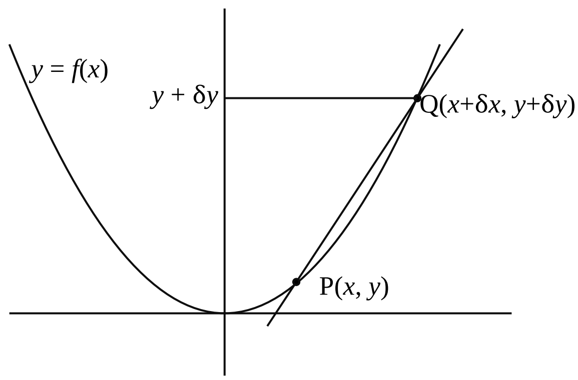
DIFFERENTIATION I

Suppose we have a smooth function *f(x)* which is represented graphically by the curve $y = f(x)$. Then we can draw a tangent to the curve at point P. It is important to be able to calculate the slope of the tangent of the curve. A graphical method can be used but this is rather imprecise, so we use the following analytical method.

We chose a second point Q on the curve which is near P and join the two points with a tangent line *PQ* called secant and calculate the slope of the line.

Then we can allow Q to approach P so that the secant swings around until it just touches the curve and become a tangent. The limit of the slope of the secant is required to find the slope of the tangent.



The gradient of the secant PQ =

$$\frac{\Delta y}{\Delta x} = \frac{\partial y}{\partial x} = \frac{f(x+\partial x) - f(x)}{x + \partial x - x}$$

$$= \frac{f(x+\partial x) - f(x)}{\partial x}$$

$\Rightarrow$ The gradient of the tangent at P($f'(x)$)

$$f'(x) = \lim \frac{f(x+\partial x) - f(x)}{\partial x}$$

$$\text{as } \partial x \longrightarrow 0$$

Example

Find the gradient of the tangent to the curve $y = x^2$.

Solution

The gradient of the tangent to the curve $y = f(x)$

$$\frac{dy}{dx}(f'x) = \lim \frac{f(x+\partial x) - f(x)}{\partial x}$$

$$\partial x \longrightarrow 0$$

$$f(x) = x^2$$

$$f(x + \partial x) = (x + \partial x)^2$$

$$= x^2 + 2x\partial x + \partial x^2$$

$$\Rightarrow f'(x) = \left[\lim \frac{x^2 + 2x\partial x + (\partial x)^2 - x^2}{\partial x} \right]$$

$$\partial x \longrightarrow 0$$

$$f'(x) = \lim(2x + \partial x)$$

$$\partial x \longrightarrow 0$$

$$= 2x$$

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\boxed{\textbf{If } y = x^n\textbf{, then } \frac{dy}{dx} = nx^{n-1} = f'(x)}$$

For example: If $y = x^4$

$$\frac{dy}{dx} = 4x^{4-1}$$

$$= 4x^3$$

Example I

Differentiate the following functions:

(a) $x^3 + 2x^2 + 3x$

(b) $4x^4 - 3x^2 + 5$

(c) $ax^2 + bx + c$

Solution

(a) $y = x^3 + 2x^2 + 3x$

$$\frac{dy}{dx} = 3x^{3-1} + 2 \times 2x^{2-1} + 3 \times 1(x^{1-1})$$

$$= 3x^2 + 4x + 3x^0$$

$$= 3x^2 + 4x + 3$$

(b) $y = 4x^4 - 3x^2 + 5$

$$\frac{dy}{dx} = 4x^{4-1} - 2 \times 3x^{2-1} + 0$$

$$= 4x^3 - 6x$$

(c) $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

Example III

Find the gradient of the curve $y = x(2 - x)$ at $x = 2$

Solution

$y = x(2 - x)$

$y = 2x - x^2$

$$\frac{dy}{dx} = 2 - 2x$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 2 - 2 \times 2$$
$$= -2$$

Example IV

Find the gradient of the curves at the given points:

(a) $y = (4x - 5)^2$ $\quad (\frac{1}{2}, 9)$

(b) $y = 3x^3 - 2x^2$ $\quad (-2, -24)$

(c) $y = (x + 2)(x - 4)$ $\quad (3, -5)$

Solution

(a) $\boldsymbol{y = (4x - 5)^2}$

$y = 16x^2 - 40x + 25$

$$\frac{dy}{dx} = 32x - 40$$
$$\left.\frac{dy}{dx}\right|_{(\frac{1}{2}, 9)} = 32 \times \frac{1}{2} - 40$$
$$= 16 - 40$$
$$= -24$$

(b) $\boldsymbol{y = 3x^3 - 2x^2}$

$$\frac{dy}{dx} = 9x^2 - 4x$$
$$\left.\frac{dy}{dx}\right|_{(-2, -24)} = 9 \times -2^2 - 4(-2)$$
$$= 36 + 8$$
$$= 44$$

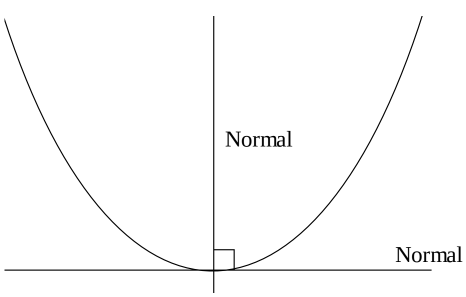
(c) $\boldsymbol{(x + 2)(x - 4)}$

$y = x^2 - 2x - 8$

$$\frac{dy}{dx} = 2x - 2$$
$$\left.\frac{dy}{dx}\right|_{(3, -5)} = 2 \times 3 - 2$$
$$= 4$$

Tangents and Normals to curves

A tangent is a line which touches a curve at only one point. A normal is a line which is perpendicular to the tangent.



Example I

Find the equations of the tangents and normal to the curve at the given points:

(a) $y = x^2$ $\quad (2, 4)$

(b) $y = 3x^2 + 2$ $\quad (4, 50)$

(c) $y = 3x^2 - x + 1$ $\quad (0, 1)$

(d) $3 - 4x - 2x^2$ $\quad (0, 1)$

Solution

(a) $\boldsymbol{y = x^2}$

$$\frac{dy}{dx} = 2x$$
$$\left.\frac{dy}{dx}\right|_{(2, 4)} = 2 \times 2$$
$$= 4$$

$\Rightarrow$ The gradient of the tangent = 4

Let n be the gradient of the normal

$$n \times 4 = -1$$
$$n = \frac{-1}{4}$$

Equation of the tangent:

$$\Rightarrow \frac{y-4}{x-2} = 4$$
$$y - 4 = 4(x - 2)$$
$$y - 4 = 4x - 8$$
$$y = 4x - 4$$

Equation of the normal:

$$\Rightarrow \frac{y-4}{x-2} = \frac{-1}{4}$$
$$4(y - 4) = 1(x - 2)$$
$$4y - 16 = x - 2$$
$$4y = x - 14$$

(b) $\boldsymbol{y = (3x^2 + 2)}$

$$\frac{dy}{dx} = 6x$$
$$\left.\frac{dy}{dx}\right|_{(4,50)} = 6 \times 4$$
$$= 24$$

Gradient of tangent = 24

Let the gradient of the normal be n

$$n \times 24 = -1$$
$$n = \frac{-1}{24}$$

Equation of the tangent:

$$\Rightarrow \frac{y-50}{x-4} = 24$$
$y - 50 = 24(x - 4)$
$y - 50 = 24x - 96$
$y = 24x - 96 + 50$
$y = 24x - 46$

Equation of the normal:

$$\Rightarrow \frac{y-50}{x-4} = \frac{-1}{24}$$
$24(y - 50) = -1(x - 4)$
$24y - 1200 = -x + 4$
$24y + x = 1204$

(c) $\boldsymbol{y = 3x^2 - x + 1}$ **(0, 1)**

$$\frac{dy}{dx} = 6x - 1$$
$$\left.\frac{dy}{dx}\right|_{(0,\,1)} = 6 \times 0 - 1 = -1$$
$$\left.\frac{dy}{dx}\right|_{(0,\,1)} = -1$$

Let the gradient of the normal be n

$n \times -1 = -1$

$n = 1$

Equation of the tangent:

$$\Rightarrow \frac{y-1}{x-0} = -1$$
$y - 1 = -x$
$y = -x + 1$

Equation of the normal:

$$\Rightarrow \frac{y-1}{x-0} = 1$$
$y - 1 = x$
$y = x + 1$

(d) $\boldsymbol{y = 3 - 4x - 2x^2}$ **(1, -3)**

$$\frac{dy}{dx} = -4 - 4x$$
$$\left.\frac{dy}{dx}\right|_{(1,\,-3)} = -4 - 4 \times 1$$
$= -8$

Let the gradient of the normal be n

$n \times -8 = -1$

$$n = \frac{1}{8}$$

Equation of the tangent:

$$\Rightarrow \frac{y - {}^{-}3}{x-1} = -8$$
$y + 3 = -8(x - 1)$
$y + 3 = -8x + 8$
$y = -8x + 5$

Equation of the normal:

$$\Rightarrow \frac{y - {}^{-}3}{x-1} = \frac{1}{8}$$
$8(y + 3) = x - 1$
$8y + 24 = x - 1$
$8y + 25 = x$

Example II

Find the coordinates of a point on $y = x^2$ at which the gradient is 2. Hence find the equation of the tangent to the curve $y = x^2$ whose gradient is 2.

Solution

y = x^2

$$\frac{dy}{dx} = 2x$$
$\Rightarrow 2x = 2$
$x = 1$

If $x = 1$, from $y = x^2$;

$y = 1^2$
$y = 1$

$\Rightarrow$ The point is (1, 1)

Equation of the tangent:

$$\frac{y-1}{x-1} = 2$$
$y - 1 = 2(x - 1)$
$y = 2x - 1$

Example III

Find the equation of the normal to the curve $y = x^2 + 3x - 2$ at the point where it cuts the x-*axis.*

Solution

$y = x^2 + 3x - 2$

$$\frac{dy}{dx} = 2x + 3$$

At the y-axis, $x = 0$

From $y = x^2 + 3x - 2$,

$\Rightarrow y = 0^2 + 3 \times 0 - 2$
$y = -2$
(0, -2)

$$\left.\frac{dy}{dx}\right|_{(0,\,-2)} = 2 \times 0 + 3$$

$$= 3$$

The gradient of the tangent = 3

Let the gradient of the normal be n

$$n \times 3 = -1$$

$$n = \frac{-1}{3}$$

$$\frac{y - {}^{-}2}{x - 0} = \frac{-1}{3}$$

$$3(y - -2) = -1(x)$$

$$3(y + 2) = -x$$

$$3y + 6 = -x$$

$$3y + x + 6 = 0$$

Example IV

Find the value of k for which $y = 2x + k$ is a normal to the curve $y = 2x^2 - 3$.

Solution

$$y = 2x + k$$

Comparing y = $2x + k$ with $y = mx + c$;

$\Rightarrow m = 2$

$\therefore$ Gradient of the normal = 2

$$y = 2x^2 - 3$$

$$\frac{dy}{dx} = 4x$$

Let the gradient of the normal be n.

$$4x \times n = -1$$

$$n = \frac{-1}{4x}$$

Since the gradient of the normal = 2,

$$\Rightarrow \frac{-1}{4x} = 2$$

$$x = \frac{-1}{8}$$

$$y = 2x^2 - 3$$

$$y = 2\left(\frac{1}{64}\right) - 3$$

$$y = \frac{2}{64} - 3$$

$$y = \frac{-190}{64} = \frac{95}{32}$$

$$\left(\tfrac{-1}{8}, \tfrac{-95}{32}\right)$$

From $y = 2x + k$

$$\frac{-95}{32} = 2 \times \frac{-1}{8} + k$$

$$\frac{-95}{32} = \frac{-1}{4} + k$$

$$k = \frac{-95}{32} + \frac{1}{4} = \frac{-87}{32}$$

Example V

Find the equations of the tangents to the curve $y = (2x - 1)(x + 1)$ at the points where the curve cuts the x-axis. Find the point of intersection of these tangents.

