

## NMC 2014 O-LEVEL PAPER 1 QUESTIONS

## SECTION A

- Qn. 1. Simplify the following expression  $\frac{(-a)^3 - (\frac{1}{a})^{-2}}{-a^2 + (-\frac{1}{a})^{-3}}$ .
- Qn. 2. Determine the number of pairs  $(x, y)$  of integer solutions for  $2^{2x} - 3^{2y} = 55$ .
- Qn. 3. A number system based on 26 uses the letters of the alphabet as its digits, with  $A = 0, B = 1, C = 2, \dots, Y = 24$ , and  $Z = 25$ . Express  $ONE + ONE$  in this system.
- Qn. 4. The harmonic mean of two numbers is defined to be the reciprocal of the average of the reciprocals of the numbers. Find  $x$  such that the harmonic mean of 4 and  $x$  is 6.
- Qn. 5. The ratio of children to adults at a party is 2 : 3. A busload of 30 more children arrives at the party, and now the ratio of children to adults is 3 : 2. How many people were at the party before the bus arrived?
- Qn. 6. It takes 852 digits to number the pages of a book consecutively. How many pages are there in the book?
- Qn. 7. Two athletes **A** and **B** exercise along the same route. **B** jogs the first half of the route at  $6km/hr$  and runs the remainder of the route at  $12km/hr$  taking a total time of  $x$  hours. **A** walks the first third of the route at  $5km/hr$  and runs the remainder of the route at  $15km/hr$  taking a total time of  $y$  hours. Determine the value of  $\frac{x}{y}$ .
- Qn. 8. If  $TEEN$  is a four-digit number such that  $TEEN_5 + TEEN_7 = TEEN_8$ , what is the value of the digit  $N$ ?
- Qn. 9. Toshiba laptops are packed either in boxes of 5 or boxes of 12. What is the smallest number of full boxes required to pack exactly 2005 Toshiba laptops?
- Qn.10.  $PQR$  is a triangle such that  $PQ = 19cm$ ,  $QR = 18cm$  and  $PR = 17cm$ . Let  $S$  be a point on  $PQ$ ,  $T$  be a point on  $PR$  and  $U$  be a point on  $ST$  such that  $QS = SU$  and  $UT = TR$ . Find the perimeter of triangle  $PST$ .

## SECTION B

- Qn.11. If  $m$  and  $n$  are non-negative real numbers such that  $m - 2n > 0$  and

$$2 \log(m - 2n) = \log m + \log n.$$

Find the ratio of  $m$  to  $n$ .

- Qn.12. Alex and John together can do job in 2 days. John and Jimmy can the same job in four days while Alex and Jimmy can do the job in  $2\frac{2}{5}$  days. Find the number of days required for Alex to do the job alone.
- Qn.13. A fruit salad consists of blueberries, raspberries, grapes, and cherries. The fruit salad has a total of 280 pieces of fruit. There are twice as many raspberries as blueberries, three times as many grapes as cherries, and four times as many cherries as raspberries. How many cherries are there in the fruit salad?

- Qn.14. In rectangle  $ABCD$ ,  $AB = 5\text{cm}$  and  $BC = 3\text{cm}$ . Points  $F$  and  $G$  are on  $CD$  with  $DF = 1\text{cm}$  and  $CG = 2\text{cm}$ , and lines  $AF$  and  $BG$  intersect at  $E$ . What is the area of triangle  $AEB$ ?
- Qn.15. The odd numbers from 1 to 17 can be used to build a  $3 \times 3$  magic square (the rows and columns have the same sum). If the 1, 5 and 13 are as shown, what is  $x$ ?

	1	
5		13
$x$		

## NMC 2014 O-LEVEL PAPER 2 QUESTIONS

## SECTION A

- Qn. 1. Given that  $k \div 2 \div 2 \div 2 \div 2$  and  $n \div 2 \div 2 \div 2$  are each equal to the smallest positive integer. Find the value of  $k + n$ .
- Qn. 2. Simplify the following expression  $(2014)^2 - (2)(2014)(2013) + (2013)^2$ .
- Qn. 3. The average of the digits of the integer 46 is 5. Including 46, how many two-digit positive integers have the average of their digits equal to 5?
- Qn. 4. A child's age, increased by 3, gives a perfect square, and when decreased by 3 the age is the square root of that perfect square. How old is the child?
- Qn. 5. When counting from 3 to 531, 53 is the 51<sup>st</sup> number counted. When counting backwards from 531 to 3, 126 is the  $n^{\text{th}}$  number counted. What is  $n$ ?
- Qn. 6. A bag contains 65 balls of the same size. There are 20 red ones, 20 green ones, 20 blue ones and another 5 that are either yellow or white. Lorinda removes balls from the bag without looking. What is the smallest number of balls that she must remove to ensure that she has 10 of the same colour?
- Qn. 7. Tacia participates at a free throw basketball contest. For each successful throw she makes, she gets 8 points and for each miss she is penalized 5 points. After 26 throws she has 0 points. How many successful throws did she make?
- Qn. 8. Mary paid shs 480 to purchase a certain number of items, but the nice vendor gave her two extra. This decreased the price per item by shs 1. How many items did she receive (including the two extra)?
- Qn. 9. Consider the positive integer  $p$  with representation  $p = 73961w$ . If  $p$  divided by 5 has a remainder of 4, and  $p$  divided by 3 has a remainder of 0, what is the value of  $w$ ?
- Qn.10. In order to estimate the value of  $x - y$  where  $x$  and  $y$  are real numbers with  $x > y > 0$ , Chantal rounded  $x$  up by a small amount, rounded  $y$  down by the same amount and then subtracted her rounded values. Which of the following statements is necessarily correct? Show your working.  
(A) Her estimate is larger than  $x - y$ . (B) Her estimate is smaller than  $x - y$ .  
(C) Her estimate equals  $x - y$ . (D) Her estimate equals  $y - x$ . (E) Her estimate is 0.

## SECTION B

- Qn.11. When a decimal point is placed between the digits of the two-digit whole number  $n$ , the resulting number is equal to the average of the digits of  $n$ . What is the value of  $n$ ?
- Qn.12. Each of Alice, Bob, Carol and Don took a test. Each of them answered atleast one question correctly and altogether they answered 67 questions correctly. Alice had more correct answers than anyone else. Bob and carol together answered 43 questions correctly. How many correct answers did Don have?

- Qn.13. Miranda bought some toy bicycles and some toy helicopters. She paid a total of  $shs201$ . She did not buy partial toy bicycles or partial toy helicopters. Each toy bicycle cost  $shs19$  and each toy helicopter cost  $shs17$ . How many of each did she buy?
- Qn.14. How many numbers between 10 and 99 have digits that differ by exactly 4?
- Qn.15. In the diagram, the rectangle is divided into nine smaller rectangles. The areas of five of these rectangles are given. Determine the area of the rectangle labelled  $R$ .

3	1	
	2	$R$
5		10

## NMC 2013 O-LEVEL PAPER 1 QUESTIONS

## SECTION A : 5 marks each

- Qn. 1. If  $xy = 2$ ,  $yz = 24$ ,  $zx = 3$  and  $x, y$  and  $z$  are all positive numbers, what is the value of  $x + y + z$ ?
- Qn. 2. In the sum below, the letter  $F = 0$  and the other letters represent the digits 1, 2, 3, 4, 5 or 6 with each digit used exactly once. The two-digit integer  $AB$  is a prime number. What is the value of  $A + B$ ?

$$\begin{array}{r} AB \\ +CD \\ \hline EFG \end{array}$$

- Qn. 3. If  $a$  and  $b$  are non zero numbers such that  $a$  and  $b$  are the two roots of  $x^2 + ax + b = 0$ . What is the value of  $a$  and  $b$ ?
- Qn. 4. Evaluate  $\sqrt[3]{\frac{1 \times 2 \times 4 + 2 \times 4 \times 8 + 3 \times 6 \times 12 + 4 \times 8 \times 16 + \dots}{1 \times 3 \times 9 + 2 \times 6 \times 18 + 3 \times 9 \times 27 + 4 \times 12 \times 36 + \dots}}$
- Qn. 5. Suppose that  $f(x + 1, y) = f(x, y) + y + 1$ ,  $f(x, 0) = x$  and  $f(x, y) = f(y, x)$  for all  $x$  and  $y$ . What is the value of  $f(12, 4)$ ?
- Qn. 6. Find all values of  $k$  such that the equation  $\log(kx) = 2 \log(x + 1)$  has exactly one root.
- Qn. 7. Four pupils are standing in front of a closed room and they want to open it. Each makes a statement. One statement is false, and the other three are true.  
Ann: "I do not have the key and Cal does not have the key."  
Ben: "I do not have the key and Ann does not have the key."  
Cal: "I do not have the key and Ben does not have the key."  
Don: "I do not have the key and Ann does not have the key."  
Who actually has the key?
- Qn. 8. What is the last digit  $y$  of the 9-digit number  $19700019y$  given that the number is prime?
- Qn. 9. For any real number  $x$ , let  $[x]$  be the greatest integer less or equal to  $x$  for example if  $x = 4.9$  then  $[x] = 4$ . Let  $m$  and  $n$  be positive real numbers such that  $m \cdot [m] = 17$  and  $n \cdot [n] = 11$ . What is the value of  $m$  and  $n$ ?
- Qn.10. A boy has as many sisters as he has brothers. Each of his sisters has twice as many brothers as she has sisters. How many children are there in the family?

