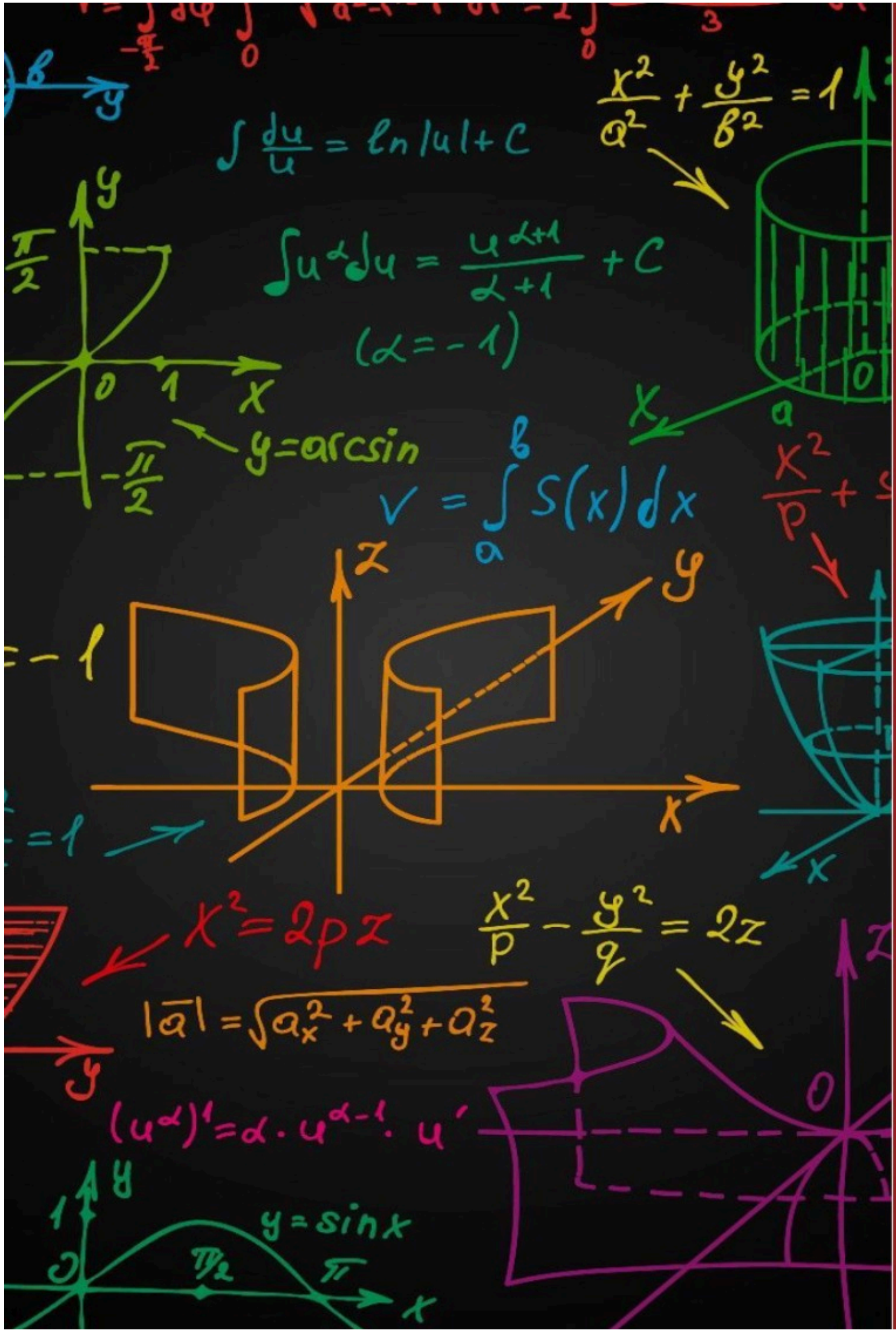

PRINCIPAL MATH (FUNDAMENTAL ITEMS) A LEVEL UACE PAPER 1 & 2

Based on the NCDC Uganda
A-Level Principal Math Syllabus

A LEVEL UACE Paper 1

Principal Mathematics

1 & 2

2ND EDITION

(Fundamental Items)

ASSESSMENT OBJECTIVE 1. ALGEBRA
ASSESSMENT OBJECTIVE 2 GEOMETRY
ASSESSMENT OBJECTIVE 3 CALCULUS
ASSESSMENT OBJECTIVE 4 DATA ANALYSIS & PROBABILTY
ASSESSMENT OBJECTIVE 5 MECHANICS

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PRINCIPAL MATHEMATICS

(2nd edition)

1&2

FUNDAMENTAL ITEMS

PAPER 1

- ASSESSMENT OBJECTIVE 1 (ALGEBRA)
- ASSESSMENT OBJECTIVE 2 (GEOMETRY)
- ASSESSMENT OBJECTIVE 3 (CALCULUS)

PAPER 2

- ASSESSMENT OBJECTIVE 4 (DATA ANALYSIS &PROBABILITY)
  ASSESSMENT OBJECTIVE 5 (MECHANICS)

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ASSESSMENT OBJECTIVE 1(ALGEBRA)

- NUMERICAL CONCEPT
- EQUATIONS AND INEQUALITIES
- PERMUTATIONS AND COMBINATIONS
- SERIES
- COMPLEX NUMBERS

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NUMERICAL CONCEPT

QUESTIONS 1–10: WITH WORKED SOLUTIONS

Question 1: District Water Infrastructure Planning

- **Scenario:** The Kisoro District Council is planning water infrastructure for a village. Five years ago, the population was 2,500; today it is 4,000. The council has a budget to build a water system capable of serving 10,000 people. However, construction will take 2 years to complete. The system must last 15 years after completion without expansion.
- **Hint:** Use exponential growth $N = N_0 e^{kt}$. Consider the timeline: $t = 0$ is today. Construction finishes at $t = 2$.
- **Task:** As a mathematics learner; (a) Determine the annual growth constant k and project the population at the time the system becomes operational (2 years from now). (b) Calculate the year in which the population will exceed the 10,000 capacity limit. (c) Evaluate the sustainability: If the system costs 50 million UGX per 1,000 people capacity, calculate the cost of the *unused capacity* (empty seats) in the first year of operation versus the cost of building a larger system now to last the full 15 years. Recommend whether to build for 10,000 or 15,000 capacity.

Worked Solution: (a) Growth Constant and Operational Population:

- Let $t = 0$ be 5 years ago. $N_0 = 2{,}500$. At $t = 5$, $N = 4{,}000$. $4{,}000 = 2{,}500e^{5k} \Longrightarrow 1.6 = e^{5k} \Longrightarrow \ln(1.6) = 5k$ $k = \frac{0.4700}{5} = 0.0940$
- Population at operation ($t = 5 + 2 = 7$ years from initial count, or 2 years from today): $N(7) = 2{,}500e^{0.0940\times7} = 2{,}500e^{0.658} \approx 2{,}500 \times 1.931 = 4{,}828$ *Alternatively from today* $(N_0 = 4000, t = 2)$: $4{,}000e^{0.094\times2} \approx 4{,}828$.
- **Answer:** $k \approx 0.094$; Population at operation $\approx 4{,}828$.

(b) Year exceeding 10,000 capacity:

- Solve for t (from today) when $N = 10{,}000$: $10{,}000 = 4{,}000e^{0.094t} \Longrightarrow 2.5 = e^{0.094t}$ $\ln(2.5) = 0.094t \Longrightarrow 0.9163 = 0.094t \Longrightarrow t \approx 9.75$ years
- Since construction takes 2 years, the system operates for $9.75 - 2 = 7.75$ years before exceeding capacity.
- **Answer:** Capacity exceeded approximately **9.75 years from today**.