Solution

$$y = (2x - 1)(x + 1)$$

$$y = 2x^2 + x - 1$$

At the x-axis, $y = 0$

$$\Rightarrow 0 = (2x - 1)(x + 1)$$

$$x = \frac{1}{2}, \qquad x = -1$$

(½, 0) and (-1, 0)

$\Rightarrow$ The curve cuts the x-axis at (½, 0) and (-1, 0)

$$y = 2x^2 + x - 1$$

$$\frac{dy}{dx} = 4x + 1$$

$$\left.\frac{dy}{dx}\right|_{(\frac{1}{2},\,0)} = 4 \times \frac{1}{2} + 1$$

$$= 3$$

$$\frac{y - 0}{x - \frac{1}{2}} = 3$$

$$y = 3x - \frac{3}{2} \quad \text{……………………..} \ (i)$$

$$\left.\frac{dy}{dx}\right|_{(-1,\,0)} = 4 \times -1 + 1$$

$$= -3$$

$$\Rightarrow \frac{y - 0}{x - {}^{-}1} = -3$$

$$y = -3(x + 1)$$

$$y = -3x - 3 \quad \text{……………………..} (ii)$$

Equating Eqn (i) and Eqn (ii);

$$\Rightarrow 3x - \frac{3}{2} = -3x - 3$$

$$6x = -3 + \frac{3}{2}$$

$$6x = \frac{-3}{2}$$

$$x = \frac{-1}{4}$$

Substituting $x = \frac{-1}{4}$ in Eqn (i);

$$y = 3 \times \frac{-1}{4} - \frac{3}{2}$$

$$y = \frac{-9}{4}$$

$\Rightarrow$ The two tangents intersect at $\left(\frac{-1}{4}, \frac{-9}{4}\right)$

Example VI

Find the coordinates of the point on $y = x^2 - 5$ at which the gradient is 3. Hence find the value of c for which the line $y = 3x + c$ is a tangent to $y = x^2 - 5$

Solution

$$y = x^2 - 5$$

$$\frac{dy}{dx} = 2x$$

$$2x = 3 \Rightarrow x = \frac{3}{2}$$

When $x = \frac{3}{2}$,

$$y = \left(\frac{3}{2}\right)^2 - 5$$

$$y = \frac{9}{4} - 5$$

$$y = \frac{-11}{4}$$

$$\left(\frac{3}{2}, \frac{-11}{4}\right)$$

$$y = 3x + c$$

$\left(\frac{3}{2}, \frac{-11}{4}\right)$ satisfies $y = 3x + c$

$$\frac{-11}{4} = 3 \times \frac{3}{2} + c$$

$$\frac{-11}{4} = \frac{9}{2} + c$$

$$c = \frac{-29}{4}$$

Example VII

A tangent to the parabola $x^2 = 16y$ is perpendicular to the line $x - 2y - 3 = 0$. Find the equation of this tangent and the coordinates of its point of contact.

Solution

$$x^2 = 16y$$

$$2x\,dx = 16\,dy$$

$$\frac{dy}{dx} = \frac{2x}{16} = \frac{x}{8}$$

$$x - 2y - 3 = 0$$

$$x - 3 = 2y$$

$$y = \frac{1}{2}x - \frac{3}{2}$$

Since the tangent is perpendicular to the line $y = \frac{1}{2}x - \frac{3}{2}$,

Let the gradient of the tangent be t.

$$t \times \frac{1}{2} = -1$$

$$t = -2$$

$$\frac{x}{8} = -2$$

$$x = -16$$

When $x = -16$,

$$-16^2 = 16y$$

$$y = 16$$

$$(-16, 16)$$

$$\Rightarrow \frac{y - 16}{x - {}^{-}16} = -2$$

$$y - 16 = -2(x + 16)$$

$$y - 16 = -2x - 32$$

$$y + 2x + 16 = 0$$

$\Rightarrow$ The equation of the tangent is $y + 2x + 16 = 0$ and the point of contact is $(-16, 16)$

Example VIII

Find the equation of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

Solution

$$y = x^3 - 6x^2 + 12x + 2$$

Comparing $y = 3x$ with $y = mx + c$ gives $m = 3$

$$\frac{dy}{dx} = 3x^2 - 12x + 12$$

$$\Rightarrow 3x^2 - 12x + 12 = 3$$

$$3x^2 - 12x + 9 = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$$x = 1 \quad \text{and} \quad x = 3$$

If $x = 1$,

$$y = 1^3 - 6 \times 1^2 + 12 \times 1 + 2$$

$y = 1 - 6 + 12 + 2$

$y = 9$

If $x = 3$, $y = 3^3 - 6 \times 3^2 + 36 + 2$

$y = 27 - 54 + 38$

$y = 11$

$\Rightarrow$ The points are (1, 9) and (3, 11)

$\Rightarrow \dfrac{y-9}{x-1} = 3$

$y - 9 = 3(x - 1)$

$y - 9 = 3x - 3$

$y = 3x + 6$

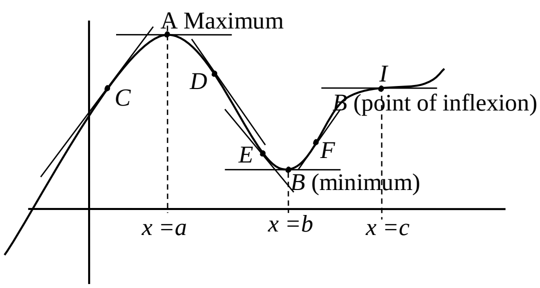
$\dfrac{y-11}{x-3} = 3$

$y - 11 = 3(x - 3)$

$y - 11 = 3x - 9$

$y = 3x + 2$

Maximum, Minimum and Inflexion points of a curve



Points A, B, and I are stationary (turning points) of the curve. We say that $f(x)$ has a maximum value at $x = a$, if $f(a)$ is greater than any value immediately preceding or following, we say that a function $f(x)$ has a minimum value at $x = b$, if $f(b)$ is less than any value immediately preceding or following.

The tangent to the curve at points A, B and C are horizontal (parallel to the x-axis).

$\Rightarrow$ The gradient of each tangent to the curve is zero;

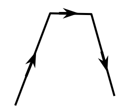
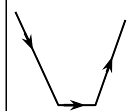
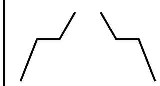
$$f(x) = 0$$

At points immediately to the left of the maximum point, C the slope of the tangent is positive. i.e. $f'(x) > 0$ while points immediately to the right at point D, the slope is negative i.e. $f'(x) < 0$.

In other words, at the maximum $f'(x)$ changes sign from + to (−).

At the minimum point, $f'(x)$ changes sign from – to +. We can see this at E and F.

Recall $f'(x) = \dfrac{dy}{dx}$.

	Maximum	Minimum	Inflexion
Sign of $\dfrac{dy}{dx}$ changes when moving through stationary values.	+ 0 − 	− + 0 	+ 0 +, − 0 − 

To locate maximum, minimum, and inflexion points of a curve without necessarily drawing the curve, we proceed as follows:

(a) Find the gradient $\dfrac{dy}{dx}$ of the curve

(b) Equate to zero the expression for $\dfrac{dy}{dx}$.

(c) Find the values of x which satisfy this equation.

(d) Consider the sign of $\dfrac{dy}{dx}$ on either sides of these points.

(e) Find the value(s) of y which correspond(s) to the values of x.

Distinguishing stationary points using the second derivative method

In order to distinguish the turning points, we find the second derivative.

If $\dfrac{d^2y}{dx^2} < 0$ at (x_1, y_2), $\Rightarrow (x_1, y_1)$ is a point of maximum

If $\dfrac{d^2y}{dx^2} > 0$ at (x_1, y_1), $\Rightarrow (x_1, y_1)$ is a minimum point;

If $\dfrac{d^2y}{dx^2} = 0$ at (x_1, y_1), $\Rightarrow (x_1, y_1)$ is a point of inflexion.

Example I

Find the coordinates of the stationary points of the curve $y = 2x^3 - 24x$ and distinguish between them.

Solution

$y = 2x^3 - 24x$

$\dfrac{dy}{dx} = 6x^2 - 24$

At stationary points, $\dfrac{dy}{dx} = 0$

$\Rightarrow 6x^2 - 24 = 0$

$x^2 - 4 = 0$

$(x + 2)(x - 2) = 0$

$x = -2$ and $x = 2$

If $x = -2$, $y = 2(-2)^3 - 24(-2)$

$y = -16 + 48$

$y = 32$

$\Rightarrow$ (-2, 32) is a stationary point.

If $x = 2$, $y = 2(2)^3 - 24(2)$

$y = 16 - 48$

$y = -32$

$\Rightarrow$ (2, -32) is a stationary point

$$\frac{dy}{dx} = 6x^2 - 24$$

$$\frac{d^2y}{dx^2} = 12x$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,\ 32)} = 12 \times -2$$

$$= -24 < 0$$

Since $\frac{d^2y}{dx^2} < 0$, $\Rightarrow$ (-2, 32) is a point of maxima.

$$\frac{d^2y}{dx^2} = 12x$$

$$\left.\frac{d^2y}{dx^2}\right|_{(2,\ -32)} = 12 \times 2$$

$$= 24 > 0$$

Since $\frac{d^2y}{dx^2} > 0$, $\Rightarrow$ (2, -32) is a point of minima.

Example II

Investigate the nature of stationary points of the following curves.

(a) $y = x(x^2 - 12)$
(b) $y = x^2(3 - x)$
(c) $y = x(x - 8)(x - 15)$
(d) $y = x^3(2 - x)$
(e) $y = 3x^4 + 16x^3 + 24x + 3$

Solution

(a) $\boldsymbol{y = x(x^2 - 12)}$

$y = x^3 - 12x$

$$\frac{dy}{dx} = 3x^2 - 12$$

At a stationary point, $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 12 = 0$

$x^2 - 4 = 0$

$x = \pm 2$

If $x = 2$, $y = x(x^2 - 12)$

$y = 2(4 - 12)$

$y = 2(-8)$

$\Rightarrow$ (2, -16) is a stationary point.

If $x = -2$, $y = -2(-2^2 - 12)$

$y = -2(4 - 12)$

$y = -2(-8)$

$y = 16$

(-2, 16) is a turning point.

$$\frac{d^2y}{dx^2} = 6x$$

$$\left.\frac{d^2y}{dx^2}\right|_{(2,\ -16)} = 6 \times 2 = 12$$

$\Rightarrow$ (2, -16) is a point of minima

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,\ 16)} = 6 \times -2 = -12 < 0$$

$\Rightarrow$ (-2, 16) is a point of maxima.

(b) $\boldsymbol{y = x^2(3 - x)}$

$y = 3x^2 - x^3$

$$\frac{dy}{dx} = 6x - 3x^2$$

At a turning point, $\frac{dy}{dx} = 0$

$6x - 3x^2 = 0$

$3x(2 - x) = 0$

$x = 0$ and $x = 2$

If $x = 0$, $y = x^2(3 - x)$

$y = 0$

$\Rightarrow$ (0, 0) is a stationary point.

If $x = 2$, $y = 2^2(3 - 2)$

$y = 4$

$\Rightarrow$ (2, 4) is a stationary point

Turning points:

$$\frac{dy}{dx} = 6x - 3x^2$$

$$\frac{d^2y}{dx^2} = 6 - 6x$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,\ 0)} = 6$$

$\Rightarrow$ (0, 0) is a point of minima

$$\left.\frac{d^2y}{dx^2}\right|_{(2,\ 4)} = 6 - 6 \times 2$$

$$= -6 < 0$$

$\Rightarrow$ (2, 4) is a point of maxima.

