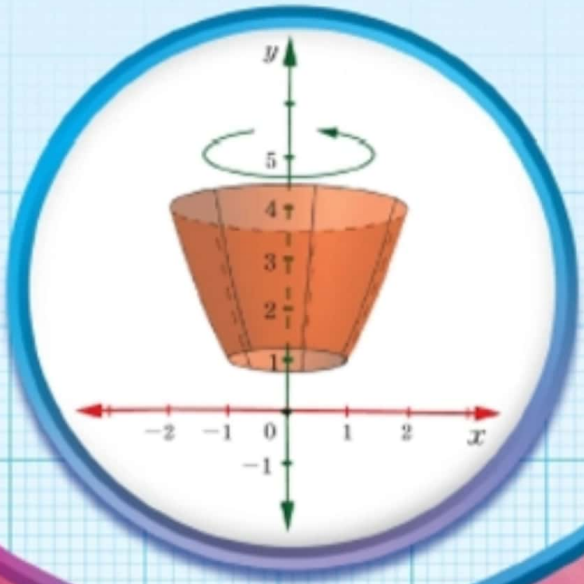
Mathematics

for Advanced Secondary Schools

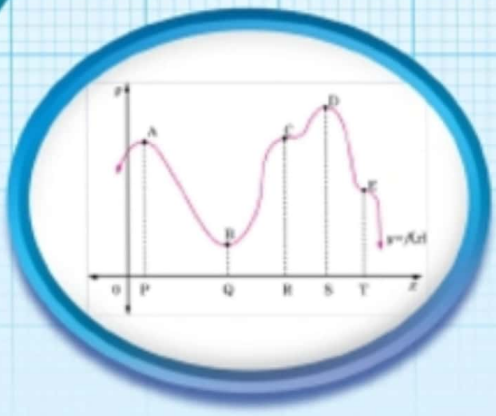
Form Five

Teacher's Guide



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



Tanzania Institute of Education

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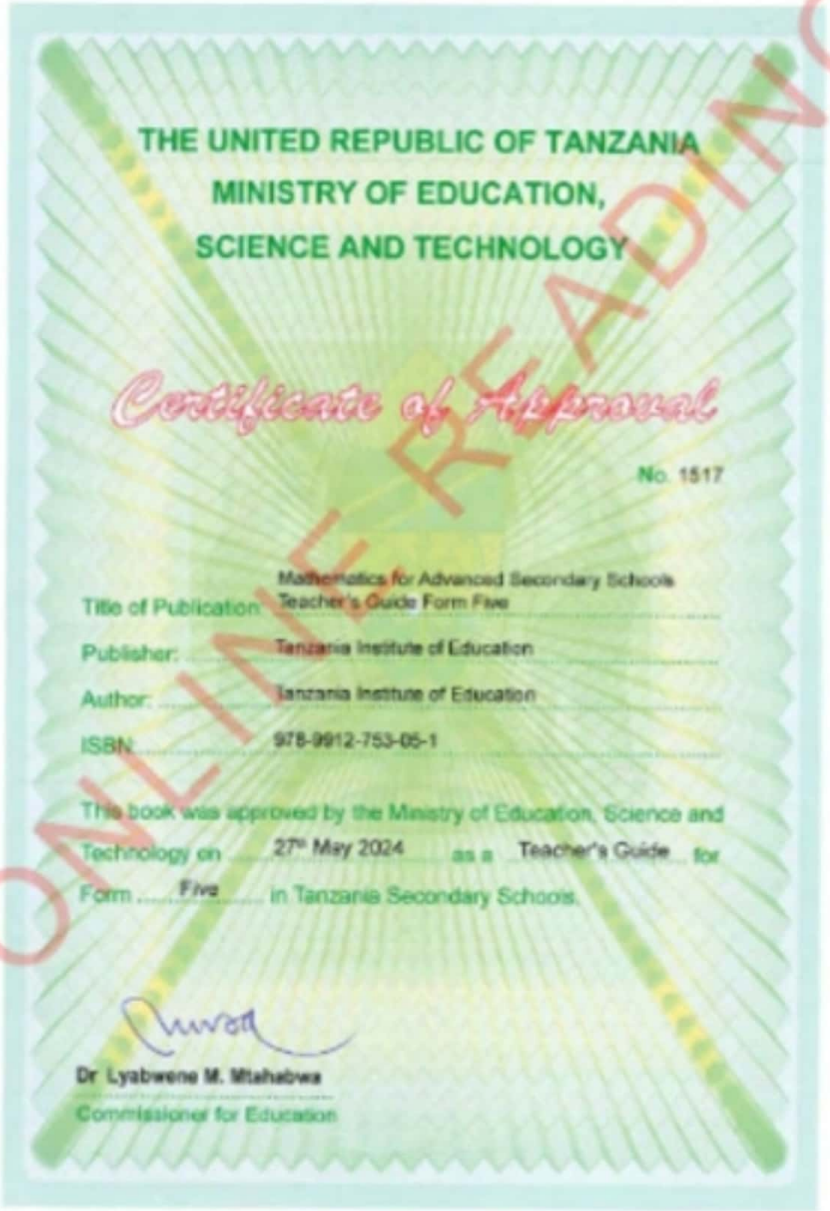

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Commissioner for Education

Tanzania Institute of Education

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Dr Aneth A. Komba
Director General
Tanzania Institute of Education

Introduction

This Teacher's Guide, *Mathematics for Advanced Secondary Schools Form Five* has been written to help the teacher to effectively guide the students to use their textbook, which is in-line with the 2023 Mathematics Syllabus for Advanced Secondary Education Form V-VI, issued by the Ministry of Education, Science and Technology (MoEST).

The guide consists of nine chapters (9), namely: Sets, Logic, Coordinate geometry I, Functions, Algebra, Trigonometry, Linear programming, Differentiation, and Integration. It contains engaging student's activities, suggested teaching and learning resources as well as teaching steps. In addition to what is provided in this Guide, you are strongly advised to use ICT tools, and adapt methods according to the resources available in your environment to enhance the teaching and learning processes.

Additional learning resources are available in the TIE e-library at https://ol.tie.go.tz or tie.go.tz

Tanzania Institute of Education

Chapter One

Sets

Introduction

In this chapter, students will learn methods of representing sets, types of sets, basic set operations, fundamental laws of the algebra of sets, and Venn diagrams. Guide the students to realize some common methods of representing set activities, types of sets, laws of the algebra of sets and operating sets using Venn diagrams in real life. Since students have already learnt the basic concepts of sets from ordinary level, the concept of sets may not be new to them. Use several activities which engage them through the use of past knowledge of the concept of sets towards developing the new knowledge of categorizing such concepts into major branches as well as applications of Mathematics. The competencies developed will enable them to perform various tasks such as organizing, creating and categorizing objects.

Students' activities

(a) Performing set operations on three related sets
(b) Simplifying set expressions
(c) Determining the number of elements of a given set

Teaching and learning resources

Real objects, pictures, playing cards, manila cards, marker pens, coloured chalks, mathematics software, flip charts, pencils, masking tapes, glue, mathematical sets, and ruler.

Methods of representing sets

Teaching steps

1. Guide students through discussion to brainstorm on the basic concept of sets from their learning experience. Ask students to visit nearby real environment and search for different types of objects.
2. Design hands on activities and ask students to discuss the three methods of representing sets, descriptive, roster method, and set builder notation. Encourage students to explore additional resources, such as online tutorials or reference books, to deepen their understanding and further practice.
3. Use Examples 1.1 and 1.2 in the Student's Book to guide students to discuss methods of representing sets.
4. Instruct students to attempt Exercise 1.1 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide constructive feedback.

Answers to Exercise 1.1

1. (a) $A = \{\text{Cubes of all positive integers}\}$ or
   $\{\text{Cubes of all natural numbers}\}$
   (b) $B = \{\text{Multiples of 3}\}$
   (c) $C = \{\text{All integers from } -4 \text{ to 4 inclusive}\}$
   (d) $D = \{\text{Domestic animals}\}$
2. (a) $A = \{x : x^2, x \in \mathbb{N}\}$ (e) $E = \{x : x \text{ is a positive even number}\}$
   (b) $B = \{x : x^3, x \in \mathbb{N}\}$ (f) $F = \{x : x \in W\}$
   (c) $C = \{x : x = -3 \text{ and } x = 3\}$.
   (d) $D = \{x : x \in \mathbb{N}\}$ (g) $\left\{x : x = \frac{n}{n+1}, \text{ for } 1 \le n \le 7, \ n \in \mathbb{N}\right\}$
3. (a) $A = \{-1, 1\}$ (b) $B = \{3, 4, 5, 6, ...\}$ (c) $C = \{1, 3\}$

Types of sets

Teaching steps

1. Guide students in groups, to discuss various types of sets, including the empty or null set, equal sets, equivalent sets, subsets, finite and infinite sets, proper and improper subsets, supersets, power sets, singleton set, and universal set.
2. Design activities to empower students' understanding of types of sets.
3. Encourage students to explore more explanations on types of sets from additional resources, such as online tutorials, or reference books in order to enrich their understanding and practice further.
4. Guide students to use Examples 1.3 to 1.13 in the Student's Book to discuss various types of sets.
5. Instruct students to attempt Exercise 1.2 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide constructive feedback if necessary.

Answers to Exercise 1.2

1. (a) Infinite set – Number of all plants on the earth is uncountable
   (b) Infinite – Real numbers between 10 and 30 are uncountable
   (c) Infinite – Non odd numbers between 10 and 20 inclusive are uncountable
2. T and R are equivalent
   T and R are equal
   H and T are unequal
   H and R are unequal
3. (a) True (b) True (c) True (d) False (e) False

4. (a) $\{\ \}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$.

   (b) $n(K) = 16$.

5. (a) $P(J) = \{\{\ \}, \{Dog\}, \{Cat\}, \{Lion\}, \{Zebra\}, \{Dog, Cat\}, \{Dog, Lion\}, \{Cat, zebra\}, \{Dog, zebra\}, \{Cat, Lion\}, \{Lion, zebra\}, \{Dog, Cat, Lion\}, \{Dog, Cat, Zebra\}, \{Cat, Lion, Zebra\}, \{Dog, Lion, Zebra\}, \{Dog, Cat, Lion, Zebra\}\}$

   (b) $n[P(J)] = 16$

6. (a) $U = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 17, 19\}$

   (b) $U = \{x : x \text{ is an integer}\} = \mathbb{Z}$.

7. (a) $A = \{\ \}$ (b) $B = \{-5\}$

Basic set operations

Teaching steps

1. Design hands on activities that allow students to recognize the basic set operations.
2. Guide students through discussion to recognize the basic set operations namely, union of sets, intersection of sets, complement of a set, difference of sets, and symmetric difference.
3. Ask students to explore more on basic set operations from additional resources such as online tutorial or reference books to foster their understanding and further practice.
4. Use Examples 1.14 to 1.23 in the Student's Book to guide students through discussion to practice basic sets operations.
5. Assign students to attempt Exercise 1.3 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide constructive feedback. Encourage

students to explore additional resources, such as online tutorials, or reference books, to enrich their understanding and practice further.

Answers to Exercise 1.3

1. $A \cup B = \{a, b, c, d, e, f, g, h\}$ $A \cap B = \{a, b, c, d, e\}$
2. $A \cup B = \{\text{Counting numbers}\}$
   $A \cap B = \{\text{Even numbers}\}$
3. $G \cup H = \{20, 25, 30, 45\}$
   $G \cap H = \{25\}$
4. $J \cup K = \{0, \Delta, 3\}$
   $J \cap K = \{\Delta\}$
5. $A \cup B = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 44, 48, 54\}$
   $A \cap B = \{12, 24, 36\}$
6. $W \cup Z = \{14, 16, 18, 20\}$
   $W \cap Z = \{\ \}$ or $\varnothing$
7. $A \cup B = \{94, 110, 120, 131, 140, 265\}$
   $A \cap B = \{94, 110\}$
8. $A \cup B = \{2, 3, 5\}$
   $A \cap B = \{3\}$
9. $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26\}$
   $A \cap B = \{6, 12, 18, 24\}$
10. $A \cup B \cup C = \{a, b, c, d, e\}$
    $A \cap B \cap C = \{\ \}$

11. $M \cap N = \{\ \}$

12. Disjoint sets

13. $M' = \{5,6,7,8,9,10,11,12\}$
    $N' = \{2,4,5,6,8,10,12\}$

14. $A = \{\text{Orange, cabbage, pineapple}\}$

15. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

16. (a) $A \cap B' = \{3,4\}$ (c) $B' \cap A' = \{9,10, 11, 12, 13\}$
    (b) $B \cup A' = \{5, 6, 7, 8, 9, 10,11,12,13\}$ (d) $A' \cap B = \{7,8\}$

17. (a) F (b) T (c) F (d) F

18. $A' = \{\text{magenta, red, blue}\}$

19. (a) $A \cap B = \{a,d,e\}$ (d) $A \cap B' = \{b,c,f\}$
    (b) $A \cup B = \{a,b,c,d,e,f\}$ (e) $A' \cap B' = \{g, h, i, j\}$
    (c) $A' \cap B = \{\ \}$ (f) $A' \cup B' = \{b, c, f, g, h, i, j\}$

20. (a) $A \cup F = \{22, 24, 26, 28, 30\}$ (b) $Y \cup W = \{2 < x < 23\}$
    $A \cap F = \{24, 26\}$ $Y \cap W = \{7 \le x \le 18\}$

21. (a) $A - B = \{13\}$ (b) $B - A = \{16,17,18\}$

22. (a) $M - K = \{32, 42\}$ (b) $K - M = \{33,35,37\}$
    (c) $(M - K) \cup (K - M) = \{32,33,35,37,42\}$

23. (a) $P - R = \{8,10,12\}$ (b) $R - P = \{14,16,18\}$
    (c) $Q - P = \{14,16,18,20,22\}$

24. (a) $B - A = \{3,5,9\}$ (b) $(A - B)' = \{1, 3, 5, 9, 11, 17, 19\}$

25. $X - Y = \{2,4,6,8,10\}$

26. $x = 9,\ y = 12$ or $x = 12,\ y = 9.$

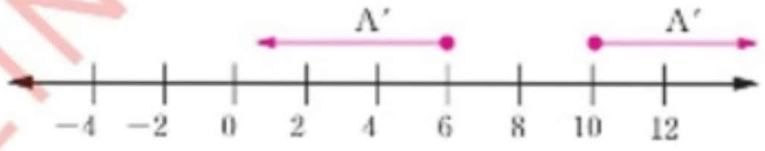
Representation of sets on a number line

Teaching steps

1. Prepare hands on activities and guide students through discussion to demonstrate representation of sets on a number line. Ask them through discussion to explore more on representation of sets on a number line using mathematics software such as Symbolab, GeoGebra, or reference books.
2. Use Table 1.1 in the Student's Book to assist students through discussion to identify forms of interval on representation of sets on a number line.
3. Use Examples 1.24 to 1.26 in the Student's Book to guide students through discussion to represent sets on a number line.
4. Assign the students to attempt Exercise 1.4 in the Student's Book. Advise students to submit their work, check the correctness of their answers and provide constructive feedback. Encourage students to explore additional resources, such as online tutorials, or reference books, to enrich their understanding and practice further.

Answers to Exercise 1.4

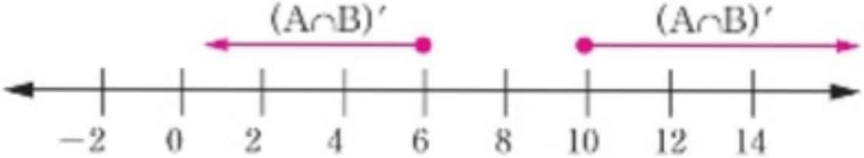
1. (a) A'



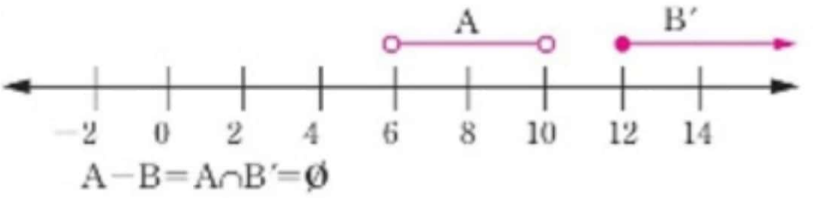
(b) B'

(c) $(A \cup B)'$

(d) $(A \cap B)'$

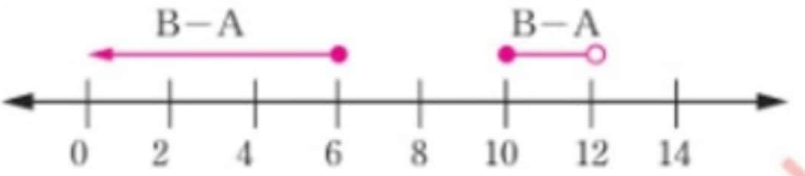


(e) $A - B$

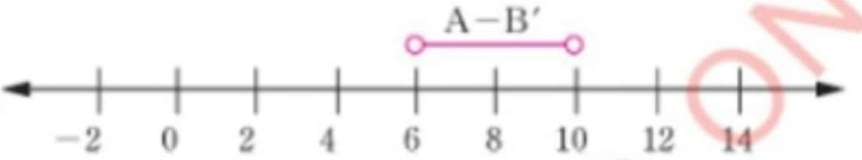


$A - B = A \cap B' = \emptyset$

(f) $B - A$

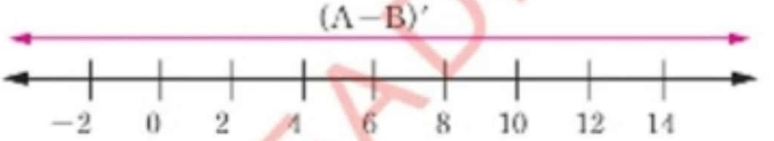


(g) $A - B'$

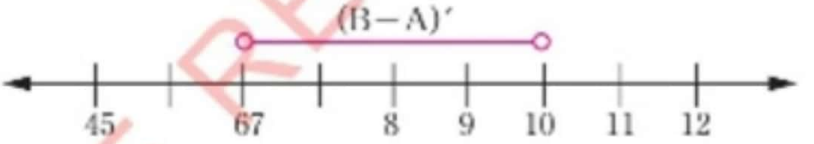


(h) $(A - B')'$

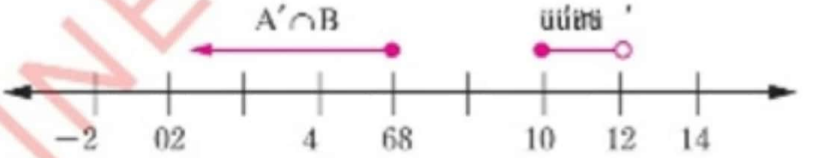
(i) $(A - B)'$



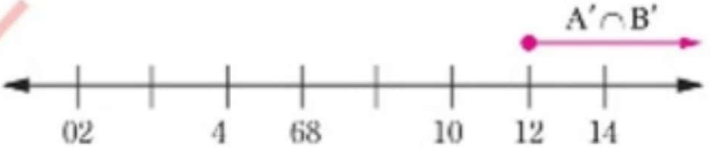
(j) $(B - A)'$



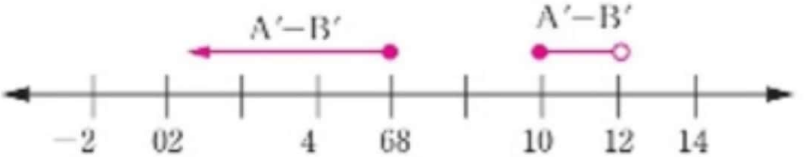
(k) $A' \cap B$



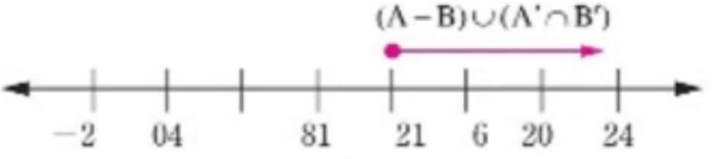
(l) üúúüü '



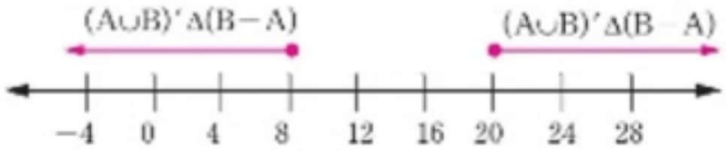
(m) $A' - B'$



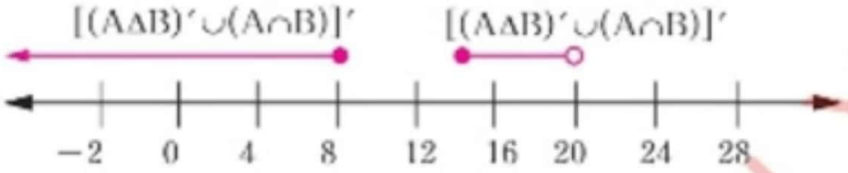
(n) $(A - B) \cup (A' \cap B')$



2. (g) $(A \cup B)' \Delta (B - A)$

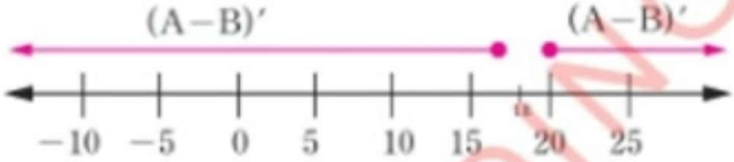


(h) $\left[(A \Delta B)' \cup (A \cap B)\right]'$

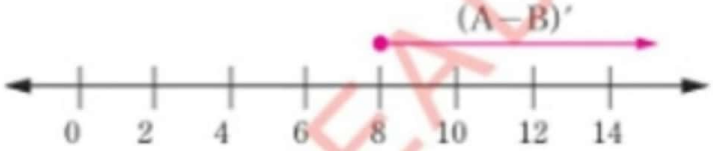


(j) $(A' \Delta B)'$

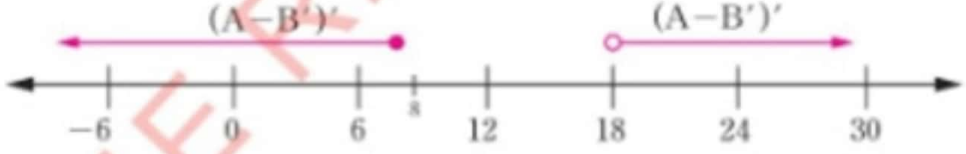
3. (a) $(A - B)'$



(b) $(B - A)'$



(c) $(A - B')'$



Fundamental laws of the algebra of sets

Teaching steps

1. Design hands on activities and guide students through discussion to engage them to recognize the fundamental laws of the algebra of sets. Ask them through discussion to explore more on fundamental laws of algebra of sets from reference books.
2. Assist them through discussion to use Table 1.2 of laws of algebra of sets. Engage students through discussion to prove the fundamental laws of algebra of sets.

3. Guide student through discussion to perform the activity in the Student's Book of identifying associative laws of sets. Ask them to design an activity of identifying any law of algebra of sets of their choice. Advise students to share their findings with other groups for more inputs.
4. Engage the students to discuss Examples 1.27 to 1.29 in the Student's Book on using fundamental laws of algebra of sets to simplify sets algebraic expressions.
5. Assign students to attempt Exercise 1.5 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide constructive feedback.

Answers to Exercise 1.5

3. (a) $A \cup (B \cap C)$ (b) $(A \cup B)'$ (c) A

(d) $\varnothing$ (e) $A' \cup B$ (f) $\varnothing$

(g) $A \cup B'$ (h) $\varnothing$ (i) B (j) $\varnothing$

Venn diagrams

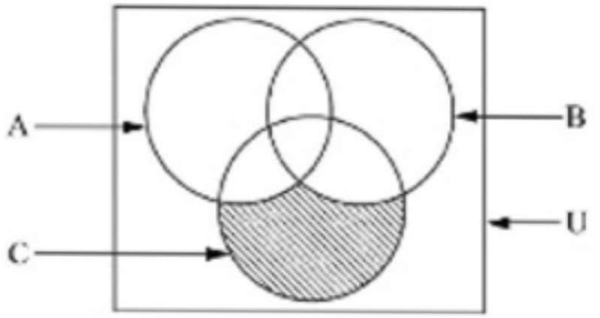
Teaching steps

1. Devise a strategy which will engage students to recognize Venn diagram of three sets. Design hands on activities that will actively engage students in demonstrating overlapping circles or ovals inscribed in a rectangle which represents a universal set. Advise students to share their work for more inputs.
2. Guide students through discussion to perform operations of sets using Venn diagrams. Assist students through discussion to depict the intersection, union, symmetric difference, relative difference, and complement of a set by shading the required region.

3. Use Example 1.30 in the Student's Book to assist students on representing sets in a Venn diagram.
4. Assign the students to attempt Exercise 1.6 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide constructive feedback.

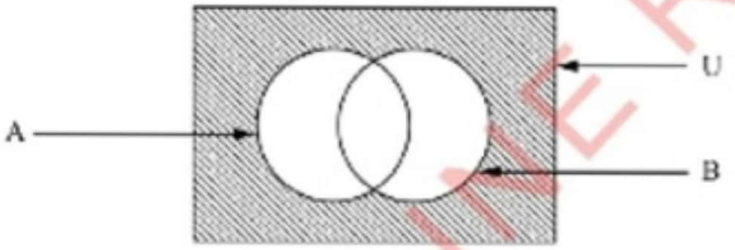
Answers to Exercise 1.6

1.



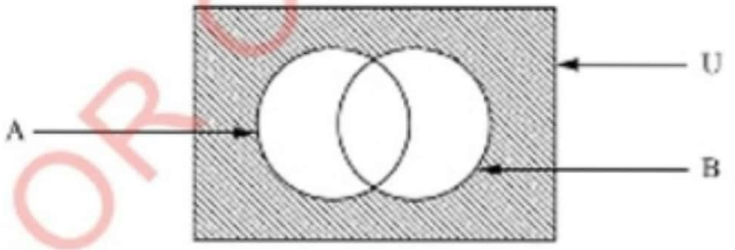
$(A \cup B)' \cap (A \cup C)$

2.



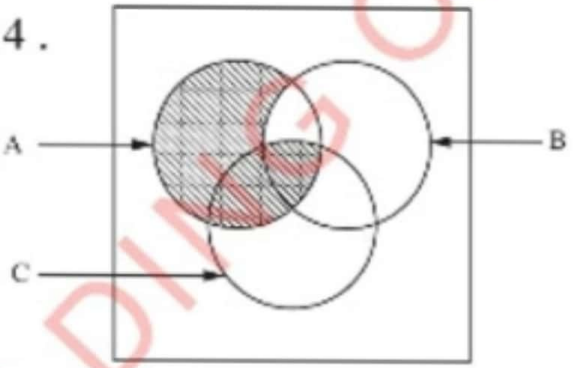
$(A \cup B)'$

3.



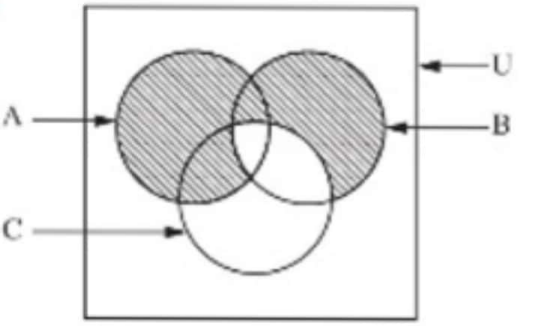
$A' \cap B'$

4.



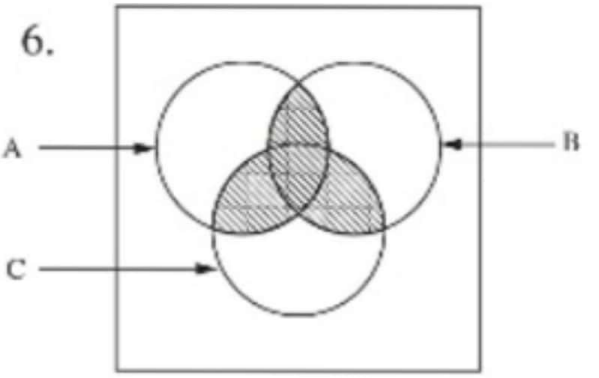
$A \cap B \cap C$

5.



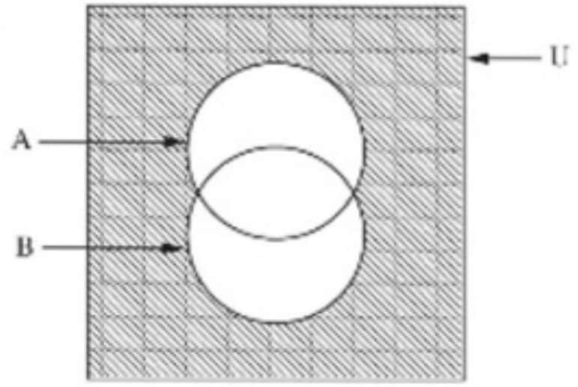
$(A - B) \cup (A - C) \cup (B - C)$

6.



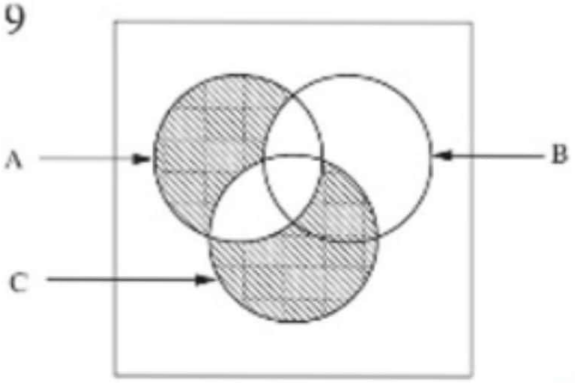
$(A \cap B) \cup (A \cap C) \cup (B \cap C)$ is shaded

7.



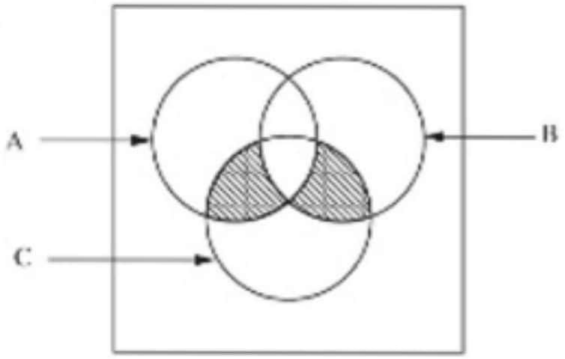
$(A \cup B)' \cap (A' \cap B')$

9



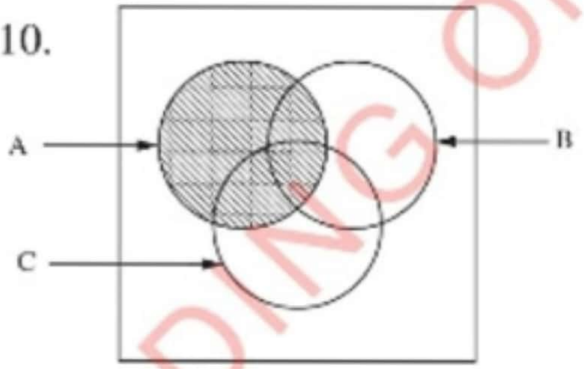
$(A - B) \Delta C$

8.



$(A \Delta B) \cap C$

10.



$A - (B \cap C)$

Number of elements/cardinality of sets

Teaching steps

1. Design hands on activities to empower students understanding on how to recognize the cardinality of sets. Assist students through discussion to derive the general formula by considering two arbitrary finite sets as shown in Figure 1.10 in the Student's Book.
2. Use Examples 1.31 and 1.32 in the Student's Book to engage students through discussion to perform calculation on cardinality of two sets.
3. Assist students through discussion to derive the cardinality of union of three sets by considering the cardinality of two sets. Allow students to share alternative methods of deriving the cardinality of the union of three sets.

4. Engage different hands on activities that will enable students to explore real life problems through Examples 1.33 to 1.38 in the student's Book.
5. For further discussions instruct students to attempt Exercise 1.7 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide constructive feedback.

Answers to Exercise 1.7

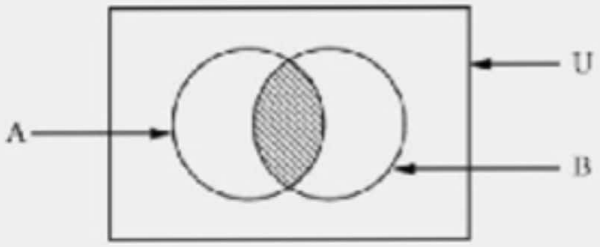
1. (a) $n(A \cap B) = 3$ (c) $n(A \cup B) = 5$
   (b) $n(A \Delta B) = 2$ (d) $n(A - B) = 1$
2. (a) $n(A \cap B) = 4$ (c) $n(A \cup B) = 10$
   (b) $n(A \Delta B) = 6$ (d) $n(A - B) = 4$
3. (a) $n(A \cap B) = 0$ (c) $n(A \cup B) = 3$
   (b) $n(A \Delta B) = 3$ (d) $n(A - B) = 1$
4. (a) $n(A \cap B) = 1$ (c) $n(A \cup B) = 3$
   (b) $n(A \Delta B) = 2$ (d) $n(A - B) = 0$
5. (a) $n(A \cap B) = 1$ (c) $n(A \cup B) = 2$
   (b) $n(A \Delta B) = 1$ (d) $n(A - B) = 1$
6. $n(A \cap B)' = 7$ 7. (a) n(A) =150 (b) n(B) = 100
8. Who had three diseases = 20 patient
9. (a) 2 (b) 119 (c) 126 (d) 129

Answers to Revision Exercise 1

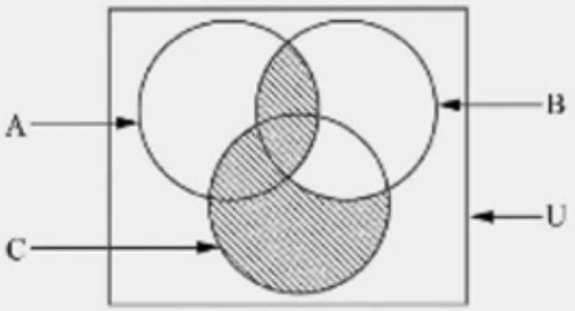
1. (a) $A \cup B \cup C = \{0, 2, 4, 6, 8, 12, 16, 18\}$ (b) $A \cap B \cap C = \{0\}$
   (c) (i) $n(A \cup B \cup C) = 8$ (ii) $n(A \cap B \cap C) = 1$

3. (a) 8 (b) 16 (c) 64

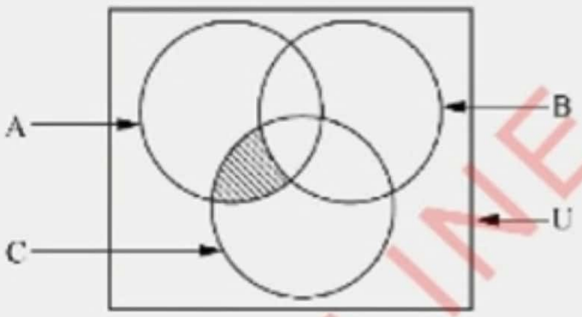
4. (i) $B - A'$



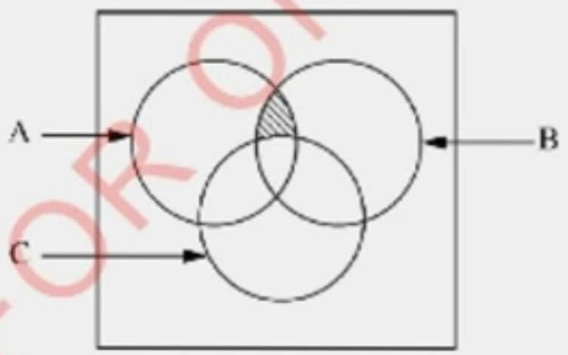
(ii) $(A \cap B) \cup (B' \cap C)$



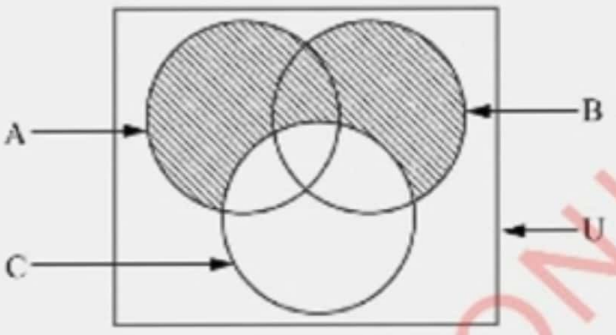
(iii) $A \cap B' \cap C$



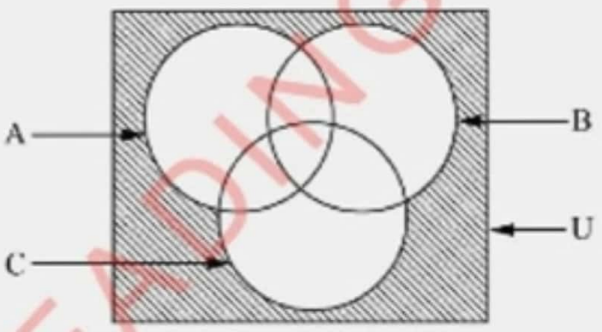
(iv) $A \cap B \cap C'$



(v) $(A \cup B) \cap C'$



(vi) $A' \cap B' \cap C'$



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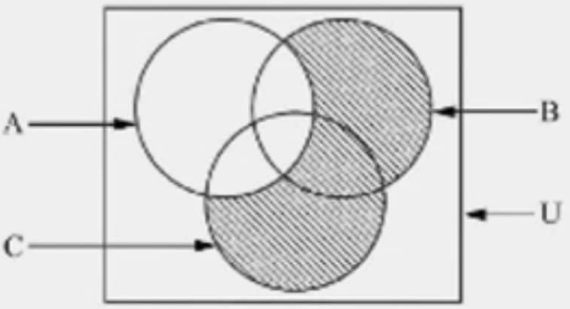
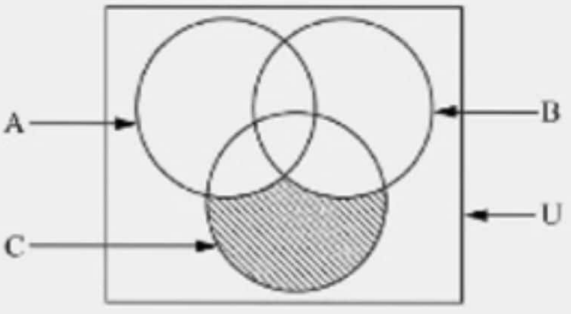
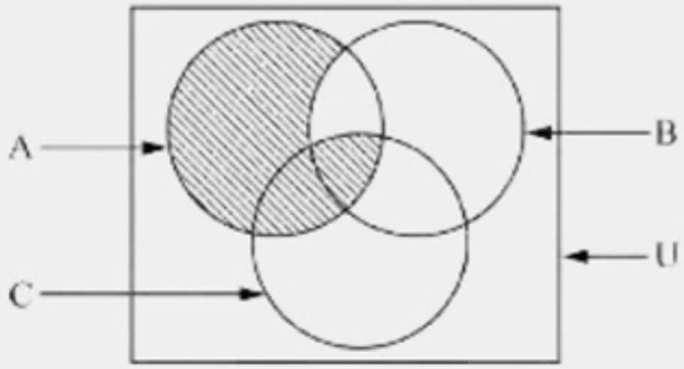


(viii) $A' \cap B' \cap C$



(ix) $A \cap (B' \cup C)$



5. (a) $(B-C)=\{b,c\}$

(b) $(A-C)=\{a, b, c\}$

(c) $(A-B)=\{a\}$

(d) $(C-B)=\{e, f, g\}$

(e) $(B-A)=\{d\}$

(f) $(C-A)=\{d, e, f, g\}$

7. (a) True (b) True (c) True (d) True
(e) False (f) False (g) False (h) False

8. (a) (i) $A=\{5, 10, 15, 20, 25, 30, 35, 40, 45\}$

$B=\{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48\}$

$A \cap B=\{20, 40\}$

$A \cup B=\{4, 5, 8, 10, 12, 15, 16, 20, 24, 25, 28, 30, 32, 35, 36, 40, 44, 45, 48\}$

(ii) $A \cap B=\{$positive integers less than 50, which are both multiples of 4 and 5$\}$

(iii) $n(A) = 9$, $n(B) = 12$, $n(A \cap B)=2$

(b) $A=\{q, r, s\}$, $B=\{q, r, p\}$, $C=\{s, t, q\}$

10. (a) U (b) $\varnothing$ (c) A (d) B (e) $A' \cup B$
(f) $A \cap B'$ (g) C (h) $X \cup Y'$ (i) $\varnothing$ (j) A B

11. (a) $\varnothing$

14. (a) 240 (b) 504 (c) 304 (d) 40

15. (a) 11 (b) 3 (c) 12 (d) 7

16. (a) 5 (b) 50 (c) 95

17. (a) 5% (b) 35% (c) 60%

18. (a) 83.8% (b) 16.2% (c) 95.6%.

19. 19. (a) 6 (b) 18 (c) 33

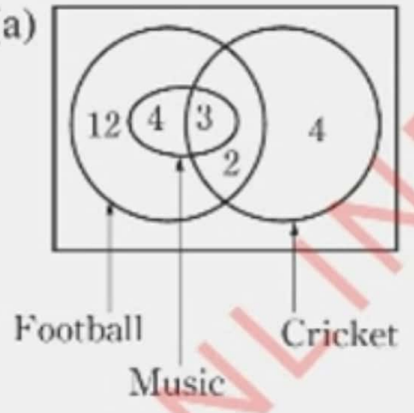
20. (a) Maximum 25, Minimum 15
(b) Maximum 10, Minimum 0.

21. (a) 5 (b) 12 (c) 43 (d) 22

22. (a) B (b) $A' \cap B'$ (c) $A \cap B$ (d) B

25. (a) 15 (b) 10 (c) 45 (d) 40.

26. (a)



(b) 25 (c) 4 (d) 5

Chapter Two

Logic

Introduction

In this chapter, students will learn about statements, logical connectives, laws of algebra of propositions, validity of the arguments, and electrical networks. The competencies developed will enable them to perform various tasks in real-life situations such as distinguishing between valid and invalid arguments, correct reasoning and making proper decisions in daily life activities, constructions of circuit diagrams in the field of electronics, and making judgments in fields of law, among many other applications.

Students' activities

(a) Explaining the concept of connectives, propositions, arguments, and electrical networks

(b) Explaining the meaning and the use of logic to analyze arguments

(c) Constructing the construction of electrical networks

Teaching and learning resources

Logical argument charts, switches, dry cells, water pipes, and logic networks animations, manila cards, pair of scissors, rubber band, flip charts, maker pens, ruler, pencils, masking tapes, glue, and straws

Concept of logic

Teaching steps

1. Devise strategies for assisting students in brainstorming and explaining the concept of logic. Assist them through discussion to relate true and false statements. Advise them to comment on the results and state any difference.
2. Assist student through discussion to perform Activity 2.1 in the student's Book on identifying a statement. Advise students to present the group responses. Engage students through discussion to identify the two types of mathematical statements, namely; simple mathematical statements and compound mathematical statements.
3. Introduce to students through discussion to recognize the truth table using Tables 2.1(a), 2.1(b), and 2.1(c). Design hands on activities which will enable students to establish the truth table when given n number of propositions.
4. Use Think-Ink-Pair-Share strategy in introducing the negation of a statement. Guide students to write statements and negate them mathematically.
5. Use Example 2.1 in the Student's Book to guide students through discussion to draw the truth table. Encourage students to explore additional resources, such as online tutorials or reference books, to enrich their understanding and further practice.
6. For further discussion, instruct students to attempt Exercise 2.1 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 2.1

1. a, b, c, e.

2. (a) Simple statement
   (b) Compound statement
   (c) Simple statement
   (e) Compound statement

3. (a)

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

(b)

p	q	r	s
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

4. (a) Pili is a woman.
   (b) It is not raining now.
   (c) Tanzanian's do not maintain their good culture.
   (d) Summer does not come after spring.
   (e) Industries are not friendly to the environment.
   (f) Tomorrow is not Saturday.

Logical connectives

Teaching steps

1. Develop hands on activities that can be used to assist students to identify logical connectives, including conjunction, disjunction, conditional (implication), and biconditional (double implications) using the truth Tables 2.3, 2.4, and 2.5.
2. Assist students through discussion if necessary to relate the truth values as T = 1 and F = 0. Also, assist them to relate logical connectives as:

(i) Conjunction is equivalent to multiplication
(ii) Disjunction is equivalent to addition
(iii) Conditional is equivalent to less or equal
(iv) Double implications is equivalent to equal

3. Use Examples 2.2 to 2.10 in the Student's Book to guide students through discussion on determining the truth values of the statements and writing statements in both symbolic form and verbal sentences.
4. Guide students to attempt Exercise 2.2 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback where necessary.

Answers to Exercise 2.2

1. (a) False (b) False (c) True

2. (a) $p \wedge q$ (c) $\sim(\sim p \vee q)$ (e) $p \vee (\sim p \wedge q)$
   (b) $p \wedge \sim q$ (d) $\sim q \wedge \sim p$

3. Let $a \equiv \sim p \leftrightarrow \sim q$ and $b \equiv \sim(p \rightarrow q)$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$	$p \rightarrow q$	$\sim(p \rightarrow q)$	$a \leftrightarrow b$
T	T	F	F	T	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	T
F	F	T	T	T	T	F	F

4. (a) $p \rightarrow q$ (b) $q \rightarrow p$ (c) $\sim(\sim p \leftrightarrow \sim q)$

5. (a) You like Physics or Chemistry but not Biology
   (b) You like Physics and Chemistry or you do not like Physics and Biology
   (c) It is not true that you like Physics but not Biology.

6.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

7. (a) $\sim (p \wedge \sim q)$ (b) $\sim p \vee \sim q$

8.

p	q	r	$\sim p$	$q \rightarrow r$	$\sim p \leftrightarrow (q \rightarrow r)$
T	T	T	F	T	F
T	T	F	F	F	T
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

9. (a) False (b) False (c) True

10. (a) $(\sim P \wedge Q) \vee (P \wedge \sim Q)$ (b) $[(P \rightarrow Q) \wedge (Q \leftrightarrow R)] \rightarrow (\sim R \wedge \sim Q)$

11. (a) Halima works hard or she is poor.
(b) It is not true that Halima is poor and she works hard.
(c) If Halima is poor, then she works hard.
(d) Halima is poor and she works hard.
(e) If Halima is not poor, then she doesn't work hard.
(f) Halima is not poor or she doesn't work hard.
(g) Halima works hard if and only if she is poor.
(h) Halima does not work hard or she is poor and she works hard.

12. (a) Swimming in the pool is dangerous.

(b) Few people have not been drawn in the swimming pool if and only if Swimming in the pool is not dangerous.

(c) If swimming in the pool is dangerous, then few people have been drawn in the Swimming pool.

13. (a)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$(p \wedge q) \wedge \sim(p \wedge q)$
T	T	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

(b)

p	q	$q \rightarrow p$	$p \rightarrow q$	$(q \rightarrow p) \leftrightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

(c)

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(d)

p	q	r	$\sim r$	$p \vee q$	$(p \vee q) \rightarrow \sim r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

(e)

p	q	r	t	$\sim p$	$\sim p \quad q$	$(\sim p \leftrightarrow q) \wedge r$	$[(\sim p \leftrightarrow q)] \wedge r \rightarrow t$
T	T	T	T	F	F	F	T
T	T	T	F	F	F	F	T
T	T	F	T	F	F	F	T
T	T	F	F	F	F	F	T
T	F	T	T	F	T	T	T
T	F	T	F	F	T	T	F
T	F	F	T	F	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F
F	T	F	T	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	T	T	F	F	T
F	F	T	F	T	F	F	T
F	F	F	T	T	F	F	T
F	F	F	F	T	F	F	T

Converse, inverse, and contrapositive

Teaching steps

1. Design hands on activities to empower students understanding the concepts of converse, inverse, and contrapositive. Assist students through discussion to prepare truth tables of the converse, inverse, and contrapositive of a conditional statement using Tables 2.7 to 2.9 in the Student's Book. Advise students to write their work on a flip chart or manila card ready for presentation or gallery walk.
2. Encourage students to explore other explanations of converse, inverse, and contrapositive from additional resources, such as online tutorials, or reference books, to enrich their understanding and practice further.
3. Guide students to discuss Examples 2.12, 2.13, and 2.14 in the Student's Book in writing the converse, inverse, and contrapositive of conditional statements.
4. Assign students to attempt Exercise 2.3 in the Student's Book. Advise them to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 2.3

1.(a) *Inverse*:If tomorrow is not Saturday, then Paul will not go to the beach.
Converse: If Paul will go to the beach, then tomorrow is Saturday.
Contrapositive: If Paul will not go to the beach, then tomorrow is not Saturday.

(b) *Inverse*: If it is not raining, then the shop is not closed.
Converse: If the shop is closed, then it is raining.
Contrapositive: If the shop is not closed, then it is not raining.

