TOPIC 6: INEQUALITIES AND GRAPHICS

Inequalities

An inequality is a statement that one number is less than the other or greater than it. It may be both greater or equal or less or equal with symbols $>$ or $<$ $\leq$ or $\geq$

Example

Find the value of x for which

$$\frac{x+3}{x-1} > 2$$

Avoid cross multiplying as you solve inequalities ensure zero on one side

$$\frac{x+3}{x-1} - 2 > 0$$

$$\frac{x+3}{x-1} - 2 > 0$$

$$\frac{x+3-2(x-1)}{x-1} > 0$$

$$\frac{x+3-2x+2}{x-1} > 0$$

$$\frac{5 - x}{x - 1} > 0$$

Get critical values by equating the numerator and the denominator to zero

$x = 1, x = 5$

Draw a table

	X<1	**1<x<5**	**x>5**
5-x	+	+	-
x-1	-	+	+
$\frac{5-x}{x-1}$	-	+	-

Since the answer should be greater than zero we take only positive ones.

1<x<5

Example

(5x-7)(X-3)<16x

$$5x^2-15x-7x+21<0$$

$$5x^2-38x+21<0$$

$$5x^2-35x-3x+21<0$$

$$5x(x-7)-3(x-7)<0 \text{ (factorizing the equation)}$$

$$(5x-3)(x-7)<0$$

Critical values

x= $\frac{3}{5}$, x=7

$-$ $x<3/5$ | $3/5<x<7$ | $x>7$ $-$

$+$ $3/5$ $-$ $x>7$ $+$

Since the inequalities is less than 0 $3/5<x<7$ is the range

Example

$$\frac{2x^2-7x-4}{3x^2-14x+11}>2$$

$$\frac{2x^2-7x-4}{3x^2-14x+11}-2>0$$

$$\frac{(2x^2-7x-4)-2(3x^2-14x+11)}{3x^2-14x+11}>0$$

$$\frac{2x^2-7x-4-6x^2+28x-22}{3x^2-14x+11}>0$$

$$\frac{-4x^2+21x-26}{3x^2-14x+11}>0$$

$$\frac{-4x^2-21x+26}{3x^2-14x+11}>0$$

But $4x^2-21x+26=4x2-8x-13(x-2)$

$$=4x()x-2+3(x-2)$$
$$=(4x-13)(x-2)$$

and $3x^2-14x+11$

$$=3x^2-3x-11x+11$$
$$=3x(x-1)-11(x-1)$$
$$(4x-11)(x-1)$$

$$\Rightarrow\frac{(4x-13)(x-2)}{(3x-11)(x-1)}<0$$

Critical value, $x=1,2,\frac{11}{13},\frac{13}{4}$

	X<1	**1<x<2**	**2<x<** $\frac{13}{4}$	$\frac{13}{4}$ **<x<** $\frac{11}{3}$	**x>** $\frac{11}{3}$
4x-13	-	-	-	+	+
x-2	-	-	+	+	+
3x-11	-	-	-	-	+

x-1	-	+	+	+	+
$\frac{(4x+3)(x-2)}{(3x-11)(x-1)}$	+	-	+	-	+

We want negative values

X:1<x<2 and $\frac{13}{4} < x < \frac{11}{3}$

GRAPHS

To sketch the curve the following should be done

1. Find the curve points
2. Find the vertical and horizontal a asymptotes
3. Finding the turning points and their nature.
4. Find the region where the curve does not pass if it exists
5. Find points where the curve crosses the horizontal asymptote only if the curve does not have turning points
6. Find the critical values and use them to draw an analysis table
7. Sketch the curve carefully and balance it

As you sketch Note the following

1. All asymptotes must be doted as you draw them
2. Lines showing the region where the curve does not exist must also be dotted.
3. However, if the lines above coincide with the axes they should be left continuous
4. The curve may cross the horizontal asymptote but only once.
5. The vertical asymptotes are never crossed by the curve
6. The region where the curve does not pass should be shaded or indicated
7. If the numerator has a higher power than the denominator the a slanting asymptote should be found using long division method
8. The curve should balance as you sketch using the analysis table and asymptotes