(c) $\boldsymbol{y = x(x - 8)(x - 15)}$

$y = x^3 - 23x^2 + 120x$

$\frac{dy}{dx} = 3x^2 - 46x + 120$

At stationary points, $\frac{dy}{dx} = 0$

$3x^2 - 46x + 120 = 0$

$x = 12, \ x = \frac{10}{3}$

If $x = 12$, $y = x(x - 8)(x - 15)$

$y = 12(12 - 8)(12 - 15)$

$y = 12(4)(-3)$

$y = -144$

$\Rightarrow (12, -144)$ is a stationary point

When $x = \frac{10}{3}$, $y = \frac{10}{3}\left(\frac{10}{3} - 8\right)\left(\frac{10}{3} - 15\right)$

$y = \frac{10}{3}\left(\frac{-14}{3}\right)\left(\frac{-35}{3}\right) = \frac{4900}{27}$

$\Rightarrow \left(\frac{10}{3}, \frac{4900}{27}\right)$ is a stationary point.

$\frac{d^2y}{dx^2} = 6x - 46$

$\left.\frac{d^2y}{dx^2}\right|_{(12, -144)} = 6 \times 12 - 46$

$= 26 > 0$

$\Rightarrow (12, -144)$ is a point of minima.

$\left.\frac{d^2y}{dx^2}\right|_{\left(\frac{10}{3}, \frac{4900}{9}\right)} = 6 \times \frac{10}{3} - 46$

$= -26 < 0$

$\Rightarrow \left(\frac{10}{3}, \frac{4900}{9}\right)$ is a point of maxima.

(d) $\boldsymbol{y = x^3(2 - x)}$

$y = 2x^3 - x^4$

$\frac{dy}{dx} = 6x^2 - 4x^3$

At stationary points, $\frac{dy}{dx} = 0$

$6x^2 - 4x^3 = 0$

$2x^2(3 - 2x) = 0$

$x = 0, \ x = \frac{3}{2}$

If $x = 0$, $y = x^3(2 - x)$

$y = 0^3(2 - 0)$

$y = 0$

$(0, 0)$ is a stationary point.

If $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^3\left(2 - \frac{3}{2}\right)$

$y = \frac{27}{8}\left(\frac{1}{2}\right) = \frac{27}{16}$

$\Rightarrow \left(\frac{3}{2}, \frac{27}{16}\right)$ is a stationary point

$\frac{d^2y}{dx^2} = 12x - 12x^2$

$\left.\frac{d^2y}{dx^2}\right|_{(0, 0)} = 0$

$\Rightarrow$ $(0, 0)$ is a point of inflexion.

$\left.\frac{d^2y}{dx^2}\right|_{\left(\frac{3}{2}, \frac{27}{16}\right)} = 12 \times \frac{3}{2} - 12\left(\frac{3}{2}\right)^2 = -9$

$\Rightarrow \left(\frac{3}{2}, \frac{27}{16}\right)$ is a point of maxima

(e) $\boldsymbol{y = 3x^4 + 16x^3 + 24x^2 + 3}$

$\frac{dy}{dx} = 12x^3 + 48x^2 + 48x$

At stationary points, $\frac{dy}{dx} = 0$

$12x^3 + 48x^2 + 48x = 0$

$12x(x + 4x + 4) = 0$

$x = 0, \quad x = -2$

If $x = 0$, $y = 3$

$\Rightarrow (0, 3)$ is a stationary point.

If $x = -2$, $y = 3(-2)^4 + 16(-2)^3 + 24(-2)^2 + 3$

$y = 48 - 128 + 96 + 3$

$y = 19$

$\Rightarrow$ $(-2, 19)$ is a stationary point.

$\frac{d^2y}{dx^2} = 36x^2 + 96x + 48$

$\left.\frac{d^2y}{dx^2}\right|_{(0, 3)} = 48 > 0$

$\Rightarrow (0, 3)$ is a point of minima.

$\left.\frac{d^2y}{dx^2}\right|_{(-2, 19)} = 36(-2)^2 + 96(-2) + 48 = 0$

$\Rightarrow (-2, 19)$ is a point of inflexion.

Example II

If $p = 4s^2 - 10s + 7$, find the minimum value of p and the values of s which gives the minimum value of p.

Solution

$p = 4s^2 - 10s + 7$

$\frac{dp}{ds} = 8s - 10$

For minimum value of p, $\frac{dp}{ds} = 0$

$8s - 10 = 0$

$s = \frac{10}{8} = \frac{5}{4}$

$p = 4s^2 - 10s + 7$

$$p_{min} = 4\left(\frac{5}{4}\right)^2 - 10\left(\frac{5}{4}\right) + 7$$

$$p_{min} = \frac{100}{16} - \frac{50}{4} + 7$$

$$p_{min} = \frac{3}{4}$$

$$\frac{dp}{ds} = 8s - 10$$

$$\frac{d^2p}{ds^2} = 8 > 0$$

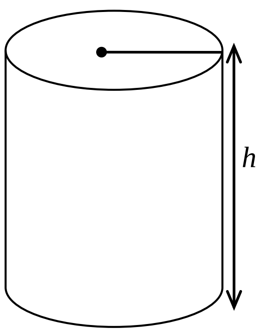
$\Rightarrow p$ is minimum when $S = \frac{5}{4}$ and the minimum value of p is $\frac{3}{4}$.

Example IV

A cylindrical can is made so that the sum of the height and the circumference of its base is 45π cm. Find the radius of the base of the cylinder if the volume of the can is maximum.

Solution

Let the radius of the base be r and the height h cm.



(Height + circumference) = 45π.

$h + 2\pi r = 45\pi$

$h = 45\pi - 2\pi r$ .... (i)

$V = \pi r^2 h$ …………………………….. (ii)

Substituting Eqn (i) in Eqn (ii);

$V = \pi r^2(45\pi - 2\pi r)$

$V = 45\pi^2 r^2 - 2\pi^2 r^3$

$$\frac{dV}{dr} = 90\pi^2 r - 6\pi^2 r^2$$

For the maximum volume, $\frac{dV}{dr} = 0$.

$90\pi^2 r - 6\pi^2 r^2 = 0$

$6\pi^2 r(15 - r) = 0$

$r = 0$ or $r = 15$

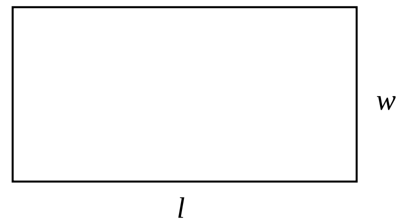
But $r \neq 0$

$\Rightarrow r = 15$ cm

Example V

Onyango wishes to fence a rectangular farm. He wants the sum of the length and the width of the farm to be 42 cm. Calculate the length and width of the farm for the area of the farm to be as maximum as possible.

Solution



Let the length and width of the rectangular farm be l and w respectively.

$l \times w = 42$

$l = 42 - w$

$A = l \times w$

$A = (42 - w)w$

$A = 42w - w^2$

$$\frac{dA}{dw} = 42 - 2w$$

For the maximum area, $\frac{dA}{dw} = 0$

$\Rightarrow 42 - 2w = 0$

$w = 21$

$l = 42 - w$

$= 42 - 21$

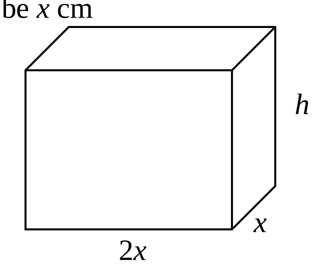
$= 21$

Example VI

The length of a rectangular block is twice its width, and the total surface area is 108 cm^2. Show that if the width of the block is x cm, the volume is $\frac{4}{3}x(27 - x^2)$. Find the dimensions of the block if the volume is maximum.

Solution

Let the width be x cm



$V = l \times w \times h$

$V = 2x \times x \times h$

$V = 2x^2 h$ ……………………………….(i)

Total surface area $A = 2(lw + wh + hl)$

$$108 = 2(2x^2 + xh + 2xh)$$
$$54 = 2x^2 + 3xh$$
$$\frac{54-2x^2}{3x} = h \text{ ………………… (ii)}$$

Substituting Eqn (ii) in Eqn (i);

$$\Rightarrow V = 2x^2\left(\frac{54-2x^2}{3x}\right)$$
$$V = 2x\left(\frac{54-2x^2}{3}\right)$$
$$V = \frac{4x}{3}(27 - x^2)$$

For the maximum volume, $\frac{dV}{dx} = 0$

$$V = \frac{4x}{3}(27 - x^2)$$
$$V = \frac{4}{3}(27x - x^3)$$
$$\frac{dV}{dx} = \frac{4}{3}(27 - 3x^2)$$

For $V_{\max}$, $\frac{dV}{dx} = 0$

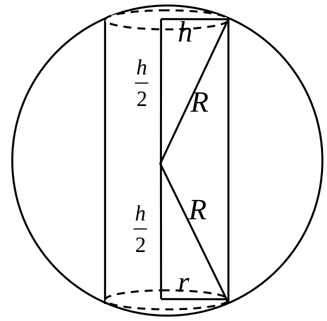
$$\frac{4}{3}(27 - 3x^2) = 0$$
$$\Rightarrow 27 - 3x^2 = 0$$
$$x^2 = 9$$
$$\Rightarrow x = 3$$
$$l = 2x$$
$$l = 6$$
$$h = \frac{54-2x^2}{3x}$$
$$= \frac{54-2(3^2)}{3\times 3}$$
$$= \frac{54-18}{9}$$
$$h = 4$$

Example VII

A cylindrical volume V is to be cut from a solid sphere of radius R. Prove that the maximum volume of the cylindar, V is $V = \frac{4\pi R^3}{3\sqrt{3}}$

Solution

Let the height of the cylinder be h



$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$
$$r^2 + \frac{h^2}{4} = R^2$$
$$r^2 = R^2 - \frac{h^2}{4}$$
$$V = \pi r^2 h$$
$$V = \pi\left(R^2 - \frac{h^2}{4}\right)h$$
$$V = \pi R^2 h - \frac{\pi h^3}{4}$$
$$\frac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2$$

For the maximum volume, $\frac{dV}{dh} = 0$

$$\pi R^2 - \frac{3}{4}\pi h^2 = 0$$
$$h^2 = \frac{4R^2}{3}$$
$$h = \frac{2R}{\sqrt{3}}$$
$$V = \pi r^2 h$$
$$V = \pi r^2\left(\frac{2R}{\sqrt{3}}\right)$$

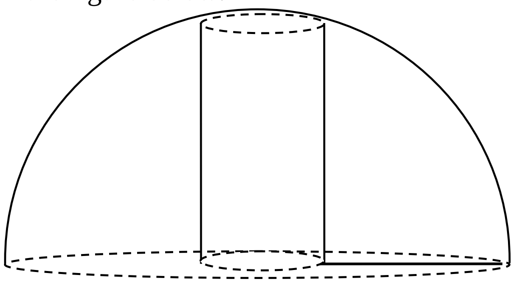
But $r^2 = R^2 - \frac{h^2}{4}$

$$\Rightarrow r^2 = R^2 - \frac{1}{4}\left(\frac{4R^2}{3}\right)$$
$$r^2 = R^2 - \frac{1}{3}R^2$$
$$r^2 = \frac{2}{3}R^2$$
$$V = \pi r^2 h$$
$$h = \frac{2R}{\sqrt{3}},\ r^2 = \frac{2}{3}R^2$$
$$V_{\max} = \pi\left(\frac{2R^2}{3}\right)\left(\frac{2R}{\sqrt{3}}\right)$$

$$V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}$$

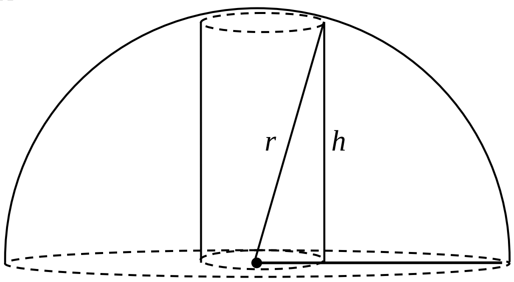
Example VIII

A cylinder is inscribed in a hemisphere of radius r as shown in the figure below.



