cos\theta$, $y = b\sin\theta$

(ii) $x = a\cot\theta$, $y = b\sec\theta$

(iii) $x = a\tan\theta$, $y = b\tan\theta$

(iv) $x = 1 - \sin\theta$, $y = 1 + \cos\theta$

(v) $x = \sin\theta + \tan\theta$, $y = \tan\theta - \sin\theta$

(vi) $x\cos\theta + y\sin\theta = a$, $x\sin\theta - y\cos\theta = b$

Solution

(i) $x = a\cos\theta$, $y = b\sin\theta$

$\frac{x}{a} = \cos\theta$, $\frac{y}{b} = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii) $x = a\cot\theta$, $y = b\,\mathrm{cosec}\theta$

$\frac{x}{a} = \cot\theta$, $= \frac{y}{b} = \mathrm{cosec}\theta$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$1 + \left(\frac{x}{a}\right)^2 = \left(\frac{y}{b}\right)^2$

$1 + \frac{x^2}{a^2} = \frac{y^2}{b^2}$

(iii) $x = a\tan\theta$, $y = b\cos\theta$

$$\frac{x}{a} = \tan\theta \,, \; \frac{y}{b} = \cos\theta \;\; \Rightarrow \frac{b}{y} = \sec\theta$$

$1 + \tan^2\theta = \sec^2\theta$

$$1 + \left(\frac{x}{a}\right)^2 = \frac{b^2}{y^2}$$

$$1 + \frac{x^2}{a^2} = \frac{b^2}{y^2}$$

(iv) $x = 1 - \sin\theta$, $\quad y = 1 + \cos\theta$
$\sin\theta = 1 - x$, $\quad y - 1 = \cos\theta$
$\sin^2\theta + \cos^2\theta = 1$
$(1 - x)^2 + (y - 1)^2 = 1$
$\Rightarrow (x - 1)^2 + (y - 1)^2 = 1$

(v) $\boldsymbol{x = \sin\theta + \tan\theta}$ …………………… (i)
$y = \tan\theta - \sin\theta$ ………………..…… (ii)
Eqn (i) + Eqn (ii);
$\Rightarrow x + y = 2\tan\theta$

$$\tan\theta = \frac{x+y}{2}$$

Eqn (i) − Eqn (ii);
$x - y = 2\sin\theta$

$$\frac{x-y}{2} = \sin\theta$$

From $\tan\theta = \frac{x+y}{2}$

$$\Rightarrow \cot\theta = \frac{2}{x+y}$$

From $\sin\theta = \frac{x-y}{2}$

$$\Rightarrow \text{cosec}\,\theta = \frac{2}{x-y}$$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$$1 + \left(\frac{2}{x+y}\right)^2 = \left(\frac{2}{x-y}\right)^2$$

$$1 + \frac{4}{(x+y)^2} = \frac{4}{(x-y)^2}$$

$\Rightarrow (x^2 - y^2)^2 = 16xy$

(vi) $x\cos\theta + y\sin\theta = a$ ..………………..(i)
$x\sin\theta - y\cos\theta = b$ ………………….. (ii)

From Eqn (i);

$$\cos\theta = \frac{a - y\sin\theta}{x} \quad \text{…………….. (iii)}$$

Substituting Eqn (iii) in Eqn (ii);

$$x\sin\theta - y\left(\frac{a - y\sin\theta}{x}\right) = b$$

$\Rightarrow x^2\sin\theta - ay + y^2\sin\theta = xb$

$(x^2 + y^2)\sin\theta = xb + ay$

$\sin\theta = \dfrac{bx + ay}{x^2 + y^2}$ …………………… (iv)

Substitute Eqn (iv) in Eqn (iii)

$\Rightarrow \cos\theta = \dfrac{a - y(\frac{bx+ay}{x^2+y^2})}{x}$

$\cos\theta = \dfrac{ax^2 + ay^2 - bxy - ay^2}{x(x^2 + y^2)}$

$\cos\theta = \dfrac{ax^2 - bxy}{x(x^2 + y^2)}$

$\cos\theta = \dfrac{ax - by}{x^2 + y^2}$

$\sin^2\theta + \cos^2\theta = 1$

$\dfrac{(bx + ay)^2}{(x^2 + y^2)^2} + \dfrac{(ax - by)^2}{(x^2 + y^2)^2} = 1$

$(bx + ay)^2 + (ax - by)^2 = (x^2 + y^2)^2$

$b^2x^2 + 2abxy + a^2y^2 + a^2x^2 - 2abxy + b^2y^2 = (x^2 + y^2)^2$

$(a^2 + b^2)x^2 + (a^2 + b^2)y^2 = (x^2 + y^2)^2$

$(x^2 + y^2)(a^2 + b^2) = (x^2 + y^2)^2$

$a^2 + b^2 = x^2 + y^2$

Proving Trigonometric Identities

(i) $\sec\theta + \text{cosec}\theta\cot\theta = \sec\theta\,\text{cosec}^2\theta$

(ii) $\sin^2\theta(1 + \sec^2\theta) = \sec^2\theta - \cos^2\theta$

(iii) $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{1}{\cos ec\theta + \cot\theta}$

(iv) $\dfrac{\tan\theta + \cot\theta}{\sec\theta + \text{cosec}\theta} = \dfrac{1}{\sin\theta + \cos\theta}$

(v) $\sec^2\theta = \dfrac{\text{cosec}\theta}{\text{cosec}\theta - \sin\theta}$

(vi) $\dfrac{1+\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

(vii) $\dfrac{1+\sin\theta}{1-\sin\theta} = (\sec\theta + \tan\theta)^2$

(viii) $\dfrac{\cot\alpha + \tan\beta}{\cot\beta + \tan\alpha} = \cot\alpha\tan\beta$

Solution

(a) secθ + cosecθ cotθ

$$= \frac{1}{\cos\theta} + \frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$$
$$= \frac{1}{\cos\theta} + \frac{\cos\theta}{\sin^2\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}$$
$$= \frac{1}{\sin^2\theta\cos^2\theta}$$
$$= \frac{1}{\cos\theta} \times \frac{1}{\sin^2\theta}$$
$$= \sec\theta\operatorname{cosec}\theta$$

(b) $\mathbf{\sin^2\theta(1 + \sec^2\theta)}$

$$= \sin^2\theta + \sin^2\theta\sec^2\theta$$
$$= \sin^2\theta + \frac{\sin^2\theta}{\cos^2\theta}$$
$$= \sin^2\theta + \tan^2\theta$$
$$= \sin^2\theta + \sec^2\theta - 1$$
$$= 1 - \cos^2\theta + \sec^2\theta - 1$$
$$= \sec^2\theta - \cos^2\theta$$

(c) $\frac{1-\cos\theta}{\sin\theta}$

$$= \frac{1-\cos\theta}{\sin\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$$
$$= \frac{1-\cos^2\theta}{\sin\theta + \sin\theta\cos\theta}$$
$$= \frac{\sin^2\theta}{\sin\theta + \sin\theta\cos\theta}$$
$$= \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\frac{\sin\theta}{\sin^2\theta} + \frac{\sin\theta\cos\theta}{\sin^2\theta}}$$
$$= \frac{1}{\operatorname{cosec}\theta + \cot\theta}$$

(d) $\frac{\tan\theta + \cot\theta}{\sec\theta + \operatorname{cosec}\theta}$

$$= \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}$$
$$= \frac{\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}} \quad = \frac{1}{\sin\theta + \cos\theta}$$

(e) $\sec^2\theta = \dfrac{1}{\cos^2\theta}$

$$\sec^2\theta = \frac{1}{1-\sin^2\theta}$$

$$= \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}}$$

$$= \frac{\text{cosec}\,\theta}{\text{cosec}\,\theta - \sin\theta}$$

(f) $\dfrac{1+\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

$$\frac{1+\sin\theta}{\cos\theta} = \frac{(1+\sin\theta)\cos\theta}{\cos\theta\times\cos\theta}$$

$$= \frac{\cos\theta + \sin\theta\cos\theta}{\cos^2\theta}$$

$$= \frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta\cos\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta$$

(g) $\dfrac{1+\sin\theta(1+\sin\theta)}{1-\sin\theta(1+\sin\theta)}$

$$= \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta}$$

$$= \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta + 2\tan\theta\sec\theta + \tan^2\theta$$

$$= (\sec\theta + \tan\theta)^2$$

(h) $\dfrac{\cot\alpha + \tan\beta}{\cot\beta + \tan\alpha}$

$$= \frac{\frac{\cos\alpha}{\sin\alpha} + \frac{\sin\beta}{\cos\beta}}{\frac{\cos\beta}{\sin\beta} + \frac{\sin\alpha}{\cos\alpha}}$$

$$= \frac{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\sin\beta}}$$

$$= \frac{\cos\alpha\sin\beta}{\cos\beta\sin\alpha}$$

$$= \frac{\cos\alpha}{\sin\alpha}\cdot\frac{\sin\beta}{\cos\beta}$$

$$= \cot\alpha\tan\beta$$

Formulae for sin(A ± B), cos(A ± B), and tan(A ± B)

sin(*A* + *B*) = sin*A*cos*B* + cos*A*sin*B*
sin(*A* – *B*) = sin*A*cos*B* – cos*A*sin*B*
cos(*A* + *B*) = cos*A*cos*B* – sin*A*sin*B*
cos(*A* – *B*) = cos*A*cos*B* + sin*A*sin*B*

tan(*A* + *B*) = $\frac{\tan A + \tan B}{1 - \tan A + \tan B}$

tan(*A* – *B*) = $\frac{\tan A - \tan B}{1 + \tan A + \tan B}$

Examples

Find the values of the following:

(a) cos(45° – 30°)
(b) cos 105°
(c) cos 75°
(d) sin(60° + 45°)
(e) sin 15°

Solution

(a) cos(45° – 35°)

= cos45° cos30° + sin45°sin30°

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{3})2\sqrt{2}}{(2\sqrt{2})(2\sqrt{2})}$$

$$= \frac{2\sqrt{2}+2\sqrt{6}}{8} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

(b) sin(30° +45°)

= sin30 cos45 + cos30 sin 45

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{2\sqrt{2}(\sqrt{3}+1)}{4} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

(c) cos 105°

= cos(60° + 45°)

= cos60 cos45 – sin60 sin45

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \qquad = \frac{(1-\sqrt{3})\sqrt{2}}{2\sqrt{2}\cdot\sqrt{2}}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

(d) cos 75°

= cos(30° + 45°)

= cos30° cos45° – sin30° sin45°

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{(\sqrt{3}-1)2\sqrt{2}}{2\sqrt{2}\times 2\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

(f) sin(60° + 45°)

= sin60 cos45 + cos60 sin45

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{(\sqrt{3}+1)2\sqrt{2}}{2\sqrt{2} \times 2\sqrt{2}}$$

$$= \frac{2\sqrt{2}(\sqrt{3}+1)}{8} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(f) sin 15°

= sin(45 – 30)

= sin45 cos30 – cos45 sin30

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{2\sqrt{3}-1}{2\sqrt{2}}$$

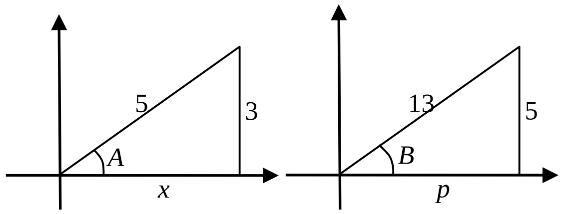
$$= \frac{(\sqrt{3}-1)2\sqrt{2}}{8} \quad = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Example II

If sin $A = \frac{3}{5}$ and sin$B = \frac{5}{13}$, where A and B are acute angles, find the values of the following:

(a) sin(*A* + *B*)
(b) cos(*A* + *B*)
(c) cot(*A* + *B*)

Solution



$x^2 + 3^2 = 5^2$ $p^2 + 5^2 = 13^2$

$x^2 + 9 = 25$ $p^2 + 25 = 169$

$x^2 = 16$ $p^2 = 144$

$x = 4$ $p = 12$

$$\Rightarrow \sin A = \frac{3}{5}; \quad \cos A = \frac{4}{5}; \tan A = \frac{3}{4}$$

$$\sin B = \frac{5}{13}; \quad \cos B = \frac{12}{13}; \tan B = \frac{5}{12}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13}$$

$$= \frac{36}{65} + \frac{20}{65}$$

$$= \frac{56}{65}$$

(b) **cos(*A* + *B*) =** $\cos A \cos B - \sin A \sin B$

$$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}$$

$$= \frac{48}{65} - \frac{15}{65}$$

$$= \frac{33}{65}$$

(c) **cot(A + B)** = $\dfrac{1}{\tan(A+B)}$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \cot(A+B) = \frac{1 - \tan A \tan B}{\tan A + \tan B}$$

$$= \frac{1 - \frac{3}{4} \times \frac{5}{12}}{\frac{3}{4} + \frac{5}{12}}$$

$$= \frac{1 - \frac{15}{48}}{7/6} = \frac{33/16}{7/6}$$

$$= \frac{33}{16} \times \frac{6}{7}$$

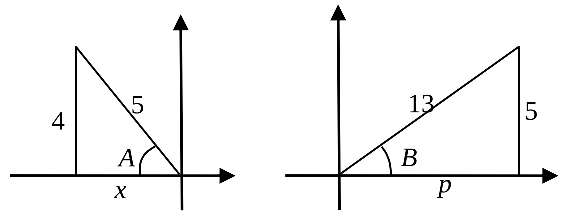
$$= \frac{99}{56}$$

Example III

If $\sin A = \frac{4}{5}$, $\cos B = \frac{12}{13}$, where *A* is obtuse and *B* is acute, find the values of:

- **(a) sin(A – B)**
- **(b) tan(A – B)**
- **(c) tan(A + B)**

Solutions



$x^2 + 4^2 = 5^2$ $p^2 + 12^2 = 13^2$

$x^2 + 16 = 25$ $p^2 + 144 = 169$

$x^2 = 9$ $p^2 = 25$

$x = 3$ $p = 5$

A is obtuse

$\Rightarrow \sin A = \frac{4}{5}$; $\cos A = \frac{-3}{5}$; $\tan A = \frac{-4}{3}$

B is acute

$\Rightarrow \sin B = \frac{5}{13}$; $\cos B = \frac{12}{13}$; $\tan B = \frac{5}{12}$

$\text{Sin}(A - B) = \sin A \cos B - \cos A \sin B$

$$= \frac{4}{5} \times \frac{12}{13} - \frac{-3}{5} \times \frac{5}{13}$$

$$= \frac{48}{65} + \frac{15}{65}$$

$$= \frac{63}{65}$$

(b) $\mathbf{\tan(A - B)} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\frac{-4}{3} - \frac{5}{12}}{1 + \frac{-4}{3} \times \frac{5}{12}}$$

$$= \frac{-7/4}{4/9} = \frac{-63}{16}$$

(c) $\mathbf{\tan(A + B)} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{\frac{-4}{3} + \frac{5}{12}}{1 - \frac{-4}{3} \times \frac{5}{12}}$$

$$= \frac{\frac{-11}{12}}{1 + \frac{20}{36}}$$

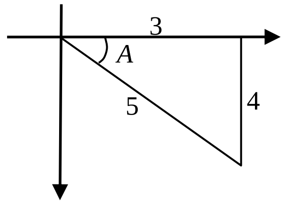
$$= \frac{\frac{-11}{12}}{\frac{56}{36}} = \frac{-33}{56}$$

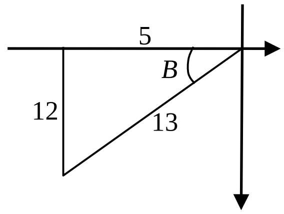
Example III

If $\cos A = \frac{3}{5}$ and $\tan B = \frac{12}{5}$; where *A* and B are reflex angles. Find the values of:

(a) sin(*A* – *B*)
(b) tan(*A* – *B*)
(c) cos(*A* + *B*)

Solutions





A and *B* are reflex

$\Rightarrow \cos A = \frac{3}{5}$; $\sin A = \frac{-4}{5}$; $\tan A = \frac{-4}{3}$

$\cos B = \frac{-5}{13}$, $\sin B = \frac{-12}{13}$; $\tan B = \frac{12}{5}$

(a) sin(*A* – *B*) = sin *A* cos *B* – cos *A* sin *B*

$$= \frac{-4}{5} \times \frac{-5}{13} - \frac{3}{5} \times \frac{-12}{13}$$

$$= \frac{20}{65} + \frac{36}{65}$$

$$= \frac{56}{65}$$

(b) tan(*A* – *B*) = $\frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\frac{-4}{3} - \frac{12}{5}}{1 + \frac{-4}{3} \times \frac{12}{5}}$$

$$= \frac{-56/15}{-11/5} \qquad = \frac{56}{33}$$

(c) cos(A + B) = cos *A* cos *B* – sin *A* sin*B*

$$= \frac{3}{5} \times \frac{-5}{13} - \frac{-4}{5} \times \frac{-12}{13}$$

$$= \frac{-15}{65} - \frac{48}{65}$$

$$= \frac{-63}{65}$$

Example IV

From the following, find the values of tan *x*

- **(a) sin(*x* + 45°) = 2cos(*x* + 45°)**
- **(b) 2sin(*x* – 45°) = cos(*x* + 45°)**
- **(c) tan(*x* – *A*) =** $\frac{3}{2}$**,** where tan *A* = 2
- **(d) sin(*x* + 30°) = cos(*x* + 30°)**

Solution

(a) sin(*x* + 45°) = 2cos(*x* + 45°)

sin*x* cos45°+cos*x* sin45° = 2(cos*x* cos45°–sin*x* sin45°)

$$\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = 2(\cos x \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\sin x)$$

$$\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = 2(\tfrac{\sqrt{2}}{2}\cos x - \tfrac{\sqrt{2}}{2}\sin x)$$

$$\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = \sqrt{2}\cos x - \sqrt{2}\sin x$$

$$\left(\tfrac{\sqrt{2}}{2} + \sqrt{2}\right)\sin x = \sqrt{2}\cos x - \frac{\sqrt{2}}{2}\cos x$$

$$\frac{3\sqrt{2}}{2}\sin x = \frac{\sqrt{2}}{2}\cos x$$

3sin x = cos x

$$\frac{3\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

3 tan x = 1

tan x = $\frac{1}{3}$

(b) 2sin(x – 45°) = cos(x + 45°)

2(sin$x$ cos45 – cos$x$sin45) = cos$x$ cos45 – sin$x$sin45

$$2\left(\sin x(\tfrac{\sqrt{2}}{2}) - \cos x(\tfrac{\sqrt{2}}{2}) \;=\; \cos x(\tfrac{\sqrt{2}}{2}) - \sin x(\tfrac{\sqrt{2}}{2})\right)$$

$$\sqrt{2}\sin x - \sqrt{2}\cos x = \tfrac{\sqrt{2}}{2}\cos x - \tfrac{\sqrt{2}}{2}\sin x$$

$$\sqrt{2}\sin x + \tfrac{\sqrt{2}}{2}\sin x = \sqrt{2}\cos x + \tfrac{\sqrt{2}}{2}\cos x$$

$$\tfrac{3\sqrt{2}}{2}\sin x = \tfrac{3\sqrt{2}}{2}\cos x$$

tan x = 1

(c) tan(x – A) = $\frac{3}{2}$, tan A = 2

$$\frac{\tan x - \tan A}{1 + \tan x \tan A} = \frac{3}{2}$$

$$\frac{\tan x - 2}{1 + 2\tan x} = \frac{3}{2}$$

2(tan x – 2) = 3(1 + 2tan x)

2tan x – 4 = 3 + 6tan x

4tan x = -7

tan x = $\frac{-7}{4}$

(d) sin(x + 30) = cos(x + 30)

sin x cos 30 + cos x sin 30 = cos x cos 30 – sin x sin 30

$$\tfrac{\sqrt{3}}{2}\sin x + \tfrac{1}{2}\cos x = \tfrac{\sqrt{3}}{2}\cos x - \tfrac{1}{2}\sin x$$

$$\tfrac{\sqrt{3}}{2}\sin x + \tfrac{1}{2}\sin x = \tfrac{\sqrt{3}}{2}\cos x - \tfrac{1}{2}\cos x$$

$$\sin x\left(\tfrac{\sqrt{3}+1}{2}\right) = \cos x\left(\tfrac{\sqrt{3}-1}{2}\right)$$

$$\frac{\sin x}{\cos x} = \frac{\frac{\sqrt{3}-1}{2}}{\frac{\sqrt{3}+1}{2}}$$

$$\tan x = \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$\tan x = \frac{(\sqrt{3}-1)(1-\sqrt{3})}{(\sqrt{3}+1)(1-\sqrt{3})}$$

$$\tan x = \frac{\sqrt{3}-3-1+\sqrt{3}}{-2}$$

$$\tan x = 2 - \sqrt{3}$$

Example V

Solve the following equations for $0° \le \theta \le 360°$

(a) $2\sin x = \cos(x + 60°)$

(b) $\cos(x + 45°) = \cos x$

(c) $\sin(x - 30°) = \frac{1}{2}\cos x$

(d) $3\sin(x + 10°) = 4\cos(x - 10°)$

Solutions

(a) $2\sin x = \cos(x + 60°)$

$$2\sin x = \cos x \cos 60° - \sin x \sin 60°$$

$$2\sin x = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$$

$$2\sin x + \frac{\sqrt{3}}{2}\sin x = \frac{1}{2}\cos x$$

$$(4 + \sqrt{3})\sin x = \cos x$$

$$\tan x = \frac{1}{4+\sqrt{3}}$$

$$x = 9.9°, 189.9°$$

(b) $\cos(x + 45°) = \cos x$

$$\cos x \cos 45° - \sin x \sin 45° = \cos x$$

$$\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x = \cos x$$

$$\frac{\sqrt{2}}{2}\cos x + \cos x + \cos x = \frac{\sqrt{2}}{2}\sin x$$

$$(\frac{\sqrt{2}}{2}+1)\cos x = \frac{\sqrt{2}}{2}\sin x$$

$$(\frac{\sqrt{2}+2}{2})\cos x = \frac{\sqrt{2}}{2}\sin x$$

$$\frac{\sqrt{2}+2}{\sqrt{2}} = \frac{\sin x}{\cos x}$$

$$\frac{\sqrt{2}+2}{\sqrt{2}} = \tan x$$

$$x = 67.5°, 247.5°$$

(c) $\sin(x + 30) = \frac{1}{2}\cos x$

$\sin x \cos 30 – \cos x \sin 30 = \frac{1}{2}\cos x$

$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{2}\cos x$

$\frac{\sqrt{3}}{2}\sin x = \cos x$

$\frac{\sin x}{\cos x} = \frac{2}{\sqrt{3}}$

$\tan x = \frac{2}{\sqrt{3}}$

$x = 49.1°, 229.1°$

(d) $2\sin(x + 10°) = 4\cos(x – 10°)$

$2(\sin x \cos 10 - \cos x \sin 10)$

$= 4(\cos x \cos 10° + \sin x \sin 10°)$

$2\sin x\cos 10–2\cos x \sin 10=4\cos x \cos 10+4\sin x\sin 10$

$2\sin x \cos 10 – 4\sin x \sin 10$

$= 4\cos x \cos 10 + 2\cos x \sin 10$

$\sin x(2\cos 10 – 4\sin 10) = \cos x(4\cos 10 + 2 \sin 10)$

$$\frac{\sin x}{\cos x} = \frac{4\cos 10 + 2\sin 10}{2\cos 10 - 4\sin 10}$$

$$\tan x = \frac{4\cos 10 + 2\sin 10}{2\cos 10 - 4\sin 10}$$

$x = 73.4°, x = 253.4°$

Example VI

If $\tan(x + 45°) = 2$, find the value of $\tan x$

Solution

$\tan(x + 45°) = 2.$

$\frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = 2$

$\frac{\tan x + 1}{1 - \tan x} = 2$

$\tan x + 1 = 2(1 – \tan x)$

$\tan x + 1 = 2 – 2\tan x$

$3\tan x = 1$

$\tan x = \frac{1}{3}$

Example VII

If $\tan(A + B) = \frac{1}{7}$ and $\tan A = 3$, find the value of $\tan B$.

$\tan(A + B) = \frac{1}{7}$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{7}$$

$\tan A = 3$

$$\frac{3 + \tan B}{1 - 3\tan B} = \frac{1}{7}$$

$7(3 + \tan B) = 1 – 3\tan B$

$21 + 7\tan B = 1 – 3\tan B$

$10\tan B = -20$

$\tan B = -2$

Example VIII

Express the following as single trigonometric ratios.

