TOPIC 2: INTEGRATION

Introduction

This is the continuation of integration from Book 1 in form five. We shall look at other methods we can use to integrate functions of x which are different from what we did in form 5.

METHOD 1

Recognizing the presence of a function and its derivative.

A learner is expected to observe carefully whether in the function given the function and derivative exists.

Example:

Find $\int x(3x^2+2)^4\,dx$

A learner has to find out the two functions which one is a derivative of the other from one above can see it clearly that x come out of derivative of $3x^2 + 2$ .Therefore $(3x^2+2)^4$ will be the function of x , the derivative .

The learner is expected to add one on the power of the function and differentiate it.

$$\frac{d}{dx}(3x^2+2)^5 = 5(3x^2+2)^4(6x) = 30x(3x^2+2)^4$$

If the learner observes carefully, it is only 30 causing the difference from the set question. Divide both sides with the 30, that is to say;-

$$\frac{d}{dx}\frac{1}{30}(3x^2+2)^5\,dx = x(3x^2+2)^4\,dx$$

The learner is expected to introduce the integral sign on both sides.

$$\int\frac{d}{dx}\frac{1}{30}(3x^2+2)^5\,dx = \int(3x^2+2)^4\,dx$$

Whenever an integral and derivative sign appear at the same point they neutralize each other and disappear

$$\frac{1}{30}(3x^2+2)^5 + C = \int(3x^2+2)^4dx$$

Example 2

$$\int Sin3xCos^2 3xdx$$

Clearly cos3x becomes a function

Sin 3x becomes a derivative.

$$\frac{d}{dx}(Cos^3 3x) = 3(Cos^2 3x)(-\sin 3x)(3)$$

$$= -9\sin 3xCos^2 3x$$

Divide through by

$$\frac{d}{dx} - \frac{1}{9}(Cos^3 3x) = Sin3xCos^2 3x$$

$$\int \frac{d}{dx} - \frac{1}{9}(Cos^3 3x)dx = \int Sin3xCos^2 3xdx$$

$$\therefore -\frac{1}{9}Cos^3 3x + C = \int Sin3xCos^2 3xdx$$

Example 3

$$\int \frac{(x+1)dx}{(x^2+2x-5)^3}$$

$$= \int (x+1)(x^2+2x-5)^{-3}dx$$

x^2+2x-5 a function and (x+1) a derivative

$$\frac{d}{dx}(x^2+2x-5)^{-2} = -2(x^2+2x-5)^{-3}(2x+2)$$

$$= -2(x^2+2x-5)^{-3}.2(x+1)$$

$$= -4(x+1)(x^2+2x-5)^{-3}$$

$$\therefore \frac{d}{dx}\frac{1}{4}(x^2+2x-5)^{-2} = -2(x+1)(x^2+2x-5)^{-3}$$

$$\int \frac{d}{dx} - \frac{1}{4}(x^2+2x-5)^{-3}dx = \int \frac{(x+1)}{x^2+2x-5}dx$$

$$-\frac{1}{4}(x^2+2x-5)^{-3}+C$$

Or

$$\frac{-1}{4(x^2+2x-5)^3}+C$$

Note: That we only divide through by a constant not variable

EXERCISE

1. $\int x^2\sqrt{(x^3+1)}dx$

2. $\int sex^2 x\tan^2 xdx$

3. $\int \frac{(x-1)}{(2x^2-4x+1)^{3/2}}dx$

METHOD 2

Change Of Variable

In this approach we change what looks to be making the question tricky and give it a letter of our choice.

If we may use the same example

EXAMPLE 1

$$\int x(3x^2+2)^4 dx$$

You may take $3x^2+2$ to be u

If $U = 3x^2 + 2$

$du = 6xdx$

$\frac{1}{6}du = xdx$

Substitute in the question above

$$\int (3x^2+2)^4(xdx) = \frac{1}{6}\int u^4 du$$

$$=\frac{1}{6}\left[\frac{u^5}{5}\right]+C$$

$$=\frac{1}{30}(3x^2+2)^5+C$$

Note that this method can also handle all question that are under Reverse Method however it can also solve those not in that category