Find the maximum volume of the cylinder in terms of r.

Solution



$x^2 + h^2 = r^2$

$x^2 = r^2 - h^2$

Volume of the cylinder, $V = \pi x^2 h$

$V = \pi(r^2 - h^2)h$

$V = \pi r^2 h - \pi h^3$

$$\frac{dV}{dh} = \pi r^2 - 3\pi h^2$$

For maximum volume, $\frac{dV}{dh} = 0$

$\pi r^2 - 3\pi h^2 = 0$

$\pi(r^2 - 3h^2) = 0$

$$\frac{r^2}{3} = h^2$$

$$h = \frac{r}{\sqrt{3}}$$

$x^2 = r^2 - h^2$

$$x^2 = r^2 - \frac{r^2}{3}$$

$$x^2 = \frac{2r^2}{3}$$

$V = \pi x^2 h$

$$x^2 = \frac{2r}{3}, \quad h = \frac{r}{\sqrt{3}}$$

$$V_{\max} = \pi \cdot \frac{2r^2}{3} \cdot \left(\frac{r}{\sqrt{3}}\right) = \frac{2\pi r^3}{3\sqrt{3}}$$

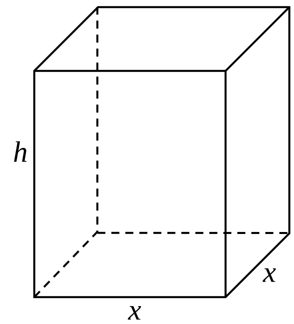
Example IX

A rectangular block has a base x cm square. Its surface area is 150 cm^2. Prove that the volume of the block is $\frac{1}{2}(75x - x^3)$.

(a) Calculate the dimensions of the block when the volume is maximum.

(b) The maximum volume.

Solution



S.A = $2(lw + wh + hl)$

$150 = 2(x^2 + xh + xh)$

$75 = (x^2 + 2xh)$

$$\frac{75 - x^2}{2x} = h$$

$V = l \times w \times h$

$V = x^2 h$

$$V = x^2\left(\frac{75 - x^2}{2x}\right)$$

$$V = \frac{x}{2}(75 - x^2)$$

$$V = \frac{1}{2}(75x - x^3)$$

$$\frac{dV}{dx} = \frac{1}{2}(75 - 3x^2)$$

For maximum volume, $\frac{dV}{dx} = 0$.

$$\frac{1}{2}(75 - 3x^2) = 0$$

$75 - 3x^2 = 0$

$x^2 = 25$

$x = 5$

$$h = \frac{75 - x^2}{2x}$$

$$h = \frac{75 - 25}{10}$$

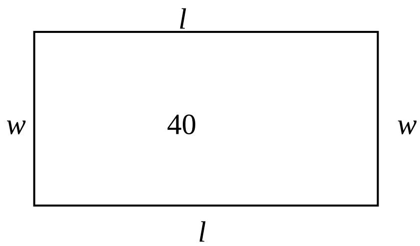
$h = 5$

Example X

(a) A variable rectangular flower garden has a constant perimeter of 40. Find the length of the side when the area is maximum.

(b) A variable rectangle has a constant area of 36 cm^2. Find the length of the sides when the perimeter is maximum.

Solution



Perimeter of the flower garden $P = 2(l + w)$

$40 = 2(l + w)$

$20 = l + w$

$l = 20 - w$

$A = lw$

$A = (20 - w)w$

$A = 20w - w^2$

$\frac{dA}{dw} = 20 - 2w$

For the maximum area, $\frac{dA}{dw} = 0$

$20 - 2w = 0$

$w = 10$

$l = 20 - w$

$l = 10$

(b) $\boldsymbol{P = 2(l + w)}$

$lw = 36$

$l = \frac{36}{w}$

$P = 2\left(\frac{36}{w} + w\right)$

$P = \frac{72}{w} + 2w$

$P = 72w^{-1} + 2w$

$\frac{dP}{dw} = -72w^{-2} + 2$

$= \frac{-72}{w^2} + 2$

For the maximum perimeter, $\frac{dP}{dw} = 0$

$\frac{-72}{w^2} + 2 = 0$

$\frac{72}{w^2} = 2$

$w^2 = 36$

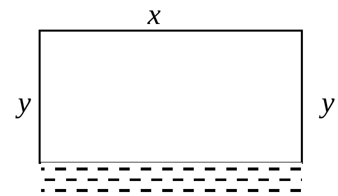
$w = 6$

$l = 6$

Example XI

Mukasa wishes to enclose a rectangular piece of land of area 1250 cm^2 whose one side is bound by a straight bank of a river. Find the least possible length of barbed wire required.

Solution



$xy = 1250$

$y = \frac{1250}{x}$

$P = x + y + y$

$P = x + 2y$

$P = x + \left(2 \times \frac{1250}{x}\right)$

$P = x + \frac{2500}{x}$

$\frac{dP}{dx} = 1 - \frac{2500}{x^2}$

For the least possible length, $\frac{dP}{dx} = 0$

$\Rightarrow 1 - \frac{2500}{x^2} = 0$

$1 = \frac{2500}{x^2}$

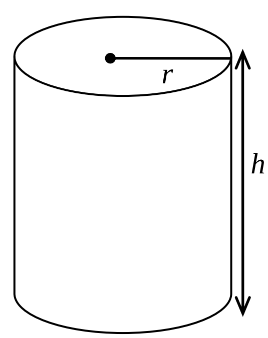
$x = 50$

$y = \frac{1250}{50} = 25$

Example XII

A closed right circular cylinder of base radius r cm and height h cm has volume of 54π cm^3. Show that S, the total surface area of the cylinder, is given by $S = \frac{108\pi}{r} + 2\pi r^2$ hence find the radius and height which makes the surface area minimum.

Solution



$V = \pi r^2 h$

$54\pi = \pi r^2 h$

$\frac{54}{r^2} = h$

Surface area of a cylinder $A = 2\pi r^2 + 2\pi rh$

$A = 2\pi r^2 + 2\pi r\left(\frac{54}{r^2}\right)$

$A = 2\pi r^2 + \frac{108\pi}{r^3}$

$\frac{dA}{dr} = 4\pi r - \frac{108\pi}{r^2}$

For the minimum surface area, $\frac{dA}{dr} = 0$

$$4\pi r - \frac{108\pi}{r^2} = 0$$

$$4\pi r^3 - 108\pi = 0$$

$$r^3 = \frac{108}{4}$$

$$r^3 = 27$$

$$r = 3$$

$$h = \frac{54}{r^2} = \frac{54}{9}$$

$$h = 6$$

Example XIII

A company that manufactures dog food wishes to pack the feed in closed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 250π cm^3 and the minimum possible surface area?

Solution

$A = 2\pi r^2 + 2\pi rh$

$V = \pi r^2 h$

$250\pi = \pi r^2 h$

$h = \frac{250}{r^2}$

$A = 2\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right)$

$A = 2\pi r^2 + \frac{500\pi}{r}$

$\frac{dA}{dr} = 4\pi r - \frac{500\pi}{r^2}$

For minimum surface area, $\frac{dA}{dr} = 0$

$4\pi r - \frac{500\pi}{r^2} = 0$

$\pi(4r^3 - 500) = 0$

$r^3 = 125$

$r = 5$ cm

$h = \frac{250}{r^2}$

$h = 10$ cm

Example (UNEB Question)

Write down the expression of the volume V and surface area S of a cylinder of radius r and height h. If the surface area S of the cylinder is kept constant, show that the volume of the cylinder will be maximum when $h = 2r$

Solution

$S = 2\pi r^2 + 2\pi rh$

$h = \frac{S - 2\pi r^2}{2\pi r}$

$V = \pi r^2 h$

$V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right)$

$V = \frac{1}{2}(Sr - 2\pi r^3)$

$\frac{dV}{dr} = \frac{1}{2}(S - 6\pi r^2)$

For maximum volume, $\frac{dV}{dr} = 0$

$\Rightarrow S - 6\pi r^2 = 0$

$S = 6\pi r^2$

$h = \frac{S - 2\pi r^2}{2\pi r}$

$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r}$

$h = 2r$

For maximum volume, $h = 2r$

Example (UNEB Question)

A right circular cone of radius r cm has a maximum volume. The sum of its vertical height h and circumference of its base is 15 cm. If the radius varies, show that the maximum volume of the cone is $\frac{125}{3\pi}$ cm^3.

Solution

The base is circular

$\Rightarrow$ The circumference of the base $= 2\pi r$

$$2\pi r + h = 15$$

$$h = 15 - 2\pi r$$

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi r^2(15 - 2\pi r)$$

$$= \frac{1}{3}\pi(15r^2 - 2\pi r^3)$$

$$\frac{dV}{dr} = \frac{\pi(30r - 6\pi r^2)}{3}$$

For maximum volume, $\frac{dV}{dr} = 0$

$$\frac{\pi}{3}(30r - 6\pi r^2) = 0$$

$$30r - 6\pi r^2 = 0$$

$$6r(5 - \pi r) = 0$$

$$5 = \pi r$$

$$r = \frac{5}{\pi}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$h = 15 - 2\pi r; \text{ But } r = \frac{5}{\pi}$$

$$h = 15 - 2\pi\left(\frac{5}{\pi}\right)$$

$$h = 5$$

$$V = \frac{\pi}{3}\left(\frac{5}{\pi}\right)^2 \times 5$$

$$V = \frac{125}{3\pi}\text{ cm}^3$$

Example

A match box consists of an outer cover open at both ends into which a rectangular box without a top. The length of the box is one and a half times the width. The thickness of the material is negligible and the volume of the match box is 25 cm^3. If the width is x cm, find in terms of x the area of the material used. Hence show that if the least area of the material is to be used to make the box, the length should be 3.7 approximately.

Solution

Area of the inner surface $= 2(lw) + 2(lh)$

$$= 2\left(\frac{3x}{2}\times x\right) + 2\left(\frac{3x}{2}\times h\right)$$

$$= 3x^2 + 3xh$$

Area of the water surface $= (lw + 2lh + 2wh)$

$$= \frac{3x}{2}\cdot x + \frac{3x}{2}\times 2\times h + 2xh$$

$$= \frac{3x^2}{2} + 5xh$$

The total surface of the match box

$$= \frac{3x^2}{2} + 5xh + 3xh + 3x^2$$

$$A = \frac{9x^2}{2} + 8xh \;\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(i)}$$

From volume $= l \times w \times h$,

$$V = \frac{3x^2 h}{2}$$

$$\Rightarrow 25 = \frac{3x^2 h}{2}$$

$$h = \frac{50}{3x^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(ii)}$$

Substituting Eqn (2) in Eqn (i);

$$A = \frac{9x^2}{2} + 8x\left(\frac{50}{3x^2}\right)$$

$$A = \frac{9x^2}{2} + \frac{400}{3x}$$

For the least area, $\frac{dA}{dx} = 0$

$$\frac{dA}{dx} = 9x - \frac{400}{3x^2}$$

$$\frac{dA}{dx} = 9x - \frac{400}{3x^2} = 0$$

$$27x^3 - 400 = 0$$

$$x^3 = \frac{400}{27}$$

$$x = \sqrt[3]{\frac{400}{27}}$$

$$l = \frac{3x}{2}$$

$$l = \frac{3}{2}\times\sqrt[3]{\frac{400}{27}}$$

$$l = 3.68403 \text{ cm}$$

$$l \approx 3.7 \text{ cm}$$

Techniques of Differentiation

Chain, Product, and Quotient rules

We can now move to some more properties involved in differentiation. To summarise, so far we have found that:

1. The derivative of a sum is a sum of its derivatives.

2. The derivative of a difference is the difference of the derivatives.

However, it turns out that:

1. The derivative of a product of derivative $f(x)g(x)$ is not a product of the derivative.

$$\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$$

2. The derivative of a quotient is not the quotient of the derivative

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$$

3. The derivative of the composition $f(x)$ is not the composition of the derivatives.

The chain, product and quotient rules tell us how to differentiate in these three situations.