Example

Sketch the curve

$$y=\frac{(x-1)(x+2)}{(x+1)(x3)}=\frac{x^2+x-2}{x^2-2x-3}$$

Cut points

When x=0

$$y=\frac{-2}{-3}=\frac{2}{3}\quad\left(0,\frac{2}{3}\right)$$

When y=0, $x=1, x=-2$

(1, 0), (-2, 0)

Vertical asymptote

Equate the denominator to zero so, x=-1, x=3 are the asymptote

Horizontal asymptote

Dived with highest power to the denominator

$$y=\frac{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{2}{x^2}}{\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{-3}{x^2}}=\frac{1+\frac{1}{x}-\frac{1}{x^2}}{1-\frac{2}{x}-\frac{3}{x^2}}$$

As $x\to+\infty$

$$y=\frac{1+\frac{1}{\infty}-\frac{2}{\infty^2}}{1-\frac{2}{\infty}-\frac{3}{\infty}}$$

$$=y=\frac{1}{1}=1$$

Y=1 is the horizontal asymptote

Turning pt

$$y = \frac{x^2 + x - 2}{x^2 - 2x - 3}$$

$$\frac{dy}{dx} = \frac{(x^2 - 2x - 3)(2x + 1) - (x^2 + x - 2)(2x - 2)}{(x^2 - 2x - 3)^2}$$

$$\frac{(2x^3 + x^2 - 4x^2 - 2x - 6x - 3) - (2x^3 - 2x^2 - 2x - 4x + 4)}{(x^2 - 2x - 3)^2}$$

$$= = \frac{-3x^2 - 2x - 7}{(x^2 - 2x - 3)^2}$$

$$= \frac{-(3x^2 + 2x + 7)}{(x^2 - 2x - 3)^2}$$

But $\mathbf{b^2 < 4ac}$ so no real roots

Therefore no turning points

So no region where the curve does not pass

Find the point where the curve crosses the horizontal asymptote when y = 1 and find ***x (*** *this is only done when there are no turning points* **)**