**SECTION B : 10 marks each**

- Qn.11. In our cafeteria, each cake costs an integer number of coins and also each sandwich costs an another integer number of coins. If every girl in the class buys one cake and every boy buys one sandwich, they will spend one coin less than if every boy buys one cake and every girl buys one sandwich. There are more girls than boys in the class. How many more girls are there?
- Qn.12. Lydia has got five real numbers  $a, b, c, d, e$  where  $a < b < c < d < e$ . She takes the sum of each pair of numbers and writes down the ten sums. The three smallest sums are 32, 36 and 37, while the two largest sums are 48 and 51. Determine  $e$ .
- Qn.13. (a) A bricklayer would take 9 hours to build a certain wall on his own and another would take 10 hours to build the same wall. If the two work together, they sometimes stop for a chat so that they lay a total of 10 fewer bricks per hour. If it takes them 5 hours to build the same wall together, what is the total number of bricks in the wall?
- (b) During a recent period of time, 11 days had some rain. A morning rain was always followed by a clear afternoon and an afternoon rain was always preceded by a clear morning. A total of nine mornings and twelve afternoons were clear. How many days had no rain at all?
- Qn.14. Two positive numbers  $x$  and  $y$  are given. If the sum  $x + y$ , the product  $xy$  and the difference  $x^2 - y^2$  are all equal. Find the value of  $y$ .
- Qn.15. Let  $ABCD$  be a rectangle and let  $P$  be a point inside the rectangle. If  $\overline{PA} = 8$ ,  $\overline{PB} = 4$  and  $\overline{PD} = 7$ . What is the value of  $\overline{PC}$ ?

## NMC 2013 O-LEVEL PAPER 2 QUESTIONS

## SECTION A

- Qn. 1. The mathematician Augustus lived his entire life during the 1800's. In the last year of his life he stated: At one time, I was  $x$  years old in the year  $x^2$ . In which year was Augustus born?
- Qn. 2. In the subtraction below  $a, b$  and  $c$  are digits and  $a$  is less than  $b$ . What is the value of  $c$ ?

$$ba - ab = c6$$

- Qn. 3. There are five sheds located on a farm. The distance between shed A and B is 2 km, between B and C is 1.5 km, between C and D is 8 km and between D and E is 3.5 km. Finally shed A is 1 km from shed E. How far is shed B from shed E?
- Qn. 4. If  $ab = 12$  and  $(a + 1)(b + 1) = 23$ , what is the value of  $(a + 2)(b + 2) + 4$ ?
- Qn. 5. Suppose that  $\frac{97}{19} = w + \frac{1}{x + \frac{1}{y}}$  where  $w, x, y$  are all integers. Find the value of  $w + x + y$ .
- Qn. 6. To make one can of orange paint, 3 parts of red paint are mixed with 2 parts of yellow paint and to make one can of green paint, 2 parts of blue paint are mixed with 1 part of yellow paint. If equal volumes of green and orange paints are mixed, What is the proportion of yellow paint in the mixture?
- Qn. 7. A restaurant can accommodate at most 400 people, which includes the guests and the waiters. Each waiter can serve a maximum of 12 guests. What is the greatest number of guests that can be served ?
- Qn. 8. In the garden of Kristin, there are two plants. One is 44 cm high, and it grows 3 cm in each 2 years. The other is 80 cm high, and it grows 5 cm in each 6 years. In how many years will the two plants be equally high?
- Qn.9. James has a three- digit code for a padlock. He has forgotten the code but he knows that if you divide the first digit by the second digit and then square the result, you get the third digit. How many three digit codes have this property?. Show your working.
- Qn.10. Each of the following five statements is either true or false. How many more statements are true given that statement (4) is true.
- (1) Statements (3) and (4) are both true.
  - (2) Statements (4) and (5) are not both false.
  - (3) Statement (1) is true.
  - (4) Statement (3) is false.
  - (5) Statements (1) and (3) are both false.

## SECTION B

- Qn.11. Find the total number of three-digit numbers that are divisible by 9 and contain no even digits.
- Qn.12. On each face of a cube there is a different number from 1 to 6. If one of the faces is at the bottom of the cube, then the sum of the numbers on all the faces except the top and bottom is 13. If another face is at the bottom, then the sum of the numbers on all the faces except the top and bottom is 12. What is the number written on the opposite side of the face labelled 1?
- Qn.13. Abel, Berna, Carol, Dan and Eva play a game of cards. If Berna gives away three cards, she will remain with the same number of cards as Eva. If Berna is instead given three more cards, she will have as many cards as Dan. Abel and Berna together have 10 cards more than the total cards for Dan and Eva. If Berna has two cards more than Carol and the total number of cards is 133, how many cards does Berna have?
- Qn.14. The diagram below is to be completed so that each box contains a whole number. The total of the numbers in the thirteen boxes is 2005 and the sum of the numbers in any three consecutive boxes is always the same. In how many different ways is it possible to complete the diagram in this way?

175												70
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- Qn.15. Let  $a$  and  $b$  be the lengths of the two shorter sides of the right angled triangle.  $C_1$  and  $C_2$  are the circles circumscribing and inscribing the triangle respectively. If the longest side  $D$  of the triangle is the diameter of  $C_1$  and  $d$  is the diameter of  $C_2$ , express  $D + d$  in terms of  $a$  and  $b$ .

## NMC 2012 O-LEVEL PAPER 1 QUESTIONS

## SECTION A

- Qn.1. Given that  $p + q = 0$  and  $p \neq 0$ , determine the value of  $\frac{p^{2012}}{q^{2012}} + 2$ .
- Qn.2. Given that  $3^2 + 4^2 + 5^2 + 12^2 = x^2 + y^2$  is satisfied by only one pair of positive integers  $(x, y)$  with  $x < y$ , determine the value of  $\sqrt{x + y}$ .
- Qn.3. A beginning algebra student mistakenly thinks that the equation  $(x - 2y)^2 = x^2 - 4y^2$  holds for all values of  $x$  and  $y$ . Determine the value of  $\frac{x}{y}$ .
- Qn.4. The number of fish that swam with me is the sum of digits of the largest integer  $x$  which satisfies the inequality  $\frac{x}{x+1} < \frac{2008}{2009}$ . How many fish swam with me?
- Qn.5. Eugene, Godfrey and Fred enjoy different water sports. If Eugene sails, then Godfrey rows. If Eugene swims, Then Fred rows. If Godfrey does not swim, then Fred sails. Who does what?
- Qn.6. A square of perimeter  $y$  is inscribed in a circle of circumference  $x$ . Find the ratio  $x : y$ .
- Qn.7. Without using a calculator, find the value of

$$\frac{1234567890}{(1234567891)^2 - (1234567890 \times 1234567892)}$$

- Qn.8. Let  $P(N)$  and  $S(N)$  denote the product and sum respectively of the digits of a positive integer  $N$ . For example,  $P(63) = 6 \times 3 = 18$  and  $S(63) = 6 + 3 = 9$ . Find all two digit numbers  $N$  for which  $N = P(N) + S(N)$ .
- Qn.9. Triangle  $PQR$  has  $\overline{PQ} = 13$ ,  $\overline{PR} = 15$  and the altitude to side  $QR$  has length 12. Find all possible lengths of the side  $QR$ .
- Qn.10. Determine the maximum possible value of the product of two numbers whose sum is 200.

## SECTION B

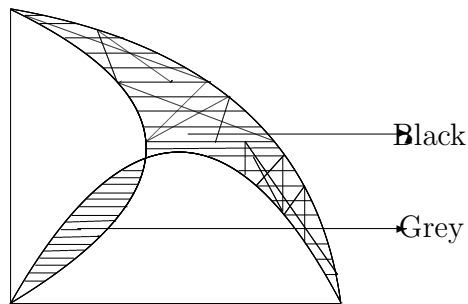
- Qn.11. (a) A jeweler has a 20 grams ring that is 60% gold and 40% silver. He melts it and adds enough gold to make it 80% gold. How many grams of gold should be added?  
 (b) If  $S = 1^2 - 2^2 + 3^2 - 4^2 + \dots + 199^2 - 200^2$ , determine the absolute value of  $S$ .
- Qn.12. (a) Given a function  $F$  such that  $F(0) = 2$  and  $F(x^2 + 1) = [F(x)]^2 + 1$  for all  $x$ , find  $F(5)$ .  
 (b) In Ruti village, one-third of the children can play football, two-thirds can ride a bicycle and one-seventh can play football and ride a bicycle (though not necessarily at the same time). Given that there are fewer than 40 children in Ruti, how many of them can neither play football nor ride a bicycle?

Qn.13. (a) Evaluate  $(1 + \frac{1}{2}) \times (1 + \frac{1}{3}) \times \dots (1 + \frac{1}{2006})$ .

(b) A bottle and a glass together hold the same amount of water as a jug. A bottle holds the same amount of water as a glass and a tankard together. Three tankards hold the same amount as two jugs. How many glasses would one tankard hold?

Qn.14. In a town, 10% of the dogs think that they are cats and 10% of the cats think that they are dogs. All the other cats and dogs are perfectly normal. When all the cats and dogs in a town are subjected to a test, 20% of the animals thought that they were cats. What percentage of them really were cats?

Qn.15. The digram shows a quarter circle with center  $O$  and two semicircles with diameters  $OA$  and  $OB$ . Calculate ratio of the area of the region shaded gray to the area of the region shaded black.



## NMC 2012 O-LEVEL PAPER 2 QUESTIONS

SECTION A

- Qn.1 What is half of  $2^{2012}$ ?
- Qn.2 The function  $f(x)$  satisfies the equation  $(x - 1)f(x) + f(\frac{1}{x}) = 1$  for all  $x \neq 0$ . Find the value of  $f(2)$ .
- Qn.3 Two fractions are removed from the following six fractions  $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \frac{1}{18}$  so that the sum of the remaining four fractions is  $\frac{2}{3}$ . What is the product of the two fractions that were removed?
- Qn.4 John takes three steps to walk the same distance as Jim walks in four steps. Each of John's steps covers 0.5 metres. How many metres does Jim travel in 24 steps?
- Qn.5 In the diagram below, the numbers 1 to 10 are placed around a circle. Amina crosses out 1, then 4 and then 7. Counting in a clockwise direction, she crosses out every third number of those remaining, until only two numbers are left. What is the sum of these two numbers?
- Qn.6 Find all the real numbers  $x$  for which the property that the median of five numbers  $x, 6, 4, 1, 9$  is equal to their mean.
- Qn.7 John and Joseph select T-shirts from a boutique. John only selects the ones with two colours and Joseph only selects the ones with seven colours. John has 5 T-shirts more than Joseph. The total number of colours found on all the selected T-shirts is 100. How many T-shirts did they select all together?
- Qn.8 Without using tables or a calculator, find the last two digits of  $123456789 \times 98765342 \times 104392831$
- Qn.9 Johann tells Kitty that the product of three positive integers is 36. Jonathan also tells Kitty the sum of the integers. However, on solving, Kitty found that the set of integers is not unique. What is the sum of these integers?
- Qn.10 In the diagram below, PX is a tangent of a circle at A. PY is a tangent of the circle at B and CD is a tangent of the circle at E. If  $\overline{PA} = 15$  units, find the perimeter of triangle PCD.