EXAMPLE 2

$$\int x\sqrt{3x-1}\ \ dx$$

Let U^2= 3x-1

2udu =3dx

$$\frac{2}{3}u\,du=dx$$

U^2+1=3x

$$\frac{u^2+1}{3}=x$$

$$\frac{2}{3}\int\left(\frac{u^2+1}{3}\right)\sqrt{u^2}du$$

$$\frac{2}{9}\int u^2(u^2+1)du$$

$$\frac{2}{9}\int u^4+u^2du$$

$$\frac{2}{9}\left(\frac{u^5}{5}+\frac{u^3}{3}\right)+C$$

$$\frac{2}{9}u^3\left(\frac{u^2}{5}+\frac{1}{3}\right)+C$$

$$\frac{2}{9}u^3\left(\frac{3u^2+5)}{15}\right)+C$$

Note that it easier to factorise before substituted

$$\frac{2}{135}(3x-1)^{3/2}(3(3x-1)+5)+C$$

$$\frac{2}{135}(3x-1)^{3/2}(9x-3+5)+C$$

$$\frac{2}{135}(3x-1)^{3/2}(9x+2)+C$$

EXAMPLE 3

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

let $\sqrt{x}$ = u

x=u^2

dx=2udu

substitute

$$2\int \frac{(Sinu)udu}{u}\sqrt{x} = 2\int Sinudu$$

= -2 Cos u + C

= -2Cos $\sqrt{x}$ +C

EXAMPLE 4

$$\int \frac{(x-2)dx}{(x+2)^3(x-6)^3}dx = \int \frac{(x-2)dx}{(x+2)(x-6)^3}dx = \int \frac{(x-2)dx}{(x^2-4x-12)^3}dx$$

Let x^2 = 4x-1 be u

(2x-4)dx = du

2(x-2) dx = du

(x-2) dx = ½ du

$$\int (x-2)(x^2-4x-12)^{-3}dx = \frac{1}{2}\int u^{-3}du$$

$= ½ \left[\frac{u^{-2}}{-2}\right] + C$

$= -\frac{1}{4}u^{2} + C$

$= -\frac{1}{4}\left(x^{2} - 4x - 12\right)^{-2} + C$

Or

$\frac{-1}{4\left(x^{2} - 4x - 12\right)^{-2}} + C$

The learner should be observant enough to see some of the tricks used in attempting the question

Questions with limits

$\int_{0}^{\frac{\pi}{2}} Sinx\sqrt{Cos\, x}\ \ dx$

Let u = $\sqrt{Cos\, x}$

U^2 = Cos x

2udu = - Sinxdx

-2udu=Sinx dx

$u = \sqrt{Cos\, x}$

X	u
$\frac{\pi}{2}$	0
0	1

Note that as you change variable also change the limits

$$-2\int_1^0 u^2 du = -2\left[\frac{u^3}{3}\right] + C$$

$$= \frac{-2}{3}(0)^3 - \frac{-2}{3}(1)^3 + C$$

$$= \frac{2}{3}$$

Observe that you don't need to go back to the function you changed.

Note that some questions may need approach of Pythagoras theorem if they have odd powers on the trigonometry used .

Use trig identities $\text{Cos}^2\text{x} + \text{Sin}^2\text{x}=1$, $1 + \text{Cos}^2\text{x}$, $1 + \tan^2\text{x}= \text{Sec}^2\text{X}$

EXAMPLE 5

$\int Sin^5 x\, dx$ Split $Sin^5 x = \text{Sin}^4\text{x Sin X}$

$= (\sin^2\text{X})^2 \sin \text{x}$

From $\text{Cos}^2\text{x} + \text{Sin}^2\text{x} =1$, It implies that $\text{Sin}^2\text{x} = 1\text{-Cos}^2\text{x}$

$Sin^5 x = (1\text{-Cos}^2\text{x})^2\ \text{Sinx}$

$Sin^5 x = (1\text{-2cos}^2\text{x} +\text{Cos}^4\text{x})\ \sin \text{x}$

$$\int Sin^5 xdx = \int \sin x - Cos^2 xSinx + Cos^4 xSinxdx$$

$$= \int Sinxdx - 2\int Cos^2 xSinx + \int Cos^4 xSinxdx$$

$$= -Cosx + \frac{2}{3}Cos^3 x - \frac{1}{5}Cos^5 x + C$$

Hint: use the approach above the generate the answer

EXERCISE

Use the same approach to prove $\int Cos^5 xd = Sinx - \frac{2}{3}Sins^3 x + \frac{1}{5}Sin^5 x + C$

If the powers on trig of *sin* and *Cos* are even the Double angle formulae is used for example $\text{Sin}^2\text{x} = ½ (1\text{-cos2x})$ and $\text{Cos}^2\text{x} = ½ (1\text{+ cos2x})$

Example

$$\int_{-2\pi}^{0} Cos^4 \frac{x}{4}\, dx$$

The learner should be careful as s/he uses the double angle formula .