(a) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

(b) $\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$

(c) $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

(d) $\frac{1}{\cos 24 \cos 15 - \sin 24 \sin 15}$

(e) $\frac{1}{2}\cos 75 + \frac{\sqrt{3}}{2}\sin 75$

(f) $\frac{1 - \tan 15}{1 + \tan 15}$

Solutions

(a) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

$= \cos 60 \cos x – \sin 60 \sin x$

$\cos(60 + x)$

$\Rightarrow \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \cos(60 + x)$

(b) $\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$

$= \frac{\tan 60 + \tan x}{1 - \tan 60 \tan x}$

$= \tan(60 + x)$

(c) $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

$= \cos 45 \sin x + \sin 45 \cos x$

$= \cos(45 – x)$

(d) $\frac{1}{\cos 24 \cos 15 - \sin 24 \sin 15}$

$= \frac{1}{\cos(24+15)}$

$= \frac{1}{\cos 39}$

= sec 39°

(e) $\frac{1}{2}\cos 75 + \frac{\sqrt{3}}{2}\sin 75$

cos 60° cos 75° + sin 60° sin 75°

cos 75° cos 60° + sin 75° sin 60°

cos(75° – 60°)

cos 15°

(f) $\frac{1-\tan 15}{1+\tan 15} = \frac{\tan 45 - \tan 15}{1+\tan 45 \tan 15}$

= tan(45 – 15)

= tan(30)

Example IX

Prove the following identities:

(i) **sin(*A* + *B*) + sin(*A* – *B*) = 2sin *A* cos *B***

(ii) **cos(A + B) – cos(A – B) = -2sin A sin B**

(iii) **tan *A* + tan *B* =** $\frac{\sin(A+B)}{\cos A \cos B}$

(iv) **tan(A + B + C) =** $\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$

Hence prove that if *A*, *B*, and *C* are angles of a triangle, then tan *A* + tan *B* + tan *C* = tan *A* tan *B* tan *C*

Solution

sin (*A* + *B*) + sin(*A* – *B*)

sin *A* cos *B* + cos *A* sin *B* + sin *A* cos*B* – cos*A* sin*B*

= 2sin *A* cos *B*

$\Rightarrow$ sin(*A* + *B*) + sin(*A* – *B*) = 2sin *A* cos *B*

(ii) cos(*A* + *B*) – cos(*A* – *B*)

cos *A* cos *B* – sin *A* sin *B* – (cos *A* cos *B* + sin *A* sin *B*)

= -2sin A sin *B*

$\Rightarrow$ cos(A + B) – cos(A – B) = -2sin A sin B

(iii) tan A + tan B

$= \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}$

$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$

$= \frac{\sin(A+B)}{\cos A \cos B}$

$$\Rightarrow \tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$$

(iv) tan(A + B + C)

Let $B + C = D$

$$\tan(A + D) = \frac{\tan A + \tan D}{1 - \tan A \tan D}$$

$$= \frac{\tan A + \tan(B+C)}{1 - \tan A \tan(B+C)}$$

$$= \frac{\tan A + \frac{\tan B + \tan C}{1 - \tan B \tan C}}{1 - \tan A\left(\frac{\tan B + \tan C}{1 - \tan B \tan C}\right)}$$

$$= \frac{\frac{\tan A - \tan A \tan B \tan C + \tan B + \tan C}{1 - \tan B \tan C}}{\frac{1 - \tan B \tan C - \tan A \tan B - \tan A + \tan C}{1 - \tan B \tan C}}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$$

Since A, B, and C are angles of a triangle, then

$A + B + C = 180°$

$\tan(A + B + C) = \tan 180°$

$\tan(A + B + C) = 0$

$$\Rightarrow \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B} = 0$$

$\tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Example (UNEB Question)

Without using tables or calculator, evaluate tan 15°

Solution

$\tan 15° = \tan(45° - 30°)$

$$= \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{(\sqrt{3} - 1)(1 - \sqrt{3})}{(\sqrt{3} + 1)(1 - \sqrt{3})}$$

$$= \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1 - 3}$$

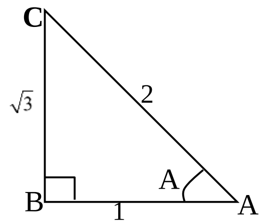
$$= \frac{2\sqrt{3} - 4}{-2} = 2 - \sqrt{3}$$

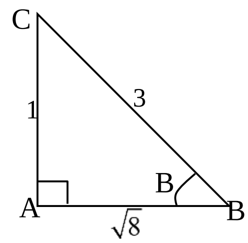
Example (UNEB Question)

The acute angles A and B are such that cos***A*** = ½, sin ***B*** = 1/3. Show without the use of tables or calculator, show that

$$\tan(A+B)=\frac{9\sqrt{3}+8\sqrt{2}}{5}$$

Solution





$$\tan B=\frac{1}{\sqrt{8}}=\frac{\sqrt{8}}{8}$$

$$=\frac{2\sqrt{2}}{8}=\frac{\sqrt{2}}{4}$$

$$\tan B=\frac{\sqrt{3}}{2}$$

From compound angle formula,

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$=\frac{\sqrt{3}+\frac{\sqrt{2}}{4}}{1-(\sqrt{3}\times\frac{\sqrt{2}}{4})}$$

$$=\frac{\frac{4\sqrt{3}+\sqrt{2}}{4}}{\frac{4-\sqrt{3}\times\sqrt{2}}{4}}$$

$$=\frac{4\sqrt{3}+\sqrt{2}}{4}\times\frac{4}{4-\sqrt{6}}$$

$$=\frac{\left(4\sqrt{3}+\sqrt{2}\right)}{\left(4-\sqrt{6}\right)}$$

$$=\frac{\left(4\sqrt{3}+\sqrt{2}\right)\left(4+\sqrt{6}\right)}{\left(4-\sqrt{6}\right)\left(4+\sqrt{6}\right)}$$

$$=\frac{16\sqrt{3}+4\sqrt{18}+4\sqrt{2}+\sqrt{12}}{16-6}$$

$$=\frac{16\sqrt{3}+4\times3\sqrt{2}+4\sqrt{2}+2\sqrt{3}}{10}$$

$$=\frac{18\sqrt{3}+16\sqrt{2}}{10}$$

$$=\frac{9\sqrt{3}+8\sqrt{2}}{5}$$

Double angle & Triple angle formulae

By writing $A = B$ in the additional formulae for sine, cosine, and tangent, we obtain the double angle formula for each of them.

$$\sin(A + B) = \sin A \cos B + \cos A \cos B$$

$\Rightarrow \sin 2A = \sin(A + A)$

$= \sin A \cos A + \cos A \sin A$

$= 2\sin A \cos A$

$$\boxed{\sin 2A = 2\sin A \cos A}$$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A + A) = \cos A \cos A - \sin A \sin A.$

$= \cos^2 A - \sin^2 A$

But $\cos^2 A = 1 - \sin^2 A$

$\Rightarrow \cos 2A = 1 - \sin^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

But when $\sin^2 A = 1 - \cos^2 A$

$\cos^2 A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

$\cos {}^2A = 2\cos^2 A - 1$ OR

$\cos {}^2A = 1 - 2\sin^2 A$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$; where $A = B$

$\tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A}$

$= \dfrac{2\tan A}{1 - \tan^2 A}$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$\sin 3A = \sin(A + 2A)$

$= \sin A \cos 2A + \cos A \sin 2A$

$= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A \cos A)$

$= \sin A - 2\sin^3 A + 2\cos^2 A \sin A$

$= \sin A - 2\sin^3 A + 2(1 - \sin^2 A)\sin A$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

$$\boxed{\sin 3A = 3\sin A - 4\sin^3 A}$$

$\cos 3A = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

$\Rightarrow$ **$\cos 3A = 4\cos^3 A - 3\cos A$**

$\tan 3A = \tan(A + 2A)$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{\frac{\tan A - \tan^3 A + 2\tan A}{1-\tan^2 A}}{\frac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\boxed{\Rightarrow \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}$$

Example I

Simplify the following expressions

(i) **2sin 17 cos 17**

(ii) $\frac{2\tan 30°}{1 - \tan^2 30°}$

(iii) **$2\cos^2 42 - 1$**

(iv) $2\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$

(v) $1 - 2\sin^2 22\frac{1}{2}°$

(vi) $\frac{2\tan \frac{1}{2}\theta}{1 - \tan^2 \frac{\theta}{2}}$

(vii) **$1 - 2\sin^2 3\theta$**

(viii) $\frac{1 - \tan^2 20}{\tan 20}$

(ix) **sec θ cosec θ**

(x) **2sin 2*A* cos 2*A***

Solutions

(i) **sin 2(17°)** = 2sin 17° cos 17°

sin 34° = 2sin 17° cos 17°

$\Rightarrow$ 2sin 17 cos 17 = sin 34

(ii) **tan(30° + 30°) =** $\frac{\tan 30° + \tan 30°}{1 - \tan 30° \tan 30°}$

tan 60° = $\frac{2\tan 30°}{1 - \tan^2 30°}$

$$\frac{2\tan 30°}{1-\tan^2 30°} = \tan 60°$$

(iii) $\mathbf{2\cos^2 42° - 1}$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2(42°) = 2\cos^2 42° - 1$

$\cos 84° = 2\cos^2 42° - 1$

$2\cos^2 42° - 1 = \cos 84°$

(iv) $\mathbf{2\sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2(\tfrac{1}{2}\theta) = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$

$\sin\theta = 2\sin\tfrac{1}{2}\theta\ \cos\tfrac{1}{2}\theta$

$\Rightarrow \sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$

$2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = \sin\theta$

(v) $\mathbf{1 - 2\sin^2 22½°}$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2(22½) = 1 - 2\sin^2 22½$

$\cos 45 = 1 - 2\sin^2 22½$

$1 - 2\sin^2 22½ = \cos 45$

(vi) $$\frac{2\tan\frac{1}{2}\theta}{1-2\tan^2\frac{1}{2}\theta} = \frac{\tan\frac{1}{2}\theta + \tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta}$$

$= \tan(\tfrac{1}{2}\theta + \tfrac{1}{2}\theta +)$

$= \tan\theta$

(vii) $1 - 2\sin^2\theta$

$\cos 2(3\theta) = 1 - 2\sin^2 3\theta$

$\cos 6\theta = 1 - 2\sin^2 3\theta$

$1 - 2\sin^2 3\theta = \cos 6\theta$

(viii) $$\frac{1-\tan^2 20}{\tan 20}$$

$\tan 40 = \tan(20 + 20)$

$$= \frac{2\tan 20}{1-\tan^2 20}$$

$$\frac{1}{\tan 40} = \frac{1-\tan^2 20}{2\tan 20}$$

$$\frac{2}{\tan 40} = \frac{1-\tan^2 20}{\tan 20}$$

$$2\cot 40 = \frac{1-\tan^2 20}{\tan 20}$$

$$\frac{1-\tan^2 20}{\tan 20} = 2\cot 40$$

(ix) $\sec\theta\,\text{cosec}\,\theta = \frac{1}{\cos\theta} \times \frac{1}{\sin\theta}$

$= \frac{1}{\cos\theta\sin\theta}$

But $\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

$\sec\theta\,\text{cosec}\,\theta = \frac{1}{\frac{1}{2}\sin 2\theta}$

$\sec\theta\,\text{cosec}\,\theta = \frac{2}{\sin 2\theta}$

$\sec\theta\,\text{cosec}\,\theta = 2\,\text{cosec}\,2\theta$

(x) 2sin 2*A* cos 2*A*

$\sin 4A = \sin 2(2A)$

$= 2\sin 2A\cos 2A$

$\Rightarrow 2\sin 2A\cos 2A = \sin 4A$

Example II

Evaluate the following without using tables or calculator:

(a) $2\sin 15° \cos 15°$
(b) $2\cos^2 75° - 1$
(c) $\cos^2 22½° - \sin^2 22½°$
(d) $\frac{1-2\cos^2 25}{1-2\sin^2 65}$
(e) $\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}}$
(f) $1-2\sin^2 67\frac{1}{2}$

Solution

(a) $2\sin 15° \cos 15° = \sin 2(15°)$

$= \sin 30°$

$= \frac{1}{2}$

(b) $2\cos^2 75° - 1 = \cos 150°$

$= -\cos 30°$

$= \frac{-\sqrt{3}}{2}$

(c) $\cos^2 22½° - \sin^2 22½°$

$= \cos(22\frac{1}{2}° + 22\frac{1}{2}°)$

$= \cos 45$

$= \frac{\sqrt{2}}{2}$

(d) $\frac{1-2\cos^2 25}{1-2\sin^2 65} = \frac{-1(2\cos^2 25 - 1)}{1-2\sin^2 65}$

$= \frac{-1(\cos 50°)}{\cos 130°}$

$= \frac{-1(\cos 50°)}{-\cos 50°} = 1$

(e) $\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}} = \tan(22\tfrac{1}{2}° + 22\tfrac{1}{2}°)$

$\tan 45° = 1$

$\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}} = \tan 45° = 1$

(f) $1-2\sin^2 67\tfrac{1}{2}$ = cos 135°

= -cos 45°

$= \frac{-\sqrt{2}}{2}$

Example III

Solve the following equations from $0 \le \theta \le 360°$

(a) **cos 2θ + cos θ + 1 = 0**

(b) **sin 2θ cos θ + sin²θ = 1**

(c) **2sin θ(5cos 2θ + 1) = 3 sin 2θ**

(d) **3cot 2**θ + cot θ = 1

(e) **4tan θ tan 2θ = 1**

Solution

(a) **cos 2θ + cos θ + 1 = 0**

$2\cos^2\theta - 1 + \cos\theta + 1 = 0$

$2\cos^2\theta + \cos\theta = 0$

$\cos\theta(2\cos\theta + 1) = 0$

$\cos\theta = 0, \quad \cos\theta = \frac{-1}{2}$

For $\cos\theta = 0, \ \theta = 90°, 270°$

For $\cos\theta = \frac{-1}{2}$, θ = 120°, 240°

$\Rightarrow$ The solutions to the equation

cos 2θ + cos θ + 1 = 0 are 90°, 120°, 240° and 270°.

(b) **sin 2θ cos θ + sin²θ = 1**

$(2\sin\theta\cos\theta)\cos\theta + \sin^2\theta = 1$

$2\cos^2\theta\sin\theta + \sin^2\theta = 1$

$2(1-\sin^2\theta)\sin\theta + \sin^2\theta = 1$

$2\sin\theta - 2\sin^3\theta + \sin^2\theta = 1$

$$2\sin^3\theta - \sin^2\theta - 2\sin\theta + 1 = 0$$

$$\sin\theta = 1, \quad \sin\theta = -1$$

$$\sin\theta = \frac{1}{2}$$

For $\sin\theta = 1$, θ = 90°

For $\sin\theta = -1$, θ = 270°

For $\sin\theta = \frac{1}{2}$, θ = 30°, 150°

$\Rightarrow$ 30°, 90°, 150°, 270° are the solutions to the equation sin 2θ cos $\boldsymbol{\theta}$ + sin$^2\boldsymbol{\theta}$ = 1

(c) 2sin $\boldsymbol{\theta}$(5cos 2$\boldsymbol{\theta}$ + 1) = 3 sin 2$\boldsymbol{\theta}$

$$2\sin\theta[5(2\cos^2\theta - 1) + 1] = 3 \cdot 2\sin\theta\cos\theta$$

$$2\sin\theta(10\cos^2\theta - 5 + 1) = 6\sin\theta\cos\theta$$

$$20\cos^2\theta\sin\theta - 8\sin\theta = 6\sin\theta\cos\theta$$

$$20\cos^2\theta\sin\theta - 8\sin\theta - 6\sin\theta\cos\theta = 0$$

$$2\sin\theta[10\cos^2\theta - 3\cos\theta - 4] = 0$$

$$\sin\theta = 0, \ \cos\theta = 0.8, \ \cos\theta = \frac{-1}{2}$$

For sin θ = 0, θ = 0°, 180°, 360°

For cos $\theta = \frac{-1}{2}$, θ = 120°, 240°

For cos θ = 0.8, θ = 36.9°, 323.1°

$\Rightarrow$ 0, 36.9, 120, 180, 240, 323.1, 360 are the solutions to the equation 2sin θ(5cos 2θ + 1) = 3 sin 2θ

(d) 3cot 2$\boldsymbol{\theta}$ + cot $\boldsymbol{\theta}$ = 1

$$\frac{3}{\tan 2\theta} + \frac{1}{\tan\theta} = 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\Rightarrow 3\left(\frac{1 - \tan^2\theta}{2\tan\theta}\right) + \frac{1}{\tan\theta} = 1$$

$$3 - 3\tan^2\theta + 2 = 2\tan\theta$$

$$3\tan^2\theta + 2\tan\theta - 5 = 0$$

$$\tan\theta = 1, \quad \tan\theta = \frac{-5}{3}$$

For tanθ = 1, θ = 45°, 225°

For tan$\theta = \frac{-5}{3}$, θ = 121°, 301°.

(e) 4tan θ tan 2$\boldsymbol{\theta}$ = 1

$$4\tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)=1$$

$$\frac{8\tan^2\theta}{1-\tan^2\theta}=1$$

$$8\tan^2\theta=1-\tan^2\theta$$

$$9\tan^2\theta=1$$

$$\tan\theta=\pm\frac{1}{3}$$

When $\tan\theta=\frac{1}{3}$, θ = 18.4°, 198.4°

When $\tan\theta=\frac{-1}{3}$, $\theta=$ 161.6°, 341.6°

t-formula

If *t* = $\tan\frac{x}{2}$**,**
$\sin x=\frac{2t}{1+t^2}$, $\qquad\cos x=\frac{1-t^2}{1+t^2}$
And if *t* = tan *x*
$\sin 2x=\frac{2t}{1+t^2}$, $\qquad\cos 2x=\frac{1-t^2}{1+t^2}$

Proof

If $t=\tan\frac{x}{2}$,

$$\sin x=\sin\left(\tfrac{x}{2}+\tfrac{x}{2}\right)$$

$$=\sin\tfrac{x}{2}\cos\tfrac{x}{2}+\cos\tfrac{x}{2}\sin\tfrac{x}{2}$$

$$=2\sin\tfrac{x}{2}\cos\tfrac{x}{2}$$

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}$$

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}$$

Dividing through by $\cos^2\frac{x}{2}$

$$\sin x=\frac{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$$

$$\sin x=\frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1}$$

$$=\frac{2t}{1+t^2}$$

$$\boxed{\sin x = \frac{2t}{1+t^2}}$$

$$\cos x = \cos(\tfrac{x}{2} + \tfrac{x}{2})$$

$$= \cos^2 \tfrac{x}{2} - \sin^2 \tfrac{x}{2}$$

$$= \cos x = \frac{\cos^2 \tfrac{x}{2} - \sin^2 \tfrac{x}{2}}{1}$$

$$\cos x = \frac{\cos^2 \tfrac{x}{2} - \sin^2 \tfrac{x}{2}}{\sin^2 \tfrac{x}{2} + \cos^2 \tfrac{x}{2}}$$

Dividing through by $\cos^2 \tfrac{x}{2}$

$$\cos x = \frac{\frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}{\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + \frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}$$

$$\cos x = \frac{1 - \tan^2 \tfrac{x}{2}}{\tan^2 \tfrac{x}{2} + 1}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

For $t = \tan x$

$$\sin 2x = \frac{2\sin x \cos x}{1}$$

$$= \frac{2\sin x \cos x}{\sin^2 x + \cos^2 x}$$

Dividing through by $\cos^2 x$

$$\sin 2x = \frac{\frac{2\sin x \cos x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}}$$

$$\sin 2x = \frac{2\tan x}{\tan^2 x + 1}$$

$$\sin 2x = \frac{2t}{1+t^2}$$

$$\cos 2x = 2\cos^2 x - 1$$

$$= \frac{2}{\sec^2 x} - 1$$

$$= \frac{2 - \sec^2 x}{\sec^2 x}$$

$$= \frac{2 - (1 + \tan^2 x)}{\sec^2 x}$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$\cos 2x = \frac{1-t^2}{1+t^2}$$

Note: The *t*-formula is used to solve equations of the form $a\cos\theta + b\sin\theta = c$

Example I

Solve the following equations for 0 ≤ θ ≤ 360°

- **(a)** $2\cos\theta + 3\sin\theta - 2 = 0$
- **(b)** $3\cos\theta - 4\sin\theta + 1 = 0$
- **(c)** $3\cos\theta + 4\sin\theta = 2$
- **(d)** $4\cos\theta\sin\theta + 15\cos 2\theta = 10$

Solution

(a) $2\cos\theta + 3\sin\theta = 2$

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}, \text{ for } t = \tan\tfrac{\theta}{2}$$

$$2\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) = 2$$

$2(1 - t^2) + 3(2t) = 2(1 + t^2)$

$2 - 2t^2 + 6t = 2 + 2t^2$

$4t^2 - 6t = 0$

$2t(2t - 3) = 0$

$t = 0$

$\Rightarrow \tan\tfrac{\theta}{2} = 0$ and $\tan\tfrac{\theta}{2} = \frac{3}{2}$

For $\tan\tfrac{\theta}{2} = 0$, $\frac{\theta}{2} = \tan^{-1}(0)$

$\frac{\theta}{2} = 0°, 180°, \ldots$

$\theta = 0, 360.$

For $\tan\tfrac{\theta}{2} = \frac{3}{2}$, $\frac{\theta}{2} = 56.3°$

$\theta = 112.6°$

$\Rightarrow$ 0°, 112.6°, and 360° are solutions to the equation $2\cos\theta + 3\sin\theta - 2 = 0$

(b) $3\cos\theta - 4\sin\theta + 1 = 0$

$$3\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right) + 1 = 0$$

$3 - 3t^2 - 8t + 1 + t^2 = 0$

$-2t^2 - 8t + 4 = 0$

$t^2 + 4t - 2 = 0$

$$t = \frac{-4 \pm \sqrt{(4)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$

$$t = \frac{-4 \pm \sqrt{16+8}}{2}$$

$$t = -2 \pm \sqrt{6}$$

$$\tan \tfrac{\theta}{2} = -2 - \sqrt{6}$$

$$\tan \tfrac{\theta}{2} = -2 + \sqrt{6}$$

For $t = -2 - \sqrt{6}$, $\tan \tfrac{\theta}{2} = -2 - \sqrt{6}$

$$\frac{\theta}{2} = 102.7,\ 282.7$$

$$\theta = 205.4°$$

When $t = \tan \tfrac{\theta}{2} = -2 + \sqrt{6}$

$$\frac{\theta}{2} = \tan^{-1}(-2 + \sqrt{6})$$

$$\frac{\theta}{2} = 24.2°$$

$$\theta = 48.4°$$

$\Rightarrow \theta = 48.4°$ and $205.4°$ are the solutions to the equation

(c) $3\cos\theta + 4\sin\theta = 2$

$$3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 2$$

$$3 - 3t^2 + 8t = 2(1 + t^2)$$

$$3 - 3t^2 + 8t = 2 + 2t^2$$

$$5t^2 - 8t - 1 = 0$$

$$t = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 5 \times (-1)}}{2 \times 5}$$

$$t = \frac{8 \pm \sqrt{64+20}}{10}$$

$$t = \frac{8 \pm \sqrt{84}}{10}$$

$$t = -0.11652$$

$$t = 1.71652$$

For $t = -0.11652$, $\tan \tfrac{\theta}{2} = -0.11652$

$$\frac{\theta}{2} = 173.4 \quad \Rightarrow \theta = 346.7°$$

$$\tan\frac{\theta}{2} = 1.71652$$

$$\frac{\theta}{2} = 59.8° \quad \Rightarrow \theta = 119.6°$$

$\Rightarrow 119.6°$ and $346.7°$ are solutions to the above equation.

(d) $4\cos\theta\sin\theta + 15\cos 2\theta = 10$

$2 \times 2\sin\theta\cos\theta + 15\cos 2\theta = 10$

$2\sin 2\theta + 15\cos 2\theta = 10$

$2\sin 2\theta + 15\cos 2\theta = 0$

Let $t = \tan\theta$

$\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$

$$2\left(\frac{2t}{1+t^2}\right)+15\left(\frac{1-t^2}{1+t^2}\right)=10$$

$4t + 15 - 15t^2 = 10 + 10t^2$

$25t^2 - 4t - 5 = 0$

$$t=\frac{4\pm\sqrt{(-4)^2-4\times 25\times(-5)}}{2\times 25}$$

$t = 0.5343$

$t = -0.3743$

For $t = 0.5343$

$\tan\theta = 0.5343$

$\theta = 28.1°$

$\theta = 208.1°$

For $t = -0.3743$, $\tan\theta = -0.3743$

$\theta = \tan^{-1}(0.3743)$

$\theta = 159.5°,\ 200.5°$

$\Rightarrow$ 28.1°, 208.1°, 159.5° and 200.5° are the solutions to the above equation

The *R*- Formula

The *R*-formula is used to solve equations of the form $a\cos\theta + b\sin\theta = c$.

$R\cos(\theta \pm \alpha) = c$
$R\sin(\theta \pm \alpha) = c$

Where $R = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$

Example I

Solve the equation $3\cos\theta + 4\sin\theta = 2$ for $0 \leq \theta \leq 360°$

Solution

$R\cos(\theta-\alpha)=2$

$R(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=2$

$R\cos\theta\cos\alpha+R\sin\theta\sin\alpha=2$

By comparison

$R\cos\theta\cos\alpha=3\cos\theta$

$R\sin\theta\sin\alpha=4\sin\theta$

$\Rightarrow R\cos\alpha=3$ ……………………….. (i)

$R\sin\alpha=4$ ……………………….. (ii)

Eqn (ii) ÷ Eqn (1);

$$\Rightarrow \tan\alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\alpha = 53.1°$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 25$$

$$R = 5$$

$$R\cos(\theta - \alpha) = 2$$

$$5\cos(\theta - 53.1) = 2$$

$$\theta - 53.1 = \cos^{-1}\left(\frac{2}{5}\right)$$

θ – 53.1° = 66.4°, 293.6°

θ – 119.5°, 346.7°

Alternatively

3cos θ + 4sin θ = 2

Rcos(θ – α) = 2

$$R = \sqrt{a^2 + b^2}$$

$$= \sqrt{(3)^2 + 4^2}$$

$$= 5$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

α = 53.1

5cos(θ – 53.1) = 2

cos(θ – 53.1°) = $\frac{2}{5}$

θ – 53.1° = 66.4°, 293.6°

θ = 119.5°, 346.7°

Example II

sin θ + $\sqrt{3}$ cosθ = 1 for 0 ≤ θ ≤ 360

Solution

$$R\sin(\theta + \alpha) = 1$$

$$R = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$$

R sin(θ + α) = 1

2sin(θ + 60°) = 1

sin(θ + 60°) = $\frac{1}{2}$

θ + 60° = sin(½)
θ + 60° = 30, 150°
θ = -30, 90°
$\Rightarrow$ θ = 90°, and 330°.

Example III
cos θ – 7sin θ = 2 for 0° ≤ θ ≤ 360°
Solution
cos θ – 7sin θ = 2

$$R\cos(\theta + \alpha) = 2$$

$$R = \sqrt{1^2 + (-7)^2} = \sqrt{50}$$

$$\alpha = \tan^{-1}\left(\frac{7}{1}\right) \qquad \Rightarrow \alpha = 81.9°$$

$$\sqrt{50}\cos(\theta + 81.9°) = 2$$

$$\cos(\theta + 81.9°) = \frac{2}{\sqrt{50}}$$

θ + 81. 9° = 73.6°, 286.4°
θ = -8.3°, 204.5°
$\Rightarrow$ θ = 204.5°, 351.7°

Example IV
Solve: 5sinθ – 12cosθ = 6
Solution
Rsin(θ – α) = 6

$$R = \sqrt{5^2 + 12^2} = 13$$

$$\alpha = \tan^{-1}\left(\frac{12}{5}\right)$$

$\alpha = 67.4°$
13sin(θ – 67.4) = 6

$$\sin(\theta - 67.4) = \frac{6}{13}$$

θ – 67.4° = 27.5°, 152.5°
θ = 94.9°, 219.9°

Example V
Solve cosθ + sinθ = secθ for 0 ≤ θ ≤ 360°
Solution

$$\cos\theta + \sin\theta = \frac{1}{\cos\theta}$$

$$\cos^2\theta + \sin\theta\cos\theta = 1 \text{.............................(i)}$$

But cos 2θ = 2cos$^2\theta$ – 1

$$\Rightarrow \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Subsituting for cos^{2}θ and sin θ cos θ in Eqn (i);
sin 2θ = 2sinθ cosθ

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$\frac{1}{2}(1+\cos 2\theta) + \frac{1}{2}\sin 2\theta = 1$$

$$\frac{1}{2}\cos 2\theta + \frac{1}{2}\sin 2\theta = \frac{1}{2}$$

$$\cos 2\theta + \sin 2\theta = 1$$

$$R\cos(2\theta - \alpha) = 1$$

$$R = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\alpha = \tan^{-1}\left(\frac{1}{1}\right) = 45°$$

$$\sqrt{2}\cos(2\theta - 45°) = 1$$

$$\cos(2\theta - 45°) = \frac{1}{\sqrt{2}}$$

$$2\theta - 45° = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$2\theta - 45° = 45°, 315°, 405°$$

$$\theta = 45°, 180°, 225°$$

Example VI

Solve the equation $4\cos\theta\sin\theta + 15\cos 2\theta = 10$

Solution

$4\cos\theta\sin\theta + 15\cos 2\theta = 10$

$$2(2\sin\theta\cos\theta) + 15\cos 2\theta = 10$$

$$2\sin 2\theta + 15\cos 2\theta = 10$$

$$R\sin(2\theta + \alpha) = 10$$

$$R = \sqrt{2^2 + 15^2}$$

$$= \sqrt{229}$$

$$\sqrt{229}\sin(2\theta + \alpha) = 10$$

$$\alpha = \tan^{-1}\left(\frac{15}{2}\right) = 82.4°$$

$$\sqrt{229}\sin(2\theta + 82.4°) = 10$$

$$\sin(2\theta + 82.4°) = \frac{10}{\sqrt{229}}$$

$$2\theta + 82.4° = \sin^{-1}\left(\frac{10}{\sqrt{229}}\right)$$

$$2\theta + 82.4° = 41.4°, 138.6°, 401.4°, 498.4°$$

$$\theta = 339.5°, 28.1°, 159.5°, 208°$$

Example VII

Show that $3\cos\theta + 2\sin\theta$ can be written as $\sqrt{13}\cos(\theta - \alpha)$. Hence find the minimum and maximum values of the function, giving the corresponding values of θ from -180° to 180°

Solution

$3\cos\theta + 2\sin\theta$

$R\cos(\theta - \alpha)$

$R = \sqrt{a^2 + b^2}$

$R = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.7°$

$\Rightarrow 3\cos\theta + 2\sin\theta = R\cos(\theta - \alpha)$

$= \sqrt{13}\cos(\theta - 33.7)$

Let $y = \sqrt{13}\cos(\theta - 33.7)$

For the maximum value of y, $\cos(\theta - 33.7) = 1$

$\Rightarrow y_{\max} = \sqrt{13}$

And for minimum value of y, $\cos(\theta - 33.7) = -1$

$\Rightarrow y_{\min} = -\sqrt{13}$

For y_{max} $\cos(\theta - 33.7°) = 1$,

$\Rightarrow \theta - 33.7° = \cos^{-1}(1)$

$\theta - 33.7° = 0, 360°$.