Chain Rule

The chain rule states that:

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

Example I

Given that $y = (x^2 + 7)^{100}$, find $\frac{dy}{dx}$

Solution

$y = (x^2 + 7)^{100}$

Let $t = x^2 + 7$

$\frac{dt}{dx} = 2x$

$\Rightarrow y = t^{100}$

$\frac{dy}{dt} = 100t^{99}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$= 100t^{99} \times 2x$

$= 200xt^{99}$

$= 200x(x^2 + 7)^{99}$.

Example II

Given that $y = (x^7 - x^2)^{42}$, find $\frac{dy}{dx}$.

Solution

$y = (x^7 - x^2)^{42}$

Let $t = x^7 - x^2 \;\Rightarrow\; y = t^{42}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$y = t^{42}$

$\frac{dy}{dt} = 42t^{41}$

$t = x^7 - x^2$

$\frac{dt}{dx} = 7x^6 - 2x$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$= 42(t^{41}) \times (7x^6 - 2x)$

$= 42(7x^6 - 2x)t^{41}$

$= 42(7x^6 - 2x)(x^7 - x^2)^{41}$

Example III

Find $\frac{dy}{dx}$ in terms of t in the following expressions:

(a) $x = t^2, \quad y = 4t - 1$

(b) $y = 3t^2 + 2t, \quad x = 1 - 2t$

(c) $x = 2\sqrt{2}, \quad y = 5t - 4$

(d) $x = \frac{1}{t}, \quad y = t^2 + 4t - 3$

(e) $x = \frac{2}{3+\sqrt{t}}, \quad y = \sqrt{t}$

Solution

(a) $x = t^2, \quad y = 4t - 1$

$y = 4t - 1$

$\frac{dy}{dt} = 4$

$x = t^2$

$\frac{dx}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 4 \times \frac{1}{2t} = \frac{2}{t}$$

(b) $y = 3t^2 + 2t, \quad x = 1 - 2t$

$\frac{dy}{dt} = 6t + 2$

$\frac{dx}{dt} = -2$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$\frac{dy}{dx} = (6t + 2) \times \frac{-1}{2}$

$\frac{dy}{dx} = -3t - 1$

(c) $x = 2\sqrt{t}$, $y = 5t - 4$

$\frac{dx}{dt} = 2 \times \frac{1}{2} t^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{t}}$

$y = 5t - 4$

$\frac{dy}{dt} = 5$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 5 \times \sqrt{t}$

$= 5\sqrt{t}$

(d) $x = \frac{1}{t}$, $y = t^2 + 4t - 3$.

$\frac{dx}{dt} = -1t^{-1-1} = \frac{1}{t^2}$

$y = t^2 + 4t - 3$

$\frac{dy}{dt} = (2t + 4)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= (2t + 4) \times -t^2$

$= -2t^3 - 4t^2$

(e) $x = \frac{2}{3 + \sqrt{t}}$, $y = \sqrt{t}$

$x = 2(3 + \sqrt{t})^{-1}$

$\frac{dx}{dt} = -2(3 + \sqrt{t})^{-2} \cdot \frac{t^{-\frac{1}{2}}}{2}$

$= \frac{-1}{(3 + \sqrt{t})^2 \sqrt{t}}$

$y = \sqrt{t}$

$\frac{dy}{dt} = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{t}} \times {}^{-}(3 + \sqrt{t})^2 \sqrt{t}$

$= \frac{{}^{-}(3 + \sqrt{t})^2}{2}$

Example IV

Find $\frac{dy}{dx}$ in terms of t if $x = at^2$ and $y = 2at$

Solution

$x = at^2$

$\frac{dx}{dt} = 2at$

$y = 2at$

$\frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$

Example V (UNEB Question)

A curve is defined by the parametric equations

$x = t^2 - t$

$y = 3t + 4$

Find the equation of the tangent to the curve at (2, 10)

Solution

$x = t^2 - t$ and $y = 3t + 4$.

$\frac{dx}{dt} = 2t - 1$

$\frac{dy}{dt} = 3$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= 3 \times \frac{1}{2t - 1}$

$= \frac{3}{2t - 1}$

At point (2, 10), $x = 2$ and $y = 10$.

$x = t^2 - t$

$y = 3t + 10$

Substituting, for $x = 2$,

$2 = t^2 - t$

$t^2 - t - 2 = 0$

$t^2 - 2t + t - 2 = 0$

$t(t - 2) + 1(t - 2) = 0$

$(t - 2)(t + 1) = 0$

Either $t - 2 = 0$,

$t = 2$

Or $t + 1 = 0$

$t = -1$

Substituting for $y = 10$,

$10 = 3t + 4$

$3t = 6$

$t = 2$

For $\frac{dy}{dx} = \frac{3}{2t - 1}$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3}{2(2)-1} = \frac{3}{4-1} = 1$$

$$\Rightarrow \frac{y-10}{x-2} = 1$$

$y - 10 = x - 2$

$y = x - 2 + 10$

$y = x + 8$

Example VI

If $x = at^2$, $y = 2at$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

Solution

$x = at^2$, $y = 2at$

$$\frac{dx}{dt} = 2at; \qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2a \times \frac{1}{2at}$$

$$= \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{-1}{t^2} \times \frac{1}{2at} = \frac{1}{2at^3}$$

Example VII

A curve is represented parametrically by

$$x = (t^2 - 1)^2; \qquad y = t^3$$

Find $\frac{dy}{dx}$

Solution

$x = (t^2 - 1)^2$, $y = t^3$

$$\frac{dx}{dt} = 2(t^2 - 1)2t$$

$$= 4t(t^2 - 1)$$

$y = t^3$

$$\frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 3t^2 \times \frac{1}{4t(t^2 - 1)}$$

$$= \frac{3t}{4(t^2 - 1)}$$

Product Rule

Consider $y = uv$, where v and u are functions of x.

$y + \partial y = (u + \partial u)(v + \partial v)$

$y + \partial y = uv + u\partial v + v\partial u + \partial u\partial v$

As $\partial u \longrightarrow 0$, $\partial v \longrightarrow 0$

$\partial u\partial v \approx 0$

$\Rightarrow \partial y + y = uv + u\partial v + v\partial u$

$\partial y = uv + u\partial v + v\partial u - y$

$\partial y = uv + u\partial v + v\partial u - uv$

$\partial y = u\partial v + v\partial u$

$$\frac{\partial y}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

As $\partial x \longrightarrow 0$

$$\frac{\partial y}{\partial x} \approx \frac{dy}{dx}, \quad \frac{\partial v}{\partial x} \approx \frac{dv}{dx} \text{ and } \frac{\partial u}{\partial x} \approx \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Example I

Differentiate the following

(a) $(x^2 + 1)(x^3 + 2)$

(b) $x^2(x + 1)^3$

(c) $(1 + x)^{3/2(x-1)5/4}$

(d) $(x-1)\sqrt{x^2+1}$

(e) $\sqrt{(x+1)(x-2)^3}$

(f) $(x-1)^2\sqrt[3]{1-2x}$

Solution

(a) $\mathbf{y = (x^2 + 1)(x^3 + 2)}$

Let $u = x^2 + 1$, $v = x^3 + 2$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^2 + 1)(3x^2) + (x^3 + 2)2x$$

$= 3x^4 + 3x^2 + 2x^4 + 4x$

$= 5x^4 + 3x^2 + 4x$

$= 5x^4 + 3x^2 + 4x$

$$\Rightarrow \frac{dy}{dx} = 5x^4 + 3x^2 + 4x.$$

(b) $\mathbf{y = x^2(x + 1)^3}$

Let $u = x^2$, $v = (x + 1)^3$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x^2 3(x+1)^2 \cdot 1 + (x+1)^3 2x$$
$$= x(x+1)^2[3x + 2(x+1)]$$
$$= x(x+1)^2(5x+2)$$
$$= x(x+1)^2(5x+2)$$

(c) $\boldsymbol{y =} (1+x)^{3/2}(x-1)^{5/4}$

$u = (1+x)^{3/2}$, $v = (x-1)^{5/4}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\Rightarrow \frac{dy}{dx} = (1+x)^{3/2} \cdot \frac{5}{4}(x-1)^{1/4} \cdot 1 + (x-1)^{5/4} \cdot \frac{3}{2}(x+1)^{1/2} \cdot 1$$
$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{1/4}(x+1)^{1/2}\left[\frac{5}{2}(x+1) + \frac{3(x-1)}{1}\right]$$
$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{1/4}(x+1)^{1/2}\left(\frac{5+5x+6(x-1)}{2}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{1/4}(x+1)^{1/2}\left(\frac{11x-6}{2}\right)$$
$$\frac{dy}{dx} = \frac{(x-1)^{1/4}(x+1)^{1/2}(11x-6)}{4}$$

(d) $y = (x-1)\sqrt{x^2+1}$

Let $u = x - 1$, $v = \sqrt{x^2+1}$

$$\frac{dy}{dx} = (x-1) \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x + \left(\sqrt{x^2+1}\right) \cdot 1$$
$$\frac{dy}{dx} = x(x-1)(x^2+1)^{-1/2} + (x^2+1)^{1/2}$$
$$\frac{dy}{dx} = (x^2+1)^{-1/2}\left[x(x-1) + x^2 + 1\right]$$
$$\frac{dy}{dx} = \frac{1}{(x^2+1)^{1/2}}\left[x^2 - x + x^2 + 1\right]$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}\left[2x^2 - x + 1\right]$$

(e) $\boldsymbol{y =} \sqrt{(x+1)(x-2)^3}$

$y = \left[(x+1)(x-2)^3\right]^{1/2}$

$y = (x+1)^{1/2}(x-2)^{3/2}$

$$\frac{dy}{dx} = (x+1)^{1/2} \cdot \frac{3}{2}(x-2)^{1/2} \cdot 1 + (x-2)^{3/2}\,\frac{1}{2}(x+1)^{-1/2}$$
$$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-1/2}(x-2)^{1/2}\left[3(x+1) + (x-2)\right]$$
$$\frac{dy}{dx} = \frac{(x-2)^{1/2}}{2(x+1)^{1/2}}\left[3x+3+x-2\right]$$
$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x-2}{x+1}}\left[4x+1\right]$$

(f) $\boldsymbol{y = (1-x)^2 \sqrt[3]{1-2x}}$

Let $u = (1-x)^2$, $v = (1-2x)^{1/3}$

$y = uv$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{dy}{dx} = (1-x)^2 \cdot \frac{1}{3}(1-2x)^{-2/3} \cdot -2 + (1-2x)^{1/3} \cdot 2(1-x)(^-1)$$
$$\frac{dy}{dx} = 2(1-x)(1-2x)^{-2/3}\left[\frac{-1}{3}(1-x) + {}^-1(1-2x)\right]$$
$$\frac{dy}{dx} = \frac{2(1-x)}{(1-2x)^{2/3}}\left[\frac{-1+x-3(1-2x)}{3}\right]$$
$$\frac{dy}{dx} = \frac{2(1-x)}{(1-2x)^{2/3}}\left[\frac{-1-3+x+6x}{3}\right]$$
$$\frac{dy}{dx} = \frac{2(1-x)}{\sqrt[3]{(1-2x)^2}}\left[\frac{7x-4}{3}\right]$$
$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{1-x}{\sqrt[3]{(1-2x)^2}}\right)(4x-8)$$

Example (UNEB Question)

Given that $R = q\sqrt{(1000-q^2)}$, find:

(a) $\dfrac{dR}{dq}$

(b) The value of q when R is maximum.