(c) Sustainability Evaluation:

- **Option A (10,000 capacity):** Cost = $10 \times 50\text{M} = 500\text{M}$ UGX. Useful life before overflow = 7.75 years.
- **Option B (15,000 capacity):** Cost = $15 \times 50\text{M} = 750\text{M}$ UGX.
  - Find time to reach 15,000: $15{,}000 = 4{,}000e^{0.094t} \Longrightarrow 3.75 = e^{0.094t} \Longrightarrow t = \frac{\ln(3.75)}{0.094} \approx 14.0$ years.
  - This covers the required 15-year lifespan (14 years is close enough, or slightly exceeds).

- **Unused Capacity Cost (Option A):** In year 1 of operation (population 4,828), unused capacity = $10{,}000 - 4{,}828 = 5{,}172$.
  - Cost of unused = $\frac{5{,}172}{10{,}000} \times 500\text{M} \approx 258.6\text{M}$ UGX.
- **Recommendation:** Option B costs 250M more upfront but lasts the full 15 years. Option A requires expansion/replacement in 7.75 years. Given inflation and construction disruption, **Option B (15,000 capacity)** is more sustainable despite higher initial unused capacity.

Question 2: Radioactive Medical Supply Chain

- **Scenario:** A hospital in Kampala uses a radioactive isotope (half-life = 8 days) for cancer treatment. A 500 g sample is delivered. The treatment protocol requires a minimum of 50 g active isotope per session. The hospital plans sessions every 3 days.
- **Hint:** Decay formula $N = N_0(0.5)^{t/T}$.
- **Task:** As a mathematics learner; (a) Calculate the mass remaining after 24 days and determine how many full treatment sessions (50 g each) can be conducted from the remaining mass. (b) Determine the exact day when the sample becomes unsafe for disposal (< 10 g). (c) The hospital considers ordering a new shipment when the mass drops to 100 g. Calculate the day this order should be placed. If delivery takes 4 days, will there be a treatment interruption?

Worked Solution: (a) Mass after 24 days and sessions: $N = 500(0.5)^{24/8} = 500(0.5)^3 = 500 \times 0.125 = 62.5$ g

- Sessions possible: $\lfloor \frac{62.5}{50} \rfloor = 1$ full session.
- **Answer:** 62.5 g remaining; **1 session**.

(b) Day when sample < 10 g: $10 = 500(0.5)^{t/8} \Longrightarrow 0.02 = (0.5)^{t/8}$ $\log(0.02) = \frac{t}{8}\log(0.5) \Longrightarrow t = 8 \times \frac{-1.69897}{-0.30103} \approx 45.15$ days

- **Answer:** Approximately **45.2 days**.

(c) Reorder point and interruption:

- Find t when $N = 100$: $100 = 500(0.5)^{t/8} \Longrightarrow 0.2 = (0.5)^{t/8}$ $t = 8 \times \frac{\log(0.2)}{\log(0.5)} \approx 8 \times 2.322 = 18.58$ days
- Order placed Day 18.58. Delivery arrives Day $18.58 + 4 = 22.58$.
- Mass at Day 22.58: $N = 500(0.5)^{22.58/8} \approx 500(0.5)^{2.82} \approx 70.7$ g.
- Since 70.7 g > 50 g (minimum for session), there is **no interruption**.
- **Answer:** Order on Day 18.6; **No interruption**.

Question 3: Village Savings Group Investment Strategy

- **Scenario:** A VSLA in Jinja has 1,000,000 UGX capital. They can invest in Option A (12% compound interest annually) or Option B (1% simple interest monthly). They need to triple their capital to build a community hall.
- **Hint:** Compound: $A = P(1 + r)^n$. Simple: $A = P(1 + rn)$. Note monthly vs annual.

- **Task:** As a mathematics learner; (a) Calculate the time required to triple the capital under Option A. (b) Calculate the time required to triple the capital under Option B. (c) If they choose Option A, but withdraw 500,000 UGX after 3 years to handle an emergency, calculate the new total time to reach 3,000,000 UGX from the *initial* start date.

Worked Solution: (a) Option A (Compound 12%): $3{,}000{,}000 = 1{,}000{,}000(1.12)^n \Rightarrow 3 = 1.12^n$ $n = \frac{\log 3}{\log 1.12} \approx 9.69$ years

- **Answer: 9.7 years**.

(b) Option B (Simple 1% monthly = 12% annually): * Simple interest rate per year = $1\% \times 12 = 12\% = 0.12$. $3{,}000{,}000 = 1{,}000{,}000(1 + 0.12n) \Rightarrow 3 = 1 + 0.12n$ $2 = 0.12n \Rightarrow n = \frac{2}{0.12} = 16.67$ years

- **Answer: 16.7 years**.

(c) Option A with Withdrawal:

- Value after 3 years: $A_3 = 1{,}000{,}000(1.12)^3 \approx 1{,}404{,}928$ UGX.
- After withdrawal: $1{,}404{,}928 - 500{,}000 = 904{,}928$ UGX.
- Time ($t_{remaining}$) to reach 3,000,000 from 904,928: $3{,}000{,}000 = 904{,}928(1.12)^{t_{rem}} \Rightarrow 3.315 = 1.12^{t_{rem}}$ $t_{rem} = \frac{\log 3.315}{\log 1.12} \approx 10.63$ years
- Total time = 3 years + 10.63 years = 13.63 years.
- **Answer: 13.6 years** total.

Question 4: Soil Acidity and Coffee Farming

- **Scenario:** Coffee farmers in Mbale face soil acidification. Optimal pH is 6.0. Current pH is 4.5. Liming can raise pH by 0.5 units per 1,000 kg of lime per hectare. However, rain lowers pH by 0.1 units per month.
- **Hint:** $pH = -\log[H^+]$. A change of 1 pH unit is a 10x change in $[H^+]$.
- **Task:** As a mathematics learner; (a) Calculate the factor by which hydrogen ion concentration $[H^+]$ must be reduced to move from pH 4.5 to 6.0. (b) If farmers apply enough lime to reach pH 6.0 instantly, calculate how many months until the soil returns to pH 4.5 due to rain. (c) To maintain pH ≥ 5.5 permanently, calculate the frequency of liming required (in months) assuming each application raises pH by 1.0 unit.

Worked Solution: (a) Reduction factor: $[H^+]_{4.5} = 10^{-4.5}, \quad [H^+]_{6.0} = 10^{-6.0}$ Factor = $\frac{10^{-4.5}}{10^{-6.0}} = 10^{1.5} = 10\sqrt{10} \approx 31.62$

- **Answer:** Concentration must reduce by factor of **31.6**.

(b) Time to return to pH 4.5: * Total drop needed = $6.0 - 4.5 = 1.5$ units. * Rate = 0.1 units/month. * Time = $1.5/0.1 = 15$ months.

- **Answer: 15 months**.

(c) Frequency for pH $\geq$ 5.5: * Target minimum = 5.5. Start after liming = 6.0 (assuming 1.0 unit raise from 5.0? No, scenario says apply to reach 6.0). * Let's assume application raises pH by 1.0 unit. To stay above 5.5, allowable drop = 0.5 units. * Time = $0.5/0.1 = 5$ months.

- **Answer:** Lime every **5 months**.

Question 5: Traditional Roofing Construction

- **Scenario:** A carpenter in Arua is building a roof for a 6m wide house. The peak is 2m above the walls. Timber comes in standard 6m lengths. Waste occurs if cut lengths are not utilized.
- **Hint:** Pythagoras $c = \sqrt{a^2 + b^2}$. Simplify surds.
- **Task:** As a mathematics learner; (a) Calculate the exact length of one rafter in simplest surd form. (b) If a 0.5m overhang is added to each rafter, calculate the new exact length. Determine if a 6m timber plank is sufficient for one rafter. (c) The carpenter needs to buy 20 rafters. If timber costs 15,000 UGX per meter sold in whole meters (round up length), calculate the total cost savings if he buys the exact surd length vs the whole meter length.