$\theta = 33.7°$

For y_{min} $\cos(\theta - 33.7°) = -1$,

$\theta - 33.7° = 180°$.

$\theta = 213.7°$

Example VII

Find the maximum and minimum values of the following expressions, stating the value of θ for which they occur (from 0° to 360°)

(g) $8\cos\theta - 15\sin\theta$

(h) $4\sin\theta - 3\cos\theta$

(i) $\sin\theta - 6\cos\theta$

(j) $\cos(\theta + 60) - \cos\theta$

Solution

(a) $8\cos\theta - 15\sin\theta$

$\boldsymbol{R}\cos(\theta - \alpha)$

$R = \sqrt{8^2 + 15^2} = 17$

$\alpha = \tan^{-1}\left(\frac{15}{8}\right) = 61.9°$

$17\cos(\theta - 61.9°)$

Let $y = 17\cos(\theta - 61.9°)$

For y_{max}, $\cos(\theta - 61.9°) = 1$

$\Rightarrow y_{max} = 17$

$\theta - 61.9° = \cos^{-1}(1)$

$\theta - 61.9° = 0, 360°$

$\theta = 61.9°$

For y_{min}, $\cos(\theta - 61.9) = -1$

$\Rightarrow y_{min} = -17$

$\theta - 61.9° = \cos^{-1}(-1)$

$\theta - 61.9° = 180°$

$\theta = 241.9°$

(b) $4\sin\theta - 3\cos\theta$

$R = \sqrt{4^2 + 3^2} = 5$

$R\sin(\theta - \alpha)$

$5\sin(\theta - \alpha)$

$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°$

$5\sin(\theta - 36.9°)$

Let $y = 5\sin(\theta - 36.9°)$

$y_{min} = -5$

$y_{max} = 5$

For y_{min}, $\sin(\theta - 36.9°) = -1$

$\theta - 36.9° = 270°$

$\theta = 306.9°$

For y_{max}, $\sin(\theta - 36.9°) = 1$

$\theta - 36.9° = 90°$

$\theta = 126.9°$

(c) $\mathbf{\sin\theta - 6\cos\theta}$

$R = \sqrt{1^2 + (-6)^2} = \sqrt{37}$

$\sqrt{37}\sin(\theta - \alpha)$

$\alpha = \tan^{-1}\left(\frac{6}{1}\right) = 80.5°$

$y = \sqrt{37}\sin(\theta - 80.1)$

$y_{max} = \sqrt{37}$ and it occurs when $\sin(\theta - 80.1) = 1$

$\theta - 80.1° = 90°$

$\theta = 170.5°$

$y_{min} = -\sqrt{37}$ and it occurs when

$\sin(\theta - 80.1) = -1$

$\theta - 80.1° = 270°$

$\theta = 350.5°$

(d) $\mathbf{\cos(\theta + 60) - \cos\theta}$

$= \cos\theta\cos 60 - \sin\theta\sin 60 - \cos\theta$

$= \frac{1}{2}\cos\theta - \sin\theta\frac{\sqrt{3}}{2} - \cos\theta$

$= \frac{-1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$

$y = -\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right]$

$y = -\left[R\cos(\theta - \alpha)\right]$

$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = 1$

$y = -[\cos(\theta - \alpha)]$

$\alpha = \tan^{-1}\left(\sqrt{3}\right) = 60°$

y = -[cos(θ – 60)]

y_{min} occurs when cos(θ – 60) = 1

θ – 60° = 0, 360

θ = 60°

y_{max} = 1 and occurs when cos(θ - 60°) = -1

θ – 60° = $\cos^{-1}$(-1)

θ = 240°

Example VIII (UNEB Question)

Solve $\cos\theta + \sqrt{3}\sin\theta = 2$ for $0 \leq \theta \leq \pi$

Solution

$\cos\theta + \sqrt{3}\sin\theta = 2$

R cos(θ – α) = 2

$R = \sqrt{1^2 + (\sqrt{3})^2} = 2$

2 cos(θ – α) = 2

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

2 cos(θ – 60°) = 2

cos(θ – 60°) = 1

θ – 60° = $\cos^{-1}$(1)

θ – 60° = 0

θ = 60°

$\theta = \frac{\pi}{3}$

Since 180 = π radians, $\Rightarrow \theta = \frac{60\pi}{180} = \frac{\pi}{3}$

Example IX (UNEB Question)

(a) Express 4cosθ – 5sinθ in the form R cos (θ + β), where R is a constant and β an acute angle. Determine the maximum value of the expression and the value of θ for which it occurs

(b) Solve the equation 4 cos θ – 5 sin θ = 2.2, for $0^0 < \theta < 360^0$.

Solution

4cosθ – 5sinθ

Rcos(θ + β)

$\beta = \tan^{-1}\left(\frac{5}{4}\right) = 51.3°$

$R = \sqrt{4^2 + 5^2} = \sqrt{41}$

$\sqrt{41}$ cos(θ + 51.3°)

Let $y = \sqrt{41}$ cos(θ + 51.3°)

$y_{max} = \sqrt{41}$ and it occurs when cos(θ + 51.3°) = 1

θ + 51.3° = 0

θ = -51.3°

$\Rightarrow$ $\theta = 308.7°$ ($0^0 < \theta < 360^0$)

4cosθ – 5sinθ = 2.2

$\Rightarrow$ $\sqrt{41}\cos(\theta + 51.3°) = 2.2$

$\cos(\theta + 51.3°) = \frac{2.2}{\sqrt{41}}$

$\theta + 51.3° = 69.9°, 290.1°$

$\theta = 18.6°, 238.8°$

Example XI (UNEB Question)

Express $y = 8\cos x + 6\sin x$ in the form R cos ($x - \alpha$) where R is positive and α is acute . Hence find the maximum and minimum values of $\frac{1}{8\cos x + 6\sin x + 15}$

Solution

$8\cos x + 6\sin x = R\cos(x - \alpha)$

$8\cos x + 6\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

By comparison

$R\cos\alpha = 8$ …...........………………… (i)

$R\sin\alpha = 6$………………..........……….(ii)

Eqn (i)^2 + Eqn (ii)^2;

$R^2 = 8^2 + 6^2 = 100$

$R = 10$

Eqn (ii) $\div$ Eqn (i)

$\tan\alpha = \frac{6}{8}$

$\alpha = 36.87^0$

Hence $8\cos x + 6\sin x = 10\cos(x - 36.87^0)$

$Now \frac{1}{8\cos x + 6\sin x + 5} = \frac{1}{10\cos(x - 36.87) + 15}$

Note: For y to be maximum, the denominator must be minimum and for y to be minimum, the denominator must be maximum.

Let $m = \frac{1}{10\cos(x - 36.87) + 15}$

$M_{max} = \frac{1}{10 \times (-1) + 15}$

$= \frac{1}{-10 + 15} = \frac{1}{5} = 0.2$

$M_{min} = \frac{1}{10 \times 1 + 15}$

$= \frac{1}{25} = 0.04$

The maximum and minimum values of $\frac{1}{8\cos x + (\sin x + 15)}$ are 0.2 and 0.04 respectively.

Factor Formula

1. **sin P + sin Q =** $2\sin(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$
2. **sin P − sin Q =** $2\cos(\frac{P+Q}{2})\sin(\frac{P-Q}{2})$
3. **cos *P* + cos Q =** $2\cos(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$
4. **cos *P* – cos Q = -** $2\sin(\frac{P+Q}{2})\sin(\frac{P-Q}{2})$

Application of the factor formula

Example 1

Express the following in factors:

(a) $\sin 7\theta + \sin 5\theta$
(b) $\sin 4x - \sin 2x$
(c) $\cos 7x + \cos 5x$
(d) cos 3A – $\cos 5A$
(e) $\sin(x + 30) + \sin(x - 30)$
(f) $\cos(x + 30) - \cos(x - 30)$
(g) $\cos \frac{3}{2}x - \cos \frac{x}{2}$
(h) $\frac{1}{2} + \cos 2\theta$
(i) $1 + \sin 2x$
(j) Sin $2(x + 40) + \sin 2(x - 40)$

Solution

(a) $\sin 7\theta + \sin 5\theta$

From $\sin P + \sin Q = 2\sin(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$

$\sin 7\theta + \sin 5\theta = 2\sin(\frac{7\theta+5\theta}{2})\cos\frac{7\theta-5\theta}{2}$

$= 2\sin 6\theta \cos\theta$

(b) $\sin 4x - \sin 2x$

From $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\sin 4x - \sin 2x = 2\cos\frac{4x+2x}{2}\sin\frac{4x-2x}{2}$

$\sin 4x - \sin 2x = 2\cos 3x \sin x$

(c) $\cos 7x + \cos 5x$

From $\cos P + \cos Q = 2\cos(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$

$\Rightarrow$ Cos $7x + \cos 5x = 2\cos(\frac{7x+5x}{2})\cos(\frac{7x-5x}{2})$

$= 2\cos 6x \cos x$

(d) $\cos 3A - \cos 5A$

From $\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\cos 3A - \cos 5A = -2\sin(\frac{3A+5A}{2})\sin(\frac{3A-5A}{2})$

$= -2\sin 4A \sin(-A)$

$= 2\sin 4A \sin A$

(e) $\sin(x + 30) + \sin(x - 30)$

$= -2\sin(\frac{(x+30)+(x-30)}{2})\cos(\frac{(x+30)-(x-30)}{2})$

$= -2\sin x \cos 30$

(f) $\cos(x + 30) - \cos(x - 30)$

$= 2\sin\left(\frac{(x+30)+(x-30)}{2}\right) \sin\left(\frac{(x+30)-(x-30)}{2}\right)$

$= 2\sin x \sin 30$

(g) $\cos\left(\frac{3x}{2}\right) - \cos\ \frac{x}{2} = {}^-2\sin\left(\frac{\frac{3x}{2}+\frac{x}{2}}{2}\right) \sin\left(\frac{\frac{3x}{2}-\frac{x}{2}}{2}\right)$

$= 2\sin x \sin\ \frac{x}{2}$

(h) $\frac{1}{2} + \cos 2\theta$

$\cos 60 + \cos 2\theta$

$= 2\cos\frac{60+2\theta}{2}\cos\frac{60-2\theta}{2}$

$= 2\cos(30 + \theta)\cos(30 - \theta)$

(i) $1 + \sin 2x$

$\sin 90 + \sin 2x$

$2\sin\frac{90+2x}{2}\cos\frac{90-2x}{2}$

$= 2\sin(45 + x)\cos(45 - x)$

(j) $\sin2(x + 40) + \sin2(x - 40)$

$= 2\sin\frac{2(x+40)+2(x-40)}{2}\cos\frac{2(x+40)-2(x-40)}{2}$

$= 2\sin 2x \cos 80$

Example II

Solve the following equations from $x = 0°$ to $360°$ inclusive.

(a) $\cos x + \cos 5x = 0$
(b) $\sin 3x - \sin x = 0$
(c) $\sin(x + 10) + \sin x = 0$
(d) $\cos(2x + 10) + \cos(2x - 10) = 0$
(e) $\cos(x + 20) - \cos(x - 70) = 0$

Solution

(a) $\cos x + \cos 5x = 0$

$2\cos\frac{x+5x}{2}\cos\frac{x-5x}{2} = 0$

$2\cos 3x \cos{}^-2x = 0$

$2\cos 3x \cos 2x = 0$

$\cos 3x \cos 2x = 0$

$\Rightarrow$ $\cos 2x = 0$ **OR**

$\cos 3x = 0$

For $\cos 2x = 0$;

$2x = \cos^{-1}(0)$

$2x = 90°, 270°, 450°, 630°, 810°$

$\Rightarrow$ $x = 45°, 135°, 225°, 315°$.

For $\cos 3x = 0$;

$3x = \cos^{-1}(0)$

$3x = 90°, 270°, 450°, 630°, 810°, 990°, 1170°$

$x = 30°, 90°, 150°, 210°, 270°, 330°$.

$\therefore$ The solutions to the equation $\cos x + \cos 5x = 0$ are 30°, 45°, 90°, 135°, 150°, 210°, 225°, 270°, 315°, 330°.

(b) $\sin 3x - \sin x = 0$

$2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2} = 0$

$2\cos 2x \sin x = 0$

$\cos 2x \sin x = 0$

$\Rightarrow 2x = \cos^{-1}(0)$

$2x = 90°, 270°, 450°, 630°, 810°, 990°$

$\Rightarrow x = 45°, 135°, 225°, 315°$

And for $\sin x = 0$;

$x = \sin^{-1}(0)$

$x = 0, 180°, 360°$

$\Rightarrow$ The solutions to the equation sin 3x – sin x = 0 are 0, 45, 135, 180, 225, 315, 360.

(c) $\sin(x + 10) + \sin x = 0$

$2\sin\left(\frac{x+10+x}{2}\right)\cos\left(\frac{x+10-x}{2}\right) = 0$

$2\sin(x + 5)(\cos 5) = 0$

$\sin(x + 5) = 0$

$x + 5 = \sin^{-1}0)$

$x + 5 = 0, 180°, 360°$

$x = 355°, 175°.$

$\Rightarrow x = 175°, 335°$ are solutions to the equation

$\sin(x + 10) + \sin x = 0$

(d) $\cos(2x + 10) + \cos(2x - 10) = 0$

$$2\cos\left(\frac{(2x+10)+(2x-10)}{2}\right)\cos\left(\frac{(2x+10)-(2x-10)}{2}\right)$$

$2\cos 2x \cos 10 = 0$

$\cos 2x = 0$

$2x = \cos^{-1}(0)$

$2x = 90°, 270°, 450°, 630°.$

$x = 45°, 135°, 225°, 315°$

$\Rightarrow$ The solutions to the equation $\cos(2x + 20) + \cos(2x - 10) = 0$ are $x = 45°, 135°, 225°$ and $315°$

(f) $\cos(x + 20) - \cos(x - 70) = 0$

$-2\sin\frac{(x+20)+(x-70)}{2}\sin\frac{(x+20)-(x-70)}{2} = 0$

$-2\sin(x - 25)\sin 45 = 0$

$\sin(x - 25) = 0$

$x - 25 = \sin^{-1}(0)$

$x - 25 = 0, 180°, 360°$

$x = 25, 205°, 385°$

Example II

Prove the following identities:

(a) $\dfrac{\cos B + \cos C}{\sin B - \sin C} = \cot\frac{B-C}{2}$

(b) $\dfrac{\cos B - \cos C}{\sin B + \sin C} = -\tan\frac{B-C}{2}$

(c) $\dfrac{\sin B - \sin C}{\sin B + \sin C} = \cot\frac{B+C}{2}\tan\frac{B-C}{2}$

(d) $\dfrac{\sin B + \sin C}{\cos B + \cos C} = \tan\frac{B+C}{2}$

Solution

(a) $\dfrac{\cos B+\cos C}{\sin B-\sin C}$

$$=\frac{2\cos(\frac{B+C}{2})\cos(\frac{B-C)}{2})}{2\cos(\frac{B+C}{2})\sin(\frac{B-C)}{2})}$$

$$=\frac{\cos\frac{B-C}{2}}{\sin\frac{B-C}{2}}$$

$$=\cot(\tfrac{B-C}{2})$$

(b) $\dfrac{\cos B-\cos C}{\sin B+\sin C}$

$$=\frac{-2\sin\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\frac{-\sin\frac{B-C}{2}}{\cos\frac{B-C}{2}}$$

$$=-\tan(\tfrac{B-C}{2})$$

$$\Rightarrow\frac{\cos B-\cos C}{\sin B+\sin C}=-\tan(\tfrac{B-C}{2})$$

(c) $\dfrac{\sin B-\sin C}{\sin B+\sin C}$

$$\frac{\sin B-\sin C}{\sin B+\sin C}=\frac{2\cos(\frac{B+C}{2})\sin(\frac{B-C}{2})}{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}$$

$$=\frac{\cos(\frac{B+C}{2})}{\sin(\frac{B+C}{2})}\times\frac{\sin(\frac{B-C}{2})}{\cos(\frac{B-C}{2})}$$

$$=\cot(\tfrac{B+C}{2})\tan(\tfrac{B-C}{2})$$

(d) $\dfrac{\sin B+\sin C}{\cos B+\cos C}$

$$=\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\tan\tfrac{B-C}{2}$$

$$\Rightarrow\frac{\sin B+\sin C}{\cos B+\cos C}==\tan\tfrac{B-C}{2}$$

Example IV

Prove the following

(a) $\sin x + \sin 2x + \sin 3x = \sin 2x(2\cos x + 1)$

(b) $\cos x + \sin 2x - \cos 3x = \sin 2x(2\sin x + 1)$

(c) $\cos\theta - 2\cos 3\theta + \cos 5\theta = 2\sin\theta\,(\sin 2\theta - \sin 4\theta)$

(d) $\sin x - \sin(x + 60) + \sin(x + 120) = 0$

(e) $1 + 2\cos 2\theta + \cos 4\theta = 4\cos^2\theta\cos 2\theta$

Solutions

(a) $\sin x + \sin 2x + \sin 3x$

$= \sin x + \sin 3x + \sin 2x$

$= 2\sin\frac{x+3x}{2}\cos\frac{x-3x}{2} + \sin 2x$

$= 2\sin 2x \cos(^{-}x) + \sin 2x$
$= 2\sin 2x \cos x + \sin 2x$
$= \sin 2x(2\cos x + 1)$
$\Rightarrow \sin x + \sin 2x + \sin 3x = \sin 2x(2\cos x + 1)$

(b) $\cos x + \sin 2x - \cos 3x$
$= \cos x - \cos 3x + \sin 2x$
$= -2\sin \frac{x+3x}{2} \sin \frac{x-3x}{2} + \sin 2x$
$= -2\sin 2x \sin(-x) + \sin 2x$
$= 2\sin 2x \sin x + \sin 2x$
$= \sin 2x[2\sin x + 1]$
$\Rightarrow \cos x + \sin 2x - \cos 3x = \sin 2x[2\sin x + 1]$

(c) $\cos\theta - 2\cos 3\theta + \cos 5\theta$
$= \cos \theta - \cos 3\theta + \cos 5\theta - \cos 3\theta$
$= -2\sin 2\theta \sin(-\theta) + {}^{-}2\sin 4\theta \sin \theta$
$= 2\sin 2\theta \sin \theta - 2\sin 4\theta \sin\theta$
$= 2\sin \theta (\sin 2\theta - \sin 4\theta)$
$\Rightarrow \cos\theta - 2\cos 3\theta + \cos 5\theta = 2\sin \theta (\sin 2\theta - \sin 4\theta)$

(d) $\sin x - \sin(x + 60) + \sin(x + 120)$
$= \sin x + \sin(x + 120) - \sin(x + 60)$
$= 2\sin(x + 60)\cos -60 - \sin(x + 60)$
$= \sin(x + 60) - \sin(x + 60)$
$= 0$
$\Rightarrow \sin x - \sin(x + 60) + \sin(x + 120) = 0$

(e) $1 + 2\cos 2\theta + \cos 4\theta$
Since $\cos 4\theta = \cos^2 2\theta - 1$,
$\Rightarrow 1 + 2\cos 2\theta + 2\cos^2 2\theta - 1$
$= 2\cos 2\theta + 2\cos^2 2\theta$
$= 2\cos 2\theta [1 + \cos 2\theta]$
$= 2\cos 2\theta [1 + 2\cos^2\theta - 1]$
$= 4 \cos^2\theta \cos 2\theta$
$\Rightarrow 1 + 2\cos 2\theta + \cos 4\theta = 4 \cos^2\theta \cos 2\theta$

Example V

Solve the following equations for values of θ from 0° to 180° inclusive

(a) $\cos \theta + \cos 3\theta + \cos 5\theta = 0$
(b) $\sin \theta - 2\sin 2\theta + \sin 3\theta = 0$
(c) $\sin \theta + \cos 2\theta - \sin 3\theta = 0$
(d) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$
(e) $\cos \frac{1}{2}\theta + 2\cos \frac{3}{2}\theta + \cos \frac{5}{2}\theta = 0$

Solution

(a) $\cos \theta + \cos 3\theta + \cos 5\theta = 0$
$\cos \theta + \cos 5\theta + \cos 3\theta = 0$
$2\cos 3\theta \cos -2\theta + \cos 3\theta = 0$
$\cos 3\theta(2\cos 2\theta + 1] = 0$

Either $\cos 3\theta = 0$ **OR**

$\cos 2\theta = -\frac{1}{2}$

For $\cos 3\theta = 0$;

$3\theta = \cos^{-1}(0)$

$3\theta = 90°, 270°, 450°, 630°, 810°, 990°$

$\theta = 30°, 90°, 150°, 210°, 270, 330°$

$\Rightarrow \theta = 30°, 90°, 150°$ (for $0° \leq \theta \leq 180°$)

For $\cos 2\theta = -\frac{1}{2}$;

$2\theta = \cos^{-1}(\frac{-1}{2})$

$2\theta = 120°, 240°.$

$\theta = 60°, 120°$

$\Rightarrow$ 30°, 60°, 90°, 120°, 150° are the solutions to the equation $\cos 3\theta + \cos 3\theta + \cos 5\theta = 0$

(b) $\sin\theta - 2\sin 2\theta + \sin 3\theta = 0$

$\sin\theta + \sin 3\theta - 2\sin 2\theta = 0$

$2\sin 2\theta \cos(-\theta) - 2\sin 2\theta = 0$

$2\sin 2\theta \cos\theta - 2\sin 2\theta = 0$

$2\sin 2\theta (\cos\theta - 1) = 0$

Either $\sin 2\theta = 0$ **OR** $\cos\theta = 1$

For $\sin 2\theta = 0$;

$2\theta = \sin^{-1}0$

$2\theta = 0°, 180°, 360°$

$\Rightarrow \theta = 0°, 90°, 180°$

(c) $\sin\theta + \cos 2\theta - \sin 3\theta = 0$

$\sin\theta - \sin 3\theta + \cos 2\theta = 0$

$2\cos 2\theta \sin -\theta + \cos 2\theta = 0$

$\cos 2\theta(-2\sin\theta + 1) = 0$

$\cos 2\theta = 0$ **OR** $\sin\theta = \frac{1}{2}$

For $\cos 2\theta = 0$

$2\theta = \cos^{-1}0$

$2\theta = 90°, 270°, 450°$

$= 45°, 135°$

For $\sin\theta = \frac{1}{2}$;

$\theta = \sin^{-1}(\frac{1}{2})$

$\theta = 30°, 150°$

$\Rightarrow$ 30°, 45°, 135°, 150° are the solutions to the equation $\sin\theta + \cos 2\theta - \sin 3\theta = 0$

(d) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

$(\sin 2\theta + \sin 6\theta) + \sin 4\theta = 0$

$2\sin 4\theta \cos -2\theta + \sin 4\theta = 0$

$2\sin 4\theta \cos 2\theta + \sin 4\theta = 0$

$\sin 4\theta (2\cos 2\theta + 1) = 0$

For $\sin 4\theta = 0$;

$4\theta = \sin^{-1}0$

$4\theta = 0, 180, 360, 540, 720$

$= 0, 45, 90, 135, 180$

For $2\cos 2\theta + 1 = 0$

$\cos 2\,\theta = -\frac{1}{2}$

$2\theta = 120°, 240°$

$\theta = 60°, 120°$

$\Rightarrow$ 0°, 45°, 60°, 90°, 120°, 135°, 180° are the solutions to the equation $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

(e) $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

$\cos\frac{1}{2}\theta + \cos\ \frac{5}{2}\theta + 2\cos\ \frac{3}{2}\theta\ = 0$

$2\cos\frac{6\theta}{2}\cos(-\theta) + 2\cos\ \frac{3\theta}{2}\ = 0$

$2\cos\frac{3\theta}{2}(\cos\theta + 1) = 0$

$\cos\frac{3\theta}{2}\ = 0$

$\frac{3\theta}{2}\ = \cos^{-1}(0)$

$\frac{3\theta}{2}\ = 90, 270, 450$

$\theta = 60, 180$

For $(\cos\theta + 1) = 0$;

$\cos\theta = {}^{-}1$

$\theta = \cos^{-1}({}^{-}1)$

$\theta = 180$

$\Rightarrow$ 60, 180 are the solutions to the equation

$\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

Example V

Prove the following identities if A, B and C are taken to be angles of a triangle.

(a) $\sin A + \sin(B - C) = 2\sin B\cos C$

(b) $\cos A - \cos(B - C) = -2\cos B\cos C$

(c) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(d) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(e) $\cos A + \cos B + \cos C - 1$

$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solutions

Sin A + sin(B – C)

$= 2\sin\frac{A+B-C}{2}\cos\frac{A-(B-C)}{2}$

$= 2\sin\frac{A+B-C}{2}\cos\frac{A+C-B}{2}$

But $A + B + C = 180$

$A + B + C - 2C = 180 - 2C$

$A + B – C = 180 – 2C$

$\dfrac{A+B-C}{2} = 90 – C$

$\Rightarrow \sin \frac{A+B-C}{2} = \sin(90 – C)$

$= \sin 90 \cos C – \cos 90 \sin C$

$= \cos C$

$A + B + C = 180$

$A + C + B – 2B = 180 – 2B$

$A + C – B = 180 – 2B$

$\cos \frac{A+C-B}{2} = \cos \frac{180-2B}{2}$

$\cos \frac{A+C-B}{2} = \cos(90 – B)$

$= \cos 90 \cos B + \sin 90 \sin B$

$= \sin B$

$\Rightarrow \quad \sin A + \sin(B – C) = 2\sin B \cos C$

(c) sin A + sin B + sin C

$= \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin C$

$= 2\sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2\sin \frac{C}{2} \cos \frac{C}{2}$

But A + B + C = 180

C = 180 – (A + B)

$\dfrac{C}{2} = \sin (90 - \frac{A+B}{2})$

$\sin \frac{C}{2} = \sin 90 \cos \frac{A+B}{2} - \cos 90 \sin \frac{A+B}{2}$

$= \cos \frac{A+B}{2}$

$\cos \frac{C}{2} = \cos(90 - \frac{A+B}{2})$

$\cos \frac{C}{2} = \cos 90 \cos \frac{A+B}{2} + \sin 90 \sin \frac{A+B}{2}$

$= \sin \frac{A+B}{2}$

$\Rightarrow 2\cos \frac{C}{2} \cos \frac{A-B}{2} + 2\cos \frac{A+B}{2} \cos \frac{C}{2}$

$= 2\cos \frac{C}{2} \cos \frac{A-B}{2} + 2\cos \frac{A+B}{2} \cos \frac{C}{2}$

$= 2\cos \frac{C}{2}\left[\cos \frac{A-B}{2} + \cos \frac{A+B}{2}\right]$

$= 2\cos \frac{C}{2} [2\cos \frac{A}{2} \cos \frac{B}{2}]$

$= 4\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

$\sin A + \sin B + \sin C = 4\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

(d) $\sin 2A + \sin 2B + \sin 2C$.