SECTION B

- Qn.11 Two men Liam and Award went for a drink. Award took a cup filled with water and drank  $\frac{1}{5}$  of its contents. When he looked aside, Liam re-filled the cup by adding alcohol and then stirred. Award drank  $\frac{1}{4}$  of the mixture. When Award again looked aside, Liam re-filled the cup with more alcohol and stirred. Award drank  $\frac{1}{3}$  of the mixture. When Award looked aside for the third time, Liam re-filled the cup with more alcohol. What percentage of the final mixture was alcohol?
- Qn.12 Each of the 9 squares shown in the figure below is to contain one number chosen from numbers 1 to 9. No number is to be repeated. Suppose the sum of the numbers in the 5 squares aligned vertically is 32 and the sum of the numbers in the horizontal squares is 20. What number goes in the shared corner square?

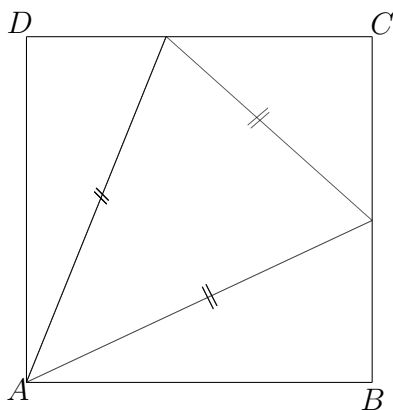
- Qn.13 David and Daniel are running at constant speeds in opposite directions around a circular track. David completes one lap every 56 seconds and meets Daniel every 24 seconds. How long does it take Daniel to complete one lap?
- Qn.14 A point  $(x, y)$  is called an integral point if both  $x$  and  $y$  are integers. Find all the points on the graph of  $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$  that are integral points. For example  $(2, -4)$  is one such point.
- Qn.15 (a) Expand  $(x - a)(x - b)(x - c) \dots (x - z)$ .
- (b) Modify the right angled triangle with sides 11, 60 and 61 by doubling the length of its shortest side but keeping its perimeter and largest angle unchanged. What is the length of the new triangle's hypotenuse?

NMC 2011 O-LEVEL PAPER 1 QUESTIONS

SECTION A

Qn.1. The real numbers  $x, y$  and  $z$  satisfy  $2^{x+y} = 10$ ,  $2^{y+z} = 20$  and  $2^{x+z} = 30$ . Find the value of  $2^x$ .

Qn.2. An equilateral triangle with sides of  $10\text{cm}$  is inscribed in a square  $ABCD$  in such a way that one vertex is at  $A$ , another vertex on  $\overline{BC}$  and one on  $\overline{CD}$ . Find the area of the square.



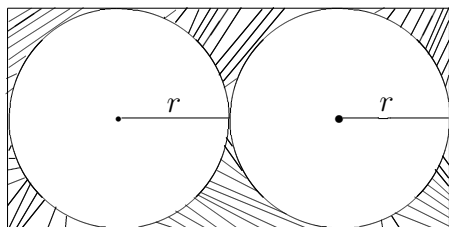
Qn.3. A prime-prime is a prime number that yields a prime when its units digit is omitted. (For example, 317 is a three-digit prime-prime because 317 is prime and 31 is prime). How many two-digit prime-primes are there?

Qn.4. Find the non-zero value of  $k$  such that the equation  $x^2 + kx + k = 0$  has only one real solution.

Qn.5. Let  $\Theta$  be defined by  $x\Theta y = x^2 - y$  and  $\oplus$  be defined by  $x \oplus y = x + y^2$ . What is the value of  $(x \oplus y)\Theta y$ ?

Qn.6. Joshua was three times as old as Janet in 1993. In 2001, Joshua was 8 years older than Janet. How old is Janet now?

Qn.7. Find the area of the shaded region if each circle has radius  $r$ .



Qn.8. Given that  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = y$ . Find the value of  $\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \frac{1}{80} + \dots$  in terms of  $y$ .

- Qn.9. The last pan cake is gone, and either Chris, Pat, Lou, Sam or Sam's dog ate it. The following is known. Each one (except Sam's dog) said they knew who ate the pan cake, Chris always tells the truth, Pat never tells the truth, Sam said it was either Pat, Chris or Lou. Lou said it wasn't me, Pat claimed Sam was a liar and Chris responded that Pat was the only liar. Who ate the last pan cake?
- Qn.10. A shopper intends to buy 3 apples and 2 oranges and will pay 5,000/ =. Changing his mind, he exchanges an orange for another apple and has to pay an additional 1,000/ =. What is the price of a single apple?

**SECTION B : 10 marks each**

- Qn.11. Two numbers are written in base  $a$  as 32 and 24. The same two numbers are written in base  $b$  as 43 and 33, respectively. What is the sum of these two numbers in base 5?
- Qn.12. Let  $a$  and  $b$  be different real numbers such that  $a^2 = 6b + 5ab$  and  $b^2 = 6a + 5ab$ , find the value of  $ab$ .
- Qn.13. If the 14 blocks below must be filled with whole numbers so that the sum of three consecutive blocks has a total of 18, find  $x$ .

			7				x				8		
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- Qn.14. A driver wishes to arrive at her destination at exactly 11 : 00 *a.m.* If she drives at 30 *kmh*, she would get there at 10 : 00 *a.m.* If she drives at 20 *kmh*, she would arrive at noon. How fast should she drive to arrive at 11 : 00 *a.m.*?
- Qn.15. Find the sum of all positive values of  $x$  for which

$$(\log_{27} x^3)^2 = \log_{27} x^6.$$

## NMC 2011 O-LEVEL PAPER 2 QUESTIONS

## SECTION A : 5 marks each

- Qn.1. Without using a calculator, evaluate,  $\sqrt{170^2 - 80^2}$ .
- Qn.2. Nancy usually leaves her cell phone ON. If her cell phone is ON but she is not using it, the battery will last for 24 hours. If she is using it constantly, the battery will last for only 3 hours. Since the last recharge, her cell phone has been ON for 9 hours, and during that time, she used it for 60 minutes. If she does not talk anymore but leaves the cell phone ON, how many more hours will the battery last?
- Qn.3. Find the product of all values  $x$  which satisfy the equation  $16^{x^2+3x-1} = 8^{x^2+3x+2}$ .
- Qn.4. If  $f\left(\frac{2x-1}{2-x}\right) = 2x$ , find  $f(4)$ .
- Qn.5. A driver travels from Mbale to Masaka for the first time and keeps stopping to take a closer look at any interesting scenery and ended up driving at an average speed of  $50 \text{ km/h}$ . On her way back she travels at an average speed of  $80 \text{ km/h}$ . What was her average speed for the whole journey?
- Qn.6. A quadrilateral has these properties: three of its angles are congruent to one another, no set of three vertices are co-linear, and its largest angle is three times as large as its smallest one. What is the value of the smallest angle in degrees?
- Qn.7. Hasnah, Robinah, and Jones each took the same 7- question True/False quiz. Hasnah and Robinah each got 6 questions right. Their answers were: *FFTTTTF* for Hasnah and *TFFTTTF* for Robinah. When Jones took the same quiz, his answers were *TTFFTTT*. How many answers did Jones get right?
- Qn.8. A triangle  $ABC$  has sides of lengths:  $\overline{AB} = \overline{BC} = \sqrt{3} \text{ cm}$  and  $\overline{BM} = \sqrt{(3/4)} \text{ cm}$  where  $M$  is the mid point of  $\overline{AC}$ . Find the value of the smallest angle of triangle  $ABC$  and the length of side  $\overline{AC}$ .
- Qn.9. A teacher asks each student in her class to draw a rectangle whose sides are whole numbers and having a perimeter of 502 cm. When marking her students' work, she discovers that the students drew all the different possible rectangles. How many different (distinct) rectangles did the students come with and what is the difference between the largest and smallest possible areas of the rectangles?
- Qn.10. In the magic square below, the sum of the numbers in each row, each column, and each diagonal is the same. Find the value of  $B$ .

19	A	14
10	B	C
D	E	11

## SECTION B : 10 marks each

- Qn.11. (a) Find a single percentage discount, that is equivalent to two successive discounts of 10% followed by 20% on a price of a commodity.

- (b) How many right angled triangles whose all sides are positive whole numbers, have the property that the area is numerically equal to it's perimeter?
- Qn.12. (a) A haunted house has six windows. In how many ways can the ghost enter the house by one window and leave by a different window.
- (b) John ordered for 4 pairs of black socks and some additional pairs of blue socks. The price of the black socks per pair is twice that of the blue socks. When the order was received by John, he found out that the number of pairs of the two colors had been interchanged. This interchange led to an increase in the bill of 50%. What is the ratio of the number of pairs of the black socks to the number of pairs of the blue socks in the original order.
- Qn.13. (a) Find the distance between the two points where the graph  $4x^2 + 4y^2 - 16y = 9$  crosses the  $y$ -axis.
- (b) Okello travels from point  $P$  to point  $Q$  in 8 minutes. Mary travels from  $Q$  to  $P$  along the same route. They start at the same time and each travels at a constant rate. If Mary reaches point  $P$  , 18 minutes after they both meet, how many minutes did the entire trip take Mary?
- Qn.14. A triangle  $ABC$  is such that  $\overline{AB} = 20 \text{ cm}$ ,  $\overline{AC} = 12 \text{ cm}$  and  $\angle ACB = 90^\circ$ . A point  $E$  on  $\overline{CB}$  is a perpendicular bisector of line  $\overline{AB}$  and bisects  $\overline{AB}$  at a point  $D$ . Find the area of the quadrilateral  $ADEC$ .
- Qn.15. (a) Without using a calculator, find the value of  $(16)^{0.32} \cdot (256)^{0.09}$ ?
- (b) A rectangle is partitioned into four smaller rectangles, three of which have areas (in  $\text{cm}^2$ ), 15, 21 and 32 as shown in the figure below. What is the area of the entire rectangle?