$$1 = \frac{x^2 + x - 1}{x^2 - 2x - 3}$$

$$x^2 - 2x - 3 = x^2 + x - 2$$

$$-3x = 1$$

$$x = \frac{-1}{3}$$

therefore, *this is* $\left(\frac{-1}{3}, 1\right)$ ***the point***

Critical values

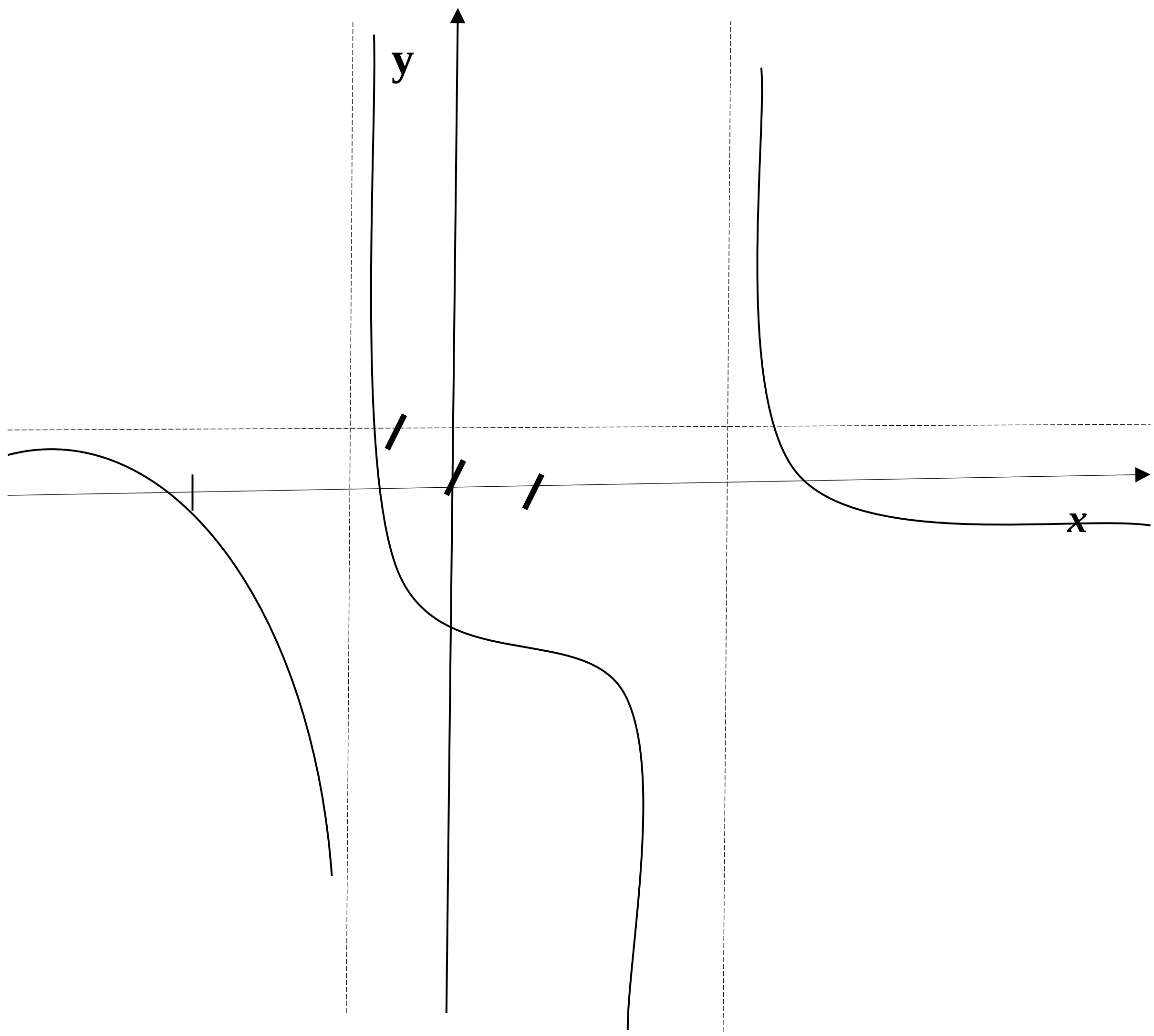
Equating numerator and denominators to zero

X=-2, -1, 1 and 3

Analysis table

	X<-2	**-2<x<y**	**-1<x<1**	**1<x<3**	**x>3**

$(x-1)(x+2)$	+	-	+	+	+
$(x+1)(x-3)$	+	+	+	-	+
y	+	-	+	-	+



(As you draw the graph first indicate the cut points and asymptotes. Join the points as they appear take note of vertical asymptote and horizontal. In the region where there is no points use the analysis tables, the asymptotes and balances the curve.)

Example

Prove that $\frac{3x-9}{x^2-x-2}$ cannot lie between certain values. Illustrate graphically

Cut pts

$x=0$, $y=\frac{9}{2}$ $\left(0,\frac{9}{2}\right)$

Where $y=0$ $x=3$ $(3,0)$

Vertical asymptote

(x-2)(x+1)=0

X=2 , x=-1

To prove that it cannot lie between two certain values

$$y(x^2-x-2)=3x-9$$
$$yx^2-yx-2y-3x+9=0$$
$$yx^2-(y+3)x+(9-2y)=0$$

For region where the curve doesn't lie

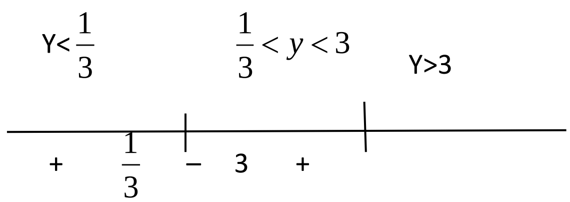
$$b^2<4ac$$

$$(y+3)^2<4(y)(9-2y)$$
$$y^2+6y+9<36y-8y^2$$
$$9y^2-30y+9<0$$
$$3y^2-10y+3<0$$

Critical values

$$3y^2-9y-y+3<0$$
$$3y(y-3)-(y-3)<0$$
$$(3y-1)(y-3)<0$$

$y = \frac{1}{3}$, 3



Therefore $\frac{1}{3} < y < 3$ is the region where the curve doesn't lie

(When you use this approach the turning points an nature are got using these bounds.)