$= 2\sin(A + B) \cos(A – B) + 2\sin C \cos C$

But $A + B + C = 180$

$C = 180 – (A + B)$

$\Rightarrow \sin C = \sin[180 – (A + B)]$

$\sin C = \sin 180 \cos (A + B) – \cos 180 \sin(A + B)$

$\sin C = \sin(A + B)$

$\cos C = \cos(180 – (A+B))$

$\cos C = \cos 180 \cos(A + B) + \sin 180 \sin(A + B)$

$= -\cos(A + B)$

$\Rightarrow$ 2sin(A + B)cos(A – B) + 2sin C cos C

= 2sin C cos (A – B) + 2sin C(-cos(A+B))

= 2sin C[cos(A – B) – cos(A + B)]

= 2sin C[-2sin A sin –B]

= 4sinA sin B sin C

$\Rightarrow$ sin 2A + sin 2B + sin 2C = 4sin A sin B sin C

(e) $\cos A + \cos B + \cos C - 1$

$\cos C = 2\cos^2 \frac{C}{2} - 1$

$\cos C = 1 - 2\sin^2 \frac{C}{2}$

$\Rightarrow 2\sin^2 \frac{C}{2} = 1 - \cos C$

$\cos A + \cos B + \cos C - 1 = \cos A + \cos B - 2\sin^2 \frac{C}{2}$

$= 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin^2 \frac{C}{2}$

A + B + C = 180

$\frac{C}{2} = 90 - \frac{A+B}{2}$

$\sin \frac{C}{2} = \sin (90 - \frac{A+B}{2})$

$\sin \frac{C}{2} = \sin 90 \cos \frac{A+B}{2} - \cos 90 \sin \frac{A+B}{2}$

$\Rightarrow \sin \frac{C}{2} = \cos \frac{A+B}{2}$

$\Rightarrow 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin \frac{C}{2} \sin \frac{C}{2}$

$= 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin \frac{C}{2} \cos \frac{A+B}{2}$

$= 2\sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{A+B}{2}$

$= 2\sin \frac{C}{2} [\cos \frac{A-B}{2} - \cos \frac{A+B}{2}]$

$= 2\sin \frac{C}{2} [-2\sin \frac{A}{2} \sin \frac{-B}{2}]$

$= 2\sin \frac{C}{2} [2\sin \frac{A}{2} \sin \frac{B}{2}]$

$= 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$\Rightarrow \cos A + \cos B + \cos C - 1 = 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Example VI **(UNEB 2007)**

Show that $\frac{\sin\theta - 2\sin 2\theta + \sin 3\theta}{\sin\theta + 2\sin 2\theta + \sin 3\theta} = -\tan^2 \frac{\theta}{2}$

Solution

$$\frac{\sin\theta - 2\sin 2\theta + \sin 3\theta}{\sin\theta + 2\sin 2\theta + \sin 3\theta}$$

$$= \frac{Sin 3\theta + \sin\theta - 2\sin 2\theta}{\sin 3\theta + \sin\theta + 2\sin 2\theta}$$

$$= \frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) - 2\sin 2\theta}{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) + 2\sin 2\theta} = \frac{2\sin 2\theta\cos\theta - 2\sin 2\theta}{2\sin 2\theta\cos\theta + 2\sin 2\theta}$$

$$= \frac{2\sin 2\theta(\cos\theta - 1)}{2\sin 2\theta(\cos\theta + 1)}$$

$$= \frac{\cos\theta - 1}{\cos + 1} = -\frac{-(1-\cos\theta)}{1+\cos\theta}$$

But $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$\cos\theta = 2\sin^2\frac{\theta}{2} - 1$

$$-\frac{-(1-\cos\theta)}{1+\cos\theta} = \frac{-(1-(1-2\sin^2\frac{\theta}{2}))}{1-2\sin^2\frac{\theta}{2}}$$

$$= \frac{-2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= -\tan^2\frac{\theta}{2}$$

Example VII (UNEB Question)

Show that $\dfrac{\sin 3\theta \sin 6\theta + \sin\theta \sin 2\theta}{\sin 3\theta \cos 6\theta + \sin\theta \cos 2\theta} = \tan 5\theta$.

Solution

$$= \frac{\sin 6\theta \sin 3\theta + \sin 2\theta \sin\theta}{\cos 6\theta \sin 3\theta + \cos 2\theta \sin\theta}$$

$\cos A - \cos B = -\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$\sin\frac{A+B}{2}\sin\frac{A-B}{2} = \dfrac{-1}{2}\left(\cos A - \cos B\right)$

$\sin 6\theta \sin 3\theta = \dfrac{-1}{2}\left(\cos A - \cos B\right)$

$\dfrac{A+B}{2} = 6\theta$

$A + B = 6\theta$ …………………….. (i)

$\dfrac{A-B}{2} = 3\theta$

$A - B = 6\theta$ ……………………… (ii)

Solving Eqn (i) and Eqn (ii) simultaneously;

$A = 9\theta$, $B = 3\theta$

$$\sin 6\theta \sin 3\theta = \frac{-1}{2}\left(\cos 9\theta - \cos 3\theta\right)$$

$$\Rightarrow \frac{\sin 3\theta \sin 6\theta + \sin\theta \sin 2\theta}{\sin 3\theta \cos 6\theta + \sin\theta \cos 2\theta}$$

$$= \frac{\frac{-1}{2}(\cos 9\theta - \cos 3\theta) + \frac{-1}{2}(\cos 3\theta - \cos\theta)}{\frac{1}{2}(\sin 9\theta - \sin 3\theta) + \frac{1}{2}(\sin 3\theta - \sin\theta)}$$

$$= \frac{\frac{1}{2}(\cos\theta - \cos 9\theta)}{\frac{1}{2}(\sin 9\theta - \sin\theta)}$$

$$\frac{-2\sin 5\theta \sin(-4\theta)}{2\cos 5\theta \sin 4\theta}$$

$$\Rightarrow \frac{2\sin 5\theta \sin(4\theta)}{2\cos 5\theta \sin 4\theta} = \tan 5\theta$$

Example VIII (UNEB Question)

If *A*, *B*, *C* are angles of the triangle, show that
cos 2*A* + cos 2*B* + cos 2*C* = -1 − 4 cos*A* cos *B* cos *C*.

Solution

cos 2A + cos 2B + cos 2C

$2\cos(A + B)\cos(A – B) + 2\cos^2 C – 1$

$= -1 + 2\cos(A + B)\cos(A – B) + 2\cos^2 C$

A + B + C = 180

A + B = (180 – C)

cos(A + B) = cos(180 – C)

cos (A + B) = cos 180 cos C + sin 180 sin C

= -cos C

$\Rightarrow$ $-1 + 2\cos(A + B)\cos(A – B) + 2\cos^2 A$

$= -1 – 2\cos C\cos(A – B) + 2\cos^2 C$

= -1 – 2cos C[cos(A – B) − cos C]

= -1 – 2cos C[cos(A – B) – cos C]

cos *C* = -cos(*A* + *B*)

= -1 – 2cos C[cos(A – B) + cos (A + B)

= -1 – 4cos A cos B cos C.

cos 2*A* + cos 2*B* + cos 2*C* = -1 − 4 cos*A* cos *B* cos *C*.

Example IX (UNEB Question)

Use the factor formula to show that $\frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A} = \tan(A+B)$

Solution

$$\frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A}$$

$$= \frac{2\sin(A+B)\cos B}{2\cos(A+B)\cos B}$$

$$= \frac{\sin(A+B)}{\cos(A+B)}$$

= tan(A + B)

$$\Rightarrow \frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A} = \tan(A+B)$$

UNEB 2008

(i) Prove that $\frac{\cos A+\cos B}{\sin A+\sin B} = \cot\frac{A+B}{2}$

(ii) Deduce that $\frac{\cos A+\cos B}{\sin A+\sin B} = \tan\frac{C}{2}$ where A, B and C are ***solution***

(i) $\frac{\cos A + \cos B}{\sin A + \sin B} = \frac{2\cos\frac{A+B}{2}\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$

$= \frac{2\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}}$

$= \cot\frac{A+B}{2}$

(ii) $A + B + C = 180^0$

$A + B = 180 - C$

$\frac{A+B}{2} = 90 - \frac{C}{2}$

$\cot\frac{A+B}{2} = \frac{\cos\left(90 - \frac{c}{2}\right)}{\sin\left(90 - \frac{c}{2}\right)}$

$= \frac{\cos 90\cos\frac{c}{2} + \sin 90\sin\frac{c}{2}}{\sin 90\cos\frac{c}{2} - \cos 90\sin\frac{c}{2}}$

$= \frac{\sin\frac{c}{2}}{\cos\frac{c}{2}}$

$= \tan\frac{c}{2}$

$\Rightarrow \frac{\cos A + \cos B}{\sin A + \sin B} = \tan\frac{C}{2}$

Example X (UNEB Question)

Solve sin$x$ – sin 4x = sin2x – sin3x for $-\pi \le x \le \pi$

Solution

sin x – sin 4x = sin 2x – sin 3x

sin 3x + sin x = sin 4x + sin 2x

$2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)$

$2\sin(2x)\cos x = 2\sin 3x\cos x$

$\sin 2x\cos x - \sin 3x\cos x = 0$

$\cos x(\sin 2x - \sin 3x) = 0$

Taking cos = 0

$x = \cos^{-1}(0)$

$x = \frac{-\pi}{2}, \frac{\pi}{2}$

Taking sin 2x – sin 3x = 0

sin 3x – sin 2x = 0

$2\cos\left(\frac{3x+2x}{2}\right)\sin\left(\frac{3x-2x}{2}\right) = 0$

$\cos\left(\frac{5}{2}x\right)\sin\left(\frac{1}{2}x\right) = 0$

Either $\cos\left(\frac{5}{2}x\right) = 0$

$$\frac{5}{2}x = \cos^{-1}(0)$$

$$\frac{5}{2}x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi$$

$$x = \pm\frac{\pi}{5}, \ \pm\frac{3\pi}{5}$$

Or $\quad \text{Sin}\left(\tfrac{1}{2}x\right) = 0$

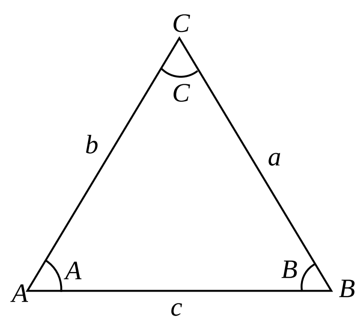
$$\frac{1}{2}x = \sin^{-1}(0) = 0, \pm\pi$$

$$x = 0^0$$

$\Rightarrow x = 0, \ -\frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{-3\pi}{5}$ are the solutions to the equation

Relationship between sides of a triangle

In a triangle ABC with angles A, B and C, we denote the side opposite these angles by their corresponding small letters a, b, and c respectively as shown in the figure below.



The sine rule

Let O be the centre of the circle circumscribing the triangle ABC with radius, *R*.

Figure I

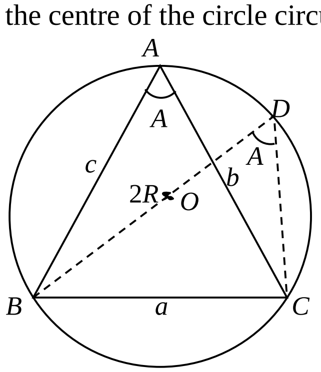


Figure II

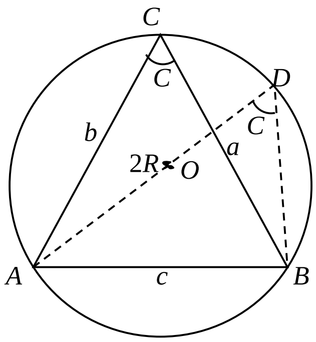
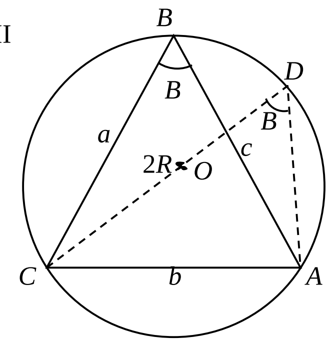


Figure III



From figure I, ∠BCD = 90°
Since this angle is subtended by the diameter,

$\Rightarrow \sin A = \frac{a}{2R}$ from figure I.

$\Rightarrow 2R = \frac{a}{\sin A}$ …………………. (i)

From figure II;

$\sin C = \frac{c}{2R}$

$2R = \frac{c}{\sin C}$ ……………………. (ii)

From figure III;

$\sin B = \frac{b}{2R}$

$2R = \frac{b}{\sin B}$ …………………….. (iii)

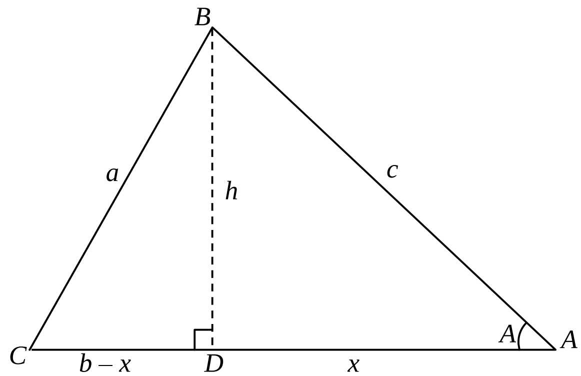
Equating equations (i), (ii), and (iii)

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

This is the sine rule

The Cosine rule

Consider a triangle ABC. Assume angle A is acute.



Considering the right-angled triangle BDA,

$x^2 + h^2 = c^2$ ……………………..…. (i)

from the right-angled triangle BCD,

$a^2 = (b - x)^2 + h^2$

$a^2 = b^2 - 2bx + x^2 + h^2$ ………….. (ii)

From Eqn (i);

$h^2 = c^2 - x^2$ ……………………..…. (iii)

Substituting Eqn (iii) in Eqn (ii)

$a^2 = b^2 - 2bx + x^2 + c^2 - x^2$

$a^2 = b^2 - 2bx + c^2$ ……………….. (iv)

From triangle ABD;

$$\cos A = \frac{x}{c}$$

$x = c \cos A$ …………..………….. (v)

Substituting Eqn (v) into (iv)

$\Rightarrow a^2 = b^2 - 2bc \cos A + c^2$

$a^2 = b^2 + c^2 - 2bc \cos A$

Application of cosine and sine rules

Example I

Prove that in a triangle ABC, $\frac{a^2 - b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Solution

From the sine rule; $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$

$$\frac{4R^2 \sin^2 A - 4R^2 \sin^2 B}{4R^2 \sin^2 C}$$

$$\frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$\frac{(\sin A + \sin B)(\sin(A - \sin B)}{\sin C \sin C}$$

$A + B + C = 180$

$C = 180 - (A + B)$

sin C = sin(180 – (A + B))

sin C = sin 180 cos (A+B) – cos 180 sin (A+B)

= sin (A + B)

$$\frac{(\sin A+\sin B)(\sin A-\sin B)}{[\sin(A+B)]^2}$$

$$= \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\cdot 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\left[2\sin\frac{A+B}{2}\cos\frac{A+B}{2}\right]\left[\sin(A+B)\right]}$$

$$= \frac{2\sin\frac{A-B}{2}\cos\frac{A-B}{2}}{\sin(A+B)}$$

$$= \frac{\sin(A-B)}{\sin(A+B)}$$

Example II

Prove that in any triangle ABC, $\frac{a^2+b^2-c^2}{a^2-b^2+c^2}$ = tan$B$ cot C

Solution

From the cosine rule;

$a^2 = b^2 + c^2 – 2bc \cos A$ ……………… (i)

$b^2 = a^2 + c^2 – 2ac \cos B$ ……………… (ii)

$c^2 = a^2 + b^2 – 2ab \cos C$ ……………… (iii)

From Eqn (i);

$2ac \cos B = a^2 + c^2 – b^2$

From Eqn (iii);

$2ab \cos C = a^2 + b^2 – c^2$

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{2ab\cos C}{2ac\cos B}$$

$$= \frac{b\cos C}{c\cos B}$$

But from the sine rule;

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$

$$\Rightarrow \frac{b\cos C}{c\cos B} = \frac{2R\sin B\cos C}{2R\sin C\cos B}$$

$$= \frac{\sin B}{\cos B}\times\frac{\cos C}{\sin C}$$

= tan B × cot C

$$\Rightarrow \frac{a^2+b^2+c^2}{a^2-b^2+c^2} = \tan B \cot C$$

Example III

Prove that in any triangle ABC, $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

Solution

$\frac{b-c}{b+c}\cot\frac{A}{2}$

From the sine rule; $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R\sin B$, $c = 2R \sin C$

$$\Rightarrow \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}\cot\tfrac{A}{2} = \frac{\sin B - \sin C}{\sin B + \sin C}\cot\tfrac{A}{2}$$

$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}}$$

But $A + B + C = 180$

$A = 180 - (B + C)$

$\frac{A}{2} = 90 - \frac{B+C}{2}$

$\cos\frac{A}{2} = \cos(90 - \frac{B+C}{2})$

$= \cos 90 \cos\frac{B+C}{2} + \sin 90 \sin\frac{B+C}{2}$

$= \sin\frac{B+C}{2}$

$\sin\frac{A}{2} = \sin(90 - \frac{B+C}{2})$

$= \sin 90 \cos\frac{B+C}{2} - \cos 90 \sin\frac{B+C}{2}$

$= \cos\frac{B+C}{2}$

$$\Rightarrow \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\sin\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\cos\frac{B+C}{2}} = \tan\frac{B-C}{2}$$

$$\Rightarrow \tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

Example IV

Prove that in any triangle ABC, $\frac{bc}{ab+ac} = \frac{\text{cosec }(B+C)}{\text{cosec}B + \text{cosec }C}$

Solution

From the sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$

$$\frac{bc}{ab+ac} = \frac{2R\sin B \cdot 2R\sin C}{(2R\sin A)(2R\sin B) + (2R\sin A)(2R\sin C)}$$

$$= \frac{4R^2\sin B\sin C}{4R^2\sin A\sin B + 4R^2\sin A\sin C}$$

$$= \frac{\sin B\sin C}{\sin A\sin B + \sin A\sin C}$$

$$= \frac{\frac{\sin B\sin C}{\sin B\sin C}}{\frac{\sin A\sin B}{\sin B\sin C} + \frac{\sin A\sin C}{\sin B\sin C}}$$

$$= \frac{1}{\frac{\sin A}{\sin C} + \frac{\sin A}{\sin B}}$$

$$= \frac{1}{\sin A(\frac{1}{\sin C} + \frac{1}{\sin B})}$$

$$= \frac{1}{\sin A(\text{cosec}B + \text{cosec }C)}$$

$$= \frac{1}{\sin A} \times \frac{1}{(\text{cosec } B + \text{cosec } C)}$$

From triangle ABC;

$A + B + C = 180$

$A = 180 – (B + C)$

$\sin A = \sin(180 – (B + C))$

$= \sin 180 \cos B + C – \cos 180 \sin(B + C)$

$= \sin (B + C)$

$$\Rightarrow \frac{1}{\sin(B+C)} \times \frac{1}{(\text{cosec } B + \text{cosec } C)} = \frac{\text{cosec } (B+C)}{\text{cosec} B + \text{cosec } C}$$

$$\Rightarrow \frac{bc}{ab+ac} = \frac{\text{cosec } (B+C)}{\text{cosec} B + \text{cosec } C}$$

Area of a triangle

Let D denote the area of a triangle ABC, then

$$D = \frac{1}{2} bc \sin A$$

$$\Rightarrow D = \frac{1}{2} bc \cdot 2 \sin \tfrac{A}{2} \cos \tfrac{A}{2}$$

$$D = bc \sin \tfrac{A}{2} \cos \tfrac{A}{2}$$

$$S = \frac{a+b+c}{2}$$

Where S is the semi perimeter.

From the cosine rule, $a^2 = b^2 + c^2 – 2bc \cos A$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = 1 – 2 \sin^2 \tfrac{A}{2}$$

$$\sin^2 \tfrac{A}{2} = \frac{1 - \cos A}{2}$$

$$\sin^2 \tfrac{A}{2} = \frac{1}{2}\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$\sin^2 \tfrac{A}{2} = \frac{1}{2}\left(\frac{a^2 - b^2 - c^2 + 2bc}{2bc}\right)$$

$$\sin^2 \tfrac{A}{2} = = \frac{1}{2}\left(\frac{a^2 - (b-c)^2}{2bc}\right)$$

$$\sin \tfrac{A}{2} = \sqrt{\frac{(a+b-c)(a+c-b)}{4bc}}$$

$a + b + c = 2s$

$a + b – c = a + b + c – 2c$

$= 2 s – 2c$

$= 2(s – c)$

$a + c - b = a + b + c - 2b$

$= 2s - 2b$

$= 2(s - b)$

$$\sin\tfrac{A}{2} = \sqrt{\frac{2(s-c)2(s-b)}{4bc}}$$

$$\sin\tfrac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

From the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = 2\cos^2\tfrac{A}{2} - 1$$

$$\cos^2\tfrac{A}{2} = \frac{1}{2}(1 + \cos A)$$

$$\cos^2\tfrac{A}{2} = \frac{1}{2}\left(\frac{2bc + b^2 + c^2 - a^2}{2bc}\right)$$

$$\cos^2\tfrac{A}{2} = \frac{1}{2}\left(\frac{(b+c)^2 - a^2}{2bc}\right)$$

$$\cos^2\tfrac{A}{2} = \frac{(b+c+a)(b+c-a)}{4bc}$$

$$\cos\tfrac{A}{2} = \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}}$$

$a + b + c = 2s$

$b + c - a = a + b + c - 2a$

$= 2s - 2a$

$= 2(s - a)$

$$\cos\tfrac{A}{2} = \sqrt{\frac{2s\cdot 2(s-a)}{4bc}} = \sqrt{\frac{s(s-a)}{bc}}$$

From the area of a triangle D;

$D = bc\sin\tfrac{A}{2}\cos\tfrac{A}{2}$

$$\cos\tfrac{A}{2} = bc\sqrt{\frac{(S-b)(S-c}{bc}}\cdot\sqrt{\frac{S(S-a)}{bc}}$$

$$= bc\frac{\sqrt{S(S-a)(s-b)(S-c)}}{bc}$$

$$= \sqrt{S(S-a)(S-b)(S-c)}$$

The area of a triangle is $\sqrt{S(S-a)(S-b)(S-c)}$

This is called the Heron formula named after the Greek Mathematician Heron

Differentiation and integration of trigonometric functions

Function	**Differentiate**	**Integrate**
Sin x	Cos x	$-\cos x$

cos x	-sinx	sin x
Sin ax	a cos ax	$\frac{-1}{a}\cos ax$
cos ax	-a sin ax	$\frac{1}{a}\sin ax$
sin 3x	3 cos 3x	$\frac{-1}{3}\cos 3x$
cos 3x	−3 sin 3x	$\frac{1}{3}\sin 3x$

Differentiation of trigonometric functions

Example I

Differentiate the following

(a) sin 6x

(b) −3 cos 5x

(c) −4 sin $\frac{3}{2}x$

(d) sin x^2

(e) 2sin $\frac{1}{2}(x + 1)$

Solutions

(a) $y = \sin 6x$

$$\frac{dy}{dx} = 6\cos 6x$$

(b) −3 cos 5x

$y = -3\cos 5x$

$$\frac{dy}{dx} = -3[5(-\sin 5x)]$$

$$\frac{dy}{dx} = 15\sin 5x$$

(c) −4 sin $\frac{3}{2}x$

$y = -4\sin\frac{3}{2}x$

$$\frac{dy}{dx} = -4\times\frac{3}{2}\cos\frac{3x}{2}$$

$$= -6\cos\frac{3x}{2}$$

(d) sin x^2

$y = \sin x^2$

$$\frac{dy}{dx} = 2x\cos x^2$$

(e) $2\sin\tfrac{1}{2}(x + 1)$

$y = 2\sin\tfrac{1}{2}(x + 1)$

$$\frac{dy}{dx} = 2 \times \frac{1}{2}\cos\tfrac{1}{2}(x+1)$$

$$\frac{dy}{dx} = \cos\tfrac{1}{2}(x+1)$$

Example II

Differentiate the following

(a) $\sin^2 x$
(b) $4\cos^2 x$
(c) $\cos^3 x$
(d) $2\sin^3 x$
(e) $3\sin^4 2x$
(f) $\sqrt{\sin 2x}$

Solutions

(a) $\sin^2 x$

$y = \sin^2 x$

$$\frac{dy}{dx} = 2\sin x(\cos x)$$

(b) $4\cos^2 x$

$y = 4\cos^2 x$

$$\frac{dy}{dx} = 8\cos x(-\sin x)$$

$$= -8\sin x\cos x$$

$$\frac{dy}{dx} = -8\sin x\cos x$$

(c) $\cos^3 x$

$y = \cos^3 x$

$$\frac{dy}{dx} = 3(\cos^2 x)(-\sin x)$$

$$\frac{dy}{dx} = -3\cos^2 x\sin x$$

(d) $2\sin^3 x$

$y = 2\sin^3 x$

$$\frac{dy}{dx} = 6\sin^2 x(\cos x)$$

$$\frac{dy}{dx} = 6\sin^2 x(\cos x)$$

(e) $3 \sin^4 2x$

$y = 3 \sin^4 2x$

$$\frac{dy}{dx} = 12\sin^3 2x(2\cos 2x)$$

$$\frac{dy}{dx} = 24\sin^3 2x\cos 2x$$

(f) $\sqrt{\sin 2x}$

$$\frac{dy}{dx} = \frac{1}{2}(\sin 2x)^{\frac{-1}{2}} \cdot 2\cos 2x$$

$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

Example II

Differentiate the following

(a) $x \cos x$

(b) $\boldsymbol{x} \sin 2x$

(c) $x^2 \sin x$

(d) $\frac{x}{\sin x}$

(e) $\frac{x^2}{\cos x}$

(f) $\frac{\cos 2x}{x}$

Solutions

(a) $y = x \cos x$

From $y = uv$;