Solution

(a) $R = q\sqrt{(1000-q^2)}$

Let $u = q$, $v = \sqrt{(1000-q^2)}$;

$$\frac{dR}{dq} = u\frac{dv}{dq} + v\frac{du}{dq}$$
$$\frac{dR}{dq} = q \cdot \frac{1}{2}(1000-q^2)^{-1/2} \times -2q + \sqrt{1000-q^2} \times 1$$
$$\frac{dR}{dq} = -q^2(1000-q^2)^{-1/2} + \sqrt{1000-q^2}$$
$$= (1000-q^2)^{-1/2}\left[(-q^2 + 1000 - q^2)\right]$$
$$= \frac{1000-2q^2}{\sqrt{1000-q^2}}$$

(b) For $R_{\max}$, $\dfrac{dR}{dq} = 0$

$$\Rightarrow \frac{1000-2q^2}{\sqrt{1000-q^2}} = 0$$

$$1000 - 2q^2 = 0$$
$$q^2 = 500$$
$$q^2 = 100 \times 5$$
$$q = \sqrt{100\times 5}$$
$$q = \pm 10\sqrt{5}$$
$$q = 10\sqrt{5} \text{ or } q = -10\sqrt{5}$$

Quotient Rule

Consider $y = \dfrac{u}{v}$, where u and v are functions of x.

$$y = \frac{u}{v}$$

$$y + \partial y = \frac{u+\partial u}{v+\partial v}$$

$$\partial y = \frac{u+\partial u}{v+\partial v} - y$$

$$\partial y = \frac{u+\partial u}{v+\partial v} - \frac{u}{v}$$

$$\partial y = \frac{(u+\partial u)(v-\partial v)}{(v+\partial v)(v-\partial v)} - \frac{u}{v}$$

$$= \frac{uv + u\partial v + v\partial u - \partial u\partial v}{v^2 - (\partial v)^2} - \frac{u}{v}$$

As $\partial v \longrightarrow 0$, $\partial u \longrightarrow 0$ and $\partial v\partial u \approx 0$, $(\partial v)^2 \approx 0$

$$\partial y = \frac{uv + v\partial u - u\partial v}{v^2} - \frac{u}{v}$$

$$\partial y = \frac{u}{u} + \frac{v\partial u - u\partial v}{v^2} - \frac{u}{v}$$

$$= \frac{v\partial u - u\partial v}{v^2}$$

$$\frac{\partial y}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

As $\partial x \longrightarrow 0$, $\dfrac{\partial y}{\partial x} \longrightarrow \dfrac{dy}{dx}$

$$\frac{\partial u}{\partial x} \longrightarrow \frac{du}{dx}$$

$$\frac{\partial v}{\partial x} \longrightarrow \frac{dv}{dx}$$

$$\boxed{\mathbf{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}}$$

Example

Differentiate the following:

(a) $\dfrac{x^2+1}{x^2-1}$ **(b)** $\dfrac{x}{\sqrt{x^2+1}}$

(c) $\sqrt{\dfrac{(x+2)^3}{x-1}}$ **(d)** $\sqrt{\dfrac{(x+1)^3}{x+2}}$

(e) $\dfrac{(1-\sqrt{x})^2}{\sqrt{x^2-1}}$ **(f)** $\dfrac{2x^2-x^3}{\sqrt{x^2-1}}$

Solutions

(a) $\dfrac{x^2+1}{x^2-1}$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = x^2 + 1$; $v = x^2 - 1$

$$\frac{dy}{dx} = \frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x[(x^2-1)-(x^2+1)]}{(x^2-1)^2}$$

$$\frac{dy}{dx} = 2x\left(\frac{-2}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

(b) $\dfrac{x}{\sqrt{x^2+1}}$

$u = x$, $v = \sqrt{x^2+1}$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1}\times(1) - x\cdot\frac{1}{2}(x^2+1)^{-1/2}\times 2x}{\left(\sqrt{x^2+1}\right)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)^{-1/2}[(x^2+1)-x^2]}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)^{-1/2}[1]}{x^2+1}$$

$$\frac{dy}{dx} = \frac{1}{(x^2+1)^{3/2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2+1)^3}}$$

(c) $y = \sqrt{\dfrac{(x+2)^3}{x-1}}$

$$y = \frac{(x+2)^{3/2}}{(x-1)^{1/2}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x-1)^{1/2} \times \frac{3}{2}(x+2)^{1/2} - (x+2)^{3/2} \times \frac{1}{2}(x-1)^{-1/2}}{\left[(x-1)^{1/2}\right]^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(x+2)^{1/2}}{(x-1)^{3/2}}(3x-3-x-2)$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x+2}{(x-1)^3}}\ (2x-5)$$

(d) $y = \sqrt{\dfrac{(x+1)^3}{x+2}}$

$$y = \frac{(x+1)^{3/2}}{(x+2)^{1/2}}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x+2)^{1/2} \cdot \frac{3}{2}(x+1)^{1/2} - (x+1)^{3/2} \cdot \frac{1}{2}(x+2)^{-1/2} \cdot 1}{[(x+2)^{1/2}]^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(x+2)^{-1/2}(x+1)^{1/2}[3(x+2)-(x+1)]}{x+2}$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(x+1)^{1/2}}{(x+2)^{3/2}}(3x+6-x-1)$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x+1}{(x+2)^3}}(2x+5)$$

$$\frac{dy}{dx} = \frac{(2x+5)}{2}\sqrt{\frac{x+1}{(x+2)^3}}$$

(e) $y = \dfrac{(1-\sqrt{x})^2}{\sqrt{x^2-1}}$

$$u = (1-\sqrt{x})^2, \quad v = \sqrt{x^2-1}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{(\sqrt{x^2-1})2(1-\sqrt{x}) \cdot \frac{-1}{2}x^{-1/2} - (1-\sqrt{x})^2\,\frac{1}{2}(x^2-1)^{-1/2} \cdot 2x}{(\sqrt{x^2-1})^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)^{-1/2}(1-\sqrt{x})x^{-1/2}\left[2(x^2-1) \cdot \frac{-1}{2\sqrt{x}} - (1-\sqrt{x})x\right]}{x^2-1}$$

$$\frac{dy}{dx} = \frac{1-\sqrt{x}}{\sqrt{x}(x^2-1)^{3/2}}\left[\frac{x^2+1-(\sqrt{x})x(1-\sqrt{x})}{\sqrt{x}}\right]$$

$$\frac{dy}{dx} = \frac{1-\sqrt{x}}{\sqrt{x}(x^2-1)^{3/2}}\left[\frac{-x^2+1-x\sqrt{x}+x^2}{\sqrt{x}}\right]$$

$$\frac{dy}{dx} = \frac{1-\sqrt{x}}{\sqrt{(x^2-1)^3}}\left[\frac{1-x\sqrt{x}}{x}\right]$$

Example (UNEB Question)

Differentiate:

(a) $(x+1)^{1/2}(x+2)^2$

(b) $\dfrac{2x^2+3x}{(x-4)^2}$

Solution

(a) $(x+1)^{1/2}(x+2)^2$

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x+1)^{1/2}2(x+2) + (x+2)^2 \cdot \frac{1}{2}(x+1)^{-1/2}$$

$$\frac{dy}{dx} = (x+1)^{-1/2}(x+2)\left[2(x+1) + \frac{1}{2}(x+2)\right]$$

$$\frac{dy}{dx} = \frac{x+2}{\sqrt{x+1}}\left[\frac{4(x+1)+x+2}{2}\right]$$

$$\frac{dy}{dx} = \frac{x+2}{\sqrt{x+1}}\left[4x+4+x+2\right]$$

$$\frac{dy}{dx} = \frac{x+2}{\sqrt{x+1}}\left[5x+6\right]$$

(b) $y = \dfrac{2x^2+3x}{(x-4)^2}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x-4)^2 \cdot (4x+3) - (2x^2+3x) \cdot 2(x-4)}{[(x-4)^2]^2}$$

$$\frac{dy}{dx} = \frac{(x-4)\left[(x-4)(4x+3) - 2(2x^2+3x)\right]}{(x-4)^4}$$

$$\frac{dy}{dx} = \frac{-19x-12}{(x-4)^3}$$

Differentiation of Implicit Functions

Example I

Find $\dfrac{dy}{dx}$ when $x^2 + 2xy + y^2 = 8$

Solution

$$\frac{d}{dx}(x^2 + 2xy + y^2) = \frac{d}{dx}(8)$$
$$2xdx + 2(xdy + ydx) + 2ydy = 0$$
$$2x+2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(2x + 2y) = -2x - 2y$$
$$\frac{dy}{dx} = \frac{-2(x+y)}{2(x+y)}$$
$$\frac{dy}{dx} = -1$$

Example II

If $x^2 - 3xy + y^2 - 2y + 4x = 0$, find $\frac{dy}{dx}$

Solution

$$x^2 - 3xy + y^2 - 2y + 4x = 0$$
$$\frac{d}{dx}(x^2 - 3xy + y^2 - 2y + 4x) = \frac{d}{dx}(0)$$
$$2x\,dx - 3(xdy + ydx) + 2ydy - 2dy + 4dx = 0$$
$$2x - 3x\frac{dy}{dx} - 3y + 2y\frac{dy}{dx} - 2\frac{dy}{dx} + 4 = 0$$
$$\frac{dy}{dx}(2y - 3x - 2) = -4 - 2x$$
$$\frac{dy}{dx} = \frac{-4-2x}{2y-3x-2}$$

Example III

Find $\frac{dy}{dx}$ when $3x^2 - 4xy = 7$

Solution

$$3x^2 - 4xy = 7$$
$$\frac{d}{dx}(3x^2 - 4xy) = \frac{d}{dx}(7)$$
$$6x\,dx - 4(x\,dy + y\,dx) = 0$$
$$6x - 4x\frac{dy}{dx} - 4y = 0$$
$$6x - 4y = 4x\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{6x-4y}{4x}$$

Example IV

If $x^2 + 3xy - y^2 = 0$, find $\frac{dy}{dx}$ at (1, 1).

Find the equation of the tangent and normal at (1, 1)

Solution

$$x^2 + 3xy - y^2 = 0$$
$$\frac{d}{dx}(x^2 + 3xy - y^2) = \frac{d}{dx}(3)$$
$$2x\,dx + 3(x\,dy + y\,dx) - 2y\,dy = 0$$
$$2x + 3x\frac{dy}{dx} + 3y - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(3x - 2y) = -2x - 3y$$
$$\frac{dy}{dx} = \frac{-2x-3y}{3x-2y}$$
$$\left.\frac{dy}{dx}\right|_{(1,\,1)} = \frac{-2-3}{3-2} = -5$$
$$\Rightarrow \frac{y-1}{x-1} = -5$$
$$y - 1 = -5(x - 1)$$
$$y - 1 = -5x + 5$$

$y = -5x + 6$ is the equation of the tangent

Let the gradient of the normal be n

$$n \times -5 = -1$$
$$n = \frac{1}{5}$$
$$\Rightarrow \frac{y-1}{x-1} = \frac{1}{5}$$
$$5(y - 1) = x - 1$$
$$5y - 5 = x - 1$$

$5y - 4 = x$ is the equation of the normal.

Example V

Find the x-stationary points of the curve $x^3 - y^3 - 4x^2 + 3y = 11x + 4$

Solution

$$x^3 - y^3 - 4x^2 + 3y = 11x + 4$$
$$\frac{d}{dx}(x^3 - y^3 - 4x^2 + 3y) = \frac{d}{dx}(11x + 4)$$
$$3x^2\,dx - 3y^2\,dy - 8x\,dx + 3dy = 11\,dx$$
$$(3 - 3y^2)\,dy = (11 - 3x^2 - 8x)\,dx$$
$$\frac{dy}{dx} = \frac{11-3x^2-8x}{3-3y^2}$$

At stationary points, $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{11-3x^2-8x}{3-3y^2} = 0$$
$$11 - 3x^2 - 8x = 0$$
$$3x^2 + 8x - 11 = 0$$
$$x = 1, \quad x = \frac{-11}{3}$$

Application of Differentiation

Small Changes

If $A(x, y)$ is a general point in the curve with equation $y = f(x)$ and $B(x+\delta x, y+\delta y)$ is a point in the curve close to A, then δx is a small increase in x and δy is a small increase in y

We know from differentiation that

$$\Rightarrow \lim_{\partial x \to 0}\left(\frac{\partial y}{\partial x}\right) = \frac{dy}{dx}$$

So when ∂x is small, we can say that $\frac{\partial y}{\partial x} \approx \frac{dy}{dx}$

$$\partial y \approx \frac{dy}{dx} \cdot \partial x$$

The approximation can be used to estimate the value of a function close to a known value $y + \delta y$ can be estimated if y is known.