(c) *Inverse*: If industries are not in harmony with the surroundings, then they are not environmentally friendly.
Converse: If the industries are environmentally friendly, then they are in harmony with the surroundings.
Contrapositive: If the industries are not environmentally friendly, then they are not in harmony with the surroundings.

(d) *Inverse*: If ABC is not an equilateral triangle, then it is not a right-angled triangle.
Converse: If ABC is a right-angled triangle, then it is an equilateral triangle.
Contrapositive: If ABC is not a right-angled triangle, then it is not an equilateral triangle.

(e) *Inverse*: If $f(x)$ is not a rational function, then it has no asymptotes.
Converse: If $f(x)$ has asymptotes, then it is a rational function.
Contrapositive: If $f(x)$ has no asymptotes, then it is not a rational function.

2. (a)

p	q	$\sim p$	$q \rightarrow \sim p$	$(q \rightarrow \sim p) \rightarrow p$
T	T	F	F	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	F

(b)

p	q	$\sim p$	$q \rightarrow \sim p$	$\sim (q \rightarrow \sim p)$	$\sim (q \rightarrow \sim p) \rightarrow \sim p$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	T	F	T

3. (a) If two vectors are not orthogonal, then their dot product is not zero.
   (b) If the dot product of two vectors is zero, then they are orthogonal.
   (c) If the dot product of two vectors is not zero, then the two vectors are not orthogonal.

4. (a) If Halima wont win, then she has no courage.
   (b) One cannot be a sailor if is not strong.
   (c) If a geometrical figure is not a rectangle, then it is not a square.

5. Converse $(q \wedge r) \rightarrow p$
   Inverse $\sim p \rightarrow \sim (q \wedge r)$
   Contrapositive $\sim (q \wedge r) \rightarrow \sim p$

6. (a) $q \rightarrow p$ (b) $\sim (q \rightarrow \sim r) \rightarrow p$
   (c) $\sim (p \leftrightarrow q) \rightarrow \sim (\sim p \leftrightarrow \sim r)$

Logic symbols dominance

Teaching steps

1. Create an activity that will actively engage students in recognizing the order of dominance of connectives in a logic expression using Table 2.10 in the Student's Book.
2. Guide students in groups to perform Activity 2.2 in the Student's Book. Advise students to present their work in a flip chart or manila cards, post them on the classroom walls and conduct gallery walk for more inputs.

Logical equivalences

Teaching steps

1. Design an activity which will motivate students towards describing logical equivalence of expressions. Assist them through discussion to realize that, a logical expression or

statement having only truth values true (T) in the last column of its truth table is a tautology.

2. Use Example 2.15 and 2.16 in the Student's Book to guide students through a discussion on logical equivalence.
3. Guide students to attempt Exercise 2.4 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 2.4

1. Equivalent
2. Not equivalent
3. Equivalent
4. Equivalent
5. Equivalent
6. Equivalent
7. Equivalent
8. Not equivalent
9. Equivalent
10. Not equivalent

Laws of the algebra of propositions

Teaching steps

1. Design activities that will engage students to recognize the laws of the algebra of propositions. Ask them through discussion to explore more information on laws of the algebra of propositions from reference books, online tutorials or additional resources to enrich their understanding.
2. Assist students through discussion to use Table 2.11 in the Student's Book, on laws of the algebra of propositions. Create activities that will engage students in proving the laws of algebra of propositions. Advise students to display their findings in flip charts or manila cards, then present to other groups for more inputs.

3. Engage students to discuss Examples 2.17, 2.18, and 2.19 in the Student's Book on applying laws of algebra of propositions to simplify or prove logical expressions.

1. Guide students to attempt Exercise 2.5 in the Student's Book. Advise them to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 2.5

1. (a) Tautology (c) Tautology (e) Tautology
   (b) Tautology (d) Not a tautology

2. $\sim(pvq)$

4. (a) $p \vee \sim q$ (b) $\sim p$ (c) p (d) T

5. T

6. (a) F (b) ϕ (c) $q \vee \sim p \vee \sim r$

Arguments

Teaching steps

1. Devise strategies for assisting students in brainstorming on the meaning of the phrase argument in logic. Assist students through discussions to identify the two parts of arguments, the initial statements, called "premises", followed by a last statement called "conclusion".

2. Assists students through discussion to perform Activity 2.3 in the Student's Book on identifying premises and conclusion of the argument. Advise students to write their work on a flip chart or manila cards ready for presentations or gallery work.

3. Design an activity that will lead students in groups to demonstrate the validity of an argument. Assist them through discussion to analyze the validity of an argument.
4. Use Examples 2.20, 2.21 and 2.22 in the Student's Book to enrich students understanding of how to use the laws of the algebra of propositions to verify the validity of an argument.
5. For further discussion, guide students to attempt Exercise 2.6 in the Student's Book. Advise them to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 2.6

1. Not valid	4. Not valid	7. Valid	9. Valid
2. Not valid	5. Not valid	8. Valid	10. Valid
3. Valid	6. Not valid		

Electrical networks

Teaching steps

1. Design hands on activities which will lead students in groups to demonstrate an electrical network. Assist students through discussion to arrange a battery, lamp, and switches to form an electrical network. Advise students to share their findings through presentations.
2. Assist students through discussion to identify the terminal and switches of an electrical networks by drawing. Guide students through discussion to check what will happen to the flow of current when the switch is ON and when the switch is OFF.
3. Create hands on activities to differentiate between series and parallel switches connections using suggested resources.

Advise students to display their work on the classroom walls for more inputs.

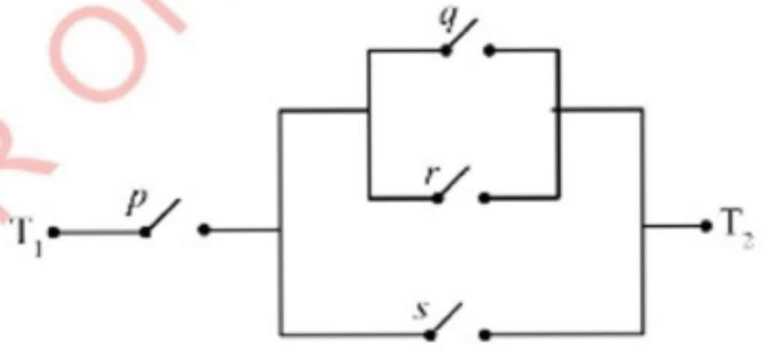
4. Assist students through discussions to use Figures 2.1 and 2.2 in the Student's Book to demonstrate open and closed switches in series and parallel connection. Also, assist them to recognize the possibility of switches connected in different states using Figure 2.3.
5. Use Examples 2.23, 2.24, and 2.25 in the Student's Book to guide students through discussion to draw electrical networks.
6. Instruct students to attempt Exercise 2.7 in the Student's Book. Advise them to submit their work, check the correctness of the answers, and Provide constructive feedback where necessary.

Answers to Exercise 2.7

1. (a) $p \wedge q$

(b) $p \wedge \sim q$

(c) $p \wedge (q \vee r) \vee s$

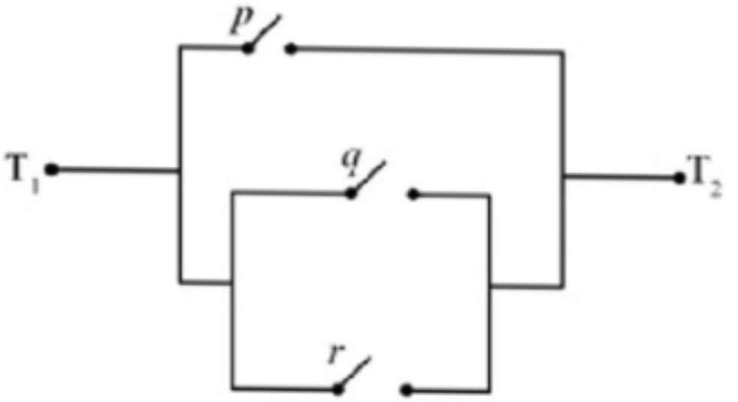


3 (a) $(p \wedge q) \vee \sim q \vee \sim r$

(b) $[\sim p \vee ((p \wedge q) \wedge \sim q)] \wedge [r \vee \sim r]$

(c) $(p \vee q \vee s) \wedge r$

5.



Construction of a compound proposition from a truth table

Teaching steps

1. Develop hands on activities that can be used to assist students to construct compound proposition from a truth table. Assist them through discussion to identify the true values in the last column of a truth table. Lead them to construct basic conjunctions for all true values in the last column.
2. Assist students through discussion to connect the conjunctions by disjunction(s).
3. Use Example 2.26 in the Student's Book to guide students through discussion to construct compound proposition from the given truth table.
4. Instruct students to attempt Exercise 2.8 in the Student's Book. Advise them to submit their work, check the correctness of their answers, and provide constructive feedback where necessary.

Answers to Exercise 2.8

1. $p \wedge \sim q$
2. $(p \wedge q) \vee (\sim p \wedge \sim q)$
3. $\sim p \wedge \sim q$
4. $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$
5. $p \wedge q$
6. $s_1 : (p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$
   $\equiv \sim q \wedge r$

   $s_2 : (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$
   $\equiv q \wedge r$

Answers to Revision Exercise 2

1. Not a mathematical statement. Because it is not possible to asign a truth value T or F to it.
2. (a) Converse: If a person is an adult, then a person is 20 years old.
   Contrapositive: If a person is not an adult, then a person is not 20 years old.
   Inverse: If a person is not 20 years old, then a person is not an adult.

   (b) *Converse:* If I have a test today, then today is Friday.
   Contrapositive: If I have no test today, then today is not Friday.
   Inverse: If today is not Friday, then I have no test.

   (c) *Converse:* If you are attractive, then you bought our clothes.
   Contrapositive: If you are not attractive, then you did not buy our clothes.

Inverse: If you did not buy our clothes, then you are not attractive.

(d) *Converse:* If today is a holiday, then it is Saturday.
Contrapositive: If today is not a holiday, then it is not Saturday.
Inverse: If today is not Saturday, then it is not a holiday.

(e) *Converse:* If Nuru pass the examination, then she is intelligent.
Contrapositive: If Nuru do not pass the examination, then she is not intelligent.
Inverse: If Nuru is not intelligent, then she will not pass the examination.

4. (a) Not a tautology (b) Tautology
(c) Tautology (d) Tautology

5. They have the same truth values (They are logically equivalent).

6. (a) Tautology (b) Tautology
(c) Not a tautology (d) Tautology
(e) Not a tautology (f) Not a tautology
(g) Tautology (h) Tautology
(i) Not a tautology (j) Tautology

7. (a), (b), (c), and (e)

9. (a) Logically equivalent (b) Logically equivalent

(c) Logically equivalent
(d) Logically equivalent

10. (d) and (f)

11. (a) T (b) T (c) $p \vee \sim q \vee r$ (d) T

12. (a) Not a tautology (b) Not a tautology

13. (a) (i) $(p \wedge q) \vee r \equiv$ Lightness is clever and polite or she is humble.

(ii) $\sim p \vee q \equiv$ Lightness is not clever or she is humble.

(b) (i) $(p \vee q) \wedge p \rightarrow \sim q$ (ii) $(q \vee r) \wedge \sim p$

(iii) $(\sim q \wedge \sim r) \rightarrow \sim p$

14. (a) Valid (b) Valid (c) Not valid (d) Valid

(e) Valid (f) Not valid (g) Valid (h) Not valid

(i) Not valid (j) Not valid

16. (a) $p \wedge (q \vee r) \wedge s$ (b) $(p \vee \sim r) \vee [(\sim p \vee q) \wedge (q \vee r)]$

(c) $p \vee q \vee r$

17. (a) $(p \wedge \sim q) \vee (\sim p \wedge q)$

(b) $(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

18. (a) Tautology (b) Not a tautology

19. $* = \wedge$

20. (a) $\sim q \wedge r : T_1$ 

(b) $q : T_1$ 

Chapter Three

Coordinate geometry I

Introduction

In this chapter, students will learn about the angle between two lines, perpendicular distance of a point from a line, locus of a moving point, ratio theorem, and circle. Guide students to recognize methods of calculating the angle between two lines, perpendicular distance of a point from a line, locus of a moving point, ratio theorem, and circle. The competencies developed will enable them to solve problems in the fields of trigonometry, calculus, and dimensional geometry. Also, in real life, coordinate geometry is used in space activities such as location of air transport, map projection, describing the position of objects, and in many other applications.

Students' activities

(a) Finding expression for an angle between two lines
(b) Identifying perpendicular distance
(c) Describing the locus of a moving point
(d) Proving the ratio theorem
(e) Deriving general equations of a circle

Teaching and learning resources

Geometrical figures, marker pens, ruler, mathematical sets, scientific calculators, Mathematical software such as Maple, GeoGebra, MATLAB, AI tools, Symbolab, manila cards, pair of scissors, rubber band, flip charts, pencils, masking tapes, glue, straws, geo-boards, graph papers, mathematical instuments, and colored chalks.

The angle between two lines

Teaching steps

1. Design hands on activities to assist students to visualize and discuss their previous understanding of coordinate geometry from their real-life experiences. Advise them to write their work on a flip chart or manila card, display them on classroom walls, and do a gallery walk for more inputs.
2. Create an activity that will lead students to demonstrate the angle of intersection between two lines. Use Figure 3.1 in the Student's Book to assist students through discussion to derive the formula for calculating the angle between lines.
3. Use Examples 3.1, 3.2, and 3.3 in the Student's Book to guide students to solve various problems involving the formula for calculating the angle between two lines. Advise students to use scientific calculators and GeoGebra to compare the answers of the examples.
4. Assign students to attempt Exercise 3.1 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 3.1

1. $109.4°$ 2. $78.7°$

3. (a) $36.9°, 40.6°, 102.5°$ (c) $17.7°, 60.3°, 102°$
   (b) $40.6°, 63.4°, 76°$

4. $x-3y+7=0$ 5. $m=\frac{18}{5}$, $n=\frac{22}{15}$, $53.6°$ or $126.4°$

6. $47.7°, 132.3°$ 7. $17\sqrt{3}x+13y=126-68\sqrt{3}$

9. $5x+y=13$ and $x-5y=13$

Perpendicular distance of a point from a line

Teaching steps

1. Guide students through discussion to use Figures 3.2 and 3.3 in the Student's Book to derive the formula for the shortest/ perpendicular distance of a point from a line.
2. Allow students through discussion to share alternative methods of deriving the formula for the perpendicular distance of a point from a line.
3. For further discussion, guide students to discuss Examples 3.4 and 3.5 in the Student's Book. Advise them to use scientific calculators and a mathematical software of their choice to compare the answers for the examples.
4. Instruct students to attempt Exercise 3.2 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 3.2

1. (a) 1 unit (b) 2.55 units (c) $\frac{3}{5}a$ units

(d) $\frac{2h+4k}{5}$ units (e) 0 units, the point lies on the line

2. $p = \frac{-9 \pm \sqrt{2}}{3}$

4. $\overline{PR} = 5.06$ units, $\overline{QS} = 0.95$ units, and $\overline{RS} = 0.32$ units

6. $k = -5$ 7. $n = -1$

8. $a = -2$ or $a = 34$

Locus

Teaching steps

1. Design an activity that will assist students to understand the concept of locus of a moving point. Use figures, models, or simulations through discussion to describe a point that moves so that it satisfies a given condition.
2. Allow students through discussion to share alternative methods of finding the locus of a moving point given a certain condition.
3. For further discussion, guide the students to discuss Examples 3.6 and 3.7 in the Student's Book, and give them constructive feedback.
4. Instruct students to attempt Exercise 3.3 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 3.3

1. (a) $x = 4$
   (b) $y = 5$
   (c) $x^2 + y^2 = 25$
   (d) $x^2 + y^2 + 4x + 6y - 51 = 0$
   (e) $x^2 + y^2 - 8x + 12 = 0$
   (f) $x^2 - 3y^2 + 32y - 64 = 0$
   (g) $3x^2 + 4y^2 + 6x - 9 = 0$
   (h) $x = \frac{3}{8}$
2. $8x - 6y - 7 = 0$
3. $y^2 = 4ax$
4. $y^2 - 3x^2 + 34x = 91$
5. $3x - 11y = 0,\ 11x + 3y = 0$

6. $73x^2 - 156x + 9y^2 - 18y - 63 = 0$
7. $3x^2 + 3y^2 + 2x + 16y + 25 = 0$
8. $8x^2 + 8y^2 - 228x - 224y + 2540 = 0$
9. $2484x^2 + 2500y^2 - 19800x - 25000y + 400 = 0$
10. $2x + 3y + 10 = 0$

Ratio theorem

Teaching steps

1. Guide students through discussion to deduce the formula for division of a line segment internally. Use Figure 3.4 in the Student's Book to demonstrate through discussion derivation of the formula for computing the coordinates of a point dividing a line segment internally.
2. Assist students through discussion to deduce the formula for dividing a line segment externally. Use Figure 3.5 in the Student's Book to demonstrate through discussion derivation of the formula for computing the coordinates of a point dividing a line segment externally.
3. Allow students through discussion to share alternative methods of deriving the formula for internal and external division of a line segment.
4. For further discussion, guide students to discuss Examples 3.8 to 3.11 in the Student's Book, and give them constructive feedback.
5. Instruct students to attempt Exercise 3.4 in the Student's Book. Advise them to submit their work, check the correctness of the answers and provide them constructive feedback where necessary.

Answers to Exercise 3.4

1. (a) $\left(-\frac{7}{5}, \frac{14}{5}\right)$, $(25, 22)$
   (b) $(-1, 3)$, no external division
   (c) $\left(-5, \frac{32}{5}\right)$, $(7, 40)$
   (d) $\left(\frac{19}{3}, 6\right)$, $(8, 6)$
2. C(20, 22)
3. 0.41 units
4. P(−1, −3), Q(2, 3)
5. $m = 3, n = 2$
6. 2:9
8. $m = 2, n = 3$
9. 1:3
10. $\left(\frac{73}{13}, \frac{64}{13}\right)$
11. $A\left(-4, \frac{-7}{3}\right)$ $B\left(\frac{-20}{7}, \frac{-9}{7}\right)$

A circle

Teaching steps

1. Design an activity that will engage students in deducing the standard equation of a circle. Use Figure 3.6 in the Student's Book to assist students through discussion to describe the locus of a point that moves such that its distance from a fixed point is constant.
2. Assist students through discussion to identify parts of a circle using GeoGebra, models, or simulations. Allow students to share alternative methods of deriving the standard equation of a circle.
3. Guide students through discussion to derive the general equation of a circle. Allow students to share alternative methods of deriving the general equation of a circle.
4. For further discussion, guide the students to discuss Examples 3.12, 3.13, and 3.14 in the Student's Book, and give them constructive feedback.

5. Instruct students to attempt Exercise 3.5 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 3.5

1. (a) $x^2+y^2-4y-12=0$
   (b) $x^2+y^2+4x-4y-17=0$
   (c) $x^2+y^2-4=0$
   (d) $4x^2+4y^2+4x-40y-299=0$
   (e) $8x^2+8y^2-32x+48y+103=0$

2. (a) Centre $(1,-2)$, radius $=\sqrt{45}$ units
   (b) Centre $(0,0)$, radius $=5$ units
   (c) Centre $(0,2)$, radius $=2$ units
   (d) Centre $(-1,0)$, radius $=2$ units
   (e) Centre $\left(\frac{1}{2},-\frac{3}{2}\right)$, radius $=1$ unit

3. $x^2+y^2-8x-8y+16=0$

4. $(x+41)^2+(y+161)^2=144$

5. $4x^2+4y^2+40x-16y+67=0$

6. Centre $\left(2,\frac{-5}{2}\right)$, radius $=\frac{3\sqrt{3}}{2}$ units

8. Centre $(5,-6)$, radius $=2\sqrt{3}$ units

9. (b) Centre $(4,1)$, radius $\frac{3\sqrt{6}}{2}$ units, diameter $=3\sqrt{6}$ units

10. $x^2+y^2+12x+8y-92=0$

Equation of a circle given the endpoints of its diameter

Teaching steps

1. Use Figure 3.7 in the Student's Book to assist students in describing a circle given the endpoints of a diameter. Guide students through discussion to deduce the formula for the equation of a circle given the endpoints of a diameter.
2. Allow students to share alternative methods of deriving the general equation of a circle given the end points of a diameter. Advise students to present their findings on a flip chart or manila cards. Guide them to make presentations for more inputs.
3. For further discussion, guide students to discuss Example 3.15 in the Student's Book, and provide them constructive feedback.
4. Design an activity which will engage students in groups to explore other examples for determining the equation of a circle given the end points of a diameter.

Equation of a circle passing through three given points

Teaching steps

1. Use Figure 3.8 in the Student's Book to assist students in describing the circle passing through three given points.
2. Allow students to share alternative methods of deriving the general equation of the circle passing through three given points. Advise them to display their findings on the manila card. Guide them to make presentations for more inputs.
3. Using Example 3.16 in the Student's Book, engage students in finding the equation of the circle.
4. Instruct students to attempt Exercise 3.6 in the Student's Book. Advise them to submit their work, check the correctness of the answers, and provide them constructive feedback.

Answers to Exercise 3.6

1. (a) $x^2+y^2+7x-5y+16=0$
   (b) $x^2+y^2-4x+4y-17=0$
   (c) $x^2+y^2+x+2y-3=0$
   (d) $x^2+y^2-x+4y-53=0$
   (e) $x^2+y^2+4x-3y=0$

2. (a) $x^2+y^2-6x+4y-12=0$
   (b) $x^2+y^2-x+4y-12=0$
   (c) $x^2+y^2-14x+10y-95=0$
   (d) $x^2+y^2-10x-8y+16=0$
   (e) $x^2+y^2-13x+3y+2=0$

3. $x^2+y^2-10x-10y+24=0$

4. $C\left(\frac{1}{2}, \frac{3}{2}\right)$, $r=\frac{9\sqrt{2}}{2}$ units

5. $x^2+y^2-2x+2y-23=0$

7. (a) $C(-4, 8)$ (b) $r=4\sqrt{10}$ units (c) $x^2+y^2-8x-16y-80=0$

9. $8x^2+8y^2-46x-35y-202=0$

Equation of tangent and normal to a circle

Teaching steps

1. Use Figure 3.9 in the Student's Book to assist students in describing the line which is tangent to a circle given the end points of a diameter. Guide students through discussion to derive the formula for the equation of a tangent to a circle at the where it touches the circle.
2. Use Figure 3.10 in the Student's Book to assist students in describing a line which is tangent to a circle. Guide them through discussion to derive the equation of a line which is normal to a circle at the point of contact with the circle.
3. Design an activity which will engage students to share alternative methods for the derivation of the equation of

tangent and normal to a circle. Advise students to present their findings on a flip chart or manila cards. Guide them to make presentations for more inputs.

4. For further discussion, guide students to discuss Examples 3.17 to 3.20 in the Student's Book, and give them constructive feedback.
5. Instruct students to attempt Exercise 3.7 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 3.7

1. (a) Tangent, $6x-2y-5=0$; Normal, $x+3y=0$
   (b) Tangent, $3x-4y+25=0$; Normal, $4x+3y=0$.
   (c) Tangent, $2x-y-9=0$; Normal, $x+2y+3=0$
   (d) Tangent, $2x+4y-7=0$; Normal, $2x-y-2=0$
   (e) Tangen, $2x+3y-22=0$; Normal, $3x-2y-7=0$
   (f) Tangent, $2x+3y=0$; Normal, $3x-2y=0$.
2. $(3, -1)$
3. $k=40$, or $k=-10$
4. $x+2y-3=0$
7. $x+y-5=0$
8. $5x+2y-21=0$, $5x+2y+28=0$
10. $4x-3y+25=0$, $4x-3y-25=0$
11. $2x+y+4=0$, $2x+y-6=0$
13. $x+2y-4=0$, $x+2y+6=0$
14. 5 units
15. $x^2+y^2+6x+10y+9=0$.

Point of intersection of circles

Teaching steps

1. Use Figure 3.11 in the Student's Book to assist students in describing two circles that touch each other externally. Guide students through group discussion to derive the condition for two circles to intersect each other externally at one point.
2. Use Figure 3.12 in the Student's Book to assist students in describing two circles that touch each other internally. Guide students through group discussion to derive the condition for two circles to intersect each other internally at one point.
3. Use Figure 3.13 in the Student's Book to assist students in describing the intersection of two circles at two distinct points. Guide students through group discussion to derive the condition for two circles to intersect each other at two distinct points.
4. Allow students to share alternative methods for deriving conditions for two circles to touch each other externally, internally, and to intersect at two distinct points.
5. For further discussion, guide students to discuss Examples 3.21, 3.22, and 3.23 in the Student's Book, and give them constructive feedback.
6. Instruct students to attempt Exercise 3.8 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 3.8

3. $y = -2x$ 4. $(0, 0)$

5. $\left(\frac{27+8\sqrt{43}}{17}, \frac{23-2\sqrt{43}}{17}\right), \left(\frac{27-8\sqrt{43}}{17}, \frac{23+2\sqrt{43}}{17}\right)$

6. $x^2 + y^2 - 4x + 2y - 20 = 0$; $(5, -5)$

Orthogonal circles

Teaching steps

1. Use Figure 3.14 in the Student's Book to assist students in describing orthogonal circles. Guide students through group discussion to derive the condition for two circles to be orthogonal.
2. Allow students to share alternative methods of derivation the condition for two circles to intersect orthogonally.
3. For further discussion, guide students to discuss Example 3.24 in the Student's Book, and give them constructive feedback.
4. Instruct the students to attempt Exercise 3.9 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback where necessary.

Answers to Exercise 3.9

3. $x^2 + y^2 - 10x - 2y + 9 = 0$

4. (a) $k = 47$ (b) $k = 34$

5. $x^2 + y^2 - 4x - 2y + 3 = 0$

6. $x^2 + y^2 - 4x - 4y + 7 = 0$

7. (a) $5x^2 + 5y^2 + 24x - 36y = 0$

10. $x^2 + y^2 = 1$, $4x^2 + 4y^2 - 15x - 4 = 0$

11. $= -32$

1. Use Figure 3.15 in the Student's Book to assist students in describing the length of a tangent from a point to a circle. Guide students through group discussion to derive the formula for determining the length of a tangent from a point to a circle. Advise them to write their work on a manila card, display them on the classroom walls, then conduct gallery walk for more inputs.
2. Use Example 3.25 in the Student's Book to guide students through discussions and apply the formula for determining the length of a tangent from a point to a circle.
3. Instruct students to attempt Exercise 3.10 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

3.10

1. 11.58 units
2. 12 cm
3. 2 units
4. $2\sqrt{3}$ units
5. 7 units
7. $x^2 + y^2 - 10x + 4y - 2 = 0$
9. $2\sqrt{11}$ units

3

1. $k = 5$
2. M(3, 6), L(6, 3), N$\left(\frac{25}{4}, 1\right)$
3. $12x^2 - 4y^2 = 3$

4. $x+2y=0$, $x+2y+10=0$

5. $x^2+y^2+6x+6y+2=0$

6. $m=2, m=8$

7. $\left(\frac{8}{5},\frac{24}{5}\right)$, (16, 12)

8. (a) $\left(\frac{18}{5},\frac{14}{5}\right)$ (b) (6, −2)

9. $4\sqrt{2}$ units

10. $-x+y+8=0$, (4, −4)

11. L(4,4), M(7, 6)

12. $\frac{6\sqrt{5}}{5}$ units

13. $x^2+y^2-12x-12y+36=0$, $x^2+y^2-2x-2y+1=0$

14. $x^2=4by$

15. 5.9 units

16. 53.13°

17. $2x-y-4=0$, $x+2y-7=0$

18. $x-3y+3=0$, $3x+y-11=0$

19. 47.7°

20. $m=-2$, $\left(\frac{1}{3},-\frac{1}{3}\right)$

21. $x^2+y^2-2x+2y-23=0$

22. $x^2+y^2+\frac{8}{5}x-\frac{16}{5}y-4=0$

centre $\left(-\frac{4}{5},\frac{8}{5}\right)$, radius = $\frac{6\sqrt{5}}{5}$ units

23. $x^2+y^2-4x+6y+8=0$

24. C(5, 6)

25. B(6, 4)

26. $x^2+y^2-2x-2y+1=0$

27. 30°

28. $6x+4y+23=0$

29. $16x^2+9y^2+24xy-156x-42y+249=0$

30. $x^2-4x+2y-3=0$

31. $x^2+y^2-10x-8y+16=0$

33. $x^2+y^2+6x-6y+8=0$

34. $x^2+y^2-4x-8y+15=0$

35. $a=2, b=0$

36. 36. (1, 2)

37. $x^2+y^2-12x-10y+36=0$

38. centre $(-2,4)$, radius = 4 units, length of tangent = 5 units

39. $2x^2+2y^2-8x-12y+25=0$

40. $y=2x$

42. $c=-32$

43. $x-y-2=0,\ x+7y\pm10=0$

44. $12x+5y+119=0;\ 12y-5x+49=0$

45. $y=2x;\ 22x+19y=0$

46. $\sqrt{37}$ units

47. $\sqrt{12}$ units

49. $4x-3y-18=0$; 13.5 square units

Chapter Four

Functions

Introduction

In this chapter, students will learn how to plot graphs of functions. Guide the students to realize some common methods of plotting graphs of polynomials, rational, composite, exponential, and logarithmic functions using graph papers and mathematical software such as Maple, GeoGebra, MATLAB, Mathematica, and AI tools. The competencies developed will enable them to design machines, make predictions, study growth relations, formulate mathematical models, develop computer programs, and solve many other real-life problems.

Students' activities

(a) Sketching graphs of polynomial functions

(b) Describing graphs of rational functions

(c) Recognizing composite functions

(d) Sketching graphs of exponential functions

(e) Sketching graphs of logarithmic functions

Teaching and learning resources

Scientific Calculator, Mathematical softwares such as Maple, GeoGebra, MATLAB, Mathematica, AI tools, Manila cards, a pair of scissors, rubber band, flip charts, marker pens, ruler, pencils, masking tapes, glue, geo-boards, graph papers, mathematical sets, and colored chalks.

Polynomial functions

Graphs of linear functions

Teaching steps

1. Assist students through group discussion to perform Activity 4.1 in the Student's Book on identifying a function from the drawn graph. Advise students to write their findings on a flip chart or manila card ready for presentations or gallery walk. Assist students through discussion to recognize the general form of a polynomial function

   $y = f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$, where $n \in \mathbb{N}$.

2. Assist students through group discussion to classify polynomial functions according to their degree.

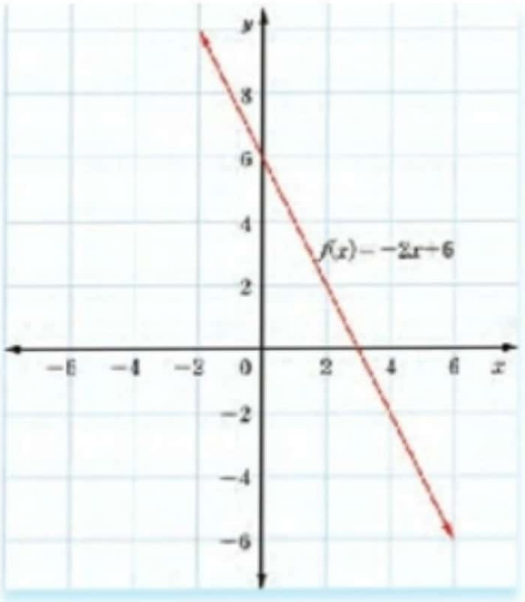
3. Use Figure 4.1 in the Student's Book to assist students in drawing the graph of $f(x) = mx + c$, $m > 0$.

4. Use Examples 4.1 and 4.2 in the Student's Book to assist students through discussions to sketch graphs of linear equations. Engage students to use GeoGebra or any other mathematical software to verify the answer to a given example.

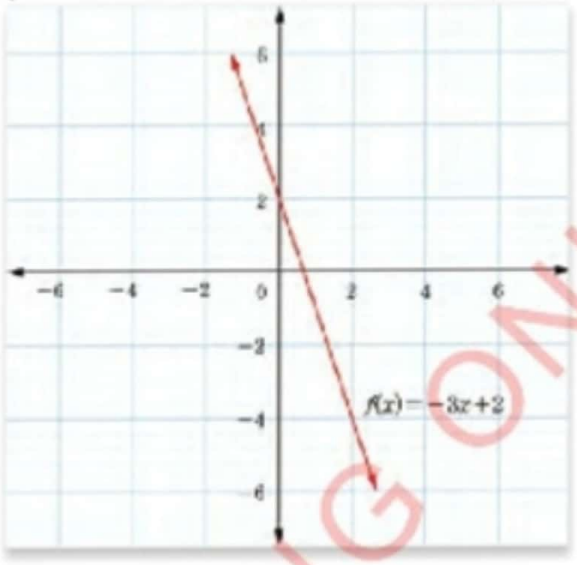
5. Instruct students to attempt Exercise 4.1 in the Student's Book. Advise them to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 4.1

1.



3.



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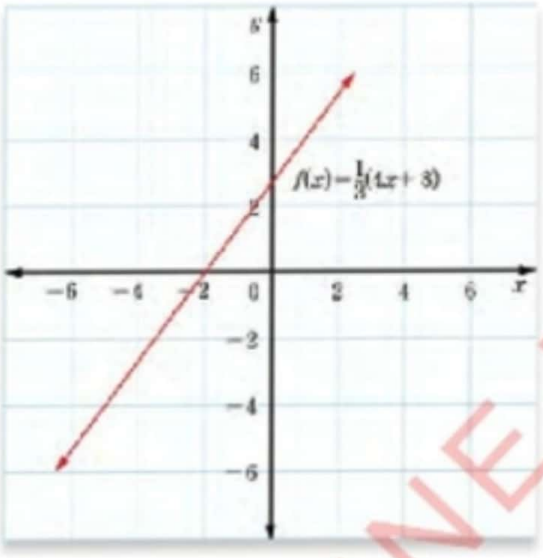

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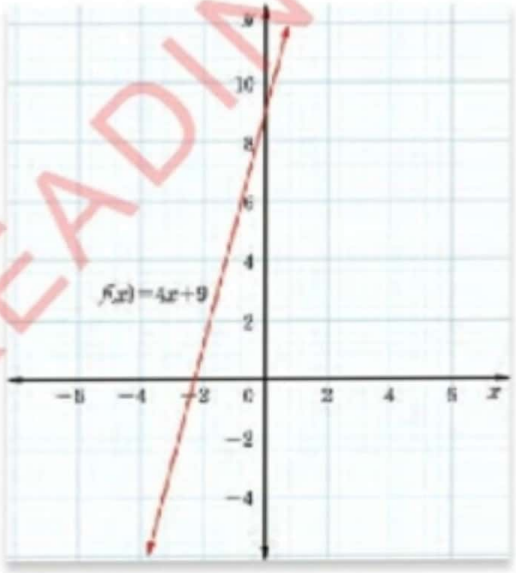

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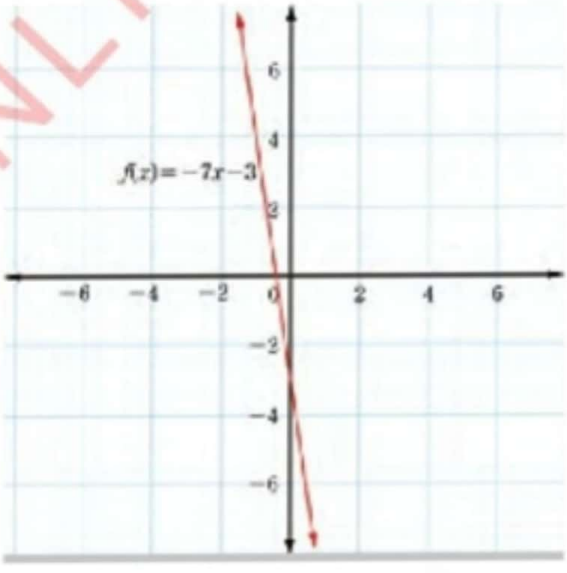

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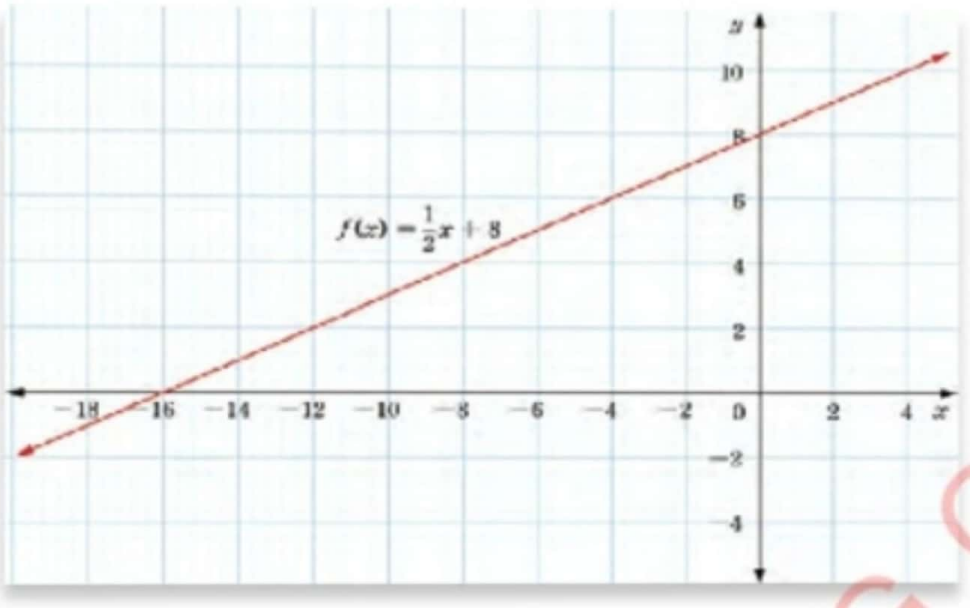

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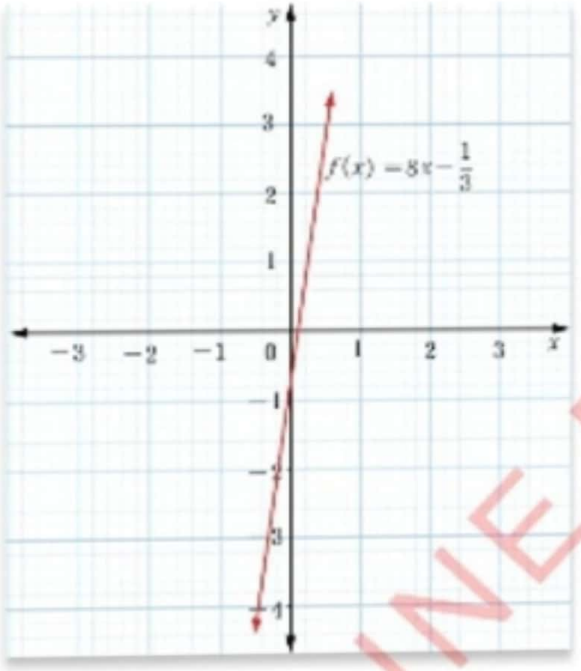
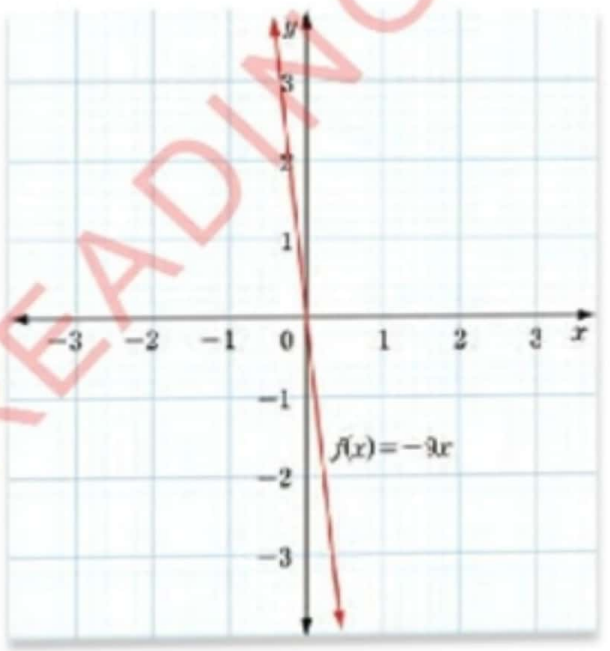
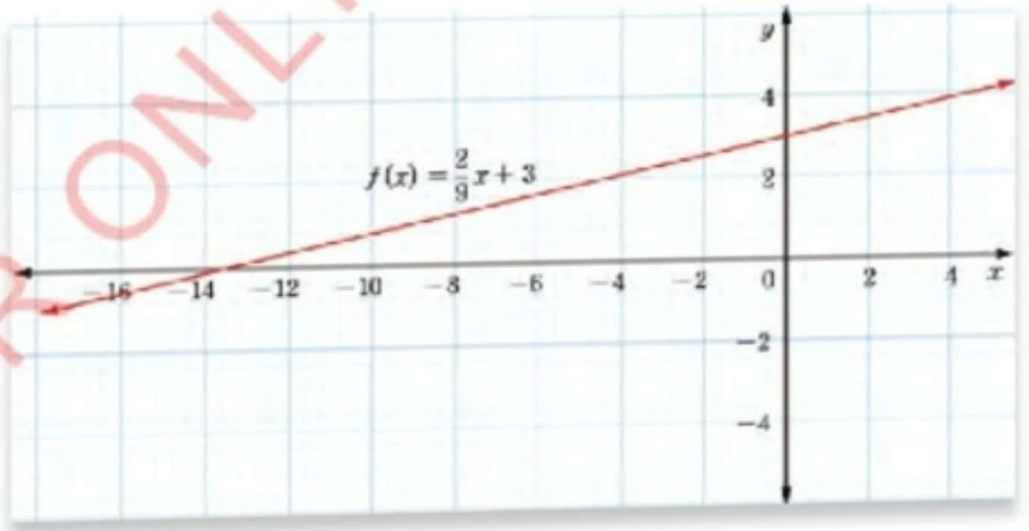


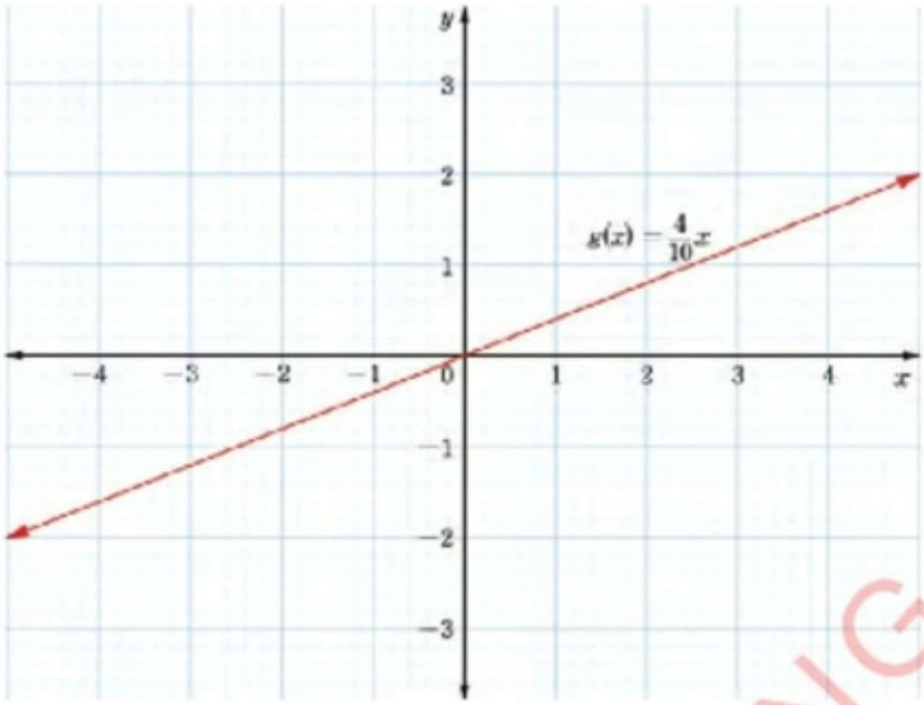
8.



9.



10.



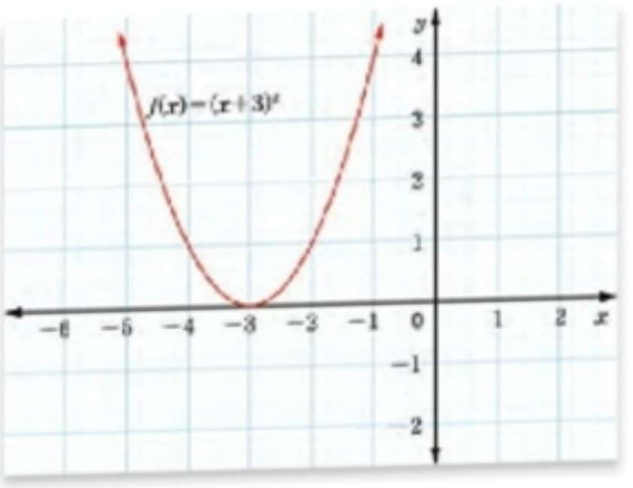
Graphs of quadratic functions

Teaching steps

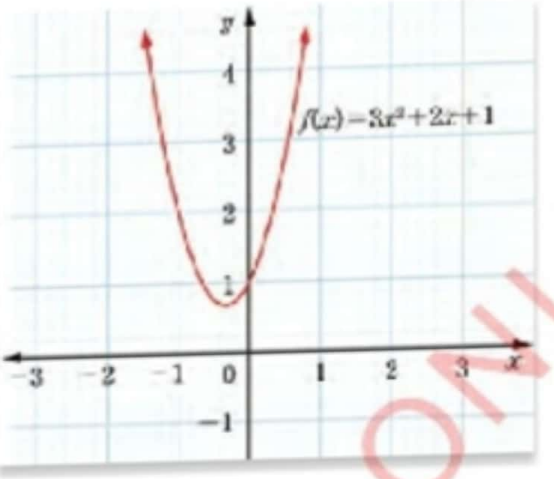
1. Using Figures 4.2(a) and 4.2(b) in the Student's Book engage students to recognize the nature of graphs of quadratic functions. Assist students to discuss procedures of sketching graphs of quadratic functions.
2. Use Examples 4.3, 4.4, and 4.5 in the Student's Book to guide students to sketch graphs of quadratic functions. Engage students to use GeoGebra or any other mathematical software to verify the answers of the given examples. Advise students to make presentations on their findings for more inputs.
3. Instruct students to attempt Exercise 4.2 in the Student's Book. Advise students to submit their work, check the correctness of the answers and provide them constructive feedback where necessary.

Answers to Exercise 4.2

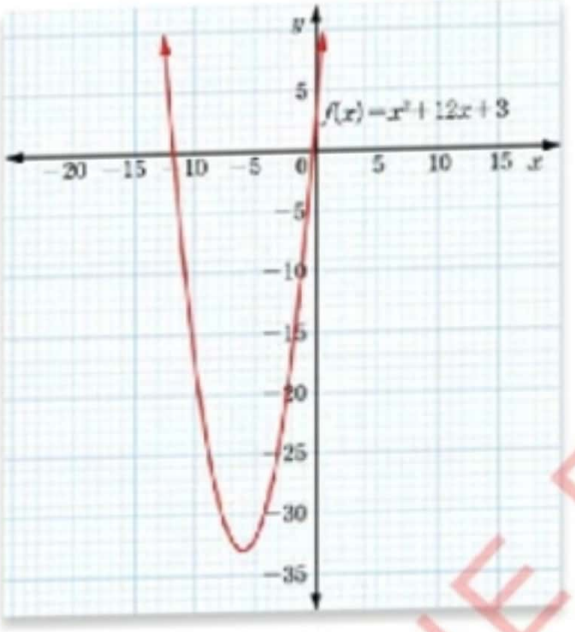
1.



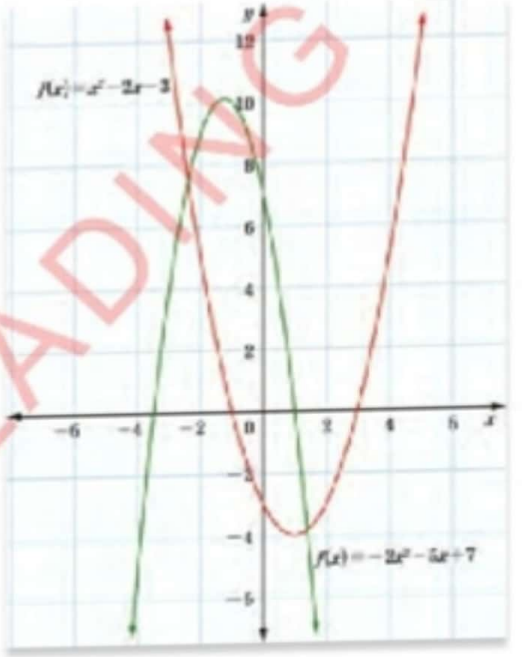
2.