Worked Solution: (a) Exact rafter length: * Base = 3m, Height = 2m. * $L = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$ m.

- **Answer:** $\sqrt{13}$ m.

(b) With overhang and sufficiency: * New Base = 3.5m = 7/2 m. * $L_{new} = \sqrt{(3.5)^2 + 2^2} = \sqrt{12.25 + 4} = \sqrt{16.25} = \sqrt{\frac{65}{4}} = \frac{\sqrt{65}}{2}$ m. * Approx value: $\sqrt{65} \approx 8.06$, so $L_{new} \approx 4.03$ m. * 6m plank is **sufficient** (4.03 < 6).

- **Answer:** $\frac{\sqrt{65}}{2}$ m; **Yes, sufficient**.

(c) Cost Savings: * Exact length per rafter ≈ 4.03 m. For 20 rafters: 80.6 m. * **Scenario 1 (Exact):** Cost = $80.6 \times 15{,}000 = 1{,}209{,}000$ UGX. * **Scenario 2 (Whole meters):** Each rafter rounded up to 5m (since 4.03 > 4). Total = $20 \times 5 = 100$ m. * Cost = $100 \times 15{,}000 = 1{,}500{,}000$ UGX. * Savings = $1{,}500{,}000 - 1{,}209{,}000 = 291{,}000$ UGX.

- **Answer:** Save **291,000 UGX**.

Question 6: Community Garden Fencing

- **Scenario:** A triangular garden in Gulu has sides $\sqrt{50}$ m, $\sqrt{18}$ m, and $\sqrt{32}$ m. Fencing wire costs 15,000 UGX per meter.
- **Hint:** Simplify surds: $\sqrt{50} = 5\sqrt{2}$, etc.
- **Task:** As a mathematics learner; (a) Calculate the exact perimeter in simplest surd form. (b) Calculate the exact cost of fencing in the form $k\sqrt{2}$ UGX. (c) If the garden area is given by Heron's formula and simplifies to $6\sqrt{11}$ m², and maize yields 0.5 kg/m², calculate the total expected yield in simplest surd form.

Worked Solution: (a) Exact Perimeter: * Sides: $5\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}$. * $P = (5+3+4)\sqrt{2} = 12\sqrt{2}$ m.

- **Answer:** $12\sqrt{2}$ m.

(b) Exact Cost: * Cost = $12\sqrt{2} \times 15{,}000 = 180{,}000\sqrt{2}$ UGX.

- **Answer:** $180{,}000\sqrt{2}$ UGX.

(c) Expected Yield: * Area = $6\sqrt{11}$ m². Yield = 0.5 kg/m². * Total = $0.5 \times 6\sqrt{11} = 3\sqrt{11}$ kg.

- **Answer:** $3\sqrt{11}$ kg.

Question 7: Physics of Water Sonar Testing

- **Scenario:** A student in Mbarara drops a stone from a 20m cliff. $g = 9.8$ m/s². She needs to calculate impact velocity and time to calibrate her sonar.
- **Hint:** $v = \sqrt{2gh}, t = \sqrt{2h/g}$.
- **Task:** As a mathematics learner; (a) Calculate the exact impact velocity in the form $a\sqrt{b}$ m/s. (b) Calculate the exact time of fall in the form $\frac{c\sqrt{d}}{e}$ seconds. (c) If air resistance reduces velocity by 15%, calculate the adjusted velocity to 1 decimal place and determine the percentage error in the sonar depth calculation if based on the vacuum theoretical velocity.

Worked Solution: (a) Exact Velocity: * $v = \sqrt{2 \times 9.8 \times 20} = \sqrt{392} = \sqrt{196 \times 2} = 14\sqrt{2}$ m/s.

- **Answer:** $14\sqrt{2}$ m/s.

(b) Exact Time: * $t = \sqrt{\frac{40}{9.8}} = \sqrt{\frac{400}{98}} = \frac{20}{\sqrt{98}} = \frac{20}{7\sqrt{2}} = \frac{20\sqrt{2}}{14} = \frac{10\sqrt{2}}{7}$ s.

- **Answer:** $\frac{10\sqrt{2}}{7}$ s.

(c) Adjusted Velocity and Error: * Theoretical $v \approx 14 \times 1.4142 = 19.80$ m/s. * Adjusted $v = 19.80 \times 0.85 = 16.83 \approx 16.8$ m/s. * Sonar depth $d = v \times t$. If v is overestimated by 15%, depth is overestimated by 15%. * **Answer:** Adjusted **16.8 m/s**; **15% error** in depth.

Question 8: Cubic Water Tank Bracing

- **Scenario:** An engineer in Soroti designs a 64 m³ cubic tank. Internal steel bracing is needed along the space diagonal.
- **Hint:** Side $x = \sqrt[3]{V}$. Diagonal $d = x\sqrt{3}$.
- **Task:** As a mathematics learner; (a) Find the exact side length. (b) Calculate the exact diagonal length in surd form. (c) If steel costs 45,000 UGX/m and welding adds a fixed 50,000 UGX per brace, calculate the exact total cost for 4 diagonal braces in the form $A\sqrt{3} + B$.

Worked Solution: (a) Side Length: * $x = \sqrt[3]{64} = 4$ m.

- **Answer:** 4 m.

(b) Diagonal Length: * $d = 4\sqrt{3}$ m.

- **Answer:** $4\sqrt{3}$ m.

(c) Total Cost for 4 Braces: * Steel cost = $4 \times (4\sqrt{3} \times 45{,}000) = 16\sqrt{3} \times 45{,}000 = 720{,}000\sqrt{3}$. * Welding cost = $4 \times 50{,}000 = 200{,}000$. * Total = $720{,}000\sqrt{3} + 200{,}000$.

- **Answer:** $720{,}000\sqrt{3} + 200{,}000$ UGX.

Question 9: Artistic Frame Design

- **Scenario:** An artist in Kampala uses the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$. Frame width is 10 cm.
- **Hint:** Length = 10ϕ.
- **Task:** As a mathematics learner; (a) Calculate the exact length in form $a + b\sqrt{5}$. (b) If a 2 cm border is added to all sides, calculate the new outer area in the form $m + n\sqrt{5}$ cm². (c) Prove mathematically that $\phi^2 = \phi + 1$ and explain how this helps in scaling the frame design without recalculating ratios.

Worked Solution: (a) Exact Length: * $L = 10 \times \frac{1+\sqrt{5}}{2} = 5(1 + \sqrt{5}) = 5 + 5\sqrt{5}$ cm.

- **Answer:** $5 + 5\sqrt{5}$ cm.

(b) New Outer Area: * New Width = $10 + 4 = 14$ cm. * New Length = $(5 + 5\sqrt{5}) + 4 = 9 + 5\sqrt{5}$ cm. * Area = $14 \times (9 + 5\sqrt{5}) = 126 + 70\sqrt{5}$ cm².

- **Answer:** $126 + 70\sqrt{5}$ cm².

(c) Proof and Explanation: * $\phi^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$. * $\phi + 1 = \frac{1+\sqrt{5}}{2} + \frac{2}{2} = \frac{3+\sqrt{5}}{2}$. * Thus $\phi^2 = \phi + 1$. * **Explanation:** This property allows the artist to scale dimensions by simply adding the previous dimension (Fibonacci-like sequence) without complex multiplication, maintaining proportionality.