15	21
32	

## NMC 2010 O-LEVEL PAPER 1 QUESTIONS

## SECTION A

Qn.1. What is the value of;

$$\log_5 \left( \frac{(125)(625)}{25} \right)?$$

Qn.2. Two non-zero real numbers,  $a$  and  $b$  satisfy the relationship  $ab = a - b$ , find the value of

$$\frac{a}{b} + \frac{b}{a} - ab?$$

Qn.3. An entertainment group consists of boys and girls. When 15 girls leave the group, the ratio of boys to girls remaining is 2 : 1. Later if 45 boys leave, the ratio of boys to girls changes to 1 : 5. Find the number of girls in the group at the beginning?

Qn.4. The total in-store price for an appliance is UGX 200,000. A television commercial advertises the same product for three easy payments each more than a quarter of the in-store price by UGX 5,000 and a one time transportation and handling charge of UGX 15,000. What percentage of the total in-store price is saved by buying the appliance from the television advertiser?

Qn.5. At the end of 1994, Walter was half as old as his grand father. The sum of the years in which they were born is 3838. How old will Walter be at the end of 2010?

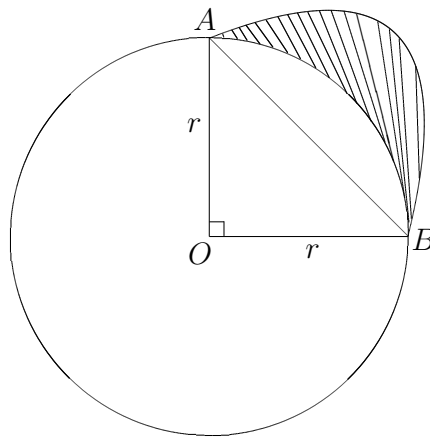
Qn.6. Each day, Jesse ate 20% of the mangoes that were in his fridge at the beginning of that day, at the end of the second day, 32 remained. How many mangoes were in the fridge originally?

Qn.7. Find the sum of all prime numbers between 1 and 100 that are both 1 greater than a multiple of 4 and 1 less than a multiple of 5?

Qn.8. Find the area of the largest triangle that can be inscribed in a semi-circle whose radius is  $r$ ?

Qn.9. A man buys a house for 20,000,000 UGX and rents it out. He puts  $12\frac{1}{2}\%$  of each months rent aside for repairs and up keep, pays 650,000 UGX a year in taxes and realizes  $5\frac{1}{2}\%$  on his investment. What is the monthly house rent?

Qn.10. The circle below has center  $O$ ,  $\overline{OA}$ , and  $\overline{OB}$  are radii and  $\angle AOB$  is a right angle. A semi circle is constructed using segment  $AB$  as it's diameter as shown above. The shaded portion of the semi circle outside circle  $O$  is called a *lune*. What is the ratio of the area of the lune to the area of the triangle?



**SECTION B : 10 marks each**

- Qn.11. (a) Okello can do a piece of work in 9 days. Oryem is 50% more efficient than Okello. Find the number of days it takes Oryem to do the same piece of work?  
 (b) If the width of a rectangle is increased by 10% and the area of the rectangle remains constant. Find the percentage by which the length would have been decreased in order to achieve this?

Qn.12. Two people  $D$  and  $M$  start at the same time to drive from Kampala to Masaka, 60 miles away.  $D$  travels 4 miles an hour slower than  $M$ .  $M$  reaches Masaka and at once turns back meeting  $D$  12 miles from Masaka. Find the speed of  $M$ .

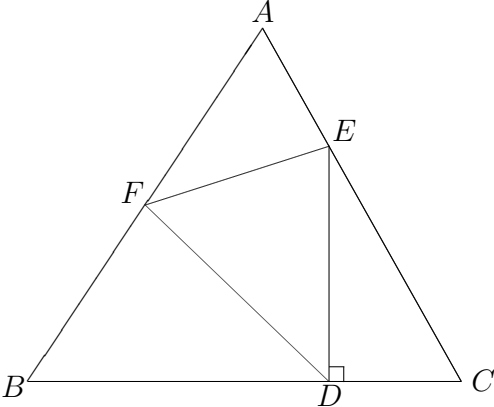
Qn.13. Let  $E(n)$  denote the sum of even digits of  $n$  for example  $E(5681) = 6 + 8 = 14$ . Find the value of

$$E(1) + E(2) + E(3) + \dots + E(100)?$$

Qn.14. The digits 2, 4, 5, 6, 8 and 9 can be distributed among the lettered squares in the array so that the sum of the entries on each of the rows and columns is the same number  $K$ . Identify all the lettered digits and hence find the value of  $K$ ?

7	$a$	$b$	1
$c$			$d$
3	$e$	$f$	10

Qn.15. Equilateral triangle  $DEF$  is inscribed in equilateral triangle  $ABC$  as shown below with  $\overline{DE} \perp \overline{BC}$ . Find the ratio of the area of  $\triangle DEF$  to the area of  $\triangle ABC$ ?



## NMC 2010 O-LEVEL PAPER 2 QUESTIONS

## SECTION A

Qn.1. If  $\frac{a}{b+c+d} = \frac{4}{3}$  and  $\frac{a}{b+c} = \frac{3}{5}$ . What is the value of  $\frac{d}{a}$ ?

Qn.2. Alice and Bob play a game involving a circle whose circumference is divided into 12 equally spaced points. The points are numbered clockwise from 1 to 12. Both start on point 12. Alice moves clockwise and Bob moves counterclockwise, In a turn of the game, Alice moves 5 points and Bob moves 9 points. The game ends when they stop on the same point. How many turns will this take?

Qn.3. If  $1 + x + x^2 + x^3 = 0$ , What is the value of  $1 + x + x^2 + x^3 + \dots + x^{2010}$ ?

Qn.4. Suppose that  $f(x) = ax + b$ , given that

$$f(f(f(x))) = 8x + 21$$

Determine the value of  $a + b$ .

Qn.5. George's company was losing money, as a result George received a pay cut of 25%. By what percentage must his new pay rate be raised to bring it back to the original level?

Qn.6. Find the value of  $x$  if

$$(\sqrt{3})^x \cdot (\sqrt{27})^x = \sqrt{3}$$

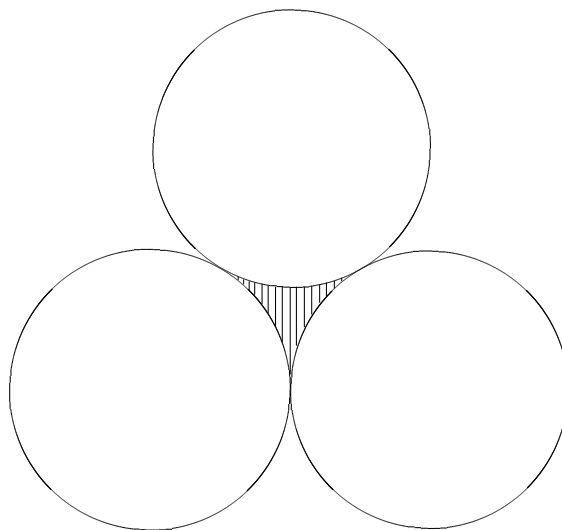
Qn.7. Find the value of  $2x + 3y$  given that  $x + y = 6$  and  $x^2 + 3xy + 2y^2 = 60$ .

Qn.8. Find the sum and product of all the solutions to the equation

$$2 \log_{10} x - \log_{10} 2x = 1.$$

Qn.9. Mark had an average score of 85 on his first ten quizzes, and 90 on his fourteen quizzes in total. What is the average score of his last 4 quizzes?

Qn.10. Three circles of radius 1 meter are mutually tangent as shown in the figure. What is the area of the shaded region?

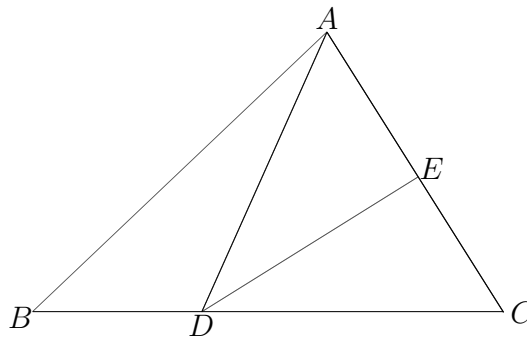


**SECTION B : 10 marks each**

- Qn.11. (a) If the letters  $a, A, b, B, c$  and  $C$  are arranged at random in a row. What is the probability that the lower case letters appear in an increasing alphabetical order?
- (b) Suppose a rectangle has an area of 3 and a diagonal of length  $\sqrt{10}$ . What is its perimeter?
- Qn.12. James and Ann leave a camp site. James biking due north and Ann biking due east. James bikes  $7\text{km/hr}$  faster than Ann. After three hours, they are  $51\text{km}$  apart. Find the sum of their speeds.
- Qn.13. Juma can run around a circular track in 40 seconds, David, running in the opposite direction meets Juma every 15 seconds. What is the average of the times David and Juma take to run around the track, expressed in seconds.
- Qn.14. A student on vacation for  $d$  days observed the following
- it rained 7 times, morning or afternoon
  - when it rained in the afternoon, it was clear in the morning
  - there were 5 clear afternoons
  - there were 6 clear mornings

Using this information, find the number of days the student was on vacation?

- Qn.15. In the figure below,  $\overline{AB} = \overline{AC}$ , angle  $BAD = 30^\circ$ , and  $\overline{AE} = \overline{AD}$ , find the value of the angle  $EDC$ .



