TOPIC 9: ESTIMATION OF ROOTS OF EQUATIONS

Some equations of the form f(x) =0 have no exact method of being solved. Various methods have been designed to find approximate solutions to the equations.

9.1 Methods of locating roots of equations:

a) **Graphical method:**

In graphical method of locating roots of the equation f(x) =0, we first write it in the form h(x) =g(x). The graphs of y=g(x) and y=h(x) are plotted or sketched on the same axes. The values of x where the graphs meet will be the point where the root lies and it's the approximate root of the equation. Alternatively: The graph of y=f(x) is plotted or sketched. The point where the root cuts the x-axis will be the approximate root of the equation.

Example 1:

Show graphically the root of the equation $x^3+5x^2-3x-4=0$, a root between x=0 and x=-1.

Soln:

Method 1

Let $f(x)=x^3+5x^2-3x-4$

x	-1	-0.8	-0.6	-0.4	-0.2	0
y	3	1.1	-0.6	-2.1	-3.2	-4

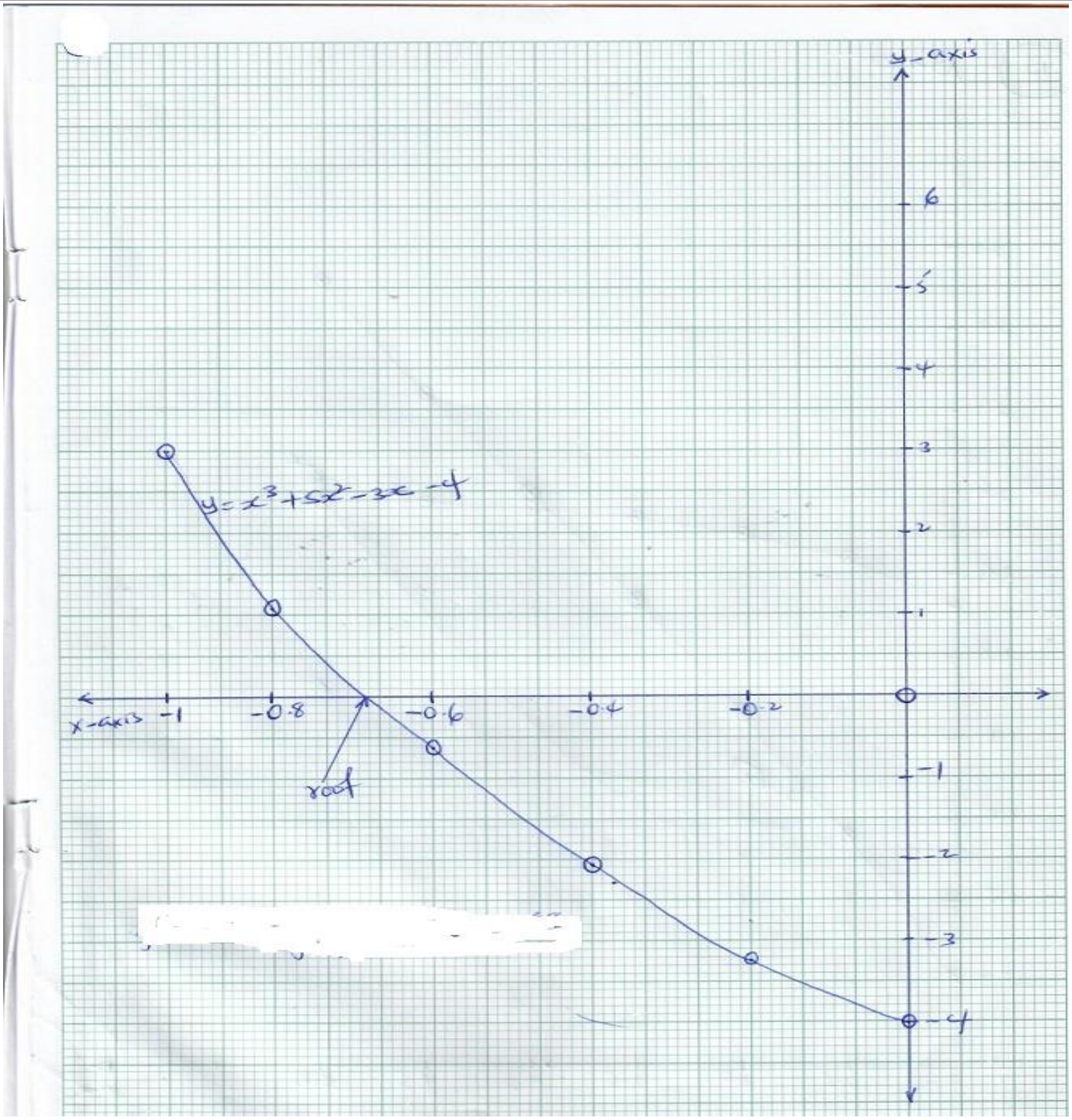
From the graph figure, below $x_0=-0.68$