Question 10: Bacteria Growth and Disinfectant

- **Scenario:** Bacteria in a lake double every 3 hours ($N = 100 \cdot 2^{t/3}$). A disinfectant reduces population by factor $\sqrt{2}$ per hour.
- **Hint:** Combine growth and decay indices.
- **Task:** As a mathematics learner; (a) Calculate the exact population after 24 hours without disinfectant. (b) If disinfectant is applied at $t = 24$ to a population of 10,000, write the expression for population $P(t)$ after t hours of treatment. (c) Determine the time required for the disinfectant to reduce the population to less than 100.

Worked Solution: (a) Population after 24 hours: * $N = 100 \cdot 2^{24/3} = 100 \cdot 2^8 = 100 \cdot 256 = 25{,}600$.

- **Answer:** 25,600 bacteria.

(b) Expression with Disinfectant: * Decay factor per hour $= \frac{1}{\sqrt{2}} = 2^{-0.5}$. * $P(t) = 10{,}000 \cdot (2^{-0.5})^t = 10{,}000 \cdot 2^{-0.5t}$.

- **Answer:** $P(t) = 10{,}000 \cdot 2^{-t/2}$.

(c) Time to reduce to < 100: * $100 = 10{,}000 \cdot 2^{-t/2} \Longrightarrow 0.01 = 2^{-t/2}$. * $\log(0.01) = -\frac{t}{2}\log(2)$. * $-2 = -\frac{t}{2}(0.301) \Longrightarrow t = \frac{4}{0.301} \approx 13.29$ hours.

- **Answer: 13.3 hours**.

Question.11: Cryptographic Key Generation via Exponential Equations

Scenario: A cybersecurity engineer is programming a quantum-resistant encryption protocol that relies on solving exponential equations for key synchronization. The system's authentication algorithm requires finding values of x that satisfy the equation:

$$2^{x+1} + 2^x - 2^{x-1} - 3^{x+1} + 3^x = 0$$

where x represents the key rotation interval in standardized time units.

Task: As a cryptographic systems analyst, solve the exponential equation to determine all valid values of x that enable secure key synchronization for the encryption protocol. *(Ans: $x = -1$, $x = 0$)*

Question 12: Pharmaceutical Dosage Calibration via Logarithmic Equations

Scenario: A clinical pharmacologist is calibrating a drug delivery system where the therapeutic index follows a logarithmic relationship. The dosage algorithm requires solving:

$$\log_2 x + \log_x 64 = 5$$

where x represents the concentration adjustment factor for optimal therapeutic effect.

Task: As a biomedical systems modeler, solve the logarithmic equation to determine the exact concentration adjustment factor x required to program the automated infusion pump. *(Ans: $x = 8$)*

Question .13: Signal Processing Filter Coefficient Determination

Scenario: A telecommunications DSP engineer is designing a digital bandpass filter where the transfer function coefficients satisfy a polynomial equation derived from logarithmic constraints. When the polynomial $f(x) = x^3 - ax + b$ is divided by $x + 1$, the remainder is 2, and $x + 2$ is an exact factor of the polynomial.

Task: As a signal integrity specialist, apply the Remainder and Factor Theorems to determine the exact integer values of coefficients a and b required to finalize the filter's coefficient matrix. *(Ans:* $a = 5$, $b = -2$*)*

Question .14: Financial Compound Interest Binomial Approximation

Scenario: A quantitative financial analyst is modeling compound interest adjustments for a sovereign wealth fund. The growth algorithm requires the binomial expansion of $\left(2 - \frac{x}{2}\right)^5$ to approximate fractional annual yield fluctuations. The model also requires precise evaluation of $(0.875)^5$ for baseline return calibration.

Task: As a portfolio risk modeler: i) Expand $\left(2 - \frac{x}{2}\right)^5$ in ascending powers of x up to the x^3 term. ii) Use your expansion to estimate $(0.875)^5$ correct to four decimal places for financial reporting compliance. *(Ans:* $32 - 40x + 20x^2 - 5x^3 + \cdots$*;* 0.5129*)*

Question 15: Aerospace Atmospheric Pressure Profiling via Logarithmic Identities

Scenario: An aerospace navigation engineer is programming an altimeter calibration matrix. Field telemetry confirms that atmospheric pressure P at altitude h satisfies logarithmic relationships. Given $\log_3 x = p$ and $\log_{18} x = q$, the system requires expressing $\log_6 3$ purely in terms of p and q for altitude conversion algorithms.

Task: As a flight systems analyst, use logarithmic change-of-base identities to derive the exact expression for $\log_6 3$ in terms of p and q to finalize the altimeter's altitude conversion matrix. *(Ans:* $\log_6 3 = \frac{q}{p-q}$*)*

Question 16: Precision Manufacturing Binomial Calibration

Scenario: A precision manufacturing facility is calibrating dimensional tolerances using binomial approximations. The calibration algorithm requires the expansion of $(2 - x)^6$ up to the x^2 term to model minor machining deviations. The quality control unit requires a precise numerical estimate of $(1.998)^6$ to finalize tolerance threshold programming.

Task: As a manufacturing systems analyst: i) Determine the first three terms of the binomial expansion of $(2 - x)^6$. ii) Use your expansion to evaluate $(1.998)^6$ correct to two decimal places for automated quality inspection calibration. *(Ans:* $64 - 192x + 240x^2$*;* 63.62*)*

Question 17: Chemical Equilibrium Concentration via Exponential Equations

Scenario: A chemical process engineer is balancing reactant concentrations in a catalytic chamber. The steady-state equilibrium constraints are governed by the exponential system:

$$2(3^{2x}) - 5(3^x) + 2 = 0$$

where x represents the normalized reaction time. The operations team requires all valid concentration pairs to maintain steady-state reaction conditions.

Task: As a reaction kinetics analyst, solve the exponential equation to determine all valid values of x that maintain steady-state reaction conditions for automated dosing calibration. *(Ans: $x = -0.6309$ or $x = 0.6309$)*

Question 18: Network Security Threshold Modeling via Logarithmic Inequalities

Scenario: A cybersecurity operations center is programming intrusion detection thresholds. The system's alert algorithm requires solving the inequality:

$$(0.8)^{-3x} > 4.0$$

where x represents the time threshold in standardized units for triggering security alerts.

Task: As a network security modeler, solve the exponential inequality to determine the minimum time threshold x required for safe system monitoring and alert calibration. *(Ans: $x > 2.07$)*

Question 19: Structural Engineering Load Distribution via Logarithmic Equations

Scenario: A civil engineering firm is validating load distribution models for a suspension bridge. The structural analysis requires solving:

$$\log_4(6 - x) = \log_2 x$$

where x represents the normalized load factor for cable tension calibration.

Task: As a structural load analyst, solve the logarithmic equation and verify the solution against physical constraints to finalize cable tension specifications. *(Ans: $x = 2$)*

Question 20: Aerospace Trajectory Root Transformation

Scenario: An orbital mechanics specialist is analyzing a spacecraft's characteristic polynomial $x^3 + x - 10 = 0$. Telemetry confirms that $x = 2$ is a valid system root. The engineering team requires the remaining complex conjugate poles' sum and product, followed by a transformed quadratic equation for their squared values, to complete the vibration damping matrix.

Task: As a flight dynamics specialist, deduce the values of $\alpha + \beta$ and $\alpha\beta$ for the remaining roots, then form the exact quadratic equation whose roots are α^2 and β^2 for stability certification. *(Ans: $\alpha + \beta = -2$, $\alpha\beta = 5$; Quadratic: $x^2 + 6x + 25 = 0$)*

Question 21: Precision Instrument Calibration via Logarithmic Equations

Scenario: A metrology laboratory is calibrating a high-precision optical sensor. The sensor's output validation requires solving:

$$\log_x 5 + 4\log_5 x = 4$$

where x represents the calibration factor for signal amplification thresholds.