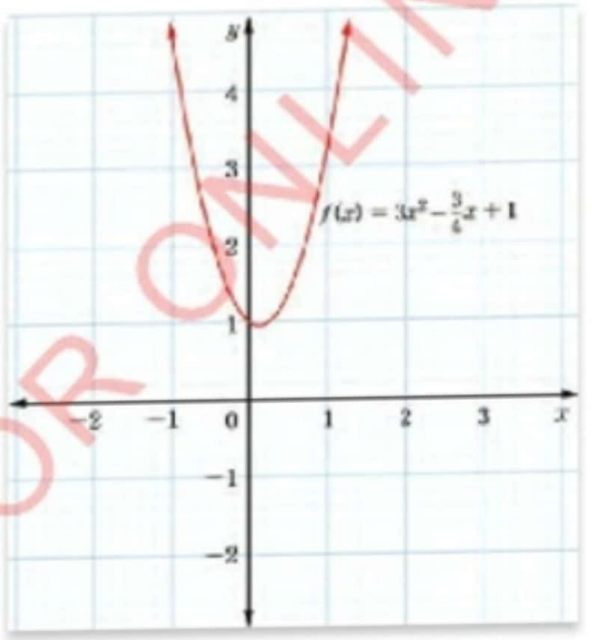
3.



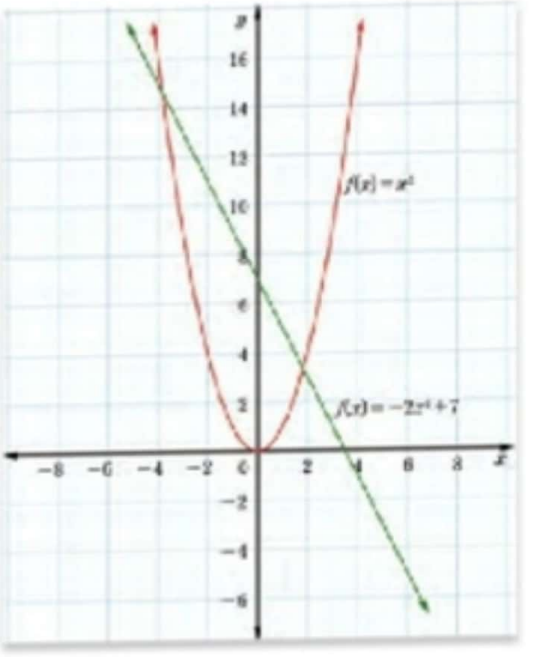
4.



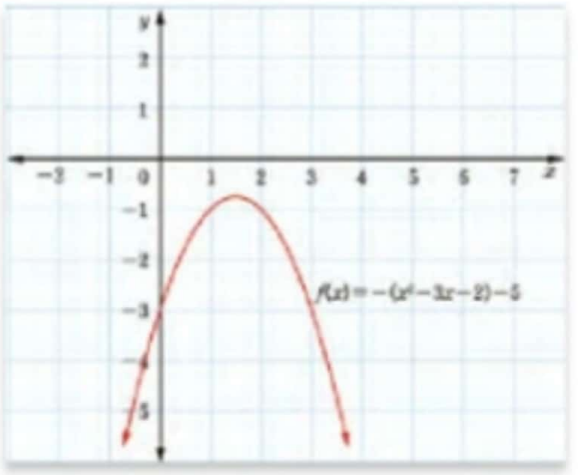
5.



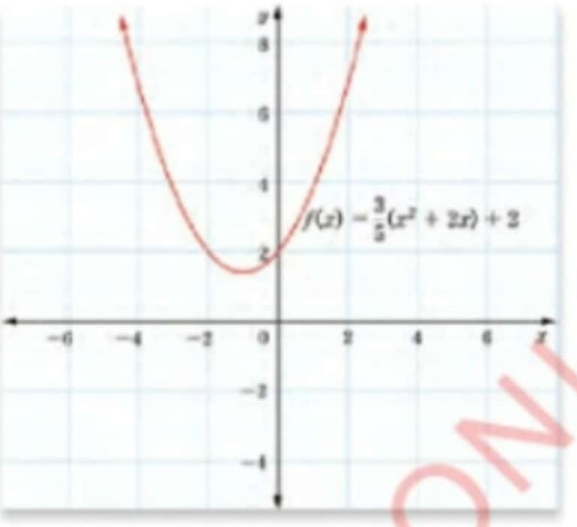
6.



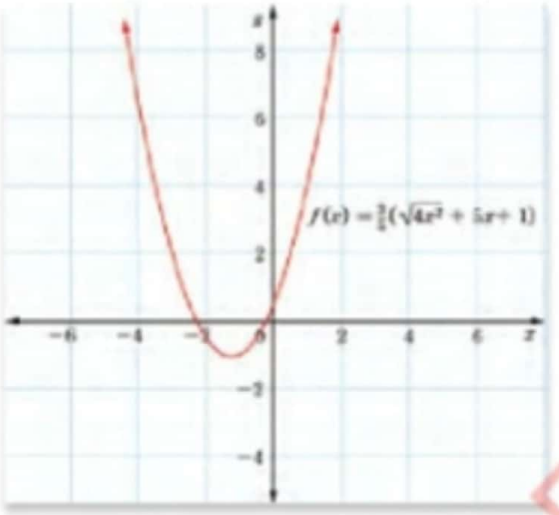
7.



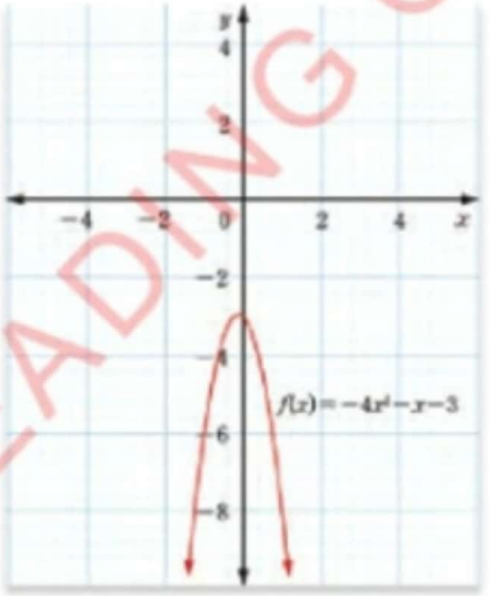
8.



9.



10.



Graphs of cubic functions

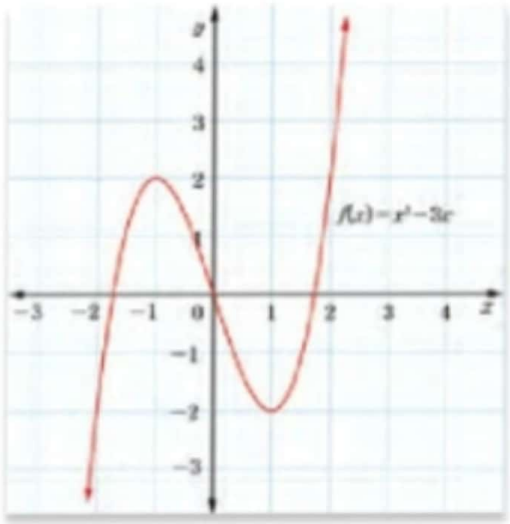
Teaching steps

1. Guide students through discussion to recognize procedures of sketching graphs of cubic functions. Allow them to compare and contrast the procedures from those of quadratic functions.
2. Use Examples 4.6, 4.7, and 4.8 in the Student's Book to guide students to sketch graphs of cubic functions. Engage students to use GeoGebra or any other mathematical software to verify the graphs of cubic functions in the given examples. Advise them to share their findings through presentations.

3. Instruct students to attempt Exercise 4.3 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback where necessary.

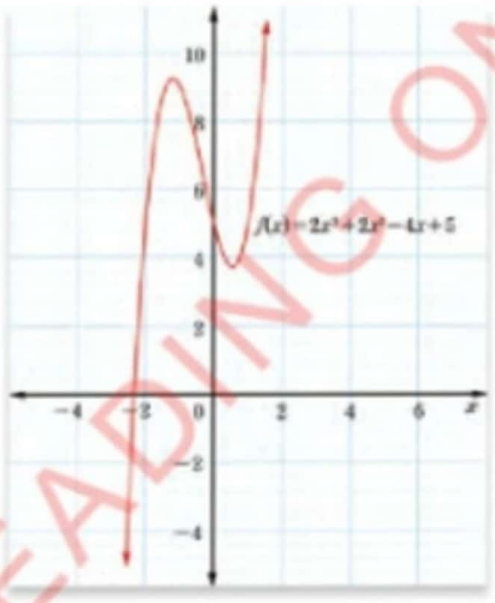
Answers to Exercise 4.3

1.



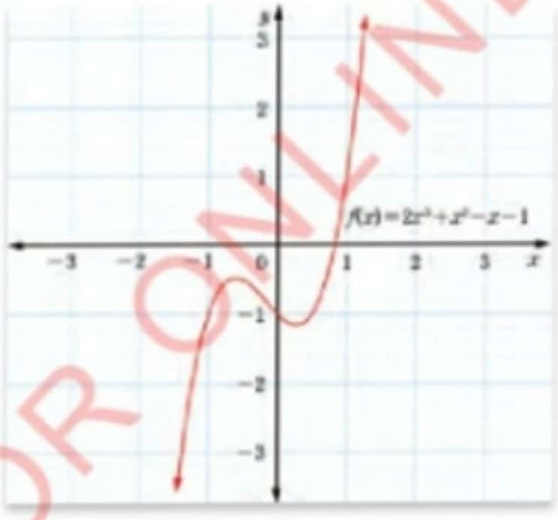
Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \in \mathbb{R}\}$

2.



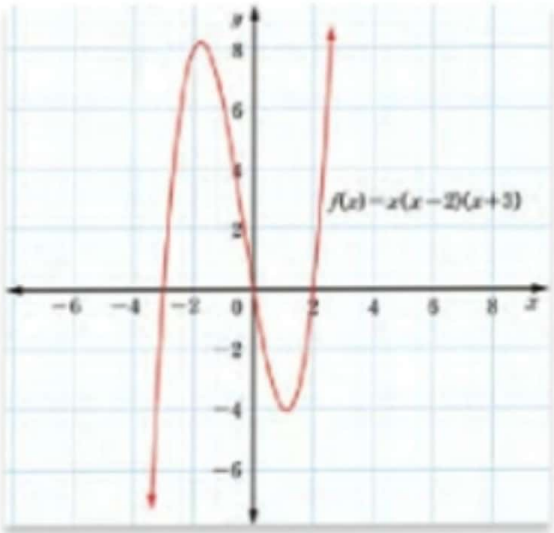
Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \in \mathbb{R}\}$

3.



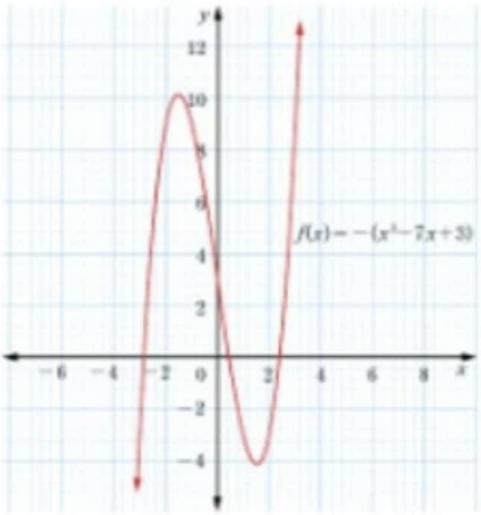
Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \in \mathbb{R}\}$

4.



Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \in \mathbb{R}\}$

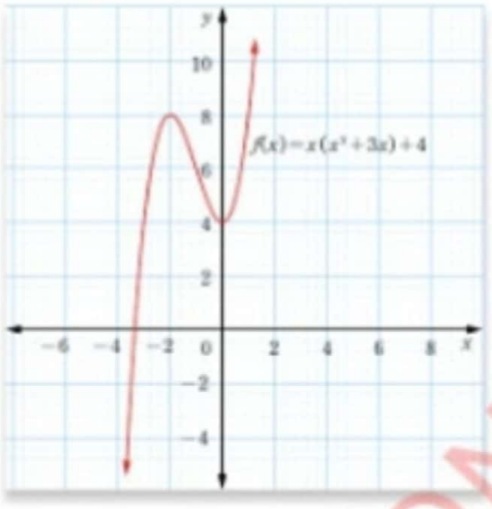
5.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

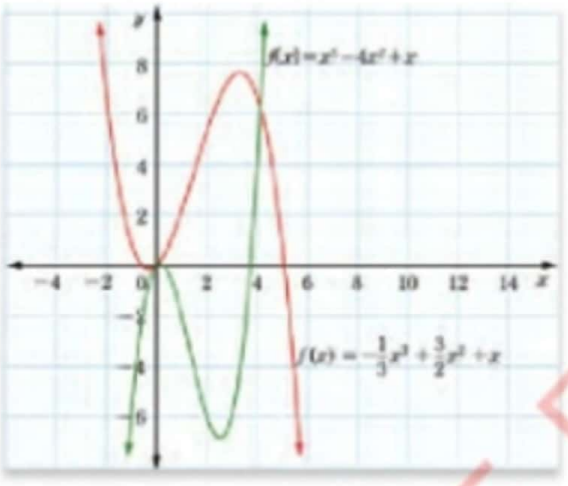
6.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \quad\}$

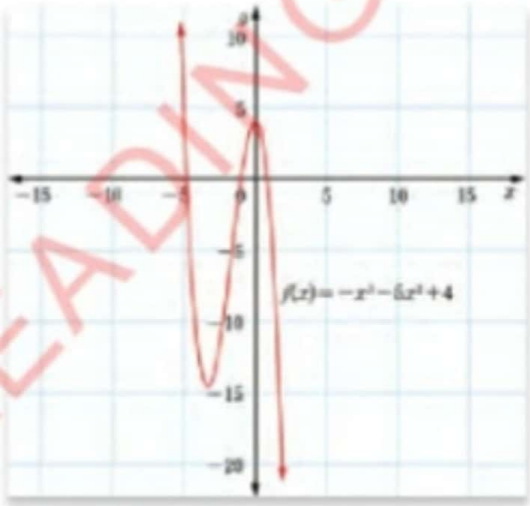
7.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

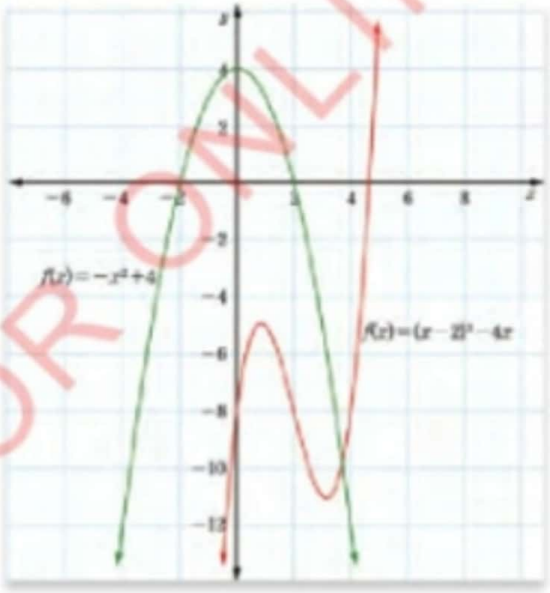
8.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

9.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

Graphs of quartic functions

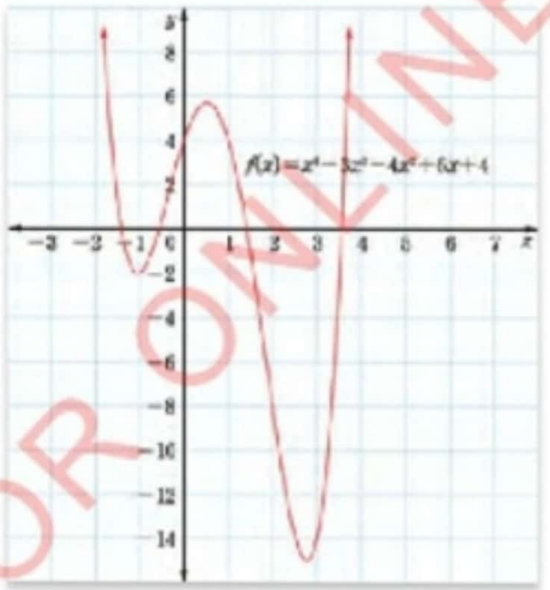
Teaching steps

1. Guide students through group discussion to use the same procedures of sketching graphs of cubic functions to sketch graphs of quartic functions.
2. Use Examples 4.9 and 4.10 in the Student's Book to guide students to sketch graphs of quartic functions. Engage students to use GeoGebra or any other mathematical software to verify the graphs of quartic functions in the given examples. Advise students to share their findings through presentations for more inputs.
3. Instruct students to attempt Exercise 4.4 in the Student's Book. Advise students to submit their work, check the correctness of the answers and provide them constructive feedback where necessary.

Answers to Exercise 4.4

1. $x = -1, 0, 1, 2$

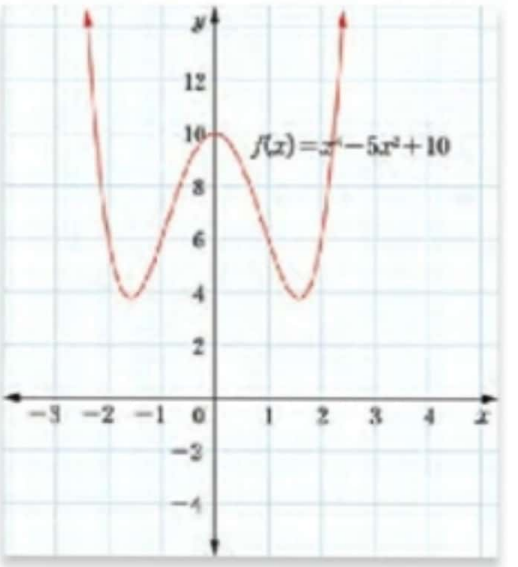
2.



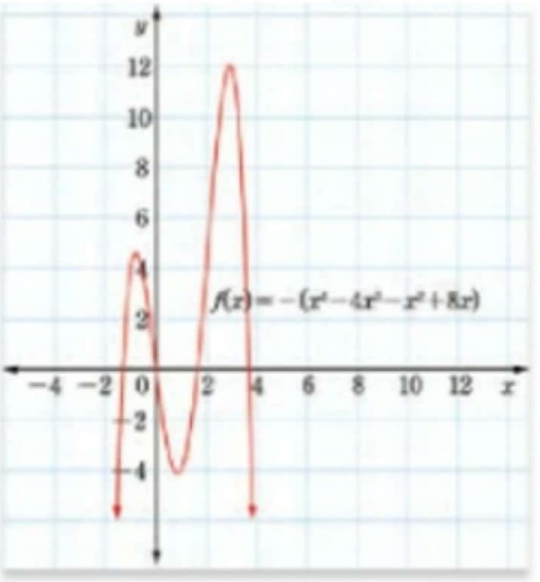
Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \geq -15\}$

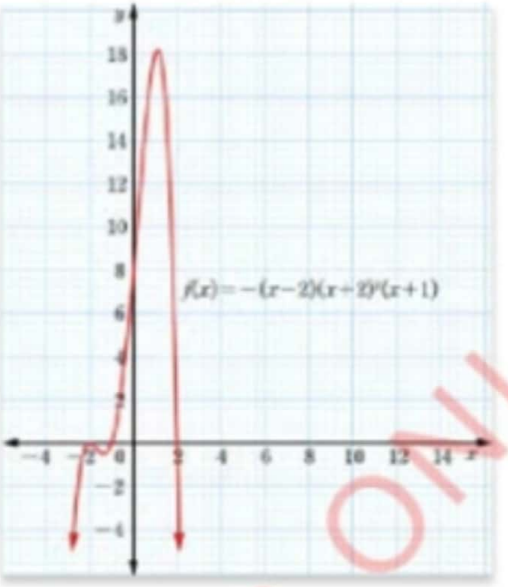
3. (a)



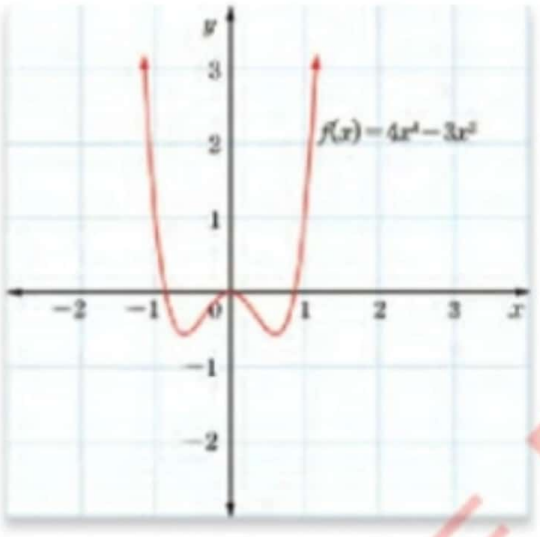
(b)



(c)



4.



5. Range: $\left\{ y : y \in \mathbb{R}, y \geq -\frac{1}{4} \right\}$

6. $f(x)$ goes to large positive ∞

7. Domain $= \{ x : x \in \mathbb{R} \}$

   Range $= \{ y : y \in \mathbb{R}, y \geq 0 \}$

8. Domain $= \{ x : x \in \mathbb{R} \}$

   Range $= \{ y : y \in \mathbb{R}, y \leq 1 \}$

9. $f(x)$ increases to $+\infty$

10. Range $= \{ y : y \in \mathbb{R}, -20.9 \leq y \leq 126 \}$

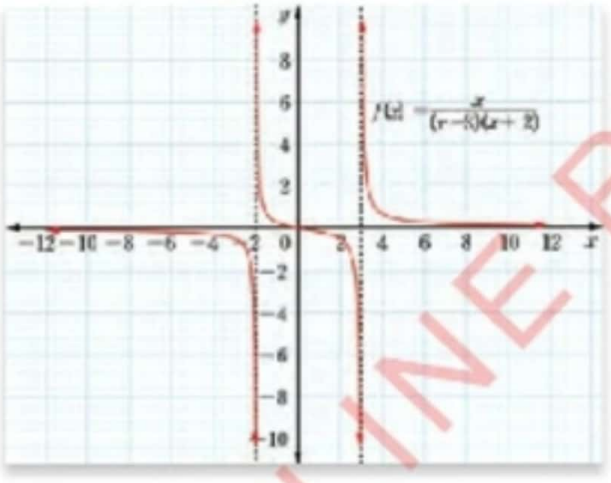
Graphs of rational functions

Teaching steps

1. Devise strategies for assisting students to brainstorm on how to recognize rational functions. Assist them in identifying the numerator and denominator of a rational function.
2. Through discussion introduce to students the concept of an asymptote. Assist students in exploring the nature of horizontal, vertical, and oblique asymptotes.

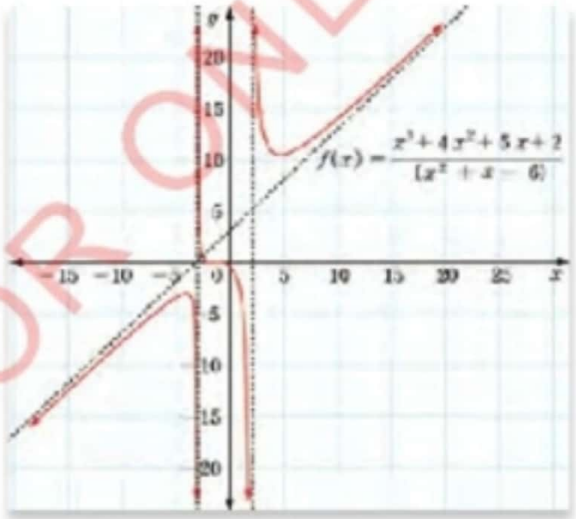
3. Guide students to discuss the steps for sketching graphs of rational functions. Allow students to share alternative steps of sketching graphs of rational functions.
4. Use Examples 4.11 to 4.17 in the Student's Book to guide students to sketch graphs of various rational functions. Engage students to use GeoGebra or any other mathematical software to verify the answers of the given examples. Advise them to make presentations for more inputs.
5. Instruct students to attempt Exercise 4.5 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback where necessary.

Answers to Exercise 4.5

1. 

Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 2\}$

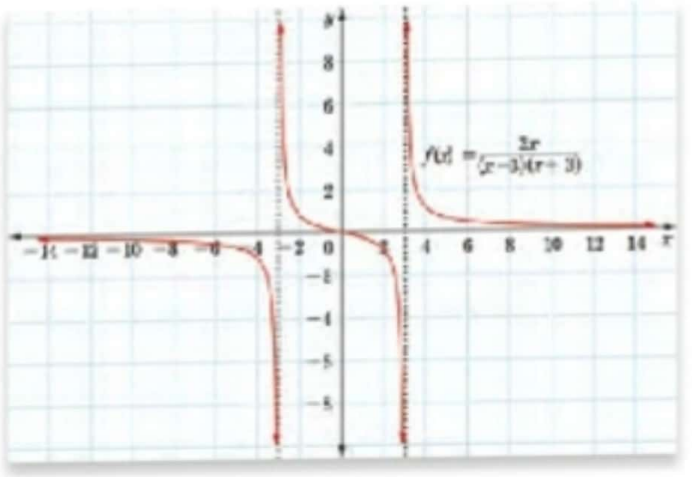
Range $= \{y : y \in \mathbb{R}\}$

2. 

Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 2\}$

Range $= \{y : y \in \mathbb{R}\}$

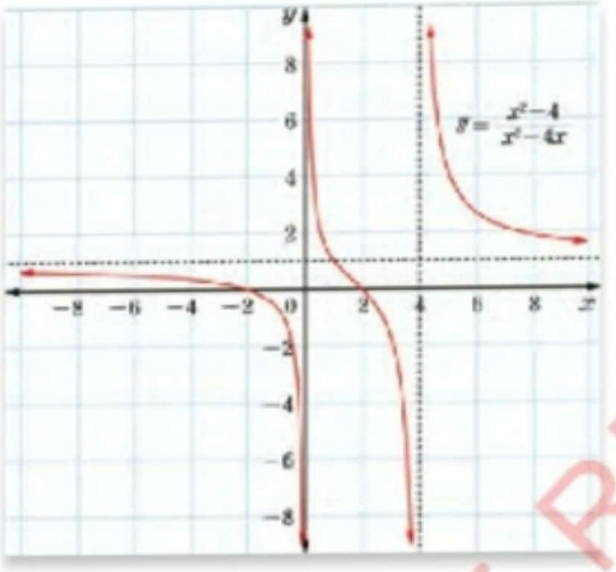
3.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $= \{y : y \in \mathbb{R}\}$

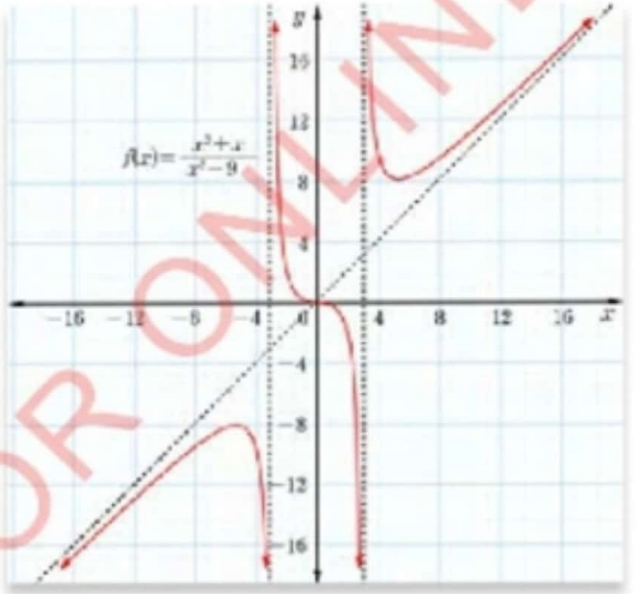
4.



Domain $= \{x : x \in \mathbb{R}, x \neq 0, x \neq 4\}$

Range $= \{y : y \in \mathbb{R}\}$

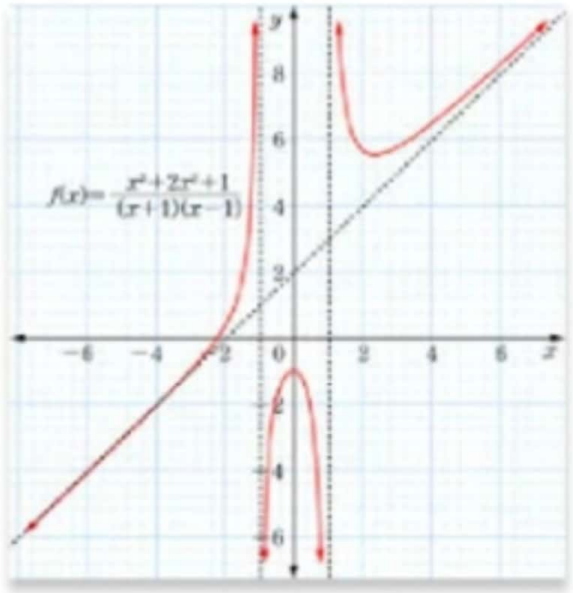
5.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $= \{y : y \in \mathbb{R}\}$

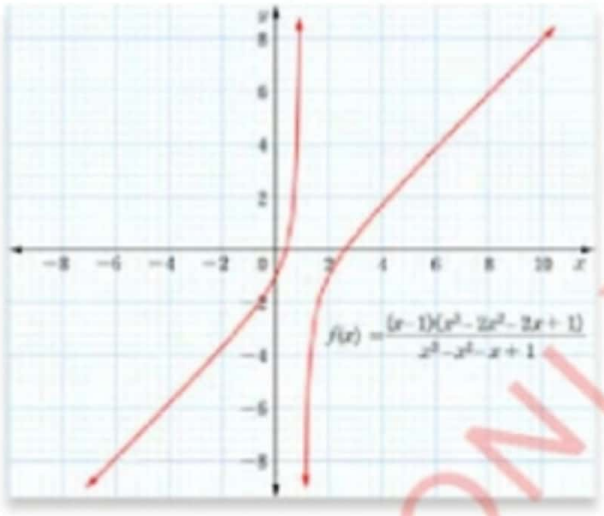
6.



Domain $=\{x: x \in \mathbb{R}, x \neq -1, x \neq 1\}$

Range $=\{y: y \in \mathbb{R}\}$

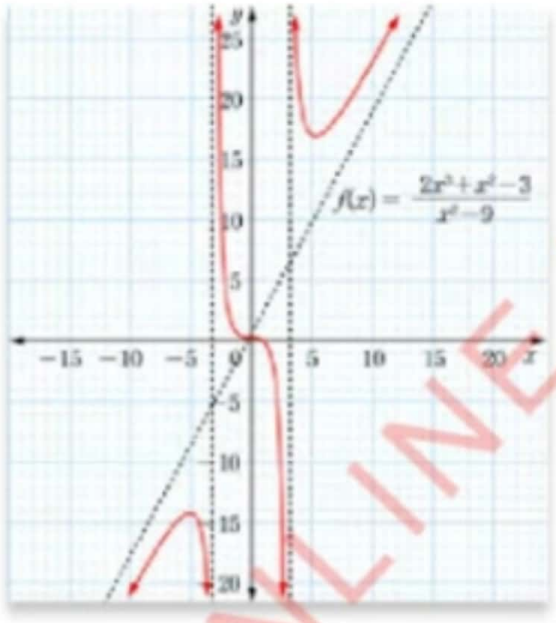
7.



Domain $=\{x: x \in \mathbb{R}, x \neq 1\}$

Range $=\{y: y \in \mathbb{R}\}$

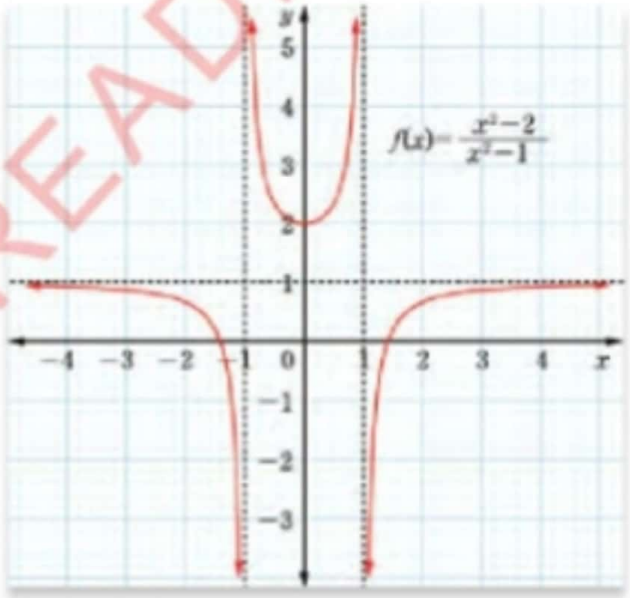
8.



Domain $=\{x: x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $=\{y: y \in \mathbb{R}\}$

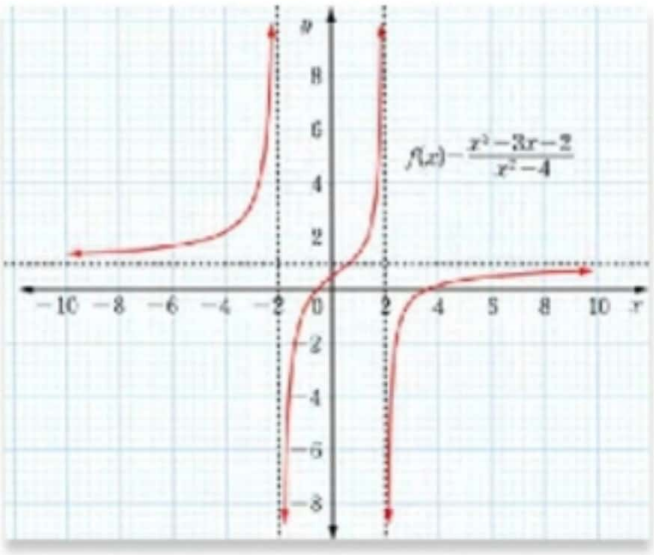
9.



Domain $=\{y: y \in \mathbb{R},\ x \neq -1,\ x \neq 1\}$

Range $=\{y: y \in \mathbb{R},\ y < 1,\ y \geq 2\}$

10.



Domain $=\{x : x \in \mathbb{R}, x \neq -2, x \neq 2\}$

Range $=\{y : y \in \mathbb{R}\}$

Composite functions

Teaching steps

1. Guide students through discussion to explain the concept of composite function. Assist students in identifying inner and outer functions of a composite function.
2. Engage students to perform Activity 4.2 in the Student's Book to verify whether the composite of two linear functions is also a linear function. Advise students to make presentations for more inputs.
3. Guide students through discussions to outline the steps to form a composite function. Through discussions ask them to decide on the inner and outer functions.
4. Use Examples 4.18 to 4.24 in the Student's Book to guide students to substitute the inner function into the variable of the outer function and assist them with further simplifications
5. Instruct students to attempt Exercise 4.6 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them feedback where necessary.

Answers to Exercise 4.6

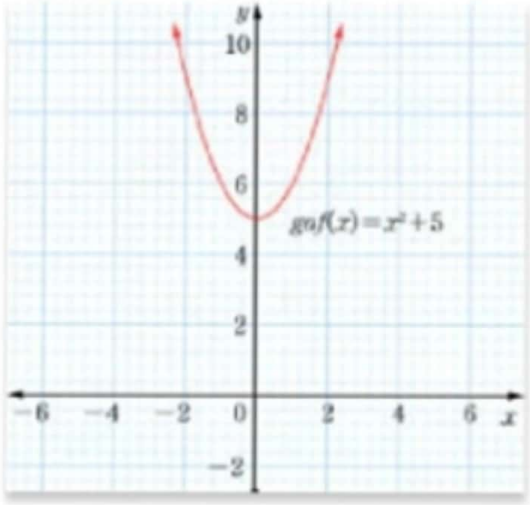
1. $f(g(x)) = -3x+17$, $g(f(x)) = -3x-1$. Not commutative.
2. (a) $f(x) = \dfrac{x^2-3x+2}{2-x}$, $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$ (b) $f(x) = 3x+2$, $f\left(\dfrac{1}{2}\right) = \dfrac{7}{2}$
3. $(g \circ f)(x) = \{(-1, 1), (0, 5)\}$ Domain $= \{x : -1, 0\}$, Range $= \{y : 1, 5\}$
4. (a) $(f \circ g)(x) = x^2+6x+7$ (c) $(f \circ g)(x) = e^{x^2}-2$
   (b) $(f \circ g)(x) = x^2-7$ (d) $(f \circ g)(x) = x$
6. -7 7. $x = \pm 5$
8. (a) 34 (b) 112 (c) 584 (d) 130
   (e) 2706 (f) x^2-2x+3
9. (a) $32x^2-312x+685$ (b) $512x^3-4576x+10149$
10. (a) $f \circ g = \{(7, 5), (-5, 8)\}$ and $g \circ f = \{(3, 10), (2, 7), (4, 3)\}$
11. $g \circ f = \{(-2, 1), (0, 3)\}$ 12. $f \circ g = \{(2, 6), (4, 7)\}$

Graphs of composite functions

Teaching steps

1. Assist students through discussion to realize that, the procedures for drawing graphs of composite functions are similar to those of other functions.
2. Use Examples 4.25 and 4.26 in the Student's Book to guide students to sketch graphs of composite functions. Engage students to use GeoGebra or any other mathematical software to verify the answers for the given examples. Advise them to share their findings for more inputs.
3. Instruct students to attempt Exercise 4.7 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback.

Answers to Exercise 4.7

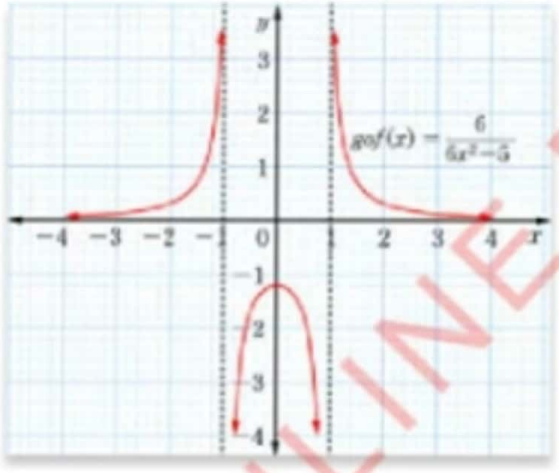
1. 

Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}, y \geq 5\}$

2. $g(x) = x - 3$ and $g(x) = -x + 1$

3. $(f \circ g)(x) = \{(3, 6)\}$

4. 

5. $\sqrt{3}x + \dfrac{10 - 2\sqrt{3}}{2\sqrt{3}}$ or $-\sqrt{3}x - \left(\dfrac{10 + 2\sqrt{3}}{2\sqrt{3}}\right)$

6. $f(x) = \dfrac{x+1}{x^2 - 2x + 5}$

7. $k = \dfrac{5}{3}$

8. $x = \pm 4$ or $x = \pm 2$

9. x-intercept $(x, y) = (1, 0)$

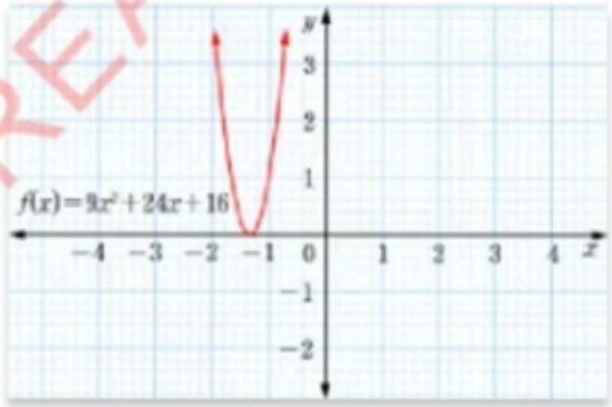
   y-intercept $(x, y) = \left(0, -\dfrac{1}{2}\right)$

   Vertical asymptote $x = -2$

   Horizontal asymptote $y = 1$

10. (a) $3x^2 + 4$

    (b) $9x^2 + 24x + 16$



11. (a) $\dfrac{1}{4}x^2 - x + 1$ (b) 7.5

12. (a) $(f \circ g)(x) = x + 9$ and $(g \circ f)(x) = x + 9$

    (b) $(f \circ g)(x) = x^2 + 2$ and $(g \circ f)(x) = x^2 + 2$

    (c) $(f \circ g)(x) = -9x + 19$ and $(g \circ f)(x) = 3 - 9x$

Graphs of exponential functions

Teaching steps

1. Guide students through discussion to recognize the nature of graphs of exponential functions. Assist students to discuss the procedures for sketching graphs of exponential functions.
2. Use Examples 4.27, 4.28, and 4.29 in the Student's Book to guide students to sketch graph of exponential functions. Engage students to use GeoGebra or any other mathematical software to verify the answers of the given examples. Advise students to make presentations from their findings for more inputs.
3. Instruct the students to attempt Exercise 4.8 in the Student's Book. Advise students to submit their work, check the correctness of the answers and provide them constructive feedback where necessary.

Answers to Exercise 4.8

1. Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > 0\}$

2. (a) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > -4\}$

   (b) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y < 0\}$

   (c) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > -3\}$

   (d) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y < 1\}$

   (e) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > 3\}$

6. (a) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > 7\}$

   (b) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > -1\}$

   (c) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > 0\}$

   (d) Domain $= \{x : x \in \mathbb{R}\}$
   Range $= \{y : y \in \mathbb{R},\ y > 1\}$

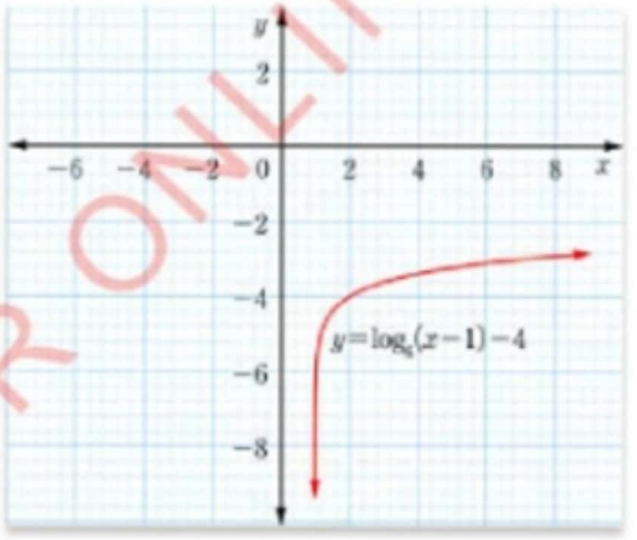
Graphs of logarithmic functions

Teaching steps

1. Guide students through discussion to recognize the nature of graphs of exponential functions. Assist students in using the same procedures of sketching graphs of exponential functions to sketch graph of logarithmic functions.
2. Design an activity that will engage students in sketching graphs of logarithmic functions.
3. Use Examples 4.30 to 4.34 in the Student's Book to guide students to sketch graphs of logarithmic functions. Engage students to use GeoGebra or any other mathematical software to verify the answers of the given examples. Advise them to make presentations for more inputs.
4. Instruct students to attempt Exercise 4.9 in the Student's Book. Advise students to submit their work, check the correctness of the answers, and provide them constructive feedback where necessary.

Answers to Exercise 4.9

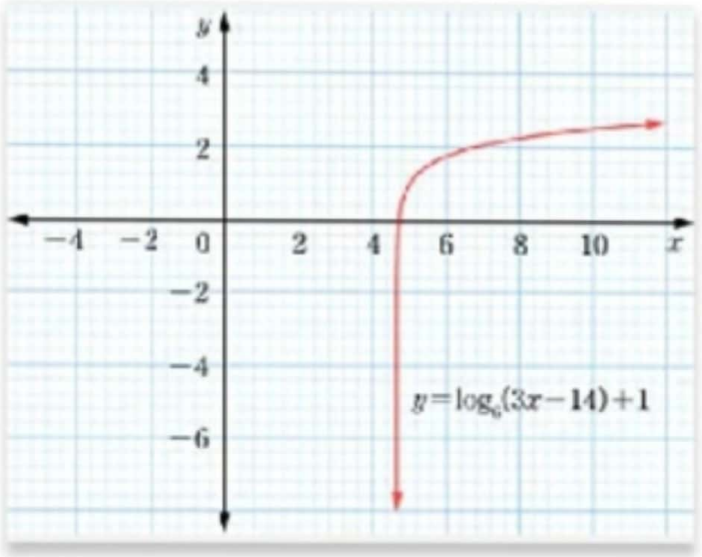
1. (a)



Domain $= \{x : x \in \mathbb{R}, x > 1\}$

Range $= \{y : y \in \mathbb{R}\}$

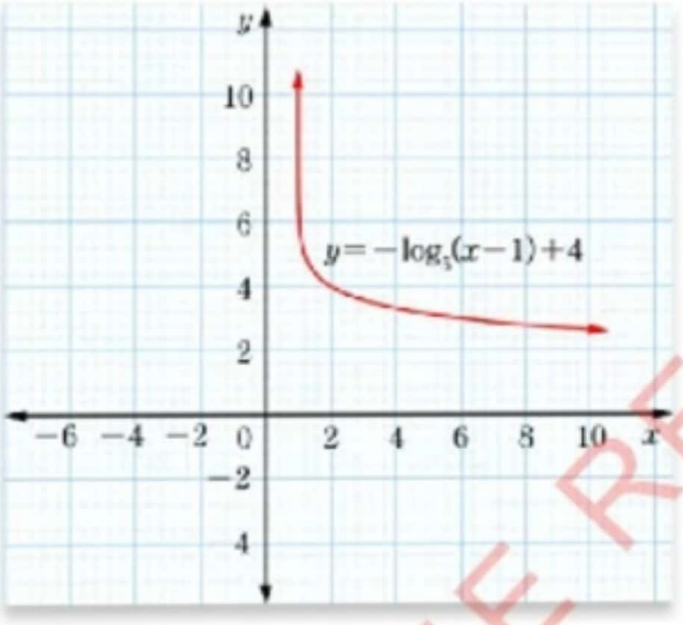
(b)



$$\text{Domain} = \left\{x : x \in \mathbb{R},\ x > \frac{14}{3}\right\}$$

$$\text{Range} = \{y : y \in \mathbb{R}\}$$

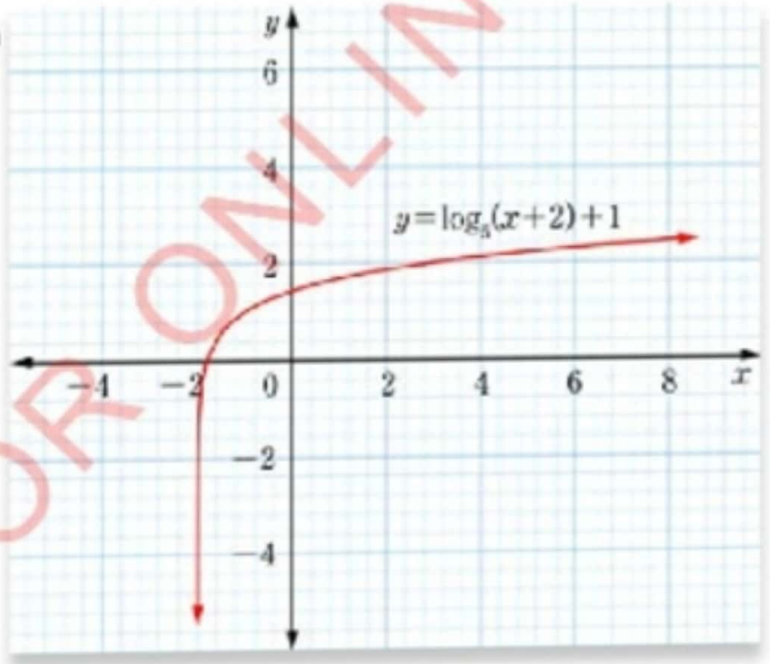
(c)



$$\text{Domain} = \{x : x \in \mathbb{R},\ x > 1\}$$

$$\text{Range} = \{y : y \in \mathbb{R}\}$$

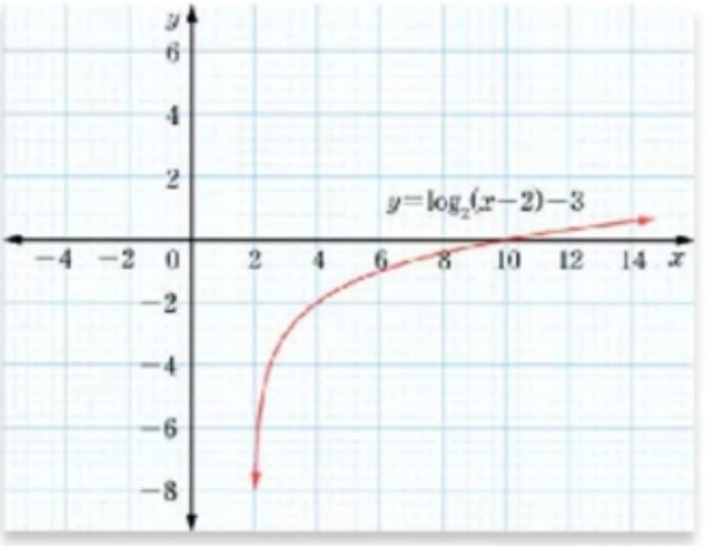
(d)



$$\text{Domain} = \{x : x \in \mathbb{R},\ x > -2\}$$

$$\text{Range} = \{y : y \in \mathbb{R}\}$$

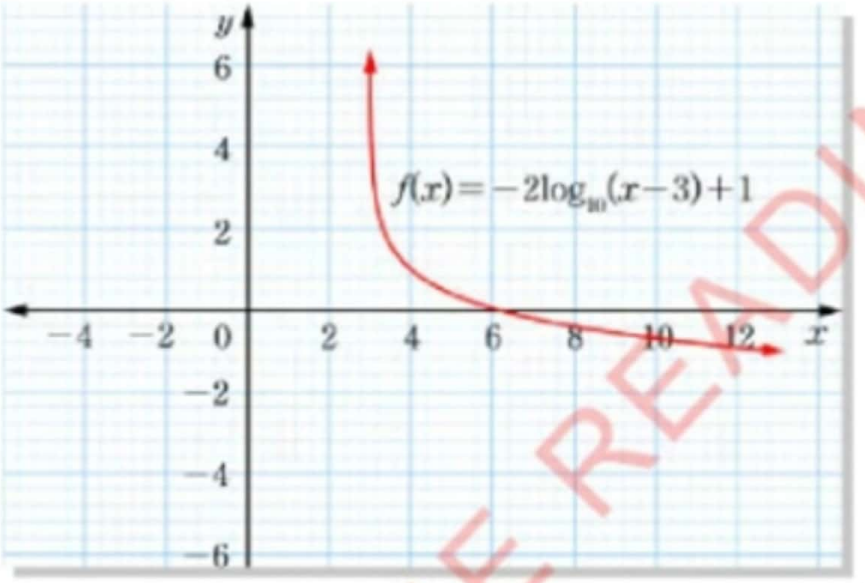
(e)



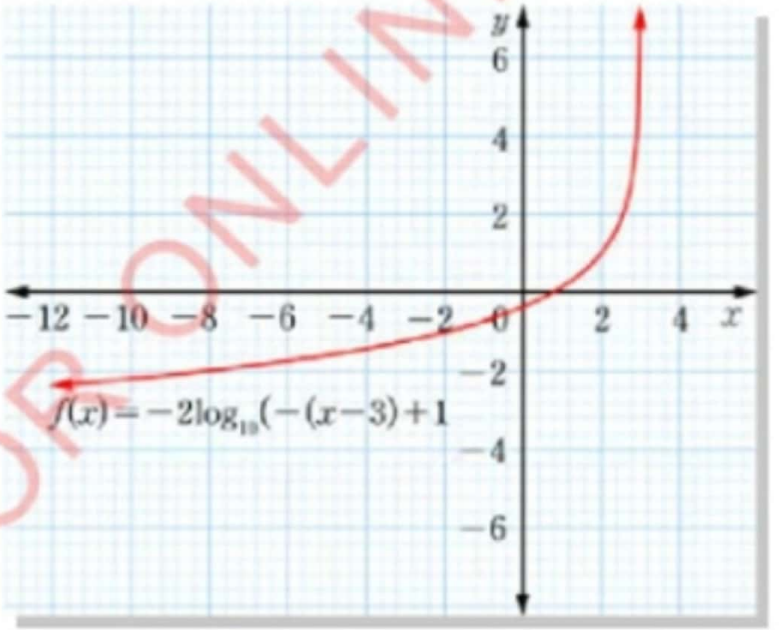
Domain $=\{x : x \in \mathbb{R},\ x > 2\}$

Range $=\{y : y \in \mathbb{R}\}$

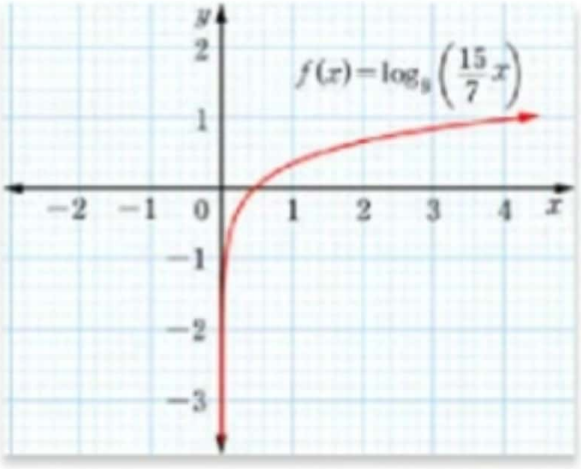
2.