Method 2

The equation above can be separated into $y_1=x^3+5x^2, and, y_2=3x+4$

The two graphs are plotted the two points where the graphs meet will be the approximate root of the equation, $x^3+5x^2-3x-4=0$

Note: When plotting a curve a minimum of four points is required.


y-axis
x-axis
$y = x^3 + 5x^2 - 3x - 4$
root
-1
-0.8
-0.6
-0.4
-0.2
6
5
4
3
2
1
-1
-2
-3
-4

Example 2:

Using a suitable graph locate roots of the equation $x^3 + x - 4 = 0$

Soln:

Method I

Let $y = x^3 + x - 4$

x	-3	-2	-1	0	1	2	3
y	-34	-14	-6	-4	-2	6	26

b) The sign change method

The sign change method of locating the root of the equation f(x) =0 between a and b. There exists a root between a and b if and only if f(a) and f(b) have opposite signs ie the function changes through a and b

Example:

Show that the equation $x^3 + 2x - 4 = 0$ has a root between $x = 1$ and $x = 2$

$f(x) = x^3 + 2x - 4$

$f(1) = 1^3 + 2(1) - 4 = -1$

$f(2) = 2^3 + 2(2) - 4 = 8$

Since $f(1).f(2) \lhd 0$ then there exists a root between $x = 1$ and $x = 2$

OR Since $f(1)$ and $f(2)$ have different signs then there exists a root between $x = 1$ and $x = 2$

Example:

Show that $x^3 - 4x^2 + 4 = 0$ has one negative root and two positive roots.

Soln:

Since the interval is not given we can use a table

Let $f(x) = x^3 - 4x^2 + 4$

X	-1	0	1	2	3	4
f(x)	-1	4	1	-4	-5	4

Sign changes

Since f(-1).f(0)< 0,$f(1).f(2) < 0$ and $f(3).f(4) < 0$. This implies that the roots of the equation above lie between -1 and 0; 1 and 2; and 3 and 4.

9.2 Finding an approximate root

This can be done using two methods

i) By graphical method (already discussed)
ii) Linear interpolation

Linear interpolation method:

If b is an approximate root of the equation $f(x) = 0$ then $f(b) = 0$. If $< b < c$, then

a	b	c
+ value	0	- value

OR

a	b	c
- value	0	+ value

Example 3:

Show that the equation $3x^2+x-5=0$ has a root between x=1 and x=2 hence use linear interpolation to determine the **initial approximation.**

Soln

Let $f(x)=3x^2+x-5$

$f(1)=3(1)^2+1-5=-1$

$f(2)=3(2)^2+2-5=9$ Since $f(1).f(2)<0$ then there exists a root between x=1 and x=2

Note: For an approximation interpolation is only used once

Then; let the initial approximation, $b=x_0$

x	1	x_0	2
f(x)	-1	0	9

$$\frac{x_0-1}{0-(-1)}=\frac{2-1}{9-(-1)}$$

$$x_0=1.1$$

9.3: Finding the root of the equation

The methods include;

1. Linear interpolation method
2. Iteration methods

Linear interpolation method: Here the linear interpolation method is used more than once until $|f(b)|\leq 0.5\times10^{-n}$ where n is the number of decimal places required.