Task: As a metrology calibration specialist, solve the logarithmic equation to determine the exact calibration factor x required to finalize the sensor's signal amplification matrix. *(Ans: $x = 5$)*

Question 22: Financial Portfolio Risk Modeling via Exponential Equations

Scenario: A quantitative risk analyst is modeling trading profit margins where key variables satisfy exponential constraints. The portfolio risk metric involves solving:

$$9^x - 3^{x+1} = 10$$

where x represents the normalized risk adjustment factor. Regulatory compliance requires exact solutions to validate algorithmic trading thresholds.

Task: As a financial risk modeler, solve the exponential equation to determine the exact risk adjustment factor x required for portfolio calibration. *(Ans: $x = 1.465$)*

Question 23: Cryptographic Key Validation via Logarithmic Systems

Scenario: A cryptographic protocol validator is verifying a key-generation algorithm defined by:

$$\log_4 x^2 - 6\log_x 4 - 1 = 0$$

where x represents the key strength parameter. Security audits require exact solutions to prevent seed generation failures.

Task: As a cryptographic systems analyst, solve the logarithmic equation to determine all valid values of x required to finalize the algorithm's key strength matrix. *(Ans: $x = 16$, $x = \frac{1}{8}$)*

Question 24: Network Bandwidth Allocation via Exponential Equations

Scenario: A cloud infrastructure architect is optimizing dual-protocol bandwidth allocation. The scaling factors x for server clusters must satisfy:

$$4^x - 2^{x+1} - 15 = 0$$

where x represents the allocation multiplier for load balancing configuration.

Task: As a network performance engineer, solve the exponential equation to determine the exact allocation multiplier x for load balancing configuration. *(Ans:* $x = 2.3219$*)*

Question 25: Precision Manufacturing Tolerance Modeling via Logarithmic Equations

Scenario: A precision manufacturing facility is calibrating dimensional tolerances using logarithmic relationships. The calibration algorithm requires solving:

$$\log_2 x - \log_x 8 = 2$$

where x represents the tolerance adjustment factor for automated quality inspection systems.

Task: As a manufacturing quality engineer, solve the logarithmic equation to determine all valid tolerance adjustment factors x required to finalize automated inspection calibration parameters. *(Ans:* $x = 8$, $x = \frac{1}{2}$*)*

EQUATIONS AND INEQUALITIES.

Question 1: The Struggling Maize Millers Cooperative

Scenario: The Bukonzo Joint Maize Millers Cooperative in Kasese has been struggling. Twenty families depend on this mill for their livelihood. Last season, they bought a new milling machine on loan, but now they're facing a crisis. Their accountant presented a grim picture: "Our weekly profit in thousands of shillings follows $P(x) = -5x^2 + 100x - 375$, where x is bags of flour processed." The cooperative chairman is worried because three weeks in a row they've made losses. Some members want to close the mill, but the women's group argues they just need to process the right amount. The cooperative has storage for only 20 bags and must process at least 2 bags weekly to maintain their supplier relationships.

Hint: Losses occur when $P(x) < 0$. They need $P(x) \geq 0$ to survive.

Task: As a mathematics learner helping this cooperative: (a) Determine the range of bags they MUST process weekly to avoid losses and save the cooperative from closure. (b) Given their storage constraint of 20 bags maximum, what is the actual feasible production range? (c) Advise the cooperative: should they continue operating or close down? Justify with your calculations.

Worked Solution:

1. **Part (a):** Set $P(x) \geq 0$ $-5x^2 + 100x - 375 \geq 0$ Divide by -5 (reverse inequality): $x^2 - 20x + 75 \leq 0$ Factor: $(x - 5)(x - 15) \leq 0$ Critical values: $x = 5$ and $x = 15$ Testing regions: Between 5 and 15 gives negative (satisfies ≤ 0) **Range to avoid losses: $5 \leq x \leq 15$ bags**

2. **Part (b):** Storage constraint: $x \le 20$ Minimum requirement: $x \ge 2$ Combining with profit requirement: $5 \le x \le 15$ **Actual feasible range: 5 to 15 bags** (storage doesn't further restrict)

3. **Part (c):** Since there IS a feasible range (5-15 bags), they should continue. Maximum profit at vertex: $x = -b/2a = -100/(2\times-5) = 10$ bags Max profit: $P(10) = -5(100) + 1000 - 375 = 125$ (125,000 UGX weekly) **Advice:** Continue operating, target 10 bags weekly for maximum profit of 125,000 UGX.

Question 2: The Village Water Tank Crisis

Scenario: In Napak District, a severe drought has hit. An NGO is installing rectangular water tanks for communities. Engineer Okello designed a tank with volume $V(x) = x^3 - 7x + 6$ cubic meters, where x is a design parameter. However, the community leaders are confused. "When we tested x = 1 meter, the tank had zero volume - that's impossible!" they complained. The NGO must manufacture 50 tanks, but if there are multiple values of x that give zero or negative volume, those tanks would be useless. The community needs tanks that hold at least 6 cubic meters to serve 200 families. The parameter x must be positive and practical (between 0 and 4 meters).

Hint: Find all roots, then determine where $V(x) \geq 6$.

Task: As a mathematics learner advising the NGO: (a) Find all values of x that would result in zero volume (design failures). (b) Determine the range of x values that give positive volume. (c) Within the practical range ($0 < x < 4$), which x values ensure the tank holds at least 6 m³?

Worked Solution:

4. **Part (a):** Solve $V(x) = 0$ $x^3 - 7x + 6 = 0$ Test $x = 1$: $1 - 7 + 6 = 0$ ✓ $(x-1)$ is a factor Divide: $(x^3 - 7x + 6) \div (x-1) = x^2 + x - 6$ Factor: $(x-1)(x+3)(x-2) = 0$ **Zero volume at: $x = 1, x = 2, x = -3$**

5. **Part (b):** Test regions for positive volume: $x < -3$: Try $x = -4$: $-64 + 28 + 6 = -30$ (negative) $-3 < x < 1$: Try $x = 0$: 6 (positive) ✓ $1 < x < 2$: Try $x = 1.5$: $3.375 - 10.5 + 6 = -1.125$ (negative) $x > 2$: Try $x = 3$: $27 - 21 + 6 = 12$ (positive) ✓ **Positive volume when: $-3 < x < 1$ OR $x > 2$**

6. **Part (c):** Practical range: $0 < x < 4$ Need $V(x) \ge 6$ $x^3 - 7x + 6 \ge 6$ $x^3 - 7x \ge 0$ $x(x^2 - 7) \ge 0$ Critical: $x = 0$, $x = \pm\sqrt{7} \approx \pm 2.646$ In range (0,4): Test regions $0 < x < 2.646$: Try $x = 1$: $1(1-7) = -6$ (negative) $x > 2.646$: Try $x = 3$: $3(9-7) = 6$ (positive) ✓ **Answer: $x \ge \sqrt{7} \approx 2.65$ meters (but $x < 4$)**

Question 3: The School Garden Food Security Project

Scenario: St. Mary's Secondary School in Soroti has a food security problem. Many students come from homes affected by drought. The school has 120 meters of chain-link fence donated by a parent, and one side of their proposed garden borders a permanent stone wall from an old building. The headteacher wants to maximize vegetable production to feed 400 students. However, the agriculture teacher warns: "We need at least 800 m^2 to grow enough sukuma wiki, beans, and maize for the school kitchen." The school bursar is concerned about cost and wants to use minimal fencing. The student council proposes different shapes, but the mathematics club argues only a rectangular design against the wall will work.