$y = \frac{1}{3}$, x=

ie

y=3 , x=

$$yx^2 - (y+3)x + (9-2y) = 0$$

$$\frac{1}{3}x^2 - \left(\frac{1}{3}+3\right)x + \left(9 - 2\left(\frac{1}{3}\right)\right) = 0$$

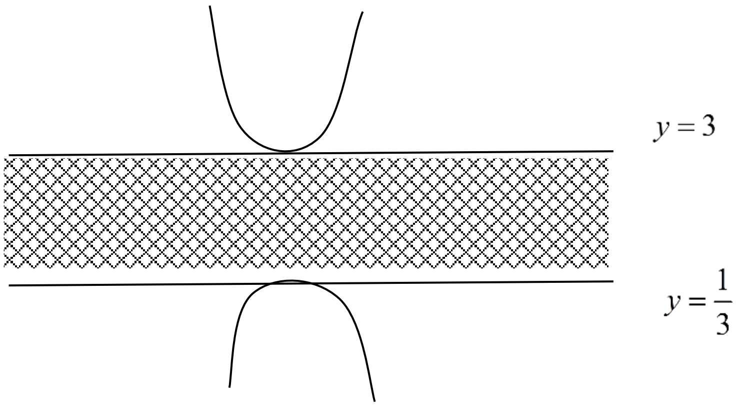
$$\frac{1}{3}x^2 - \frac{10}{3}x + \frac{25}{3} = 0$$

$$x^2 - 10x + 25 = 0$$

$$(x-5)^2 = 0$$

$$x = 5$$

$\left(5, \frac{1}{3}\right)$ is a turning point and since $\frac{1}{3}$ is a lower bound then it turns maximally



$$3x^2 - (3+3)x + (9 - 2(3)) = 0$$

$$3x^2 - 6x + 3 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

$\therefore$ (1, 3) is a turning point

and min because of upper bound

Also the horizontal asymptote can be got using the coefficient behind x^2 and equating it to zero

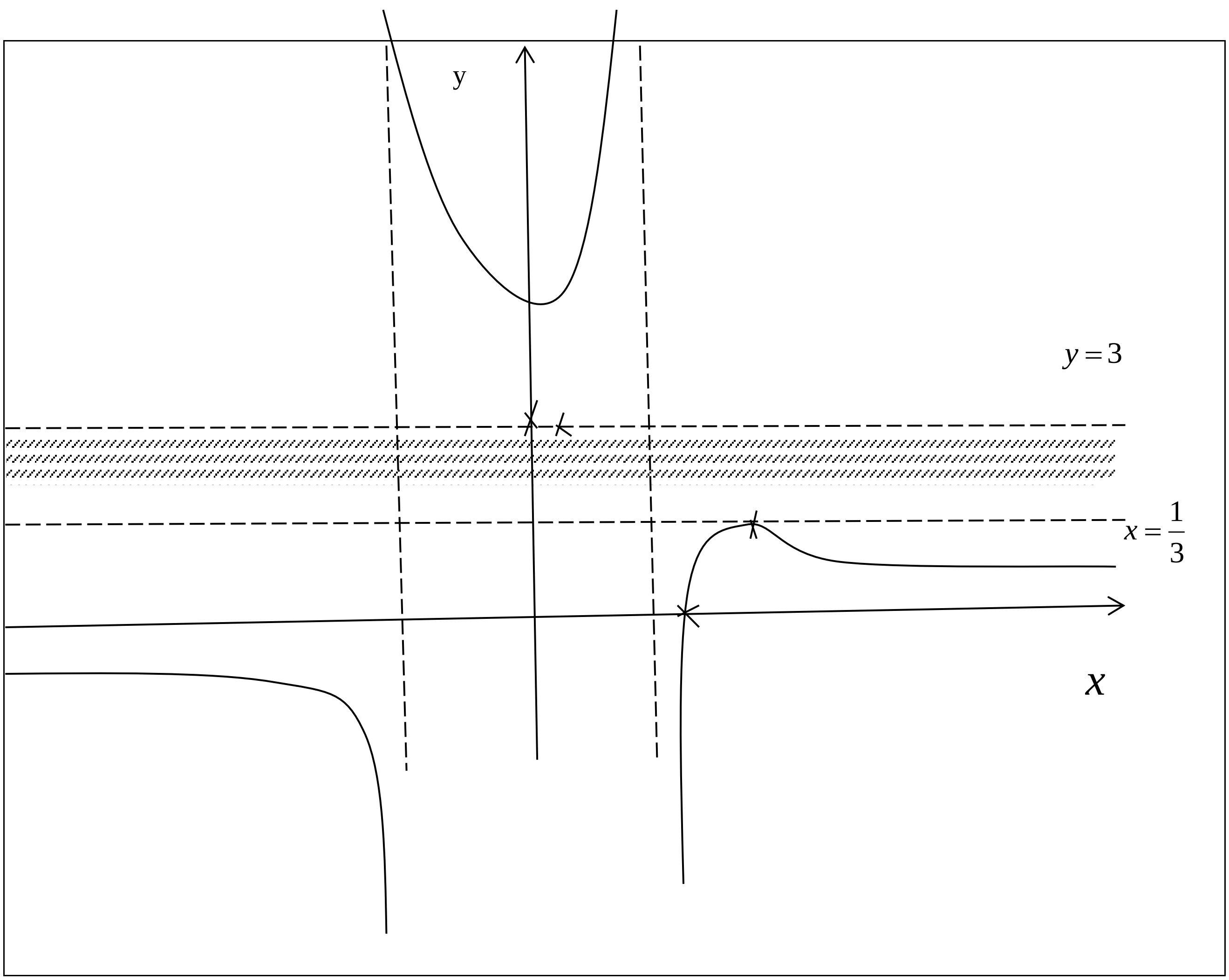
y = 0 is a horizontal asymptote.