Example 4: Show that the equation $3x^2+x-5=0$ has a root between x=1 and x=1.5 hence use linear interpolation to determine the root of the equation correct to 2 decimal places.

Soln

Since the accurate root is required linear interpolation should be done more than once.

Let $f(x)=3x^2+x-5$

$f(1)=3(1)^2+1-5=-1$

$f(2)=3(1.5)^2+1.5-5=3.25$ Since $f(1).f(1.5)<0$ then there exists a root between x=1 and x=1.5

Since x=1.5 gives a value far 0 then another value e.g 1.25 (average of 1and 1.5) can be used

$f(1.25)=3(1.25)^2+1.25-5=0.9375$

Either (a)

1	x_0	1.25
-1	0	3.25

OR (b) it is also ok to use 1 and 1.5

1	x_0	1.5
-1	0	9

From a) $\frac{x_0-1}{0--1}=\frac{1.25-1}{0.9375--1}, x_0=1.12903$

Testing:

$f(1.12903)=3\times 1.12903^2+1.12903-5=-0.0468$

Since $f(1.12903).f(1.25)<0$ then

1.12903	x_1	1.25
-0.0468	0	0.9375

$$\frac{x_1 - 1.12903}{0 - -0.0468} = \frac{1.25 - 1.12903}{0.9375 - -0.0468}$$

$x_1 = 1.13478$

Testing:

$f(1.13478) = 3(1.13478)^2 + 1.13478 - 5 = -0.00204$

Since $|f(1.13478| = 0.00204 < 0.005$

$\therefore Root = 1.13$

ITERATIVE METHODS OF SOLVING $f(x) = 0$

Iteration means logical steps. An iterative method is therefore a numerical method which is step by step of obtaining an estimate to a value of the root such that the successive approximations from the sequence converge to a value of the root. The methods include:

i) **the general iterative method**
ii) **the Newton Raphson iterative method**

General iterative method

Here the iterative formula to be used in estimating the root of the equation is obtained by rearranging the terms of the equation and written in the form

$X_{n+1} = g(x_n)$ Where n=0, 1, 2, 3.....

Example 5: Given the equation $x^3 + x - 5 = 0$. We can generate a number of formulas as shown.

a)

$$x^3 + x - 5 = 0$$
$$x = 5 - x^3$$
$$X_{n+1} = 5 - x^3$$

b) $$x^3 = 5 - x$$
$$x = \sqrt[3]{5-x}$$
$$X_{n+1} = \sqrt[3]{5-x_n}$$

c) $$x^2 + 1 - \frac{5}{x} = 0$$
$$x = \sqrt{\frac{5}{x} - 1}$$
$$X_{n+1} = \sqrt{\frac{5}{x_n} - 1}$$

Note: Not all the iterative formulas obtained by rearrangements of terms are suitable to estimate a root therefore test for suitability of a general iterative formula

Test for convergence

Mehod 1

Example 6:

The iterative formulas below are formed by rearranging the equation $x^3 - 10 = 0$

a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$ and b) $X_{n+1} = \frac{10}{x_n^2}$. Starting with an approximation of $x_0 = 2$ use each formula three times to determine the most suitable formula for solving the equation, hence state the root of the equation correct to significant figures.

Solution:

Formula a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$ Formula b) $X_{n+1} = \frac{10}{x_n^2}$

$$X_1 = \frac{2(2)^3 + 10}{3(2)^2} = 2.1667$$

$$X_2 = \frac{2(2.1667)^3 + 10}{3(2.1667)^2} = 2.1544$$

$$X_3 = \frac{2(2.1544)^3 + 10}{3(2.1544)^2} = 2.1544$$

$$X_1 = \frac{10}{2^2} = 2.5$$

$$X_2 = \frac{10}{2.5^2} = 1.6$$

$$X_3 = \frac{10}{1.6^2} = 3.9063$$

Formula a) is the suitable formula because it produces a convergent sequence hence the root is 2.154.