Hint: Three sides need fencing: $2w + l = 120$. Area $= l \times w$.

Task: As a mathematics learner helping the school: (a) Express the garden area as a function of width only. (b) Find the dimensions that give maximum possible area. (c) Can this design meet the 800 m^2 minimum requirement? If yes, what range of widths achieves at least 800 m^2?

Worked Solution:

7. **Part (a):** Let width = w (perpendicular to wall) Length = l (parallel to wall) Constraint: $2w + l = 120 \rightarrow l = 120 - 2w$ Area $A = l \times w = (120 - 2w)w = 120w - 2w^2$ **$A(w) = -2w^2 + 120w$**

8. **Part (b):** Complete the square: $A(w) = -2(w^2 - 60w) = -2[(w - 30)^2 - 900] = -2(w - 30)^2 + 1800$ Maximum at w = 30 meters Length = 120 - 2(30) = 60 meters **Max Area = 1800 m^2 with dimensions 30m × 60m**

9. **Part (c):** Need $A(w) \geq 800$ $-2w^2 + 120w \geq 800$ $-2w^2 + 120w - 800 \geq 0$ Divide by -2: $w^2 - 60w + 400 \leq 0$ Factor: $(w - 20)(w - 40) \leq 0$ Range: $20 \leq w \leq 40$ Check feasibility: If w = 20, l = 80. If w = 40, l = 40. **Yes, achievable! Width between 20m and 40m gives at least 800 m^2**

Question 4: The Nakasero Market Price War

Scenario: At Nakasero Market in Kampala, three women traders - Mama Sarah (rice), Mama Joseph (beans), and Mama Grace (sugar) - are in a difficult situation. A new supermarket opened nearby, selling cheaper imports. The three women pool their resources to buy in bulk, but they're confused about pricing. Last week:

- Sarah bought 2kg rice + 1kg beans + 1kg sugar for 10,000 UGX
- Joseph bought 1kg rice + 2kg beans + 1kg sugar for 9,000 UGX
- Grace bought 1kg rice + 1kg beans + 2kg sugar for 11,000 UGX

They suspect the wholesaler is cheating them with inconsistent prices. They need to know the actual price per kg of each item to negotiate fairly and compete with the supermarket. If rice costs more than 3,000 UGX/kg, they can't compete.

Hint: Set up three equations with three unknowns.

Task: As a mathematics learner helping these traders: (a) Formulate the system of equations representing their purchases. (b) Solve to find the actual price per kg of rice, beans, and sugar. (c) Advise them: Can they compete with the supermarket given the rice price constraint?

Worked Solution:

10. **Part (a):** Let r = rice price, b = beans price, s = sugar price (in thousands) 2r + b + s = 10 ...(1) r + 2b + s = 9 ...(2) r + b + 2s = 11 ...(3)
11. **Part (b):** Subtract (2) from (1): (2r - r) + (b - 2b) + (s - s) = 10 - 9 r - b = 1 → r = b + 1 ...(4)

    Subtract (2) from (3): (r - r) + (b - 2b) + (2s - s) = 11 - 9 -b + s = 2 → s = b + 2 ...(5)

    Substitute (4) and (5) into (2): (b+1) + 2b + (b+2) = 9 4b + 3 = 9 4b = 6 → b = 1.5

    From (4): r = 1.5 + 1 = 2.5 From (5): s = 1.5 + 2 = 3.5

    Prices: Rice = 2,500 UGX/kg, Beans = 1,500 UGX/kg, Sugar = 3,500 UGX/kg
12. **Part (c):** Rice costs 2,500 UGX/kg, which is LESS than 3,000 UGX. **Yes, they CAN compete!** Their rice price is competitive.

Question 5: The Boda Boda Stage Profit Crisis

Scenario: The Kisenyi Boda Boda stage has 50 riders, but they're facing a crisis. Fuel prices have doubled, and many riders are leaving. The stage chairman, Mr. Mukasa, analyzed their daily profit: $P(n) = -50n^2 + 1000n - 2000$, where n is the number of active riders and P is profit in thousands of UGX. Currently, only 8 riders remain active. The stage needs at least 15,000 UGX daily profit to pay the stage rent, security, and maintain the parking area. If they can't make this profit, the landlord will evict them and convert the stage into a parking lot for cars. Some riders suggest recruiting more members, but others worry too many riders will reduce individual earnings.

Hint: Find the vertex for maximum profit and solve $P(n) \geq 15$.

Task: As a mathematics learner advising the stage: (a) What number of riders gives maximum profit, and what is that maximum? (b) What is the minimum number of riders needed to achieve at least 15,000 UGX profit? (c) Given they currently have 8 riders, what should Mr. Mukasa do to save the stage?

Worked Solution:

13. **Part (a):** $P(n) = -50n^2 + 1000n - 2000$ Vertex at $n = -b/2a = -1000/(2\times-50) = -1000/-100 = 10$ riders Max profit: $P(10) = -50(100) + 10000 - 2000 = -5000 + 10000 - 2000 = 3000$ **Maximum: 10 riders give 3,000,000 UGX daily profit**
14. **Part (b):** Need $P(n) \geq 15$ (in thousands) $-50n^2 + 1000n - 2000 \geq 15$ $-50n^2 + 1000n - 2015 \geq 0$ Divide by -5: $10n^2 - 200n + 403 \leq 0$

    Use quadratic formula: $n = [200 \pm \sqrt{(40000 - 16120)}]/20$ $n = [200 \pm \sqrt{23880}]/20 = [200 \pm 154.5]/20$ $n_1 = 354.5/20 \approx 17.7$ $n_2 = 45.5/20 \approx 2.3$

    Need between 3 and 17 riders (approximately)
15. **Part (c):** Currently 8 riders. $P(8) = -50(64) + 8000 - 2000 = -3200 + 8000 - 2000 = 2800$ Current profit: 2,800,000 UGX (above 15,000 requirement)

However, maximum is at 10 riders with 3,000,000 UGX. **Advice:** Recruit 2 more riders to reach 10 total. This maximizes profit at 3 million UGX daily, well above the survival threshold.

Question 6: The Community Health Center Drug Distribution

Scenario: The Kitgum Community Health Center serves 5,000 people across 12 villages. They receive monthly supplies of three essential medicines: Malaria tablets (M), Antibiotics (A), and Pain relievers (P). The storage constraints are:

- Total storage capacity: 1000 boxes
- Malaria tablets take 2 units of space, Antibiotics take 3 units, Pain relievers take 1 unit
- Total space available: 2400 units
- The health workers need twice as many Malaria tablets as Antibiotics due to high malaria rates

Last month, they ran out of malaria medication, causing a crisis. The center must optimize their ordering to serve the community effectively.

Hint: $M + A + P = 1000$, $2M + 3A + P = 2400$, $M = 2A$

Task: As a mathematics learner helping the health center: (a) Set up the system of equations. (b) Solve for the optimal number of boxes of each medicine. (c) Verify this fits all constraints.

Answer: (a) $M+A+P=1000$, $2M+3A+P=2400$, $M=2A$; (b) $M=400$, $A=200$, $P=400$; (c) Total=1000✓, Space=800+600+400=1800≤2400✓

Question 7: The Village Savings Group Investment

Scenario: The "Tulime Tusobole" (We Can Farm) savings group in Bushenyi has 30 women members. They've saved 2 million UGX and want to invest in a group project. They're considering buying a posho mill. Their financial advisor projects the annual profit as $P(x) = -2x^2 + 40x - 150$, where x is the number of bags of maize processed monthly (in hundreds) and P is profit in hundred-thousands of UGX. The women need at least 500,000 UGX monthly profit to share dividends and reinvest. However, they can only store 1500 bags maximum, and must process at least 200 bags monthly to maintain supplier relationships.