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x(-\sin x) + \cos x$$

$$\frac{dy}{dx} = -x \sin x + \cos x$$

(b) $\boldsymbol{x} \sin 2x$

$y = x \sin 2x$

$$\frac{dy}{dx} = x .2\cos 2x + \sin 2x.1$$

$$\frac{dy}{dx} = 2x \cos 2x + \sin 2x$$

(c) $x^2 \sin x$

$y = x^2 \sin x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x^2\cos x + (\sin x)\ 2x$$

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$$

(d) $\frac{x}{\sin x}$

$$y = \frac{x}{\sin x}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x \cdot 1 - x\cos x}{(\sin x)^2}$$

$$\frac{dy}{dx} = \frac{\sin x - x\cos x}{(\sin x)^2}$$

(e) $\frac{x^2}{\cos x}$

$$\boldsymbol{y} = \frac{x^2}{\cos x}$$

From $y = \frac{u}{v}$;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x^2}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \cdot 2x - x^2(-\sin x)}{(\cos x)^2}$$

$$\frac{dy}{dx} = \frac{2x\cos x + x^2 \sin x}{\cos^2 x^2}$$

(f) $\frac{\cos 2x}{x}$

$$y = \frac{\cos 2x}{x}$$

From $y = \frac{u}{v}$;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x \cdot {}^{-}2\sin 2x - \cos 2x}{x^2}$$

$$\frac{dy}{dx} = \frac{{}^{-}2x\sin 2x - \cos 2x}{x^2}$$

Derivatives of tan *x*, cot *x*, sec *x*, and cosec *x*

(i) $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x)$

$\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

Proofs

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\Rightarrow \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\text{cosec}^2 x$$

$$\frac{d}{dx} = -\text{cosec}^2 x$$

(iii) $\frac{d}{dx}$ **(sec x)**

$$= \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$\frac{dy}{dx} = \frac{\cos x \cdot 0 - 1(^{-}\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$

$= \sec x \tan x$

$$\Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\Rightarrow \frac{d}{dx}\text{cosec } x = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \cdot 0 - 1 \cdot \cos x)}{(\sin x)^2}$$

$$\frac{-\cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$= -\cot x \text{ cosec } x$

Example I

Differentiate the following

(a) $\tan 2x$

(b) $\cot 3x$

(c) $2\text{cosec}\,\frac{1}{2}x$

(d) $-\tan(2x + 1)$

(e) $\frac{1}{3}\sec(3x - 2)$

(f) $\tan\sqrt{x}$

Solution

(a) $\tan 2x$

$y = \tan 2x$

$$\frac{dy}{dx} = 2\sec^2 2x$$

(b) $\cot 3x$

$y = \cot 3x$

$$\frac{dy}{dx} = 3(-\text{cosec}^2\, 3x)$$

$= -3\text{cosec}^2 3x$

(c) $2\text{cosec}\,\frac{1}{2}x$

$y = 2\text{cosec}\,\frac{1}{2}x$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{2}\left(-\text{cosec}\,\tfrac{1}{2}x \cot \tfrac{1}{2}x\right)$$
$$\frac{dy}{dx} = \left(-\text{cosec}\,\tfrac{1}{2}x \cot \tfrac{1}{2}x\right)$$

(d) $-\tan(2x + 1)$

$\boldsymbol{y} = -\tan(2x + 1)$

$$\frac{dy}{dx} = -2\sec^2(2x + 1)$$

(e) $\frac{1}{3}\sec(3x - 2)$

$y = \frac{1}{3}\sec(3x - 2)$

$$\frac{dy}{dx} = \frac{1}{3} \cdot 3\sec(3x-2)\tan(3x-2)$$
$$\frac{dy}{dx} = \sec(3x - 2)\tan(3x - 2)$$

(f) $\tan\sqrt{x}$

$y = \tan\sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}}\sec^2\sqrt{x}$$
$$\frac{dy}{dx} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Example II

Differentiate the following:

(a) $x \tan x$
(b) $\sec x \tan x$
(c) $x^2 \cot x$
(d) $3x \text{ cosec } x$
(e) $\text{cosec } x \cot x$
(f) $\frac{\tan x}{x}$

Solutions

(a) $x \tan x$

$y = x \tan x$

$$\frac{dy}{dx} = x\sec^2 x + (\tan x).1$$
$$\frac{dy}{dx} = x\sec^2 x + \tan x$$

(b) $\sec x \tan x$

$y = \sec x \tan x$

$$\frac{dy}{dx} = \sec x \sec^2 x + \tan x \cdot (\sec x \tan x)$$
$$\frac{dy}{dx} = \sec^3 x + \tan^2 x \sec x.$$

(d) 3*x* cosec *x*

$y = 3x \operatorname{cosec} x$

$$\frac{dy}{dx} = 3x(-\operatorname{cosec} x \cot x) + \operatorname{cosec} \cdot 3$$
$$\frac{dy}{dx} = -3x \operatorname{cosec} x \cot x + 3\operatorname{cosec} x$$

(e) cosec *x* cot *x*

$y = \operatorname{cosec} x \cot x$

$$\frac{dy}{dx} = \operatorname{cosec} x \cdot -\operatorname{cosec}^2 x + (\cot x)(-\cot x \operatorname{cosec} x)$$
$$\frac{dy}{dx} = \operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

Example III

Differentiate the following

(a) $\tan^2 x$
(b) $\sec^2 x$
(c) $3\operatorname{cosec}^2 x$
(d) $-\tan^2 2x$
(e) $\frac{1}{2}\cot^2 3x$
(f) $\sqrt{\tan x}$
(g) $-2\operatorname{cosec}^4 x$

Solution

(a) $\tan^2 x$

$y = \tan^2 x$

$$\frac{dy}{dx} = 2\tan x(\sec^2 x)$$
$$\frac{dy}{dx} = 2\sec^2 x \tan x$$

(b) $\sec^2 x$

$y = \sec^2 x$

$$\frac{dy}{dx} = 2\sec x(\sec x \tan x)$$
$$\frac{dy}{dx} = 2\sec^2 x \tan x$$

(c) $3\operatorname{cosec}^2 x$

$y = 3\operatorname{cosec}^2 x$

$\frac{dy}{dx} = 3 \times 2\text{cosec}\, x(-\text{cosec}\, x \cot x)$

$\frac{dy}{dx} = -6\text{cosec}^2 x \cot x$

(d) $-\tan^2 2x$

$y = -\tan^2 2x$

$\frac{dy}{dx} = -2(\tan 2x)(2\sec^2 2x)$

$\frac{dy}{dx} = -4\sec^2 2x \tan 2x$

(e) $\frac{1}{2}\cot^2 3x$

$y = \frac{1}{2}\cot^2 3x$

$\frac{dy}{dx} = \frac{1}{2} \times 2 \cot 3x(-3\text{cosec}^2 3x)$

$= -3\text{cosec}^2 3x \cot 3x$

(f) $\sqrt{\tan x}$

$\frac{dy}{dx} = \frac{1}{2}(\tan x)^{\frac{-1}{2}} \cdot \sec^2 x$

$\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$

(g) $-2\text{cosec}^4 x$

$y = -2\text{cosec}^4 x$

$\frac{dy}{dx} = -8\text{cosec}^3 x(-\text{cosec}\, x \cot x)$

$\frac{dy}{dx} = 8\text{cosec}^4 x \cot x$

Integration of Trigonometric functions

Integration is the process of obtaining a function from its derivative

Note: $\int \cos ax\, dx = \frac{1}{a}\sin(ax) + c$

$\int \sin ax\, dx = \frac{-1}{a}\cos(ax) + c$

Example I

Integrate the following

(a) $\cos 3x$
(b) $\sin 3x$
(c) $\cos(3x - 1)$

(d) $\sin(2x + 1)$
(e) $6 \cos 4x$

Solution

(a) $\cos 3x$

$y = \cos 3x$

$$\int y\,dx = \int \cos 3x\ dx$$

$$= \frac{1}{3}\sin 3x + c$$

$$\int \cos 3x\,dx = \frac{1}{3}\sin 3x + c$$

(b) $\int \sin 3x\ dx = \frac{1}{3}\sin 3x + c$

$$= \frac{-1}{3}\cos 3x + c$$

(c) $\int \cos(3x - 1)\ dx = \frac{1}{3}\sin(3x - 1) + c$

(d) $\int \sin(2x + 1)\,dx = \frac{-1}{2}\cos(2x + 1) + c$

(e) $\int 6\cos 4x\ dx = 6\int \cos 4x\ dx$

$$= 6\left[\frac{1}{4}\sin 4x\right] + c$$

$$= \frac{3}{2}\sin 4x + c$$

Example

Integrate the following

(a) $\sec^2 2x$
(b) $3\sec x \tan x$
(c) $-\text{cosec}^2 \tfrac{1}{2}x$
(d) $\frac{1}{3}\text{cosec } 3x \cot 3x$
(e) $2\sec^2 x \tan x$
(f) $\frac{\sin x}{\cos^2 x}$
(g) $\frac{1}{\sin^2 2x}$
(h) $\frac{\cos 2x}{\sin^2 2x}$

Solution

Note:	$\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \int \sec^2 x \; dx = \tan x + c$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\Rightarrow \int \sec x \tan x \; dx = \sec x + c$$

$$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$$

$$\Rightarrow \int \text{cosec}^2 x \; dx = -(\cot x) + c$$

$$\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$$

$$\Rightarrow \int \text{cosec}\, x \cot x \; dx = -\text{cosec}\, x + c$$

(a) $\int \sec^2 2x \; dx$

Let $u = 2x$

$du = 2dx$

$dx = \frac{du}{2}$

$$\int \sec^2 2x \; dx = \int \sec^2 u \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int \sec^2 u \; du$$

$$= \frac{1}{2}\tan u + c$$

$$= \frac{1}{2}\tan(2x) + c$$

(b) $\int 3\sec x \tan x \, dx$

$$= 3\int \sec x \tan x \, dx$$

$$= 3 \sec x + c$$

(c) $\int {}^{-}\text{cosec}^2 \tfrac{1}{2}x \; dx$

Let $u = \frac{1}{2}x$

$\frac{du}{dx} = \frac{1}{2}$

$dx = 2\,du$

$$\int -\text{cosec}^2 \tfrac{1}{2}x \; dx = \int -\text{cosec}^2 u(2du)$$

$$= 2\int -\text{cosec}^2 u$$

$$= 2[\cot u] + c$$

$$= 2\cot \tfrac{1}{2}x + c$$

(d) $\int \tfrac{1}{3}\text{cosec}\; 3x \cot 3x \, dx$

Let $u = 3x$

$du = 3\ dx$

$$dx = \frac{du}{3}$$

$$\int \tfrac{1}{3}\text{cosec } 3x \cot 3x\, dx = \frac{1}{3}\int \text{cosec } 3x \cot 3x\ dx$$

$$= \frac{1}{3}\int \text{cosec } u \cot u \cdot \frac{du}{3}$$

$$= \frac{1}{9}\int \text{cosec } u \cot u\ du$$

$$= \frac{1}{9}(\text{-cosec } u) + c$$

$$= \frac{-1}{9}\text{cosec } 3x + c$$

(e) $\int 2\sec^2 x \tan x\ dx$

Consider $\frac{d}{dx}(\sec^2 x) = 2\sec x(\sec x \tan x)$

$= 2\sec^2 x \tan x$

$$\Rightarrow \int 2\sec^2 x \tan x\ dx = \sec^2 x + c$$

(f) $\int \frac{\sin x}{\cos^2 x} dx$

$$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx$$

$$= \int \tan x \sec x\ dx$$

$= \sec x + c$

(g) $\frac{1}{\sin^2 2x} = \int \text{cosec}^2 2x\ dx$

Let $u = 2x$

$du = 2\ dx$

$$dx = \frac{du}{2}$$

$$\int \text{cosec}^2 2x\ dx = \int \text{cosec}^2 u \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int \text{cosec}^2 u$$

$$= \frac{-1}{2}\cot u + c$$

$$= \frac{-1}{2}\cot 2x + c$$

(h) $\frac{\cos 2x}{\sin^2 2x}$

$$= \int \frac{\cos 2x}{\sin 2x} \cdot \frac{1}{\sin 2x}\, dx$$

$$= \int \cot 2x \operatorname{cosec} 2x\, dx$$

Let $u = 2x$

$du = 2\, dx$

$$dx = \frac{du}{2}$$

$$\int \cot u \operatorname{cosec} u \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \operatorname{cosec} u \cot u\, du$$

$$= \frac{1}{2}(-\operatorname{cosec} u) + c$$

$$= \frac{-1}{2} \operatorname{cosec} 2x + c$$

Example III

Evaluate the following

(a) $\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

(b) $\int_{\frac{-\pi}{3}}^{\frac{\pi}{6}} \sec^2 x\, dx$

(c) $\int_0^{\pi} \sin^2 x\, dx$

Solution

(a) $\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

$$= \left[\frac{-1}{2} \cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{-1}{2} \cos 2\left(\tfrac{\pi}{2}\right) - \frac{-1}{2} \cos 0$$

$$= \frac{-1}{2}(-1) + \frac{1}{2}$$

$$= 1$$

(b) $\int_{\frac{-\pi}{3}}^{\frac{\pi}{6}} \sec^2 x\, dx$

$$= \left[\tan x\right]_{\frac{-\pi}{3}}^{\frac{\pi}{6}}$$
$$= \tan\left(\tfrac{\pi}{6}\right) - \tan\left(\tfrac{-\pi}{3}\right)$$
$$= \frac{1}{\sqrt{3}} - \left(-\sqrt{3}\right)$$
$$= \frac{1}{\sqrt{3}} + \sqrt{3}$$
$$= \frac{\sqrt{3}}{3} + \sqrt{3} = \frac{4\sqrt{3}}{3}$$

(c) $\int_0^{\pi} \sin^2 x \, dx$

From $\cos 2x = 1 - 2\sin^2 x$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$
$$= \frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - (0 - 0)\right]$$
$$= \frac{1}{2}\left[\pi\right]$$
$$= \frac{1}{2}\pi$$

Example

A particle moves in a straight line such that its velocity in m/s after passing through a fixed point O is $3\cos t - 2\sin t$. Find:

(a) Its distance from O after $\tfrac{1}{2}\pi$ s
(b) Its acceleration after π s
(c) The time when its velocity is first zero.

Solution

$V = 3\cos t - 2\sin t$

$$\frac{dS}{dt} = 3\cos t - 2\sin t$$

$dS = (3\cos t - 2\sin t)\, dt$

$S = 3\sin t + 2\cos t + c$

When $t = 0$, $S = 0$

$0 = 3\sin(0) + 2\cos(0) + c$

$-2 = c$

$\Rightarrow S = 3\sin t + 2\cos t - 2.$

When $t = \tfrac{1}{2}\pi$,

$S = 3\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} - 2$

$S = 3 - 2$

$S = 1$ m

$V = 3\cos t - 2\sin t$

$a = \dfrac{dV}{dt} = -3\sin t - 2\cos t$

$a = \left.\dfrac{dV}{dt}\right|_{\pi} = -3\sin\pi - 2\cos\pi$

$= 2$ m/s^2

$\Rightarrow a = 2$ m/s^2

$V = 3\cos t - 2\sin t$

$3\cos t - 2\sin t = 0.$

$R = \cos(t + \alpha) = 0$

$R = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\sqrt{13}\cos(t + \alpha) = 0$

$\sqrt{13}\cos(t + 33.7) = 0$

$\cos(t + 33.7) = 0$

$t + 33.7 = \cos^{-1}0$

$t + 33.7 = 90$

t = 56.3

$t = \dfrac{56.3\pi}{180}$

$t = 0.983$ s

Revision Exercise

1. Solve the following for all values of x from 0° to 360°.

   (a) $\sin x = \frac{1}{2}$

   (b) $\cos x = \frac{-1}{2}$

   (c) $\tan x = 1$

   (d) $\tan x = -1$

   (e) $\sin x = \frac{-\sqrt{3}}{2}$

   (f) $\cos x = \frac{1}{\sqrt{2}}$

2. Solve the following equations for values of x from -180° to 180°

   (a) $\sin x = \frac{-1}{2}$

   (b) $\cos x = \frac{1}{2}$

   (c) $\sin x = \frac{\sqrt{3}}{2}$

(d) $\tan x = \sqrt{3}$

(e) $\cos x = \frac{-1}{\sqrt{2}}$

(f) $\cos x = \frac{-\sqrt{3}}{2}$

3. Solve the following equations for all values of x from 0° to 360°

(a) $\sin x = \frac{-1}{2}$

(b) $\cos x = -0.7$

(c) $\tan x$ -0.75

(d) $\cos^2 x = \frac{1}{4}$

(e) $\sin x = 2\cos x$

(f) $2\sin x - 3\cos x = 0$

(g) $\sin 2x = \frac{-\sqrt{3}}{2}$

(h) $\cos 2x = \frac{1}{2}$

(i) $\sin(x + 20) = \frac{-\sqrt{3}}{2}$

(j) $\tan(x - 30) = 1$

(k) $3(\cos x - 1) = -1$

(l) $\sin x (1 - 2\cos x) = 0$

(m) $\cos x(2\sin x + \cos x) = 0$

(n) $2\sin x \cos x + \sin x = 0$

(o) $4\sin x \cos x = 3\cos x$

(p) $4\cos^2 x + \cos x = 0$

(q) $\tan x = 4 \sin x$

(r) $(2\sin x - 1)(\sin x + 1) = 0$

(s) $2\sin^2 x - \sin x - 1 = 0$

(t) $2\tan^2 x - \tan x - 6 = 0$

(u) $2\tan x - \frac{1}{\tan x} = 1$

Solve the following equations for all values of x from -180° to 180°

1. $\cos^2 x = \frac{3}{4}$
2. $\sin 2x = 2\cos 2x$
3. $\cos(x - 20) = \frac{-1}{\sqrt{2}}$
4. $\cos x(\sin x - 1) = 0$
5. $3\sin^2 x = 2\sin x \cos x$

6. $2\cos^2 x - 5\cos x + 2 = 0$

7. Factorise the expression $6\sin\theta\cos\theta + 3\cos\theta + 4\sin\theta + 2$. Hence solve $6\sin\theta\cos\theta + 3\cos\theta + 4\sin\theta + 2 = 0$ for $-180° \leq 180°$

8. Factorise the equation $3\sin\theta\cos\theta - 3\sin\theta + 2\cos\theta - 2$. Hence solve $3\sin\theta\cos\theta - 3\sin\theta + 2\cos\theta = 2$.

9. Without using tables or calculator, find the values of:

(a) sec 45°	(g) cosec 330°
(b) cot 45°	(h) sec 240°
(c) cosec 30	(i) cot -135°
(d) sec 60°	(j) sec -60°
(e) cosec 135°	(k) sec(-120°)
(f) sec 120°	(l) cosec 315°

10. Simplify the following expression:
    (a) $\sqrt{(1-\sin A)(1+\sin A)}$
    (b) $\operatorname{cosec}\theta\tan\theta$
    (c) $\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta}$
    (d) $\cot\theta\sqrt{1-\cos^2\theta}$

11. Prove the following identities
    (a) $\sin\theta\tan\theta + \cos\theta = \sec\theta$
    (b) $\operatorname{cosec}\theta - \sin\theta = \cot\theta\cos\theta$
    (c) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2\theta = 2$
    (d) $(\sin\theta + \operatorname{cosec}\theta)^2 = \sin^2\theta + \cot^2\theta + 3\theta$
    (e) $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$
    (f) $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$
    (g) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$
    (h) $\dfrac{\operatorname{cosec}\theta}{\cos\theta + \tan\theta} = \cot\theta$
    (i) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$
    (j) $\dfrac{\sin\theta}{1-\cos\theta} + \dfrac{\sin\theta}{1+\cos\theta} = 2\operatorname{cosec}\theta$
    (k) $\cos^4 x - \sin^4 x = \cos^2 x$
    (l) $\cos A + \cos(B + C) = 0$
    (m) $\dfrac{\cos^2\theta}{1+\cot^2\theta} = 2\cos\theta$

12. Prove the following identities:
    (a) $2\operatorname{cosec} 2\theta = \operatorname{cosec}\theta\sec\theta$
    (b) $\tan A + \cot A = 2\text{coesec } 2A$
    (c) $\dfrac{1+\tan^2 A}{2-\tan^2 A} = \sec 2A$

(d) $\cot 2A = \operatorname{cosec} 2A - \tan A$

(e) $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

(f) $\tan\theta - \cot\theta = -2\cot 2\theta$

13. Prove the following identities:

(a) $\frac{1+\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$

(b) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(c) $\frac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

14. Eliminate θ from each of the following pairs of relationships

(a) $x = \sin\theta$, $y = \cos\theta$

(b) $x = 3\sin\theta$, $y = \operatorname{cosec}\theta$

(c) $5x = \sin\theta$, $y = 2\cos\theta$

(d) $x = 3 + \sin\theta$, $y = \cos\theta$

(e) $x = 2 + \sin$, $\cos\theta = 1 + y$.

15. Solve the following equations for all values of θ from -180° to 180°.

(a) $4 - \sin\theta = 4\cos^2\theta$

(b) $\sin^2\theta + \cos\theta + 1 = 0$

(c) $5 - 5\cos\theta = 3\sin^2\theta$

(d) $8\tan\theta = 3\cos\theta$

(e) $\sin^2\theta + 5\cos^2\theta = 0$

(f) $1 - \cos^2\theta = -2\sin\theta\cos\theta$

16. Solve the following equations from 0° to 360°

(a) $\sec\theta = 2$

(b) $\cot 2\theta = \frac{-2}{5}$

(c) $3\cot\theta = \tan\theta$

(d) $2\sin\theta = -3\cot\theta$

(e) $2\sec^2\theta - 3 + \tan\theta = 0$

17. If $A + B + C = 180°$, prove that

$$\cos 2A + \cos 2B + \cos 2C = 1 - \cos A\cos B\cos C$$

18. Prove that sin 3A = 4sinAsin(60 + A)sin(60 – A)

19. Show that in a triangle ABC, if $2S = a + b + c$, then

$$1 - \tan\tfrac{A}{2} + \tan\tfrac{B}{2} = \frac{C}{A}$$

20. Prove that in any triangle ABC,

$$(a + b + c)(\tan\tfrac{A}{2} + \tan\tfrac{B}{2}) = 2c\cot\tfrac{C}{2}.$$

21. Prove that in any triangle

$$ABC,\ \frac{a+b-c}{a+b+c} = \tan\tfrac{A}{2}\tan\tfrac{B}{2}$$

22. From a point A, a light wind due to north of A has an elevation α from a point B, due west of A. The angle of elevation is β. Prove that the angle of elevation from the midpoint of *AB*. is $\tan^{-1}\left(\frac{2}{\sqrt{3\cot^2\alpha+\cot^2\beta}}\right)$
23. Solve: 4cos α – 3sin α = 2
24. Solve the equation $15\cos 2\theta + 20\sin 2\theta + 7 = 0$
25. Find all the possible values of *x* that satisfy $\tan^{-1}3x+\tan^{-1}x=\frac{\pi}{4}$
26. Prove that $\frac{\sin 8\theta\cos\theta-\sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta-\sin 3\theta\sin 4\theta}=\tan 2\theta$
27. Solve the equation $2\cos^2(x-\frac{\pi}{2})-3\cos(x-\frac{\pi}{2})$ = 0 for 0 ≤ *x* ≤ 2π.
28. Solve $\cos^4x + \sin^4x = \frac{7}{8}$ for $0 \le x \le \frac{\pi}{2}$.
29. Find the value of *x* for $3\cos^2x - 8\cos x + 4 = 0$
30. Show that $\left(\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2=\frac{1-\cos\theta}{1+\cos\theta}$
31. Prove that $\frac{\tan\theta}{1-\cot\theta}+\frac{\cot\theta}{1-\tan\theta}=1+\tan\theta+\cot\theta$
32. Solve the equation cos *x* – cos 4*x* = cos 2*x* – cos 3*x* for $-\pi \le x \le \pi$.
33. Given that *y* = 4cos *x* – 6sin *x*. Express *y* in the form *R*cos(*x* + *α*), where *R* is a constant. Find the maximum and minimum value of *y*.
34. Express (45° + *x*) in terms of tan *x*. Hence or otherwise express tam 75° in the form $a + b\sqrt{3}$.
35. Given sin *x* = $\frac{-4}{5}$, where 180° ≤ *x* ≤ 270, find without using tables or calculator the value of tan 3*x*.
36. Show that:
    (a) $\tan^{-1}\frac{1}{3}+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$
    (b) $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$
    (c) $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$
37. Solve the equation
    (a) $\tan^{-1}(2x + 1) + \tan^{-1}(2x - 1) = \tan^{-1}2$
    (b) $\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = 32$
    (c) $\cos^{-1}x + \cos^{-1}x\sqrt{8} = \frac{\pi}{2}$
    (d) $2\sin\frac{x}{2}+\sin^{-1}x\sqrt{2}=\frac{\pi}{2}$
38. Without using tables or calculator, evaluate $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}$

6. $\pm 1, \pm i, \pm\frac{1}{\sqrt{2}}(1-i), \frac{1}{\sqrt{2}}(1+i)$

7. (i) $-1 + i\sqrt{3}$, 2, $2\pi/3$; $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, 1, $\pi/6$

(ii) $1 \pm 2i$; (a) $p^2 = q - 4$, (b) $2p = q + 5$

8. (a) $-3\sqrt{3} + 5i$; $2\sqrt{13}$, 2.38 rad, (b) 5, –20 .

9. (i) $-1 - i$, $3\pi/4$, (ii) $2 - i$, 2; -10.

10.(b) -1, $\frac{1}{2} \pm \frac{3}{2}i$, (c) $-1 - 3i$, $\frac{1}{2}(1 - 3\sqrt{3}) + \frac{1}{2}(3 + \sqrt{3})i$

11. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$; $1 \pm i\sqrt{3}$;, $\frac{5}{2} \pm \frac{\sqrt{3}}{2}i$; 28

12. (ii) $x^3 - 3abx + a^3 + b^3$; $\sqrt[3]{2} - \sqrt[3]{4}$, $\omega\sqrt[3]{2} - \omega^2\sqrt[3]{4}$, $\omega^2\sqrt[3]{2} - \omega\sqrt[3]{4}$

13. (i) $\pm(3 - 2i)$, (ii) $3 \pm i\sqrt{3}$.

14. (i) (a) 7/5, -4/5; (b) $2 \pm i$

15. (i) $1 - i$, $-1 \pm 2i$, (ii) $\frac{1}{2}(3 + i\sqrt{3})$; $\pi/6$, $2\pi/3$

16. (i) 2, 32, $\pi/3$, $-\pi/3$; $-16\sqrt{3}$, (ii) $2 + 3i$

17. (i) -1, (ii) $1 + 2i$, $2\sqrt{5}$

18. (i) $\sqrt{2}$, $-\pi/4$; $\frac{1}{2} + \frac{1}{2}i$, $8i$.

20. (i) $2 - i$, $3 - 4i$; (2, -1)

21. (ii) $3x^2 + 3y^2 + 10x + 3 = 0$.

22. (i) $x = 1, y = 2$ or $x = -1, y = -2$

23. (i) a) 13, -23°, (b) 1, 90°; (ii) 12;

(iii) $|z + 1 + i| \geq 4$

24. $2\sin\theta$; $\frac{1}{2}\text{cosec}\theta, -\theta$; $y = -\frac{1}{2}$. 25. $2\sqrt{2} - 2$.