Note: Substitution at every stage must be seen

Method 2

Second test for suitability of an iterative formula. If $|g^1(x_0)| < 1$, it implies that the iterative formula is suitable otherwise not

Example 7: The iterative formulas below are formed by rearranging the equation $x^3 - 10 = 0$

a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$ and b) $X_{n+1} = \frac{10}{x_n^2}$. Taking $x_0 = 2$, **without iteration**, deduce the most suitable formula for solving the equation.

Solution:

Formula a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$

$$\Rightarrow g(x) = \frac{2x^3 + 10}{3x^2} = \frac{2x}{3} + \frac{10}{3x^2}$$

$$g^1(x) = \frac{2}{3} - \frac{20}{3x^3}$$

$$g^1(2) = \frac{2}{3} - \frac{20}{24} = \frac{1}{6}$$

$$|g^1(2)| = \frac{1}{6} < 1$$

Formula b) $X_{n+1} = \frac{10}{x_n^2}$

$$\Rightarrow g(x) = \frac{10}{x^2}$$

$$g^1(x) = \frac{-20}{x^3}$$

$$\left|g^1(2)\right| = \left|\frac{-20}{2^3}\right| = 2.5 > 1$$

Since $\left|g^1(2)\right| < 1$ in a) then formula a) given by $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$ is the more suitable formula for solving the equation.

Example 8: Show that the general iterative formula for solving the equation $x^3 - x - 1 = 0$ is $X_{n+1} = \sqrt{1 + \frac{1}{x_n}}$. Hence taking 1.2 as the first approximation, find the root of the equation correct to 2 decimal places

Solution:

$$x^3 - x - 1 = 0$$

$$x^3 = x + 1$$

$$x^2 = 1 + \frac{1}{x}$$

$$x = \sqrt{1 + \frac{1}{x}}$$

$$X_{n+1} = \sqrt{1 + \frac{1}{x_n}}$$

$$x_1 = \sqrt{1 + \frac{1}{1.2}} = 1.354$$

$$|x_1 - x_0| = 0.154$$

$$x_2 = \sqrt{1 + \frac{1}{1.354}} = 1.3185$$

$$|x_2 - x_1| = 0.0355$$

$$x_3 = \sqrt{1 + \frac{1}{1.3185}} = 1.3261$$

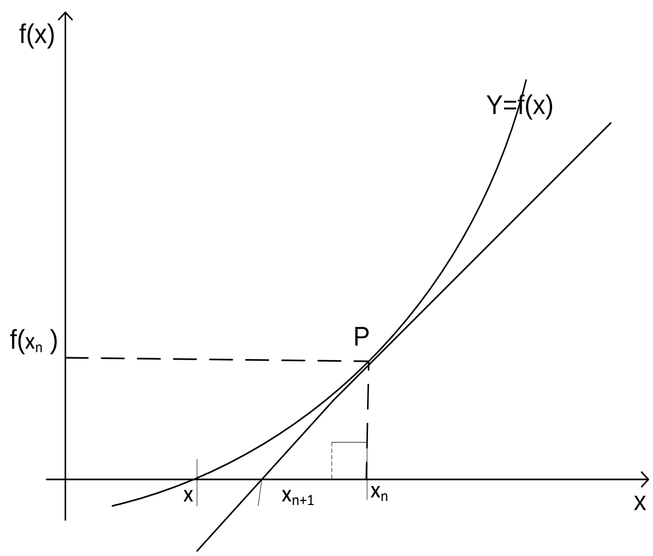
$$|x_3 - x_2| = 0.0076$$

$$x_4 = \sqrt{1 + \frac{1}{1.3261}} = 1.3244$$

$|x_4 - x_3| = 0.0017 \le 0.005$, Hence the root is 1.32

The Newton Raphson method (N.R.M)