Hint: Solve $P(x) \geq 5$ and consider constraints.

Task: As a mathematics learner advising the women's group: (a) Find the range of bags that gives at least 500,000 UGX profit. (b) What is the optimal number of bags to process for maximum profit? (c) Can they meet their profit goal within their storage constraints?

Answer: (a) 500 ≤ bags ≤ 1500; (b) 1000 bags for max profit of 850,000 UGX; (c) Yes, feasible range is 500-1500 bags

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Question 8: The Secondary School Construction Project

Scenario: Oyam Secondary School is expanding. They need to build classrooms, a library, and a laboratory. The contractor presented cost estimates:

- 2 classrooms + 1 library + 1 lab cost 180 million UGX
- 1 classroom + 2 libraries + 1 lab cost 200 million UGX
- 1 classroom + 1 library + 2 labs cost 220 million UGX

The school has 300 million UGX from the government and community contributions. They need at least 4 classrooms, 1 library, and 1 laboratory. The headteacher is confused about whether they can afford everything and what the individual costs are.

Hint: Set up three equations to find individual costs.

Task: As a mathematics learner helping the school: (a) Determine the cost of one classroom, one library, and one laboratory. (b) Calculate the cost of their minimum requirement (4 classrooms, 1 library, 1 lab). (c) Can they afford this with 300 million UGX? If not, what should they do?

Answer: (a) Classroom=40M, Library=60M, Lab=80M; (b) 4(40)+60+80=300M; (c) Exactly affordable with 300M, no room for extras

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Question 9: The Youth Farming Cooperative Land Division

Scenario: Twenty unemployed youth in Tororo formed a farming cooperative. They leased 12 hectares of land to grow maize and beans. The land has varying quality:

- High-quality land yields 3 tons/ha of maize or 2 tons/ha of beans
- Low-quality land yields 1 ton/ha of maize or 1.5 tons/ha of beans
- 8 hectares are high-quality, 4 hectares are low-quality

They have a contract to supply at least 15 tons of maize and 10 tons of beans to a buyer. However, labor constraints mean they can only cultivate a maximum of 10 hectares this season. They need to decide how to allocate their limited labor and land.

Hint: Let x = high-quality hectares used, y = low-quality hectares used.

Task: As a mathematics learner advising the cooperative: (a) Formulate inequalities for the maize and bean requirements. (b) Determine if they can meet both requirements with only 10 hectares. (c) Recommend the optimal land allocation.

Answer: (a) $3x+y\geq15$, $2x+1.5y\geq10$, $x+y\leq10$, $x\leq8$, $y\leq4$; (b) Yes, feasible region exists; (c) Use 6ha high-quality for maize, 2ha high-quality for beans, 2ha low-quality for beans

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Question 10: The Village Bridge Safety Assessment

Scenario: The district engineer inspected the old wooden bridge in Kabale that connects 3 villages to the main road. She determined the bridge's load capacity follows $L(w) = -w^2 + 30w - 176$, where w is vehicle weight in tons and L is the safety margin. If $L < 0$, the bridge will collapse. Currently, boda bodas (0.3 tons), motorcycles (0.5 tons), cars (1.5 tons), and light trucks (up to 8 tons) use the bridge. The community needs to know which vehicles can safely cross and what the maximum safe weight is.

Hint: Find where $L(w) \geq 0$.

Task: As a mathematics learner assessing bridge safety: (a) Find the range of vehicle weights that can safely cross. (b) What is the maximum safe weight? (c) Can a 10-ton truck carrying harvest cross safely? What about a 5-ton truck?

Answer: (a) $8 \leq w \leq 22$ tons; (b) Maximum safe load is 22 tons; (c) 10-ton truck: YES (in range), 5-ton truck: NO (below minimum - bridge needs some weight for stability)

Questions 11-30: Additional Community Scenarios

Question 11: The Parish Church Roof Project

Scenario: St. Peter's Parish in Arua needs to replace their church roof. The contractor quotes depend on the roof angle θ. The cost function is $C(\theta) = 5\theta^2 - 180\theta + 2000$ (in thousands UGX), where θ is in degrees. The roof must be between 15° and 45° for architectural and safety reasons. **Task:** (a) Find the angle that minimizes cost, (b) Calculate minimum cost, (c) Is a 30° roof more expensive than optimal? **Answer:** (a) θ = 18°; (b) 380,000 UGX; (c) Yes, 30° costs 650,000 UGX

Question 12: The Cooperative Milk Collection

Scenario: A dairy cooperative in Mbarara collects milk from 50 farmers. The daily collection volume $V(d) = -2d^3 + 30d^2 - 100d + 150$, where d is the day of the month. They need at least 200 liters daily to supply the processor. **Task:** (a) Find when volume drops below 200L, (b) Which days are problematic, (c) How many days meet the requirement? **Answer:** (a) Solve $V(d) < 200$; (b) Days 1-3 and 25-30; (c) Approximately 20 days

Question 13: The Market Stall Rental

Scenario: Kampala City Council charges stall rentals based on area. A quadratic pricing model: $R(A) = 0.5A^2 - 40A + 1000$ UGX daily, where A is area in m². Vendors want to minimize rent but need at least 20m² to operate. **Task:** (a) Find optimal stall size, (b) Minimum rent, (c) Is 50m² affordable? **Answer:** (a) 40m²; (b) 200 UGX/day; (c) 50m² costs 350 UGX - affordable

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Question 14: Pharmaceutical Dosage Parameter Calibration

Scenario: A clinical pharmacology team is calibrating a multi-component infusion protocol. The dosage coefficients x, y, and z for three active compounds must satisfy a system of linear constraints derived from metabolic clearance rates and tissue distribution limits:

$$\begin{cases} x + 2y - 3z = 0 \\ 3x + 3y - z = 5 \\ x - 2y + 2z = 1 \end{cases}$$

Task: As a biomedical systems analyst, solve the simultaneous equations to determine the exact dosage coefficients x, y, and z required to program the automated infusion pump. *(Ans:* $x = 1$*,* $y = 1$*,* $z = 1$*)*

Question 15: Structural Stress Function Coefficient Determination

Scenario: A civil engineering firm is modeling the stress distribution function $f(p) = ap^2 + bp + c$ along a cantilever beam, where p represents the distance from the support joint in meters. Material testing reveals:

- Dividing by $p - 1$ leaves a residual stress of 1 unit.
- Dividing by $p - 2$ leaves a residual stress of 1 unit.
- Dividing by $p + 1$ leaves a residual stress of 25 units.

Task: As a structural mechanics engineer, determine the exact values of coefficients a, b, and c, then fully factorize the expression to identify critical reinforcement points along the beam. *(Ans:* $a = 4$*,* $b = -12$*,* $c = 9$*; factors:* $(2p - 3)$ *and* $(2p - 3)$*)*

Question 16: Signal Processing Filter Decomposition

Scenario: A telecommunications DSP engineer is isolating harmonic distortion in a transmission line. The raw signal polynomial $2x^3 + 5x^2 - 4x - 3$ must be expressed relative to a standard noise-canceling filter response $x^2 + x - 2$ in the form:

$$(x^2 + x - 2) \cdot Q(x) + Ax + B$$

Task: As a signal integrity specialist, determine the quotient polynomial $Q(x)$ and the constants A and B to finalize the digital filter's attenuation matrix. *(Ans:* $A = -3$*,* $B = 3$*,* $Q(x) = 2x + 3$*)*

Question 17: Control System Eigenvalue Transformation

Scenario: An automation engineer is redesigning a feedback control loop. The original system eigenvalues α and β satisfy $\alpha + \beta = 2$ and $\alpha^3 + \beta^3 = 26$. The new control architecture requires solving for the exact original eigenvalues to map stability margins.