3.

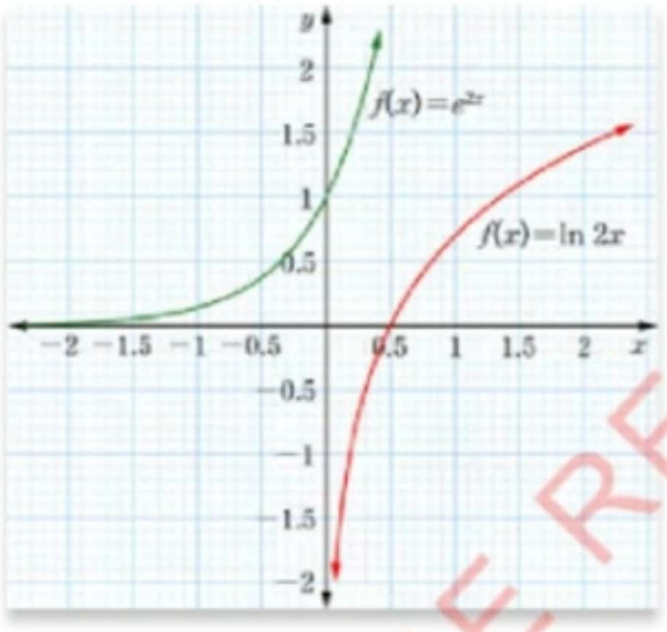


4.

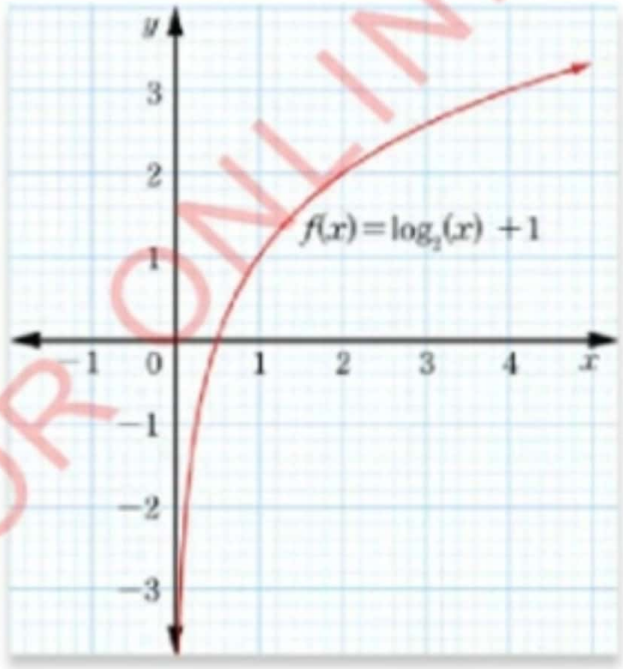


Domain=$\{x : x \geq 0\}$, Range =$\{y : y \in \mathbb{R}\}$

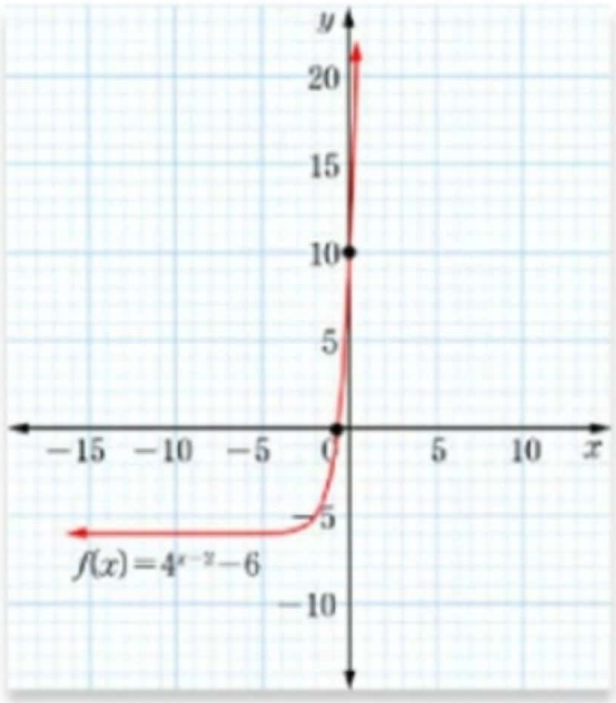
5.



6.



7.



(a) Domain=$\{x : x \in \mathbb{R}\}$,

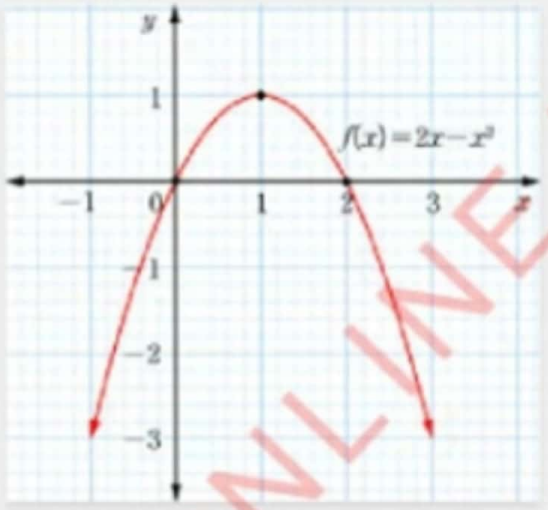
Range =$\{y : y > -6\}$

(b) x-intercept $=(-0.7, 0)$

y-intercept $=(0, 10)$

Answers to Revision Exercise 4

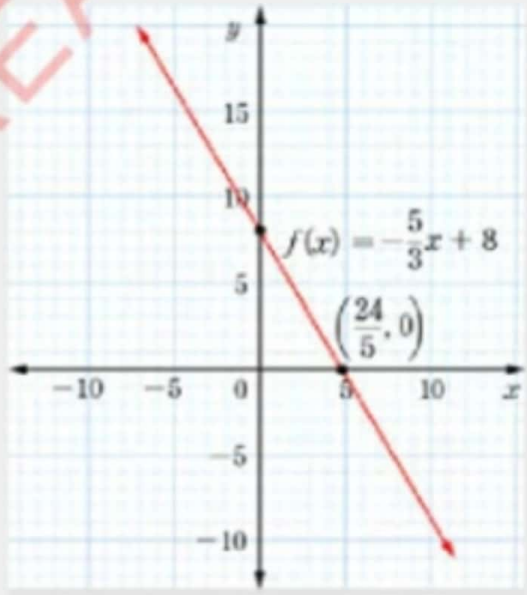
1.



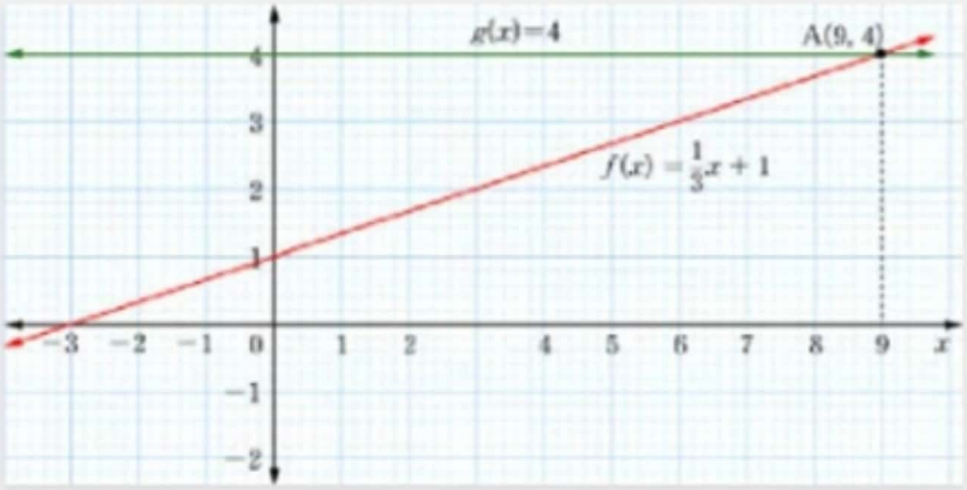
Domain=$\{x : x \in \mathbb{R}\}$,

Range =$\{y : y \leq 1\}$.

2. (a)

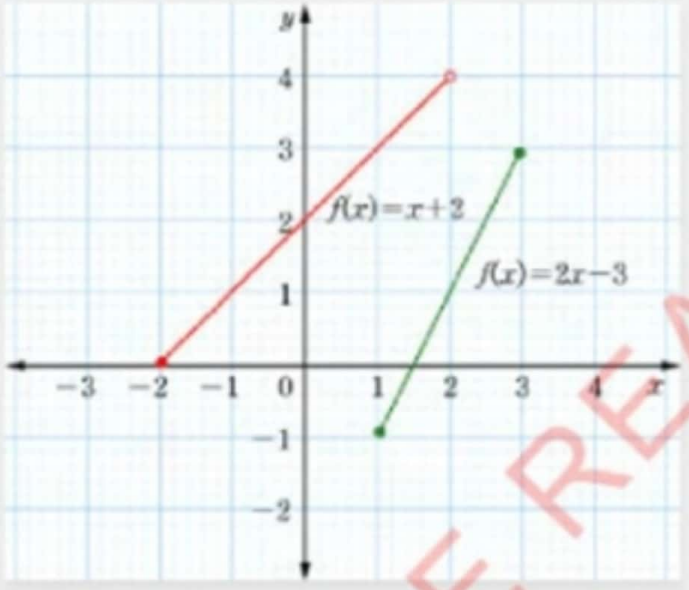


(b)

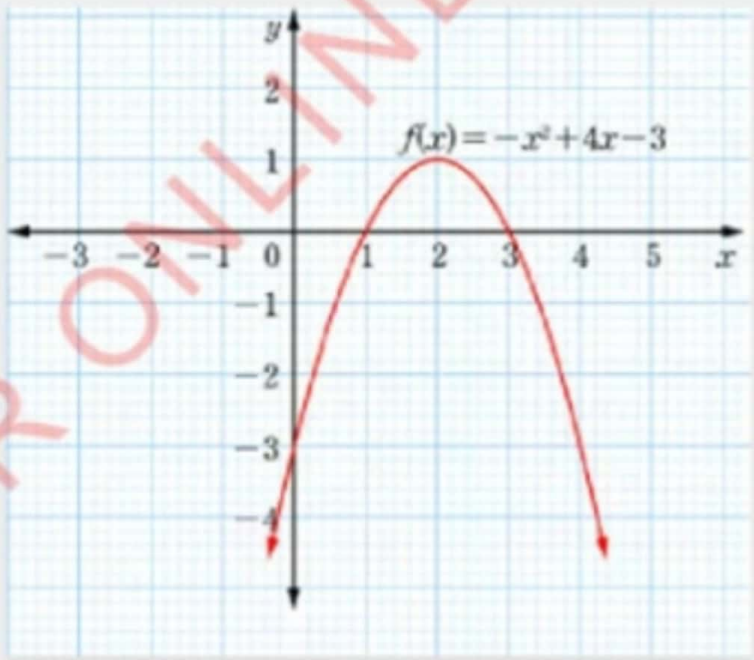


The point is (9, 4).

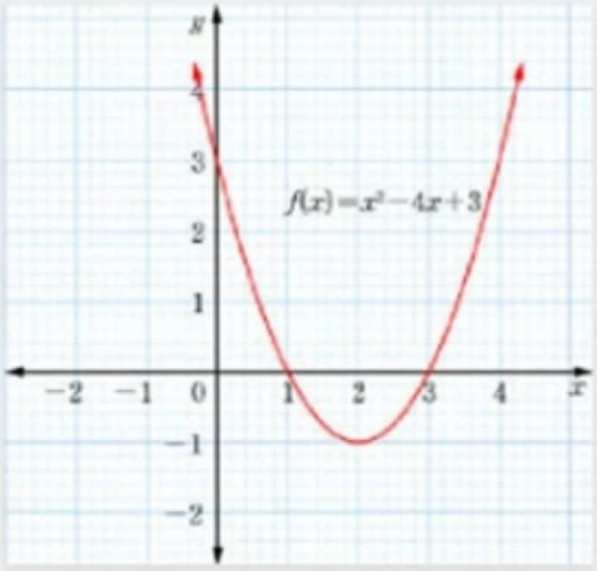
(c)



3. (a)



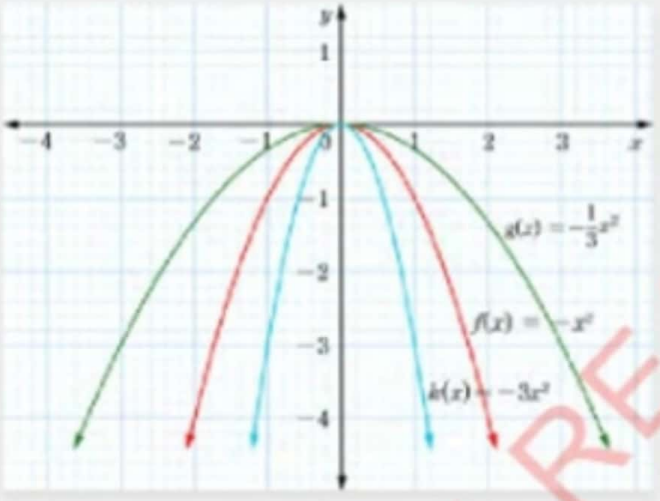
(b)



Domain=$\{x : x \in \mathbb{R}\}$,

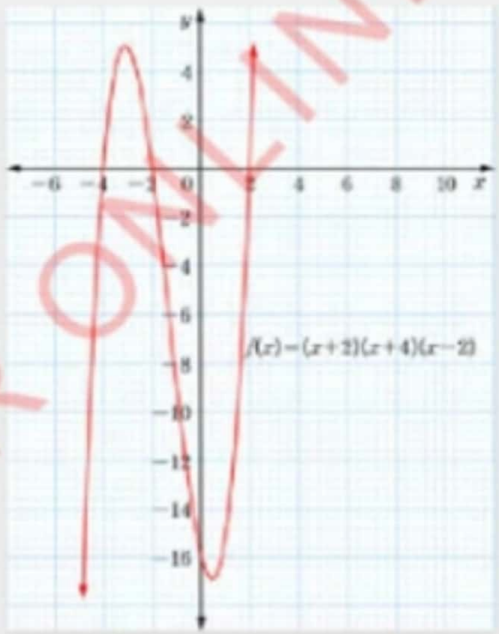
range =$\{y : y \geq -1\}$

(c)



4. Domain =$\{x : x = 1 \text{ and } x = 2\}$

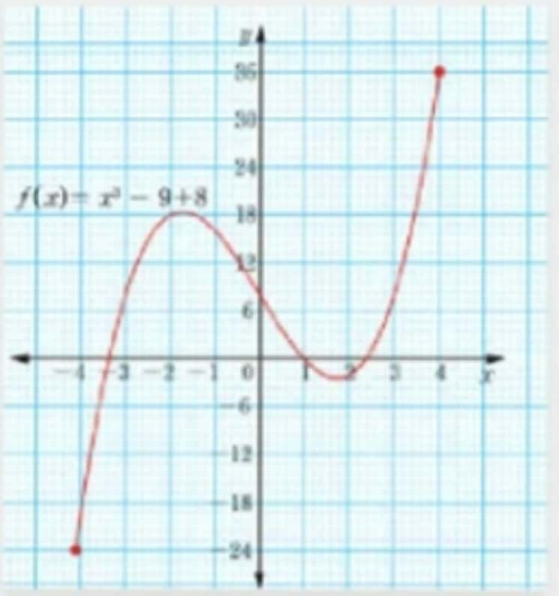
5. (a)



Domain =$\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

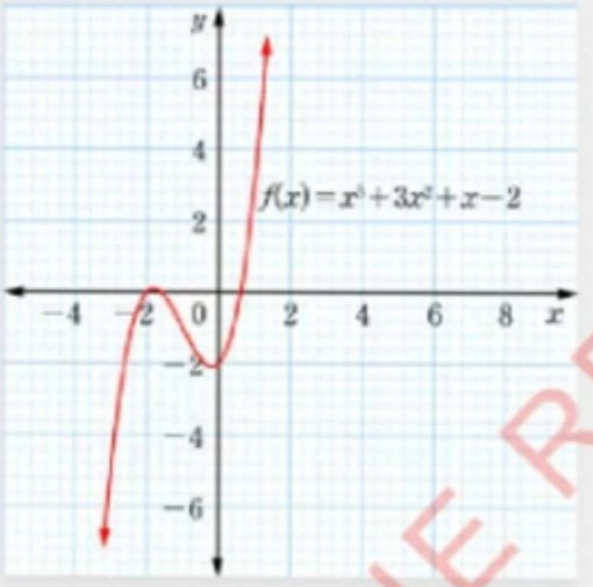
(b)



Domain $=\{x : x \in \mathbb{R}, -4 \leq x \leq 4\}$

Range $=\{y : y \in \mathbb{R}, -20 \leq y \leq 36\}$

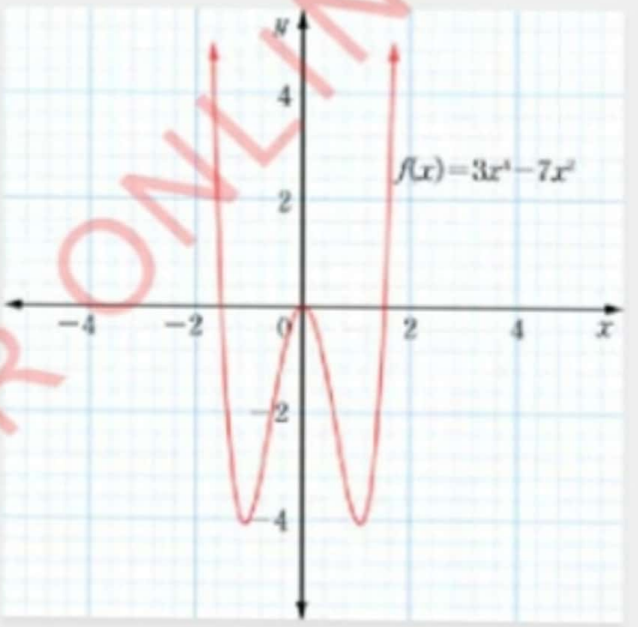
(c)



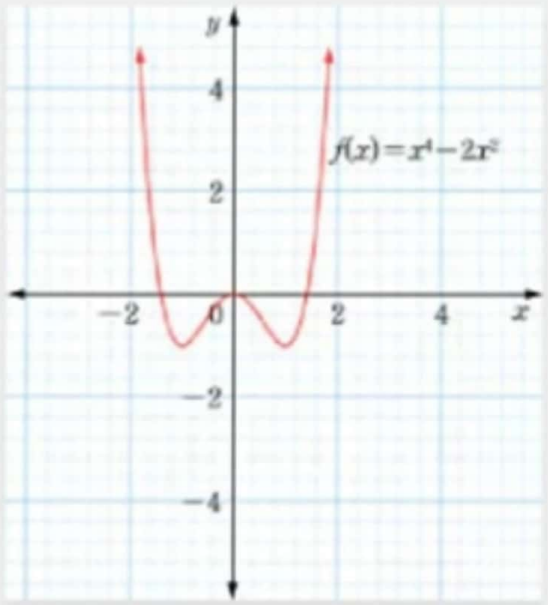
Domain $=\{x : x \in \mathbb{R}\}$

Range $=\{y : y \in \mathbb{R}\}$

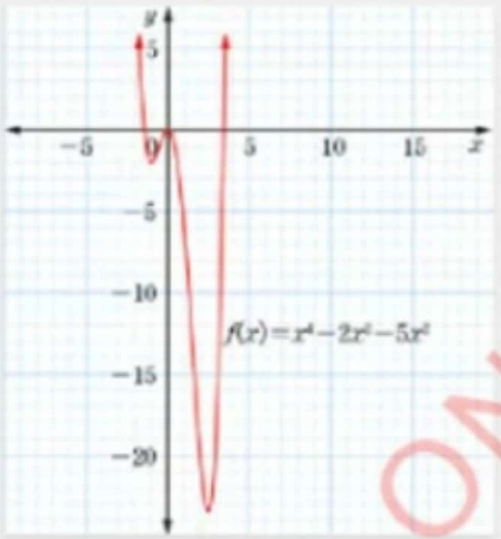
6. (a)



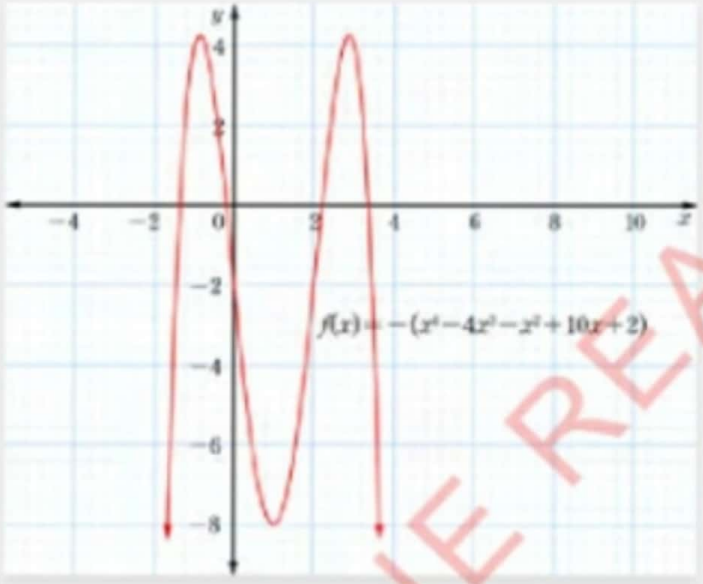
(b)



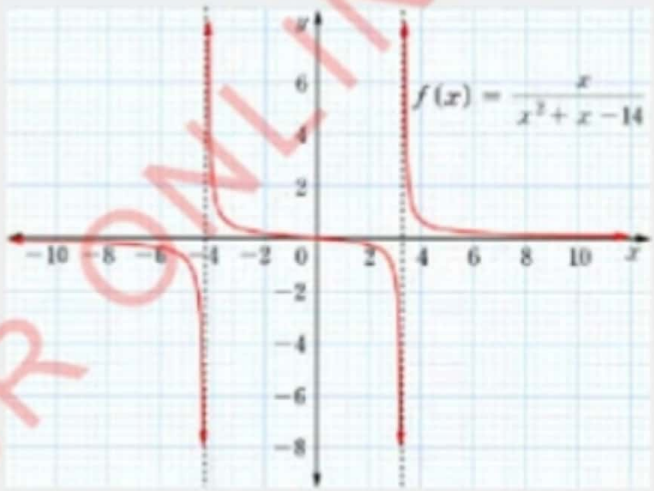
(c)



(d)



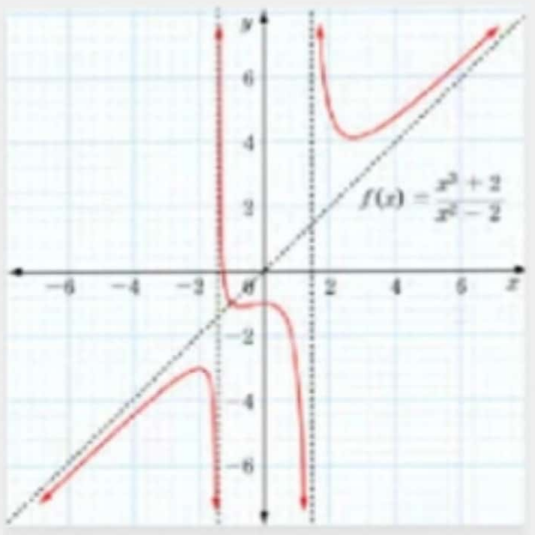
7. (a)



Domain $= \{x : x \in \mathbb{R},\ x \neq -4.27,\ x \neq 3.27\}$

Range $= \{y : y \in \mathbb{R}\}$

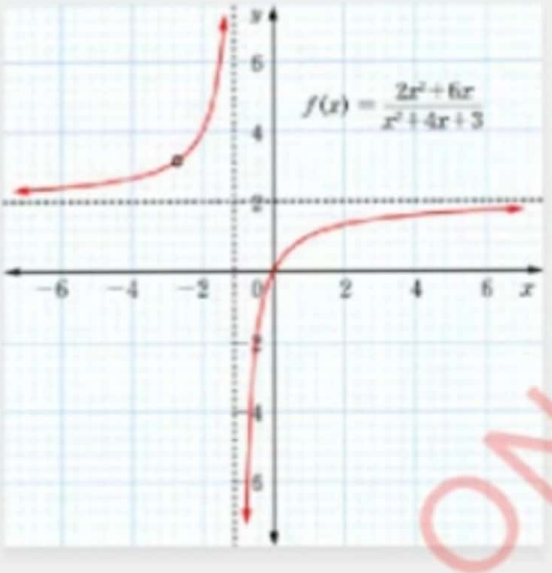
(b)



$\text{Domain} = \{x : x \in \mathbb{R}, x \neq \pm\sqrt{2}\}$

$\text{Range} = \{y : y \in \mathbb{R}\}$

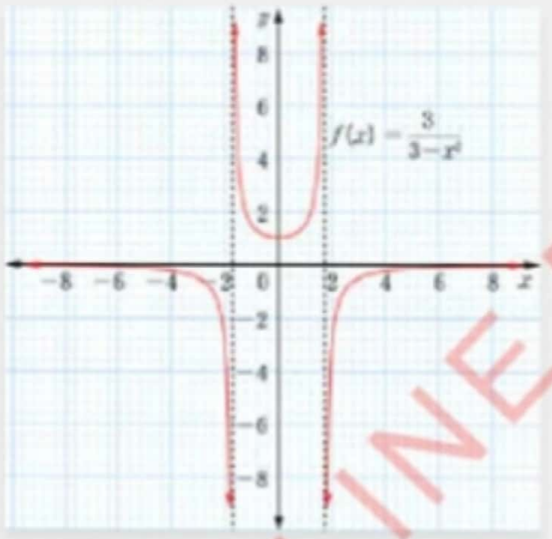
(c)



$\text{Domain} = \{x : x \in \mathbb{R}, x \neq -3, x \neq -1\}$

$\text{Range} = \{y : y \in \mathbb{R}, y \neq 2\}$

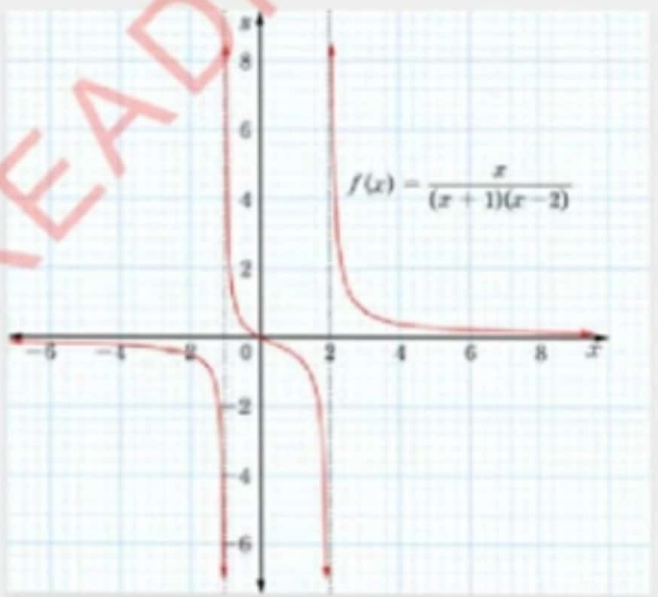
(d)



$\text{Domain} = \{x : x \in \mathbb{R}, x \neq \pm\sqrt{3}\}$

$\text{Range} = \{y : y \in \mathbb{R}, y < 0, y \geq 1\}$

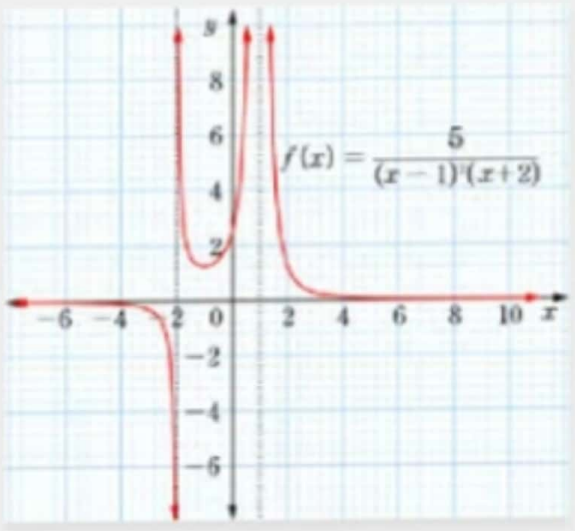
(e)



$\text{Domain} = \{x : x \in \mathbb{R}, x \neq -1, x \neq 2\}$

$\text{Range} = \{y : y \in \mathbb{R}\}$

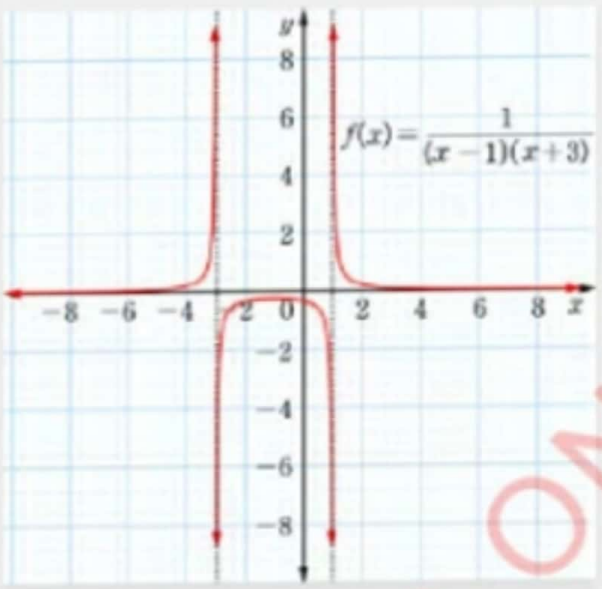
(f)



Domain $=\{x: x\in\mathbb{R},\ x\neq -2, x\neq 1\}$

Range $=\{y: y\in\mathbb{R},\ y\neq 0\}$

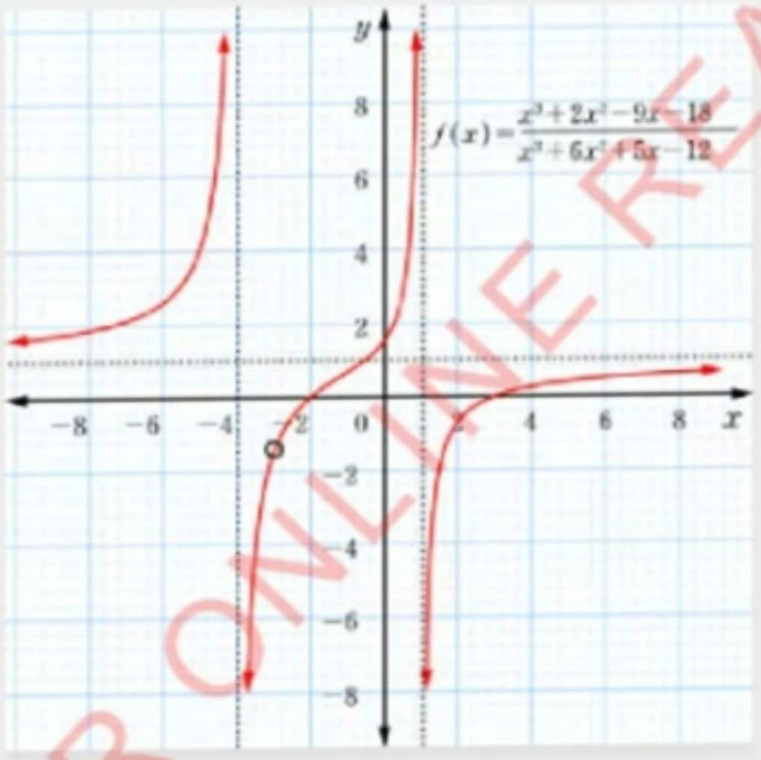
(g)



Domain$=\{x: x\in\mathbb{R}, x\neq -3,\ x\neq 1\}$

Range $=\left\{y: y\in\mathbb{R},\ y<-\frac{1}{4},\ y>0\right\}$

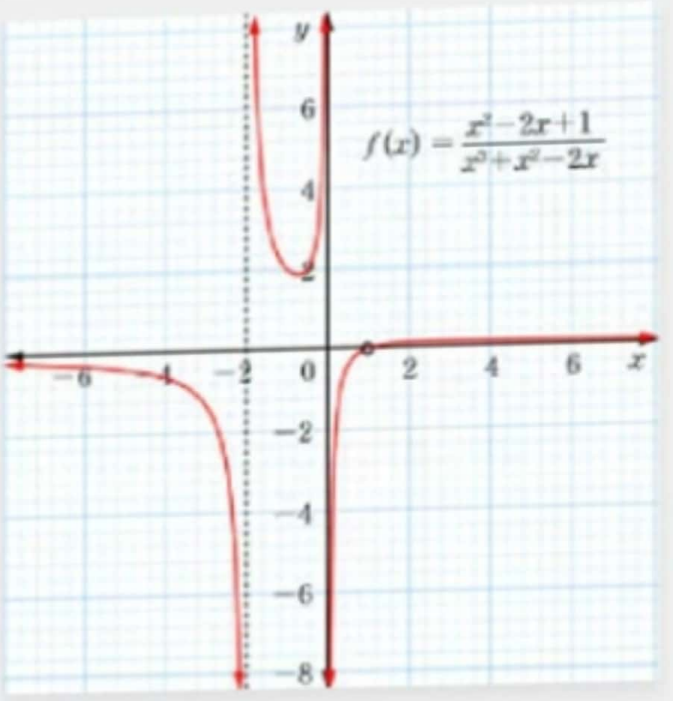
(h)



Domain $=\{x: x\in\mathbb{R},\ x\neq -4, -3, x\neq 1\}$

Range $=\{y: y\in\mathbb{R}\}$

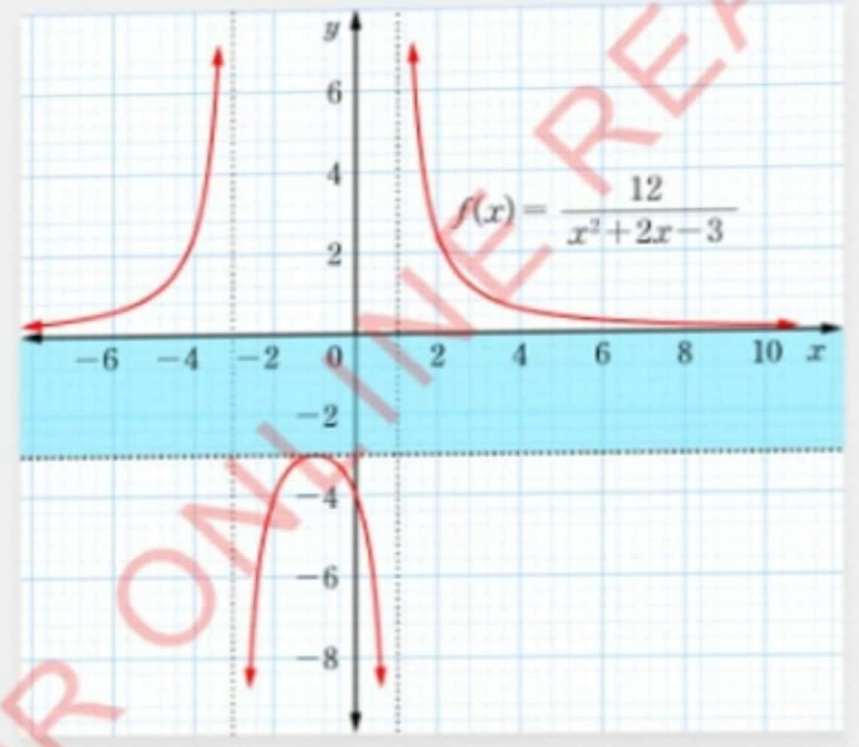
(i)



Domain=$\{x : x \in \mathbb{R}, x \neq -2,\ x \neq 0, x \neq 1\}$

Range=$\{y : y \in \mathbb{R},\ y \geq 1.8 \text{ and } y \leq 0.3\}$

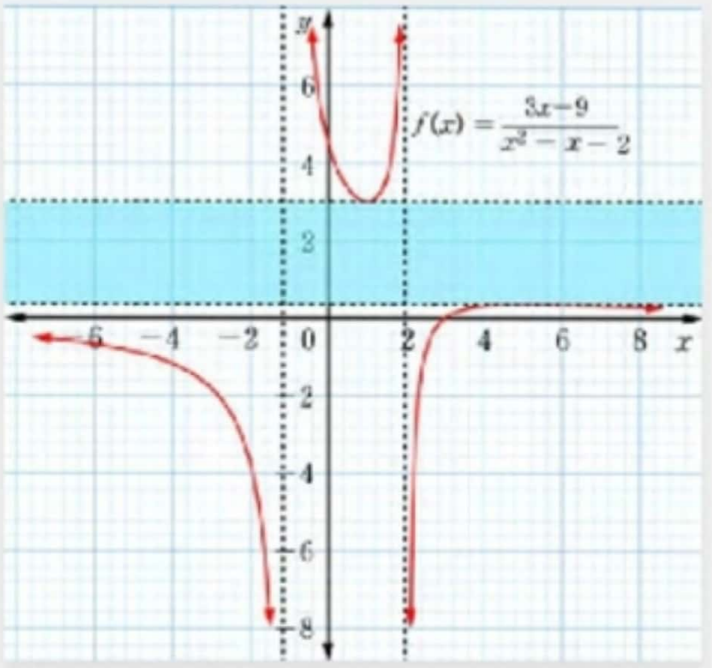
(j)



Domain=$\{x : x \in \mathbb{R}, x \neq -3,\ x \neq 1\}$

Range=$\{y : y \in \mathbb{R},\ y \leq -3 \text{ or } y > 0\}$

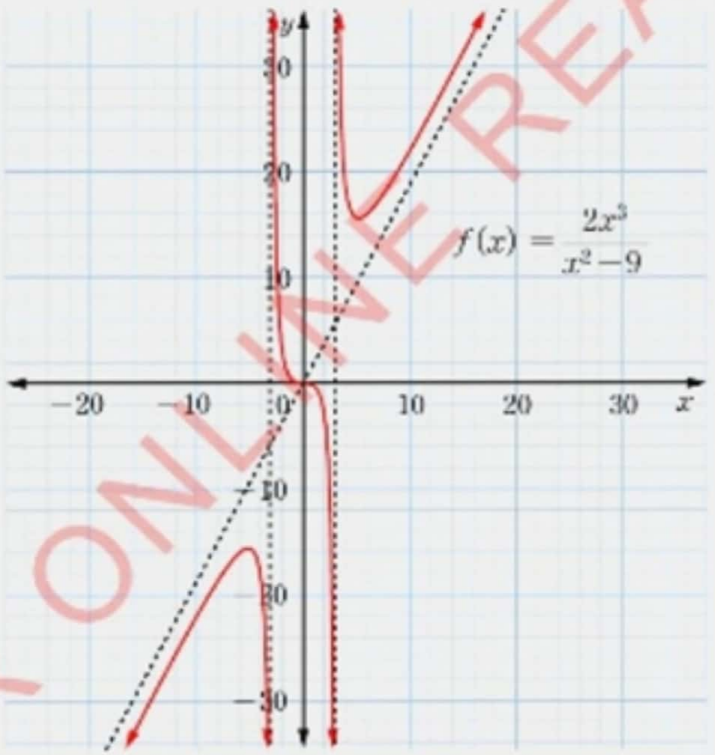
(k)



Domain=$\{x : x \in \mathbb{R}, x \neq -1,\ x \neq 2\}$

Range=$\left\{y : y \in \mathbb{R},\ y \geq 3,\ y \leq \frac{1}{3}\right\}$

(m)



Domain=$\{x : x \in \mathbb{R}, x \neq -3,\ x \neq 3\}$

Range=$\{y : y \in \mathbb{R}\}$

8. (a) $f \circ g(x) = \{(2,-2), (3, 2)\}$ (b) $g \circ f(x) = \{(3, 1), (4, 6)\}$

9. (a) $f \circ g(x) = \{(7,12), (-1,19), (9,15)\}$ (b) $g \circ f(x) = \{(1,3), (2,4)\}$

10. (a) $f \circ g(x) = x^2 - 6x + 10$ (b) $g \circ f(x) = x^2 + 2x - 2$

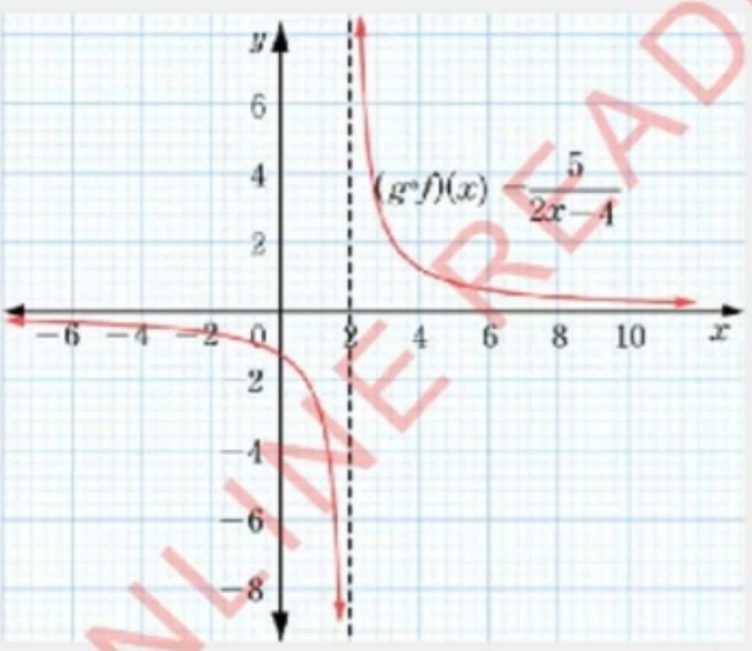
(c) $g \circ f \circ h(x) = \cos^2 x + 2\cos x - 2$

11. $f(x) = \pm 3x - \dfrac{5}{4}$

13. (a) $(f \circ g)(x) = \dfrac{-2x+14}{x-2}$ (c) $(f \circ g)(3) = 8$

(b) $(g \circ f)(x) = \dfrac{5}{2x-4}$ (d) $(g \circ f)(x) = \dfrac{5}{4}$

(e)



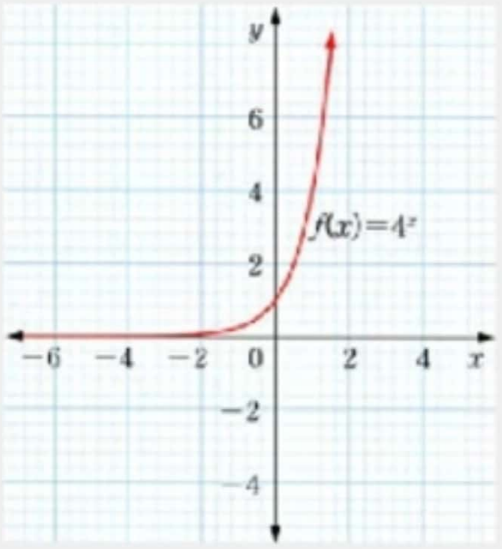
Domain $= \{x : x \in \mathbb{R},\ x \neq 2\}$

Range $= \{y : y \in \mathbb{R},\ y \neq 0\}$

14. $x = \pm 2.91$ 15. (a) $\dfrac{3-3x}{x^2+4x+4}$ (b) $\dfrac{1}{4}$

16. (a) 2 (b) 4

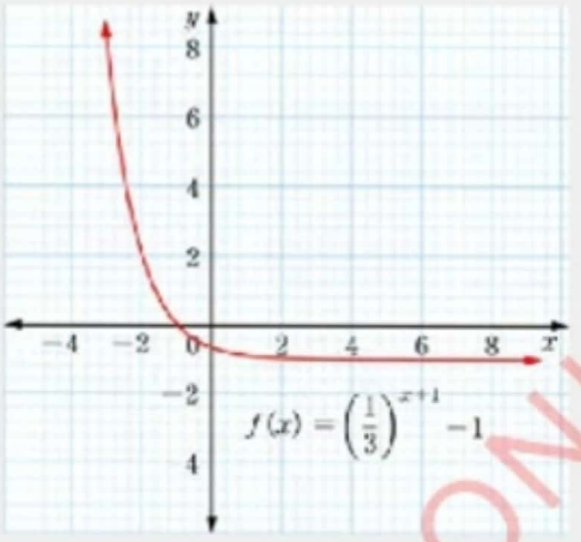
17. (b)



