SENIOR FIVE TERM 1

TOPIC 4/6: Partial Fractions

Competency: The learner decomposes rational expressions into partial fractions useful in integral calculus and real-world context.

Contents

Partial fractions

There are three established types of partial fractions depending on the nature of the denominator.

Denominators with linear factors, e.g. 3x – 1, x + 2 and 3x – 4

Each linear factor (ax + b) in the denominator has a corresponding partial fraction of the form $\frac{c}{(ax+b)}$, where a, b and c are constants.

Example 1

(a) Express each of the following in partial fraction. Hence find the integral of each with respect to x.

(i) $\frac{x-1}{(x+1)(x-2)}$

Solution

Let $\frac{x-1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

Multiplying by (x+1)(x-2)

⇨ x – 1 = A(x – 2) + B(x+1)

then we find the values of A and B

Putting x = 2:

1 = 3B, => B = $\frac{1}{3}$

Putting x = -1:

-2 = -3A,

=> A = $\frac{2}{3}$

$\therefore \frac{x-1}{(x+1)(x-2)} = \frac{\frac{2}{3}}{(x+1)} + \frac{\frac{2}{3}}{(x-2)}$

$$= \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

(ii) $\frac{1}{x^3-9x}$

Solution

$\frac{1}{x^3-9x} = \frac{1}{x(x^2-9)} = \frac{1}{x(x-3)(x+3)}$

⇨ $\frac{1}{x^3-9x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying through with $x(x-3)(x+3)$

1= $A(x^2-9) + B(x^2+3x) + C(x^2-3x)$

Putting x = 0;

1 = -9A

=> $A = -\frac{1}{9}$

Putting x = 3; 1 = 18B => $B = \frac{1}{18}$

Putting x = -3; 1 = 18C => $C = \frac{1}{18}$

⇨ $\frac{1}{x^3-9x} = -\frac{1}{9x} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}$

(iii) $\frac{2x+1}{(x-1)(3x^2+7x+2)}$

Solution

$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{2x+1}{(x-1)(x+2)(3x+1)}$

$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{1}{(3x+1)}$

Multiplying by $(x-1)(x+2)(3x+1)$

2x + 1= A(x+2)(3x+1)+B(x-1)(3x+1)+C(x-1)(x+2)

Putting x = 1; 3 = 12A => $A = \frac{1}{4}$

Putting x = -2; -3 = 15B => $B = -\frac{1}{5}$

Putting x = $-\frac{1}{3}$; $\frac{1}{3} = -\frac{20}{9}$C => $C = -\frac{3}{20}$

$\therefore \frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{1}{4(x-1)} - \frac{1}{5(x+2)} - \frac{3}{20(3x+1)}$

Hence,

$\int \frac{2x+1}{(x-1)(3x^2+7x+2)} dx$

$= \frac{1}{4}\int \frac{1}{(x-1)} dx - \frac{1}{5}\int \frac{1}{(x+2)} dx - \frac{3}{20}\int \frac{1}{(3x+1)} dx$

$= \frac{1}{4}In(x-1) - \frac{1}{5}In(x+2) - \frac{3}{20}In(3x+1)$

$= \frac{1}{20}In\frac{(x-1)^5}{(x+2)^4(3x+1)^3}$

(iv) $\frac{2x^2-x+1}{(x^2-1)(x+2)}$

Solution

$\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{2x^2-x+1}{(x+1)(x-1)(x+2)}$

⇨ $\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Multiplying through by (x+1)(x-1)(x+2)

$2x^2 - x+1$= A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)

Putting x = -1; 4 = -2A => $A = -2$

Putting x = 1; 2 = 6B => $B = \frac{1}{3}$

Putting x = -2; 11 = 3C => $C = \frac{11}{3}$

$\therefore \frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{1}{3(x-1)} - \frac{2}{(x+1)} + \frac{11}{3(x+2)}$

Exercise 1

Express the following in partial fraction

(i) $\frac{x^2+1}{x^3+4x^2+3x}$ $\left[\frac{1}{3x} - \frac{1}{(x+1)} + \frac{5}{3(x+3)}\right]$

Denominators with linear factors

Quadratic factors

Each quadratic factors (ax^2+bx +c) has a corresponding partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where a, b, c and A and B are constants.

Example 2

Express each of the following fractions in partial fraction.

(a) $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)}$

Solution

Let $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{A}{x-6} + \frac{Bx+c}{x^2+3x+5}$

Multiplying through by (x – 6)(x^2+3x+5)

$7x^2$+2x-28= A(x^2+3x+5)+ (Bx +C)(x – 6)

Putting x = 6; 236= 59A, => A = 4

Equating coefficients of x^2

7 = A + B

7 = 4 + B; => B = 3

Equating constants

-28 = 5A – 6C

-28 = 20 – 6C

C = 8

$\therefore \frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{4}{x-6} + \frac{3x+8}{x^2+3x+5}$

(b) $\frac{2x-1}{(x-1)(x^2+1)}$

Solution

Let $\frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

Multiplying through by $(x-1)(x^2+1)$

2x – 1 = A(x^2+1)+ (Bx + C)(x-1)

Putting x = 1; 1 = 2A => A =$\frac{1}{2}$

Putting x= 0; -1 = A –C => C = $\frac{3}{2}$

Putting x= -1; 2A + 2B – 2C => B = $-\frac{1}{2}$

$\therefore \frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x^2+1)}$

$\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{3-x}{2(x^2+1)}$

Note the values of x = 0 and x = -1 are conveniently chosen, but the constants B and C by expansion of the expression and equating constants, i.e.