Consider an equation f(x) =0, when the graph is plotted it cuts the x-axis at a point (x,0) where x is the root of the equation. A tangent at $P[x_n, f(x_n)]$ is drawn where x_n is an approximation to the root. The tangent meets the x-axis at $(x_{n+1}, 0)$, where x_{n+1} is a better approximation since it is closer to x. Graphically we have



f(x)=0, and x_n is the initial approximation to the root. The gradient of the tangent at P is $f^1(x_n)$ and at the same time as $\tan\theta$ where

$$\tan\theta = \frac{f(x_n) - 0}{x_n - x_{n+1}} = f^1(x_n)$$

Rearranging the equation above we get

$x_{n+1} = x_n - \dfrac{f(x_n)}{f^1(x_n)}$; n = 0, 1, 2,3………. This expression is the Newton Raphson formula

Example 9:

a) Show that the Newton Raphson formula for solving the equation $2x^2 - 6x - 3 = 0$ is $x_{n+1} = \dfrac{2x_n^2 + 3}{4x_n - 6}$, Where n = 0,1,2,…………

b) Show that the positive root for $2x^2 - 6x - 3 = 0$ lies between 3 and 4.
Find the root correct to 2 decimal places

Solution

a) Let $f(x) = 2x^2 - 6x - 3$

$f^1(x) = 4x - 6$

Then $X_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$

$$X_{n+1} = x_n - \frac{2x_n^2 - 6x_n - 3}{4x_n - 6}$$

$$X_{n+1} = \frac{4x_n^2 - 6x_n - 2x_n^2 + 6x_n + 3}{4x_n - 6}$$

$$\therefore X_{n+1} = \frac{2x_n^2 + 3}{4x_n - 6}$$

b)

$$f(3) = 2(3)^2 - 6 \times 3 - 3 = -3$$

$$f(4) = 2(4)^2 - 6 \times 4 - 3 = 5$$

Since $f(4).f(3) < 0$ then there exists a root between 3 and 4

Since $|f(3)| < |f(4)|$ then $x_0 = 3$

$$x \;= \frac{2(3)^2 + 3}{4(3) - 6} = 3.5$$

$$x_2 = \frac{2(3.5)^2 + 3}{4(3.5) - 6} = 3.4375$$

$$x_3 = \frac{2(3.4375)^2 + 3}{4(3.4375) - 6} = 3.4365$$

$$|x_3 - x_2| = |3.4365 - 3.4375| = 0.002 < 0.005$$

$$Root = 3.44$$

Example 10:

Show that the Newton –Raphson Iterative formula for finding the cube root of a number N is given by $X_{n+1}=\frac{1}{3}\left(2x_n+\frac{N}{x_n^2}\right); n=0,1,2.......$ hence find the $(96)^{\frac{1}{3}}$ correct to 3 decimal places

Solution

Let $x=\sqrt[3]{N}$

$$x^3=N$$
$$x^3-N=0$$

$$f(x)=x^3-N$$

$$f^1(x)=3x^2$$

$$X_{n+1}=x_n-\left(\frac{x_n^3-N}{3x_n^2}\right)$$

$$=\frac{3x_n^3-x_n^3-N}{3x_n^2}=\frac{1}{3}\left(2x_n+\frac{N}{x_n}\right)$$; where n=0,1,2,.....

$$X_{n+1}=\frac{1}{3}\left(2x_n+\frac{N}{x_n^2}\right);$$ n=0,1,2,……………………..

$$\sqrt[3]{96}=?$$
$$\sqrt[3]{64}=4<\sqrt[3]{96}=?<\sqrt[3]{125}=5$$

$$f(x)=x^3-N$$
$$f(4)=4^3-96=-32$$
$$f(5)=5^3-96=29$$
$$Since, |f(5)|<|f(4)|, then, x_0=5$$

$$x_1 = \frac{1}{3}\left(2(5) + \frac{96}{5^2}\right) = 4.6133$$

$$x_2 = \frac{1}{3}\left(2(4.6133) + \frac{96}{4.6133^2}\right) = 4.57911$$

$$x_3 = \frac{1}{3}\left(2(4.57911) + \frac{96}{4.57911^2}\right) = 4.5789$$

$$|x_3 - x_2| = 0.00021 < 0.0005,$$

$$\therefore Root = 4.579$$

Example 11:

a) Draw on the same axes the graphs of $y = 2 - e^{-x}$ and $y = \sqrt{x}$ for $2 \le x \le 5$

b) Determine from your graphs the interval within which the root of the equation $e^{-x} + \sqrt{x} - 2 = 0$ lies. Hence, use Newton Raphson's method to find the root of the equation correct to 3 decimal places.