Domain $=\{x : x \in \mathbb{R}\}$

Range $=\{y : y > 0\}$

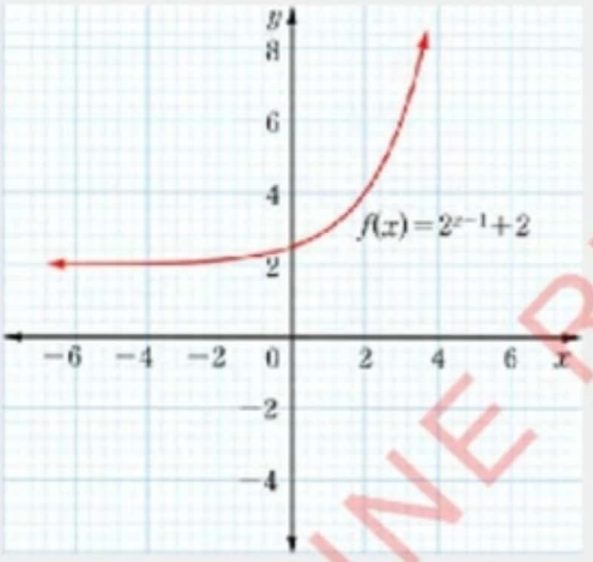
(c)



Domain $= \{x : x \in \mathbb{R}\}$

Range$=\{y : y > -1\}$

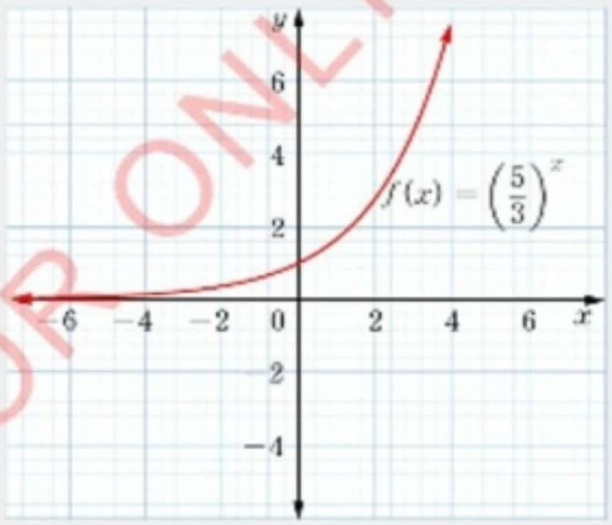
(d)



Domain$=\{x : x \in \mathbb{R}\}$

Range$=\{y : y > 2\}$

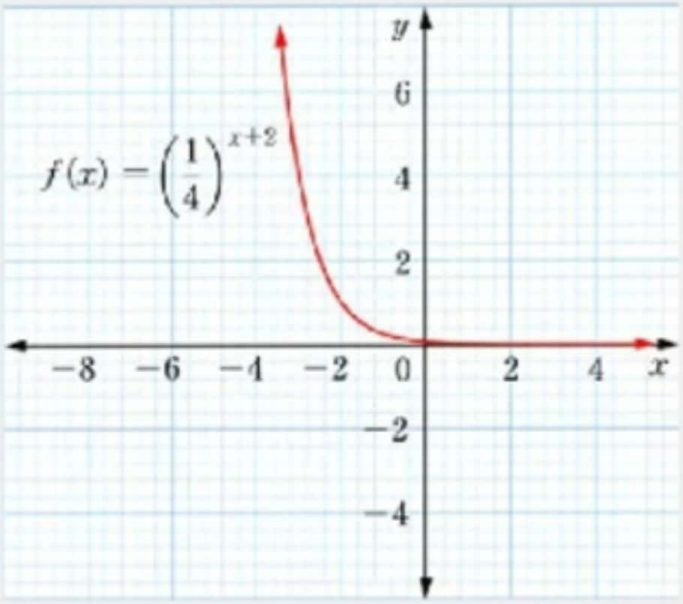
(e)



Domain$=\{x : x \in \mathbb{R}\}$

Range $= \{y : y > 0\}$

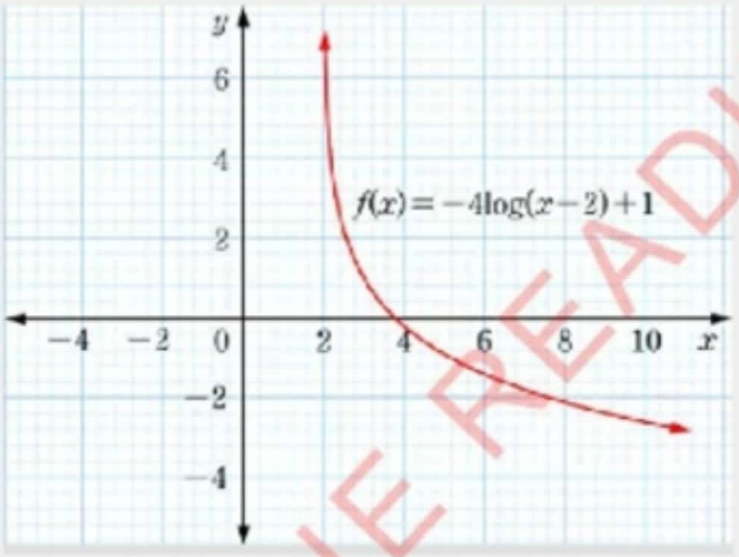
(f)



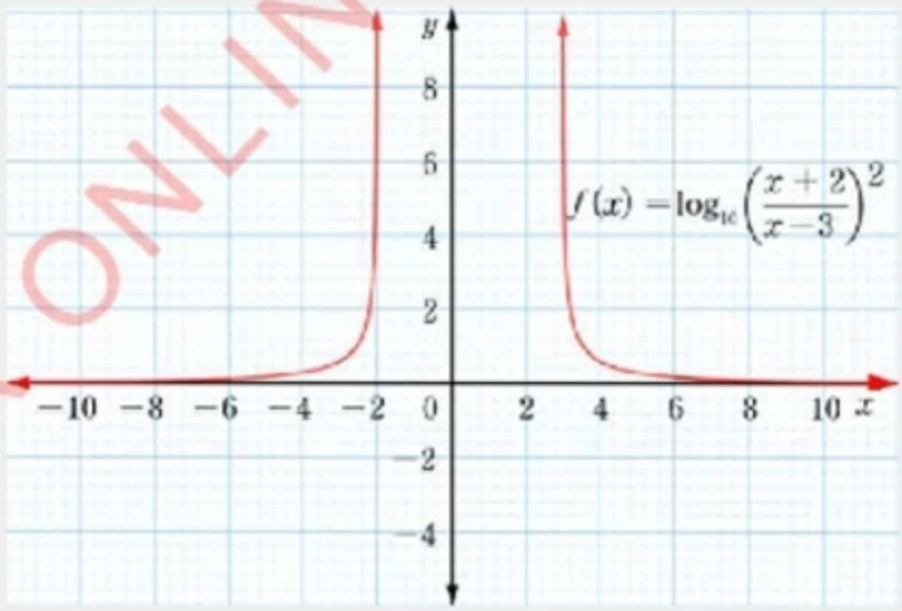
Domain=$\{x : x \in \mathbb{R}\}$

Range=$\{y : y \geq 0\}$

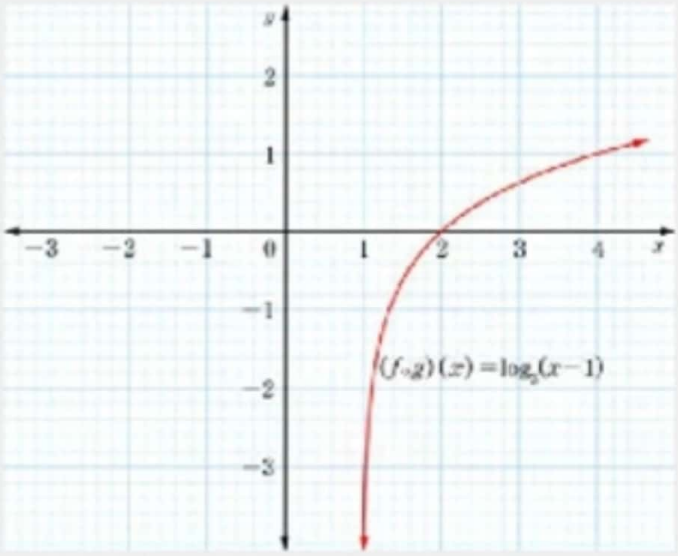
18. (a)



(b)



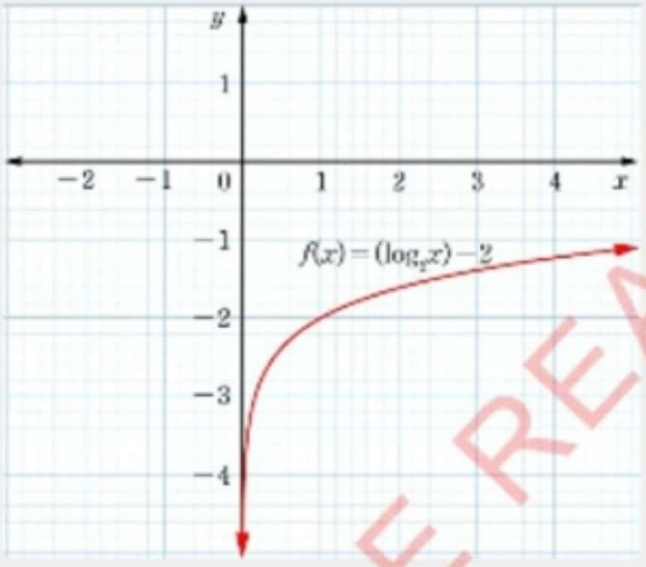
19. (a)



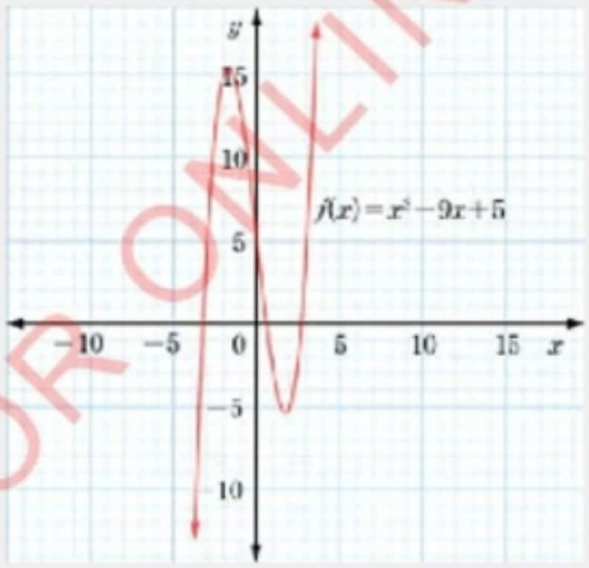
$\text{Domain} = \{x : x \in \mathbb{R},\ x > 1\}$

$\text{Range} = \{y : y \in \mathbb{R}\}$

(b)



20.



Chapter Five

Algebra

Introduction

In this chapter, students will learn about sequences and series, proof by mathematical induction, roots of polynomial functions, remainder and factor theorems, operations on polynomials, inequalities, matrices, binomial theorem, and partial fractions. Guide the students to realize some common methods of finding sums of series of squares and cubes of natural numbers, solving roots of quadratic and rational inequalities, operating with 3×3 *matrices, forming partial fractions, proof by mathematical induction, and expanding expressions using the binomial theorem. The competencies developed will enable students in solving various real life problems including solving problems related to landscape designing, computer programming, real estate planning, business and finance management, geometry, budgeting, scheduling of activities, cooking, shopping, and in many other fields, enable them to perform various tasks such as organizing, creating and categorizing objects.*

Students' activities

(a) Recognising sequences and series

(b) Proofing by mathematical induction

(c) Establishing the relationship between roots of polynomial functions and their coefficients

(d) Determining the value of a function using the remainder theorem

(e) Determining the factors of a polynomial function using the factor theorem

(f) Performing various operations on polynomial functions

(g) Describing the concept of inequalities

(h) Exploring the concept of 3×3 matrices

(i) Applying the binomial theorem to solve problems

(j) Expressing rational fractions in partial fractions

Teaching and learning resources

Scientific calculators, Flip chart, marker pens, ruler, mathematical software such as Maple, GeoGebra, MATLAB, and AI tools

Sequences and series

Teaching steps

1. Guide students in groups to discuss the concept of a sequence through Activity 5.1 in the Student's Book and other resources as suggested. Advise them to present the group responses.
2. Use Examples 5.1 and 5.2 in the Student's Book to guide students on how to find the next terms in a sequence.
3. Develop hands on activities that can be used to assist students to realize the concept of series. Introduce to students the sigma notation $\sum$.
4. Guide students through Think-Ink-Pair-Share strategy to use the sigma notation to the sum of finite and infinite series in the Student's Book.
5. Assist students when they face challenges, and clear their doubts and misconceptions through Examples 5.3 to 5.6 in the Student's Book and discuss with them the summation of finite and infinite series using sigma notation.

6. In groups, discuss how to involve students in deducing the formula for the sum of the first n natural numbers through Activity 5.2 in the Student's Book. Furthermore, through discussion, ask students to deduce the sum of the squares of the first n natural numbers. Use Examples 5.7 and 5.8 in the Student's Book, to assist students in deducing the sum of the given expressions.
7. Instruct students through Activity 5.3 in the Student's Book to derive the formula for the sum of the cubes of the first n natural numbers. Advice students to present their findings on flip chart, post them on classroom walls and allow them to do gallery walk for more inputs. Use Example 5.9 in the Student's Book, to assist students in deducing the sum of the given expressions.
8. Require students to attempt Exercise 5.1 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.1

1. (a) $\sum_{k=1}^{5}(3k+5)$ (b) $\sum_{k=1}^{14}2k$ (c) $\sum_{k=1}^{7}32\left(\frac{1}{2}\right)^{k-1}$

   (d) $\sum_{k=1}^{11}k^2$ (e) $\sum_{k=1}^{6}\frac{1}{2^k}$ (f) $\sum_{k=1}^{n}(-1)^k k^2$

2. (a) $4+5+6+7+8+9$ (b) $29+70+145$

   (c) $-2-10-24-44-70-102-140$

   (d) $-3-7-15-31-63-127-255$

   (e) $\frac{1}{4}+\frac{1}{2}+1+2+4+8+16+32+64+128$

3. (a) 156 (b) $\frac{63}{2}$ (c) $\frac{42}{5}$ (d) 380 (e) 1

4. (a) $2n(n+1)$ (b) $\frac{1}{3}n(n+1)(n+5)$ (c) $\frac{1}{12}n(n+1)(3n^2+23n+46)$

5. (a) $\sum_{k=1}^{4} k(2k-1)$ (b) $\sum_{k=1}^{5}(-x)^{k-1}$ (c) $\sum_{k=1}^{\infty} ky^k$ (d) $\sum_{k=1}^{8}(3k-7)$

   (e) $\sum_{k=1}^{4}\frac{1}{2k+1}$ (f) $\sum_{k=1}^{4}(3k-2)(3k+1)$

6. (a) 4, 8, 16 (b) 2, 6, 12 (c) $2, \frac{1}{2}, \frac{4}{15}$

   (d) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}$ (e) 1, −4, 9 (f) 18, 300, 1,134

7. $\frac{1}{6}n(n+1)(3n^2+1\text{ln}-140)$, $\frac{1}{6}(7,997,000)$

8. 48,124,960 9. 1,871,040

Proofs by mathematical induction

Teaching steps

1. In groups, guide students to discuss the concept of proof by mathematical induction through Activity 5.4 in the Student's Book and other resources as suggested. Advise them to write their work on a flip chart ready for presentations or gallery walk.
2. Use Examples 5.10, 5.11, and 5.12 in the Student's Book to guide students to discuss proof by mathematical induction.
3. Assign students to attempt Exercise 5.2 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Roots of polynomial functions

Teaching steps

1. Develop hands on activities that can be used to assist students to realize the concept of general form of the polynomial

$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$ where a_n, a_{n-1}, a_{n-2}, a_1, and a_0 are real numbers and n is a natural number. Instruct students to discuss the name of the polynomial when;

(i) $n = 1$ (linear)

(ii) $n = 2$ (quadratic)

(iii) $n = 3$ (cubic)

(iv) $n = 4$ (quartic)

(v) $n = 5$ (quintic)

2. Guide students through jigsaw to discuss roots of polynomials. Devise strategies for assisting students in brainstorming on the concept of roots or zeros of a polynomial function.
3. Use Example 5.13 in the Student's Book to guide students to find roots of polynomial functions. Prepare an activity which will engage students in performing more examples on finding roots of polynomial functions from other resources as suggested.
4. Guide students to discuss the relationships between roots of a quadratic equation and its coefficients. Use Examples 5.14 to 5.16 in the Student's Book to guide students to find the sum and product of roots and to formulate a quadratic equation when the roots are given.
5. Guide students to discuss the relationships between roots of a cubic equation and its roots. Use Examples 5.17 and 5.18 in the Student's Book to assist students in finding the roots of a cubic equation, when the roots are given.
6. Require students to attempt Exercise 5.3 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.3

1. $\frac{2}{1-q}, \frac{-1}{1-q}$ 2. (a) 6 (b) $-\frac{117}{8}$

3. (a) $x^2-5x-3=0$ (b) $3x^2-10x-4=0$

5. $\sqrt{\frac{140}{17}}$ 6. $9x^2+55x+6=0$

7. (a) $4x^2+40x+51=0$ (d) $3x^2+37x+12=0$
   (b) $4x^2-37x+9=0$ (e) $2x^2-7x=0$
   (c) $x^2-5x-6=0$ (f) $2x^2+25x+72=0$

10. $4x^3-13x^2+48x-64=0$

11. (a) $x^3-3x^2-2x-32=0$ (b) $x^3+4x^2-x-11=0$

12. (a) $-\frac{1}{12}$ (b) 23 (c) 64 (d) $-\frac{1}{12}$ (e) $-\frac{5}{12}$ (f) 18

Remainder theorem, factor theorem, and operations on polynomials

Teaching steps

1. Using the Think-Ink-Pair-Share strategy, guide students to explore the concept of the remainder theorem using both the Student's Book and other resources as suggested.
2. Use Examples 5.19, 5.20, and 5.21 in the Student's Book to guide students in discussing various problems involving the remainder theorem.
3. Devise a strategy which engages students to explore the concept of factor theorem using the Student's Book and other resources as suggested.
4. Use Examples 5.22 and 5.23 in the Student's Book to guide students in discussing problems involving factor theorem.

5. In groups, design hands on activities for introducing the concept of mathematical operations such as addition, subtraction, multiplication, and division of polynomial functions. Ask students to perform Activity 5.5 in the Student's Book, which highlights some operations on polynomials. Advise them to present the group responses.
6. Assist students when they face challenges, and clear their doubts and misconceptions through Examples 5.24 to 5.27 in the Student's Book which illustrate the mathematical operations of addition and subtraction.
7. Design group activities which engage students in discussing Example 5.28 in the Student's Book. Explore more examples on multiplication of polynomials using other resources as suggested.
8. Design an activity which you think will introduce students to the concept of division of polynomials by considering two methods, namely the long division method and the synthetic division method.
9. Use Examples 5.29, 5.30, and 5.31 in the Student's Book to assist students in discussing various problems involving the long division and synthetic division method for polynomials.
10. Require students to attempt Exercise 5.4 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.4

1. (a) $-43x^5+86x^4-56x^3+12x^2-24x+20$

   (b) $7x^5-58x^4+8x^3-12x^2-24x+120$

2. $a=\frac{41}{7}$, $b=-\frac{114}{7}$, and $c=-\frac{648}{7}$

3. 890

4. $e=9,\ f=-2\ ,g=-11$

5. (a) $x^4-4x^3-2x^2+12x+9$ (b) $-3x^4+7x^3-6x^2+3x-1$

6. (a) $m=10,\ n=-7,\ p=4$

7. (a) quotient x^2+2x+1, remainder 3
   (b) quotient $x^3-8x+10$, remainder -11
   (c) quotient x^5-2x^4+2,remainder -5
   (d) quotient $x-13$, remainder 64
   (e) quotient $x^3+4x^2+12x+50$, remainder 206.
   (f) quotient $2x^3-x^2+x-1$, remainder 2.

8. (a) quotient $2x^3+2x+8$ remainder $-x+6$.
   (b) quotient x^2-x-1, remainder $2x+2$.
   (c) quotient $2x^3-x^2+x-1$, remainder 2.
   (d) quotient $x^3+ax^2+a^2x+a^3$, remainder 0.

9. (a) 2 (b) 7 (c) 319 (d) 2549 (e) 131

10. $t=30$, no factors

Inequalities

Teaching steps

1. Design hands activities to enhance students' understanding of the concept of inequalities using the Student's Book and other resources as suggested. Advise students to share their work with other groups for more inputs.
2. Guide students through discussions to perform Activity 5.6 in the Student's Book on recognizing the solution of a quadratic inequality. Engage students to use scientific calculators to obtain the real roots of quadratic equations. Advise students to write their work on a flip chart or manila sheet ready for presentations or for gallery walk.

3. Use Examples 5.32, 5.33, and 5.34 in the Student's Book to guide students to discuss the solutions of quadratic inequalities. Engage students to use scientific calculators to obtain the real roots of quadratic equations.
4. In groups, devise strategies for assisting students in brainstorming and explaining the steps for solving various rational inequalities in the Student's Book.
5. Develop hands on activities that can be used to assist students to obtain the solution of rational inequalities. Engage students to explore properties of rational inequalities in the Student's Book.
6. Use Examples 5.35 and 5.36 in the Student's Book to guide students in discussing the solutions of rational inequalities. Ask students to use mathematical software to compare the answers of the examples.
7. Through discussions guide students on the concept of absolute value inequalities. Engage students to explore definitions 1 and 2 as well as types of absolute value inequalities in the Student's Book.
8. Use Examples 5.37, 5.38, and 5.39 in the Student's Book to guide students to discuss the solutions of absolute value inequalities. Ask students to use mathematical softwares to compare the answers of the examples.
9. Require students to attempt Exercise 5.5 in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.5

1. $\left\{x \in \mathbb{R} : -\frac{1}{3} < x < 7\right\}$

2. $\left\{x \in \mathbb{R} : \frac{5}{3} < x < 11, x \neq 4\right\}$

3. $\left\{x \in \mathbb{R} : x > \frac{3}{2} \text{ or } x < -\frac{7}{2}\right\}$
4. $\left\{x \in \mathbb{R} : x < -\frac{1}{2} \text{ or } x > \frac{3}{2}\right\}$
5. $\{x \in \mathbb{R} : -4 < x < 5\}$
6. $\{x \in \mathbb{R} : x > 3 \text{ or } x < -1\}$
7. $\{x \in \mathbb{R} : -1 < x < 1\}$
8. $\{x \in \mathbb{R} : -1 < x < 0 \text{ and } x > 1\}$
9. $\{x \in \mathbb{R} : x \leq -1 \text{ or } 0 < x \leq 1\}$
10. $\{x \in \mathbb{R} : x \leq 3 \text{ or } x \geq 5\}$
11. $\{x \in \mathbb{R} : 0 \leq x \leq 5\}$
12. $\left\{x \in \mathbb{R} : x \geq -1 \text{ or } x \leq -5 \text{ or } -2 < x < \frac{-3}{2}\right\}$
13. $\left\{x \in \mathbb{R} : \frac{-3}{2} < x < 1\right\}$
14. $\{x \in \mathbb{R} : 1 < x < 3 \text{ or } x < -1\}$
15. $\{x \in \mathbb{R} : -1 < x < 0 \text{ or } 0 < x < 1\}$
16. $\left\{x \in \mathbb{R} : \frac{2}{3} < x < 1 \text{ or } \frac{3}{2} < x < 4\right\}$
17. $\left\{x \in \mathbb{R} : -1 < x < \frac{1}{3} \text{ or } 1 < x < 2\right\}$
18. $\left\{x \in \mathbb{R} : x < \frac{-1}{2} \text{ or } x > \frac{-3}{7}\right\}$
19. $\{x : x \in \mathbb{R}\}$

Matrices

Teaching steps

1. Think-In-Pair-Share on practices of introducing matrices through their prior knowledge. Ask students through discussion to perform previous Activity 5.7 in the Student's Book to recognize the number of elements, rows, columns, and order of a matrix. Advise them to submit their work and provide constructive feedback.
2. Create relevant hands on activities which will arouse students' interest on the concept of types of matrices, multiplications of 3×3 matrices, and transpose of 3x3 matrices. Engage students to use scientific calculators or Maple to obtain the transpose of a matrix and multiplication of 3×3 matrices.
3. Use Examples 5.40 to 5.43 in the Student's Book to guide students to discuss multiplication and transpose of a 3×3 matrices.

Engage students in groups to use scientific calculators or Maple to verify answers to the given examples. Advise them to share their work for more inputs.

4. Assign students to attempt Exercise 5.6 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.6

2. $\begin{pmatrix} 25 & 37 & 53 \\ 119 & 31 & 55 \\ 68 & -9 & 0 \end{pmatrix}$

4. (a) $\begin{pmatrix} -5890 \\ -190 \\ 9310 \end{pmatrix}$ (b) Not possible

(c) Not possible

5. $c = -50$, $d = 4$, $e = 9$.

6. $\begin{pmatrix} 1220 \\ 620 \\ 10 \end{pmatrix}$

8. (a) $\begin{pmatrix} -11 & -15 & -10 \\ 13 & 15 & 8 \\ -31 & -36 & -23 \end{pmatrix}$

(b) $\begin{pmatrix} -11 & -15 & -10 \\ 13 & 15 & 8 \\ -31 & -36 & -23 \end{pmatrix}$

9. $\begin{pmatrix} 2580 \\ 2170 \\ 2292 \end{pmatrix}$ where, 2580, 2170, and 2292 are the total points from the departments of Geography, Chemistry, and Biology, respectively.

10. (a)
$$\begin{array}{l} \\ 1^{st}\text{ student} \\ 2^{nd}\text{ student} \\ 3^{rd}\text{ student} \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{pmatrix} 8 & 12 & 16 \\ 5 & 9 & 10 \\ 15 & 18 & 12 \end{pmatrix} \end{array}$$

(b) The total sum of money spent by the 1st, 2nd and 3rd students are Tshs 18,000, Tshs 12,000 and Tshs 22,500, respectively.

Minors, cofactors, and determinant of a 3×3 matrix

Teaching steps

1. Guide students in groups to search and discuss the meaning of minors and cofactors of a 3×3 matrix using the Student's Book and other resources as suggested.
2. Use Example 5.44 in the Student's Book to guide students to discuss the minors and cofactors of a 3×3 matrices. Engage students in groups to use scientific calculators or Maple to verify the answers of the given examples. Advise students to share their work for more inputs.
3. Design an activity for exploring the concept of determinant of a 3×3 matrix in the Student's Book. Furthermore, create group activities which will involve students in outlining the steps that can be used in determining the determinant of a 3×3 matrix. Advise students to share their work with other groups for more inputs.
4. In groups, guide students to discuss Examples 5.45 and 5.46 in the Student's Book.
5. Require students to attempt Exercise 5.7 in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and give them constructive feedback.

Answers to Exercise 5.7

1. (a) -532 (b) 380

2. (a) $\begin{pmatrix} -16 & 8 & -34 \\ -14 & -46 & 63 \\ -24 & 12 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & -9 & -10 \\ 1 & -12 & -5 \\ -10 & -5 & 0 \end{pmatrix}$

3. (a) $\begin{pmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 15 & -3 & -21 \\ -6 & -18 & 2 \\ -30 & 6 & 10 \end{pmatrix}$

4. (a) -212 (b) 25 9. (a) $m = 4$ (b) $r = -10$

10. (a) $\begin{pmatrix} 17 & 2 & -6 \\ -4 & 4 & 6 \\ -2 & 6 & 9 \end{pmatrix}$ (b) 144

Solution of systems of linear simultaneous equations by Cramer's rule

Teaching steps

1. Create an activity which will engage students in groups to discuss and present the system of linear simultaneous equations involving a 3×3 matrix of the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

2. Design activities that will engage students in determining the unknown values of x, y, and z using Cramer's rule. Lead the discussion using the formulae:

$x = \frac{|A_x|}{|A|}$, $y = \frac{|A_y|}{|A|}$, and $z = \frac{|A_z|}{|A|}$ where $|A|$ is the determinant of the coefficient matrix.

3. Use Example 5.47 in the Student's Book to guide students to find the unknown values of x, y, and z. Engage students in groups to use scientific calculators or Maple to verify the answers obtained for the given example. Give comments on the answers.

Adjoint and Inverse of a 3×3 matrix

Teaching steps

1. Guide students in groups to discuss the concept of adjoint of a matrix A as the transpose of the matrix of cofactors of matrix A. Design learning activities that will help students to generate the elements of the adjoint of matrix using the formula:
   $\text{Adj}(\text{A})=(\text{cof}(\text{A}))^{\text{T}}$, where $\text{cof(A)}=\left((-1)^{i+j}\text{M}_{ij}\right)$.
2. Assist students to discuss the inverse of a 3×3 matrix. Guide students to discuss properties of the inverse of a matrix as stipulated in the Student's Book. Ask students to search for more properties of the inverse of the 3×3 matrix from the suggested resources.
3. Guide students to discuss the inverse of a 3×3 of a matrix A using the formula $\text{A}^{-1}=\dfrac{\text{Adj(A)}}{|\text{A}|}$ where Adj(A) is the abbreviation for adjoint.
4. Use Example 5.48 in the Student's Book to guide students to find the inverse of a matrix by finding determinant, minors, cofactors, and adjoint of a given matrix. Engage students to use a scientific calculator or Maple to verify the inverse of a matrix from the given example. Advise students to share their work with other groups for more iputs.

Solution of systems of linear equations using inverse matrix method or determinant method

Teaching steps

1. Guide students in groups to consider the 3×3 system of linear simultaneous equations of the form:
   $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$. Lead them to write the system

in the form $\mathrm{A}\underline{x} = \underline{d}$ where $\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

2. Guide students to pre-multiply the equation $\mathrm{A}\underline{x} = \underline{d}$ by A^{-1} on both sides:
   $\mathrm{A}^{-1}\mathrm{A}\underline{x} = \mathrm{A}^{-1}\underline{d}$. But $\mathrm{A}^{-1}\mathrm{A} = \mathrm{I}$ where I is an identity matrix. Assist them to multiply on both sides to obtain, $\mathrm{I}\underline{x} = \mathrm{A}^{-1}\underline{d}$
3. Use Example 5.49 in the Student's Book to guide students to find the unknown values of x, y, and z using the inverse matrix method. Engage students to use scientific calculators or Maple to verify the answers of the given example.
4. Require students to attempt Exercise 5.8, in the student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.8

1 (a) $\mathrm{R}^{-1} = \frac{1}{100}\begin{pmatrix} 2 & 10 & 26 \\ 16 & -20 & 8 \\ 22 & 10 & -14 \end{pmatrix}$ (b) $x = 1,\ y = -2,\ z = 3$

2. (a) $(x, y, z) = (3, 1, 4)$ (b) $(x, y, z) = (3, 2, 1)$
   (c) $(x, y, z) = (-4.714, -2.071, -5.86)$ (d) $(x, y, z) = (-1, 2, 3)$
   (e) $(x, y, z) = (8, -11, 3)$.

3. (a) $\begin{pmatrix} 8 & 8 & 8 \\ -7 & 11 & -1 \\ -5 & 1 & 13 \end{pmatrix}$ (b) $x = 1,\ y = 3,\ z = -2$

4. $\frac{1}{15}\begin{pmatrix} -4 & 11 & 0 \\ -7 & 8 & 0 \\ 27 & -3 & 15 \end{pmatrix}$

5. $(a, b, c) = (2, 5, 1)$

6. $(I_1, I_2, I_3) = (1.3, -4.5, -3.7)$

7. $(k, m, n) = (-5, 10, 2)$

8. $q = 40,000,\ r = 75,\ s = 30$

9. 5 units of simple, 8 units of medium, 8 units of complex

10. $(x, y, z) = (2, 3, 1)$ units

Binomial theorem

Teaching steps

1. Guide students in groups to discuss the binomial theorem using the Student's Book and other resources as suggested. Assist students in performing Activity 5.8 in the student's Book on identifying the coefficients of a binomial expression.
2. Lead students to use Activity 5.8 to deduce the assumptions for the binomial expression $(a+b)^n$. Guide students to generate elements of the triangular array in Pascal's triangle.
3. Use Examples 5.50 to 5.52 in the Student's Book to guide students to expand and simplify various binomial expressions. Engage students to use mathematical software to verify the generated expressions for the given examples.

Binomial expansion

Teaching steps

1. Guide students in groups to discuss the approach that can be used to obtain the expansion which is given by,

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{1\times 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3 + \ldots + na^1b^{n-1} + b^n$$

where n is a positive integer.

2. Use Examples 5.53 to 5.56 in the Student's Book to guide students on how to obtain expansions of the given expressions. Engage students to use mathematical software to verify the given expansion for the given examples.
3. Instruct students to attempt Exercise 5.9 in the student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.9

1. (a) $32x^5 + 240x^4z + 720x^3z^2 + 1080x^2z^3 + 810xz^4 + 243z^5$

   (b) $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$

   (c) $32x^5 + 160x^3 + 320x + \frac{320}{x} + \frac{160}{x^3} + \frac{32}{x^5}$

2. (a) $140\sqrt{2}$ (b) 24476 (c) $40\sqrt{2}$ (d) 98 .
3. $1 + 10x + 55x^2 + 210x^3 + \cdots$; 1.106
4. (a) 1.0743 (b) 1279.2 (c) 7272.2
5. $m = -9$ and $n = 46$
7. $32 + 80y + 80y^2 + 40y^3 + 10y^4 + y^5$; 32.08008
8. $1024 + 1280y + 720y^2 + 240y^3$; 1159
9. $4 - 28c + 85c^2 - 146c^3 + 155c^4 + \cdots$
10. $81a^4 - 54a^3b + \frac{27}{2}a^2b^2 - \frac{3}{2}ab^3 + \frac{1}{16}b^4$; 757335.0625

Binomial expansion for fractional and negative indices

Teaching steps

1. Guide students in groups to discuss the modified binomial theorem when n is negative or a fraction, the modified binomial theorem is given by,

$$(1+x)^n = \frac{1}{0!} + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots$$

2. In groups, ask students to discuss convergence of the series expansion. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.
3. Use Examples 5.57 to 5.60 in the student's Book, and give clear instructions for each example in the Student's Book to guide students on how to obtain the series expansion of the given expressions. Engage students to use mathematical software to verify the given expansion for the given examples. Advise them to share their findings.
4. Require students to attempt Exercise 5.10 in the student's Book. Advise them to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.10

3. $1+2x+\frac{5}{2}x^2+\cdots;\ -\frac{1}{3}<x<\frac{1}{3}$ 4. $-\frac{2}{3}<y<\frac{2}{3}$

5. $1-a-\frac{7}{2}a^2+\cdots;\ -\frac{1}{3}<a<\frac{1}{3}$ 6. $1-4y-24y^2-224y^3;\ 2.499$

8. $1+\frac{1}{3}t-\frac{1}{9}t^2+\cdots;\ 2.080$

10. $-\frac{1}{4}-\frac{1}{2}n-\frac{3}{16}n^2-\cdots;\ -1<n<1$

The general term of a binomial expansion

Teaching steps

1. Guide students in groups to discuss the general term of a binomial expansion $(a+b)^n$ which is given by $T_{r+1} = {}^nC_r a^{n-r}b^r$.

Assist students to explore further conditions of the general term. That is, if it is of the form:

(i) $(a-b)^n$, it becomes $T_{r+1} = (-1)^r \, {}^nC_r a^{n-r} b^r$

(ii) $(1+x)^n$, it becomes $T_{r+1} = {}^nC_r x^r$ and,

(iii) $(1-x)^n$, it becomes $T_{r+1} = (-1)^r \, {}^nC_r x^r$.

2. Use Examples 5.61, 5.62, and 5.63 in the student's Book and give clear instructions for each example to guide students to obtain the general term and the term independent of x for the given expressions. Engage students to use mathematical software to verify the given expansion for the given examples. Advise them to share their findings.

Middle terms in a binomial expansion

Teaching steps

1. Use Think-Ink-Pair-Share strategy to discuss the two conditions for the middle terms in the binomial expansion. That is,

   (i) If n is even, the middle term is $\left(\frac{n}{2}+1\right)^{\text{th}}$ term

   (ii) If n is odd, the middle terms are $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$.

2. Use Examples 5.64 and 5.65 in the student's Book and give clear instructions for each example to guide students to find the middle term/terms from the given expressions.
3. Require students to attempt Exercise 5.11 in the student's Book. Advise them to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.11

2. $n = 55$ 3. 27 4. $r = 7$

5. (a) ${}^{9}C_{r}(-1)^{r}\dfrac{2^{18-3r}}{5^{9-2r}}y^{9-2r}$ (b) ${}^{5}C_{r}(2y)^{5-r}\left(\dfrac{-1}{y}\right)^{r}$

   (c) $(-1)^{r}\,{}^{12}C_{r}y^{24-3r}$ (d) $(-1)^{r}\,{}^{6}C_{r}t^{12-2r}s^{r}$

8. $p = 3, q = 5, n = 6$

9. (a) ${}^{15}C_{10}\left(\dfrac{1}{6}\right)^{5}$ (b) 45 (c) $\dfrac{17}{54}$ (d) −3432

10. (a) 14 (b) −16

Partial fractions

Teaching steps

1. Guide students in groups to discuss the concept of partial fractions by relating it to arithmetic fractions and rational expression. Ask students to discuss how to form fractions. Encourage students to explore additional resources, such as online tutorials or reference books.
2. Guide students to discuss the decomposition of rational functions whose denominators consist of non-repeated linear factors. Use Example 5.66 in the Student's Book to assist students in expressing the rational expression as a sum of partial fractions.
3. Use Think-Ink-Pair-Share strategy to discuss the decomposition of rational functions whose denominators consist of non-repeated irreducible quadratic factors. Use Example 5.67 to assist students to write the rational expression into partial fractions.

4. Devise strategy that will engage students on the decomposition of rational functions whose denominators consist of repeated linear or quadratic factors. Use Example 5.68 in the Student's Book to assist students in expressing rational expression as partial fractions.
5. Guide students to discuss the decomposition of rational functions when the degree of the numerator is equal to or higher than the degree of the denominator. Use Example 5.69 and 5.70 in the Student's Book to assist students in expressing the rational expression in terms of partial fractions.
6. Require students to attempt Exercise 5.12 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials or reference books, to deepen their understanding and practice further.

Answers to Exercise 5.12

1. $\frac{4}{x+2}-\frac{2}{x+1}$

2. $\frac{2}{x-3}-\frac{4}{x-1}$

3. $\frac{5}{12(x+1)}+\frac{34}{3(x-2)}-\frac{43}{4(x-3)}$

4. $\frac{3}{13(x+1)}+\frac{5x-17}{13(x^2+3)}$

5. $1-\frac{14}{x}+\frac{12x-6}{x^2-x+1}$

6. $\frac{8x+13}{3(x^2+2)}+\frac{1}{3(x+1)}$

7. $y-5+\frac{2}{y+1}+\frac{1}{y+2}-\frac{3}{y+3}$

8. $-\frac{3}{2(x-1)}+\frac{1}{(x-1)^2}+\frac{2}{x-2}-\frac{1}{2(x-3)}$

9. $\frac{16}{3(x+1)}+\frac{-13+11}{3(x^2-x+1)}$

10. $\frac{36}{5(t^2+2)}+\frac{8}{5\sqrt{3}(t+\sqrt{3})}-\frac{8}{5\sqrt{3}(t-\sqrt{3})}$

Summation of series using partial fractions

Teaching steps

1. Guide students in groups to discuss the concept of telescopic series. Ask students to explore the behavior of a telescopic series through additional resources, such as online tutorials or reference books.
2. Use Examples 5.71 and 5.72 in the Student's Book, to guide students to find the sum of the series using partial fractions.
3. Direct students to attempt Exercise 5.13, in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 5.13

1. $\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}$; $s_\infty = 0$

3. $2-\frac{1}{n+1}-\frac{2}{n+2}$; $s_\infty = 2$

4. $\frac{1}{2n-1}-\frac{1}{2n+1}$; $\frac{1}{2}-\frac{1}{2n+1}$

5. $\frac{1}{2}+\frac{n+3}{(2n+1)(2n+3)}$

6. $\frac{1}{n}-\frac{1}{n+2}$

7. $\frac{1}{4(2k-1)}-\frac{1}{4(2k+3)}$

9. $\frac{3}{8}-\frac{2n+1}{(2n+2)(2n+6)}$

10. (a) $\frac{1}{6}-\frac{n+2}{(n+3)(n+4)}$ (b) $\frac{11}{18}-\frac{1}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$

(c) $\frac{1}{9}-\frac{1}{9(n+1)}$

Answers to Revision Exercise 5

3. (a) $x^3+9x^2+45x+7=0$ (b) $x^3+x^2-17x-49=0$

(c) $x^3-6x^2+14x-8=0$

4. Factors: $x+1, x-1$, and x^2+4, $\dfrac{1}{x+1}-\dfrac{2}{x-1}+\dfrac{3x+2}{x^2+4}$ 5. $k=2\sqrt{5}-2$

6. $\dfrac{3}{1+x}+\dfrac{4-3x}{1+x^2}$; $7-6x-x^2+7x^4-6x^5$, $a=7$, $b=-6$, $c=-1$, $d=0$, $e=7$, $f=-6$

7. $n=15$ 8. $\dfrac{2}{(x-2)^6}+\dfrac{5}{(x-2)^7}+\dfrac{4}{(x-2)^8}$

11. $\left\{x: x<-2, -1\le x\le\dfrac{2}{3} \text{ or } x>4\right\}$

13. (a) $\dfrac{\sqrt{3}}{36\left(x-\sqrt{3}\right)}-\dfrac{\sqrt{3}}{36\left(x+\sqrt{3}\right)}-\dfrac{1}{6\left(x^2+3\right)}$.

(b) $\dfrac{2}{x}-\dfrac{2x}{x^2+3}+\dfrac{2-5x}{\left(x^2+3\right)}$.

14. $\begin{pmatrix}11 & -4 & 1\\ -25 & 9 & -2\\ 15 & -5 & 1\end{pmatrix}$ 15. $x=30$, $y=20$, $z=-60$

16. $x=\dfrac{1}{2}$, $y=1$, and $z=\dfrac{3}{2}$

17. (a) $1-3x+6x^2-10x^3+\cdots$ (b) $1-\dfrac{2}{3}x^2-\dfrac{1}{9}x^4-\dfrac{4}{81}x^6+\cdots$

18. $\dfrac{21875}{128}x^4$ 20. $\dfrac{3}{3x-1}-\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}$, Coefficient $=n(-1)^n-3^{n+1}$

21. (a) 1.00499 (b) 0.9933 22. $n=11$; 55, 165, and 330.

24. (a) 0 (b) $(y-x)(z-x)(z-y)$

25. $2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}-\ldots,\ -4<x<4.$

29. $1+2x+3x^2+4x^3+\cdots$

30. $x=1,\ y=-4,\ z=-4$ 31. (a) $\begin{pmatrix}19 & 31\end{pmatrix}$ (b) $\begin{pmatrix}-6 & -4 & 13\\ -1 & -2 & 3\\ 3 & -6 & -10\end{pmatrix}$

33. $\frac{2}{9}+\frac{11}{27}x+\frac{20}{27}x^2;\ -\frac{3}{2}<x<\frac{3}{2}$

36. (a) $k=-6,\ k=6$ (b) $k<-6$ or $k>6$

37. (a) $x<-5$ or $x>\frac{1}{5}$ (b) $2<x<3$ or $x>8$

38. $n\left(n^2+6n+11\right);\ 296,616$

39. (b) 3 carton of concentrated product, 1 carton of diluted product and 2 cartones of dry product.

41. (a) (i) $-28x^4+51x^3-71x^2+62x-35$

(ii) $8x^4-7x^3-7x^2+29x+48$

(b) $m=17,\ n=15,\ p=19,\ q=-16,\ r=6,\ t=-4$

42. (a) $10x^5-4x^4-80x^3+52x^2-8x$ (b) $\frac{1}{10}x+4+\frac{8x+2}{10x^2-4x}$

(c) x^3-10x^2-4x+2 (d) $2x^6+68x^4-72x^3+48x^2-64x+8$

43. (a) $x^5+4x^4+12x^3+48x^2+194x+668;$ Remainder $=2,675$

(b) $x^5+4x^4+12x^3+48x^2+194x+668;$ Remainder $=2,675$

44. (a) $\left\{x \in \mathbb{R} : -2 \leq x \leq \frac{-3}{2}\right\}$ (b) $\{y \in \mathbb{R} : 1 < y < 3\}$

(c) No solution

45. (a) $p = 12,\ q = 9,\ r = 5,\ s = \frac{50}{3}$ (b) $x - 1$ is a factor of $p(x)$

46. (a) $\begin{bmatrix} 99 & 11 & -121 \\ -108 & 3 & 156 \\ -54 & -15 & 111 \end{bmatrix}$ (d) $|\mathrm{N}| = 297$

(b) $\begin{bmatrix} 99 & -11 & -121 \\ 108 & 3 & -156 \\ -54 & 15 & 111 \end{bmatrix}$ (e) $\frac{1}{297}\begin{bmatrix} 99 & 108 & -54 \\ -11 & 3 & 15 \\ -121 & -156 & 111 \end{bmatrix}$

(c) $\begin{bmatrix} 99 & 108 & -54 \\ -11 & 3 & 15 \\ 121 & 156 & 111 \end{bmatrix}$

47. $(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3) = (5, 4, 6)$ 48. $(m, n, p) = (-4, 8, 12)$

49. (b) $\frac{224}{27}x^{20}$ and $\frac{448}{9}x^{16}$

50. (a) $\frac{1}{2(2n+1)} - \frac{1}{2(2n+3)}$ (c) $\frac{90}{4669}$

(b) $\frac{1}{6} + \frac{1}{2(2n+3)}$ (d) $\frac{1}{6}$

Chapter Six

Trigonometry

Introduction

In this chapter, students will learn about trigonometric ratios, trigonometric identities, compound angle formulae, double angle formulae, trigonometric equations, factor formulae, radians and small angles, trigonometric functions, and inverse trigonometric functions. Guide students to recognize some common methods of identifying trigonometic ratios, simplifying small angles, deriving compound angles, and factor formulae. Also, guide students to solve trigonometric functions, trigonometric equations, inverse trigonometric functions, draw graphs of inverse trigonometric functions, and finding domain, and range. The competencies developed will enable them in various real life situations such as in solving problems related to astronomy, navigation, architecture, oceanography, and in the drawing of maps.

Students' activities

(a) Identifying trigonometric ratios
(b) Deducing compound angle and factor formulae
(c) Applying trigonometric equations in solving problems
(d) Recognizing small angles and their applications
(e) Recognizing trigonometric functions
(f) Sketching graphs of trigonometric functions
(g) Determining the domain and range of trigonometric functions
(h) Finding solutions of inverse trigonometric functions
(i) Sketching graphs of inverse trigonometric functions

(j) Determining the domain and range of inverse trigonometric functions

Teaching and learning resources

Scientific calculator, Mathematical software such as Maple, GeoGebra, Symbolab, MATLAB, Mathematica, AI tools, Manila cards, pairs of scissors, rubber band, flip charts, marker pens, ruler, pencils, masking tapes, glue, geo-boards, graph papers, mathematical sets, and colored chalks.

Trigonometric ratios

Teaching steps

1. Guide students through discussions to recall their previous knowledge of trigonometric ratios. Assist students in groups to deduce the formulae of trigonometric ratios through Activity 6.1 in the Student's Book and other resources as suggested. Advise them to present their findings in a flip chart or manila cards, post them on the classroom walls, and do gallery walk for improvements.
2. Guide students through discussion on how to use Figure 6.1 of the right-angled triangle in the Student's Book to derive Pythagoras' theorem. Lead them to recognize the reciprocals of the trigonometric functions; sine, cosine, and tangent.
3. Use Examples 6.1, 6.2, and 6.3 in the Student's Book to guide students to use the trigonometric ratios and the Pythagoras' theorem to solve the given problems. Engage students in groups to use mathematical software to verify answers from the given examples. Advise them to share their findings through presentations.
4. Require students to attempt Exercise 6.1 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback where necessary.