(Since there is turning points so no need to find the point where it crosses the horizontal asymptote)

Find the critical values

X = 3, 2,-1

	X<-1	-1<x<2	2<x<3	x>3
3(x-3) (x+1)(x-2) Y	+ - -	- - +	+ - -	+ + +



Note that the curve only cross the horizontal asymptote one and never crosses the vertical asymptote

Example

Find the turning points and their nature of the curve

$$y = \frac{3(x-2)}{x(x+6)} = \frac{3x-6}{x^2+6x}$$

$$\frac{dy}{dx} = \frac{(x^2+6x)(3)-(3x-6)(2x+6)}{(x^2+6x)^2}$$

$$= \frac{(3x^2+18x)-(6x^2+18x-12x-36)}{(x^2+6x)^2}$$

$$= \frac{-3x^2+12x+36}{(x^2+6x)^2}$$

$$= \frac{-3(x^2-4x-12)}{(x^2+6x)^2}$$

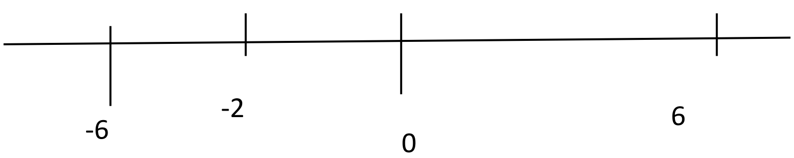
At turning pt $\frac{dy}{dx} = 0$

$$x^2-4x-12=0$$
$$(x-6)(x+2)=0$$
$$x=6, x=-2$$

L R L R

X	-3	-2	-1	1	6	7
$\frac{dy}{dx}$	–	0	+	+	0	–

(As you pick the figure to test for the turning point take care of the vertical asymptotes.)



Pick any figure less than -6 on left hand side of x=-2 and on the right hand side it should be greater than -2 but less than zero and for x=6 on the left hand side anything greater than 6. Then test using those figures.