VECTORS

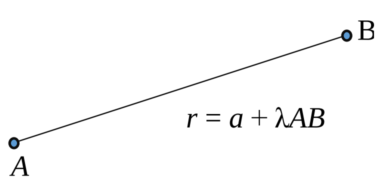
Straight line in space

A straight line is uniquely determined in space if either; we know one point on the straight line and its direction or two points on the straight line.

Vector equation of a line

The vector equation of a line is given by

$r = \boldsymbol{a} + \lambda AB$



$r = \boldsymbol{a} + \lambda AB$

$r = a + \lambda d$

Where; a = any point on the line

d = directional vector of the line.

The Cartesian equation is given by;

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c} = \lambda$$

Where a, b and c are direction vectors

Example 1

Find the vector and Cartesian equation of a line passing through $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and is parallel to $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

Solution;

$r = a + \lambda d$

$$r = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Cartesian equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$x = 3 + 3\lambda$
$y = -1 - \lambda$
$z = 2 + 2\lambda$
$\frac{x-3}{3} = \lambda, \frac{y+1}{-1} = \lambda, \frac{z-2}{2} = \lambda$
$\frac{x-3}{3} = \frac{y+1}{-1} = \frac{z-2}{2} = \lambda$

Example II
Find the vector and the Cartesian equation of a line passing through A(3, 4, -7) and B(1, -1, 6)
Solution

$\boldsymbol{r} = a + \lambda d$
$d = AB = OB - OA$
$\begin{pmatrix}1\\-1\\6\end{pmatrix} - \begin{pmatrix}3\\4\\-7\end{pmatrix} = \begin{pmatrix}-2\\-5\\13\end{pmatrix}$
$\boldsymbol{r} = \begin{pmatrix}3\\4\\-7\end{pmatrix} + \lambda\begin{pmatrix}-2\\-5\\13\end{pmatrix}$(vector equation of line)
$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\4\\-7\end{pmatrix} + \lambda\begin{pmatrix}-2\\-5\\13\end{pmatrix}$
$x = 3 - 2\lambda$
$y = 4 - 5\lambda$
$z = -7 + 13\lambda$
$\frac{x-3}{-2} = \lambda$
$\frac{y-4}{-5} = \lambda$
$\frac{z+7}{13} = \lambda$

Cartesian equation
$\frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z+7}{13} = \lambda$

Example III
Find the vector and Cartesian equation of a line passing through (2, -1, 1) and is parallel to the line whose equation
$\frac{x-3}{2} = \frac{y+1}{7} = \frac{z-2}{-3} = \lambda$
Solution
Since the lines are parallel, it implies that they have the same parallel vectors.

$\boldsymbol{r} = a + \lambda d$
$\boldsymbol{r} = \begin{pmatrix}2\\-1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\7\\-3\end{pmatrix}$

Cartesian equation:
$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\7\\-3\end{pmatrix}$
$x - 2 = 2\lambda \Rightarrow \frac{x-2}{2} = \lambda$
$y + 1 = 7\lambda \Rightarrow \frac{y+1}{7} = \lambda$
$z - 1 = -3\lambda \Rightarrow \frac{z-1}{-3} = \lambda$
$\frac{x-2}{2} = \frac{y+1}{7} = \frac{z-1}{-3} = \lambda$

Example III
Find the vector and Cartesian equations of the a line passing through the following points
(a) 5, -4, 6) and (3, 7, 2)
(b) (3, 4, -7) and (5, 1, 6)
Solution

$r = a + \lambda AB$
$r = a + \lambda d$
$\boldsymbol{d} = \begin{pmatrix}3\\7\\2\end{pmatrix} - \begin{pmatrix}5\\-4\\6\end{pmatrix} = \begin{pmatrix}-2\\11\\-4\end{pmatrix}$

(a) $\boldsymbol{r} = \begin{pmatrix}5\\-4\\6\end{pmatrix} - \lambda\begin{pmatrix}-2\\11\\-4\end{pmatrix}$

$\frac{x-5}{-2} = \frac{y+4}{11} = \frac{z-6}{-4} = \lambda$

(b) A(3, 4, -7) and B(5, 1, 6)

$r = a + \mu d$
$\boldsymbol{d} = \begin{pmatrix}5\\1\\6\end{pmatrix} - \begin{pmatrix}3\\4\\-7\end{pmatrix}$
$\boldsymbol{d} = \begin{pmatrix}2\\-3\\13\end{pmatrix}$
$\boldsymbol{r} = \begin{pmatrix}3\\4\\-7\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\13\end{pmatrix}$
$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z+7}{13} = \lambda$

Example IV
Find the coordinates of the point where the line joining the points (2, 3, 1) and (3, -4 -5) meets the *x-y* plane

$\boldsymbol{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-7\\-6\end{pmatrix}$

$x = 2 + \lambda$
$y = 3 - 7\lambda$
$z = 1 - 6\lambda$

For the line to meet the *x-y* plane, *z* = 0

$0 = 1 - 6\lambda$

$\lambda = \frac{1}{6}$

$x = 2 + \frac{1}{6}$

$x = \frac{13}{6}$

$y = 3 - \frac{7}{6}$

$y = \frac{11}{6}$

The coordinates are $\left(\frac{13}{6}, \frac{11}{6}, 0\right)$

Example V

Show that 4**i** – **j** – 12**k** lies on the line

$$\boldsymbol{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} + \lambda(\boldsymbol{i} - \boldsymbol{2}j + 4\boldsymbol{k}$$

Solution

$\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

$(4, -1, 12)$

$\begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$4 = 2 + \lambda \Rightarrow \lambda = 2$

$-1 = 3 - 2\lambda \Rightarrow \lambda = 2$ and

$12 = 4 + 4\lambda \Rightarrow \lambda = 2$

∴ The point lies on the line since the values of μ are the same.

Example V

The points A, B, C have position vectors $\begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix}$. Find which of the three points lie in the line $\boldsymbol{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Solution

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

For *A*, $\boldsymbol{r} = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$-4 = -1 + 3\lambda \Rightarrow \lambda = -1$

$5 = 4 - \lambda \Rightarrow \lambda = -1$

$-1 = 1 + 2\lambda \Rightarrow \lambda = -1$

$\Rightarrow \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$ lies on the line.

For B, $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$5 = -1 + 3\lambda \Rightarrow \lambda = 2$

$2 = 4 - \lambda \Rightarrow \lambda = 2$

$3 = 1 + 2\lambda \Rightarrow \lambda = 1$

Since the values of λ are not the same, point B does not lie on the line.

For C, $\begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix}$

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$8 = -1 + 3\lambda \Rightarrow \lambda = 3$

$1 = 4 - \lambda \Rightarrow \lambda = 3$

$7 = 1 + 2\lambda \Rightarrow \lambda = 3$

⇒ Since the vales of λ are the same, point C lies on the line.

Angle between two lines

The angle between two lines is the angle between their directional vectors

Consider two lines L_1 and L_2 with vector equations
$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{b} + \mu \boldsymbol{d}_2$ respectively
The angle between the two lines is given by the formula $\frac{d_1 \cdot d_2}{|d_1||d_2|}$

Examples

1. Find the angle between the lines;

$\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k} + \lambda(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$

$\boldsymbol{r} = 5\boldsymbol{i} - 2\boldsymbol{j} + \mu(3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$

$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\,\boldsymbol{\theta}$

$$cos\,\boldsymbol{\theta} = \frac{a.b}{|a||b|}$$

$$\cos\theta = \frac{d_1.d_2}{|d_1||d_2|}$$

$$d_1 = \begin{pmatrix}1\\2\\2\end{pmatrix} \qquad d_2 = \begin{pmatrix}3\\2\\6\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}1\\2\\2\end{pmatrix}.\begin{pmatrix}3\\2\\6\end{pmatrix}}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+2^2+6^2}}$$

$$\cos\theta = \frac{3+4+12}{\sqrt{9}\sqrt{49}}$$

$$\cos\theta = \frac{19}{21}$$

$$\theta = \cos^{-1}\left(\frac{19}{21}\right)$$

$$\theta = 25.2°$$

Example II

Find the angles between the lines

$$\frac{x+4}{3} = \frac{y+1}{5} = \frac{z+3}{4} \;\&\; \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

Solution

$$\cos\theta = \frac{d_1.d_2}{|d_1||d_2|}$$

$$d_1 = \begin{pmatrix}3\\5\\4\end{pmatrix}, \quad d_2 = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}3\\5\\4\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix}}{\sqrt{3^2+5^2+4^2}\sqrt{1^2+1^2+2^2}}$$

$$\cos\theta = \frac{3+5+8}{(\sqrt{50})\sqrt{6}}$$

$$\cos\theta = \frac{16}{\sqrt{300}}$$

$$\theta = \cos^{-1}\left(\frac{16}{\sqrt{300}}\right)$$

$$\theta = 22.5°$$

Example III

Find the acute angle between the lines:

$$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z-2}{-1} \text{ and } \frac{1-x}{2} = \frac{y-3}{1} = \frac{z-7}{2}$$

Solution

$$\Rightarrow \frac{x-1}{2} = \frac{y+2}{1} = \frac{z-2}{-1} \text{ and } \frac{x-1}{-2} = \frac{y-3}{1} = \frac{z-7}{2}$$

$$\Rightarrow \frac{x-1}{-2} = \frac{y-3}{1} = \frac{z-7}{2}$$

$$\boldsymbol{d_1} = \begin{pmatrix}2\\1\\-1\end{pmatrix} \text{ and } \boldsymbol{d_2} = \begin{pmatrix}-2\\1\\2\end{pmatrix}$$

$$\cos\theta = \frac{d_1 \cdot d_2}{|d_1|\cdot|d_2|}$$

$$\cos\theta = \frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}.\begin{pmatrix}-2\\1\\2\end{pmatrix}}{\sqrt{2^2+1^2+(-1)^2}\,\sqrt{(-2)^2+1^2+(2)^2}}$$

$$\cos\theta = \frac{-4+1-2}{\sqrt{6}\,.\sqrt{9}}$$

⇒ The acute angle between the two lines is 47.1°

Example IV

Find the angle between the lines:

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-2}{1} = \lambda \text{ and } \frac{x-5}{1} = \frac{y-1}{1} = \frac{z}{2} = \mu$$

Solution

$$\boldsymbol{d_1} = \begin{pmatrix}3\\2\\1\end{pmatrix} \text{ and } \boldsymbol{d_2} = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\cos\theta = \frac{d_1 \cdot d_2}{|d_1|\cdot|d_2|}$$

$$\cos\theta = \frac{\begin{pmatrix}3\\2\\1\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix}}{\sqrt{3^2+2^2+(1)^2}\,\sqrt{(1)^2+1^2+(2)^2}}$$

$$\cos\theta = \frac{3+2+2}{\sqrt{14}\,.\sqrt{6}}$$

θ = 40.2°

⇒ The acute angle between the two lines is 40.2°

Note: If two lines are perpendicular, then $(\boldsymbol{d_1}.\boldsymbol{d_2}) = \boldsymbol{0}$

Point of Intersection of two Lines

Example

Find the point of intersection of the lines

$$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} \;\&\; \frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$$

Solution

$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} = \lambda$ ………………. (i)

$\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1} = \mu$ …………… (ii)

From equation (i)

$x = \lambda$ ……………………………… (iii)

$\frac{y+2}{2} = \lambda$

$y + 2 = 2\lambda$

$y = 2\lambda - 2$ ……………………… (iv)

$\frac{z-5}{-1} = \lambda$

$z - 5 = -\lambda$

$z = -\lambda + 5$ ……………………… (v)

From equation (ii)

$x = -\mu + 1$ ……………………… (vi)

$y + 3 = -3\mu$

$y = -3\mu - 3$ … … … … … … … … … … (vii)

$z = \mu + 4$ ………………………….. (viii)

$\lambda = -\mu + 1$ … … … … … … … … … … (*)

$2\lambda - 2 = -3\mu - 3$

$2\lambda + 3\mu = -1$ … … … … … … … … . . (**)

Substituting Eqn (*) in Eqn (**)

$2(1-\mu) + 3\mu = -1$

$2 - 2\mu + 3\mu = -1$

$2 + \mu = -1$

$\mu = -3$

$\lambda = -\mu + 1$

$\lambda = 3 + 1$

$\lambda = 4$

Equating Eqn (v) and Eqn (viii)

$-\lambda + 5 = \mu + 4$

$-4 + 5 = -3 + 4$

$1 = 1$

The two lines intersect

$x = 4$

$y = 2\lambda - 2$

$y = 8 - 2$

$y = 6$

$z = -4 + 5$

$z = -4 + 5 = 1$

The point of intersection of the lines is (4, 6, 1)

Example II

Find the point of intersection of the line

$\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$\boldsymbol{r} = -\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k} + \mu(-2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

Solution

From $\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -2 + \lambda \\ 3 - \lambda \end{pmatrix}$ …………………… (1)

$\boldsymbol{r} = -\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k} + \mu(-2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$(\boldsymbol{r}) = \begin{pmatrix} -1 - 2\mu \\ 3 + \mu \\ 1 + 2\mu \end{pmatrix}$$

$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} -1 - 2\mu \\ 3 + \mu \\ 7 + 2\mu \end{pmatrix}$ … … … … … … … . (2)

Equating the corresponding *x* components:

$1 + 2\lambda = -1 - 2\mu$

$2\lambda + 2\mu = -2$

$\lambda + \mu = -1$ … … … … … … … … … … … … (3)

Equating the corresponding *y* components:

$-2 + \lambda = 3 + \mu$

$\lambda - \mu = 5$ … … … … … … … … … … … … … (4)

Equating the corresponding *z* component;

$3 - \lambda = 7 + 2\mu$

$2\mu + \lambda = -4$ ……………………….. (5)

Eqn (3) –eqn (4)

$2\mu = -6$

$\mu = -3$

From Eqn (4)

$\lambda - (-3) = 5$

$\lambda = 2$

Substituting $\lambda = 2$ and $\mu = -3$ in Eqn (5);

$\Rightarrow$ The two lines intersect at (5, 0, 1)

Example III

Find the point of intersection of the lines

$x - 2 = \frac{y+3}{4} = \frac{z-5}{2}$ & $\frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-2}$

Solution

$x - 2 = \frac{y+3}{4} = \frac{z-5}{2} = \lambda$ … … … … … … (*)

$\frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-2} = \mu$ … … … … … … (**)

From equation (*)

$x - 2 = \lambda$

$x = 2 + \lambda$ … … … … … … … … … … … … … (1)

$y + 3 = 4\lambda$

$y = 4\lambda - 3$ … … … … … … … … … … … … … (2)

$z - 5 = 2\lambda$

$z = 2\lambda + 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

From equation (**)

$x - 1 = -\mu$

$x = 1 - \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

$y - 8 = \mu$

$y = \mu + 8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$

$z - 3 = 2\mu$

$z = 2\mu + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$

Equating the corresponding components

$2 + \lambda = 1 - \mu$

$\mu + \lambda = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$

$\mu + 8 = 4\lambda - 3$

$\mu - 4\lambda = -11 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$

Eqn(8) - (7)

$-5\lambda = -10$

$\lambda = 2$

Substitute λ = 2 in Eqn (8)

$\mu - 4 \times 2 = -11$

$\mu = -3$

∴ The point of intersection is (4, 5, 9)

PLANES

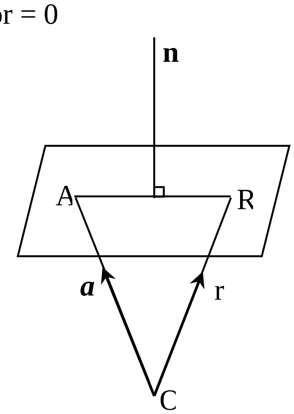
A plane is a surface which contains at least three non-collinear points. If two points are taken then the lines joining the two lines lies completely on the surface of the plane.

A plane is completely known if we know one point that lie on the plane and then the normal to the plane.

Equation of a Plane

Suppose a plane P passes through a point A with a position vector ***a*** and is perpendicular to vector **n**. Let **r** be any point (*x*, *y*, *z*) in the plane.

If two lines are perpendicular, dot product of their direction vector = 0



$AR.n = 0$

$(AO + OR).n = 0$

$(-\boldsymbol{a} + \boldsymbol{r}).n = 0$

$(-n.\boldsymbol{a} + n.\boldsymbol{r}) = 0$

$n.\boldsymbol{a} = n.\boldsymbol{r}$

Equation of a plane is given by $\mathbf{n}.\boldsymbol{r} = \mathbf{n}.\boldsymbol{a}$

Where **n** = normal and **a** = the point that lies on the plane.

Example I

Find the equation of a plane passing through (1, 2, 3), and is perpendicular to vector $\boldsymbol{4i} + 5\boldsymbol{j} + 6\boldsymbol{k}$

Solution

$\mathbf{n}.\mathbf{r} = \mathbf{n}.\boldsymbol{a}$

$$\begin{pmatrix}4\\5\\6\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix}.\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$4x + 5y + 6z = 4 + 10 + 18$

$4x + 5y + 6z = 32$

Example II

Find the equation of a plane which contains A with position vector $\boldsymbol{3i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ and is perpendicular to $\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$.

Solution

$n.\boldsymbol{r} = n.\boldsymbol{a}$

$$\begin{pmatrix}1\\2\\-2\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\-2\end{pmatrix}.\begin{pmatrix}3\\4\\2\end{pmatrix}$$

$x + 2y - 2z = 3 + 8 - 4$

$x + 2y - 2y = 7$

Example III

Find the equation of a plane passing through a point A with a position vector $-2\boldsymbol{i} + 4\boldsymbol{k}$ and is perpendicular to the vector $\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$.

Solution

$\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\boldsymbol{a}$

$$\begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}-2\\0\\4\end{pmatrix}$$

$x + 3y - 2z = -2 + 0 - 8$

$x + 3y - 2z = -10$

$x + 3y - 2z + 10 = 0$

Angle between two planes

The angle between two planes is the angle between their normals

$$\boxed{\cos\theta = \frac{n_1.n_2}{|n_1||n_2|}}$$

Example I

Find the angle between the planes $2x + 3y + 5z = 7$, $3x + 4y - z = 8$

Solution

$$n_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, n_2 = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$$

$$\cos\theta = \frac{n_1.n_2}{|n_1||n_2|}$$

$$\cos\theta = \frac{\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}}{\sqrt{2^2 + 3^2 + 5^2}.\sqrt{3^2 + 4^2 + 1^2}}$$

$$\cos\theta = \frac{6 + 12 - 5}{\sqrt{38}.\sqrt{26}} = \frac{13}{\sqrt{38}.\sqrt{26}}$$

$$\theta = \cos^{-1}\frac{13}{\sqrt{38}.\sqrt{26}}$$

$$\theta = 65.6°$$

Example II

Find the angle between the planes $3x - 3y - z = 0$ and $x + 4y - 2z = 4$

Solution

$$n_1 = \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}, n_2 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$\cos\theta = \frac{n_1 n_2}{|n_1||n_2|}$$

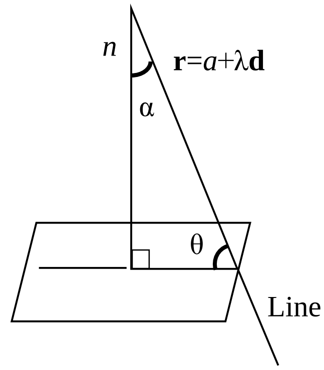
$$\cos\theta = \frac{\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}}{\sqrt{3^2 + (-3)^2 + (-1)^2}.\sqrt{1^2 + 4^2 + (-2)^2}}$$

$$\cos\theta = \frac{3 - 12 + 2}{\sqrt{19}.\sqrt{21}} = \frac{-7}{\sqrt{21}.\sqrt{19}}$$

$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{21}.\sqrt{19}}\right)$$

$$\theta = 69.5°$$

Angle between a line and a plane



$n.d = |\quad||d|\cos\alpha$

$\theta + 90° + \alpha = 180°$

$\theta + \alpha = 90°$

$\alpha = 90° - \theta$

$$n.d = |n||d|\cos(90° - \theta)$$

$$n.d = |n||d|\sin\theta$$

$$\sin\theta = \frac{n.d}{|n||d|}$$

$$\boxed{\boldsymbol{\sin\theta = \frac{n.d}{|n||d|}}}$$

Example

Find the angle between the lines $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ and the plane $2x - y + z = 4$

Solution

$$\sin\theta = \frac{n.d}{|n||d|}$$

$$\sin\theta \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{1^2 + (-1)^2 + 1^2}.\sqrt{2^2 + (-1)^2 + 1^2}}$$

$$\sin\theta = \frac{2 + 1 + 1}{\sqrt{3}.\sqrt{6}}$$

$$\sin\theta = \left(\frac{4}{\sqrt{18}}\right)$$

$$\theta = \sin^{-1}\left(\frac{4}{\sqrt{18}}\right)$$

$$\theta = 70.5°$$

Find the acute angle between the line $\frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-2}$ and $7x - y + 5z = -5$

Solution

$$\sin\theta = \frac{n.d}{|n||d|}$$

$$\sin\theta \frac{\begin{pmatrix}5\\-1\\1\end{pmatrix}\begin{pmatrix}7\\-1\\5\end{pmatrix}}{\sqrt{5^2+(-1)^2+1^2}.\sqrt{7^2+(-1)^2+5^2}}$$

$$\sin\theta = \frac{35+1+5}{\sqrt{27}.\sqrt{75}}$$

$$\sin\theta = \left(\frac{41}{\sqrt{2025}}\right)$$

$$\theta = \sin^{-1}\left(\frac{41}{\sqrt{2025}}\right)$$

$$\theta = 65.7°$$

Solution

Find the angle between the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$ and $x+y+z=12$

Solution

$$\sin\theta = \frac{n.d}{|n||d|}$$

$$\sin\theta \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}2\\5\\-1\end{pmatrix}}{\sqrt{1^2+1^2+1^2}.\sqrt{2^2+5^2+1^2}}$$

$$\sin\theta = \frac{2+5-1}{\sqrt{3}.\sqrt{30}}$$

$$\sin\theta = \left(\frac{6}{\sqrt{90}}\right)$$

$$\theta = \sin^{-1}\left(\frac{6}{\sqrt{90}}\right)$$

$$\theta = 39.2°$$

Point of intersection of a line and a plane

Example I

Find the point of intersection of the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$ and $x+y+z=19$

Solution

$$\frac{x+1}{5} = \frac{y-3}{-1} = \frac{z+1}{1} = \lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

From (*)

$x+1=2\lambda$

$x=2\lambda-1$………………. (1)

$y-3=5\lambda$

$y=3+5\lambda$…………………. (2)

$z+1=-\lambda$

$z=-1-\lambda$……………….... (3)

$x+y+z=12$

$(2\lambda-1)+(5+3\lambda)+(-\lambda-1)=12$

$4\lambda=16$

$\lambda=4$

From equation (1)

$x=2(4)-1=7$

From equation (2)

$y=5(4)+3=23$

From equation (3)

$z=-1-4=-5$

∴ The point of intersection $(7, 23, -5)$

Example II

Find the point of intersection of the line $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$ and the plane $3x+4y+2z=25$

Solution

$$\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4} = \lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

$x=5\lambda$………………………….. (1)

$y+2=2\lambda$

$y=2\lambda-2$……………………... (2)

$z-1=4\lambda$

$z=4\lambda+1$…………………….. (3)

$3x+4y+2z=25$

$3(5\lambda)+4(2\lambda-2)+2(4\lambda+1)=25$

$15\lambda+8\lambda-8+8\lambda+2=25$

$31\lambda=25+6$

$31\lambda=31$

$\lambda=1$

$x=5, \quad y=2-2=0, \quad z=5$

∴ The point of intersection = (5, 0, 5)

Example

Find the point of intersection of the line; $\frac{x+2}{-1} = \frac{y-2}{2} = z-4$ and the plane $2x-y+3z=10$

Solution

$$\frac{x+2}{-1} = \frac{y-2}{2} = z-4 = \lambda$$

$x=-\lambda-2$…………….. (1)

$y=2\lambda+2$……………... (2)

$z=\lambda+4$………………. (3)

$2x-y+3z=10$

$2(-\lambda-2)-(2\lambda+2)+3(\lambda+4)=10$

$-2\lambda-4-2\lambda-2+3\lambda+12=10$

$-4\lambda+3\lambda+6=10$

$-\lambda=4$

$\lambda=-4$

$x=-4-2=-6, \quad y=-8+2=-6,$

$z=-4+4=0$

The point of intersection (-6, -6, 0)

Perpendicular distance of a point from a plane

The perpendicular distance of a point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is given by the formula;

$$D = \left|\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Example

Find the distance of a point $(-2, 0, 6)$ from the plane $2x - y + 3z = 21$

Solution

$$D = \left|\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$x_1, y_1, z_1 = (-2, 0, 6)$

Comparing $ax + by + cz + d = 0$ with $2x - y + 3z - 21 = 0$;

$a = 2, \quad b = -1, \quad c = 3, \quad d = -21$

$$D = \left|\frac{-4 + 0 + 18 - 21}{\sqrt{2^2 + (-1)^2 + 3^2}}\right|$$

$D = \frac{-7}{\sqrt{4+1+9}} = \frac{-7}{\sqrt{14}}$ Units

Line of intersection of two planes

Two planes intersect in a line

Examples I

Find the line of intersection of the planes 2x + 3y + 4z = 1 and x + y + 3z=0

Solution

$2x + 3y + 4z = 1$

$x + y + 3z = 0$

Let $z = \lambda$

$2x + 3y = 1 - 4\lambda$…………………….. (1)

$x + y = -3\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$ (2)

Eqn (2) ×2

$2x + 2y = -6\lambda$ ………………………. (3)

Eqn (1) – Eqn (3);

$y = 1 + 2\lambda$

$\frac{y - 1}{2} = \lambda$

From Eqn (2);

But $y = 1 + 2\lambda$

$x + y = -3\lambda$

$x + 1 + 2\lambda = -3\lambda$

$x + 1 = -3\lambda - 2\lambda$

$x + 1 = -5\lambda$

$\frac{x + 1}{-5} = \lambda$

$$\frac{x + 1}{-5} = \frac{y - 1}{2} = z = \lambda$$

Example II

Find the line of intersection of planes 2x + 3y – z = 4 and x – y + 2z = 5.

Solution

$2x + 3y - z = 4$

$x - y + 2z = 5$

Let $z = \lambda$

$2x + 3y - \lambda = 4$

$x - y + 2\lambda = 5$

$2x + 3y = 4 + \lambda$ ……………… (i)

$x - y = 5 - 2\lambda$……………….. (ii)

Multiply Eqn (ii) by 3;

$3x - 3y = 15 - 6\lambda$ …………… (iii)

Eqn (iii) + Eqn (i);

$5x = 19 - 5\lambda$

$5\lambda = -x + 19$

$\lambda = -x + \frac{19}{5}$

$\lambda = \frac{\left(x - \frac{19}{5}\right)}{-1}$

Multiply Eqn (ii) by 2;

$2x - 2y = 10 - 4\lambda$ …………… (iv)

Eqn (iv) – Eqn (i);

$-5y = 6 - 5\lambda$

$5\lambda = -6 + 5y$

$\lambda = \frac{-6}{5} + y$

$\lambda = \frac{y - \frac{6}{5}}{1}$

$$\frac{x - \frac{19}{5}}{-1} = \frac{y - \frac{6}{5}}{1} = z = \lambda$$

Example

Find the Cartesian equation of a line of intersection of the lines.