Answers to Exercise 6.1

1. $x = 8.1\text{cm}$, $h = 12.8676\text{cm}$
2. $\overline{VW} = 20$ units, $\overline{UV} = 40$ units, $\overline{UW} = 20\sqrt{5}$ units,
3. $\overline{BD} = 13.4\text{cm}$
4. $\overline{JK} = 10$ cm, $\overline{KL} = 10$ cm, $\overline{JL} = 14.14$ cm
5. $\text{cosec}\, D = \frac{169}{119}$, $\sec D = \frac{169}{120}$
6. (a) $1\frac{1}{12}$ (b) $2\frac{2}{5}$
7. $\hat{M} = 45°$, $\hat{N} = 45°$
8. $5\sqrt{5}$ cm, 53.13° and 126.87°
9. $\overline{PR} = 13.97$ cm
10. 407 metres
11. $h = 46.86$ metres

Trigonometric identities

Teaching steps

1. Design an activity to enhance students' understanding on deducing Pythagorean identities using Figure 6.2 in the Student's Book. Allow them through group discussions to share alternative methods of deducing Pythagorean identities.
2. Use Examples 6.4 to 6.10 in the Student's Book to engage students in applying the Pythagorean identities to find solutions of various trigonometric ratios problems. Engage students to use mathematical software to verify the solutions to the given examples.
3. Instruct students to attempt Exercise 6.2 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.2

1. (a) $\cos y$ (b) $\cos\theta$ (c) (d) $\sec\theta$

3. $\theta = -172.8°, -7.2°, 14.5°, 165.5°, 187.2°$

4. (a) $x^2 + y^2 = 1$ (b) $(x-2)^2 + (y+1)^2 = 1$
   (c) $xy = 3$

5. (a) $\theta = 60°, 120°, 240°, 300°$ (b) $\theta = 70.5°, 289.5°$
   (c) $\theta = 26.6°, 45°, 225°, 206.6°$

Compound angle formulae

Teaching steps

1. Using Think-Ink-Pair-Share Strategy, engage students in deriving compound angle formulae as illustrated in Figure 6.3 in the Student's Book.
2. Assist students through discussion to summarize the compound angle formulae in terms of sum and difference. Encourage students to explore additional resources, such as online tutorials or reference books, to enrich their understanding of compound angle formulae.
3. Use Examples 6.11 to 6.16 in the Student's Book to engage students to apply the compound angle formulae to find solutions of various trigonometric problems. Engage students to use mathematical software to verify the solutions of the given examples.
4. Require students to attempt Exercise 6.3 in the Student's Book. Advise students to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.3

1. (a) $\frac{56}{65}, \frac{33}{65}, \frac{56}{33}$ (b) $-\frac{36}{325}, \frac{323}{325}, -\frac{36}{323}$

2. (a) $\frac{204}{325}, -\frac{253}{325}, -\frac{204}{253}$ (b) $\frac{56}{65}, \frac{33}{65}, \frac{56}{33}$

6. (a) $157\frac{1}{2}°, 337\frac{1}{2}°$ (b) $49.1°, 229.1°$ (c) $56.5°, 236.5°$

8. (a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{3}$ (c) $\frac{\sqrt{6}+\sqrt{2}+6}{8}$

Double and half angle formulae

Teaching steps

1. Design an activity which will engage students to deduce the double angle formulae from the compound angle formulae. Allow students to share alternative methods of obtaining the double angle formulae.
2. Use Examples 6.17 to 6.22 in the Student's Book to engage students in applying the double angle formulae to find solutions of the given trigonometric problems. Engage students to use mathematical software to verify the solutions of the given examples.
3. Guide students through discussion to derive the trigonometric ratios of half angles. Assist them in deriving other identities that involve half angles. Allow students to share alternative methods of deriving half angle formulae.
4. Use Examples 6.23 to 6.25 in the Student's Book to guide students in applying half angle formulae in finding solutions of the given trigonometric problems. Engage students to use mathematical software to verify the solutions of the given examples.

5. Encourage students to explore additional resources, such as online tutorials or reference books, to enrich their understanding of the double and half angle formulae.
6. Instruct students to attempt Exercise 6.4 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.4

3. (a) 2 (c) $1-\sin x$ (b) $\cot\theta$ (d) $\cot x$

5. (a) $x+2y^2=0$ (b) $x=2y^2+4y+1$
   (c) $y=\dfrac{4x\left(1-x^2\right)}{1-6x^2+x^4}$ (d) $\dfrac{(x-4)^2}{9}+\dfrac{(y-7)^2}{81}=1$

6. (a) $22.5°, 112.5°, 202.5°, 292.5°$
   (b) $45°, 121°, 225°, 301°$
   (c) $14.47°, 165.53°$
   (d) $0°, 48.18°, 180°, 311.82°, 360°$

8. (a) $\dfrac{336}{527}, -\dfrac{336}{625}, -\dfrac{625}{527}$ (b) $-\dfrac{28560}{239}, -\dfrac{28560}{28561}, \dfrac{28561}{239}$

10. (a) $-\dfrac{\sqrt{3}}{3}$ (b) $\dfrac{\sqrt{3}}{2}$
    (c) $-\dfrac{1}{2}$ (d) $\sqrt{\dfrac{4-\sqrt{6}-\sqrt{2}}{8}}$

11. (a) 1 (a) $\dfrac{\sqrt{2}}{2}$ (b) $2\sqrt{2}$

Trigonometric equations of the form $a\cos\theta + b\sin\theta = c$

Teaching steps

1. Guide students through discussion to solve equations of the form $a\cos\theta + b\sin\theta = c$ by using -formulae. Assist them in using the double angle formulae to express $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan\theta$.
2. Assist students through discussion to perform Activity 6.2 in the Student's Book on deducing the t-formulae for $\cos\theta$ and $\tan\theta$. Guide students through discussion to use Figure 6.4 in the Student's Book of the right-angled triangle to deduce the t-formula for $\sin\theta$, $\cos\theta$, and $\tan\theta$. Advise students to share their work with other groups for more inputs.
3. Use Examples 6.26, 6.27, and 6.28 in the Student's Book to guide students in applying the t-formulae to find solutions of the given trigonometric equations. Engage students to use mathematical software of their choice to verify the answers of the given examples. Advise them to share their work through presentation.
4. Guide students through discussion to express $a\cos\theta + b\sin\theta$ in the form $R\cos(\theta - \alpha)$ or $R\cos(\theta + \alpha)$. Assist them to find the maximum and minimum values of trigonometric equations of the form $a\cos\theta + b\sin\theta = c$.
5. Use Examples 6.29 and 6.30 to guide students to express $a\cos\theta + b\sin\theta$ in the form of $R\cos(\theta + \alpha)$ or $R\cos(\theta - \alpha)$ and use them to solve the given problems.
6. Guide students through group discussion to express $a\cos\theta + b\sin\theta$ in the form $R\sin(\theta + \alpha)$ or $R\sin(\theta - \alpha)$. Assist them to determine the maximum or minimum values of trigonometric equations of the form $a\cos\theta + b\sin\theta = c$.
7. Use Examples 6.31 to 6.35 in the Student's Book to find the solutions of the given trigonometric problems. Engage students

to use scientific calculators and mathematical software to verify the solutions of the given examples. Advise students to share their findings through presentations.

8. Require students to attempt Exercise 6.5 in the Student's Book. Advise them to submit their work, check the correctness of their answers, and give them constructive feedback.

Answers to Exercise 6.5

1. (a) $0°, 112.62°, 360°$
   (b) $53.1°, 323.1°$
   (c) $48.4°, 205.34°$
   (d) $119.56°, 346.7°$
   (e) $114.2°, 335.7°$
   (f) $188.4°, 319.72°$
   (g) $101.34°, 355.4°$
   (h) $36.9°, 241.92°$
   (i) $80.72°, 234.04°$
   (j) $36.87°, 126.87°$

2. (a) $-30°, 90°$
   (b) $76.71°, -150.45°$
   (c) $-4.9°, -129.88°$
   (d) $-20.52°, 28.11°, 159.48°$
   (e) $-24.3°, 114.29°$
   (f) $0°, 45°, 180°$
   (g) $-8.3°, -155.43°$
   (h) $-180°, -106.85°, -90°, -16.85°, 0°, 73.16°, 90°, 163.16°, 180°$
   (i) $-40.72°, 72.6°$
   (j) $51.33°, 128.67°$

3. (a) Maximum $\sqrt{2}$, minimum $-\sqrt{2}$
   Maximum occurs at $45°$
   Minimum occurs at $225°$
   (b) Maximum $\sqrt{2-\sqrt{3}}$, minimum $-\sqrt{2-\sqrt{3}}$ Maximum occurs at $255°$ Minimum occurs at $75°$
   (c) Maximum 5, minimum -5 Maximum occurs at $143.13°$
   Minimum occurs at $306.87°$

(d) Max. $\sqrt{\frac{201}{4}+7\sqrt{3}}$, min. $-\sqrt{\frac{201}{4}+7\sqrt{3}}$,
Maximum occurs at $83.42°$, Minimum occurs at $263.42°$

(e) Maximum $\sqrt{10}$, minimum $-\sqrt{10}$,
Maximum occurs at $71.57°$, Minimum occurs at $251.57°$

(f) Maximum 5, minimum -5, Maximum occurs at $36.87°$,
Minimum occurs at $216.87°$

(g) Maximum 17, minimum -17, Maximum occurs at $118.07°$,
Minimum occurs at $293.96°$

(h) Maximum $\sqrt{34}$, minimum $-\sqrt{34}$, Maximum occurs at $30.96°$, Minimum occurs at $210.96°$

(i) Maximum $\sqrt{37}$, minimum $-\sqrt{37}$,
Maximum occurs at $99.46°$, Minimum occurs at $279.46°$

(j) Maximum $\frac{1}{2}$, minimum $-\frac{1}{2}$,
Maximum occurs at $45°$, Minimum occurs at $135°$.

4. Maximum $\sqrt{13}$, minimum $-\sqrt{13}$ Maximum occurs at $33.69°$ Minimum occurs at $-146.31°$

5. $\cos\theta+2\sin\theta\equiv\sqrt{5}\sin(\theta+\alpha)$ Maximum $\sqrt{5}$, minimum $-\sqrt{5}$,
Maximum occurs at $63.43°$, Minimum occurs at $-116.565°$

General solutions

Teaching steps

1. Guide students through discussion to recognize the general solution of trigonometric equations. Assist them in using Figure 6.5 in the Student's Book, to find the general solution for the cosine function.

2. Guide students to perform Activity 6.3 in the Student's Book, on finding the general solution for trigonometric sine and tangent functions using the same procedures. Assist them to summarize the general solutions in terms of radians and degrees. Advise them to share their findings through presentations for improvement.
3. Use Examples 6.36, 6.37, and 6.38 in the Student's Book to guide students to apply the general solution to the given trigonometric equations. Engage students to use mathematical software of their choice to verify the answers for the given examples.
4. Require students to attempt Exercise 6.6 in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.6

1. $\frac{\pi}{2}n+(-1)^n\frac{\pi}{12}$
2. $\frac{\pi}{5}n+\frac{\pi}{20}$
3. $2\pi n\pm\frac{\pi}{4}$
4. $2.251+2\pi n;\ 6.0928+2\pi n$
5. $\pi n+1.6512;\ 1.769+\pi n$
6. $1.0616+n\pi+1.9148+n\pi$
7. $\pi n+(-1)^n\frac{\pi}{2}$ or $\pi n-(-1)^n\frac{\pi}{6}$
8. $1.7+2\pi n;\ 5.563+2\pi n$
9. $0.864+n\pi;\ 2.972+n\pi$
10. $2\pi n+\frac{11}{6}\pi.$

Factor formulae

Teaching steps

1. Using Think-Ink-Pair-Share Strategy, engage students in deducing the factor formulae from the compound angle formulae in the Student's Book.

2. Assist students through group discussions to summarize the factor formulae. Encourage them to explore additional resources, such as online tutorials or refence books to enrich their understanding of factor formulae.
3. Use Examples 6.39 to 6.44 in the Student's Book to guide students to apply the factor formulae in finding solutions of the given trigonometric problems. Engage students to use mathematical software to verify the solutions of the given examples.
4. Instruct students to attempt Exercise 6.7 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.7

1. (a) $2\sin 45°\cos 5°$ (b) $2\cos 45°\sin 25°$
   (c) $2\cos 40°\cos 15°$ (d) $2\sin 55°\sin 20°$
2. (a) $-\frac{1}{2}(\cos 95° - \cos 15°)$ (b) $\frac{1}{2}(\sin 165° - \sin 55°)$
   (c) $\frac{1}{2}(\sin 70° + \sin 10°)$ (d) $\frac{1}{2}(\cos 85° + \cos 15°)$.
3. (a) $\theta = \frac{\pi}{4}$ or $\theta = \pi n \pm \frac{\pi}{3}$
   (b) $x = \pi n \pm \frac{\pi}{4}$ or $x = 2\pi n \pm \frac{2\pi}{3}$
   (c) $x = \frac{1}{4}\pi n$, *or* $\frac{2}{3}\pi n \pm \frac{2}{9}\pi$
   (d) $\theta = \frac{n\pi}{3}$ or $\frac{\pi n}{4} - (-1)^n \frac{\pi}{8}$
   (e) $x = \frac{n\pi}{4}$ or $2\pi n \pm \frac{\pi}{3}$
   (f) $x = \pi n + (-1)^n \frac{\pi}{4}$ or $x = \frac{\pi}{3}n \pm \frac{\pi}{12}$

6. $\frac{a}{b}, \frac{2a-b+a^2b}{1-a^2+2ab}$

7. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$

8. (a) $22.5°, 60°, 67.5°, 112.5°, 120°, 157.5°$

   (b) $50°, 140°$

   (c) $20°, 90°, 100°, 140°$ (d) $0°, 45°, 90°, 120°, 135°, 180°$

10. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, 2\pi$

11. (a) $\tan\frac{5\lambda}{2}$ (b) $-126°, -54°, 18°, 90°, 162°$

12. (a) $\tan\frac{17A}{2}$ (b) $\cot\frac{2}{3}\pi$

Radians

Teaching steps

1. Design an activity which will assist students to understand the concept of radian. Assist them in using Figure 6.6 in the Student's Book, which describes an arc of a circle, to deduce the formula for converting degrees into radians and vice-versa.
2. Use Examples 6.45 to 6.48 in the Student's Book to guide students to apply the formula for converting degrees into radians and vice-versa of the given problems. Engage students to use scientific calculators and mathematical software to verify answers from the given examples. Advise them to share their findings through presentations.
3. Require students to attempt Exercise 6.8 in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.8

1. (a) $\frac{\pi}{6}$ (b) $\frac{4\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $-\frac{3\pi}{4}$ (e) $\frac{2\pi}{9}$
   (f) $\frac{3\pi}{2}$ (g) $\frac{7\pi}{4}$ (h) $\frac{9\pi}{4}$
2. (a) $60°$ (b) $300°$ (c) $210°$ (d) $-120°$
   (e) $540°$ (f) $252°$ (g) $382.5°$ (h) $360°$
3. (a) $\frac{\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{\pi}{4}, \frac{5\pi}{4}$ (c) $\frac{\pi}{4}, \frac{3\pi}{4}$ (d) $\frac{7\pi}{6}, \frac{11\pi}{6}$
   (e) $\frac{3\pi}{4}, \frac{5\pi}{4}$ (f) $\frac{\pi}{3}, \frac{4\pi}{3}$
4. (a) $28.07°$ (b) $98.55°$ (c) $-135.22°$
   (d) $-48.70°$ (e) $251.53°$ (f) $-291.06°$
5. (a) 1.14 (b) 0.559 (c) 1.48
6. $\frac{7\pi}{12}, \frac{\pi}{4}$
7. (a) $\frac{7\pi}{8}$ (b) $\frac{11\pi}{36}$ (c) $\frac{5\pi}{9}$
8. (a) 0.71π (b) $\frac{2\pi}{3}$ (c) $\frac{3\pi}{4}$
9. $360°, 135°$
10. (a) $-64.3°$ (b) $270°$ (c) $-114.6°$ (d) $439.3°$

Approximating small angles

Teaching steps

1. Through discussions guide students how to apply the concept of small angle approximations to approximate values of the principal trigonometric functions.
2. Through discussion assist students how to perform Activity 6.4 in the Student's Book and recognize geometrical properties of small angles. Advise students to share their findings for improvement.

3. Guide students through discussion to use Figure 6.7 in the Student's Book to deduce the small angle formulae for sine, cosine and tangent.
4. Use Examples 6.49, 6.50, and 6.51 in the Student's Book to guide students to apply the small angle formula of sine, cosine, and tangent in simplifying expressions and proving trigonometry identities. Engage students in the use of mathematical software to verify answers for the given examples. Advise them to share their findings through presentations.
5. Require students to attempt Exercise 6.9 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.9

1. (a) $1+5\theta$ (b) 4 (c) $\frac{1}{1+2\theta}$ (d) $\frac{4\theta}{1-2\theta^2}$

   (e) $\frac{1}{4}\sqrt{2}\left(\theta^2+2\theta-2\right)$ (f) $-\frac{1}{2}\theta$

2. 5 3. 0.294 6. $8\theta^2$

7. $\frac{2}{3}\left[\left(1-2\theta^2\right)\sin 3\alpha+2\theta\cos 3\alpha\right]$ 9. $\frac{18\theta^2}{1+2\theta+6\theta^2}$

10. (a) $1+2\theta-\theta^2$ (b) $\tan\alpha+\theta\sec^2\alpha$

Trigonometric functions

Teaching steps

1. Design an activity that will motivate students to understand the concept of trigonometric functions.
2. Guide students through discussion to analyze the domain and range of a trigonometric function. Encourage students to explore additional resources, such as online tutorials or

reference books, to enrich their understanding of the domain and range of trigonometric functions.

3. Use Examples 6.52 and 6.53 in the Student's Book to guide students to determine the domain and range of trigonometric functions. Engage students to use mathematical software to verify answers of the given examples. Advise students to share their findings for improvement.
4. Instruct students to attempt Exercise 6.10 in the Student's Book. Advise them to submit their work, check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.10

1. Domain $= \left\{ x \in \mathbb{R} : x \neq (2n+1)\frac{\pi}{2}, \text{n} \in \mathbb{Z} \right\}$

   Range $= \{ y \in \mathbb{R} : \text{except } -1 < y < 1 \}$

2. Domain $= \left\{ x \in \mathbb{R} : x \neq \frac{\pi}{4} + \frac{\pi n}{2}, \text{n} \in \mathbb{Z} \right\}$, Range $= \{ y \in \mathbb{R} \}$

3. Domain $= \{ x \in \mathbb{R} : x \neq \pm\pi, \pm 2\pi, \pm 3\pi, ... \}$

   Range $= \{ y : y \in \mathbb{R} \}$

4. Domain $= \{ x \in \mathbb{R} : x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, ... \}$

   Range $= \{ y \in \mathbb{R} : y \in (-\infty, -1] \cup [1, \infty) \}$

5. Domain $= \left\{ x \in \mathbb{R} : x \neq \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, ... \right\}$

   Range $= \{ y \in \mathbb{R} : \text{except } -1 < y < 1 \}$

6. Domain $= \{ x \in \mathbb{R} : -\pi \leq x \leq \pi \}$, Range $= \{ y \in \mathbb{R} : -2 \leq y \leq 2 \}$

7. Domain $= \left\{ x \in \mathbb{R} : x \neq \frac{\pi}{4} + \frac{\pi n}{2}, n \in \mathbb{Z} \right\}$, Range $= \{ y : y \in \mathbb{R} \}$

8. Domain $= \left\{ x \in \mathbb{R} : x \neq (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z} \right\}$

   Range $= \left\{ y \in \mathbb{R} : \text{except } -\frac{1}{2} < y < \frac{1}{2} \right\}$

9. Domain $= \{ x \in \mathbb{R} : 0 \leq x \leq \pi \}$, Range $= \{ y \in \mathbb{R} : -6 \leq y \leq 6 \}$

10. Domain $= \left\{ -2\pi \leq x \leq 2\pi,\ x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \right\}$

    Range $= \{ y : y \in \mathbb{R} \}$

Graphs of sine and cosine functions

Teaching steps

1. Guide students through discussion to draw graphs of sine and cosine functions. Assist students to perform Activity 6.5 in the Student's Book on drawing graphs of f (θ) = sin θ. Ask them to draw graphs of other trigonometric functions and explore their properties.
2. Use Examples 6.54 and 6.55 in the Student's Book to guide students to draw graphs of trigonometric functions and explore their properties. Engage students to use GeoGebra or any other mathematical software to verify the answers for the given examples.

Inverse trigonometric functions

Teaching steps

1. Introduce to students through discussion the inverse trigonometric sine, cosine, tangent, cotangent, secant, and cosecant. Assist them in identifying the domain of sine, cosine, and tangent functions.

2. Guide students through discussion to recognize the inverse of reciprocal trigonometric functions. Assist them in deriving the relationship between trigonometric functions and the inverse of the reciprocal of trigonometric functions.
3. Use Examples 6.56 and 6.60 in the Student's Book to guide students to apply the inverse of trigonometric function in solving equations, simplifying expressions, and proving trigonometry identities of the given problems. Engage students in the use of mathematical software to verify the answers for the given examples. Advise them to share their findings through presentation.
4. Assign students to attempt Exercise 6.11 in the Student Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.11

1. (a) $\frac{\sqrt{2}-\sqrt{6}}{4}$ (b) $\frac{\pi}{4}$ (c) $\sqrt{3}$ (d) 1

2. (a) $\frac{4}{3}$ (b) $-\frac{13}{85}$ (c) $\frac{87}{425}$ (d) $\frac{56}{65}$

4. (a) $\sqrt{1-x^2}$ (b) $\frac{x}{\sqrt{1+x^2}}$

5. (a) $x=0.7862$ (b) $x=\pm 1,\ x=\frac{1}{2}$ (c) $x=0.2808$
   (d) $x=\sqrt{6-4\sqrt{2}}$ (e) $\frac{1}{3}$ (f) $\frac{1}{6}$
   (g) $\pm\sqrt{2}$ (h) 0 (i) $\pm\sqrt{2}$ (j) $\pm\frac{3}{4}$

10. 0

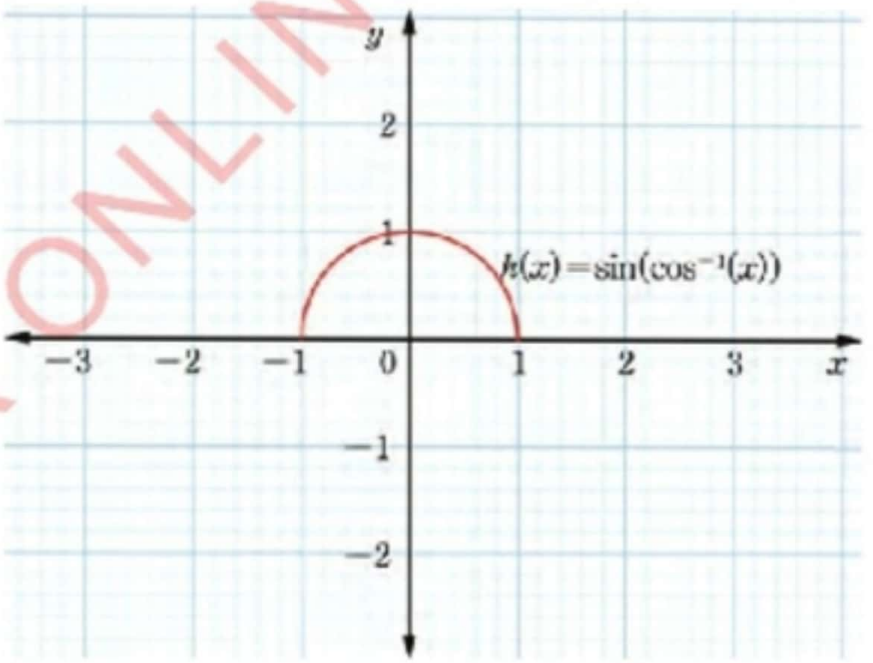
Graphs of inverse trigonometric functions

Teaching steps

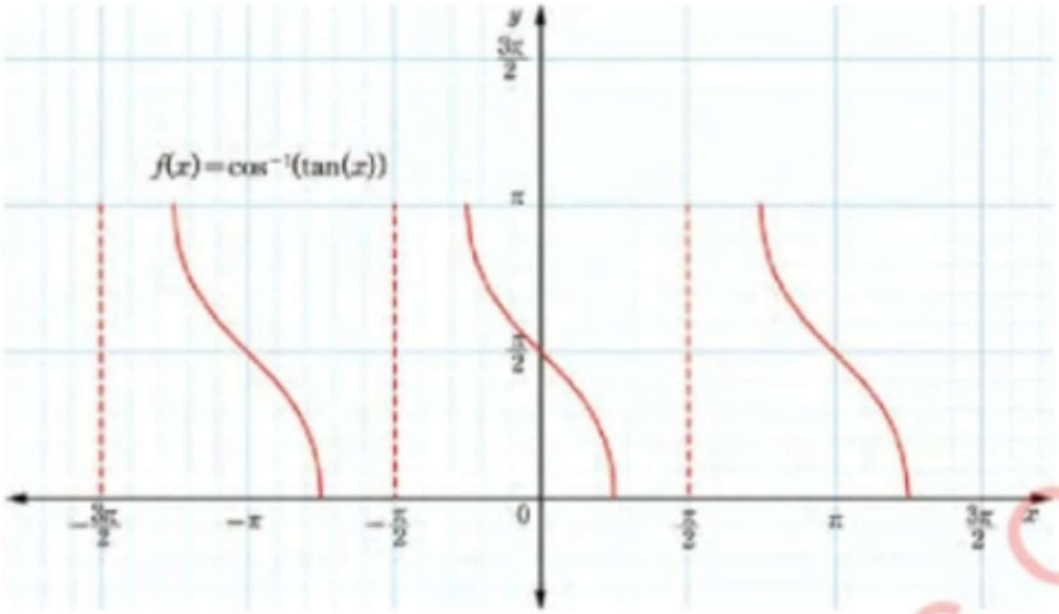
1. Guide students through discussion to draw graphs of inverse trigonometric functions. Assist them in interchanging the roles of y and θ when drawing the graph of inverse trigonometric functions.
2. Assist students through discussion to perform Activity 6.6 in the Student's Book on drawing the graph of the inverse of sine. Ask them to draw graphs of other inverse trigonometric functions of their choice.
3. Use Examples 6.61 and 6.62 in the Student's Book to guide students to draw graphs of inverse trigonometric functions and explore their properties. Engage students to use GeoGebra or any other mathematical software to verify the answers of the given examples.
4. Require students to attempt Exercise 6.12 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 6.12

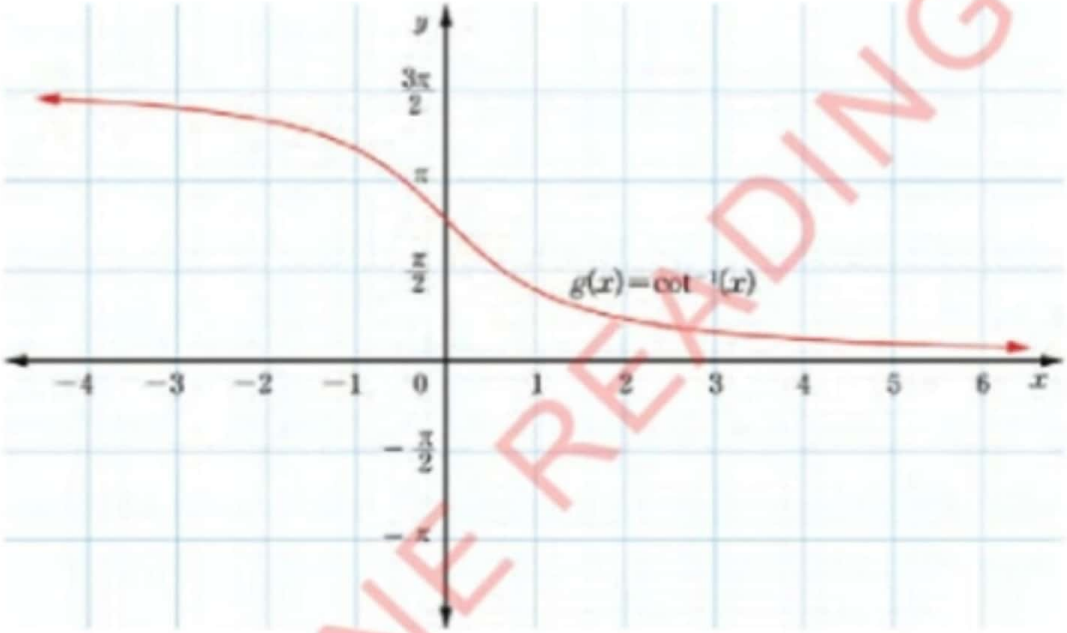
1. (a)



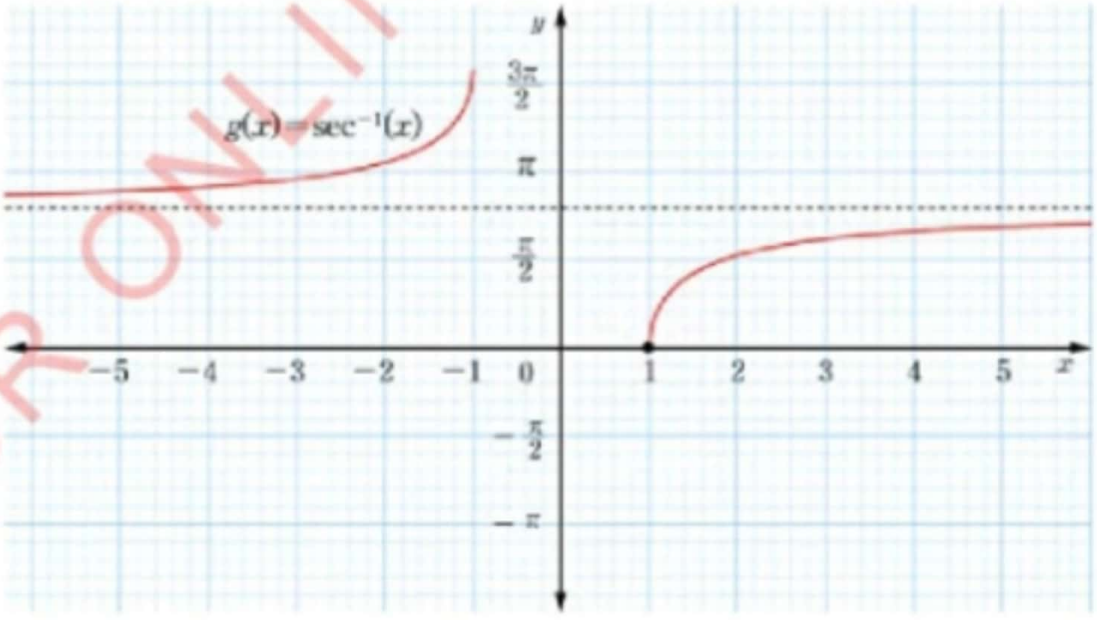
(b)



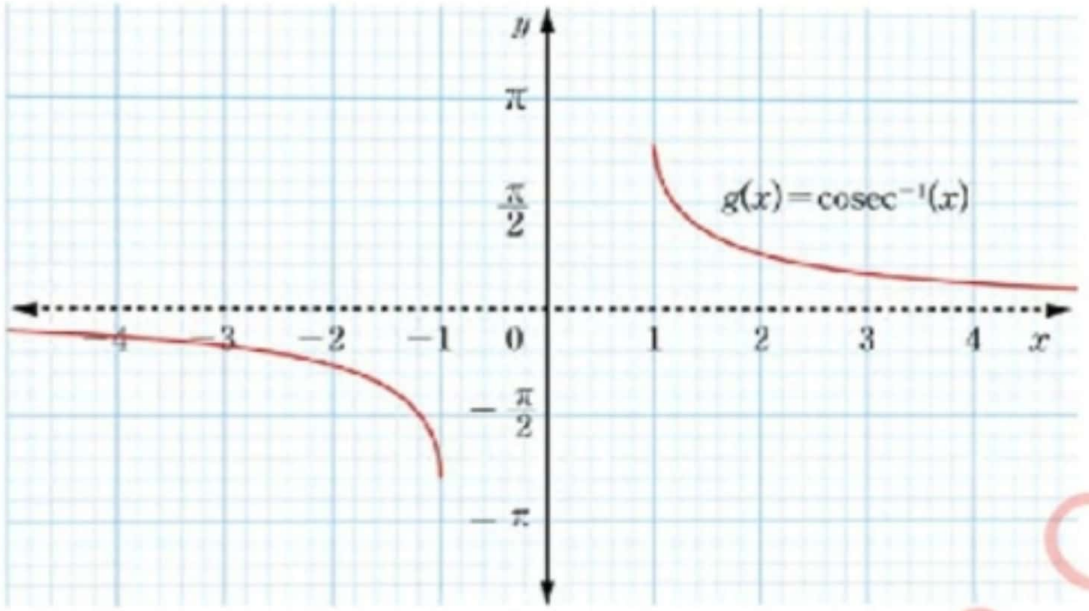
(c)



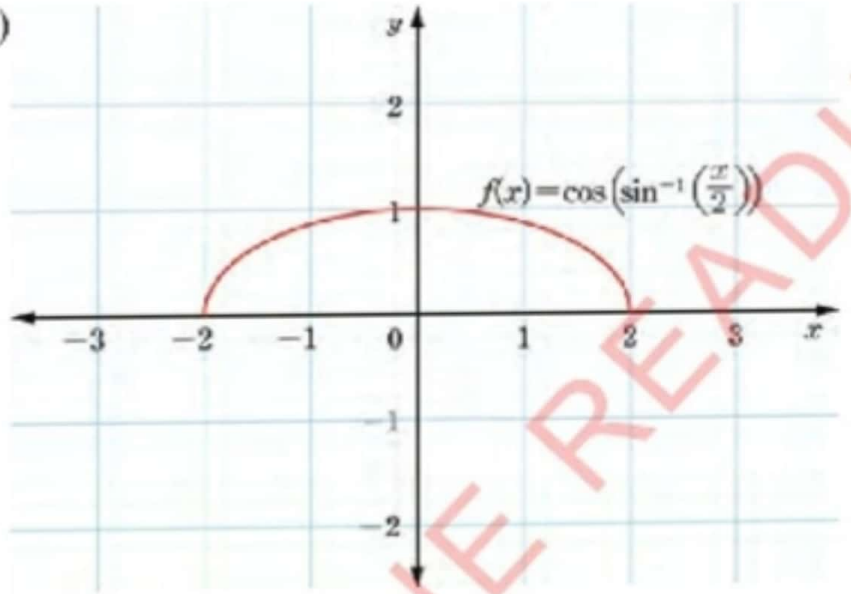
(d)



(e)



(f)



2. (a) Domain $= \{x \in \mathbb{R} : -1 \leq x \leq 1\}$

Range $= \left\{y \in \mathbb{R} : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$

(b) Domain $= \{x : x \in \mathbb{R}\}$

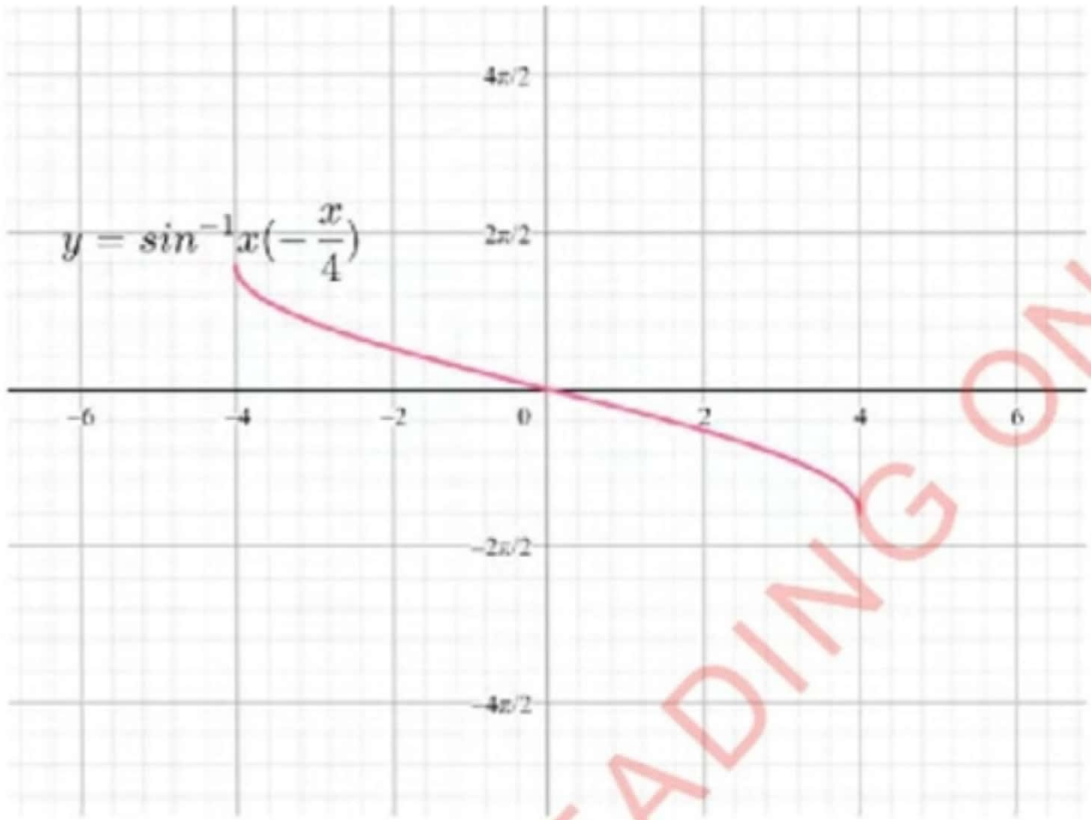
Range $= \{y \in \mathbb{R} : 0 < y < \pi\}$

(c) Domain $= \{x : x \in \mathbb{R}\}$

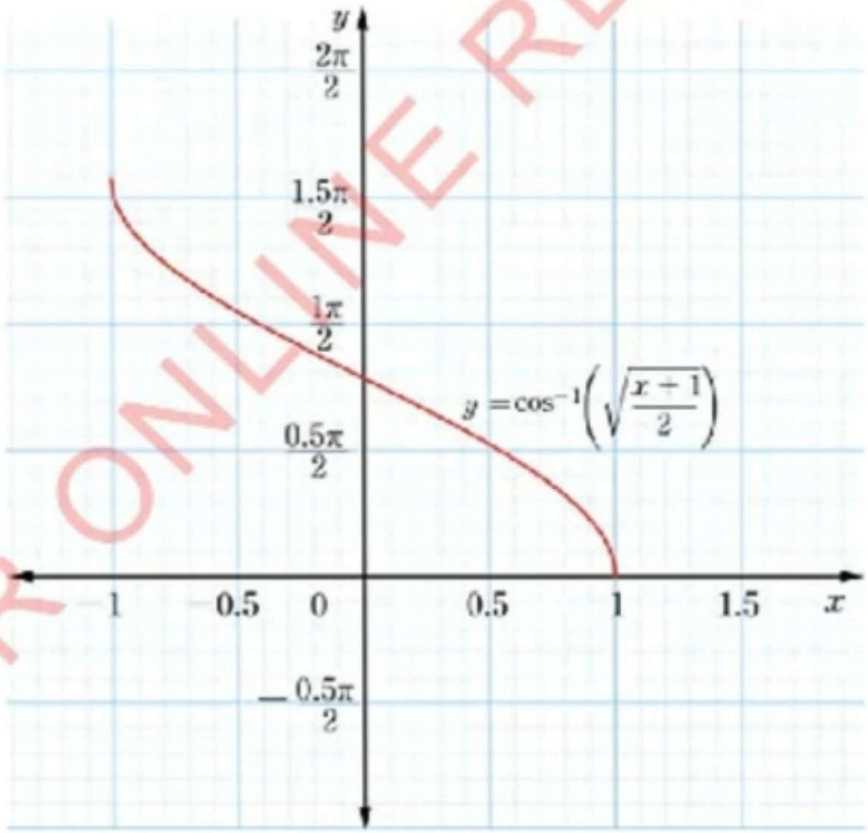
Range $= \left\{y \in \mathbb{R} : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

3. (a) $0 < \cot^{-1} x < \pi$

4.



5.



Answers to Revision Exercise 6

1. (a) $-\left(\dfrac{15\sqrt{135}+168}{315-8\sqrt{135}}\right)$ (b) $-\left(\dfrac{315-8\sqrt{135}}{408}\right)$

   (c) $\dfrac{15\sqrt{135}-168}{408}$

2. (a) $\left(\dfrac{x-9}{4}\right)^2+\left(\dfrac{y-7}{15}\right)^2=1$ (b) $y=\dfrac{4x^2-46x+94}{x^2-10x+16}$

   (c) $x=2\left(1-y^2\right)$ (d) $yx^3+12x^2-256=0$

3. (a) $2\tan 2\theta$ (b) $-\dfrac{3\sin 3\theta}{4}$ (c) $-\tan x$ (d) $\dfrac{\cos 3y}{\cos 8y}$

5. (a) $460°$ (b) $316.55°$ (c) $429.7°$ (d) $-150°$

6. (a) $\dfrac{73}{72}\pi$ (b) $\dfrac{19}{30}\pi$

7. (a) -3π (b) $-\dfrac{35}{6}\pi$ (c) $\dfrac{9641}{14400}\pi$

8. (a) $9\theta+2$ (b) $\dfrac{4}{3}$ (c) $-\dfrac{9}{8}$ (d) $6\theta^2$

10. (a) $-144.1°, -58.5°, 35.9°, 121.5°$ (b) $139.26°, -107.38°$

11. (a) $2\pi n,\ 2\pi n+\dfrac{\pi}{2},\ \dfrac{\pi(1-2n)}{5}$

    (b) $\theta=2\pi n$, or $2\pi n+\dfrac{1}{2}\pi$

    (c) $\theta=\dfrac{2}{5}\pi n+\dfrac{1}{10}\pi$, or $\theta=-\dfrac{2}{3}\pi n+\dfrac{1}{6}\pi$

    (d) $\theta=360°n+69.2°$ or $\theta=360°n-32.3°$

(e) $\frac{4\pi n \pm \pi}{a+b}, \frac{4\pi n \pm \pi}{a-b}$

(f) $\theta = 360°n - 19°47'$ or $\theta = 360°n - 47°35'$

(g) $\theta = \frac{\pi n}{3}$ or $\theta = \pi n \pm \frac{\pi}{3}$

(h) $\theta = 180°n \pm 60°$ or $\theta = 180°n \pm 24.09°$

12. (a) $(x, y) = (15°, 75°); (75°, 15°); (195°, 255)$
$(15°, 255); (195°, 75°).$

(b) $(x, y) = (45°, 165°)$ or $(x, y) = (165°, 45°)$

13. $\frac{52}{39}$

14. 27.2°, 152.8°, 207.2°, 332.8°

15. 26.6°, 206.6°

16. (a) 103.1°, 330° (b) 36.9°, 270°
(c) 0°, 112.6° (d) 28.1°, 208.1°, 159.5°, 339.5°
(e) 76.7°, 209.6° (f) 90°, 330°

17. (a) 13, 67.4° (b) 5, 53.13°
(c) 10, 53.13° (d) $\sqrt{2}$, 45°

18. (a) $\sqrt{2}$, 45°; $-\sqrt{2}$, 225° (b) 5, 53.1°; −5, 233.1°
(c) 17, 298.1°; −17, 118.1° (d) $\sqrt{37}$, 170.5°; $-\sqrt{37}$, 350.5°

19. Maximum = 5, $x = 53.13°$
Minimum = −5, $x = -126.9°$

20. (a) $-\frac{1}{2}\sin\alpha$ (b) $\frac{2}{3}(1+\theta)$ (c) $\frac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$

21. $3-\frac{5}{2}\cos\left(2x-\tan^{-1}\frac{3}{4}\right)$,

Maximum $= 5.5$

Minimum $= 0.5$

22. $22.5°, 112.5°$

24. $t=\frac{1}{2}$ or $t=\frac{1}{p}$, $53.12°$ and $60°$

25. (a) $2\cos\left(\frac{23}{2}\theta\right)\sin\left(\frac{9}{2}\theta\right)$ (b) $2\sin(11\theta)\cos(2\theta)$

(c) $-2\sin\left(\frac{11}{10}\theta\right)\sin\left(\frac{43}{10}\theta\right)$ (d) $2\cos\left(\frac{7}{2}\theta\right)\cos\theta$

28. $x=\frac{1}{2}, x=\frac{1}{3}$ 35. $-360°, -90°, 0°, 360°$

36. $0°, 10.32°, 60°, 70.32°, 120°, 130.32°, 180°,$
$190.32°, 240°, 250.32°, 300°, 310.32, 360°$

37. $\tan 2x=\frac{2t}{1-t^2}$, $x=\pi n, \frac{2\pi}{3}+\pi n, \frac{\pi}{3}+\pi n$

39. (a) $\pm\sqrt{\frac{4+\sqrt{15}}{4}}$ (b) $\pm\sqrt{\frac{4-\sqrt{15}}{8}}$ (c) $\pm\sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}}}$

Chapter Seven

Linear programming

Introduction

In this chapter, students will learn about formulation and graphical solutions of linear transportation problems. Guide students to understand some common methods of formulation of linear programming problems of transportation with two sources and two destinations, and with two sources and three destinations by engaging students to discuss some real-life situations. The competencies developed will enable them to decide, allocate, select, schedule, and possibly evaluate resources to optimize the available resources, especially in the fields of agriculture, economics, finance, business, engineering, energy, manufacturing, and transportation, among many other applications.

Students' activities

(a) Applying linear programming to solve transportation problems that involve two sources and two destinations

(b) Applying linear programming to solve transportation problems that involve two sources and three destinations

Teaching and learning resources

Mathematical softwares such as Maple, GeoGebra, MATLAB, Symbolab, Mathematical sets, scientific calculators, AI tools, manila cards, flip charts, pairs of scissors, rubber bands, marker pens, ruler, pencils, masking tapes, glue, graph papers, and coloured chalks.

Formulation of linear programming problems

Teaching steps

1. Guide students through discussion to formulate linear programming problems. Assist them in identifying the requirements when formulating a real life problem into mathematical algebraic equations or inequalities.
2. Assist students through discussion to identify the decision variables that determine the output. Ask them to use letters such as x, y, and z as variables when formulating an inequality.
3. Guide students through discussion to formulate the constraints. Assist them in connecting the decision variables under certain restrictions or limitations. Lead them to recognize the non-negativity value of constraints for all linear programming problems.
4. Introduce students through discussion to the graphical solution of linear programming problems. Lead them to treat the inequalities as linear equations to determine the region satisfying all the constraints.
5. Guide students through discussion to identify the following characteristic terminologies:
   (i) Optimal problem
   (ii) Feasible region
   (iii) Feasible solution
   (iv) Optimal solution
   (v) Optimal value
   (vi) Optimal point
   (vii) Objective function
6. Assist students through discussion to follow steps 1 to 6 in the Student's Book for solving graphically linear programming problems.

Transportation problems

Teaching steps

1. Design an activity that can be used to engage students in understanding the concept of a transportation problem. Assist them in recognizing only a balanced transportation problem.
2. Assist students through discussions to use Figures 7.1(a) and (b) in the Student's Book to demonstrate the transportation of goods / products from two sources to two destinations and from two sources to three destinations, respectively.

Formulation of transportation problems

Teaching steps

1. Guide students through discussion to formulate the transportation problem. Lead them through discussion to use Figure 7.2 in the Student's Book to describe the transportation of goods/ products from two sources to two destinations.
2. Assist students through discussion to use Figure 7.3 in the Student's Book to describe the transportation of goods/ products from two sources to three destinations.
3. Guide students through discussion to formulate the objective function and the constraints from the corresponding decision variables.
4. Use Examples 7.1 and 7.2 in the Student's Book to guide students to formulate transportation problems. Encourage students to explore additional resources, such as online tutorials, or reference books, to enrich their understanding and further practice.
5. Instruct students to attempt Exercise 7.1 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 7.1

1. Min $f = 6000x + 5000y + 25200000$

Subject to: $x \le 1100$

$y \le 1200$

$x + y \ge 500$

$x + y \le 1500$

$x \ge 0,\ y \ge 0$

2. Min $w = 1200x + 800y + 1682000$

Subject to: $x \le 60$

$y \le 60$

$x + y \le 160$

$x + y \ge 112$

$x \ge 0,\ y \ge 0$

3. Min $w = -(2500x + 2000y) + 575000$

Subject to: $x \le 30$

$y \le 25$

$x + y \le 40$

$x + y \ge 20$

$x \ge 0,\ y \ge 0$

4. Min $w = 25000x + 35000y + 2100000$

Subject to: $x \le 30$

$y \le 25$

$x + y \ge 10$

$x + y \le 50$

$x \ge 0,\ y \ge 0$

5. Min $f = 500x + 1407500$

Subject to: $x \le 40$

$y \le 55$

$x + y \le 80$

$x + y \ge 30$

$x \ge 0,\ y \ge 0$

6. Min $f = -(30 + 10y) + 82000$

Subject to: $x \le 8$

$y \le 5$

$x + y \ge 4$

$x + y \le 6$

$x \ge 0,\ y \ge 0$

7. Min $w = 4x + 2y + 11000$

Subject to: $x \le 2000$

$y \le 1500$

$x + y \ge 2000$

$x + y \le 3000$

$x \ge 0,\ y \ge 0$

8. Min $w = 1000x + 7000y + 190000$

Subject to: $x \le 5$

$y \le 5$

$x + y \ge 4$

$x + y \le 8$

$x \ge 0,\ y \ge 0$

9. Min $w = 400x + 300y + 380000$
Subject to: $x \leq 500$

$$y \leq 600$$
$$x + y \geq 500$$
$$x + y \leq 900$$
$$x \geq 0,\ y \geq 0$$

Graphical solutions of transportation problems

Teaching steps

1. Guide students through discussion to find solutions of transportation problems graphically. Remind them that graphs of transportation problems are drawn similar to the way of solving linear programming problems.
2. Assist students through discussion to identify the feasible region and the corner points after the graph is drawn. Lead them to use the corner points to obtain the optimal solution of the given transportation problem.
3. Use Examples 7.3 and 7.4 in the Student's Book to guide students to formulate transportation problems. Encourage students to explore additional resources, such as online tutorials, or reference books, to deepen their understanding and further practice.
4. Require students to attempt Exercise 7.2 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 7.2

1.