Example I

Given that $y = 3x^2 + 2x - 4$. Use small changes to find the small change in y when x increases from 2 to 2.02.

Solution

$y = 3x^2 + 2x - 4$

$\frac{dy}{dx} = 6x + 2$

$\partial y = \frac{dy}{dx} \cdot \partial x$

$\partial x = (2.02 - 2) = 0.02$

$x = 2; \quad \partial x = 0.02$

$\partial y = \frac{dy}{dx} \cdot \partial x$

$\partial y = (6x + 2)\ \partial x$

$\partial y = [(6 \times 2) + 2] \times 0.02$

$\partial y = 0.28$

Example II

Use small changes to estimate $\sqrt{101}$

Solution

$y = \sqrt{101}$

$y + \partial y = \sqrt{x + \partial x}$

$y = \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$x = 100, \quad \partial x = 1$

$\partial y = \frac{dy}{dx} \cdot \partial x$

$\partial y = \frac{1}{2\sqrt{x}} \cdot (\partial x)$

$\partial y = \frac{1}{2\sqrt{100}} \cdot (1)$

$\partial y = \frac{1}{20}$

$\partial y = 0.05$

$y + \partial y = \sqrt{x + \partial x}$

$x = 100, \quad y = \sqrt{x}$

$y = \sqrt{100} = 10$

$10 + 0.05 = \sqrt{100 + 1}$

$10.05 = \sqrt{101}$

Example III

In an experiment, the diameter x of a metal is measured and the volume V cm^3 is calculated using the formula $V = \frac{1}{6}\pi x^3$. If the diameter is found to be 10 cm with a possible error of 0.1cm, estimate the possible error in the volume calculated.

Solution

$V = \frac{1}{6}\pi x^3$

$\partial y = \frac{dy}{dx} \cdot \partial x$

$\partial x = 0.1, x = 10$

$\partial y = \frac{1}{2}\pi x^2 \cdot (0.1)$

$\partial y = \frac{1}{2}\pi (10)^2 \times (0.1)$

$= 5\pi$

Hence the possible error in the volume is 5π cm^3

Example IV

Find the approximate value of $\sqrt[3]{1003}$

Solution

$y = \sqrt[3]{x}$

$x = 1000, \ \partial x = 3$

$y + \partial y = \sqrt[3]{1000 + 3}$

$y + \partial y = \sqrt[3]{1003}$

$y = \sqrt[3]{x}$

$y = x^{1/3}$

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

$$\partial y = \frac{dy}{3x^{2/3}} \cdot 3$$

$$= \frac{1}{3(1000)^{2/3}} \cdot 3$$

$$= \frac{3}{300} = 0.01$$

$$y + \partial y = \sqrt[3]{1003}$$

$$y = \sqrt[3]{x}$$

$$y = \sqrt[3]{1000} = 10$$

$$10 + 0.01 = \sqrt[3]{1003}$$

$$10.01 = \sqrt[3]{1003}$$

Example I

Use small changes to find the cube root of 1005

Solution

$$y = \sqrt[3]{1005}$$

$$y + \partial y = \sqrt[3]{x + \partial x}$$

$$y = x^{1/3}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

x = 100, $\partial x = 5$

$$y + \partial y = \sqrt[3]{x + \partial x}$$

$$y = x^{1/3}$$

$$y = 1000^{1/3}$$

$$y = 10$$

$$y + \partial y = \sqrt[3]{1005}$$

$$10 + \partial y = \sqrt[3]{1005}$$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

$$\partial y = \frac{1}{3x^{2/3}} \cdot 5$$

$$\partial y = \frac{1}{3(1000)^{2/3}} \times 5$$

$$\partial y = \frac{5}{300}$$

$$\partial y = 0.016667$$

$$10 + 0.016667 = \sqrt[3]{1005}$$

$$10.016667 = \sqrt[3]{1005}$$

$$\sqrt[3]{1005} = 10.01667$$

Example

Use small changes to find $\sqrt{627}$.

Solution

$$y = \sqrt{x}$$

$$y + \partial y = \sqrt{x + \partial x}$$

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$$

$$\partial y = \frac{dy}{dx} \cdot \partial x$$

x = 625, $\partial x = 2$

$$\partial y = \frac{1}{2 \times \sqrt{625}} \times 2$$

$$\partial y = \frac{1}{25}$$

$$y = \sqrt{x}$$

$$y = \sqrt{625}$$

$$y = 25$$

$$25 + \frac{1}{25} = \sqrt{625 + 2}$$

$$\sqrt{625} = 25 + 0.04$$

$$\sqrt{625} = 25.04$$

Percentage Small Changes

An error of 3% is made in measuring the radius of the sphere. Find the percentage error in the volume.

Solution

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\partial r = \frac{3}{100} r$$

$$\partial V = \frac{\partial V}{\partial r} \cdot \partial r$$

$$\partial V = 4\pi r^2 \cdot \frac{3}{100} r$$

$$\partial V = \cdot \frac{12\pi r^3}{100}$$

$$\frac{\partial V}{V} \times 100 = \frac{\frac{12\pi r^3}{100}}{V} \times 100 = \frac{\frac{12\pi r^3}{100}}{\frac{4}{3}\pi r^3} \times 100$$

$$= 9\%$$

Example II

The height of a cylinder is 10 cm and the radius is 4 cm. Find the approximate percentage increase in the volume when the radius increases from 4 to 4.02 cm.

Solution

$$V = \pi r^2 h$$

$$\frac{dV}{dr} = 2\pi rh$$

$$\partial V = \frac{\partial V}{\partial r} \cdot \partial r$$

$$\partial r = 4.02 - 4 = 0.02$$

$$\partial V = 2\pi rh(0.02)$$

$$\partial V = 2\pi \times 10 \times 4(0.02)$$

$$\partial V = 1.6\pi$$

$$V = \pi r^2 h$$

$$V = \pi(4)^2 \times 10$$

$$V = 160\pi$$

Percentage increase in the volume is $\frac{\partial V}{V} \times 100$

$$= \frac{1.6\pi}{160\pi} \times 100 = 1\%$$

Example III

The period T of a simple pendulum is calculated from the formula $T = 2\pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum and g is the acceleration due to gravity constant. find the percentage change in the period caused by lengthening the pendulum by 2%.

Solution

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$T = \frac{2\pi l^{1/2}}{g^{1/2}}$$

$$\frac{dT}{dl} = \frac{\pi l^{-1/2}}{g^{1/2}} = \frac{\pi}{g^{1/2} l^{1/2}}$$

$$\partial T = \frac{dT}{dl} \cdot \partial l$$

$$\partial T = \frac{\pi}{g^{1/2} l^{1/2}} \cdot \frac{2}{100} l$$

$$\partial T = \frac{2\pi}{100 g^{1/2}} l^{1/2}$$

$$\partial T = \frac{2\pi}{100}\sqrt{\frac{l}{g}}$$

Percentage change in period = $\frac{\partial T}{T} \times 100$

$$= \frac{\frac{2\pi}{100}\sqrt{l/g}}{2\pi\sqrt{l/g}} \times 100$$

$$= 1\%$$

Example

An error of 2.5% is made in measuring the area of a circle. What is the percentage error in the circumference?

Solution

$$\partial A = \frac{\partial A}{\partial r} \cdot \partial r$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\partial A = \frac{2.5A}{100}$$

$$\partial A = 2\pi r \cdot \partial r$$

$$\frac{2.5}{100} A = 2\pi r \ \partial r$$

$$\frac{2.5}{100} \pi r^2 = 2\pi r \ \partial r$$

$$\partial r = \frac{1.25r}{100}$$

$$C = 2\pi r$$

$$\frac{\partial C}{\partial r} = 2\pi$$

$$\partial C = \frac{\partial C}{\partial r} \cdot \partial r$$

$$\partial C = 2\pi \times \frac{1.25}{100} r$$

$$\partial C = \frac{2.5\pi r}{100}$$

Percentage error in circumference = $= \frac{\partial C}{C} \times 100$

$$= \frac{\frac{2.5\pi r}{100}}{2\pi r} \times 100 = 1.25\%$$

Example

If l is the length of a pendulum and t is the time of a complete swing, it is known that $l = kt^2$. The length of the pendulum is increased by x%. x is so small. Find the corresponding increase in the time of the string.

Solution

$$l = kt^2$$

$$\frac{dl}{dt} = 2kt$$

$$\partial l = \frac{dl}{dt} \cdot \partial t$$

$$\partial l = 2kt \cdot \partial t$$

$$\frac{x}{100} l = 2kt \cdot \partial t$$

$$\partial t = \frac{\frac{xl}{100}}{2kt} = \frac{xl}{200kt}$$

$$\partial t = \frac{x(kt^2)}{200kt} = \frac{xt}{200}$$

Percentage increase in time = $\frac{\partial t}{t} \times 100$

$$= \frac{\frac{xt}{200}}{t} \times 100 = \frac{x}{2}\%$$

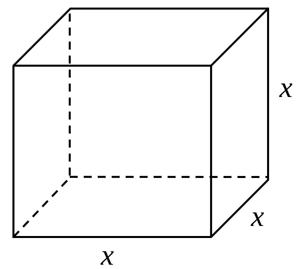
Rates of Change

Application of derivatives

Example I

A side of a cube is increasing at a rate of 6cm/s. Find the rate of increase in the volume of the cube when the length of the side is 8cm.

Solution



$V = x^3$

$$\frac{dV}{dx} = 3x^2$$

$$\frac{dx}{dt} = 6 \text{ cm/s}$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\frac{dV}{dt} = 3x^2 \times 6$$

$$\frac{dV}{dt} = 18x^2$$

$$\left.\frac{dV}{dt}\right|_{x=8} = 18 \times 8^2$$

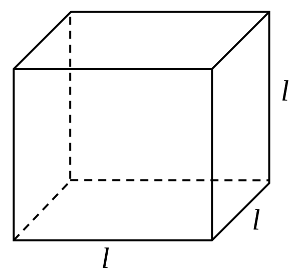
$$= 1152$$

$$\frac{dV}{dt} = 1152 \text{ cm}^3\text{/s}$$

Example II

The volume of a cube is increasing at a rate of 2 cm^3/s. Find the rate of change of the side of the base when the length is 3 cm.

Solution



$V = l^3$

$$\frac{dV}{dt} = 2\text{cm}^3\text{/s}$$

$$\frac{dV}{dl} = 3l^2$$

$$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$$

$$\frac{dV}{dt} = 3l^2 \times \frac{dl}{dt}$$

$$2 = 3l^2 \frac{dl}{dt}$$

$$\frac{dl}{dt} = \frac{2}{3l^2}$$

$$\left.\frac{dl}{dt}\right|_{l=3} = \frac{2}{3\times 3^2} = \frac{2}{3\times 9}$$

$$\frac{dt}{dl} = \frac{2}{27} \text{ cm/s}$$

Example III

The area of the circle is increasing at a rate of 3cm^2/s. Find the rate of change of the circumference when its radius is 2cm.