**NMC 2014 O-LEVEL PAPER 1 SOLUTIONS**  
**SECTION A**

1.

$$\frac{(-a)^3 - \left(\frac{1}{a}\right)^{-2}}{-a^2 + \left(-\frac{1}{a}\right)^{-3}} = \frac{(-a)^3 - \frac{1}{\left(\frac{1}{a}\right)^2}}{-a^2 + \frac{1}{\left(-\frac{1}{a}\right)^3}} = \frac{-a^3 - a^2}{-a^2 - a^3} = \frac{-a^3 - a^2}{-a^3 - a^2} = 1.$$

2. The left hand side can be factorized to give  $(2^x - 3^y)(2^x + 3^y) = 55$  so either  $2^x - 3^y = 1$  and  $2^x + 3^y = 55$  or  $2^x - 3^y = 5$  and  $2^x + 3^y = 11$ . Solving simultaneously the first two equations gives  $2^x = 28$  and  $3^y = 27$ , which does not have an integer solution for  $x$ . The second possibility gives  $2^x = 8$  and  $3^y = 3$ , which has one integer solution  $(x, y) = (3, 1)$ .
3. *BDAI*. The rightmost position is  $4 + 4 = 8 = I$ . The next position is  $13 + 13$ , for which we enter  $0 = A$  and carry 1. The next position is  $1 + 14 + 14$ , for which we enter  $3 = D$  and carry  $1 = B$  to the leftmost position.
4. Let the numbers be  $x$  and  $y$ .

$$\text{Harmonic mean of } x \text{ and } y = \frac{1}{\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)} = \frac{4xy}{2x + 2y} = \frac{2xy}{x + y}$$

$$6 = \frac{2 \times x \times 4}{x + 4}$$

$$\begin{aligned} 6x + 24 &= 8x \\ x &= 12. \end{aligned}$$

5. Let  $A$  represent the number of adults while  $C$  represent the number of Children. Then  
Initially,  $\frac{C}{A} = \frac{2}{3} \implies C = \frac{2}{3}A$ .  
After the bus load, we have  $\frac{C+30}{A} = \frac{3}{2} \implies C + 30 = \frac{3}{2}A$ .  
Substituting for  $C$  in the equation gives

$$\frac{2}{3}A + 30 = \frac{3}{2}A$$

$$\frac{5A}{6} = 30 \implies A = 36.$$

Therefore  $C = \frac{2}{3} \times 36 = 2 \times 12 = 24$ . Thus the number of people at the party before the bus arrived are  $24 + 36 = 60$ .

6. From pages 1 through 9, we use 9 digits. From page 10 through 99, we use  $((99 - 10) + 1) \times 2 = 90 \times 2 = 180$  digits which makes a total of  $9 + 180 = 189$  digits for pages 1 through 99. That leaves  $852 - 189 = 663$  digits remaining to be used. These are obtained by going  $\frac{663}{3} = 221$  pages beyond page 99 which makes up to page  $99 + 221 = 320$ . Therefore there are 320 pages of the book.
7. Let the length of the route be  $d$  km. Then  $B$  jogs  $\frac{d}{2}$  km at 6 km/h and runs  $\frac{d}{2}$  km at 12 km/h. Since the total time taken was  $x$  hours, then

$$x = \frac{\text{Distance}}{\text{Speed}} = \frac{\frac{d}{2}}{6} + \frac{\frac{d}{2}}{12} = \frac{d}{12} + \frac{d}{24} = \frac{2d + d}{24} = \frac{3d}{24} = \frac{d}{8}.$$

A walks  $\frac{d}{3}$  km at 5 km/h and runs for  $\frac{2}{3}d$  km at 15 km/h.

$$y = \frac{\text{Distance}}{\text{Speed}} = \frac{\frac{d}{3}}{5} + \frac{\frac{2d}{3}}{15} = \frac{d}{15} + \frac{2d}{45} = \frac{d}{9}.$$

Thus

$$\frac{x}{y} = \frac{d}{8} \div \frac{d}{9} = \frac{d}{8} \times \frac{9}{d} = \frac{9}{8}.$$

8. Note that all three digits must be either 0, 1, 2, 3 or 4. Further more,  
 $TEEN_5 = 125T + 25E + 5E + N = 125T + 30E + N$   
 $TEEN_7 = 343T + 49E + 7E + N = 343T + 56E + N$   
 $TEEN_8 = 512T + 64E + 8E + N = 512T + 72E + N$   
 Therefore,  $125T + 30E + N + 343T + 56E + N = 512T + 72E + N$ , from which  $44T - 14E = N$ .  
 Suppose  $T = 1$ , then  $E = 3$  and  $N = 2$ . If  $T = 2, 3$  or  $4$ , then  $N$  must be greater than 4 for  $E$  to be less than 5 which would be a contradiction. Hence  $N = 2$ .
9. In order to have the smallest number of full boxes, we must have as many boxes of 12 laptops as possible. Suppose the laptops are packed in the boxes of 12, then 167 boxes can be used and 1 laptop remains. This is not the right choice as 1 laptop cannot fill any box. If 166 of 12 are used, then  $1 + 12 = 13$  laptops will remain and these cannot fill up the entire box. If 165 of 12 are used, then  $1 + 12 + 12 = 25$  laptops will remain and these can fill up to 5 boxes that contain 5 laptops each. Thus, the minimum number of full boxes are 165 big boxes + 5 small boxes = 170 full boxes.
10. Perimeter of  $\triangle PST = PS + ST + PT = PS + (SU + UT) + PT = PS + SQ + TR + PT = (PS + SQ) + (PT + TR) = PQ + PR = 19\text{cm} + 17\text{cm} = 36\text{cm}$ .

## SECTION B

11.

$$\begin{aligned} 2\log(m - 2n) &= \log m + \log n \\ \log(m - 2n)^2 &= \log mn \implies m^2 - 4mn + 4n^2 = mn \end{aligned}$$

$$\frac{m^2}{n^2} - 5\frac{m}{n} + 4 = 0 \implies \left(\frac{m}{n}\right)^2 - 5\left(\frac{m}{n}\right) + 4 = 0.$$

$$y^2 - 5y + 4 = 0 \text{ where } \frac{m}{n} = y.$$

$$y = \frac{+5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2} = 4 \text{ or } 1.$$

Suppose  $y = 1$ , then  $\frac{m}{n} = 1 \implies m = n$ . This means that  $m - 2n = n - 2n = -n$  which contradicts the fact that  $m$  and  $n$  are non-negative real numbers. Therefore  $y$  cannot be equal to 1. Suppose  $y = 4$ , then  $\frac{m}{n} = 4 \implies m = 4n$ . Hence  $m - 2n = 4n - 2n = 2n > 0$ . Hence the ratio  $m : n = 4 : 1$ .

12. Let  $a, b, c$  denote the respective number of days it would take Alex, John and Jimmy to complete the job if he was working alone. Then  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$  represents the fractions of the job done by each in one day. Since it takes Alex and John together 2 days to do the whole job, it implies they both do half the job in one day. We thus have the following equations.

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} \tag{1}$$

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{4} \tag{2}$$

$$\frac{1}{a} + \frac{1}{c} = \frac{5}{12} \tag{3}$$

Subtracting Equation (2) from Equation (1) gives  $\frac{1}{a} - \frac{1}{c} = \frac{1}{4}$ . This added to Equation (3) gives  $\frac{2}{a} = \frac{8}{12} \implies 8a = 24$  so that  $a = 3$ . Therefore it would take Alex 3 days to do the job alone.

13. For each blueberry in the fruit salad there are 2 raspberries, 8 cherries and 24 grapes. Thus there are  $1 + 2 + 8 + 24 = 35$  pieces of fruit for each blueberry. Because  $280 = 35 \times 8$ , it follows that there are a total of 8 blueberries,  $8 \times 2 = 16$  raspberries,  $8 \times 8 = 64$  cherries, and  $8 \times 24 = 192$  grapes in the fruit salad. Thus there are 64 cherries.

14. Let  $H$  be on the line  $AB$  such that  $EH$  is perpendicular to  $AB$  and  $I$  be the intersection of  $EH$  and  $DC$ . Then  $FG = 2\text{cm}$ . Since  $\triangle ABE$  is similar to  $\triangle FGE$ , we have that  $\frac{FG}{AB} = \frac{EI}{EH} = \frac{EI}{EI+IH}$ . Thus  $\frac{2}{5} = \frac{EI}{EI+3}$  so that  $EI = 2$ . Hence  $EH = EI + IH = 2 + 3 = 5$ . The area of

$$\triangle ABE = \frac{1}{2}AB \times EH = \frac{1}{2} \times 5 \times 5 = \frac{25}{2} = 12.5\text{cm}^2.$$

15. The sum of odd numbers from 1 to 17 is  $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81$ . Since the rows and columns have the same sum, the sum in any row or column is  $\frac{81}{3} = 27$ .

	1	
5	$p$	13
$x$	$q$	$r$

From the magic square, we have

$$5 + p + 13 = 27 \implies p = 27 - 18 = 9.$$

$$1 + p + q = 27 \implies q = 27 - 1 - 9 = 17.$$

$$x + q + r = 27 \implies x + r = 27 - 17 = 10.$$

Since  $x + r = 10$ , then either  $x = 3$  or  $r = 7$  or  $x = 7$  and  $r = 3$ . Suppose  $x = 3$ , then we would have to use 7 in the bottom right square which would require another 7 in the top right square. This eliminates  $x = 3$  since it would require that we use 7 twice in the magic square. Therefore the value of  $x$  is 7.

## NMC 2014 O-LEVEL PAPER 2 SOLUTIONS

## SECTION A

**Qn1.**  $k \div 2 \div 2 \div 2 \div 2 = 1 \implies k \div 16 = 1$  so that  $k = 16$ . Also  $n \div 2 \div 2 \div 2 = 1 \implies n \div 8 = 1$  so that  $k = 8$ . Therefore  $k + n = 16 + 8 = 24$ .

**Qn2.**  $(2014)^2 - (2)(2014)(2013) + (2013)^2 = (2014 - 2013)^2 = 1^2 = 1$ .

**Qn3.** For the average of two digits to be 5, their sum must be 10. The two-digit positive integers whose digits sum to 10 are 19, 28, 37, 46, 55, 64, 73, 82, 91 of which there are 9.

**Qn4.** If the child's age is  $x$ , then  $x+3 = (x-3)^2$ , which simplifies to  $x^2 - 7x + 6 = 0$  or  $(x-1)(x-6) = 0$ , so  $x = 1$  or  $x = 6$ . However, if  $x = 1$ , then  $x - 3 = -2$ , which is not the square root of 4, since the child's must be positive. Thus  $x = 6$  implying that the child's age is 6 years.

**Alternatively,**

By trial and error, the age is 3 less than a perfect square, so the only possibilities for a child are 1 or 6 or 13 and 6 is the only one that satisfies the other condition.