Note that the angle in question has to be doubled as us bring in the identity

$$Cos^4 \frac{x}{4} = \left(Cos^2 \frac{x}{4}\right)^2$$

$$Cos^2 \frac{x}{4} = \frac{1}{2}\left(1 + Cos\frac{x}{2}\right)$$

$$Cos\frac{x}{4} = \left(\frac{1}{2}\left(1 + Cos\frac{x}{2}\right)\right)^2$$

$$= \frac{1}{4}\left(1 + 2Cos\frac{x}{2} + Cos^2\frac{x}{2}\right)$$

And $Cos^2 \frac{x}{2} = \frac{1}{2}Cosx)$

$$= Cos^4 \frac{x}{4} = \frac{1}{4}\left(1 + Cos\frac{x}{2} + \frac{1}{2}(1 + Cosx\right)$$

$$= \frac{1}{4}\left(1 + \frac{1}{2} + Cos\frac{x}{2} + \frac{1}{2}Cosx\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + Cos\frac{x}{2} + \frac{1}{2}Cosx\right)$$

$$\int Cos^4 \frac{x}{4} dx \;\; = \frac{1}{4}\int\left(\frac{3}{2} + Cos\frac{x}{2} + \frac{1}{2}Cosx\right) dx$$

$$= \left[\frac{3}{8}x + \frac{2}{4}Sin\frac{x}{2} + \frac{1}{8}Sinx\right]_{-2\pi}^{0}$$

$$= 0 - \left[\frac{3}{8}(-2\pi + 0)\right]$$

$$= \frac{3}{4}\pi$$

Questions leading to inverse trigonometric solutions

$$\int \frac{dx}{\sqrt{a^2 - b^2x^2}}$$

A learner should change the variable with intention of creating 1-sin^{2}x or 1-cos^{2}x in the root .

Let $b^2x^2 = a^2\sin^2\theta$ or $a^2\cos^2\theta$

bx $= a\sin\theta$

bdx $= \cos\theta d\theta$

dx $= \frac{a}{b}\cos\theta d\theta$

$$\frac{a}{b}\int \frac{\cos\theta d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \frac{a}{b}\int \frac{\cos\theta d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \frac{a}{b}\int \frac{\cos\theta d\theta}{\sqrt{a^2\cos^2\theta}} = \frac{a}{b}\int \frac{\cos\theta d\theta}{\sqrt{a^2\cos^2\theta}}$$

$$= \frac{a}{b}\int d\theta = \frac{1}{b}\theta + C$$

Make θ a subject from above

bx= asin θ

$$\frac{b}{a}x = \sin\theta$$

$$\sin^{-1}\left(\frac{b}{a}x\right) = \theta$$

$$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

And the form $\int \frac{dx}{a^2 + b^2x^2}$

Generate $1 + \tan^2\theta$ to simplify that is to say;-

Let $b^2x^2 = a^2\tan^2\theta$

bx = atan θ

bdx = $a\sec^2\theta\, d\theta$

dx= $\frac{a}{b}Sec^2\theta d\theta$

$$\frac{a}{b}\int\frac{Sec^2\theta d\theta}{a^2+a^2\tan^2\theta}=\frac{a}{b}\int\frac{Sec^2\theta d\theta}{a^2(1+\tan^2\theta)}$$

$$=\frac{a}{b}\int\frac{Sec^2\theta d\theta}{a^2Sec^2\theta}$$

$$=\frac{1}{ab}\theta+C$$

From the above $bx=\tan\theta$

$$\frac{b}{a}x=\tan\theta$$

$$\tan^{-1}\left(\frac{b}{a}x\right)=\theta$$

$$\int\frac{dx}{\sqrt{a^2+b^2x^2}}=\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right)+C$$

