

LOGARITHMS, SURDS AND INDICES (INTRODUCTION)

Item 1

A small rural community depends on two wells, A and B. Due to agricultural runoff, the nitrate concentration in Well A is x (mg/L), and in Well B is y (mg/L). Water is safe if nitrate concentration ≤ 10 mg/L. The following is known: $\log_8 x + \log_{64} y^2 = 2$ and $\log_x 8 + \log_4 y = \frac{5}{3}$

Task

Find the values of x and y to determine if either well is unsafe and make any Recommendations.

Item 2

An architect, Lydia, is designing a regular hexagonal pavilion for a school in Entebbe. She observes that for a prototype pavilion, the total boundary length is *directly proportional to the square of the side length*. The prototype pavilion has side length $s = \frac{4+\sqrt{14}}{3}$ metres.

Lydia plans to construct a larger pavilion with each side measuring 18 metres, while maintaining the same proportional relationship.

To pave the floor, she uses identical hexagonal tiles. Each tile has a side length of $\frac{\sqrt{75}}{5}$ metres.

During excavation, a smart underground device reveals a hidden coordinate system (x, y) governed by the following logarithmic relations:

$$\log_2(x + 4) + \log_2(y) = 5, \quad \log_2(y - 2) = \log_2(x - 1).$$

Lydia realizes that the solution (x, y) determines how the tiles should be arranged within the pavilion.

Tasks

- Determine the constant of proportionality in its simplest surd form using the prototype pavilion, and hence find the exact perimeter of the new pavilion.
- Simplify the tile side length $\frac{\sqrt{75}}{5}$, determine the exact area of one tile, and calculate the number of tiles required to cover the pavilion floor.
- Solve the system of logarithmic equations to find (x, y) :

$$\log_2(x + 4) + \log_2(y) = 5, \quad \log_2(y - 2) = \log_2(x - 1).$$

- If x represents the number of tiles along one direction and y represents the number of rows, determine whether the pavilion can be completely tiled without cutting any tiles. Justify your answer.

Item 3

In a crowded refugee camp, three groups a , b , c represent daily contact rates among children, adults, and elderly. Epidemiologists define:

$$x = \log_a(bc), \quad y = \log_b(ca), \quad z = \log_c(ab)$$

They also observe that the infection rate satisfies:

$$\log_c(a + b) = 2 \log_c a + \log_c \left(1 + \frac{3b}{a} + \frac{b^2}{a^2} \right)$$

$$\log(a + b) = \log 3 + \frac{1}{2}(\log a + \log b)$$

Tasks

- Prove that $x + y + z = xyz - 2$ and $\log(a + b) - 1 = \log a + \log b$.
- If $x = 2$, find a, b and c hence determine the order in which priorities should be given.

Item 4

A sound engineer is testing two speakers in a concert hall. The intensity of Speaker A is $I_A = I_0 \cdot 2^{3/2}$ and the intensity of Speaker B is $I_B = I_0 \cdot 3^{1/2}$, where I_0 is the reference intensity (a constant).

The sound intensity level in decibels (dB) for any intensity I is given by $L = 10 \log_{10} \left(\frac{I}{I_0} \right)$.

Tasks

a) Write the ratio $\frac{I_A}{I_B}$ in simplest surd form.

b) Find the difference $L_A - L_B$ in terms of \log_2 and \log_3 , hence show that $L_A - L_B = 5[\log I_A + \log 6 - \log I_B]$.

c) The engineer wants to combine the two speakers. The total intensity is $I_{\text{total}} = I_A + I_B$.

Express I_{total}/I_0 in the form $\sqrt{m} + \sqrt{n}$, where m and n are integers. Then write $\log_{10} \left(\frac{I_{\text{total}}}{I_0} \right)$ as a single logarithm with its surd.

Item 5

Three neighboring villages p, q, r share food resources. An aid worker defines:

$$a = \log_p(qr), \quad b = \log_r(pq), \quad c = \log_q(pr)$$

She also notes seasonal changes: $\log_a \left(1 + \frac{1}{6} \right) = l$, $\log_a \left(1 + \frac{1}{10} \right) = m$, $\log_a \left(1 + \frac{1}{18} \right) = n$. The food stock x (in tons) satisfies: $\log_5 x - \log_x 25 = \log_5 (390625x)$

Tasks

a) Prove $abc = a + b + c + 2$ and $\log_a \left(1 + \frac{41}{190} \right) = l + m - n$.

b) If there are 5000 people, and it is known that 0.3 tons are enough for 1800 people, is there enough food?