Solution:

Let $y_1 = 2 - e^{-x}$ **and** $y_2 = \sqrt{x}$

x	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**
y_1	**1.86**	**1.92**	**1.95**	**1.97**	**1.98**	**1.99**	**1.99**
y_2	**1.41**	**1.58**	**1.73**	**1.87**	**2.00**	**2.12**	**2.24**

From the graph, the roots of the equation lies between x=3.5 and x=4.0.

For the hence part:

$$f(x) = e^{-x} + \sqrt{x} - 2, \Rightarrow f'(x) = e^{-x} + \frac{1}{2\sqrt{x}} = \frac{2e^{-x}\sqrt{x} + 1}{2\sqrt{x}}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2,3...............$$

$$x_{n+1} = x_n - \frac{\left(e^{-x_n} + \sqrt{x_n} - 2\right) \times \sqrt{x_n}}{2e^{-x_n}\sqrt{x_n} + 1}$$

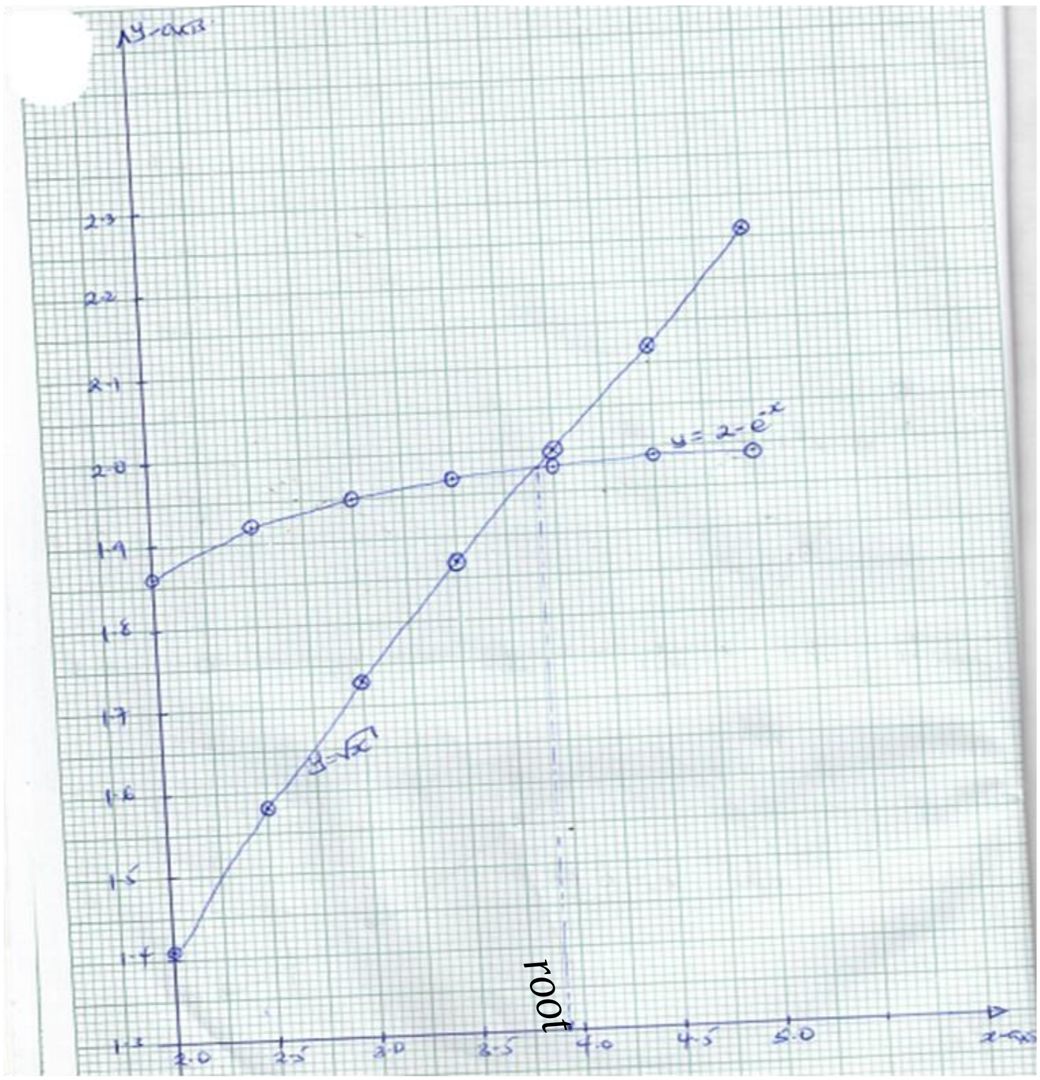
The initial approximation of the root $x_0 = 3.925$