Solution

$$\frac{dA}{dt} = 3$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$3 = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{3}{2\pi r}$$

$$\left.\frac{dr}{dt}\right|_{r=2} = \frac{3}{2\pi \times 2}$$

$$\frac{dr}{dt} = \frac{3}{4\pi}$$

$$C = 2\pi r$$

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dC}{dt} = 2\pi \times \frac{3}{4\pi}$$

$$\frac{dC}{dt} = \frac{6}{4}$$

$$\frac{dC}{dt} = 1.5 \text{cm/s}$$

Example III (UNEB Question)

A spherical balloon is inflated such that the rate at which its radius is increasing is 0.5cm/s. Find the rate at which:

(a) the volume is increasing at the instant when r = 5.0cm

(b) the surface area is increasing when r = 8.5 cm

Solution

$$V = \frac{4}{3}\pi r^3, \quad \frac{dr}{dt} = 0.5 \text{ m/s}$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 2\pi r^2$$

$$\left.\frac{dV}{dt}\right|_{r=5} = 2\pi(5)^2 = 50\pi \text{ cm}^2\text{/s}$$

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 8\pi r \times 0.5$$

$$\frac{dA}{dt} = 4\pi r$$

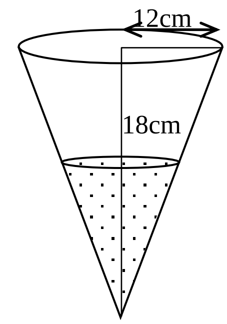
$$\left.\frac{dA}{dt}\right|_{r=8.5} = 4\pi(8.5)$$

$$\left.\frac{dA}{dt}\right|_{r=8.5} = 34\pi \text{ cm}^2\text{/s}$$

Example IV

A hollow circular cone is held vertex downwards beneath a tap leaking at a rate of 2cm^3/s. Find the rise of water level when the level is 6 cm. Given that the height of the cone is 18 cm and its radius is 12 cm.

Solution



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = 2 \text{ cm}^3\text{/s}$$

$$\frac{r}{h}=\frac{12}{18}=\frac{2}{3}$$

$$r=\frac{2}{3}h$$

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\tfrac{2}{3}h\right)^2 h$$

$$V=\frac{1}{3}\pi\left(\frac{4h^2}{9}\right)h$$

$$V=\frac{4}{27}\pi h^3$$

$$\frac{dV}{dh}=\frac{12}{27}\pi h^2$$

$$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt}=\frac{12\pi h^2}{27}\times\frac{dh}{dt}$$

$$2=\frac{12\pi h^2}{27}\times\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{54}{12\pi h^2}$$

$$h=6$$

$$\frac{dh}{dt}=\frac{54}{12\pi(6)^2}$$

$$\frac{dh}{dt}=\frac{54}{432\pi}=\frac{1}{8\pi}\ \text{cm/s}$$

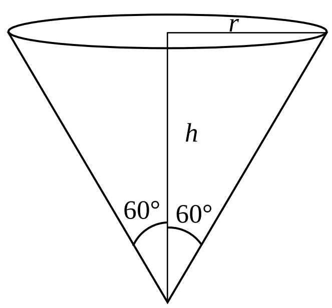
Example V

An inverted right circular cone of vertical angle 120° is collecting water from a tap at a steady rate of 18π cm^3/min. Find:

(a) the depth of the water after 12 minutes

(b) the rate of increase of the depth at this instant.

Solution



Volume of the cone $V=\frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt}=18\pi\ \text{cm}^3/\text{min}$$

$$1\ \text{min}\longrightarrow 18\pi\ \text{cm}^3$$

$$12\ \text{min}\longrightarrow x\ \text{cm}^3$$

$$x=12\times 18\pi$$

$$=216\pi\ \text{cm}^3$$

$$\tan 60=\frac{r}{h}\Rightarrow\sqrt{3}=\frac{r}{h}$$

$$r=\sqrt{3}h$$

$$V=\frac{1}{3}\pi r^2 h$$

$$V=\frac{1}{3}\pi(\sqrt{3}\ h)^2 h$$

$$V=\pi h^3$$

$$V=\pi h^3$$

$$216\pi=\pi h^3$$

$$216=h^3$$

$$h=6\ \text{cm}$$

$$V=\pi h^3$$

$$\frac{dV}{dh}=3\pi h^2$$

$$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$18\pi=3\pi h^2\times\frac{dh}{dt}$$

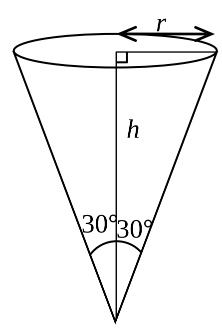
$$\frac{dh}{dt}=\frac{18}{3h^2}$$

$$\left.\frac{dh}{dt}\right|_{h=6}=\frac{18}{3\times 6^2}=\frac{1}{6}\ \text{cm/min}$$

Example VI

An inverted cone with vertical angle of 60° is collecting water leaking from a tap at a rate of 2cm^3/s. If the height of water collected is 10cm, find the rate at which the depth is decreasing at that instant.

Solution



$$\tan 30=\frac{r}{h}$$

$$\frac{1}{\sqrt{3}}=\frac{r}{h}\quad\Rightarrow h=\sqrt{3}\ h$$

$$r=\frac{h}{\sqrt{3}}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{1}{3}\pi\left(\frac{h^2}{3}\right)h$$

$$= \frac{1}{9}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{3}\pi h^2$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$0.2 = \frac{1}{3}\pi h^2 \cdot \frac{dh}{dt}$$

$$\frac{0.6}{\pi h^2} = \frac{dh}{dt}$$

When h = 10,

$$\frac{dh}{dt} = \frac{0.6}{\pi h^2}$$

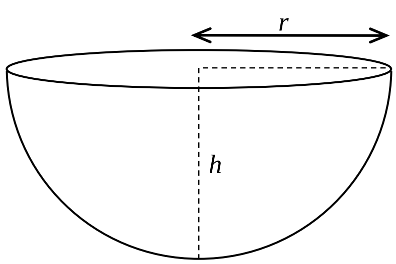
$$\frac{dh}{dt} = \frac{0.6}{\pi(10)^2}$$

$$\frac{dh}{dt} = \frac{0.6}{100\pi}$$

$$\frac{dh}{dt} = \frac{6}{1000\pi}\text{ cm/s}$$

Example

A hemispherical bowl is being filled with water at a uniform rate when the height of water is h cm. The volume is $\pi(rh^2 - \frac{1}{3}h^3)\,\text{cm}^3$, r being the radius of the sphere. Find the rate at which the water level is rising when it is half-way to the top, given that r = 6 and the bowl fills in 1 minute.



$$V = \pi(rh^2 - \tfrac{1}{3}h^3)$$

When it is full, $r = h$

$$V = \pi(h^3 - \tfrac{1}{3}h^3)$$

$$V = \frac{2\pi h^3}{3}$$

$$r = h = 6$$

$$V = \frac{2}{3}\pi \times 6^3$$

$$= 144\pi\ \text{cm}^3$$

$$\frac{dV}{dt} = 144\pi\,\text{cm}$$

(*Because the bowl fills in a minute*)

When the bowl is not full, $r \neq h$

r = 6 cm

$$V = \pi(rh^2 - \tfrac{1}{3}h^2)$$

$$V = \pi(6h^2 - \tfrac{1}{3}h^2)$$

$$\frac{dV}{dh} = \pi(12h - h^2)$$

When h = 3, $\frac{dV}{dh} = \pi(36 - 9)$

$$\frac{dV}{dh} = 27\pi$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt} = 27\pi \times \frac{dh}{dt}$$

$$144\pi = 27\pi \times \frac{dh}{dt}$$

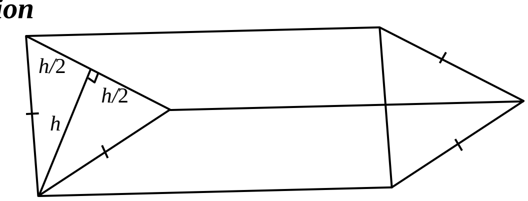
$$\frac{dh}{dt} = \frac{144\pi}{27\pi} = \frac{48}{9}$$

$$\frac{dh}{dt} = \frac{16}{3}\text{ cm/min} = \frac{4}{45}\text{ cm/s}$$

Example

A horse trough has a triangular cross-section area of height 50 cm and base 60cm and height 2m long. A horse is drinking steadily and when the water level is 5cm below the top, it is being lowered at a rate of 1cm/min. Find the rate of consumption in litres per minute.

Solution



h = 50

$$V = \left(\frac{1}{2}\times b \times h\right)\times l$$

l = 200 cm

$$V = \frac{1}{2}\times b\times h\times 200$$

V = 100bh

$$\frac{h}{b/2} = \frac{50}{30}$$

$$\frac{2h}{b} = \frac{5}{3}$$

$$2h=\frac{5}{3}b$$

$$b=\frac{6h}{5}$$

$$V=100\left(\frac{6h}{5}\right)h=120h^2$$

$$\frac{dV}{dh}=240h$$

$$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt}=240h\times 1$$

$$\frac{dV}{dt}=240h$$

$$\frac{dV}{dt}=(240\times 20)=4800\text{ cm}^3/\text{min}$$

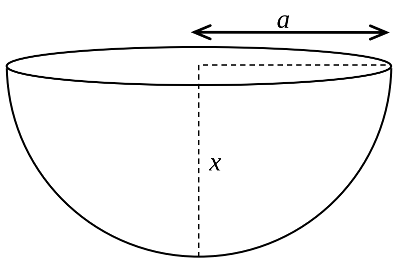
$$\frac{dV}{dt}=4.8\text{ litres/minute}$$

Example (UNEB Question)

A hemispherical bowl of radius *a* cm is initially full of water. The water runs out through a small hole at the bottom of the bowl at a constant rate such that it empties the bucket in 24 s. Given that when the depth of water is *x* cm and the volume of water is $\frac{1}{3}\pi x^2(3a-x)$ cm^3, show that the depth of water at that instant is decreasing at a rate of $a^3(36(2a-x))^{-1}$. Find how long it will take for the depth of water to be $\frac{1}{3}a$ cm and the rate at which the depth is increasing at that instant.

Solution

$$V=\frac{1}{3}\pi x^2(3a-x)$$



When it is full of water, $x = a$

$$V=\frac{1}{3}\pi a^2(3a-a)$$

$$V=\frac{1}{3}\pi a^2(2a)$$

$$V=\frac{2}{3}\pi a^3$$

Because it empties in 24s

$$24\text{s}\longrightarrow\frac{2}{3}\pi a^3\text{ cm}^3$$

$$1\text{ s}\longrightarrow x$$

$$x=\frac{2}{72}\pi a^3\text{ cm}^3/\text{s}$$

$$\frac{dV}{dt}=\frac{\pi a^3}{36}\text{cm}^3/\text{s}$$

When $x\neq a$

$$V=\frac{1}{3}\pi x^2(3a-x)$$

$$\frac{dV}{dt}=\frac{dV}{dx}\cdot\frac{dx}{dt}$$

$$\frac{\pi a^3}{36}=(2\pi ax-\pi x^2)\frac{dx}{dt}$$

$$\frac{dx}{dt}=\frac{\pi a^3}{36\pi x(2a-x)}$$

$$\frac{dx}{dt}=a^3[36x(2a-x)]^{-1}$$

$$h=\frac{1}{3}a$$

$$V=\frac{1}{3}\pi x^2(3a-x)$$

$$V=\frac{1}{3}\pi\left(\tfrac{1}{3}a\right)^2(3a-\tfrac{1}{3}a)$$

$$V=\frac{1}{3}\pi\left(\frac{a^2}{9}\right)\left(\frac{8a}{3}\right)$$

$$V=\frac{8\pi a^3}{81}$$

Volume of water in the bowl = $\frac{8\pi a^3}{81}$

Volume of the water emptied

$$=\frac{2\pi a^3}{3}-\frac{8\pi a^3}{81}=\frac{46\pi a^3}{81}$$

$$\frac{dV}{dt}=\frac{\pi a^3}{36}$$

$$1\text{ s}\longrightarrow\frac{\pi a^3}{36}$$

$$x\text{ s}\longrightarrow\frac{46\pi a^3}{81}$$

$$x=\frac{\frac{46\pi a^3}{81}}{\frac{\pi a^3}{36}}=\frac{46\pi a^3}{81}\times\frac{36}{\pi a^3}$$

x = 20.4445 cm