To \ From	D_1	D_2
C_1	80	0
C_2	40	10

Minimum cost is 2,500

2.

To \ From	Misungwi	Ngudu
Mabuki	20	0
Misasi	5	10

Minimum cost is Tshs 5,450

3.

To \ From	H_1	H_2
P	0	500
Q	500	100
R	400	0

Minimum cost is Tshs 53,000

4.

To \ From	M_1	M_2
A	0	5
B	5	0
C	3	1

Minimum cost is Tshs 1,550

5.

From / To	A	B
Malaga	0	15,000
Falulu	20,000	0
Tina	10,000	5,000

Minimum cost is Tshs 3,200,000

6.

From / To	S_1	S_2
P_1	5,000	30,000
P_2	20,000	0
P_3	25,000	0

Minimum cost is Tshs 3,200,000

7.

From / To	W_1	W_2
C_1	10	50
C_2	50	0
C_3	40	0

Minimum cost is Tshs 5,100,000

8.

From / To	B_1	B_2
C_1	5	15
C_2	0	15
C_3	20	0

Minimum cost is Tshs 8,750

9.

From / To	L_1	L_2
Msigwa	35,000	0
Asha	45,000	15,000

Minimum cost is Tshs 1,015,000,000

Answers to Revision Exercise 7

1. (a) Min $w = 1000x - 4000y + 3718000$

(b) $x \quad 100$

$y \quad 100$

$x + y \geq 80$

$x + y \leq 160$

$x \geq 0,\ y \geq 0$

2. Min $w = 21000x - 2200y + 1456700$

Subject to: $x \leq 12000$

$y \leq 15000$

$x + y \geq 2000$

$x + y \leq 20000$

$x \geq 0,\ y \geq 0$

3. Min $f = 150x + 170y + 2872000$

Subject to: $x \leq 3000$

$y \leq 2400$

$x + y \geq 1600$

$x + y \leq 4200$

$x \geq 0,\ y \geq 0$

4. Min $f = 15x + 5y + 3200000$

Subject to: $x \leq 25000$

$y \leq 20000$

$x + y \geq 15000$

$x + y \leq 40000$

$x \geq 0,\ y \geq 0$

5. (a) 0 bags of groundnuts from T_1 to M_1

600 bags of groundnuts from T_1 to M_2

700 bags of groundnuts from T_2 to M_1

(b) Minimum cost is 51,500 Tanzanian shillings

6. (a) Min $f = 25000x + 16000y + 101650000$

Subject to: $x \leq 180$

$y \leq 140$

$x + y \geq 150$

$x + y \leq 200$

$x \geq 0,\ y \geq 0$

(b) 10 tonnes from Dar es Salaam to Kibaha

140 tonnes from Dar es Salaam to Chalinze

170 tonnes from Morogoro to Kibaha

0 tonnes from Morogoro to Chalinze

7. The minimum cost is Tshs 44,000. The transportation is as follows;

500 litres from N to D

3,000 litres from N to E

3,500 litres from N to F

4,000 litres from Q to D

0 litres from Q to E

0 litres from Q to F

8. The minimum cost is Tshs 64,000. The crates should be supplied as follows;

0 crates from W_1 to R_1

200 crates from W_1 to R_2

700 crates from W_1 to R_3

600 crates from W_2 to R_1

300 crates from W_2 to R_2

0 crates from W_2 to R_3

9.

From / To	A	B
P_1	70	0
P_2	90	30

Minimum Transportation cost is Tshs 40,600

10.

	A	B	C
Warehouse 1	50	150	0
Warehouse 2	100	0	200

The minimum cost is Tshs 236,000

11. (a) The objective function is **Min** $z = (40x + 20y) + 45200$

(b) Inequalities associated to the transportation problems are

$x + y \leq 160$

$x + y \geq 50$

$x \leq 70$

$y \leq 120$

$x \geq 0,\ y \geq 0$

(c) The problem is not balanced since total supply from sources is not equal to the total demand.

12. (a) **Min** $f = 400x + 240y + 5120000$

Subject to : $x \leq 4000$

$y \leq 3000$

$x + y \geq 6000$

$x + y \leq 4000$

$x \geq 0,\ y \geq 0$

(b) The problem is balanced since total demand is equal to total supply. That is;

Total supply is $6000 + 3000 = 9000$ bricks.

Total demand is $4000 + 3000 + 2000 = 9000$ bricks.

Chapter Eight

Differentiation

Introduction

In this chapter, students will learn about derivatives, differentiation of functions, applications of differentiation in real life problems, Taylor's and Maclaurin's series, and an introduction to partial derivatives of functions. Guide students to realize some common methods of differentiation: first principles, power rule, chain rule, product rule, quotient rule, and partial derivatives. Also, guide students to determine the nature of stationary points, rates of change between quantities, small changes in quantities, and series expansions of different functions. The competencies developed are applicable in various real life situations such as in business, science, engineering, economics, building and constructions, population modeling, and dynamical systems, among many other applications.

Students' activities

(a) Deducing the first principles method of differentiation
(b) Deriving the power rule method of differentiation
(c) Deriving the chain rule method of differentiation
(d) Deriving the product rule method of differentiation
(e) Deriving the quotient rule for differentiation
(f) Recognizing partial derivative
(g) Determining the nature of stationary points, rates of change between quantities, small changes in quantities, and series expansions of different functions

Teaching and learning resources

Mathematical software such as Maple, GeoGebra, MATLAB, AI tools, Mathematica, Symbolab, flip charts, graphs, pencils, coloured chalks, mathematical sets, manila cards, marker pens, masking tapes, glue, pairs of scissors, and rubber bands.

Derivatives

Teaching steps

1. Guide students through discussion in introducing the concept of the derivative of a function. Assist them in using Figure 8.1in the Student's Book to describe the small increase from the graph of a straight line and write the slope in terms of functional notation.
2. Assist students through discussion to recognize the appropriate notations when writing the derivative of a function. Encourage students to explore additional resources, such as online tutorials or reference books, to enrich their understanding and further practice.
3. In groups, engage students to recognize the various notations which are used to denote derivatives. Advise students to share their findings.

Differentiation of a function from first principles

Teaching steps

1. Guide students through discussion to derive the formula for differentiation of a function from first principles by using the slope of a secant line. Assist them in using Figure 8.2 in the Student's Book to describe a curve of the function $y = f(x)$ with a secant line passing through the points Q and P.
2. In groups, introduce students to the concept of limit as applied in derivatives. Assist them in recognizing the limit of a function $f(x)$ as x approaches a certain value.

3. Guide students through discussion to use the notation of limit to describe the behaviour of graphs as the variable approaches a certain point.
4. Assist students through discussion to perform Activity 8.1 in the Student's Book, on the application of limits in finding derivatives of functions. Ask them to discuss how to use the gradient of a secant line to approximate the slope of the tangent line.
5. Guide students through discussion to continue with the process of differentiation until the first principle of differentiation is understood to be $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
6. Ask students through group discussions to perform Activity 8.2 in the Student's Book on illustrating the concept of the derivative of a function from first principles. Advise students to present their findings in manila cards, post them on the classroom walls, then conduct galley walk for improvements.
7. Use Examples 8.1 to 8.4 in the Student's Book, to guide students to find the derivative of each of the given functions. Engage students to use mathematical software to verify answers for the given examples. Advise students to share their findings.
8. Require students to attempt Exercise 8.1 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback.

8.1

1. (a) $f'(x) = 5$
   (b) $f'(t) = 3t^2 - 2t$
   (c) $f'(x) = -3$
   (d) $f'(x) = 4x - 3$
   (e) $f'(v) = 2$
   (f) $f'(x) = (3-x)^{-2}$
   (g) $f'(t) = 5kt^4$
   (h) $f'(x) = (2x)^{-\frac{1}{2}}$

2. (a) $f'(x)=5;\ f'(1)=5$ (d) $f'(x)=2x;\ f'(3)=6$

   (b) $f'(x)=3x^2-2x;\ f'(2)=8$

   (c) $f'(x)=1;\ f'(2)=1$ (e) $f'(x)=-2x^{-3};\ f'(4)=-\frac{1}{32}$

3. $f'(x)=5-4x;\ f'(3)=-7;\ f'(-1)=9$

4. (a) $y=1080;\ \frac{dy}{dx}=540$ (c) $y=1;\ \frac{dy}{dx}=-1$

   (b) $y=2;\ \frac{dy}{dx}=9$

Differentiation of a function

Teaching steps

1. Guide students through discussion to explore differentiation of function as a process of finding the derivative of a function or rate of change of one variable with respect to another variable.
2. Instruct students through discussion to recognize a differentiable function at a given point. Assist students to identify functions that cannot be differentiated.

Derivatives of polynomial functions

Teaching steps

1. Guide students through discussion to deduce the power rule for differentiating the polynomial function $f(x)=x^n$ from first principles.
2. Assist students through discussion to obtain the power rule formula as $\frac{d}{dx}\left(x^n\right)=f'(x)=nx^{n-1}$. Allow students to share alternative methods of deducing the power rule.

3. Guide students through discussion to attempt Examples 8.5, 8.6, and 8.7 in the Student's Book, and assist them in applying the power rule of differentiation. Engage students to use mathematical software to verify the solutions for the functions in the given examples. Advise them to share their findings.
4. Instruct students to attempt Exercise 8.2 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and practice further.

Answers to Exercise 8.2

1. (a) $f'(1) = -2$ (b) $f'(0) = 0$
   (c) $f'(-1) = 20$ (d) $f'(-2) = -37$
2. (a) $f'(x) = \frac{1}{2}ax - 3b$ (b) $f'(x) = \frac{9}{25}x^{\frac{4}{5}} - \frac{4}{25}x^{-\frac{6}{5}}$
   (c) $f'(x) = \frac{3}{2}\sqrt{x} - \frac{2}{\sqrt{x}}$
3. (a) $f'(x) = 4x - 12$ (b) $g'(x) = \frac{3}{4}(2x - 3)$
   (c) $h'(x) = \frac{2}{3}(x - 1)$
4. $a = 2$ 5. $\left(-2, \frac{1}{2}\right)$ 6. $x = \pm\frac{2\sqrt{3}}{3}$
7. $t = 0, t = -1$; $x'(2) = 36$ 8. $h'(1) = 9$
9. $4x - 8x^{-3}$; $15\frac{7}{8}$ 10. $\frac{dy}{dx} = -200(x + 2)$; -200

Derivative of product of polynomials

Teaching steps

1. Guide students through discussion to deduce the rule for differentiating the product of two functions u and v of the independent variable x.
2. Assist students through discussion to obtain the product rule $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$. Allow students to share alternative methods of deducing the product rule.
3. Use Examples 8.8 and 8.9 in the Student's Book to guide students to apply the product rule to differentiate the given functions. Engage students to use mathematical software to verify the solutions of the given problems. Advise them to share their findings.
4. Instruct students to attempt Exercise 8.3 in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica or reference books, to enrich their understanding and further practice.

Answers to Exercise 8.3

1. (a) $f'(x) = 4x^3 + 6x^2 - 2$

   (b) $f'(s) = 24s^3 + 60s^2 + 64s - 32$

   (c) $f'(y) = -96y^3 + 36y^2 + 96y - 27 - 3y^{-4} - 2y^{-2}$

   (d) $f'(t) = 70t^4 - 32t^3 + 12t^2 + 30t - 22$

2. $x = \dfrac{-6 \pm \sqrt{46}}{5}$

3. $\dfrac{dy}{dx} = \dfrac{17\sqrt{3}}{6}$

4. (a) $f'(x) = \dfrac{5x^2 - 23}{2\sqrt{x-3}}$

(b) $g'(x) = 2x^5\left[6(x+2)^5 + 5x(x+2)^4\right]$

(c) $-3x^{-4}(1+x)^{\frac{1}{2}} + \dfrac{1}{2}x^{-3}(1+x)^{-\frac{1}{2}}$

(d) $k'(x) = -9x^8 + 8x^7 + 6x^5 - 5x^4 - 6x^2$

3. $g'(x) = 2(4x-1)(x+4)^{-3}$

7. (a) 1 (b) 7, −7

8. $f'(r) = -6r^5 - 4r^3 + 2r$

9. $\dfrac{dz}{dy} = 6y^2 - 10y - 4$

10. $f'(u) = 2(u+1)(3u+1)(u-1)^3$; $f'(-5) = -24192$

1. Design a strategy that will engage students to deduce the rule for the derivative of the quotient of two functions u and v of an independent variable x.

2. Assist students through discussion to obtain the quotient rule $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$. Allow students to share alternative methods of deducing the quotient rule.

3. Use Examples 8.10 and 8.11 in the Student's Book to guide students to apply the quotient rule to differentiate the given functions. Engage students to use mathematical software and scientific calculators to verify the solutions for the given Examples. Advise them to share their findings.

4. Instruct students to attempt Exercise 8.4 in the Student's Book. Advise them to submit their work; check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica or reference books, to enrich their understanding and further practice.

Answers to Exercise 8.4

1. (a) $-\dfrac{16}{(3x-2)^2}$ (b) $-\dfrac{4(x+1)}{(x-1)^3}$ (c) $-\dfrac{2\left(x^2+x+2\right)}{\left(x^2-2\right)^2}$

2. (a) $(1+x)^{-\frac{1}{2}}(3+x)^{-\frac{3}{2}}$; $f'(2)=0.052$

   (b) $\dfrac{3}{2(x+3)^2}\sqrt{\dfrac{x+3}{x}}$; $f'(2)=0.095$

   (c) $\dfrac{1}{2}x\left(10x^2+33x-12\right)(x+3)^{-\frac{3}{2}}$; 8.408

3. (a) $-\dfrac{1}{t^4}$ (b) $1-\dfrac{3}{q^2}$ (c) $\dfrac{2}{3}$

4. $\dfrac{9}{4}$ 5. $x=0$ and $x=-2$

9. $\dfrac{-4s^2+20s-22}{(s-3)^2(s-4)^2}$, $-\dfrac{208}{9}$

The chain rule

Teaching steps

1. Design a strategy which will engage students in deducing the chain rule for differentiating of a composite function $y = f(u)$ where u itself is a function of x.
2. Assist students through discussion to obtain the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. Allow students to share alternative methods of deducing the chain rule.
3. Use Examples 8.12, 8.13, and 8.14 in the Student's Book to guide students to apply the chain rule to differentiate the given functions. Engage students to use scientific calculators and mathematical software to verify the solutions for the given examples. Advise them to share their findings.
4. Instruct students to attempt Exercise 8.5 in the Student's Book. Advise students to submits their work; check the correctness of their answers and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica or reference books, to enrich their understanding and practice further.

Answers to Exercise 8.5

1. (a) $f'(x) = 2x^3\left(x^4 - 2\right)^{-\frac{1}{2}}$ (b) $f'(x) = -3x^2\left(x^3 + 1\right)^{-2}$

2. (a) $\frac{dy}{dx} = 32x\left(x^2 + 1\right)^{15}$

   (b) $\frac{dy}{dz} = -6\left(3z^2 + 8z - 3\right)\left(z^3 + 4z^2 - 3z - 3\right)^{-7}$

(c) $\frac{dy}{dx}=18x\left(3x^2-4\right)^2$ (d) $\frac{dy}{dx}=x\left(x^2+5\right)^{-\frac{1}{2}}$

3. (a) $f'(t)=(t-3)\left(t^2-6t+7\right)^{-\frac{1}{2}}$

(b) $f'(t)=-21(3-t)^{20}$

(c) $g'(z)=\frac{3}{2}z^2\left(3-z^3\right)^{-\frac{3}{2}}$

(d) $z'=\frac{2}{5}\left(x+\frac{1}{x}\right)^{-\frac{3}{5}}\left(1-\frac{1}{x^2}\right)$

(e) $y'=2.9\left(t^3-\sqrt{t}\right)^{1.9}\left(3t^2-\frac{1}{2\sqrt{t}}\right)$

5. (a) $f'(t)=-\frac{8t^3+9t^2-5}{3\sqrt[3]{\left(2t^4+3t^3-5t+6\right)^4}}$ (b) $f'(-1)=0.06189$

6. $r'=\frac{r^4+1}{r^2\sqrt{r^4-1}}$ 8. $x'=\frac{1}{3u^{\frac{3}{2}}\left(1-\frac{2}{\sqrt{u}}\right)^{\frac{2}{3}}}$

7. $\theta'=-10(4-2\theta)^4$ 9. $\frac{15}{16}$

Differentiation of implicit functions

Teaching steps

1. Use Think-Ink-Pair-Share strategy to recognize an implicit function. Assist students to write an implicit function in terms of both dependent and independent variables.
2. Guide students through discussions to use the product rule and chain rule to deduce the implicit differentiation formula of the form, $\frac{d}{dx}\left(x^m y^n\right)=nx^m y^{n-1}\frac{dy}{dx}+mx^{m-1}y^n$. Allow students to

share alternative methods of deriving the formula for implicit differentiation of implicitly defined functions.

3. Use Examples 8.15, 8.16, and 8.17 in the Student's Book to guide students and assist them in finding solutions to the given problems by using implicit differentiation. Allow students to share alternative methods of finding solution by the quotient rule. Engage them in using mathematical software and scientific calculators, to verify the solution of the given Examples.
4. Instruct students to attempt Exercise 8.6 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books to enrich their understanding and for further practice.

Answers to Exercise 8.6

1. (a) $3y^2\frac{dy}{dx}$ (b) $1+\frac{dy}{dx}$ 3. $\frac{5}{2(x-2)(x+3)}$

4. $\frac{108(x+2)^3}{\sqrt[3]{25(3x+6)^4}}$ 7. $\frac{\sqrt{2}}{4}$ 8. $\frac{2}{3}$

9. $y'=\frac{x}{\sqrt{3x^2-27}}$ 10. $z'=\pm\frac{4}{3}$

Further implicit differentiation of functions

Teaching steps

1. Assist students through discussion to differentiate expressions without rearranging the terms of the expression. Lead them to differentiate each terms by applying the chain rule and the product rule.

2. Use Examples 8.18 to 8.20 in the Student's Book to guide students through discussion to differentiate implicit functions. Engage students to use mathematical software to verify the solutions of the given Examples.
3. Instruct the students to attempt Exercise 8.7 in the Student's Book. Advise students to submit their work. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica or reference books, to enrich their understanding and practice further.

Answers to Exercise 8.7

1. (a) $\frac{dy}{dx}=-\frac{4x}{y\left(1+x^2\right)^2}$ (b) $\frac{dy}{dx}=\frac{1-2y}{2x+2y-1}$

(c) $\frac{dy}{dx}=\frac{6x^2y^4-8x^3y^5-4x^2y^3-y}{12x^4y^4-12x^3y^3+4x^3y^2+x}$ (d) $\frac{dy}{dx}=-\frac{y^2+2x}{3y^2+2xy}$

2. $\frac{dy}{dx}=\frac{1-2xy^3}{3x^2y^2}$ 3. $\frac{dy}{dx}=\frac{13}{9}$ 5. $\frac{dy}{dx}=\frac{y^2-2x-2xy}{x^2+2y-2xy}$

6. $\frac{dz}{dx}=\frac{6z-3-2x}{2z-6x-2}$ 9. $\frac{dy}{dx}=\frac{y^2-2x-2xy}{x^2-2xy+2y}$

10. $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$

13. (a) $\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$ (b) $\frac{dy}{dx}=\frac{4x\sqrt{xy}-y}{2\sqrt{xy}+x}$

(c) $\frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$

Derivatives of trigonometric functions

Teaching steps

1. Guide students through discussion to identify trigonometric functions that involve sine, cosine, tangent, secant, cosecant, and cotangent with their units in radians.
2. Design a strategy which will engage students to deduce the derivatives of sine, cosine, and tangent from first principles.
3. Use Examples 8.21 to 8.26 in the Student's Book to guide students through discussion to differentiate trigonometric functions. Allow students to share alternative methods of differentiating trigonometric functions. Engage them to use mathematical software to verify the solutions of the given Examples.
4. Instruct students to attempt Exercise 8.8 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.8

1. (a) $2\cos 2x$

(b) $2x\sin x + x^2\cos x$

(c) $-\frac{2}{x^2}\cos\left(\frac{2}{x}\right)$

(d) $-12\sin 2x$

(e) $2x\sin\left(1-x^2\right)$

(f) $-\frac{1}{x^4}\left(x\sin x + 3\cos x\right)$

(g) $\cos x + \cos 2x$

(h) $\frac{\sqrt{x+2}}{2(x+2)}\sec^2\left(\sqrt{x+2}\right)$

(i) $4\sec^2 4x$

(j) $12x\cos\left(3x^2-1\right)\sin\left(3x^2-1\right)$

(k) $\dfrac{1}{1+\cos x}$

(l) $\dfrac{-\sin x+\cos x+1}{(1+\cos x)^2}\sin\left(\dfrac{1-\sin x}{1+\cos x}\right)$

2. (a) $2\pi-2$ (b) 4

5. $\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}$

7. $\dfrac{\theta\cos\sqrt{(\theta^2-1)}}{\sqrt{(\theta^2-1)}}$

10. (a) $-\dfrac{y^2}{2xy-2\sin 2y}$ (b) $\dfrac{\sin\left(x+y^2\right)-3x^2}{4y^3-2y\sin\left(x+y^2\right)}$

(c) $\sec^2 x$ (d) $\dfrac{2\sec^2 x}{(1-\tan x)^2}$

Derivatives of inverse trigonometric functions

Teaching steps

1. Guide students through discussion to identify inverse trigonometric functions of sine, cosine, tangent, secant, cosecant, and cotangent with their units in radians.
2. Use Activity 8.3 in the Student's Book to engage students in recognizing derivatives of inverse trigonometric functions.
3. Use Examples 8.27 to 8.32 in the Student's Book to guide students through discussion to differentiate inverse trigonometric functions. Allow students to share alternative methods of differentiating inverse trigonometric functions. Engage students to use mathematical software to verify the solutions of the examples.

4. Instruct students to attempt Exercise 8.9 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.9

2. $-\frac{1}{1+x^2}$

4. $\frac{x^2}{(a^2-x^2)\sqrt{a^2-x^2}}$

5. $\frac{1-(1+x^2)y^3\cos x}{(1+x^2)(1+3y^2\sin x)}$

6. $-\frac{1}{1+x^2}$

9. $2x\sin^{-1}x+\frac{x^2}{\sqrt{1-x^2}}$

10. (a) $\frac{5}{\sqrt{1-25x^2}}$ (b) $-\frac{\sqrt{2}}{\sqrt{1-2x^2}}$

Computer packages for differentiating polynomial and trigonometric functions

Teaching steps

1. Assist students through discussion to recognize mathematical softwares such as Maple, MATLAB, and other packages that are used for differentiating polynomial and trigonometric functions.
2. Use Examples 8.33 to 8.35 in the Student's Book to guide students and assist them in differentiating functions by using the Maple software.
3. Use Examples 8.36 to 8.38 in the Student's Book to guide students and assist them in differentiating functions by using the MATLAB software.
4. Guide students through discussions to differentiate various polynomial and trigonometric functions by using Maple and MATLAB softwares from suggested resources.

5. Direct students to attempt Exercise 8.10 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them feedback where necessary.

Answers to Exercise 8.10

1. (a) $4x(5x-16)(x-8)^2$ (b) $2\sec^2 2x+10\tan 5x\sec^2 5x$

   (c) $-\dfrac{\left(x^2+6x+4\right)}{\left(x^2-4\right)^2}$ (d) $-\left(3x^2+2\right)\sin\left(x^3+2x+1\right)$

   (e) $7\cos(7x+4)$

2. (a) $2\cos 2x\cos 4x-4\sin 2x\sin 4x$

   (c) $\dfrac{2}{\sqrt{1-4x^2}}-\dfrac{2}{\sqrt{2-x^2}}$

   (d) $3\sec(3x+6)\tan(3x+6)$

   (e) $-\dfrac{(7x+29)}{15}(2x+1)^{-\frac{4}{5}}(x-8)^{-\frac{5}{3}}$

   (f) $2\text{cosec}2x\cot 2x\text{cosec}^2(\text{cosec}2x)$

Derivatives of logarithmic functions

Teaching steps

1. Guide students through discussions to deduce derivatives of logarithmic functions. Assist them to change logarithmic form into exponential form and logarithms base b into logarithms base e.
2. For further practice, engage students through discussion to perform Activity 8.4 in the Student's Book on recognizing derivatives of logarithmic functions. Advise students to share their findings through presentations.

3. Use Examples 8.39 to 8.45 in the Student's Book to guide students and assist them in finding derivatives of logarithmic functions. Engage students to use mathematical software to verify the solutions of the given problems.

4. Require students to attempt Exercise 8.11 in the Student's Book. Advise them to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.11

1. (a) $y' = \dfrac{5}{5x-4}$ (b) $y' = \ln x$ (c) $y' = 2\sin(\ln x)$

2. $y' = \dfrac{2}{(x+3)(x+5)}$ 3. $y' = \dfrac{2x}{x^2+3} + \dfrac{3x^2}{x^3+2}$

4. (a) $f'(x) = \dfrac{1}{\sqrt{1+x^2}}$ (b) $f'(x) = \dfrac{4}{x} - \dfrac{6}{3x-5}$

   (c) $f'(x) = (\sin x)^x [\ln(\sin x) + x\cot x]$

   (d) $f'(x) = \dfrac{6x}{2+x^2} - \dfrac{12x^2}{1-x^3}$

5. (a) $y' = (\sin x)^{\tan x}\left(1 + \sec^2 x \ln \sec x\right)$

   (b) $y' = \dfrac{2x}{x^2+3} - 7\,\text{cosec}2x$ (c) $y' = \dfrac{2x^2-4}{x\left(x^2-4\right)}$

   (d) $y' = x^x(1+\ln x)$

6. $f'(x) = \dfrac{2\text{cosec}2x}{\ln \tan x}$ 7. $2\sec 2\theta$

11. $y' = \dfrac{3}{4(\ln 2)^2}$ 12. $\dfrac{dy}{dx} = \dfrac{1}{2}$

Derivatives of exponential functions

Teaching steps

1. Guide students through discussions to deduce derivatives of exponential functions. Engage students in groups to perform Activity 8.5 on recognizing the derivative of an exponential function. Advise students to share their findings through presentations.
2. Use Examples 8.46 to 8.49 in the Student's Book to guide students and assist them on how to find derivatives of exponential functions. Engage students to verify the solutions by using mathematical software.
3. Assign students to attempt Exercise 8.12 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.12

1. (a) $f'(x)=2xe^{x^2}$ (b) $g'(x)=e^x\left(1-12x-6x^2\right)$

   (c) $h'(x)=e^x\left(\frac{1}{x}+\ln x\right)$ (d) $f^1(x)=x^3 9^x \ln 9+9^x 3x^2$

2. $y'=\left(\cos x-\sin^2 x\right)e^{\cos x}$

5. $f'(x)=2xe^x+x^2e^x$

6. (a) $y'=4\sin(8x)e^{\sin^2 4x}$ (b) $y'=(2\ln 3)3^{2x}$

   (c) $y'=(1+x\ln 3)3^x$ (d) $x'=\sec^2\theta e^{\tan\theta}$

   (e) $x'=3t^2+6te^{-3t^2}+\sec t\tan t$ (f) $v'=(2u+2)e^{u^2+2u-8}$

10. $y'=\frac{x^3\ln 2x}{e^x\sin x}\left[\frac{3}{x}+\frac{1}{x\ln 2x}-1-\cot x\right]$

The second derivative of a function

Teaching steps

1. Guide students through discussions to find the second derivative of a function. Assist them through discussion to write the second derivative of a function using various notations.
2. Use Examples 8.50 to 8.51 in the Student's Book to guide students and assist them to find the second derivative of a function. Engage students to verify the solution by using a mathematical software of your choice.
3. Introduce students through discussion to recognize parametric functions. Guide students to deduce derivatives of parametric functions using the chain rule.
4. Use Examples 8.52, 8.53, and 8.54 in the Student's Book to guide students and assist them to find derivatives of parametric functions. Engage students to verify the solutions by using a mathematical software of their choice.
5. Assign students to attempt Exercise 8.13 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.13

2. $\frac{dy}{dx}=\frac{x\cos x-2\sin x}{x^3}$; $\frac{d^2y}{dx^2}=\frac{\left(6-x^2\right)\sin x-4x\cos x}{x^4}$

3. (a) $\frac{dy}{dx}=\frac{\left(2t+1\right)^2}{t^2}$ (b) $\frac{d^2y}{dx^2}=\frac{-2\left(2t+1\right)^3}{3t^3}$

   (c) $\frac{d^2y}{dx^2}=\frac{9}{4}$

4. (a) $\frac{dy}{dx} = -\tan t$ (b) $\frac{d^2y}{dx^2} = -\frac{1}{2}e^{-t}\sec^3 t$ (c) $-\frac{1}{2}$

6. (a) -1 (b) 48

7. $\frac{dy}{dx} = \frac{1}{5}$ 9. $\frac{2\ln(x)-3}{x^3}$; $\frac{2\ln(2)-3}{8}$ 10. 1

Applications of differentiation

1. Design a strategy that will engage students to use applications of differentiations to solve problems on small changes.
2. For further understanding, use Examples 8.55 to 8.58 in the Student's Book to guide students and assist them to find solutions of problems on small changes.
3. Instruct students to attempt Exercise 8.14 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback.

8.14

1. 0.18
2. (a) 3.001 (c) 2.005 (b) 127.25
3. $0.03x^3\text{cm}^3$ 7. $1.8\pi\ \text{cm}^2$
4. $80\pi\ \text{m}^3$; $16\pi\ \text{m}^2$ 8. 1.25%
5. $40\pi\ \text{cm}^3$ 10. $\frac{1}{2}y\%$
6. 0.015

Problems on rates of change

Teaching steps

1. Devise a strategy that will assist students to solve problems on rates of change involving distance, velocity, and acceleration.
2. For further understanding, use Examples 8.59 to 8.62 in the Student's Book to guide students and assist them to find solutions of problems that involve rates of change.
3. Require students to attempt Exercise 8.15 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.15

1. 32π cm^2/s
2. 157.5 m
3. $\frac{1}{8\pi}$ cm / s
4. 384π cm^3/min
5. 192π cm^2/s
6. 1.5 cm/s
7. 0.04cm^2 / s
8. 4.096cm, 0.3576cm / min
9. 3cm^2 / s
11. $v = 15$ m/s, $a = 12$ m/s^2
12. (a) $v(t) = 3t^2 - 24t + 36$, $a(t) = 6t - 24$
    (b) when $t = 2$ and $t = 6$
    (c) $t = 4$
    (d) $2 < t < 6$

Turning points and points of inflexion of a curve

Teaching steps

1. Guide students through discussions to brainstorm on turning points and points of inflexion of a curve.
2. For further practices, engage students in Activity 8.6 on identification of turning points and points of inflexion. Advise students to present their findings on a flip chart, then allow them to make presentations for further improvement.
3. Instruct students to use Figure 8.3 in the Student's Book to investigate the concepts of turning points and points of inflexion. Furthermore, allow students to discuss the important steps for determining turning points.
4. Use Examples 8.63, 8.64 and 8.65 in the Student's Book to guide students to determine the turning points of the given functions.
5. Assign students to attempt Exercise 8.16 in the Student's Book. Advise them to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.16

1. $(-1, 3)$ and $\left(\frac{1}{2}, -\frac{15}{4}\right)$ 2. $(0, 0)$ and $(4, -32)$

3. $(5, -210)$ and $(-1, 6)$

4. (a) $(1, -9)$ (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ (c) $(1, 2)$ and $(-1, -2)$

5. (a) $\left(\frac{1}{3}, \frac{76}{27}\right)$ and $(-1, 4)$ (b) $(-2, 0)$ and $\left(\frac{2}{3}, \frac{256}{27}\right)$

   (c) $(2, -8)$ (d) $(3, 4)$

6. k = 2; Turning points (1, −3) and $\left(\frac{1}{3}, -\frac{77}{27}\right)$

7. $(0,-9)$ 10. $y = x^3 - 3x + 5$

Real life problems on maximum and minimum values

Teaching steps

1. Guide students through discussion to follow steps 1 to 5 in the Student's Book for classifying stationary points. Assist them in determining the nature of the stationary points.
2. Use Examples 8.66 to 8.70 in the Student's Book to guide students to use the conditions for maximum and minimum values to find solutions of the given examples.
3. Assign students to attempt Exercise 8.17 in the Student's Book. Advise them to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.17

1. $48m^2$ 2. $x = \frac{3}{2}; y = \frac{243}{4}cm^3$ 3. 20

4. Maximum point $\left(\frac{2}{3}, 5\sqrt{13}\right)$

5. Maximum point $\left(\frac{7}{3}, \frac{4}{27}\right)$, Minimum point $(3,0)$

6. (a) Maximum point $\left(\frac{4}{3}, \frac{4}{27}\right)$ minimum point $(2, 0)$; x-intercepts $= 1$ and 2, y-intercept $= -4$

(b) Point inflexion $(0, 0)$ minimum point $(3, -27)$;

x-intercept $= 0$ and 4, y-intercept $= 0$

(c) Maximum point $(2, 44)$ minimum point $(-3, -81)$;

x-intercept $= 3.6$ and -5.1, y-intercept $= 0$

(d) Maximum point $\left(\frac{1}{3}, \frac{1}{81}\right)$, infexion point $(0, 0)$;

x-intercept $= 0$ and $\frac{5}{12}$, y-intercept $= 0$

(e) Maximum point $(0, -4)$; x-intercept $= 2$, and -2;

y-intercept $= -4$

(f) Maximum point $\left(-\frac{1}{6}, \frac{49}{54}\right)$, minimum point $\left(\frac{1}{2}, \frac{3}{2}\right)$;

x-intercept $= 1$, y-intercept $= -1$

7. $\frac{2x^2 + 24}{x}$ cm, $8\sqrt{3}$ cm 8. $10.98\ \text{m}^2$

9. $r = \sqrt{\frac{200}{3\pi}}$ cm, $h = 20\sqrt{\frac{2}{3\pi}}$ cm 10. $1250\,\text{m}^2$

Taylor and Maclaurin series

Teaching steps

1. Introduce to students through discussion Taylor's series. Assist them in deducing Taylor's series by performing successive differentiation of $f(x)$ about $x = a$.
2. Use Examples 8.71, 8.72, and 8.73 in the Student's Book to guide students to apply Taylor's series to find solutions for the given problems.

3. Guide students through discussion to deduce that Maclaurin's series is a special case of Taylor's series when $a = 0$.
4. Use Examples 8.74 and 8.75 in the Student's Book to guide students to apply Maclaurin's series to find solutions of the given problems.
5. Assign students to attempt Exercise 8.18 in the Student's Book. Advise students to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.18

2. $\frac{4}{5}+\frac{3}{5}x-\frac{2}{5}x^2-\frac{1}{10}x^3+\cdots$

3. (a) $4x-8x^2+\frac{64}{3}x^3-64x^4+\cdots$

(b) $1+\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{5}{24}\left(x-\frac{\pi}{2}\right)^4+\cdots$

(c) $\sqrt{2}-\sqrt{2}\left(x+\frac{\pi}{4}\right)+\frac{3}{\sqrt{2}}\left(x+\frac{\pi}{4}\right)^2-\frac{11}{3\sqrt{2}}\left(x+\frac{\pi}{4}\right)^3+\frac{19}{4\sqrt{2}}\left(x+\frac{\pi}{4}\right)^4+\cdots$

(d) $\frac{1}{2}-\frac{\sqrt{3}}{2}x-\frac{1}{4}x^2+\frac{1}{4\sqrt{3}}x^3+\frac{1}{48}x^4+\cdots$

4. $-\frac{1}{2}<x<\frac{1}{2}$

5. (a) $1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots$

(b) $x-\frac{1}{3!}x^3+\cdots$

(c) $1+(\ln a)x+\frac{(\ln a)^2}{2}x^2+\frac{(\ln a)^3}{6}x^3+\frac{(\ln a)^4}{24}x^4+\cdots$

(d) $1+2x-2x^2+4x^3-10x^4+\cdots$

6. $1-4\theta^2+\frac{16}{3}\theta^4-\frac{128}{45}\theta^6+\cdots$

7. $1+\frac{3}{2}x+\frac{9}{8}x^2+\frac{9}{16}x^3+\cdots$

9. $1-\frac{1}{2}h^2+\frac{1}{24}h^4+\cdots$

Introduction to partial derivatives

Teaching steps

1. Introduce students through discussion to the concept of partial derivatives of a function $z = f(x, y)$ where the two variables x and y are independent of each other.
2. Guide students through discussion to identify functions of two independent variables. Lead them to deduce the partial derivative of z with respect to x and the partial derivative of z with respect to y.
3. Assist students through discussion to recognize various notations used for writing partial derivatives of functions.
4. Use Examples 8.76 to 8.79 in the Student's Book to guide students to apply partial derivatives to find solutions for the given examples.
5. Assign students to attempt Exercise 8.19 in the Student's Book. Advise them to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 8.19

1. $\frac{\partial p}{\partial V} = -k\frac{T}{V^2}$; $\frac{\partial p}{\partial T} = k\frac{1}{V}$

2. $\frac{\partial z}{\partial x} = 2xy^3$, $\frac{\partial z}{\partial y} = 3x^2y^2$

3. $\frac{\partial z}{\partial x} = \frac{1}{y^2} + \frac{2y}{x^3}$ and $\frac{\partial z}{\partial y} = -\frac{2x}{y^3} - \frac{1}{x^2}$

4. (a) $2x\sin\left(x^3y\right) + 3x^2y\left(x^2 + 4y^2\right)\cos\left(x^3y\right)$ (b) $\frac{x^2 + 2xy - y^2}{(x+y)^2}$

   (c) $6(3x + y)$

5. (a) $\frac{\partial f}{\partial x} = 3yx^2 + 2y^2x,\ \frac{\partial f}{\partial y} = x^3 + 2x^2y$

(b) $\frac{\partial f}{\partial x} = 2x + 3y,\ \frac{\partial f}{\partial y} = 3x$

(c) $\frac{\partial f}{\partial x} = 4x^3y^3 + 16xy + 5,\ \frac{\partial f}{\partial y} = 3x^4y^2 + 8x^2 + 4y^3$

(d) $\frac{\partial f}{\partial x} = \frac{2x}{y} - \frac{y^2}{x^2},\ \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + \frac{2y}{x}$

6. (a) $\frac{\partial^2 z}{\partial x^2} = -x^2\sin(x-2y) + 4x\cos(x-2y) + 2\sin(x-2y)$

(b) $\frac{\partial^2 z}{\partial y^2} = -4x^2\sin(x-2y)$

(c) $\frac{\partial^2 z}{\partial x \partial y} = -4x\cos(x-2y) + 2x^2\sin(x-2y)$

(d) $\frac{\partial^2 z}{\partial y \partial x} = -4x\cos(x-2y) + 2x^2\sin(x-2y)$

7. $\frac{\partial^2 z}{\partial x^2} = -\frac{x}{\left(y^2 - x^2\right)^{\frac{3}{2}}},\ \frac{\partial^2 z}{\partial y^2} = \frac{2x^3y - 4xy^3}{2\left(y^4 - y^2x^2\right)^{\frac{3}{2}}}$

8. $\frac{\partial^2 z}{\partial x^2} = -0.71$

9. $\frac{\partial^2 z}{\partial y^2} = -6$

10. (a) $\frac{\partial T}{\partial l} = \frac{\pi}{\sqrt{gl}}$ (b) $\frac{\partial T}{\partial g} = -\frac{\pi}{g}\sqrt{\frac{l}{g}}$

Answers to Revision Exercise 8

1. (a) $\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}$ (b) $-2\sin 2x$

   (c) $x\cos x+\sin x$ (d) $-5\sin(5x+2)$

3. (a) $5\cos x+4\sin x$ (b) $\dfrac{xe^{2y}-8x^3\ln\sin 2y}{4x^4\cot 2y-e^{2y}\left(1+x^2\right)}$

   (c) $\dfrac{\cos x-\sin x}{2(1+\sin x\cos x)}$ (d) $\dfrac{4-3x}{\sqrt{1+8x-3x^2}}$

4. (a) $\tan 2\theta$ (b) $-\sec^3 2\theta$ (c) 0.4142

5. (a) $\dfrac{-2y^3-6xy}{6xy^2+3x^2}$

   (b) $\dfrac{y\left(x^2+y^2+2\right)+2x^3\sec^2\left[\ln\left(x^2+y^2+2\right)\right]}{x\left(x^2+y^2+2\right)-2x^2y\sec^2\left[\ln\left(x^2+y^2+2\right)\right]}$

   (c) $-\left(\dfrac{2x^2\ln x+x^2+y^2}{2xy\ln x}\right)$

   (d) $\dfrac{40xy^2+2y^2\sec^2 2x-3x^2\sin y}{x^3\cos y-2y\tan 2x-40x^2y}$

6. (a) $4e^x\left(1+e^x\right)^3$ (c) $e^x\left(x^2-3\right)+2xe^x$ (e) $-5e^{1-x}$

   (b) $\dfrac{1-x}{e^x}$ (d) $\dfrac{1-3\ln x}{x^4}$

7. (a) $2xe^{4x}+4x^2e^{4x}$ (b) $\dfrac{e^x-1}{2\sqrt{e^x-x}}$

(c) $6x^5 - 16x^3 + 2$ (d) $\dfrac{6x}{\left(1+x^2\right)^2}$

8. $40\sqrt{2}$ m

9. (a) $(2+x)^2 e^{-x} = 4 - x^2 + \dfrac{1}{3}x^3$

(b) $\log_e y = (y-1) - \dfrac{1}{2}(y-1)^2 + \dfrac{1}{3}(y-1)^3$

(c) $\dfrac{\cos x}{\sqrt{1-x}} = 1 + \dfrac{1}{2}x - \dfrac{1}{8}x^2 + \dfrac{1}{16}x^3 + \dfrac{49}{384}x^4$

11. (a) 0 12. $\dfrac{902}{1215}$

13. (a) $\sin(x+h) = \sin x + \dfrac{h\cos x}{1!} - \dfrac{h^2 \sin x}{2!} - \dfrac{h^3 \cos x}{3!} + \dfrac{h^4 \sin x}{4!} + \cdots$

(b) $\sin\left(\dfrac{\pi}{6}+h\right) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}h - \dfrac{1}{4}h^2 - \dfrac{\sqrt{3}}{12}h^3 + \dfrac{1}{48}h^4 + \cdots$

(c) $\tan\left(\dfrac{\pi}{3}+h\right) = \sqrt{3} + 4h + 4\sqrt{3}h^2 + \dfrac{40}{3}h^3 + \dfrac{44}{3}h^4 + \cdots$

(d) $\tan\left(\dfrac{\pi}{6}+x\right) = \dfrac{\sqrt{3}}{3} + \dfrac{4}{3}x + \dfrac{4}{3\sqrt{3}}x^2 + \dfrac{8}{9}x^3 + \dfrac{4}{3\sqrt{3}}x^4 + \cdots$

14. (a) $\dfrac{-2x\sin 2x - \cos 2x}{2x^2}$ (d) $\dfrac{-9}{2\sqrt{5-9x}}$

(b) $16x + 4$ (e) $4\cos 4\theta + 2\sin 2\theta$

(c) $6(1-6x)^{-2}$

16. (a) $\ln 3, \dfrac{1}{3}$ (b) $\ln 3 + \dfrac{1}{3}x - \dfrac{1}{9}x^2$

17. Turning points: $(0, -1)$

18. (a) 0.9316 (b) 0.1745

19. $\frac{1}{14}$ cm/s 20. 16 cm 21. $\frac{1}{28\pi}$ cm/s

23. (a) $-706.8583\,\text{cm}^2$ (b) $-4417.8647\,\text{cm}^3$

24. (a) $\frac{-\sin x}{2+\cos x}+\frac{\cos x}{3-\sin x}$ (b) $-\frac{1}{x^2+1}$

(c) $2\sec 2x$ (d) $\frac{-4-x^3}{2x^3\sqrt{x^3+1}}$

25. (b) $x+\frac{1}{3}x^3+\cdots$

26. (a) $1-\frac{9}{2}x^2+\frac{27}{8}x^4$ (b) $2x-\frac{4}{3}x^3+\frac{4}{15}x^5$

(c) $81+216t+216t^2$

27. (a) $\frac{-3x^2-14x-27}{\left(x^2-9\right)^2}$ (b) $3e^{3x}(x-4)^6+6\left(1+e^{3x}\right)(x-4)^5$

(c) $\frac{-\sec^2 x}{\sqrt{1-(1+\tan x)^2}}$ (d) $\frac{4\sec^2 x+e^x\sqrt{1+y}-4\ln\sin y}{4x\cot y-\frac{1}{2}e^x\frac{\sqrt{1+y}}{1+y}}$

28. (a) $v=-5$ m/s (b) -4.25 m/s, 5 m/s^2
$a=-4$ m/s^2

29. (a) $\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{25-x^2-y^2}}$; $\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{25-x^2-y^2}}$

(b) $\frac{\partial z}{\partial x} = (2x + y)e^{x^2+xy}$; $\frac{\partial z}{\partial y} = xe^{x^2+xy}$

(c) $\frac{\partial z}{\partial x} = 3x^2 + 2y$; $\frac{\partial z}{\partial y} = 2x + 4y^3$

(d) $\frac{\partial z}{\partial x} = \frac{3x^2}{y^2} + \frac{y^3}{x^2}$; $\frac{\partial z}{\partial y} = -\frac{2x^3}{y^3} - \frac{3y^2}{x}$

(e) $\frac{\partial z}{\partial x} = 4x - 6y$; $\frac{\partial z}{\partial y} = -6x + 2y$

(f) $\frac{\partial z}{\partial x} = 2\cos(2x + 3y)$; $\frac{\partial z}{\partial y} = 3\cos(2x + 3y)$

30. 1.85 cm

31. Length $l_1 = \frac{100\pi}{4+\pi}$ cm and $l_2 = \frac{400}{4+\pi}$ cm

32. $\frac{-3}{\pi}$ m/min

33. −86 m/s

34. Velocity (v) $= (-4t + 10)$ m/sec, acceleration (a) $= -4$ m/s^2

35. (a) $\frac{\partial z}{\partial x} = 16$ (b) $\frac{\partial^2 z}{\partial x \partial y} = -8$

37. $r = \sqrt[3]{\frac{63}{\pi}}$ cm; $h = \left(\sqrt[3]{\frac{63}{\pi}}\right)^2$ cm

38. $(-1, 2)$ maximum; $(1, 2)$ minimum and $(0, 0)$ point of inflexion

39. 0.16%

40. $\frac{d^2y}{dx^2} = -\frac{2}{3}$

Chapter Nine

Integration

Introduction

In this chapter, students will learn about the inverse process of differentiation, integration of functions, and applications of integration. Guide the students to learn the common methods of integration: integration by parts, substitution method, inspection method, and using partial fractions. Also, guide students to calculate the area under a curve, the area enclosed between two curves, the area of a sector, the volume of a solid of revolution, and the length of an arc. The competencies developed have many real life applications such as determining the total cost and total revenue of goods produced, finding displacement, velocity and acceleration of moving bodies, moment of inertia of vehicles, and the rate of a chemical reaction. It is also used in building constructions, and in graphical representations where three-dimensional models are demonstrated in the analysis of spread of infectious diseases, among many other applications.

Students' activities

(a) Integrating functions by the substitution method

(b) Integrating functions by parts

(c) Integrating rational functions using partial fractions

(d) Applying integration to determine the area under a curve and the area enclosed between two curves, volume of a solid of revolution, and length of an arc

Teaching and learning resources

Mathematical software such as Maple, GeoGebra, MATLAB, Symbolab, Mathematical sets, scientific calculators, AI tools, manila cards, flip charts, marker pens, pairs of scissors, rubber bands, pencils, ruler, masking tapes, glue, geo-boards, and coloured chalks.

The inverse process of differentiation

Teaching steps

1. Guide students in groups to describe integration as the inverse process of differentiation. Ask students the meaning of the inverse process of differentiation. Encourage students to explore additional resources, such as online tutorials or reference books, to enrich their understanding and further practice.
2. Using the Student's Book guide students to explore the the opposite of the derivative of a function as the anti-derivative of a function.
3. Instruct students individually, to demonstrate the integral, $\int f(x)dx$ where $\int$ is integral notation, dx the differential variable and $f(x)$ is the function to be integrated.
4. Use Example 9.1 in the student's Book, to guide students to find the anti-derivative of each of the given integrals.
5. Guide students through explorations to investigate some useful standard integrals in Table 9.1 in the student's Book. Ask students to explore additional resources, such as online tutorials or reference books.
6. Guide students to discuss the two types of integrals, namely; indefinite integrals and definite integrals. Instruct students that, the presentation will start with indefinite integral.