EXAMPLE

$$\int\frac{dx}{\sqrt{16-9x^2}}$$

Let $9x^2=16Sin^2\theta$

$$3x^2=4Sin\theta$$

$$3dx=4Cos\theta d\theta$$

$$dx=\frac{4}{3}Cos\theta d\theta$$

$$\frac{4}{3}\int\frac{Cos\theta d\theta}{\sqrt{16-16Sin^2\theta}}=\frac{4}{3}\int\frac{Cos\theta d\theta}{\sqrt{16(1-Sin^2\theta)}}=\frac{4}{3}\int\frac{Cos\theta d\theta}{\sqrt{16Cos^2\theta}}$$

$$\frac{4}{3}\int\frac{Cos\theta d\theta}{4Cos\theta}=\frac{1}{3}\int d\theta==\frac{1}{3}\theta+C$$

But 3x = 4Sin θ

$$\frac{3}{4}x = Sin\theta$$

$$Sin^{-1}\left(\frac{3}{4}x\right) = \theta$$

$$\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}Sin^{-1}\left(\frac{3}{4}x\right) + C$$

EXAMPLE

$$\int \frac{dx}{3+4x^2}$$

Let $4x^2 = 3\tan^2\theta$

$$2x = \sqrt{3}\tan\theta$$

$$2dx = \sqrt{3}Sec^2\theta d\theta$$

$$dx = \frac{\sqrt{3}}{2}Sec^2\theta d\theta$$

Substituting

$$\frac{\sqrt{3}}{2}\int \frac{Sec^2\theta d\theta}{3+3\tan^2\theta}$$

$$\frac{\sqrt{3}}{2}\int \frac{Sec^2\theta d\theta}{3+3\tan^2\theta} = \frac{\sqrt{3}}{2}\int \frac{Sec^2\theta d\theta}{3Sec^2\theta}$$

$$= \frac{\sqrt{3}}{6}\int d\theta = \frac{\sqrt{3}}{6}\theta + C$$

But $2x = \sqrt{3}\tan\theta$

$$\frac{2}{\sqrt{3}}x = \tan\theta$$

$$\tan^{-1}\left(\frac{2}{\sqrt{3}}x\right) = \theta$$

$$\int \frac{dx}{3+4x^2} = \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{2}{\sqrt{3}}x\right) + C$$

Example

$$\int \frac{dx}{\sqrt{9-3(x+1)^2}}$$

$$3(x+1)^2 = 9Sin^2\theta$$

$$\sqrt{3}(x+1) = 3Sin\theta$$

$$\sqrt{3}dx = 3Cos\theta d\theta$$

$$dx = \frac{3}{\sqrt{3}}Cos\theta d\theta$$

$$\frac{3}{\sqrt{3}}\int \frac{Cos\theta d\theta}{\sqrt{9-9Sin^2\theta}}$$

$$\frac{3}{\sqrt{3}}\int \frac{Cos\theta d\theta}{\sqrt{9(1-Sin^2\theta)}} = \frac{3}{\sqrt{3}}\int \frac{Cos\theta d\theta}{3Cos\theta d\theta}$$

$$\frac{1}{\sqrt{3}}\int d\theta = \frac{1}{\sqrt{3}}\theta + c$$

But $\sqrt{3}(x+1) = 3Sin\theta$

$$\frac{\sqrt{3}}{3}(x+1) = Sin\theta$$

$$= \frac{1}{3}Sin^{-1}\left(\frac{\sqrt{3}}{3}(x+1)\right) + C$$

Example

$$\int \frac{dx}{5+3(x-2)^2}$$

Let $3(x-2)^2 = 5\tan^2\theta$

$$\sqrt{3}(x-2)=\sqrt{5}\tan\theta$$

$$\sqrt{3}dx=\sqrt{5}Sec^2\theta d\theta$$

$$dx=\frac{\sqrt{5}}{\sqrt{3}}Sec^2\theta d\theta$$

$$\frac{\sqrt{5}}{\sqrt{3}}\int\frac{Sec^2\theta d\theta}{5+5\tan^2\theta}$$

$$\frac{\sqrt{5}}{\sqrt{3}}\int\frac{Sec^2\theta d\theta}{5(1+\tan^2\theta)}=\frac{\sqrt{5}}{\sqrt{3}}\int\frac{Sec^2\theta d\theta}{5Sec^2\theta}$$