**Qn5.** The number 531 is the 1<sup>st</sup> number counted when proceeding backwards from 531 to 3. In turn, 530 is the 2<sup>nd</sup> number, 529 is the 3<sup>rd</sup> number and  $x$  is the  $(531 - x + 1)^{th}$  number. Therefore 126 is the  $(532 - 126)^{th}$  number, which is the 406<sup>th</sup> number.

**Qn6.** To avoid having ten balls of the same colour, Tacia can remove up to nine of each of the colours red, green, blue, plus the five other balls, making a maximum total of 32 balls. Thus in order to ensure ten of the same colour, she needs to remove  $32 + 1 = 33$  balls.

**Qn7.** Let  $b$  represent baskets and  $m$  represent missed. Then  $8b - 5m = 0$  and  $b + m = 26$ . Solving simultaneously for  $b$  and  $m$  gives  $b = 10$ . Thus he had 10 successful throws.

**Qn8.** Let  $x$  be the number of items that he received, then

$$\frac{480}{x-2} - \frac{480}{x} = 1$$

This simplifies to  $960 = x(x-2)$ , yielding  $x = 1 + \sqrt{961} = 1 + 31 = 32$ .

**Alternatively**

Let  $x$  be the number of items that she would have received if the vendor had not given her extra two items, then

$$\frac{480}{x} - \frac{480}{x+2} = 1$$

This simplifies to  $960 = x(x+2)$ , yielding  $x = \frac{-2 \pm \sqrt{3844}}{2} = -2 \pm 32 = 30$  since the negative value is ignored as she did not buy negative items. Therefore including the two items, Mary received  $x + 2 = 30 + 2 = 32$  items.

**Qn9.**  $73961a$  is divisible by 3 if  $7+3+9+6+1+w$  is a multiple of 3. This implies that the possible values of  $w$  are 1, 4 or 7. Since  $73961w$  divided by 5 has a remainder of 4, then  $w$  is either  $0+4=4$  or  $5+4=9$ . Since  $739619$  is not a multiple of 3, it implies  $w$  must be 4.

**Qn10.** Consider  $x$  and  $y$  as points on the real number line, with  $x$  necessarily to the right of  $y$ . Then  $x-y$  is the distance between  $x$  and  $y$ . Chantal's rounding moved  $x$  to the right and moved  $y$  to the left. Therefore the distance between them increased, and her estimate is larger than  $x-y$ .

To see that the other answer choices are not correct, let  $x = 2.9$  and  $y = 2.1$ , and round each by 0.1. Then  $x-y = 0.8$  and Chantal's estimated difference is  $(2.9+0.1) - (2.1-0.1) = 1.0$ .

### SECTION B

**Qn11.** Suppose that  $n$  has digits  $AB$ . Then  $n = 10A + B$ . The average of the digits of  $n$  is  $\frac{A+B}{2}$ . Putting a decimal point between the digits of  $n$  is equivalent to dividing  $n$  by 10, so the resulting number is  $\frac{10A+B}{10}$ . So we want to determine  $A$  and  $B$  so that

$$\frac{A+B}{2} = \frac{10A+B}{10}$$

$$10A+B = 5(A+B)$$

$$5A = 4B$$

Since  $A$  and  $B$  are digits such that  $5A = 4B$ , then  $A = 4$  and  $B = 5$  is the only possibility. Therefore,  $n = 45$ . (We can quickly check that the average of the digits of  $n$  is 4.5, the number obtained by putting a decimal point between the digits of 45.)

#### Alternatively,

When we compute the average of two digits, the result is either an integer or a half-integer (that is a decimal number of the form  $x.5$ ). Therefore, the possible averages are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0, 8.5, 9.0. (Note that 0.0 is not possible since the smallest two-digit whole number is 10 and the average of digits 1 and 0 which gives the smallest average of the digits is 0.5). From this list, the only one equal to the average of the two digits forming it is 4.5. Therefore  $n = 45$  (formed by removing the decimal point from 4.5).

**Qn12.** Since one of Bob and Carol had at least 22 questions answered correctly, Alice must have had at least 23. Thus, the three of them together had at least  $43 + 23 = 66$  questions answered correctly. Thus Don answered at most  $67 - 66 = 1$  and since everyone had exactly one correct answer, he had exactly one correct answer.

**Qn13.** Suppose that Fiona bought  $x$  toy bicycles and  $y$  toy helicopters. Then  $19x + 17y = 201$ . Since  $x$  and  $y$  are non-negative integers, then  $19x \leq 201$  so  $x \leq 10$ . Suppose  $x = 10$ , then  $17y = 201 - 19x = 11$ , which does not have an integer solution because 11 is not divisible by 17.

Suppose  $x = 9$ , then  $17y = 201 - 19x = 30$ , which does not have an integer solution.

Suppose  $x = 8$ , then  $17y = 201 - 19x = 49$ , which does not have an integer solution.

Suppose  $x = 7$ , then  $17y = 201 - 19x = 68$ , so  $y = 4$ . Therefore,  $19(7) + 17(4) = 201$ , and so Fiona buys 7 toy bicycles and 4 toy helicopters.

(We can check that  $x = 0, 1, 2, 3, 4, 5, 6$  do not give values of  $y$  that work.)

**Qn14.** Let the tens digit be  $t$  and the units digit be  $u$ , so  $t - u = \pm 4$ . Also  $1 \leq t \leq 9$  and  $0 \leq u \leq 9$ . (Note that the tens digit cannot be zero.) If  $t = u + 4$ , then  $0 \leq u \leq 5$  which gives six numbers. If  $t = u - 4$ , then  $5 \leq u \leq 9$ , giving another five numbers, for a total of 11.

**Qn15.** Looking at the middle column of rectangles, each has the same width, so the ratio of their areas equals the ratio of their heights. Thus,  $AB : BC = 1 : 2$ . Looking at the rectangles in the first column, the area of the middle rectangle must be twice the area of the top rectangle, or 6. Thus,  $BC : CD = 6 : 5$  by the reasoning above. So, looking at the third column,  $R : 10 = 6 : 5$  or  $R = 12$ .

	x	y	z
A	3	1	
B		2	$R$
C			10
D	5		

**Alternatively**

Let the width of the first column be  $x$ .

Since the area of the top left rectangle is 3, the height of the first row is  $\frac{3}{x}$ .

Since the area of the bottom left rectangle is 5, the height of the third row is  $\frac{5}{x}$ .

Since the height of the first row is  $\frac{3}{x}$  and the area of the top middle rectangle is 1, the width of the middle column is  $\frac{3}{x}$ . Thus, the height of the middle row is  $\frac{6}{x}$ , since the area of the middle rectangle is 2.

Since the height of the third row is  $\frac{5}{x}$  and the area of the bottom right rectangle is 10, then the width of the third column is  $2x$ .

Since the rectangle labelled  $R$  has height  $\frac{6}{x}$  and width  $2x$ , then it has area 12.

**Alternatively**

We label some of the lengths in the diagram.

	x	y	z
a	3	1	
b		2	$R$
c	5		10

From the given information,  $ax = 3, ay = 1, by = 2, bz = R, cx = 5$  and  $cz = 10$ . We want to determine  $bz$ . But  $bz = \frac{(ax)(by)(cz)}{(ay)(cx)} = \frac{(3)(2)(10)}{(1)(5)} = 12$ , so  $R = 12$ .

NMC 2013 O-LEVEL PAPER 1 SOLUTIONS

SECTION A : 5 marks each

- $y = \frac{2}{x}, y = \frac{24}{z}, z = \frac{3}{x}$ . This means  $\frac{24}{z} = \frac{2}{x} \implies z = 12x$ . Also  $12x = \frac{3}{x} \implies x^2 = \frac{1}{4}$ .  
 $x = \pm\frac{1}{2}$ . This then implies that  $y = 4$  and  $z = 3 \times 2 = 6$ .  
 Therefore  $x + y + z = \frac{1}{2} + 4 + 6 = 10\frac{1}{2}$ .
- Since  $AB$  is prime,  $B$  is either 1 or 3. Also  $A + C \leq 11$  so  $E = 1$ . It then follows that  $B = 3$ .  
 Now, since  $D \leq 6$ ,  $B + D < 10$  and  $A + C = 10$ . We deduce that  $A$  and  $C$  must be 4 and 6 in some order but since 63 is not prime,  $AB = 43 \implies A = 4$ .  
 Therefore  $A + B = 4 + 3 = 7$
- Since  $a$  and  $b$  are the roots of the equation, then  $x - a$  and  $x - b$  are factors of the quadratic equation. Therefore

$$x^2 + ax + b = (x - a)(x - b)$$

$$x^2 + ax + b = x^2 + (a + b)x + ab$$

equating coefficients  $b = ab$  and  $a = -a - b$ .

since  $b \neq 0 \implies a = 1$  and so  $b = -2$

ALTERNATIVELY

we substitute  $a$  and  $b$  in the quadratic equation to get  $2a^2 + b = 0$  and  $b^2 + ab + b = 0$  implying  $b = -2a^2$  and  $b = -a - 1 \implies -2a^2 - a - 1 = 0$ . Solving this quadratic equation gives WWait abit

- $\sqrt[3]{\frac{1 \times 2 \times 4 + 2 \times 4 \times 8 + 3 \times 6 \times 12 + 4 \times 8 \times 16 + \dots}{1 \times 3 \times 9 + 2 \times 6 \times 18 + 3 \times 9 \times 27 + 4 \times 12 \times 36 + \dots}} = \sqrt[3]{\frac{(1 \times 2 \times 4)(1 + 8 + 27 + 64 + \dots)}{(1 \times 3 \times 9)(1 + 8 + 27 + 64 + \dots)}} = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$ .
- $f(x + 1, y) = f(x, y) + y + 1, f(x, 0) = x$  and  $f(x, y) = f(y, x)$  for all  $x$  and  $y$   
 $f(12, 0) = 12 = f(0, 12)$ . Then,  
 $f(1, 12) = f(0, 12) + 12 + 1 = f(0, 12) + 13 = 12 + 13 = 25$ .  
 $f(2, 12) = f(1, 12) + 12 + 1 = f(1, 12) + 13 = 25 + 13 = 38$ .  
 $f(3, 12) = f(2, 12) + 12 + 1 = f(2, 12) + 13 = 38 + 13 = 51$ .  
 $f(4, 12) = f(3, 12) + 12 + 1 = f(3, 12) + 13 = 51 + 13 = 64$ .  
 Hence  $f(4, 12) = f(12, 4) = 64$ .
- 
- Consider the following table of statements of 4 statements depending on who has the key.