7. Use Examples 9.2 and 9.3 in the student's Book to guide students to find solutions of the given integrals following all the procedures indicated in the Student's Book. Engage students to use scientific calculators and mathematical software to verify the answers of the given examples.
8. Require students to attempt Exercise 9.1 in the student's Book. Advise the student to submit their work; check the correctness in their answers, and provide them constructive feedback.

Answers to Exercise 9.1

1. (a) $x^6 + c$ (b) $x^{-} + c$ (c) $3x^{-7} + c$ (d) $x^{\frac{1}{2}} + c$
   (e) $x^{\frac{2}{3}} + c$ (f) $\frac{3}{2}x^{-2} + c$
2. (a) $\frac{1}{3}x^3 + c$ (b) $5e^x + c$ (c) $\frac{1}{4}\ln|x| + c$
   (d) $2\sin\theta + c$ (e) $-\frac{1}{2}\cos\theta + c$ (f) $4\ln|\sec\theta| + c$
3. (a) $-\frac{3}{x} + c$ (b) $2x^{\frac{1}{2}} + c$ (c) $\frac{12}{5}t^{\frac{5}{4}} + c$
   (d) $-2e^x + c$ (e) $10\cos\theta + c$ (f) $\frac{5}{4}x^{\frac{8}{5}} + c$

Integration of simple functions

Teaching steps

1. Guide students to identify integrals which involve simple functions with mathematical operations on simple polynomials, fractions, simple trigonometric functions, and exponential functions. Ask students to use the rules of mathematical operations such as scalar multiplication, addition, and subtraction of integral functions to demonstrate that,

(i) $\int af(x)\,dx = a\int f(x)\,dx$

(ii) $\int[af(x)\pm bg(x)]dx = a\int f(x)\,dx \pm b\int g(x)\,dx$ where a and b are constants.

2. Use Examples 9.4 to 9.7 in the Student's Book to guide students to find the solution of some simple functions of the given integrals following all procedures as indicated in the Student's Book. Engage students to use scientific calculators and mathematical software to verify the answers of the given examples. Advise them to share their findings for more inputs
3. Require students to attempt Exercise 9.2 in the student's Book. Advise them to submit their work; check the correctness of their answers, and provide them constructive feedback.

Answers to Exercise 9.2

1. $\frac{5}{4}x^4+\frac{8}{3}x^3-\frac{3}{2}x^2+5x+c$
2. $-\frac{3}{2}x^4+3x^3+2x^2-3x+c$
3. $\frac{2}{5}x^{\frac{5}{2}}+x^2+3x+c$
4. $\frac{48}{7}t^{\frac{7}{4}}-\frac{18}{7}t^{\frac{7}{2}}+c$
5. $\frac{1}{12}x^4+\frac{5}{3}x^3-\frac{7}{2}x^2+2x+c$
6. $-\frac{1}{2}t^{-2}-\frac{2}{5}t^5+7\ln|t|+t+c$
7. $2x^{\frac{1}{2}}-2x^{\frac{3}{2}}+\frac{6}{5}x^{\frac{5}{2}}-\frac{2}{7}x^{\frac{7}{2}}+c$
8. $-\frac{3}{x}+\frac{\sin x}{2}+c$
9. $\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{3}x^{\frac{1}{2}}+c$
10. $8\ln|x|+\frac{5}{x}-\frac{3}{x^2}+c$
11. $\ln|t|+\frac{5}{4}t^4+\frac{2}{t^2}+c$
12. $\ln|x|-\frac{\cos x}{5}-\frac{\sin x}{3}+c$
13. $6x-x^2+c$
14. $\frac{1}{3}x^3+2x^2+x+c$

15. $\frac{1}{5}x^5 - x^2 - 2x^{\frac{3}{2}} + c$

16. $t^4 - t^3 - t^2 + t + c$

17. $11e^x + c$

18. $\frac{y^2}{2} - \frac{2}{y} + c$

19. $2\sin\theta - 2\cos\theta + c$

22. $\frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{3}{2}x^{-\frac{2}{3}} + \frac{1}{x} + c$

20. $\frac{4}{3}\sin x + c$

23. $\ln|t| + \frac{5}{4}t^4 + \frac{2}{t^2} + c$

21. $x^4 + \frac{7}{3}x^3 - \frac{3}{2}\sin x + c$

24. $-\frac{1}{r} + \frac{1}{3r^3} + \frac{3r^2}{2} + c$

Techniques of integration

Introduce students to common techniques for integration. Assist them in identifying the most common techniques of integration which include integration by substitution, by inspection, by parts, and by partial fractions.

Integration by substitution method

Teaching steps

1. Guide students to discuss the steps for integrating a given function by the substitution method in the Student's Book. Assist them in using the substitution method to integrate integrals of the form:

   (a) $\int (ax \pm b)^n dx$ (b) $\int \sqrt[m]{(ax \pm b)^n}\, dx$ (c) $\int \frac{dx}{ax \pm b}$

   (d) $\int \sin(ax \pm b)\, dx$ (e) $\int e^{ax \pm b} dx$, where a and b and constants.

2. Use Examples 9.8 to 9.11 in the student's Book, to guide students to find solutions of the integrals by the substitution method. Engage students to use scientific calculators and mathematical softwares to verify the answers of the given

integrals in the given examples. Advise students to share their findings for more inputs.

3. Direct students to attempt Exercise 9.3 in the student's Book. Advise them to submit their work; check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.3

1. $\frac{1}{54}(6x-9)^9+c$
2. $-\frac{1}{5}(1-2x)^{\frac{5}{2}}+c$
3. $-\frac{1}{6}(3t-1)^{-2}+c$
4. $-3\ln|1-x|+c$
5. $12e^{\frac{x}{3}}+c$
6. $-\frac{1}{3}\ln|4-3t|+c$
7. $-2\sin(1-3x)+c$
8. $-\frac{1}{7}\ln|5-7x|+c$
9. $\frac{1}{2}\ln|\sec(2\theta+1)|+c$
10. $\frac{\ln|2x-1|}{2}+c$
11. $-\frac{e^{3-4x}}{14}+c$
12. $-10\cos\left(\frac{x}{2}-1\right)+c$
13. $\frac{e^{2x+2}}{2}+c$
14. $\frac{1}{a}\sin(ax\pm b)+c$
15. $\frac{1}{a}\ln|\sec(ax\pm b)|+c$

Integration by the inspection method

Teaching steps

1. Guide students to discuss the form of integration that requires the inspection method. Assist students through discussion to evaluate integrals where the function is a simple polynomial, trigonometric, logarithmic, or exponential function.
2. Use Examples 9.12 and 9.13 in the student's Book, to guide students to find solutions of the integrals by the inspection method. Engage students to use scientific calculators and

mathematical softwares to verify solutions of the given integrals in the given examples.

3. Require students to attempt Exercise 9.4 in the student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.4

1. $\frac{1}{2}\left(x^3+x^2-3\right)^2+c$
2. $-\frac{1}{2}\left(1+e^{-t}\right)^2+c$
3. $\frac{4}{9}\left(3t^3-1\right)^2+c$
4. $\ln\left|1+e^x\right|+c$
5. $\ln\left|\ln|x|\right|+\ln|x|+c$
6. $-\frac{1}{2}\sin^2\theta+c$ or $\frac{1}{2}\cos^2\theta+c$
7. $-\ln|\cos x-\sin x|+c$
8. $\frac{x^3}{2}+\frac{6}{5}x^{\frac{5}{2}}+c$
9. $\ln|\cos\theta+\sin\theta|+c$
10. $\ln|x|+\frac{1}{2}\left[\ln|x|\right]^2+c$
11. $\ln\left|3+4x+9x^2-6x^3\right|+c$
12. $\frac{1}{4}\left[\ln\left(|x+2|\right)\right]^2+c$
13. $\frac{2}{3}\ln\left|1+x^{\frac{3}{2}}\right|+c$
14. $x-2\sqrt{x}+2\ln\left|\sqrt{x}+1\right|-3+c$
15. $\frac{1}{3}\left(x^2-1\right)^{\frac{3}{2}}+c$
16. $\frac{1}{2}e^{x^2+4x}+c$
17. $\frac{1}{6}\sin(3e^{x^2})+c$
18. $t-2\ln\left|1+e^t\right|+c$
19. $-\frac{1}{x\ln x}+c$
20. $-\frac{1}{\ln x}+c$
21. $e^{\tan^{-1}x}+c$
22. $\ln\left|\tan(x+1)\right|+c$

Integration by parts

Teaching steps

1. Guide students to discuss the type of integrands (which consists two functions) for integrals that can be determined by integrating it by parts. Assist them through discussion to identify products of two functions, where the left part of the integrand is considered as the first function and its right part as the second function.
2. Lead students through discussion to derive the method of integration by parts given $u(x)$ and $v(x)$ are any two differentiable functions. Assist students in obtaining the integration by parts method which is given by,

$$\int u(x)\frac{d}{dx}v(x)dx = u(x)v(x) - \int v(x)\frac{d}{dx}u(x)dx \text{ or } \int udv = uv - \int vdu$$

3. Guide students to choose the function whose derivative is easily integrated. Assist them to use the order of preference ILATE, where I stands for inverse trigonometric functions, L stands for the logarithmic functions, A stands for algebraic functions, T stands for the trigonometric functions, and E stands for exponential functions.
4. Engage students through group discussions to apply integration by parts for integrals of the form:
   (a) $\int x^n \sin(ax)\,dx$ or $\int x^n \cos(ax)\,dx$
   (b) $\int x^n \ln(ax)$, where a is a constant
   (c) $\int x^n \sin(ax)dx$ or $\int x^n \cos(ax)dx$, where a is a constant
   (d) $\int e^{ax}\sin(bx)\,dx$ *or* $\int e^{ax}\cos(bx)\,dx$, where a and b are constants
   (e) $\int x^n \sin^{-1} x\,dx$, $\int x^n \cos^{-1} x\,dx$ *or* $\int x^n \tan^{-1} x\,dx$

5. Design strategies for assisting students in groups to perform Examples 9.14 to 9.22 in the Students Book. Engage students in groups to use mathematical software to verify answers for the integrals in the given examples. Advise students to post their solutions in manila cards, display them on the classroom walls, and allow them to conduct gallery walk. Finally, share observations for improvement.
6. Introduce student to the reduction formula which is an extension of the method of integration by parts. Ask students through group discussions to find solutions of complex integration problems that rely on recurrence relations and cannot be integrated directly. Allow them to share their findings.
7. In groups, design an engaging strategy that will assist students to solve Examples 9.23, 9.24, and 9.25 in the Student's Book. Engage students to use mathematical software to verify the solutions obtained in the given examples. Advise students to share their findings through presentations.
8. Require students to attempt Exercise 9.5 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.5

1. $\frac{1}{2}e^{2x}\left(x^2-x+\frac{1}{2}\right)+c$

2. $x\tan^{-1}(3x)-\frac{1}{6}\ln(1+9x^2)+c$

3. $x\cos^{-1}\left(\frac{1}{2}x\right)-\sqrt{4-x^2}+c$

4. $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$

5. $-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+c$ or $\frac{1}{2}x(\sin x)^2+\frac{1}{4}\cos x\sin x-\frac{1}{4}x+c$

6. $x^2\sin x+2x\cos x-2\sin x+c$

7. $\frac{1}{4}x^4\ln|x|-\frac{1}{16}x^4+c$

8. $\frac{1}{2}\left(x^2-2x\right)\ln|2x|+x-\frac{x^2}{4}+c$

9. $x\ln|3x|-x+c$

10. $\frac{1}{5}e^{2x}(2\cos x+\sin x)+c$

16. $-e^{-3x}\left(\frac{1}{3}x^2+\frac{2}{9}x+\frac{2}{27}\right)+c$

11. $-\frac{1}{5}e^{-x}\left(2\cos(2x)+\sin(2x)\right)+c$

12. $\frac{2}{5}e^{\frac{1}{2}x}\left(2\sin x+\cos x\right)+c$

17. $x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}+c$

13. $\frac{1}{10}e^{4x}\left(2\sin 2x-\cos 2x\right)+c$

18. $\frac{1}{3}x^3\ln|x|-\frac{1}{9}x^3+c$

14. $\frac{1}{13}e^{2x}\left(3\sin 3x+2\cos 3x\right)+c$

19. $2\sqrt{x}\ln|x|-4\sqrt{x}+c$

15. $\theta\tan\theta+\ln|\cos\theta|+c$

20. $-\frac{1}{6x^6}\ln|x|-\frac{1}{36x^6}+c$

21. $\frac{1}{3}(1+x)^3\ln|3x|-\frac{1}{3}\ln|x|-\frac{1}{9}x^3-\frac{1}{2}x^2-x+c$

23. $I_n=x^ne^x-nI_{n-1}$, $I_4=x^4e^x-4x^3e^x+12x^2e^x-24xe^x+24e^x+c$

Integration using partial fractions

Teaching steps

1. Guide students to discuss the case where the integrand is a proper rational function. Assist them to express a rational function in terms of partial functions. Lead them to integrate the partial fractions separately.
2. Assist students through discussion to consider the case where an integrand is an improper rational function in which case the

numerator is divided by the denominator and the fraction is expressed as a sum of the quotient and proper fraction. Lead them to intergrate the partial fractions.

3. Instruct students to use Table 9.2 in the Student's Book to see some forms of rational functions, their partial fractions, and corresponding integrals.
4. Use Examples 9.26 to 9.30 in the Student's Book to engage students and assist them in finding solutions of integrals that involve rational functions. Engage students in groups to use mathematical software to verify the solutions the integrals in the given examples. Advise them to share their findings for more inputs.
5. Require students to attempt Exercise 9.6 in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.6

1. $\ln\left|\frac{(x+5)^2}{x-2}\right|+c$

2. $\frac{1}{36}\ln|x-1|+\frac{1}{6(x+5)}-\frac{1}{36}\ln|x+5|+c$

3. $\ln\left|\frac{x-1}{(9+x^2)^{\frac{1}{2}}}\right|-\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$

4. $\ln\left|\frac{s^2}{(s-1)^2}\right|-\frac{1}{s-1}-\frac{1}{s}+c$

5. $\ln\left|\frac{(2x+1)^{\frac{3}{2}}}{x-2}\right|+\frac{2}{2-x}+c$

6. $\ln\left|x^2(x+2)^3\right|+\frac{1}{x}+c$

7. $\frac{2}{t}-\frac{2t}{t^2+3}+\frac{2-5t}{\left(t^2+3\right)^2}+c$

8. $\ln\left|(y-3)^{\frac{1}{5}}y^2(y+2)^{\frac{9}{5}}\right|+c$

9. $\ln\left|\frac{(x+2)^3}{(x+1)^2}\right|+c$

10. $\ln\left|\frac{x(x-1)}{x+1}\right|+c$

11. $\ln\left|\left(\frac{t-1}{t+4}\right)^{\frac{1}{5}}\right|+c$

12. $-\frac{1}{x-1}-\frac{2}{(x-1)^2}+c$

13. $-\frac{1}{4(x+3)^4}+\frac{9}{5(x+3)^5}-\frac{x-3}{(x+3)^6}+c$

14. $x-\frac{1}{2}\tan^{-1}(x)+\ln\left|\left(\frac{x-1}{x+1}\right)^{\frac{1}{4}}\right|+c$

15. $5\ln|x-3|+\frac{2}{x-3}+c$

16. $\ln\left|\frac{t^{\frac{2}{5}}}{(t+5)^{\frac{2}{5}}}\right|+c$

17. $\ln\left|\left(\frac{x-4}{x+4}\right)^{\frac{15}{8}}\right|+x+c$

18. $3x+\frac{1}{3}x^3+\frac{4}{3}\ln|x-1|-\frac{13}{3}\ln|x+2|+c$

19. $\ln\left|r\left(\frac{r-1}{r+1}\right)^{\frac{1}{2}}\right|+c$

20. $\ln\left|\frac{(x+2)^4}{(x+5)^3}\right|+c$

21. $\frac{x^2}{2}+\frac{1}{4x}-\frac{1}{4x^2}-2x+\frac{1}{8}\ln x+\frac{31}{8}\ln(x+2)+c$

Integration of trigonometric functions

Teaching steps

1. Guide students through discussion to share some common trigonometric identities before introducing integration of trigonometric functions.

2. Assist students through discussion to find solutions of integrals of the form:

   (a) $\int \sin ax \cos bx\, dx$, $\int \sin ax \sin bx\, dx$, or $\int \cos ax \cos bx\, dx$ where a and b are constants. Instruct students to use Table 9.3 in the Student's Book to identify factor formulae and the corresponding integrals.

   (b) $\int \sin^n x\, dx$; or $\int \cos^n x\, dx \sin^n x\, dx$; or $\int \cos^n x\, dx$

   (c) $\int \tan^n x\, dx$ or $\int \sec^n x\, dx$

3. Formulate strategies for assisting students in groups to study carefully Examples 9.31 to 9.39 in the Student's Book. Allow students to share alternative methods of solving integrals of trigonometric functions. Engage students in group to use mathematical software to verify the solutions of the integrals in the examples.

4. Require students to attempt Exercise 9.7 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.7

1. $\frac{1}{2}\left(\sin x-\frac{1}{7}\sin 7x\right)+c$

2. $\frac{1}{16}(-4\cos 2x-\cos 8x)+c$

3. $\frac{1}{20}(5\sin 2x+\sin 10x)+c$

4. $\frac{1}{12}(3\cos 2x-\cos 6x)+c$

5. $\frac{1}{24}\cos 6\theta-\frac{1}{16}\cos 4\theta-\frac{1}{8}\cos 2\theta+c$

6. $\frac{1}{12}(3\sin 2\theta-\sin 6\theta)+c$

7. $-\cos 2x+\frac{1}{3}\cos^3 2x-\frac{1}{10}\cos^5 2x+c$

8. $\frac{1}{8}[4x-2-\sin(4x-2)]+c$

9. $\frac{1}{96}\sin(12x+4)+\frac{1}{12}\sin(6x+2)+\frac{3}{8}x+c$

10. $\frac{1}{96}(\sin 12x-8\sin 6x+36x)+c$

11. $\frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+c$

12. $\frac{3}{4}(x-1)-\frac{1}{2}\sin(2x-2)+\frac{1}{16}\sin(4x-4)+c$

13. $\sin x-\frac{1}{3}\sin^3 x+c$

14. $\cos x\left(\frac{2}{3}\cos^2 x-\frac{1}{5}\cos^4 x-1\right)+c$

15. $\frac{1}{2}\sin\theta-\frac{1}{4}\sin 2\theta+c$

16. $\cos^3 x\left(\frac{1}{5}\cos^2 x-\frac{1}{3}\right)+c$

17. $\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+c$

18. $\frac{1}{64}(8x-\sin 8x)+c$

19. $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+c$

20. $-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+c$

21. $\frac{1}{12}\tan^4(3\theta)+\frac{1}{18}\tan^6(3\theta)+c$

22. $\frac{1}{4}\sec 2x\tan 2x+\frac{1}{4}\sec 2x\tan 2x+c$

Integration by trigonometric substitutions

Teaching steps

1. Guide students through discussion to recognize integrals of rational functions whose denominators cannot be factorized and cannot be expressed in the form $x^2 - a^2$. Help students to recognize common useful trigonometric substitutions which involve tangent, sine, or cosine.

2. Assist students through discussion to find solutions of integrals of the form:

   (a) $\int \frac{1}{x^2 + a^2} dx$ (b) $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ (c) $\int \sqrt{a^2 - x^2}\, dx$,

   where a is a constant

3. In groups, formulate an engaging strategy that will assist students to solve Examples 9.40 to 9.46 in the Student's Book. Allow students to share alternative methods of solving integrals which involve trigonometric substitutions. Engage students to use mathematical softwares and scientific calculators to verify the solutions of the given examples. Advise them to share their solutions through presentations.

4. Direct students to attempt Exercise 9.8 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.8

1. $\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+c$

2. $\frac{\sqrt{5}}{45}\tan^{-1}\left(\frac{\sqrt{5}x}{9}\right)+c$

3. $3\sqrt{3}\tan^{-1}(\frac{x}{\sqrt{3}})+c$

4. $\frac{1}{2}\tan^{-1}\left(\frac{z+1}{2}\right)+c$

5. $\frac{1}{3}\ln\left|\frac{\sqrt{9+4y^2}-3}{y}\right|+c$

6. $-\frac{\sqrt{4+z^2}}{4z}+c$

7. $\tan^{-1}x+c$

8. $\frac{1}{2}\ln\left(y^2+9\right)+\tan^{-1}\frac{y}{3}+c$

9. $\frac{1}{2}\tan^{-1}\left(\frac{4y}{7}\right)+c$

10. $\frac{\sqrt{2}}{2}\sin^{-1}\sqrt{2}x+c$

11. $6\sin^{-1}(\frac{2}{3}x)+c$

12. $\sin^{-1}(\frac{\sqrt{5}}{5}x)+c$

13. $x\sqrt{1-4x^2}+\frac{1}{2}\sin^{-1}(2x)+c$

14. $5\left(\sin^{-1}\left(\frac{10}{\sqrt{10}}(x+2)\right)\right)+$
$\frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{\sqrt{10}}{10}(x+2)\right)\right)+c$

15. $-\frac{\sqrt{9-x^2}}{9x}+c$

16. $\frac{1}{2}\left(\sin^{-1}(x+1)+\frac{1}{2}\sin\left(2\sin^{-1}(x+1)\right)\right)+c$

17. $\frac{9}{2}\left[\sin^{-1}\frac{1}{3}(x-2)+\frac{1}{9}(x-2)\sqrt{5+4x-x^2}\right]+c$

18. $\frac{1}{3}(t^2+9)^{\frac{3}{2}}-9\sqrt{t^2+9}+c$

19. $2\sin^{-1}(\frac{1}{2}x)-\sin(2\sin^{-1}(\frac{1}{2}x))+c$

20. $\frac{1}{8}(\sin^{-1}x-\frac{1}{4}\sin(4\sin^{-1}x))+c$

21. $\frac{1}{2}\tan^{-1}(\frac{x-2}{2})+c$

22. $\frac{2}{5}\sqrt{5}\tan^{-1}\left(\frac{\sqrt{5}}{5}(x+5)\right)+c$

23. $\sin^{-1}(\frac{1}{2}x+1)+c$

24. $-2\sqrt{9-x^2}-7\sin^{-1}(\frac{1}{3}x)+c$

25. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}(x-2)\right)+c$

26. $\dfrac{2s^2+9}{8(4s^2+9)^{\frac{1}{2}}}+c$

27. $7\ln\left|\sqrt{x^2+1}+x\right|+\dfrac{\sqrt{2}}{16}\left[\sin^{-1}\left(\sqrt{2}x\right)\right]+\dfrac{1}{2}\sin\left[2\sin^{-1}\left(\sqrt{2}x\right)\right]+c$

5. Guide students through discussion to recognize integrals whose solutions are obtained by applying the trigonometric substitutions $t=\tan x$ or $t=\tan\dfrac{x}{2}$.
6. Assist students through discussion to solve integrals of the form:
   (a) $\displaystyle\int\frac{1}{a+b\sin^2 x+c\cos^2 x}$
   (b) $\displaystyle\int\frac{1}{a+b\sin x+c\cos x}dx$
7. In groups, formulate a strategy that will engage students to perform Examples 9.47 to 9.51 in the Student's Book. Allow students to share alternative methods of finding solutions for the integrals that involve trigonometric substitutions. Engage students in groups to use mathematical softwares and scientific calculators to verify solutions for the integrals in the examples in the Student's Book. Advise them to share their solutions through presentations.
8. Direct students to attempt Exercise 9.9 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.9

1. $\tan\frac{1}{2}x+c$

2. $\frac{2}{3}\tan^{-1}\left[\frac{1}{3}\tan(\frac{1}{2}\theta)\right]+c$

3. $\dfrac{-1}{1+\tan x}+c$

4. $\dfrac{\sqrt{2}}{2}\tan^{-1}\left(\dfrac{\sqrt{2}}{2}\tan x\right)+c$

5. $\ln\left|\tan\left(\dfrac{x}{2}\right)\right|+c$

6. $\tan^{-1}(\sin x)+c$

7. $-\ln\left|1+\cos^2 x\right|+c$

8. $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right)+c$

9. $\dfrac{1}{6}\ln\left|\dfrac{3\tan x+1}{3\tan x-1}\right|+c$

10. $\frac{1}{2}\tan\theta+$

11. $\ln\left|\sec^2\left(\dfrac{x}{2}\right)\right|+c$

12. $\ln\left|\tan(\frac{x}{2})\right|+c$

13. $\frac{1}{4}\ln\left|\tan(4x)+\sec(4x)\right|+c$

14. $\sin^{-1}(\frac{1}{2}\sin x)+c$

15. $-\ln\left|\tan x+\sec x\right|+2\sin x+c$

16. $\dfrac{1}{2}\ln\left|\dfrac{\tan\frac{x}{4}+2}{\tan\frac{x}{4}-2}\right|+c$

17. $\dfrac{1}{3}\left(\ln\left|\dfrac{\tan\frac{x}{2}}{3}+1\right|-\ln\left|\dfrac{\tan\frac{x}{2}}{3}-1\right|\right)+c$

18. $2\ln\left|\tan(\frac{}{4})\right|+$

19. $\dfrac{1}{3}\tan\left(\dfrac{3}{2}y\right)+c$

20. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left[\dfrac{1}{\sqrt{3}}\left(2\tan\dfrac{x}{2}\right)-1\right]+c$

21. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left[\dfrac{1}{\sqrt{3}}\left(2\tan\dfrac{x}{2}+1\right)\right]+c$

22. $\tan^{-1}\left[\frac{1}{2}\tan\left(\frac{\theta}{4}\right)\right]+c$

23. $\frac{1}{3}\ln\left|\frac{\cos x}{\cos x-3}\right|+c$

24. $\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)+1}{\tan\left(\frac{\theta}{2}\right)+3}\right|+c$

25. $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}\tan\left(\frac{x}{3}\right)\right]+c$

26. $\ln\left|\frac{2\sin x-4}{2\sin x-2}\right|+c$

27. $\tan^{-1}(\tan x+1)+c$

28. $-\theta+\frac{10\sqrt{3}}{3}\tan^{-1}\left[\sqrt{3}\tan\left(\frac{\theta}{2}\right)\right]+c$

29. $\ln\left|\tan(\frac{x}{2})+2\right|+c$

30. $\sin x\sqrt{1+\sin x}-\frac{1}{3}(1+\sin x)^{\frac{3}{2}}+\sqrt{1+\sin x}+c$

Integration by splitting the numerator

Teaching steps

1. Guide students through discussion to recognize integrals whose integrand is;
   (a) A rational function with a quadratic denominator that cannot be factorized.
   (b) A rational function with trigonometric functions in both the numerator and denominator.
2. Assist students through discussion to solve integrals of the form:

(a) $\int\frac{ax+b}{cx^2+dx+e}dx$

(b) $\int\frac{a\cos x\pm b\sin x}{p\cos x\pm q\sin x}dx$

(c) $\int\frac{a\cos x\pm b\sin x+c}{p\cos x\pm q\sin x}dx$

(d) $\int\frac{a\cos x\pm b\sin x}{p\cos x\pm q\sin x+r}dx$

where a, b, c, d, e, p, q and r are constants.

3. In groups, formulate an engaging strategy that will assist students to perform Examples 9.52 to 9.55 in the Student's Book. Allow students to share alternative methods of solving the integrals which involve splitting the numerator.
4. Engage students in groups to use scientific calculators and mathematical softwares to verify the solutions of Examples 9.53 to 9.55 in the Student's Book. Advise them to share their solutions through gallery walk.
5. Require students to attempt Exercise 9.10 in the Student's Book. Check the correctness of their answers; and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.10

1. $\frac{1}{2}\ln\left|x^2+5\right|+\frac{7}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$
2. $\frac{1}{4}\ln\left|4x^2+12x+10\right|+\frac{1}{2}\tan^{-1}(2x+3)+c$
3. $\frac{1}{2}\theta-\frac{1}{2}\ln\left|\cos\theta+\sin\theta\right|+c$
4. $-\frac{7}{10}\ln\left|\sin\theta+3\cos\theta\right|-\frac{1}{10}\theta+c$
5. $\ln\left|x^2+2x+10\right|+\frac{1}{3}\tan^{-1}\left(\frac{1}{3}(x+1)\right)+c$
6. $\ln\left|y+3\right|-\frac{1}{y+3}+c$
7. $\frac{3}{2}\ln\left|u^2+2u+5\right|+\frac{5}{2}\tan^{-1}\left(\frac{1}{2}(u+1)\right)+c$

8. $\frac{11}{17}\ln\left|\cos\theta+4\sin\theta\right|-\frac{10}{17}\theta+c$

9. $\frac{3}{5}\ln\left|3+\cos x+2\sin x\right|+\frac{4}{5}x-\frac{12}{5}\tan^{-1}\left[\tan\left(\frac{x}{2}\right)+1\right]+c$

10. $\frac{1}{2}x+\frac{5}{2}\ln\left|\cos x+\sin x\right|+c$

11. $\frac{3}{13}\ln\left|2\cos\theta+5\sin\theta\right|+\frac{2}{13}\theta+c$

12. $\frac{3}{106}\ln\left|7\cos 2x-2\sin 2x\right|-\frac{37}{53}x+c$

Integrals of exponential functions of the form $\int a^x dx$ where a is a positive constant

Teaching steps

1. Guide students through discussion to recognize integrals of this form where x is a variable and a is a constant, $a>0$, $a\neq 1$.
2. Assist students through discussion to solve integrals of the form:
   $\int a^x dx$, where a is a constant.
3. Use Examples 9.56 and 9.57 in the Student's Book to guide students to solve integrals of exponential functions. Allow students to share alternative methods of solving integrals of exponential functions.
4. Engage students in groups to use scientific calculators and mathematical softwares to verify the solutions of Examples 9.56 and 9.57 in the Student's Book. Advise them to share their findings through presentations or gallery walk.

Integrals of logarithmic functions of the form $\int \log ax\,dx$, where a is a constant.

Teaching steps

1. Guide students through discussion to recognize integrals of this form where a is a constant.
2. Assist students through discussion to solve integral of the form: $\int \log ax\,dx$, where a is a constant.
3. Use Examples 9.58 and 9.59 in the Student's Book to guide students to solve integrals of logarithmic functions. Allow students to share alternative methods of solving integrals of logarithmic functions.
4. Engage students in groups to use scientific calculators and mathematical softwares to verify the solutions of Examples 9.58 and 9.59 in the Student's Book. Advise them to share their findings through presentation or gallery walk.
5. Direct students to attempt Exercise 9.11 in the Student's Book. Advise them to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.11

1. $\frac{7^x}{\ln 7}+c$
2. $\frac{(\ln x)^2}{2\ln 10}+c$
3. $\frac{8^x}{\ln 2}+c$
4. $-\frac{x^2}{\ln 10}+c$
5. $\frac{3^{2x}}{2\ln 3}+c$
6. $-\frac{6^{1-2y}}{2\ln 6}+c$
7. $\frac{-1}{4^x \ln 4}+c$
8. $\frac{-9^{3-7x}}{7\ln 9}+c$
9. $\frac{10^{x-1}}{\ln 10}+c$

10. $\frac{x}{\ln 10}(\ln|3x|-1)+c$

11. $\frac{1}{2\ln 2}(\ln|x|)^2+c$

12. $\frac{2^x}{\ln 2}+\frac{x}{\ln 10}(\ln|5x|-1)+c$

13. $\frac{(2x+1)\ln|2x+1|}{2\ln 10}-\frac{x}{\ln 10}+c$

14. $\frac{\left(x^2+x-1\right)\left(\ln\left|x^2+x-1\right|\right)}{\ln 10}+c$

15. $\frac{x}{6\ln 10}(\ln|2x|-1)+c$

16. $\frac{x}{\ln 10}(\ln|\frac{3}{4}x|-1)+c$

17. $-\frac{(\frac{1}{3})^x}{\ln 3}+c$

18. $\frac{\tan x}{\ln 10}(\ln|\tan x|-1)+c$

19. $\frac{x}{\ln 2}(\ln|x|-1)+c$

20. $\frac{(\ln|3x|)^2}{4\ln 5}+c$

Definite integrals

Teaching steps

1. Design an activity that can be used to assist students to recognize the definite integral of a function over a specified interval.
2. Use Examples 9.60 to 9.64 in the Student's Book to guide students on how to evaluate a definite integral over a specified interval.
3. Engage students to use scientific calculators or mathematical softwares to verify the solutions of Examples 9.60 to 9.64 in the Student's Book. Advise them to share their solutions through presentations.
4. Encourage students to solve more examples of evaluating definite integrals of polynomial, trigonometric, exponential,

logarithmic, and rational functions from the suggested resources.

5. Ask students to attempt Exercise 9.12 in the Student's Book. Advise students to submit their work. Check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, Symbolab, Mathematica or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.12

1. 16
2. $\frac{52}{3}$
3. 32
4. 180
5. $\frac{15}{2}$
6. 2
7. $\ln 2$
8. $\frac{2\sqrt{2}}{3}$
9. $-\frac{12}{\pi}$
10. $\frac{55}{4}$
11. $\ln 2+\frac{69}{4}$
12. $\frac{113}{24}$
13. 7174089
14. $\frac{9\sqrt{3}-1}{5}$
15. $\frac{5}{128}$
16. $3\ln 2$
17. $3(e-1)$
18. $\frac{1}{3}(2\ln 2-\ln 3)$
19. 2
20. $\frac{-3}{7}\ln 2$
21. 0
22. $\frac{1}{2}\ln 7$
23. $\frac{e^{12}-1}{4e}$
24. $10\left[\cos(1)-\cos\left(\frac{\pi-4}{4}\right)\right]$
25. 2
26. $\frac{1}{2}(1-e^{-2})$
27. $\frac{16\sqrt{2}-4\sqrt[4]{2}}{9}$

28. $\frac{2}{15}$ 29. $\ln(\frac{23}{5})$ 30. $\frac{2}{3}$

31. 1 32. 6 33. 1

34. $\frac{3}{2}$ 35. $\frac{40}{3}$ 36. $\frac{(\ln 2)^2}{4}$

37. $\frac{3\pi}{8}+1$ 38. $3+4\ln 2$ 39. 9

40. $\frac{1}{6}$ 41. $\frac{e^5-1}{2}$ 42. $2\ln 2-2\ln 3$

43. $\frac{3}{10}$ 44. $12\ln 3-8$ 45. $\frac{1}{36}-\frac{7}{36e^6}$

46. $\frac{26}{3}\ln 2-\frac{59}{18}$ 47. $-\frac{\ln 2-\ln 3}{2}$ 48. $3\ln 3-5\ln 2$

49. $\frac{3\sqrt{3}}{8}$ 50. $\frac{203}{480}$ 51. 0

52. $\frac{2}{35}$ 53. $\frac{2}{15}$ 54. π

55. $\frac{5}{2}$ 56. $\frac{\pi}{4}$ 57. $\frac{1}{2}(\tan^{-1}(\frac{3}{2})-\frac{\pi}{4})$

58. $\frac{4}{3}\tan^{-1}(\frac{1}{3})$ 59. $\frac{\pi}{4}$ 60. $6-4\sqrt{2}-7\sin^{-1}(\frac{1}{3})$

Applications of integration

The area under a curve and the area enclosed between two curves

Teaching steps

1. Guide students through a jigsaw to describe the area under a curve and the area enclosed between two curves. Encourage students to explore additional resources, such as online tutorials or reference books, to enrich their understanding and for further practice.

2. Assist students through discussions to calculate the area under a curve and the x-axis through the following procedures:
   (i) Guide students to sketch and identify the area under a curve
   (ii) Determine the limits of integration if not given
   (iii) Set up the definite integral
   (iv) Integrate to get the area
3. Ask students to investigate the area lying above the x-axis (which will have a positive value) and the area lying below the x-axis (which will have a negative value). Ask them through brainstorming, if is it possible for the required area to lie above and below the x-axis at the same time.
4. Guide students through discussion to follow the same procedures when calculating the area under a curve and the y-axis.
5. Use Examples 9.65 to 9.69 in the Student's Book to guide students and assist them in calculating the under a curve and the x-axis and the area under a curve and the y-axis. Allow students to share alternative methods of calculating the area under a curve and the x-axis. Advise them to share their findings through presentation.
6. Formulate a strategy which will engage students to identify the area of a region enclosed between two curves. Assist them through discussion to determine the integral of the difference between the two functions using common limits.
7. Use Examples 9.70 and 9.71 in the Student's Book to guide students and assist them in calculating the area of a region enclosed between two curves. Allow students to share alternative methods of calculating the area enclose between two curves.

8. Using Figures 9.8 and 9.9 in the Student's Book guide students through discussions to derive the formula for calculating the length of an arc of a function in the Cartesian coordinate system. Allow students to share alternative methods of deriving the formula for calculating the length of an arc.
9. Assist students through discussion to develop formulae for calculating the length of an arc for functions given by parametric equations and when given in polar coordinates.
10. Use Examples 9.72 to 9.75 in the Student's Book to guide students and assist them in calculating the length of an arc expressed in Cartesian coordinate system, parametric, and in polar form. Engage students to use mathematical softwares to verify the solutions of Examples 9.72 to 9.75 in the students Book.
11. Require students to attempt Exercise 9.13 in the Student's Book. Advise students to submit their work and check the correctness of their answers, and provide them constructive feedback. Encourage students to explore additional resources, such as online tutorials, or reference books, to deepen their understanding and for further practice.

Answers to Exercise 9.13

1. $\frac{1}{3}$ square units
2. $\frac{1}{2}ab\pi$ square units
3. $\frac{\pi}{2}$ square units
4. $\frac{9}{2}(1-e^{-2})$ square units
5. 18 square units
6. $\frac{80}{\pi}$ square units
7. 3π square units
8. $\left(\frac{1}{2}-\frac{4\pi-3\sqrt{3}}{24}\right)$ square units

9. $\frac{45}{4}$ square units
11. $\sqrt{257}$ units
12. 4.44 units
13. 12.0397 units
14. $\frac{\pi}{2}r$ units
15. 21.2563 units
16. 2π units
17. $78\frac{2}{3}$ square units
18. $\frac{3\pi a}{2}$ square units

Volume of a solid of revolution

Teaching steps

1. Guide students through jigsaw to describe what happens when the plane in Figure 9.10 (a) in the Student's Book bounded by the curve, the x-axis, and the lines $x = a$ and $x = b$ is rotated through a complete revolution about the x-axis.
2. Assist students through discussion to study Figure 9.10 (b) in the Student's Book and observe the solid generated by rotating the plane about the x-axis. Encourage students to explore additional resources, such as online tutorials, simulations, or reference books, to enrich their understanding and for further practice.
3. Assist students through discussions to use Figures 9.11(a) and 9.11(b) in the Student's Book, to derive the formula for calculating the volume of the solid of revolution generated after rotating the plane figure a complete revolution about the x-axis.
4. Guide students through discussion to investigate the volume of the solid of revolution generated when the strip rotates completely about the x-axis. Allow students to share alternative methods of deriving the formula for the volume of the solid of revolution.

5. Guide students through discussion to follow the same procedures when deriving the formula for the volume of the solid of revolution generated after rotating a complete revolution about the y-axis.
6. Use Examples 9.76 to 9.79 in the Student's Book, to guide students and assist them in calculating the volume of the solid revolution. Engage students to use mathematical softwares to verify the solutions in Examples 9.76 to 9.79 in the Student's Book.
7. Formulate strategies of enhancing students understanding of how to determine the volume of the solid of revolution about any line using the disk, washer, and shell method.
8. Use Examples 9.80 to 9.84 in the Student's Book, to guide students and assist them in calculating the volume of a solid of revolution about any line. Allow students to share alternative methods of calculating the volume of a solid of revolution about any line.
9. Guide students through discussions using Figure 9.13 in the Student's Book to derive the formula for calculating the area of a sector. Allow students to share alternative methods of deriving the formula for calculating the area of a sector.
10. Use Examples 9.85 and 9.86 in the Student's Book to guide students and assist them in calculating the area of a sector.
11. Require students to attempt Exercise 9.14 in the Student's Book. Advise them to submit their work; check the correctness of their answers, and provide them feedback. Encourage students to explore additional resources, such as online tutorials, or reference books, to enrich their understanding and for further practice.

Answers to Exercise 9.14

1. $\frac{96}{5}\pi$ cubic units

2. $\frac{93}{5}\pi$ cubic units

3. $\frac{25}{8}\pi^2$ cubic units

4. 16π cubic units

5. $\frac{64}{3}\pi$ cubic units

6. $\frac{80}{\pi}$ cubic units

7. (a) $\frac{4\pi ab^2}{3}$ cubic units (b) $\frac{4\pi a^2 b}{3}$ cubic units

8. $\frac{16\pi}{5}$ cubic unit

9. 2500π cubic units

10. $\frac{404\pi}{5}$ cubic unit

11. (a) $\frac{62\pi}{5}$ cubic units (b) 30π cubic units

(c) $\frac{423\pi}{80}$ cubic units (d) $\frac{10\pi}{3}$ cubic units

12. $\frac{\pi}{2}\left(e^2-1\right)$ cubic units

13. $\frac{3\pi}{10}$ cubic units

14. 9π cubic units

15. $\left(2-\frac{\pi}{4}\right)$ square units

16. $\frac{\pi}{24}$ square units

17. $\frac{\pi}{8}$ square units

18. $\frac{4\pi-7\sqrt{3}}{16}$ square units

Answers to Revision Exercise 9

1. (a) $\frac{2}{9}(3x-1)^{\frac{3}{2}}+c$ (b) $\frac{x^5}{5}+x^3+c$

(c) $\frac{1}{2}\ln\left|e^{2x}+1\right|+c$ (d) $\frac{1}{5}x^5+3x^4+16x^3+32x^2+c$

(e) $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$ (f) $\frac{x^3}{3}\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{2}\ln\left(1+x^2\right)+c$

2. $\frac{\pi}{12}$ 3. $\frac{\ln 3}{4}$

4. (a) $-\frac{116}{15}$ (b) $\frac{2}{3}\tan^{-1}\left[\frac{1}{3}\tan\left(\frac{\theta}{2}\right)\right]+c$

(c) $\frac{1}{5}(\theta+1)(2\theta-3)^{\frac{3}{2}}+c$ (d) $x^2e^x-2xe^x+2e^x+c$

5. 0.1794

7. (a) $\frac{1}{4}\ln 3$ (b) $-\frac{\sqrt{-x^2+4}}{4x}+c$ (c) $\frac{1}{2}(\sin x-\frac{1}{7}\sin 7x)+c$

(d) $-\frac{e^x\cos x}{2}+\frac{e^x\sin x}{2}+c$ (e) $\frac{2}{\sqrt{23}}\tan^{-1}\left[\frac{4}{\sqrt{23}}\tan\left(\frac{\theta}{2}\right)+\frac{1}{\sqrt{23}}\right]+c$

(f) $\frac{1}{2}\tan^{-1}\left(\frac{\theta-3}{2}\right)+c$ (g) 2.3752 (h) $\frac{1}{6\ln 24}24^{6x}+c$

9. (a) $\sin(x^2-5)+c$ (b) $-\cos(x^3-2x^2+1)+c$

(c) $-2\sqrt{2}\cos\left(\frac{x}{2}\right)+c$ (d) $\frac{1}{3}\left(\frac{\sin^9(3z)}{9}-\frac{2\sin^{11}(3z)}{11}+\frac{\sin^{13}(3z)}{13}\right)+c$

(e) $\frac{1}{2}\left[\cos^3 2t\sin 2t+\frac{3}{8}\left(2t-\frac{1}{4}\sin 8t\right)\right]+c$ (f) $\frac{1}{3}\cos(\cos 3x)+c$

(g) $\frac{1}{3}\left(\frac{\tan^3 3y}{3}+\frac{2\tan^5 3y}{5}+\frac{\tan^7 3y}{7}\right)+c$

(h) $\frac{1}{2}\left(-\frac{1}{11}\cos 11t-\frac{1}{5}\cos 5t\right)+c$ (i) $\ln|\sec x|-\sec^2 x+\frac{1}{4}\sec^4 x+c$

(j) $\frac{3}{2}\left[-\frac{1}{5}\cos^5\left(\frac{2x}{3}\right)+\frac{1}{7}\cos^7\left(\frac{2x}{3}\right)\right]+c$ (k) $e^{4\sin x}\quad c$

(l) $\frac{-32-5\sqrt{2}}{252\sqrt{2}}$ (m) 0 (n) $\frac{1}{6}(\sin x-\cos x)^6+c$

10. $I_6=-\frac{1}{6}\cos x\sin^5 x+\frac{5}{6}\left[-\frac{1}{4}\sin^3 x\cos x+\frac{3}{8}\left(x-\frac{1}{2}\sin 2x\right)\right]+c$ and

$I_7=\frac{1}{7}\cos^7 x-\frac{3}{5}\cos^5 x+\cos^3 x-\cos x+c$

11. (a) $\frac{1}{2}\left(\tan^{-} x+\frac{x}{1+x}\right)+c$ (d) $2\left(\frac{1}{3}(x-1)^{\frac{3}{2}}+\sqrt{x-1}\right)+c$

(b) $\frac{2}{3}(x+1)^{\frac{3}{2}}+\frac{2}{3}x^{\frac{3}{2}}+c$

(e) $\frac{1}{2}\ln\left|x^2+4x+8\right|-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+2)\right)+c$

(c) $\frac{2}{3}(1+\ln x)^{\frac{3}{2}}+c$ (f) $\frac{\pi\sqrt{3}}{9}$

11. (a) $\frac{1}{2\ln 2}-\frac{1}{e}$ (b) $-\frac{1}{2}\left(\frac{1}{2}\ln 3-1\right)$ (c) $\frac{7}{6}+\ln 2-\ln 3$ (d) $\frac{1}{2}$

14. (a) $-5\sec^{-1}\frac{1}{5}x+\sqrt{x^2-25}+c$ (b) $\frac{4\sqrt{3}\pi-\pi+2\ln 2-4}{6}$

(c) $\frac{1}{2\ln 10}$ (d) $\frac{\pi}{4}$

15. (a) $\frac{1}{4}\left(\sin^{-1}\left(\sin^2 x\right)\right)+\frac{1}{2}\sin\left(2\sin^{-1}\left(\sin^2 x\right)\right)+c$

(b) $2e^{\sqrt{x-2}}+c$ (c) $-\frac{2}{3^{\sqrt{x}}\ln 3}+c$

(d) $\frac{1}{2}\ln\left|x^2+2x+5\right|+c$ (e) $\frac{1}{2}\ln\left|x^2+1\right|+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$

(f) $\frac{x^2}{2}+x+2\ln\left|x+2\right|+\ln\left|x-3\right|+c$

(g) $\ln\left|x-1\right|+\frac{1}{x-1}-\frac{1}{2}\ln\left|x^2+1\right|+\tan^{-1}x+c$

(h) $\tan^{-1}(x)-\frac{1}{2(x^2+1)}+c$ (i) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+3}{4}\right)+c$

(j) $1+\ln\left|x\right|-\ln\left|1+\ln\left|x\right|\right|+c$ (k) $-\frac{2}{3(x^3+1)}+c$

(l) $\frac{1}{3}(x+1)^{\frac{3}{2}}+\frac{1}{3}(x-1)^{\frac{3}{2}}+c$ (m) $\frac{1}{2\ln 7}7^{2x}+c$

16. (a) $\frac{13}{10}\ln 3+\frac{11}{20}\pi$ (b) $\frac{1}{2}\ln\left|\theta^2+16\right|+8\tan^{-1}\left(\frac{\theta}{4}\right)+c$

(c) $\frac{1}{4}\ln\left|\sec 4\theta+\tan 4\theta\right|+c$

(d) $\frac{3}{2}\ln\left|\cos x+\sin x\right|+\frac{1}{2}x+4\sqrt{2}\ln\left[\frac{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}\right]+c$

17. (a) $\frac{35}{256}\pi$ (b) $\frac{128}{315}$

19. (a) $\frac{1328\sqrt{3}}{2835}$ (b) $\frac{17}{36}+\frac{3}{32}\ln 3$

21. (b) $\frac{1}{3}$ 22. $\ln(1+\sqrt{2})$ units

23. (a) 1 square unit (b) $\pi(e-1)$ cubic units

24. $\left(-37e^{-5}+2\right)$ units

26. The total sales for 4 months is 3702.80.

27. $\frac{2\pi+\sqrt{3}+8}{16}$ square units

28. $\frac{17}{6}$ units

30. (a) P(4, 16), Q(−6.25, −25) (b) $\left(68-45\ln 3\right)$ square units.

31. (a) $21\frac{1}{3}$ square units

32. (a) $\frac{8\pi-9\sqrt{3}}{64}$ square units (b) $\frac{1}{48}\left(99\pi+32\sqrt{2}+66\right)$ square units

33. $\frac{56\pi}{3}$ cubic units

34. $\frac{32}{5}\pi$ cubic units

35. $\frac{19}{3}\pi$ cubic units

37. $\frac{a^2}{4k}\left(e^{2\pi k}-e^{-2\pi k}\right)$ square units

38. $\frac{32}{3}$ square units

40. (a) 768π cubic units (b) 8 square units

Project work

Provide a recommended guideline then, supervise students to identify any mathematical related problem in their community and ask them to carry out a project to address it.