X= -2

$$y = \frac{3(-2-2)}{-2(-2+6)} = \frac{-12}{-8} = \frac{3}{2}$$

When x=6

$$y = \frac{3(6-2)}{6(6+6)} = \frac{12}{72} = {}^{2}\!/_{12} = {}^{1}\!/_{6}$$

$$\left(-2.\frac{3}{2}\right) \text{ min, } \left(6,\frac{1}{6}\right) \text{ max}$$

State the region where the curve doesn't pass using the y coordinates of the turning points

$$\frac{1}{6} < y < \frac{3}{2}$$

Cut points

When x=0 , y is not defined there fore the curve doesn't cross the y- axis

y=0 , x=2 (2 , 0)

Vertical asymptote

X=0, x=-6

Horizontal asymptote

$$y = \frac{\frac{3x}{x^2} - \frac{6}{x^2}}{\frac{x^2}{x^2} + \frac{6x}{x^2}} = \frac{\frac{3}{x} - \frac{6}{x^2}}{1 + \frac{6}{x}} \text{ as } x \to +\infty$$

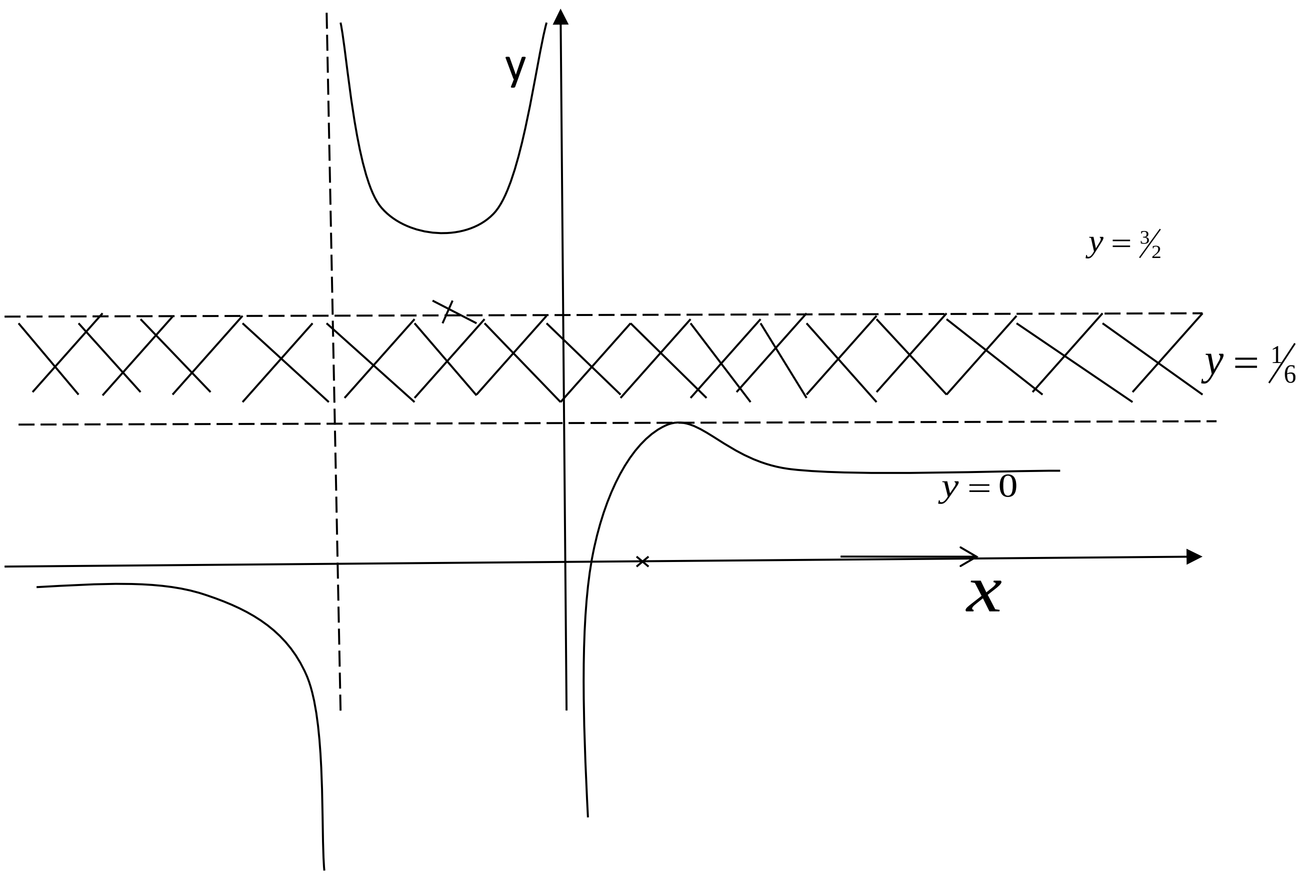
$$y = \frac{0}{1} = 0$$

y=0 is a horizontal asymptote

Critical values

x=2, x=0, x=-6

	X<-6	-6<x<0	x>0		
3(x-2)	-	-	-		
x(x+6)	+	-	+		
y	-	+	-		



X=-6 X=0

Example

Determine the nature of the turning points of the curve

$$y=\frac{x^2-6x+5}{2x-6}$$

State the asymptotes and sketch the curve.

Since the numerator has a higher power than the denominator the curve will have aslanting asymptote.