Ann has the key	F	F	T	F
Ben has the key	T	F	F	T
Cal has the key	F	T	F	T
Don has the key	T	T	T	F

Hence Don has the key.

8. Let  $N = 19700019y$ . Clearly,  $y$  cannot be even or 5. If  $y$  is a multiple of 3, then  $N$  is divisible by 3 which can be checked by summing its digits and obtain a multiple of 3. As 197 is an apparent divisor of  $N$  if  $y = 7$ , then the only possibility is when  $y = 1$ .
9. We observe that  $m.[m] = 17 \implies 4 < m < 5$  so that  $[m] = 4$ . Therefore  $m = \frac{17}{4}$ . Similarly,  $n.[n] = 17 \implies 3 < n < 4$  so that  $[n] = 3$ . Thus  $n = \frac{11}{3}$
10. Suppose there are  $b$  boys and  $g$  girls in the family. Then, each boy has  $(b - 1)$  brothers and  $g$  sisters. Hence  $b - 1 = g \dots \dots (1)$   
 Each girl has  $b$  brothers and  $g - 1$  sisters. This implies  $b = 2(g - 1)$ . we can substitute for  $b$  in the equation (1) to get  $2g - 2 - 1 = g$   
 $\implies g = 3$  and so  $b = 4$ .  
 The total number of children in the family is  $b + g = 4 + 3 = 7$ .

**SECTION B : 10 marks each**

11. Suppose there are  $x$  girls and  $y$  boys in the class and that a cake costs  $p$  coins and a sandwich costs  $q$  coins. we know that

$$\begin{aligned}
 xp + yq &= yp + xq - 1 \\
 (xp - yp) + (yq - xq) &= -1 \\
 p(x - y) - q(x - y) &= -1 \\
 (x - y)(p - q) &= -1.
 \end{aligned}$$

Since both factors are integers, one must be +1 and the other -1. We also know that  $x > y$ . This implies  $x - y = 1$ . Therefore there is only one additional girl.

12. Out of the ten sums 32, 36, 37,  $\dots$ , 48, 51 the largest one is  $d + e$  and the second largest one is  $c + e$ . Therefore  $d + e = 51$  and  $c + e = 48$ .  
 Furthermore,  $a + b$  is the smallest sum and  $a + c$  the second smallest, so  $a + b = 32$  and  $a + c = 36$ .  
 The third sum could be either  $a + d$  or  $b + c$ .  
 However, we know that  $a + d = (a + c) + (d + e) - (c + e) = 36 + 51 - 48 = 39$ .  
 Apparently,  $a + d$  is not the third smallest sum, so  $b + c = 37$ .  
 Combining the things we have got so far yields  
 $2e = 2(c + e) - (a + c) - (b + c) + (a + b) = 2(48) - 36 - 37 + 32 = 55$   
 This implies that  $2e = 55 \implies e = \frac{55}{2}$ .

13. (a) Let the first bricklayer lay  $x$  bricks per hour and the second bricklayer  $y$  bricks per hour. The total number of bricks in the wall =  $9x = 10y$ . Working together, they lay  $x + y - 10$  bricks per hour. The total number of bricks in the wall =  $5(x + y - 10)$ .  
 Now,  $5(x + y - 10) = 9x \implies 5y - 4x = 50 \dots\dots\dots$ (i).  
 Also  $5(x + y - 10) = 10y \implies -5y + 5x = 50 \dots\dots\dots$  (ii).  
 Solving equations (i) and (ii) simultaneously gives  $x = 100$  and  $y = 90$ . The total number of bricks in the wall is 900 bricks.
- (b) Suppose the time period is  $t$  days during which it rains on  $x$  mornings and  $y$  afternns. It never rains on both morning and afternoon of the same day.  
 The number of rain days =  $x + y = 11 \dots\dots\dots$  (i)  
 The number of clear mornings =  $t - x = 9 \dots\dots\dots$  (ii)  
 The number of clear afternoons =  $t - y = 12 \dots\dots\dots$  (iii)  
 Adding equations (i), (ii) and (iii) gives  $2t = 32 \implies t = 16$ .  
 The total number of rain free days is  $t - 11 = 5$  days.
14. It is given that  $x, y > 0$  hence  $0 \neq x + y = x^2 - y^2 = (x + y)(x - y)$ . Divide through  $(x + y)$  to get  $x - y = 1 \implies x = y + 1$ .  
 We also know that  $xy = x + y$  hence  $0 = xy - y - x = x(y - 1) - y = (y + 1)(y - 1) - y = y^2 - y - 1$   
 so  $y$  is positive and  $y^2 = y + 1$  and solving this quadratic equation gives  $b = \frac{1 + \sqrt{5}}{2}$ .
15. Sketch

Drop a perpendicular from  $P$  to side  $AB$  and let  $R$  be the point of intersection. Let  $\overline{AR} = x$  and  $\overline{BR} = y$ . Similary, drop a perpendicular from  $P$  to side  $AD$  and let  $S$  be the point of intersection .  
 Let  $AS = u, DS = v \implies y^2 + u^2 = \overline{PB}^2 = 4^2 = 16$ .

Also,  $u^2 + x^2 = \overline{PA}^2 = 8^2 = 64$ .

$x^2 + v^2 = \overline{PD}^2 = 7^2 = 49$ .  
 Hence  $\overline{PC}^2 = y^2 + v^2 = (y^2 + u^2) + (x^2 + v^2) + (u^2 + x^2) = 16 + 49 - 64 = 1$ . Thus,  $\overline{PC} = 1$ .

## NMC 2013 O-LEVEL PAPER 2 SOLUTIONS

## SECTION A : 5 marks each

- The difference between the two plants' height is 36 cm currently. The smaller plant grows 9cm in 6 years, and the taller plant grows 5cm in 6 years. Thus, the smaller plant can make up 4cm in each 6 years. Hence, it will take  $(\frac{36}{4} = 9) \times 6$  years, that is, 54 years for the two plants to be equally high.
- Among the given distances, the largest one equals the sum of all the others:  $8 = 2 + 1.5 + 3.5 + 1$ . This is possible only if the sheds are along a line. The order of the sheds must be  $C, B, A, E, D$ . Therefore,  $\text{distant}(B, E) = \text{distant}(B, A) + \text{distant}(A, E) = 2 + 1 = 3\text{km}$ .
- The sum of the numbers on all the faces is 21. The first observation implies that the sum of the numbers on a couple of opposite sides is 8. This couple can be  $(6, 2)$  or  $(5, 3)$ . The second observation implies that the sum of the numbers on another couple of opposite sides is 9. That couple can be  $(6, 3)$  or  $(5, 4)$ . Combining the above information, the only possibility is that  $(6, 2)$  and  $(5, 4)$  are on opposite sides. Hence,  $(3, 1)$  is the third couple of opposite sides, therefore the answer is 3.
- The mathematician Augustus lived his entire life during the 1800's. In the last year of his life he stated: Once I was  $x$  years old in the year  $x^2$ . In which year was Augustus born?.
- section B.** Find the total number of three-digit numbers that are divisible by 9 and contain no even digits ?  
A number is divisible by 9 if and only if the sum of its digits is divisible by 9, so we need to find three odd digits whose sum is 9 or 27. (The sum of three odd numbers is odd, so 18 is impossible.) The possibilities for the digits are:
  - 1, 1, 7 which can be arranged to give three different numbers which are 117, 171 and 711.
  - 1, 3, 5 which can be arranged to give six different numbers. These are 135, 153, 315, 351, 513 and 531.
  - 3, 3, 3 which gives only one number, 333.
  - 9, 9, 9 which also gives one number 999. The total is therefore  $3 + 6 + 1 + 1 = 11$ .
- One can of orange paint contains  $\frac{3}{5}$  of a can of red paint and  $\frac{2}{5}$  of a can of yellow paint. One can of green paint contains  $\frac{2}{3}$  of a can of blue paint and  $\frac{1}{3}$  of a can of yellow paint. When orange and green are mixed, we have  $(\frac{2}{5} + \frac{1}{3})$  of a can of yellow paint in a total of 2 cans, so the proportion is  $(\frac{2}{5} + \frac{1}{3})/2 = \frac{11}{30}$ .
- A restaurant can accommodate at most 400 people, which includes the guests and the waiters. Each waiter can serve a maximum of 12 guests. What is the greatest number of guests that can be served ?

**solution**

We need to know how many sets of 13 people (12 guests and 1 waiter) can be formed from 400 people. Now  $400 = 30 \times 13 + 10$ . But those last 10 people in the remainder also need a waiter. Therefore 31 waiters are required, which means  $400 - 31 = 369$  guests can be served.

8. Nine numbers are written in ascending order. The middle number is also the average of the nine numbers. The average of the five largest numbers is 68 and the average of the five smallest numbers is 44. What is the value of the middle number?

**Solution**

Let the sum of all the numbers be  $S$ , so the middle number is  $\frac{1}{9}S$ . The sum of the five largest numbers is  $5 \times 68 = 340$ , and the sum of the five smallest numbers is  $5 \times 44 = 220$ . If we add these sums together, then we have counted the middle number twice, so  $S + \frac{1}{9}S = 340 + 220$ , that is  $\frac{10}{9}S = 560$ . Thus  $S = \frac{560 \times 9}{10} = 504$ . Therefore the middle number is  $\frac{1}{9}S = \frac{1}{9} \times 504 = 56$ .

9. Tammy, John, and Martha were all born at noon on January 19th, but in different years. When Tammy was four years old, John was three times as old as Martha. When Martha was twice as old as Tammy, John was five times as old as Tammy. How old was Tammy when John was twice as old as Martha?