-1 = A –C => C = $\frac{3}{2}$

2 = C – B

B = $\frac{3}{2} - 2 = -\frac{1}{2}$

Thus, $\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{(x-1)} + \frac{1}{(x^2+1)} + \frac{x}{(x^2+1)}$

(c) $\frac{3+3x}{x^3-1}$

Solution

Note memorize the identities

$x^3 - 1 = (x-1)(x^2+x+1)$

$x^3 + 1 = (x - 1)(x^2 - x + 1)$

Then

$\frac{3+3x}{x^3-1} = \frac{3+3x}{(x-1)(x^2+x+1)}$

$Let\, \frac{3+3x}{x^3-1} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$

Multiplying through by (x – 1)(x^2+x+1)

3+3x = A(x^2+x+1) + (Bx+C)(x – 1)

Putting x = 1, 6 = 3A, => A = 2

By expanding and equating coefficients

x^2: A + B = 0, => B = 0 – 2 = -2

x^0: A – C = 3, => C = 2 – 3 = -1

$\therefore \frac{3+3x}{x^3-1} = \frac{2}{(x-1)} - \frac{2x+1}{(x^2+x+1)}$

(d) $\frac{x^2}{x^4-1}$

Solution

$\frac{x^2}{x^4-1} = \frac{x^2}{(x-1)(x+1)(x^2+1)}$

Let$\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1)}$

By multiplying through by (x – 1)(x+1)(x^2+1)

x^2= A(x+1)(x^2+1)+ B(x -1)(x^2+1) + (Cx+D)(x^2-1)

By equating coefficients

x^3: A + B + C = 0 .... (i)

x^2: A – B + D= 1 ....(ii)

x^1: A + B – C = 0 ....(iii)

x^0: A – B – D = 0 ....(iv)

Eqn. (ii) – Eqn. (iv)

2D = 2 => D = $\frac{1}{2}$

Eqn.(i)+ (iii)

2A+ 2B = 0 .... (v)

Eqn. (ii) + Eqn. (iv)

2A – 2B = 1.............................(vi)

Eqn. (v)+ Eqn. (vi)

4A = 1 => A = $\frac{1}{4}$

Eqn. (v)

B = $-\frac{1}{4}$

Eqn. (i)

C = 0

$\therefore \frac{x^2}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x^2+1)}$

Exercise 2

Express the following fractions in partial fractions

(i) $\frac{x^2+6}{(x^2+4)(x^2+9)}$ $\left[\frac{2}{5(x^2+4)} + \frac{3}{5(x^2+9)}\right]$

Repeated factors

Each repeated factor $(ax^2+ b)^n$ in the denominator has corresponding partial fraction of the form:$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$, where a, b, A_i are constants (i = 1, 2, ….n)

Example 3

Express each of the follow in partial fraction

(i) $\frac{4x-9}{(x-3)^2}$

Solution

Let$\frac{4x-9}{(x-3)^2}$ =$\frac{A}{x-3} + \frac{B}{(x-3)^2}$

Multiplying through by $(x-3)^2$

4x – 9 = A(x-3) + B = Ax – 3A + B

Equating coefficients

x^1: x = 4

x^0: -3A + B = 4; B = 3

$\therefore \frac{4x-9}{(x-3)^2}$ =$\frac{4}{x-3} + \frac{3}{(x-3)^2}$

(ii) $\frac{3x-14}{x^2-8x+16}$

Solution

$\frac{3x-14}{x^2-8x+16} = \frac{3x-14}{(x-4)^2}$

Let $\frac{3x-14}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$

Multiplying through by $(x-4)^2$

$3x - 14 = A(x-4) + B = Ax - 4A + B$

Equating coefficients

x^1: x = 3

x^0: -4A + B = -14; B = -2

$\therefore \frac{3x-14}{(x-3)^2} = \frac{3}{x-3} - \frac{2}{(x-4)^2}$

(iii) $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

Solution

Let $\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

Multiplying through by $(x-2)(x-1)^2$

$2x^2-5x+7 = A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Putting x = 1: 4 = -C, => C = -4

Putting x = 2: A = 5

Putting x = 0, 7 =A – 2B – 2C; B = -2

$\therefore \frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{5}{x-2} - \frac{2}{x-1} - \frac{4}{(x-1)^2}$