Note that.

$$\frac{dy}{dx}=\frac{(2x-1)(2x-6)-(x^2-6x+5)(2)}{(x^2-6x+5)^2}$$

$$= \frac{(4x^2 - 12x - 2x + 6) - (2x^2 - 12x + 10)}{(x^2 - 6x + 5)^2}$$

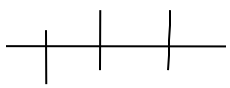
$$= \frac{2x^2 - 2x - 4}{(x^2 - 6x + 5)^2}$$

At turning point

$$\frac{dy}{dx} = 0$$

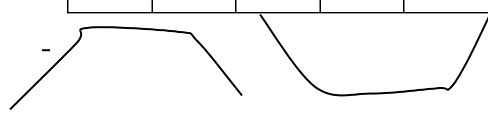
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

$$x = 2, \quad x = -1$$



L R L R

-2	-1	0	1	2	3
+	0	-	-	0	+



$$x = 2$$
$$y = \frac{4 - 12 + 5}{4 - 1} = \frac{-3}{3} = -1$$

(2, -1) , (-1, 0)

min max

$-1 < y < 0$ is region where the curve doesn't pass

Cut point

When $y = 0$

$$x^2 - 6x + 5 = 0$$
$$(x-5)(x-1) = 0$$

(5,0) , (1,0)

$x = 5$, $(x = 1)$

When $x = 0$, $y = -5$

$(0, -5)$

Vertical asymptote

$$2x - 1 = 0$$
$$x = \frac{1}{2}$$

Slanting asymptote

$$\begin{array}{r|l} & \frac{1}{2}x - \frac{11}{4} \\ 2x-1 & x^2 - 6x + 5 \\ & x^2 - \frac{1}{2}x \\ \hline & \frac{-11}{2}x + 5 \\ & \frac{-11}{2}x + \frac{11}{4} \\ \hline & \frac{9}{4} \end{array}$$

$y = \frac{1}{2}x - \frac{11}{4}$ is slanting asymptote (i.e. y= Quotient is taken to be the asymptote)