$$\frac{\sqrt{5}}{5\sqrt{3}}\int d\theta=\frac{\sqrt{5}}{5\sqrt{3}}\theta+C$$

But $\sqrt{3}(x-2)=\sqrt{5}\tan\theta$

$$\frac{\sqrt{3}}{\sqrt{5}}(x-2)=\tan\theta$$

$$\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}(x-2)\right)=\theta$$

$$=\frac{\sqrt{3}}{5\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}(x-2)\right)+C$$

$$=\frac{\sqrt{15}}{15}\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}(x-2)\right)+C$$

Questions which require completion of squares

Example

$$\int=\frac{dx}{\sqrt{12x-2x^2-9}}$$

Let $f(x)=12x-2x^2-9$

$$f(x)=-2x^2+12x-9$$

$$\frac{f(x)}{-2} = x^2 - 6x + \frac{9}{2}$$

$$\frac{f(x)}{-2} = x^2 - 6x + (-3)^2 + \frac{9}{2} - (-3)^2$$

$$\frac{f(x)}{-2} = (x-3)^2 - \frac{9}{2}$$

$$\frac{f(x)}{-2} = (x-3)^2 + \frac{9}{2}$$

$$f(x) = 9 - 2(x-3)^2$$

$$\int = \frac{dx}{\sqrt{12x - 2x^2 - 9}} = \int \frac{dx}{\sqrt{9 - 2(x-3)^2}}$$

Let $2(x-3)^2 = 9Sin^2\theta$

$$\sqrt{2}(x-3) = 3Sin\theta$$

$$\sqrt{2}dx = 3Cos\theta d\theta$$

$$\frac{3}{\sqrt{3}}\frac{Cos\theta d\theta}{\sqrt{9 - 9Sin^2\theta}} = \frac{3}{\sqrt{3}}\frac{Cos\theta d\theta}{\sqrt{9(1 - Sin^2\theta)}} = \frac{3}{\sqrt{3}}\frac{Cos\theta d\theta}{\sqrt{9Cos^2\theta}} = \frac{3}{\sqrt{3}}\frac{Cos\theta d\theta}{3Cos\theta}$$

$$= \frac{1}{\sqrt{3}}\int d\theta = \frac{1}{\sqrt{3}}\theta + C$$

$$= \frac{1}{\sqrt{3}}Sin^{-1}\frac{\sqrt{2}}{3}(x-3) + C$$

Example

$$\int \frac{dx}{4x^2 - 8x + 7}$$

Pull out the quadratic function

$$f(x) = 4x^2 - 8x + 7$$

$$\frac{f(x)}{4} = x^2 - 4x + \frac{7}{4}$$

Complete squares

$$\frac{f(x)}{4} = x^2 - 2x + (-1)^2 + \frac{7}{4} - (-1)^2$$

$$\frac{f(x)}{4} = (x-1)^2 + \frac{3}{4}$$

$$f(x) = 3 + 4(x-1)^2$$

$$\int \frac{dx}{4x^2 - 8x + 7} = \int \frac{dx}{3 + 4(x-1)^2}$$

Let $4(x-1)^2 = 3\tan^2\theta$

$$2(x-1) = \sqrt{3}\tan\theta$$

$$2dx = \sqrt{3}Sec^2\theta d\theta$$

$$dx = \frac{\sqrt{3}}{2}Sec^2\theta d\theta$$

$$\frac{2}{\sqrt{3}}(x-1)\tan\theta$$

$$\tan^{-1}\left(\frac{2}{\sqrt{3}}(x-1)\right) = \theta$$

$$\frac{3}{2}\int \frac{Sec^2\theta d\theta}{3 + 3\tan^2\theta}$$

$$\frac{\sqrt{3}}{2}\int \frac{Sec^2\theta d\theta}{3(1+\tan^2\theta)} = \frac{\sqrt{3}}{2}\int \frac{Sec^2\theta d\theta}{3Sec^2\theta} = \frac{\sqrt{3}}{2}\int d\theta$$

$$\frac{\sqrt{3}}{2}\theta + C = \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x-1)\right) + C$$