$$x_1 = 3.925 - \frac{\left(e^{-3.925} + \sqrt{3.925} - 2\right) \times 2\sqrt{3.925}}{2e^{-3.925} \times \sqrt{3.925} + 1} = 3.92168$$

$$x_2 = 3.92168 - \frac{\left(e^{-3.92168} + \sqrt{3.92168} - 2\right) \times 2\sqrt{3.92168}}{2e^{-3.92168} \times \sqrt{3.92168} + 1} = 3.9212$$

$$|x_2 - x_1| = |3.9212 - 3.92168| = 0.00048 < 0.0005$$

$$\therefore root = 3.921$$



Note:

- The axes must be labeled
- The graphs must be labeled

- **Use of uniform scale**

ASSIGNMENT 9.1.12

1. (a) Using the same graph, Show that the curves $2e^x$ and $4-x^2$ have two real roots.
   (b) Using the Newton – Raphson formula thrice, find the positive root of the equation $2e^x + x^2 = 4$, giving your answer correct to two decimal places.

2. Show that the equation $2x^3 - \frac{18}{x} + 2 = 0$ has two real roots in the interval $|x| < 3$. Use linear interpolation to find the least root , correct to one decimal place.

3. (a) Derive the iterative formula based on Newton- Raphson process for obtaining the reciprocal of a number N.

(b) Hence, using N-R process once, and taking x_0 to be 0.1, find the approximate value of $\frac{1}{11}$, correct to 2 significant figures.

4. a) Locate graphically the smallest positive real root of $\sin x - \ln x = 0$

b) Use Newton Raphson method to approximate this root of the equation in a) above correct to 3 dps.

5. By plotting the graphs of $y = \ln x$ and $y = 2 - x$, Show that there is a common root in the interval 1<x<2, obtain from your graph the approximate root of the equation $\ln x + x - 2 = 0$, correct to one decimal place.

6. Show graphically that the equation $\pi \sin x = x$ has three real roots. (hint: Plot graphs of y=sinx and $y = \frac{x}{\pi}$. Use intervals of π)

7. If α is an approximate root of the equation $x^2 = n$. Show that the iterative formula for finding the root reduces to: $\frac{\frac{n}{\alpha} + \alpha}{2}$ hence, taking $\alpha = 4$, estimate $\sqrt{17}$, correct to 3 decimal places.

8. (a) Show that the root of the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ lies between 1 and 2.

(b) Use the Newton Raphson method to find the root of the equation above correct to 2 dps.

9. (a) If a is an approximate root of the equation $x^5 - b = 0$, Show the second approximation is given by $\frac{4a + b/a^4}{5}$

b) Show that the positive real root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8. hence use the formula in a) above to determine the root to three decimal places