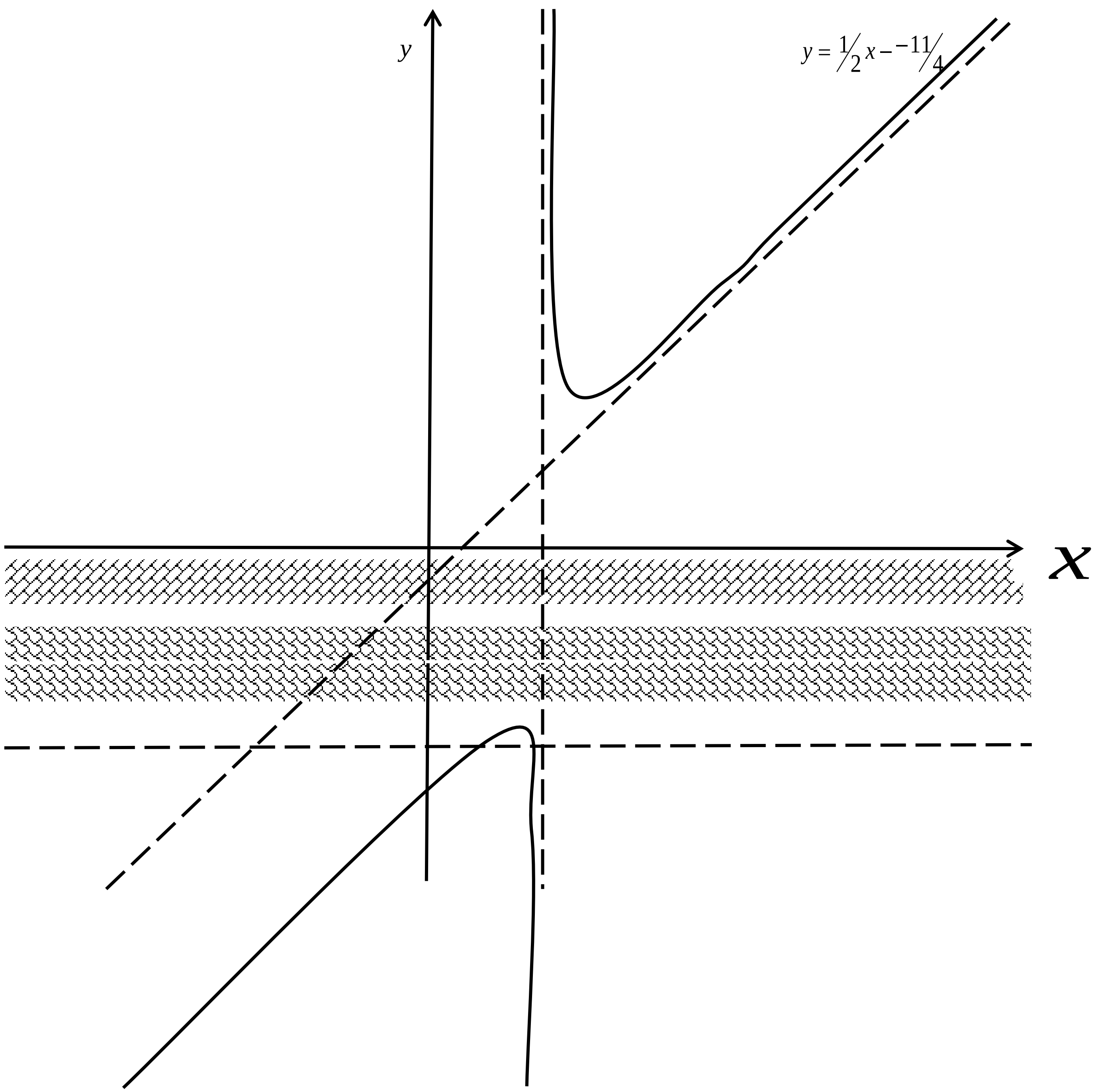
Analysis table

Critical values

$$X = \frac{1}{2}, X = 1, X = 5$$

	X<-$\frac{1}{2}$	$\frac{1}{2}$<x<1	1<x<5	x>5

(x-1) (x-5)	+	+	-	+
2x-1	-	+	+	+
y	-	+	-	+



$$X = \frac{1}{2}$$

Exercise

Solve the inequalities

1. $\frac{4-x}{x+2} < 3$
2. $(4x-3)(x+1) > 2$
3. $\frac{(x-1)(x-3)}{(x+1)(x-2)} > 0$

4. Find the turning points and the nature of the curve.

$$y=\frac{2x^2-9x+4}{x^2-2x+1}$$

State the asymptote and sketch the curve.

5. Find the region where the curve doesn't lie

$$y=\frac{x^2+1}{x^2-x-2}$$

State the asymptotes and sketch the curve

6. A curve has the equation $y=\dfrac{2}{1+x^2}$

(a). Determine the nature of the turning point on the curve

(b) Find the asymptotes and sketch the curve.

7. Sketch the curve

$$y=\frac{(x-1)(x+3)}{(x-2)(x+2)}$$

8. Sketch the curve

$$y=\frac{3x+3}{x(3-x)}$$

9. Sketch the curve

$$y=\frac{(x-5)(x-1)}{x^2-2x-3}$$

10. Sketch the curve

$$y=\frac{x^2-x-6}{x-1}$$