(iv) $\frac{7x+2}{3x^3+x^2}$

Solution

$\frac{7x+2}{3x^3+x^2} = \frac{7x+2}{x^2(3x+1)}$

Let $\frac{7x+2}{x^2(3x+1)} = \frac{A}{(3x+1)} + \frac{B}{x} + \frac{C}{x^2}$

Multiplying through by $x^2(3x+1)$

$7x+2 = Ax^2 + Bx(3x+1)+C(3x+1)$

Putting x = 0; c = 2

Putting x = $-\frac{1}{3}$; $\frac{A}{9} = 2 - \frac{7}{3}$ => A = -3

Putting x= -1; -5 = A + 2B- 2C, => B = 1

$\therefore \frac{7x+2}{x^2(3x+1)} = \frac{-3}{(3x+1)} + \frac{1}{x} + \frac{2}{x^2}$

Expressing improper fractions into partial fractions

Improper fractions are those whose index of the numerator is equal to or greater than that of the denominators.

They are first changed to proper fraction by long division or otherwise, before being integrated.

Example 4

Express the following improper fractions into partial fractions

(a) $\frac{5x^2-71}{(x+5)(x-4)}$

Solution

$\frac{5x^2-71}{(x+5)(x-4)} = \frac{5x^2-71}{x^2+x-20}$

Using long division

```
              5
x²+x-20 ) 5x²+0x-71
         - 5x²+5x-100
              -5 +29
```

=> $\frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-5x+29}{x^2+x-20}$

Let $\frac{-5x+29}{(x+5)(x-4)} = \frac{A}{x+5} + \frac{B}{x-4}$

Multiplying through by (x+5)(x-4)

-5x + 29 = A(x – 4)+ B(x + 5)

Putting x = 4, B = 1

Putting x = -5; A = -6

$\therefore \frac{-5x+29}{(x+5)(x-4)} = \frac{-6}{x+5} + \frac{1}{x-4}$

Hence $\frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-6}{x+5} + \frac{1}{x-4}$

(b) $\frac{3-2x}{1+x}$

Solution

$\frac{3-2x}{1+x} = \frac{-2x+3}{x+1}$

Using long division

$$\begin{array}{r} -2 \\ x+1 \overline{) -2x+3} \\ \underline{-2x-2} \\ 5 \end{array}$$

$\therefore \frac{3-2x}{1+x} = -2 + \frac{5}{x+1}$

Revision exercise on partial fractions

Express the following into partial fraction

(i) $\frac{8x}{x^2-4x-12}$ $\left[\frac{6}{x-6} + \frac{2}{x+2}\right]$

(ii) $\frac{x^4-x^3+x^2+1}{x^3+x}$ $\left[x - 1 + \frac{1}{x} + \frac{x-1}{x^2+1}\right]$

(iii) $\frac{5x-1}{2x^2+x} - 10$ $\left[\frac{3}{2x+5} + \frac{1}{x-2}\right]$

(iv) $\frac{2x^2-7x+1}{(2x+1)(2x-1)(x-2)}$

$\left[\frac{1}{2x+1} + \frac{2}{3(2x-1)} - \frac{1}{3(x-2)}\right]$

(v) $\frac{6x+7}{(x^2+2)(x+3)}$ $\left[\frac{x+3}{x^2+2} - \frac{1}{x+3}\right]$

(vi) $\frac{5x+7}{(x+1)^2(x+2)}$ $\left[\frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{3}{x+2}\right]$

(vii) $\frac{2x^3+3x^2-x-4}{x^2(x+1)}$ $\left[2 + \frac{3}{x} + \frac{4}{x^2} - \frac{2}{x+1}\right]$

(viii) $\frac{2x^2-x+14}{(4x^2-1)(x+3)}$ $\left[\frac{-3}{(2x+1)} + \frac{2}{(2x-1)} + \frac{1}{(x+3)}\right]$

(ix) $\frac{x^2+1}{x^3+4x^2+3x}$ $\left[\frac{1}{3x} + \frac{-1}{(x+1)} + \frac{5}{3(x+3)}\right]$

(x) $\frac{x^2}{x^4-1}$ $\left[\frac{-1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x^2+1)}\right]$

(xi) $\frac{6x}{(x-2)(x+4)^2}$ $\left[\frac{1}{3(x-2)} - \frac{1}{3(x+4)} + \frac{4}{(x+4)^2}\right]$

(xii) $\frac{x^2-4}{(x+1)^2(x-5)}$ $\left[\frac{5}{12(x+1)} + \frac{1}{2(x+1)^2} + \frac{7}{12(x-5)}\right]$

(xiii) $\frac{3x^2+x+1}{(x-2)(x+1)^3}$

$\left[\frac{5}{9(x-2)} + \frac{5}{9(x+1)} + \frac{4}{3(x+1)^2} + \frac{1}{(x+1)^3}\right]$

(xiv) $\frac{8x}{x^2-4x-12}$ $\left[\frac{6}{x-6} + \frac{2}{x+2}\right]$

.

Thank you Dr. Bbosa Science