Item 6

A young engineer, Amina, is tasked with designing a smart greenhouse in Wakiso district. She decides to base the rectangular foundation on a special proportion known as the *Eco Ratio*, defined as:

The ratio of the perimeter of the rectangle to three times its width.

She is given a prototype greenhouse with dimensions

$$\frac{3 + \sqrt{13}}{2} \text{ metres (width) and } 2 \text{ metres (length).}$$

Amina wants her actual greenhouse to have a width of 30 metres, while maintaining the same Eco Ratio.

To cover the greenhouse floor, she plans to use rectangular solar tiles. Each tile has a width of

$$\frac{\sqrt{27}}{3} \text{ metres.}$$

However, during excavation, workers discover an old encoded panel beneath the site. The panel contains a coordinate grid (x, y) and the following system of exponential equations:

$$3^{x+1} + 3^y = 108, \quad 3^y = 3^{x-1} + 18.$$

Amina suspects that the solution (x, y) determines how the tiles should be arranged.

Tasks

(a) Determine the Eco Ratio in its simplest form using the prototype greenhouse, then find the exact length of Amina's greenhouse in its simplest form.

- (b) Simplify the tile width $\frac{\sqrt{27}}{3}$, determine the exact length of each tile using the Eco Ratio, and calculate the total number of tiles needed to cover the greenhouse.
- (c) Solve the system of exponential equations to find the coordinates (x, y) :

$$3^{x+1} + 3^y = 108, \quad 3^y = 3^{x-1} + 18.$$

- (d) If x represents the number of dozens of tiles in each box and y represents the number of boxes of tiles, determine whether the greenhouse can be fully tiled without buying any more tiles. Justify your answer.

Item 7

In a farming village, a group of young people use a solar-powered computer system to keep track of how much information (in gigabytes) they store every month. The amount of data after t months is given by the rule

$$P(t) = 2^{0.5t+3},$$

one evening, the group checks a computer report and finds the following log entry;

$$\log P(t) = \log 8 + \log 10^{\sqrt{t}}.$$

Later, one of the students, finds another part of the report that includes this strange expression

$$\sqrt{6 + \sqrt{2}} + \sqrt{22 - 4\sqrt{10}}.$$

He is to simplify it into a difference of two different square roots.

- (a) Help the group to interpret the log entry by giving them the exact values of t to the nearest years.
- (b) Write down the simplified expression as the summation of two positive square roots hence write down the product of the numbers in the square roots.

Item 8

A chemical reaction is monitored using a sensor. The intensity I of the reaction after t seconds is modelled by $I = \frac{6\sqrt{2} \cdot 4^{t/3}}{\sqrt{2^t}}$.

- a) Simplify the expression for I , writing it in the form $I = A \cdot 2^{kt}$, where A and k are constants to be determined.
- b) The reaction becomes unstable when the intensity exceeds 96. Find the time t at which instability first occurs.
- c) The sensor records the reaction strength S , defined by $S = \log_{10}(I - 4)$. Find the value of S at the time when instability first occurs.
- d) Due to calibration limits, the sensor only functions when

$$S \geq 1.$$

Find the range of values of t for which the sensor readings are valid.

- e) Explain why the model is not valid for all real values of t , making reference to both the algebraic form of I and the logarithmic expression for S .

Item 9

A civil engineer, Daniel, is designing a rectangular solar farm in Mukono district. The layout must follow a special design rule:

The total boundary length of the rectangle is equal to four times its length plus twice its width.

He is given a prototype solar field with dimensions

$$\frac{5 + \sqrt{21}}{2} \text{ metres (length) and } 3 \text{ metres (width).}$$

Daniel plans to construct a larger solar farm with a width of 36 metres, while maintaining the same design rule.

To cover the farm, he uses rectangular solar panels. Each panel has a width of

$$\frac{\sqrt{48}}{4} \text{ metres.}$$

While preparing the land, an underground digital sensor reveals a hidden coordinate system (x, y) governed by the following exponential relations:

$$4^x + 4^{y-1} = 80, \quad 4^y = 4^{x-2} + 255.$$

A message appears on the sensor screen:

“Only a precise balance of dimensions and growth reveals the true energy grid.”

Daniel realizes that the solution (x, y) determines how the solar panels should be arranged.

Tasks

- Using the prototype solar field, express the design rule in its simplest form, then determine the exact length of the new solar farm.
- Simplify the panel width $\frac{\sqrt{48}}{4}$, determine the exact length of each panel using the same design rule, and calculate the total number of panels required.
- Solve the system of exponential equations to find (x, y) :

$$4^x + 4^{y-1} = 80, \quad 4^y = 4^{x-2} + 255.$$

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