Task: As a control systems modeler, use the relationship between roots and coefficients to form and solve the underlying quadratic equation, determining the exact values of α and β. *(Ans:* $\alpha = 3, \beta = -1$ *or* $\alpha = -1, \beta = 3$*)*

Question 18: Manufacturing Batch Consistency Verification

Scenario: A quality assurance auditor is validating consistency between two production batches modeled by quadratic characteristic equations $x^2 + bx + c = 0$ and $x^2 + px + q = 0$. Regulatory standards require a mathematical proof that if both batches share exactly one identical failure threshold (common root), their parameters must satisfy a specific invariant relationship.

Task: As a reliability validation specialist, prove that when the equations share a common root, the parameters must satisfy $(c-q)^2 = (b-p)(cp-bq)$. *(Ans: Proof established via elimination of the common root)*

Question 19: Chemical Equilibrium Concentration Balancing

Scenario: A chemical process engineer is balancing reactant concentrations x and y in a catalytic chamber. The equilibrium constraints are governed by the non-linear system:

$$\frac{6}{x} - \frac{1}{y} = 1 \quad \text{and} \quad x(5-x) = 2y$$

Task: As a reaction kinetics analyst, solve the simultaneous non-linear equations to determine all valid concentration pairs (x, y) that maintain steady-state reaction conditions. *(Ans: $(1, -1)$ and $(\frac{5}{9}, 3)$)*

Question 20: Network Bandwidth Allocation Scaling

Scenario: A cloud infrastructure architect is optimizing dual-protocol bandwidth allocation. The scaling factors x and y for two server clusters must satisfy:

$$2^x + 4^y = 12 \quad \text{and} \quad 3^x - 2^{2y} = 16$$

Task: As a network performance engineer, solve the simultaneous exponential equations to determine the exact allocation multipliers x and y for load balancing configuration. *(Ans: $x = 2$, $y = 1$)*

Question 21: Cryptographic Key Polynomial Validation

Scenario: A cybersecurity developer is validating a key-generation algorithm defined by $f(x) = x^3 - ax + b$. Security audits reveal that when divided by $x + 1$, the remainder is 2, and $x + 2$ is an exact factor of the polynomial.

Task: As a cryptographic systems analyst, apply the Remainder and Factor Theorems to determine the exact integer values of a and b required to finalize the algorithm's polynomial seed. *(Ans: $a = 5$, $b = -2$)*

Question 22: Sensor Output Operational Range Mapping

Scenario: A precision instrumentation engineer is calibrating a non-linear pressure sensor. The output voltage y relative to input pressure x follows the rational function $y = \frac{2x^2-3}{x+4}$. The engineering team requires the exact mathematical range of y for all real input values x to program saturation alert thresholds.

Task: As a metrology systems specialist, determine the range of possible values of y for real x by analyzing the discriminant of the rearranged quadratic form. *(Ans:* $y \leq -16 - 2\sqrt{58}$ *or* $y \geq -16 + 2\sqrt{58}$*)*

Question 23: Aerospace Pole Placement Transformation

Scenario: A flight dynamics engineer is transforming system stability poles. The original quadratic characteristic equation $x^2 - 2x + 10 = 0$ has roots α and β. The control software requires a new quadratic equation whose roots are transformed to $\frac{1}{\alpha+2}$ and $\frac{1}{\beta+2}$.

Task: As an orbital control specialist, derive the exact coefficients of the transformed quadratic equation without explicitly solving for α and β. *(Ans:* $324x^2 + 1 = 0$*)*

Question 24: Manufacturing Production Window Optimization

Scenario: An industrial operations manager is defining feasible production windows under resource constraints. The operational limits are defined by the inequalities $y > x - 5$ and $0 < y < 6$. Within this feasible region, the production balance must satisfy the system $xy + 2x = 5$ and $9x = y + 6$.

Task: As a production planning analyst: i) Shade the unwanted regions on a coordinate plane to identify the feasible operational window. ii) Solve the simultaneous system algebraically to find exact production balance points that lie within the constraints. *(Ans:* $(1,3)$ *lies within constraints;* $(-\frac{5}{9}, -1)$ *falls outside feasible window)*

Question 25: Structural Resonance Damping Calibration

Scenario: A civil engineering firm is designing a vibration-damping foundation modeled by $P(x) = x^3 + 4ax^2 + bx + 3$. Structural codes require the polynomial to be exactly divisible by $(x-1)^2$ to eliminate harmonic resonance at the critical frequency $x = 1$.

Task: As a structural dynamics engineer: i) Use the condition of divisibility by $(x-1)^2$ to determine the exact values of a and b. ii) Fully solve $P(x) = 0$ to identify all resonance frequencies that require additional damping treatment. *(Ans:* $a = -\frac{1}{4}$*,* $b = -5$*; roots:* $x = 1$ *(repeated),* $x = -7$*)*

Question 26: Radioactive Containment Clearance Timing

Scenario: A nuclear safety officer is calculating clearance times for a containment chamber. The residual contamination index decays according to the inequality $(0.8)^{-3x} > 4.0$, where x represents time in standardized units. The facility must remain sealed until the inequality condition is safely exceeded.

Task: As a radiation safety modeler, solve the exponential inequality to determine the minimum time threshold x required for safe chamber access. *(Ans: $x > 2.07$)*

Question 27: Algorithmic State Dependency Validation

Scenario: A computational mathematician is verifying state transition dependencies in a distributed ledger system. Given that α and β are roots of $x^2 + px + q = 0$, the system requires an expression for $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q. Additionally, the algorithm must detect when one system state is the exact square of another.

Task: As a cryptographic protocol validator: i) Express $(\alpha - \beta^2)(\beta - \alpha^2)$ purely in terms of p and q. ii) Deduce that for one root to be the square of the other, the parameters must satisfy $p^3 - 3pq + q^2 + q = 0$. *(Ans: i) $-p^3q + 2pq^2 + q^3 + q^2$; ii) Condition proven)*

Question 28: Multi-Axis Robotic Kinematics Resolution

Scenario: A robotics technician is calibrating a 3-axis positioning arm. The joint alignment parameters x, y, and z must satisfy the spatial constraint system:

$$\begin{cases} x + 2y - 2z = 0 \\ 2x + y - 4z = -1 \\ 4x - 3y + z = 11 \end{cases}$$

Task: As a motion control systems analyst, solve the simultaneous linear equations to determine the exact calibration coordinates x, y, and z for end-effector precision alignment. *(Ans: $x = 3$, $y = 1$, $z = 2$)*

Question 29: Cryptographic Key Validation & Root Dependency Analysis

Scenario: A cybersecurity architect is validating a polynomial-based key generation algorithm. The algorithm's stability depends on the roots α and β of the characteristic quadratic equation $x^2 + px + q = 0$. The security protocol requires a mathematical verification of how asymmetric root transformations affect the system's error-detection polynomial.

Task: As a cryptographic validation specialist: i) Express the product $(\alpha - \beta^2)(\beta - \alpha^2)$ purely in terms of the coefficients p and q. ii) Deduce the exact algebraic condition relating p and q that must hold true if one cryptographic state is exactly the square of the other, ensuring key synchronization integrity.

(Ans: i) $-p^3q + 2pq^2 + q^3 + q^2$; ii) $p^3 - 3pq + q^2 + q = 0$)

Question 30: Engineering System Eigenvalue Calibration

Scenario: A control systems engineer is analyzing a dual-mass vibration damper. The system's natural frequencies α and β satisfy the relationships $\alpha + \beta = 2$ and $\alpha^3 + \beta^3 = 26$.