$2x - 3y - z = 1$

$3x + 4y + 2z = 3$

Let $x = \lambda$

$-3y - z = 1 - 2\lambda$ …………… (i)

$4y + 2z = 3 - 3\lambda$ …………… (ii)

Eqn (i) × 2

$$-6y - 2z = 2 - 4\lambda \text{ .... (iii)}$$

Eqn (iii) + Eqn (ii)

$$-2y = 5 - 7\lambda$$

$$-2y - 5 = -7\lambda$$

$$\frac{-2y - 5}{-7} = \lambda$$

$$\frac{-2\left(y + \frac{5}{2}\right)}{-7} = \lambda$$

Eqn (i) × 4

$$\Rightarrow -12y - 4z = 4 - 8\lambda \text{ .... (iv)}$$

Eqn (ii) × 3

$$12y + 6z = 9 - 9\lambda \text{ .... (v)}$$

Eqn (iv) + Eqn (v)

$$2z = 13 - 17\lambda$$

$$\frac{2z - 13}{-17} = \lambda$$

$$\frac{2\left(z - \frac{13}{2}\right)}{-17} = \lambda$$

$$x = \frac{\left(y + \frac{1}{2}\right)}{\frac{7}{2}} = -\frac{\left(z - \frac{13}{2}\right)}{\frac{17}{2}} = \lambda$$

$$x = \frac{\left(y + \frac{1}{2}\right)}{\frac{7}{2}} = \frac{\left(z - \frac{13}{2}\right)}{-\frac{17}{2}} = \lambda$$

Equation of a Plane

Given three points on the plane, we can find the equation of a plane;

Example I

Find the Cartesian equation of a plane passing through A (0, 3, -4) B (2, -1, 2) and C (7, 4, -1)

Solution

Let the normal $= \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$AB = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$$

$$AC = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = 0$$

$$2p - 4q + 6r = 0$$

$$p - 2q + 3r = 0 \text{ .... (i)}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$7p + q + 3r = 0 \text{ .... (ii)}$$

From (i)

$$p = 2q - 3r \text{ .... (iii)}$$

$$\Rightarrow 7(2q - 3r) + q + 3r = 0$$

$$14q - 21r + q + 3r = 0$$

$$15q - 18r = 0$$

$$5q - 6r = 0$$

$$5q = 6r$$

$$q = \tfrac{6}{5}r \text{ .... (iv)}$$

$$\Rightarrow p = 2\left(\frac{6r}{5}\right) - 3r$$

$$p = \frac{12}{5}r - 3r$$

$$p = -\frac{3}{5}r$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -3r/5 \\ 6r/5 \\ r \end{pmatrix} = \frac{r}{5}\begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$$

$$\therefore n = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$$

$$n.\boldsymbol{r} = n.\boldsymbol{a}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} . \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$$

$$-3x + 6y + 5z = 0 + 18 - 20$$

$$-3x + 6y + 5z = -2$$

$$3x - 6y - 5z - 2 = 0$$

Example II

Find the equation of a plane passing through points P(4, 2, 3), Q(5, 1, 4) and R(-2, 1, 1).

Solution

Let the normal to the plane be $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$PQ = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$PR = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$p - q - r = 0 \text{ .... (i)}$$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}-6\\-1\\-2\end{pmatrix}=0$$

$-6p-q-2r=0$

$6p+q+2r=0$ ………………… (ii)

From Eqn (i);

$p=q-r$

$6(q-r)+q+2r=0$

$6q-6r+q+2r=0$

$7q-4r=0$

$7q=4r$

$q=\frac{4r}{7}$

$p=\frac{4r}{7}-r$

$\mathrm{p}=\frac{-3\mathrm{r}}{7}$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}\frac{-3r}{7}\\\frac{4r}{7}\\r\end{pmatrix}=\frac{r}{7}\begin{pmatrix}-3\\4\\7\end{pmatrix}$$

$$n=\begin{pmatrix}-3\\4\\7\end{pmatrix}$$

$\mathrm{n.r}=\mathrm{n.a}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}-3\\4\\7\end{pmatrix}=\begin{pmatrix}-3\\4\\7\end{pmatrix}.\begin{pmatrix}4\\2\\3\end{pmatrix}$$

$-3x+4y+7z=-12+8+21$

$-3x+4y+7z=17$

$3x-4y-7z+17=0$

Example III

Find the equation of the planes passing through the following points:

(i) A (0, 2, -4) B (2, 0, 2) C (-8, 4, 0)

Solution

Let the normal $n=\begin{pmatrix}p\\q\\r\end{pmatrix}$

$$AB=\begin{pmatrix}2\\0\\2\end{pmatrix}-\begin{pmatrix}0\\2\\-4\end{pmatrix}=\begin{pmatrix}2\\-2\\6\end{pmatrix}$$

$$AC=\begin{pmatrix}-8\\4\\0\end{pmatrix}-\begin{pmatrix}0\\2\\-4\end{pmatrix}=\begin{pmatrix}-8\\2\\4\end{pmatrix}$$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}2\\-2\\6\end{pmatrix}=0$$

$2p-2q+6r=0$

$p-q+3r=0$ …………………… (i)

$$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}-8\\2\\4\end{pmatrix}=0$$

$-8p+2q+4r=0$

$-4p+q+2r=0$ …………………… (ii)

$p-q+3r=0$

$p=q-3r$

$-8(q-3r)+2q+4r=0$

$-8q+24r+2q+4r=0$

$-6q+28r=0$

$6q=28r$

$q=\frac{14r}{3}$

$p=\frac{14r}{3}-3r=\frac{5r}{3}$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}5r/3\\14r/3\\r\end{pmatrix}=\frac{r}{3}\begin{pmatrix}5\\14\\3\end{pmatrix}$$

$$n=\begin{pmatrix}5\\14\\3\end{pmatrix}$$

$n.\boldsymbol{r}=n.\boldsymbol{a}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}5\\14\\3\end{pmatrix}=\begin{pmatrix}5\\14\\3\end{pmatrix}\begin{pmatrix}0\\2\\-4\end{pmatrix}$$

$5x+14y+3z=0+28-12$

$5x+14y+3z-16=0$

(ii) A (-1, 0, 1), B(3, 3, -2), C(-1, 1, 1)

Let the normal $=\begin{pmatrix}p\\q\\r\end{pmatrix}$

$$\mathrm{AB}=\begin{pmatrix}3\\3\\-2\end{pmatrix}-\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}4\\3\\-3\end{pmatrix}$$

$$AC=\begin{pmatrix}-1\\1\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}\boldsymbol{p}\\\boldsymbol{q}\\\boldsymbol{r}\end{pmatrix}.\begin{pmatrix}4\\3\\-3\end{pmatrix}=0$$

$4p+3q-3r=0$ ………………… (i)

$$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}0\\1\\0\end{pmatrix}=0$$

$q=0$

Substitute $q=0$ in Eqn (i);

$4p=3r$

$p=\frac{3r}{4}$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}3r/4\\0\\r\end{pmatrix}=\frac{r}{4}\begin{pmatrix}3\\0\\4\end{pmatrix}$$

$$n=\begin{pmatrix}3\\0\\4\end{pmatrix}$$

$$n.\boldsymbol{r}=n.\boldsymbol{a}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}3\\0\\4\end{pmatrix}=\begin{pmatrix}3\\0\\4\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$3x+4z=-3+4$$

$$(3x+4z=1)$$

$$3x+4z-1=0$$

Example IV

Find the Cartesian equation of a plane containing the point (1, 3, 1) and it's parallel to vectors (1, -1, -3) and (2, 1, -3)

Solution

$$AB=\begin{pmatrix}1\\-1\\3\end{pmatrix}\text{and } AC=\begin{pmatrix}2\\1\\-3\end{pmatrix}$$

Let the normal $=\begin{pmatrix}p\\q\\r\end{pmatrix}$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}1\\-1\\3\end{pmatrix}=0$$

$p-q+3r=0$ ……………….. (i)

$$\begin{pmatrix}p\\q\\r\end{pmatrix}\begin{pmatrix}2\\1\\-3\end{pmatrix}=0$$

$2p+q-3r=0$ ……………… (ii)

$$p=q-3r$$

$$2(q-3r)+q-3r=0$$

$$2q-6r+q-3r=0$$

$$3q-9r=0$$

$$q=3r$$

$$p=3r-3r$$

$$p=0$$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}0\\3r\\r\end{pmatrix}$$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=r\begin{pmatrix}0\\3\\1\end{pmatrix}$$

$$\mathbf{r}\cdot\mathbf{n}=\mathbf{n}\cdot\mathbf{a}$$

$$n=\begin{pmatrix}0\\3\\1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}0\\3\\1\end{pmatrix}=\begin{pmatrix}0\\3\\1\end{pmatrix}.\begin{pmatrix}1\\3\\1\end{pmatrix}$$

$$3y+z=10$$

Example V

Find the Cartesian equation of the plane passing through the points A(1, 0, -2), B (3, -1, 1) parallel to the line

$\boldsymbol{r}=3\boldsymbol{i}+(2\alpha-1)\boldsymbol{j}+(5-\alpha)\boldsymbol{k}$

Solution:

$$\boldsymbol{r}=3\boldsymbol{i}+2\alpha\boldsymbol{j}-\boldsymbol{j}+5\boldsymbol{k}-\alpha\boldsymbol{k}$$

$$\boldsymbol{r}=3\boldsymbol{i}-\boldsymbol{j}+5\boldsymbol{k}-\alpha(\boldsymbol{0j}+\boldsymbol{2j}-\boldsymbol{k})$$

$$AB=\begin{pmatrix}3\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\0\\-2\end{pmatrix}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$$

$$AB=\begin{pmatrix}2\\-1\\3\end{pmatrix}$$

$$AC=\begin{pmatrix}0\\2\\-1\end{pmatrix}$$

$$\begin{pmatrix}2\\-1\\3\end{pmatrix}.\begin{pmatrix}p\\q\\r\end{pmatrix}=0$$

$2p-q+3r=0$ …………………….. (i)

$$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$$

$2q-r=0$………………………… (ii)

From Eqn (ii);

$\Rightarrow r=2q$

$$2p-q+3(2q)=0$$

$$2p-q+6q=0$$

$$2p+5q=0$$

$$p=\frac{-5}{2}q$$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}\frac{-5q}{2}\\q\\2q\end{pmatrix}=\frac{q}{2}\begin{pmatrix}-5\\2\\4\end{pmatrix}$$

$$n=\begin{pmatrix}-5\\2\\4\end{pmatrix}$$

$$n.\boldsymbol{r}=n.\boldsymbol{a}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}-5\\2\\4\end{pmatrix}=\begin{pmatrix}-5\\2\\4\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$-5x+2y+4z=-5-8$$

$$(-5x+2y+4z=-13)$$

$$5x-2y-4z-13=0$$

Example VI

Find the equation of the plane containing line

$r = \begin{pmatrix}1\\-1\\0\end{pmatrix} + t\begin{pmatrix}-2\\1\\-1\end{pmatrix}$ and is parallel to the line

$r = \begin{pmatrix}2\\1\\1\end{pmatrix} + S\begin{pmatrix}-1\\1\\2\end{pmatrix}$

$$AB = \begin{pmatrix}-2\\1\\-1\end{pmatrix}, \quad AC = \begin{pmatrix}-1\\1\\2\end{pmatrix}, \quad n = \begin{pmatrix}p\\q\\r\end{pmatrix}$$

$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}-2\\1\\-1\end{pmatrix} = 0$;

$-2p + q - r = 0$

$\Rightarrow 2p - q + r = 0$ ……………….. (i)

$\begin{pmatrix}p\\q\\r\end{pmatrix}\begin{pmatrix}-1\\1\\2\end{pmatrix} = 0$

$-p + q + 2r = 0$……………….. (ii)

From Eqn (i);

$r = -2p + q$

$\Rightarrow p - q - 2(q - 2p) = 0$

$p - q - 2q + 4p = 0$

$5p - 3q = 0$

$p = \frac{3q}{5}$

$r = -2\left(\frac{3q}{5}\right) + q$

$r = \frac{-q}{5}$

$n = \begin{pmatrix}p\\q\\r\end{pmatrix} = \begin{pmatrix}\frac{3q}{5}\\q\\\frac{-q}{5}\end{pmatrix} = \frac{q}{5}\begin{pmatrix}3\\5\\-1\end{pmatrix}$

$n.\boldsymbol{r} = n.\boldsymbol{a}$

$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}3\\5\\-1\end{pmatrix} = \begin{pmatrix}3\\5\\-1\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$3x + 5y - z = 3 - 5 + 0$

$3x + 5y - z = -2$

Example VII

Find the Cartesian equation of the plane formed by the lines $\mathbf{r} = -2\mathbf{i} + 5\mathbf{j} - 11\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ and

$\mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \lambda(4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$

Solution

$Let\ n = \begin{pmatrix}p\\q\\r\end{pmatrix} \Rightarrow \begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}3\\1\\3\end{pmatrix} = 0$

$3p + q + 3r = 0$ …………….. ….(i)

$\begin{pmatrix}p\\q\\r\end{pmatrix}.\begin{pmatrix}4\\2\\5\end{pmatrix} = 0$

$4p + 2q + 5r = 0$ ……………… (ii)

From Eqn (i);

$q = -3p - 3r$

$4p + 2(-3p - 3r) + 5r = 0$

$4p - 6p - 6r + 5r = 0$

$-2p - r = 0$

$r = -2p$

$q = -3p - 3(-2p)$

$q = 3p$

$\begin{pmatrix}p\\q\\r\end{pmatrix} = \begin{pmatrix}p\\3p\\-2p\end{pmatrix} = p\begin{pmatrix}1\\3\\-2\end{pmatrix}$

$n = \begin{pmatrix}1\\3\\-2\end{pmatrix}$

$n.\boldsymbol{r} = n.\boldsymbol{a}$

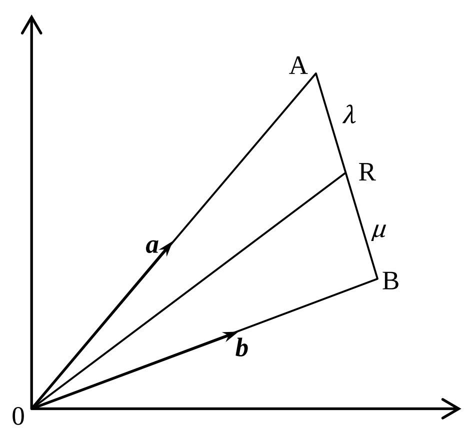
$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}-2\\5\\-11\end{pmatrix}$

$x + 3y - 2z = -2 + 15 + 22$

$x + 3y - 2z = 35$

INTERNAL AND EXTERNAL DIVISIONS

Let A and B be points in space with position vectors A and B.



Let R be a point on a line segment AB dividing AB internally in the ratio of $\lambda : \mu$

$\mathbf{OR = OA + AR}$

$OR = \boldsymbol{a} + \frac{\mu}{\lambda + \mu}AB$

$= \boldsymbol{a} + \frac{\boldsymbol{\lambda}}{\lambda + \mu}(\boldsymbol{b} - \boldsymbol{a})$

$$OR = \frac{a\lambda + a\mu + b\lambda - a\lambda}{\lambda + \mu}$$

$$OR = \frac{a\mu + b\lambda}{\lambda + \mu}$$

Example I

Given that; $OP = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}, OQ = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Find the coordinates of R such that PR :RQ = 1:2

$$\boldsymbol{r} = \frac{\boldsymbol{a}\mu + \boldsymbol{b}\lambda}{\lambda + \mu}$$

$$OR = 2\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$OR = \frac{\begin{pmatrix} 9 \\ -6 \\ 12 \end{pmatrix}}{3}$$

$$OR = \frac{1}{3}\begin{pmatrix} 0 \\ -6 \\ 12 \end{pmatrix}$$

$$R = (3, -2, 4)$$

Example II

The points A $\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$ and B $\begin{pmatrix} 7 \\ 6 \\ 1 \end{pmatrix}$ form a line segment which is divided externally in the ratio of 4:-1. Find the coordinates of T

$$(OT) = \frac{-1\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + 4\begin{pmatrix} 7 \\ 6 \\ 1 \end{pmatrix}}{-1 + 4}$$

$$OT = \frac{\begin{pmatrix} -2 + 28 \\ 1 + 24 \\ -6 + 4 \end{pmatrix}}{3}$$

$$= \left(\frac{1}{3}\right)\begin{pmatrix} 26 \\ 25 \\ -2 \end{pmatrix}$$

$$OT = \left(\frac{26}{3}, \frac{25}{3}, -\frac{2}{3}\right)$$

Example III

Find the position vectors $\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix}$, Find the position vectors of C which divides AB externally in the ratio of 5:-3

Solution:

$$\frac{-3\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + 5\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix}}{5 + \ -3}$$

$$\frac{\begin{pmatrix} -9 \\ 6 \\ -15 \end{pmatrix} + \begin{pmatrix} 45 \\ 5 \\ -5 \end{pmatrix}}{2}$$

$$\frac{\begin{pmatrix} -9 + 45 \\ 6 + 5 \\ -15 - 5 \end{pmatrix}}{2}$$

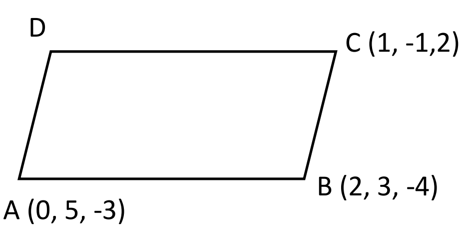
$$\frac{1}{2}\begin{pmatrix} 36 \\ 11 \\ -20 \end{pmatrix}$$

$$OC = \begin{pmatrix} 18 \\ 11/2 \\ 10 \end{pmatrix}$$

$$C = \left(18, \frac{11}{2}, -10\right)$$

Example IV

Given that A(0, 5, -3), B(2, 3, -4) and C(1, -1, 2). Find the coordinates of D if ABCD is a rectangle or parallelogram.



$$AB = DC$$

$$(OB - OA) = (OC - OD)$$

$$OD = OC + OA - OB$$

$$OD = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

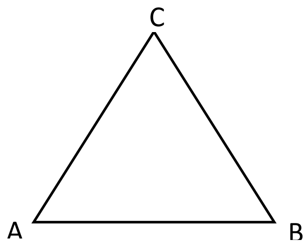
$$OD = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

$$OD = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$D = (-1, 1, 3)$$

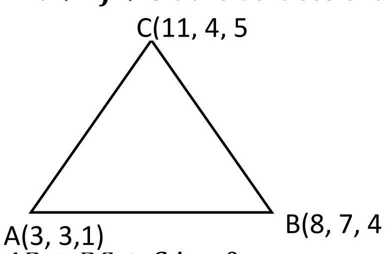
Proving that three points are vertices of a triangle

Give a triangle ABC with vertices
$A = (x_1 y_1 z_1)$ B (x_2, y_2, z_2) C (x_3, y_3, z_3)



$$\boldsymbol{AB + BC + CA = 0}$$
$$\boldsymbol{OB - OA + OC - OB + OA - OC = 0}$$

Example
Show that $3\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}, 8\boldsymbol{i} + \boldsymbol{7j} + 4\boldsymbol{k}$ **and** $\boldsymbol{11i} + \boldsymbol{4j} + \boldsymbol{5k}$ are vertices of a triangle



$$AB + BC + CA = 0$$

$$OB - OA + OC - OB + OA - OC$$

$$= \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 11 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 11 \\ 4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \\ -4 \end{pmatrix}$$

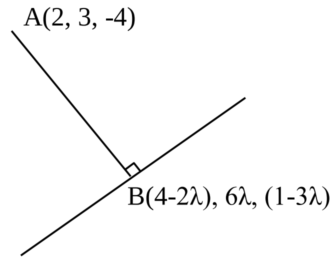
$$\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

Length and the equation of the perpendicular drawn from the point
Example I
Find the equation and length of the perpendicular drawn from a point (2, 3, -4) to the line

$$\frac{4 - x}{2} = \frac{y}{6} = \frac{1 - z}{3}$$

Solution

$$\frac{4 - x}{2} = \frac{y}{6} = \frac{1 - z}{3}$$
$$\Rightarrow \frac{x - 4}{-2} = \frac{y}{6} = \frac{z - 1}{-3}$$



$$r = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2\lambda \\ 6\lambda \\ -1 - 3\lambda \end{pmatrix}$$

$$AB.\begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$AB = OB - OA$$

$$= \begin{pmatrix} 4 - 2\lambda - 2 \\ 6\lambda - 3 \\ 1 - 3\lambda - 4 \end{pmatrix} = \begin{pmatrix} 2 - 2\lambda \\ 6\lambda - 3 \\ 5 - 3\lambda \end{pmatrix}$$

$$\begin{pmatrix} 2 - 2\lambda \\ 6\lambda - 3 \\ 5 - 3\lambda \end{pmatrix} \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$(-2(2 - 2\lambda) + 6(6\lambda - 3) - 3(5 - 3\lambda) = 0$$

$$-4 + 4\lambda + 36\lambda - 18 - 15 + 9\lambda = 0$$

$$36\lambda + 9\lambda + 4\lambda - 18 - 15 - 4 = 0$$

$$49\lambda = 37$$

$$\lambda = \frac{37}{49}$$

$$AB = \begin{pmatrix} 2 - 2\left(\frac{37}{49}\right) \\ 6\left(\frac{37}{49}\right) - 3 \\ 5 - 3\left(\frac{37}{49}\right) \end{pmatrix}$$

$$AB = \begin{pmatrix} 24/49 \\ 75/49 \\ 134/49 \end{pmatrix}$$

$$r = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} \frac{24}{49} \\ \frac{75}{49} \\ \frac{134}{49} \end{pmatrix}$$

Equation of the perpendicular

$$\frac{x-2}{72/49} = \frac{y-3}{-69/49} = \frac{z-4}{186/49}$$

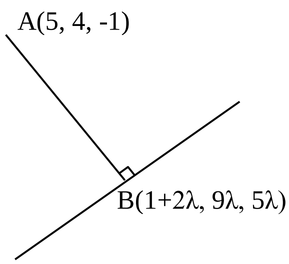
Length of the perpendicular AB

$$AB = \sqrt{\left(\frac{24}{49}\right)^2 + \left(\frac{75}{49}\right)^2 + \left(\frac{134}{49}\right)^2}$$

AB = 3.1719 units

Find the length and equation of the perpendicular drawn from a point (5, 4, -1) to the line; $\boldsymbol{r} = \boldsymbol{i} + \lambda(2\boldsymbol{i} + 9\boldsymbol{j} + 5\boldsymbol{k})$

Solution



$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\lambda \\ 9\lambda \\ 5\lambda \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ 9\lambda \\ 5\lambda \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 + 2\lambda - 5 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix} = \begin{pmatrix} 2\lambda - 4 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix}$$

$$AB.d = 0$$

$$d = \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 2\lambda - 4 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix} = 0$$

$$(2(2\lambda - 4) + 9(9\lambda - 4) + 5(5\lambda + 1) = 0$$

$$4\lambda - 8 + 81\lambda - 36 + 25\lambda + 5 = 0$$

$$81\lambda + 25\lambda + 4\lambda - 8 + 5 - 36 = 0$$

$$110\lambda = 39$$

$$\lambda = \frac{39}{110}$$

$$AB = \begin{pmatrix} 2\left(\frac{39}{110}\right) - 4 \\ 9\left(\frac{39}{110}\right) - 4 \\ 5\left(\frac{39}{110}\right) + 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-362}{110} \\ \frac{-89}{110} \\ \frac{-305}{110} \end{pmatrix}$$

$$|AB| = \sqrt{\left(\frac{-362}{110}\right)^2 \left(\frac{-89}{110}\right)^2 \left(\frac{-305}{110}\right)^2}$$

$|AB| = 4.379$ units

Equation of the perpendicular bisector is

$$r = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} \frac{-362}{110} \\ \frac{-89}{110} \\ \frac{305}{110} \end{pmatrix}$$

$$\frac{x-5}{-362/110} = \frac{y-4}{-89/110} = \frac{z+1}{305/110} = \mu$$

Shortest Distance between Parallel Planes

Example I

Find the perpendicular distance between two parallel planes;

$$2x + 5y - 14z = 30$$
$$2x + 5y - 14z = -15$$

Solution

$$r.\hat{n} = d_1$$

Plane 1

$$r.\left(\frac{2\boldsymbol{i} + 5\boldsymbol{j} - 14\boldsymbol{k}}{15}\right) = \frac{30}{15}$$

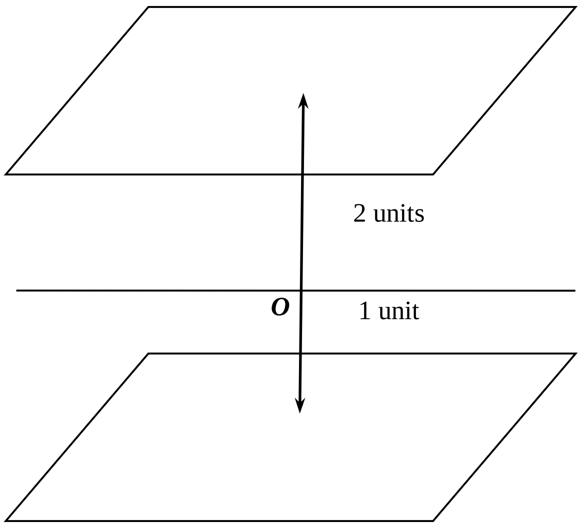
$$r.\left(\frac{2\boldsymbol{i} + 5\boldsymbol{j} - 14\boldsymbol{k}}{15}\right) = 2$$

$r.\left(\frac{2\boldsymbol{i}+5\boldsymbol{j}-14\boldsymbol{k}}{15}\right)=2$

Plane 2

$r.2\boldsymbol{i}+5\boldsymbol{j}-14\boldsymbol{k}=-15$

$r.\left(\frac{2\boldsymbol{i}+5\boldsymbol{j}-14\boldsymbol{k}}{15}\right)=-1$



Example II
Find the perpendicular distance between two parallel planes;
$x+2y-z=-4$ *and* $x+2y-z=3$
$r.\hat{n}=d_1$
For plane 1

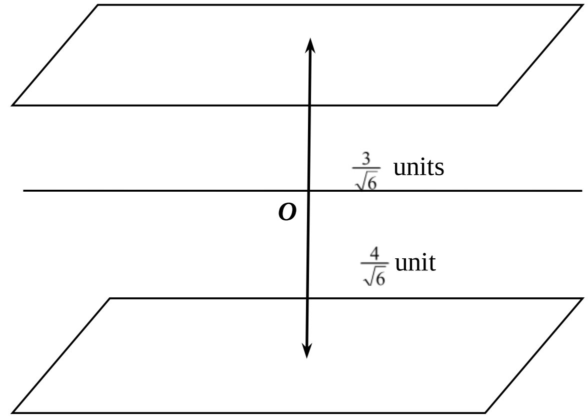
$r.(\mathbf{i}+2\mathbf{j}-\mathbf{k})=-4$

