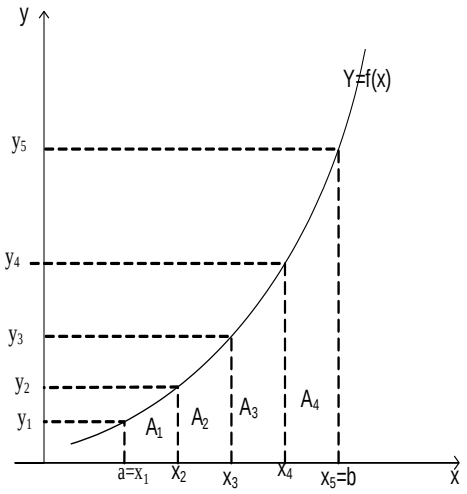
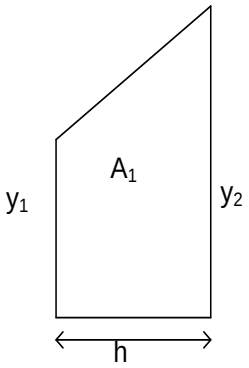
Topic 7: Numerical integration

The trapezium rule: It is a numerical method used to estimate the area under a curve or to estimate the value of an integral where Calculus can't be applicable.

Note: Accurate area under the curve or a value of an integral is obtained by calculus, i.e integration.

Therefore the error in the value obtained by trapezium rule = value obtained by calculus –value by trapezium rule. Consider a function $y = f(x)$, if the area under the curve is divided into equal trapezoids each of width, h





Total area $= \int_a^b f(x)dx = A_1 + A_2 + A_3 + A_4$

$$A_1 = \frac{1}{2}h(y_1 + y_2)$$

$$A_1 = \frac{1}{2}h(y_2 + y_3)$$

$$A_3 = \frac{1}{2}h(y_3 + y_4)$$

$$A_4 = \frac{1}{2}h(y_4 + y_5)$$

Then the total area $= \frac{1}{2}h(y_1 + 2(y_2 + y_3 + y_4) + y_5)$

$$= \frac{1}{2}h\{y_1 + y_5 + 2(y_2 + y_3 + y_4)\}$$

Hence the general trapezium rule formula can be written as

$$\int_a^b f(x)dx = \frac{1}{2}h\{y_1 + y_n + 2(y_2 + y_3 + y_4 + ... + y_{n-1})\}$$

Note:

1. **The areas $A_1, A_2, A_3, and A_4$ are called strips or sub-intervals**
2. **The x-values, x_1, x_2, x_3, x_4 and x_5 are called ordinates.**
   $x_1 = a, x_2 = a + h, x_3 = x_2 + h$
3. **Number of subintervals =number of ordinates -1**
4. **$h = \frac{b-a}{n}$, where n=number of subintervals or strips.**
5. **h must remain as a fraction if it does not give an exact decimal**
6. **When dealing with trigonometric functions, leave the calculator in radians**

Example 1:

a) Use trapezium rule with 5 subintervals to estimate $\int_0^1 5^{2x}\,dx$, and give your answer correct to 3 decimal places.

b) i) Find the exact value of the $\int_0^1 5^{2x}\,dx$ correct to 3 decimal places

ii) Calculate the percentage error in your estimation in a) above

iii) Suggest how the percentage error may be reduced

soln

Note: y values must be atleast 4 decimal places since the final answer is required to 3 decimal places

a) $h = \frac{1-0}{5} = 0.2$

x	y
0	1
0.2	1.9036
0.4	3.6239
0.6	6.8986
0.8	13.1327
1	25
Sub-totals	26 51.1176

Then $\int_0^1 5^{2x}\,dx = \frac{1}{2}(0.2)\{26 + 2(51.1176)\}$

$=7.71176$

≈ 7.712

b) i) $\int_0^1 5^{2x}\,dx = \left[\frac{5^{2x}}{2\ln 5}\right]_0^1 = \frac{5^2}{2\ln 5} - \frac{1}{2\ln 5} = 7.456$

Error $= |7.456 - 7.712| = o.256$

ii) Percentage error$= \frac{0.256}{7.456} \times 100 = 3.4\%$

iii) Percentage error can be reduced by increasing the number of subintervals

Example:

Use trapezium rule with 6 ordinates to estimate $\int_0^{\frac{\pi}{3}} \tan x dx$ correct to 3 decimal places

soln

$$h = \frac{\frac{\pi}{3} - 0}{6-1} = \frac{\pi}{15}$$

x	y
0	0
$\frac{\pi}{15}$	0.2126
$\frac{2\pi}{15}$	0.4452
$\frac{3\pi}{15}$	0.7263
$\frac{4\pi}{15}$	1.1106
$\frac{\pi}{3}$	1.7321
Sub-totals	1.7321 2.4947

$$\int_0^{\frac{\pi}{3}} \tan x dx = \frac{1}{2}\left(\frac{\pi}{15}\right)\{1.732 + 2(2.4947)\} = 0.703873 \approx 0.704$$

ASSIGNMENT 7.1.10

1. (a) Use trapezium rule with six strips to estimate $\int_0^{\pi} x \sin x dx$ correct to 2dp.

   (b) Determine the percentage relative error in your estimation.

2. Use trapezium rule to estimate the approximate value of $\int_0^1 \frac{1}{1+x^2} dx$ using 6 ordinates correct to 3 decimal places.

3. (a) Use the trapezium rule with six ordinates to evaluate, $\int_0^1 \theta \sin \theta d\theta$ correct to three decimal places.
(b) Find the actual value of, correct to three decimal places.
(c) Calculate the relative error made in (a) above and state how you would reduce such an error.

4. Use trapezium rule with 6- ordinates to estimate the value of $\int_0^{\frac{\pi}{4}} (t + \sin t) dt$ corret to three decimal places.