$r.\frac{(\mathbf{i}+2\mathbf{j}-\mathbf{k})}{\sqrt{6}}=\frac{-4}{\sqrt{6}}$

For plane 2

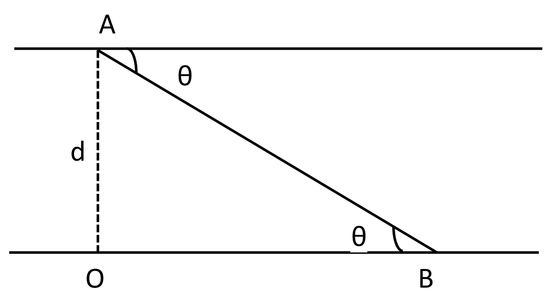
$r.(\mathbf{i}+2\mathbf{j}-\mathbf{k})=3$

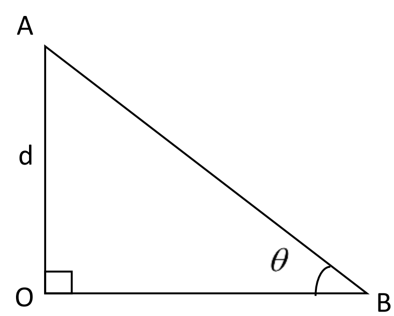
0 $r.\frac{(\mathbf{i}+2\mathbf{j}-\mathbf{k})}{\sqrt{6}}=\frac{3}{\sqrt{6}}$



$$=\frac{3}{\sqrt{6}}+\frac{4}{\sqrt{6}}=\frac{7}{\sqrt{6}}units$$

Shortest distance between two parallel lines





Distance between a point A and line B
$d=AB\sin\theta$

Example I
Find the shortest distance between the following pairs of parallel lines

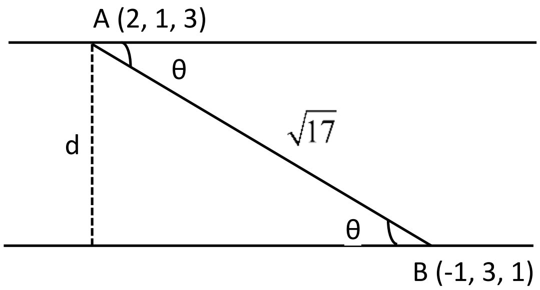
$$\frac{x-2}{1}=\frac{y-1}{-1}=\frac{z-3}{2}$$
and

$$\frac{x+1}{1}=\frac{y-3}{-1}=\frac{z-1}{2}$$

$$AB=\sqrt{(2+1)^2+(1-3)^2+(3-1)^2}$$
$$AB=\sqrt{17}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

$$\begin{pmatrix}-1\\3\\1\end{pmatrix}-\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}-3\\2\\-2\end{pmatrix}$$



$$\cos\theta=\frac{AB.d}{|AB|.d}$$

$$\cos\theta=\frac{\begin{pmatrix}-3\\2\\-2\end{pmatrix}.\begin{pmatrix}1\\-1\\2\end{pmatrix}}{\sqrt{17}\sqrt{6}}$$

$$\theta=\cos^{-1}\left(\frac{-9}{\sqrt{100}}\right)$$

$$\theta=26.8°$$

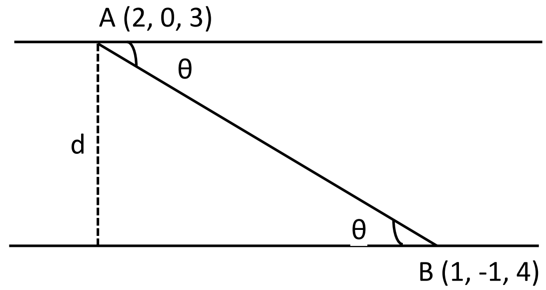
$$\sin 26.8°=\frac{d}{\sqrt{17}}$$
$$d=1.859\,units$$

Example II

Find the distance between the following pairs of parallel lines

$$r=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$$
$$r=\begin{pmatrix}1\\-1\\4\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

Solution



$$AB=\sqrt{(2-1)^2+(0+1)^2+(3-4)^2}$$
$$AB=\sqrt{1+1+1}$$
$$AB=\sqrt{3}$$
$$\cos\theta=\frac{2}{\sqrt{18}}$$
$$\theta=\cos^{-1}\left(\frac{2}{\sqrt{18}}\right)$$
$$\theta=61.9°$$
$$\sin\theta=\frac{d}{\sqrt{3}}$$
$$\sin 61.9°=\frac{d}{\sqrt{3}}$$
$$d=\sqrt{3}\sin 61.9°$$
$$d=1.52789\,units$$

SKEW LINES

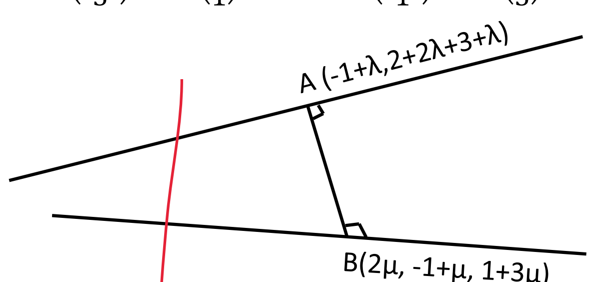
These are lines which are neither parallel nor perpendicular

Shortest distance between two skew lines

Example I

Find the shortest distance between the following skew lines

$$\boldsymbol{r}=\begin{pmatrix}-1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}\text{ and }\boldsymbol{r}=\begin{pmatrix}0\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$$



$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

$$\begin{pmatrix} 2\mu-(1+\lambda) \\ -1+\mu-(2+2\lambda) \\ 1+3\mu-(3+\lambda) \end{pmatrix} - \begin{pmatrix} 2\mu-\lambda+1 \\ \mu-2\lambda-3 \\ 3\mu-\lambda-2 \end{pmatrix}$$

$$\begin{pmatrix} 2\mu-\lambda+1 \\ \mu-2\lambda-3 \\ 3\mu-\lambda-2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$2\mu-\lambda+1+2\mu-4\lambda-6+3\mu-\lambda-2=0$$

$$7\mu-6\lambda=7 \text{.......................(1)}$$

$$\begin{pmatrix} 2\mu-\lambda+1 \\ \mu-2\lambda-3 \\ 3\mu-\lambda-2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$4\mu-2\lambda+2+\mu-2\lambda-3+9\mu-3\lambda-6=0$$

$$14\mu-7\lambda-7=0$$

$$14\mu-7\lambda=7 \text{........................(2)}$$

$$\mu=\frac{-1}{5}, \lambda=-\frac{7}{5}$$

$$AB = \begin{pmatrix} 2 \\ -0.4 \\ -1.2 \end{pmatrix}$$

$$AB = \sqrt{2^2+(-0.4)^2+(-1.2)^2} = 2.3664 units$$

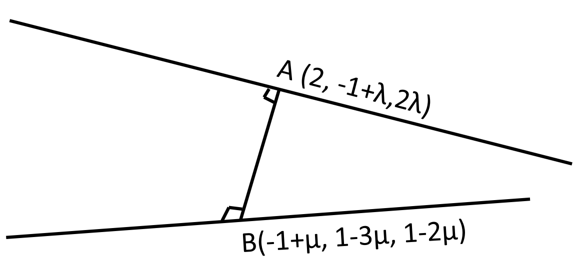
Example II

Find the shortest distance between the following pairs of skew lines

$$\frac{x-2}{0}=\frac{y+1}{1}=\frac{z}{2} \text{ and } \frac{x+1}{1}=\frac{y-1}{-3}=\frac{z-1}{-2}$$

Solution

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\begin{pmatrix} (-1+\mu)-2 \\ (1-3\mu)-(-1+\lambda) \\ (1-2\mu)-2\lambda \end{pmatrix} = \begin{pmatrix} \mu-3 \\ -3\mu-\lambda+2 \\ 1-2\mu-2\lambda \end{pmatrix}$$

$$\begin{pmatrix} \mu-3 \\ -3\mu-\lambda+2 \\ 1-2\mu-2\lambda \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$-3\mu-\lambda+2+2-4\mu-4\lambda=0$$

$$-7\mu-5\lambda-4=0$$

$$7\mu+5\lambda-4=0$$

$$7\mu+5\lambda=4 \text{..........................(1)}$$

$$\begin{pmatrix} \mu-3 \\ -3\mu-\lambda+2 \\ 1-2\mu-2\lambda \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = 0$$

$$\mu-3+9\mu+3\lambda-6-2+4\mu+4\lambda$$

$$14\mu+7\lambda-11=0$$

$$14\mu+7\lambda=11 \text{......................(2)}$$

$$AB = \begin{pmatrix} -12/7 \\ -6/7 \\ 3/7 \end{pmatrix}$$

$$\mu=\frac{9}{7}, \lambda=-1$$

$$AB = \sqrt{\left(\frac{-12}{7}^2\right)+\left(\frac{-6}{7}\right)^2+\left(\frac{3}{7}\right)^2}$$

$$AB = \sqrt{\frac{144}{49}+\frac{36}{49}+\frac{9}{49}}$$

$$AB = \frac{3\sqrt{21}}{7} units$$

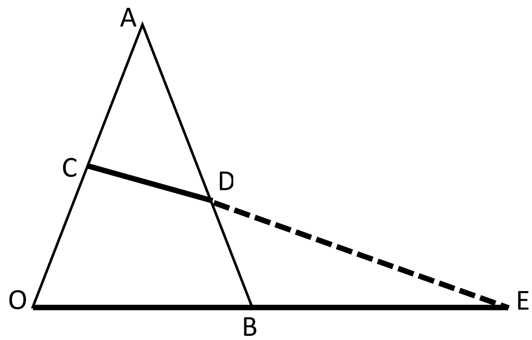
Vector Geometry

Example I

Triangle OAB has OA=**a,** OB=**b**. C is a point on OA such that OC= $\frac{2}{3}$ **a**. D is a mid point of AB when CD is produced, it meets OB at E such that DE = nCD and BE=**k**b. Express BE, DE in terms of;

a) **n, a** and **b**

b) **k, b** and **a**. Hence find the values of **n** and **k.**



$$\overrightarrow{DE} = n\overrightarrow{CD}$$

$$\overrightarrow{DE} = n\left[\overrightarrow{CA} + \overrightarrow{AD}\right]$$

$$\overrightarrow{DE} = n\left[\frac{1}{3}\mathbf{a} + \overrightarrow{AD}\right]$$

$$\overrightarrow{DE} = n\left[\frac{1}{3}\mathbf{a} + \frac{1}{2}\overrightarrow{AB}\right]$$

$$\overrightarrow{DE} = \frac{1}{3}n\mathbf{a} + \frac{1}{2}n\mathbf{b} - \frac{1}{2}n\mathbf{a}$$

$$\overrightarrow{DE} = \frac{-1}{6}n\mathbf{a} + \frac{1}{2}n\mathbf{b}..................(1)$$

$$\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE}$$

$$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB} + k\mathbf{b}$$

$$\overrightarrow{DE} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) + k\mathbf{b}$$

$$\overrightarrow{DE} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} + k\mathbf{b}$$

$$\overrightarrow{DE} = \left(\frac{1}{2} + k\right)\mathbf{b} - \frac{1}{2}\mathbf{a}$$

$$\frac{-1}{2}\mathbf{a} = -\frac{1}{6}n\mathbf{a}$$

$$\frac{1}{2} = \frac{1}{6}n$$

$$6 = 2n$$

$$n = 3$$

$$\left(\frac{1}{2} + k\right)\mathbf{b} = \frac{1}{2}n\mathbf{b}$$

$$\frac{1}{2} + k = \frac{1}{2} \times 3$$

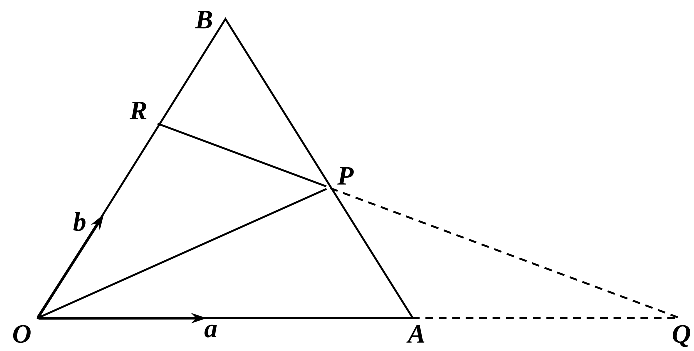
$$k = \frac{3}{2} - \frac{1}{2} = 1$$

Example II

Given that OA is **a** and OB=**b** point R is on OB such that OR:RB=4:1. Point P is on AB such that BP:PA=2:3. When RP and OA are both produced, they meet at Q. Find OR and OP in terms of **a** and **b**

ii) OQ in terms of **a**

Solution



$$\overrightarrow{OR} = \frac{4}{5}OB \qquad \Rightarrow \overrightarrow{OR} = \frac{4}{5}\mathbf{b}$$

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$$

$$\overrightarrow{OP} = \mathbf{b} + \frac{2}{5}\overrightarrow{BA}$$

$$\overrightarrow{OP} = \mathbf{b} + \frac{2}{5}(a - b)$$

$$\overrightarrow{OP} = \frac{1}{5}(3\mathbf{b} + 2\mathbf{a})$$

$$\overrightarrow{OQ} = \lambda\overrightarrow{OA} = \lambda\mathbf{a}$$

$$\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ}$$

$$\overrightarrow{OQ} = \frac{4}{5}\mathbf{b} + \mu\overrightarrow{RP}$$

$$\overrightarrow{OQ} = \frac{4}{5}\mathbf{b} + \mu\left(\frac{-4}{5}\mathbf{b} + \frac{1}{5}\left(2\mathbf{a} + 3\mathbf{b}\right)\right)$$

$$\overrightarrow{OQ} = \left(\frac{4}{5} - \frac{1}{5}\mu\right)\mathbf{b} + \frac{2}{5}\mu\mathbf{a}$$

$$\frac{4}{5}-\frac{1}{5}\mu=0$$

$$\mu=4$$

$$\lambda=\frac{2}{5}\mu$$

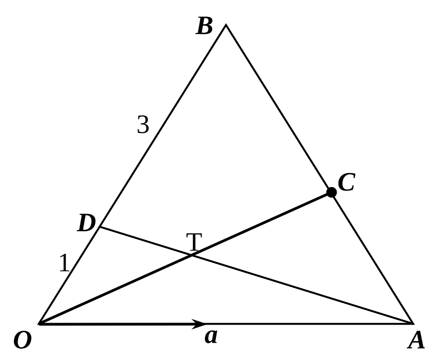
$$\lambda=\frac{8}{5}$$

OQ = $\frac{8}{5}\mathbf{a}$

Example III

O, A and B are non collinear points OA = a, OB = b, C is midpoint of AB, D is a point on OB such that $\mathbf{OD} = \frac{1}{4}\mathbf{OB}$. T is a point of intersection of OC and AD. Find the vector OT in terms of *a* and *b*.

Solution



OT = λOC

OC = OB + BC

$= \mathbf{b} + \frac{1}{2}\text{BA}$

$= \mathbf{b} + \frac{1}{2}(\mathbf{a}-\mathbf{b})$

$\mathbf{OC} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$

$$\mathbf{OT}=\lambda\left(\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$$

$$\mathbf{OT}=\frac{1}{2}\lambda\mathbf{a}+\frac{1}{2}\lambda\mathbf{b}\text{..........................(i)}$$

OT = OA + AT

$= \mathbf{a} + \mu\,\mathbf{AD}$

AD = AO + OD

$= \mathbf{a} + \frac{1}{4}\mathbf{b}$

$\mathbf{OT} = \mathbf{a} + \mu(\mathbf{a} + \frac{1}{4}\mathbf{b})$

$\mathbf{OT} = \mathbf{a} - \mu\,\mathbf{a} + \frac{1}{4}\mu\,\mathbf{b}$

$\mathbf{OT} = (1-\mu)\mathbf{a} + \frac{1}{4}\mu\,\mathbf{b}$ ……..….. (ii)

Equating components of vectors **a** and **b** in Eqns (i) and (ii);

$$\frac{1}{2}\lambda=1-\mu\text{.......................(iii)}$$

$$\frac{1}{2}\lambda=\frac{1}{4}\mu\text{.........................(iv)}$$

From Eqn (iv);

$$2\lambda=\mu$$

$$\Rightarrow\frac{\lambda}{2}=1-2\mu$$

$$\frac{5\lambda}{2}=1$$

$$\lambda=\frac{2}{5}$$

$$\mu=\frac{4}{5}$$

$$\mathbf{OT}=\frac{2}{5}\left(\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$$

$$\mathbf{OT}=\frac{1}{5}(\mathbf{a}+\mathbf{b})$$

Revision Exercise

1. In a triangle *ABC*, the altitudes from B and C meet the opposite sides at E and F respectively. BE and CF intersect at O. Taking O as the origin, use the dot product to prove that *AO* is perpendicular to BC

   (b) Find the point of intersection of the line $\frac{x}{5}=\frac{y+2}{2}=\frac{z-1}{4}$ with the plane
   $3x + 4y + 2z - 25 = 0$

   (c) Find the angle between the line $\frac{x+4}{8}=\frac{y-2}{2}=\frac{z+1}{-4}$ and the plane $4x + 3y + 1 = 0$

2. (a) Show that the equation of the plane through points A with position vector 2i + 2k perpendicular to the vector i + 3j – 2k is $x + 3y - 2z + 10 = 0$

   (b) (i) Show that the vector 2i – 5j + 3.5k is perpendicular to the line **r** = 2i – j + λ(4i + 3j + 2k)

(ii) Calculate the angle between the vector 3i – 2j + k and the line in (b)(i) above.

3. A point P has coordinates (1, -2, 3) and a certain plane has the equation $x + 2y + 2z = 8$. The line through P parallel to the line $\frac{x}{3} = \frac{y+1}{-1} = \frac{z+1}{-2}$ meets the plane at a point Q.

4. (a) The line through A(1, -2, 2) and perpendicular to the plane $4x - y + 2z + 12 = 0$ meets the plane in point B. Find the coordinates of B.
(b) Given that the vectors a**i** – 2**j** + **k** and 2a**i** + a**j** – 4**i** are perpendicular, find the values of a.

5. Find the equation of the plane through the point (1, 2, 3) and perpendicular to the vector **r** = 4i + 5j + k.

6. (a) The vertices of a triangle are P(2, -1, 5), Q(7, 1, -3) and R(13, -2, 0). Show that $\angle PQR = 90°$. Find the coordinates of S if PQRS is a rectangle.
(b) Find the equation of the line through A(2, 2, 5) and B(1, 2, 3)
(c) If the line in (b) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ at P, find the:
(i) coordinates of P,
(ii) angle between the two lines

7. The position vector of points P and Q are 2i – 3j and 3i – 7j + 12k respectively. Determine the length of PQ. PQ meets the plane $4x + 5y - 2z = 5$ at point S. Find:
(a) the coordinates of *S*,
(b) the angle between PQ and the plane.

8. (a) Find the angle between the line **r** = 3k + λ(7i – j + 4k) and the plane **r** · (2i – 5j – 2k) = 8
(b) Show that the lines with vector equations
$\mathbf{r}_1 = (1 + 4\lambda)i + (1 - \lambda)j + (2\lambda)k$, and
$\mathbf{r}_2 = (5 + 3\mu)i + (2\mu)j + (2 - 5\mu)k$.
intersect at right angles and give the position vector of the point of intersection.

9. Find the equation of the line with directrix vector **d** which passes through the point with position vector **a** given that
(a) **a** = $i + 2j - k$, **d** = $3i - k$
(b) **a** = $4i - 3k$, **d** = $i - 3j + 3k$

10. Find the vector equation of the line which passes through the points with (a) position vectors $3i - 3j + k$ and $-2j + j + k$.
(a) position vector $i + 4j$ and $3i - j + 2k$,
(b) coordinates (0, 6, -6) and (5, -7, 2)
(c) coordinates (0, 0, 0) and (5, -2, 3)

11. Write down in parametric form the vector equations of the planes through the given points parallel to the given pairs of vectors.
(a) (1, -2, 0); i + 3j and –j + 2k
(b) the origin; 2i – j and –i + 2j – 7k
(c) (3, 1, -1); j and i + j + k.

12. Find a vector equation for the plane passing through the points with position vectors $2k$, $i - 3j + k$ and $5i + 2j$.

13. Find the vector equation of the plane through the points A(1, 0, -2) and B(3, -1, 1) which is parallel to the line with vector equation **r** = $3i + (2\lambda - 1)j + (5 - \lambda)k$. Hence find the coordinates of the point of intersection of the plane and the line **r** = $\mu i + (5 - \mu)j + 2\mu - 7)k$.

14. Find a vector equation for the line joining the points
(a) (2, 6) and (5, 2)
(b) (-1, 2, -3) and (6, 3, 0).

15. (a) Points A and B have coordinates (4, 1) and (2, -5) respectively. Find a vector equation for the line which passes through A and perpendicular to the line AB.
(b) Points P and Q have coordinates (3, 5) and (-3, -7) respectively. Find a vector equation for the line which passes through the point P and which is perpendicular to the line PQ

16. Find a vector equation for the perpendicular bisector of the points:
(a) (6, 3) and (2, -5)
(b) (7, -1) and (3, -3)

17. Points P, Q and R have position vectors 4i – 4j, 2i + 2j, and 8i + 6j respectively.
(a) Find a vector equation for the line L_1 which is the perpendicular bisector to the points P and Q
(b) Find a vector equation for the line L_2 which is the perpendicular bisector to the points A and R.
(c) Hence find the position vector of the point where L_1 and L_2 meet.

18. Two lines L_1 and L_2 have equations

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \text{ and } L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Show that L_1 and L_2 are concurrent (meet at a common point) and find the position vector of their point of intersection.

(b) Find the angle between L_1 and L_2.

19. Points P, Q, and R have coordinates (-1, 1), (4, 6) and (7, 3) respectively.

(a) Show that the perpendicular distance from the point R to the point PQ is $3\sqrt{2}$.

(b) Deduce that the area of the triangle PQR is 15 sq.units.

20. Points A, B and C have position vectors –i + 3j + 9k, 5i + 6j – 4k and 4i + 7j + 5k respectively. P is the point on AB such that $\overrightarrow{AP} = \lambda \overrightarrow{AB}$. Find:

(a) $\overrightarrow{AB}$

(b) $\overrightarrow{CP}$

(c) Find the perpendicular distance from the point C to the line AB.

21. Two lines L_1 and L_2 have vector equations

$\mathbf{r}_1 = (2 - 3\lambda)\mathrm{i} + (1 + \lambda)\mathrm{j} + 4\lambda\mathrm{k}$

$\mathbf{r}_2 = (-1 + 3\lambda)\mathrm{i} + 3\mathrm{j} + (4 - \lambda)\mathrm{k}$ respectively. Find:

(a) the position vector of their common point of intersection.

(b) the angle between the lines.

22. Find the equation of the plane containing points P(1, 1, 1), Q(1, 2, 0) and (-1, 2, 1).

23. Find the equation of the plane containing point (4, -2, 3) and parallel to the plane $3x - 7z = 12$

24. Show that the point with position vector 7i – 5j – 4k lies in the plane r = 4i + 3j + 2k + λ(i – j – k) + μ(2i + 3j + k). Find the point at which the line $x = y - 1 = 2z$ intersects the plane $4x - y + 3z = 8$.

25. Find the parametric equations for the line through the point (0, 1, 2) that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

26. Find the distance between the parallel planes $z = x + 2y + 1$ and $3x + 6y - 3z = 4$

27. Two planes are given by the parametric equations

$x = r + 3$ and $x = 1 + r + s$

$y = 3s$ and $y = 2 + r$

$z = 2r$ and $z = -3 + 5$

Find the Cartesian equation of the intersection point.

28. The equation of a plane P is given by $r \cdot \begin{pmatrix} 2 \\ 6 \\ 9 \end{pmatrix} = 33$, where r is the position vector of P. find the perpendicular distance from the plane to the origin.

29. The line through point P(1, -2, 3) and parallel to the line $\frac{x}{3} + \frac{y+1}{-1} = z + 1$ meets the plane $x + 2y + 27$ 8 at Q. find the coordinates of Q.

30. (a) Find the angle between the plane $x + 4y - z = 72$ and the line $\mathbf{r} = 9\mathrm{i} + 6\mathrm{j} + 8\mathrm{k}$.

(b) obtain the equation of the plane that passes through (1, -2, 2) and perpendicular to the line $\frac{x-9}{4} = \frac{y-6}{-1} = \frac{z-8}{1}$

(c) Find the parametric equations of the line of intersection of the plane $x + y + z = 4$ and $x - y + 2z + 2 = 0$

31. Find the point of intersection of the three planes $2x - y + 3z = 4$, $3x - 2y + 6z = 3$ and $7x - 4y + 5z = 11$.

32. Find the Cartesian equation of the plane with parametric vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

33. Find the Cartesian equation of the plane containing the point with position vector $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and parallel to the vectors $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

34. Find the Cartesian equation of the plane containing the points with position vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$.

35. Find the perpendicular distance from the plane $\mathbf{r}.(2\mathrm{i} - 14\mathrm{j} + 5\mathrm{k}) = 10$ to the origin.

36. Find the position vector of the point where the line $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ meets the plane

$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = 15$.

37. Two lines have vector equations $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

and $\mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Find the position vector of the point of intersection of the two lines and the Cartesian equation of the plane containing the two lines.

38. The position vector of points P and Q are 3i - j + 2k and 2i + 2j + 3k, respectively. Find the acute angle between PQ and the line $1 - x = \frac{y-3}{2} = \frac{4-z}{4}$.

(b) Find the point of intersection of the line $x - 2 = 2y + 1 = 3 - z$ and the plane $x + 2y + z = 3$.

(c) Find the equation of the plane through the origin parallel to the lines $\mathbf{r}$ = 3i + 3j – k + s(i – j – 2k) and $\mathbf{r}$ = 4i – 5j – 8k + t(3i + 7j – 6k)

39. (a) The points A and B have position vectors $\mathbf{a}$ = 2i – j + 6k and $\mathbf{b}$ = 7i – 6j + k respectively. Find the coordinates of a point P which divides the vector AB in the ratio:

(i) 4:1

(ii) 1:4

40. (b) Find the Cartesian equation of the plane through the origin parallel to the lines

$x - 3 = 3 - y = \frac{z+1}{-2}$ and

$\frac{x-4}{3} = \frac{y+5}{7} = \frac{x+8}{-6}$

(c) Find the angle between the line $1 - x = \frac{y-3}{2} = \frac{4-z}{4}$ and the plane $2x - 3y - 2z + 5 = 0$.

41.(a) Determine the unit vector perpendicular to the plane containing the points A(0, 2, -4), B(2, 0, 2) and C(-8, 4, 0).

(b) Find the equation of the plane in (a) above

(c) Show that the point T(5, -4, 3) lies on the plane in (a) above.

(d) Write down the equation in the form $\mathbf{r} = a + \lambda b$ of the perpendicular through the point P(3, 4, 2) to the plane in (a) above.

(e) If the perpendicular meets the plane in (a) above at N, determine vector NP.