**Solution sec B** Let  $j$  denote John's age when Tammy was 4, and let  $m$  denote Martha's age when Tammy was 4. Let  $a$  be the number of years after Tammy was 4 when Martha was twice as old as Tammy and John was five times as old as Tammy. Let  $b$  be the number of years after Tammy was 4 when John was twice as old as Martha. Then we have the following equations: (i)  $j = 3m$ , (ii)  $m + a = 2(4 + a)$ , (iii)  $j + a = 5(4 + a)$ , and (iv)  $j + b = 2(m + b)$ . We want to know the value of  $4 + b$ . From (iv),  $b = j - 2m$  so that, from (i),  $b = m$ . From (ii),  $m = 8 + a$ . Using (i) in (iii), we also have  $3m = 20 + 4a$ . So  $b = m = 4m = 4(8 + a) = 20 + 4a = 12$ . Hence,  $4 + b = 16$ .

10. Each of the following five statements is either true or false. How many of these statements are true given that statement (4) is true.
- (1) Statements (3) and (4) are both true.
  - (2) Statements (4) and (5) are not both false.
  - (3) Statement (1) is true.
  - (4) Statement (3) is false.
  - (5) Statements (1) and (3) are both false.

**solution**

If Statement (4) is true, then Statement (3) is false. Hence, (3) and (4) cannot both be true. It follows that Statement (1) cannot be true. Hence, both Statement (1) and Statement (3) must be false. We see then that Statement (4) and Statement (5) are true. It follows that Statement (2) is true. The answer is 3. Thus three statements are true.

11. The two shortest sides of a right triangle have lengths  $\sqrt{3}$  and 2. Let  $\alpha$  be the smallest interior angle of this triangle. What is the value of  $\sin \alpha$ ?

**Solution**

The side opposite to angle  $\alpha$  has length  $\sqrt{3}$  and the hypotenuse of the triangle is  $\sqrt{(\sqrt{3})^2 + 2^2} =$

$\sqrt{7}$ . Therefore  $\sin \alpha = \frac{\sqrt{3}}{\sqrt{7}} = \sqrt{\frac{3}{7}}$ .

**NMC 2012 O-LEVEL PAPER 1 SOLUTIONS**  
**SECTION A**

1. Given that  $p + q = 0$  implies that  $p = -q$  and  $\frac{p}{q} = -1$ .

Therefore,

$$\left(\frac{p}{q}\right) + 2 = (-1)^{2012} + 2 = 1 + 2 = 3.$$

2. Using Pythagoras Theorem,

$$3^2 + 4^2 + 5^2 + 12^2 = 5^2 + 13^2 = x^2 + y^2.$$

This implies that  $x = 5$  and  $y = 13$ . Therefore,  $\sqrt{x+y} = \sqrt{5+13} = \sqrt{18} = 3\sqrt{2}$ .

- 3.

$$\begin{aligned} (x - 2y)^2 &= (x - 2y)(x - 2y) \\ &= x^2 - 4xy + 4y^2 \\ &= x^2 - 4y^2 + 4y^2 - 4xy + 4y^2 \\ &= x^2 - 4y^2 - 4xy + 8y^2. \end{aligned} \tag{4}$$

The mistaken expansion has only the first two terms in equation (4). This therefore implies that  $-4xy + 8y^2 = 0$  or  $4xy = 8y^2$  or  $y(4x - 8y) = 0$  which implies that  $y = 0$  or  $2y = x$ .

Therefore,  $\frac{x}{y} = 2$  if  $y \neq 0$ . Otherwise, it is not defined.

- 4.

$$\begin{aligned} \frac{x}{x+1} < \frac{2008}{2009} &\Rightarrow 2009x < 2008(x+1) \\ &\Rightarrow 2009x < 2008x + 2008 \\ &\Rightarrow x < 2008. \end{aligned}$$

Therefore  $x = 200$  and the number of fish which swam with me are  $2 + 0 + 0 + 7 = 9$ .

5.
  - If Eugene sails, then Godfrey rows. This means that Godfrey does not swim. If Godfrey does not swim, then Fred sails which is a problem, because if Eugene sails, then Fred cannot sail.
  - If Eugene swims then Fred rows. Since Godfrey does not swim because Eugene does, the Fred must also sail, which is not yet a solution.

- Hence Eugene rows. If Godfrey sails, then he does not swim; so Fred must also sail which is also a problem.

Therefore, the only solution is that Eugene rows, Godfrey swims and Fred sails.

6. Let the side of the square be  $s$  and the radius of the circle be  $r$  as shown in Figure. Using Pythagoras Theorem, we have that

$$s^2 + s^2 = (2r)^2 = 4r^2; \Rightarrow s = \sqrt{2}r.$$

Perimeter of the square is  $y = 4s = 4\sqrt{2}r$  and circumference of the circle is  $x = 2\pi r$ . Therefore,  $\frac{x}{y} = \frac{2\pi r}{4\sqrt{2}r} = \frac{\pi}{2\sqrt{2}} = \frac{\pi\sqrt{2}}{4r}$ . The required ratio is  $x : y = \pi\sqrt{2} : 4r$ .

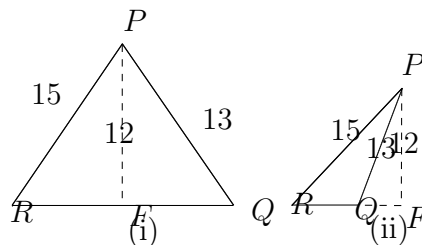
7. Let  $n = 1234567890$ . Then,  $1234567891 = n + 1$  and  $1234567892 = n + 2$ . Therefore,

$$\begin{aligned} \frac{1234567890}{(1234567891)^2 - (1234567890 \times 1234567892)} &= \frac{n}{(n+1)^2 - n(n+2)} \\ &= \frac{n}{(n^2 + 2n + 1) - (n^2 + 2n)} \\ &= \frac{n}{1} \\ &= n = 1234567890. \end{aligned}$$

8. Let the two digit number  $N = 10a + b$  where  $a$  and  $b$  are digits. Then,  $P(N) = ab$  and  $S(N) = a+b$ . Using  $N = P(N)+S(N)$  we have that  $10a+b = ab+a+b$  or  $10a+b = (b+1)a+b$ . Comparing coefficients we get  $10 = b + 1$  implying that  $b = 9$ .

Therefore, the numbers must have the second digit as 9. These are 19, 29, 39, 49, 59, 69, 79, 89 and 99.

9. Let the foot of the altitude be  $F$  as shown in the Figure 15b.



Using Pythagoras Theorem in  $\triangle PRF$

$$\begin{aligned} FR^2 &= PR^2 - PF^2 \\ &= 15^2 - 12^2 = 81. \end{aligned}$$

Therefore,  $\overline{FR} = 9$ .

Likewise, using Pythagoras Theorem in  $\triangle PQF$ ,

$$\begin{aligned} FQ^2 &= PQ^2 - PF^2 \\ &= 13^2 - 12^2 = 25. \end{aligned}$$

Therefore  $\overline{FQ} = 5$ .

The two figures depict the possible positions of  $F$ . Using (i),  $\overline{QR} = \overline{QF} + \overline{FR} = 5 + 9 = 14$ . The second possibility is depicted in Figure 15b(ii). Here  $\overline{QR} = \overline{FR} - \overline{QF} = 9 - 5 = 4$ .

10. Let the numbers be  $x$  and  $y$  and let  $p = xy$ . We are given that  $x + y = 200$  or  $y = 200 - x$ .

**Method I.**

$$\begin{aligned} p &= xy \\ &= x(200 - x) \\ &= -x^2 + 200x \\ &= -(x^2 - 200x) \\ &= -((x - 100)^2 + 100^2) \\ &= 10,000 - (x - 100)^2. \end{aligned}$$

The maximum value of  $p$  is 10,000 and occurs when  $x = 100$ .

**Method 2**

Then  $p = x(200 - x) = 200x - x^2 \implies \frac{dp}{dx} = 200 - x$  and  $\frac{d^2p}{dx^2} = -2$ .

$$\frac{dp}{dx} = 0 \text{ when } 200 - 2x = 0 \implies x = 100$$

When  $x = 100$ ,  $\frac{d^2p}{dx^2} = -2 < 0$  so a local maximum occurs at  $x = 100$  and so the two numbers are  $x = 100$  and  $200 - 100 = 100$ . The maximum possible value of the product is  $100 \times 100 = 10000$

## Section B

11. (a) Let
- $x$
- be the number of grams of gold to be added.

$$\text{Gold concentration} = \frac{\text{weight of gold}}{\text{total weight}}$$

$$0.80 = \frac{20 \times 0.6 + x}{20 + x}$$

$$12 + x = 16 + 0.8x \implies 0.2x = 4$$

$$\text{Hence } x = 20\text{g}$$

- (b)

$$\begin{aligned} S &= 1^2 - 2^2 + 3^2 - 4^2 + \dots + 199^2 - 200^2 \\ &= (1 - 2)(1 + 2) + (3 - 4)(3 + 4) + \dots + (199 - 200)(199 + 200) \\ &= -3 - 7 - 11 - \dots - 399 \\ &= -1[3 + 7 + 11 + \dots + 391 + 395 + 399]. \end{aligned}$$

Let

$$T = 3 + 7 + 11 + \dots + 391 + 395 + 399 \quad (5)$$

$$= 399 + 395 + 391 + \dots + 11 + 7 + 3 \quad (6)$$

Summing the two equations gives

$$2T = 402 + 402 + 402 + \dots + 402 = 100 \times 402 = 40200 \implies T = 20100.$$

Therefore,

$$S = -T = -20100 \text{ and } |S| = 20100.$$

12. (a)

$$F(1) = F(0^2 + 1) = [F(0)]^2 + 1 = 2^2 + 1 = 5.$$

$$F(2) = F(1^2 + 1) = [F(1)]^2 + 1 = 5^2 + 1 = 26.$$

$$F(5) = F(2^2 + 1) = [F(2)]^2 + 1 = 26^2 + 1 = 677$$

- (b) Let
- $x$
- be the number of children in the village. we are given that
- $\frac{1}{3}x$
- can play football
- $\frac{2}{3}x$
- can ride a bicycle and
- $\frac{1}{7}x$
- can ride a bicycle and play football. This information implies that the number of children in the village must be a common multiple of 3 and 7. The only common multiple of 3 and 7 which is less than 40 is 21.

13. (a)

$$\begin{aligned} \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \dots \times \left(1 + \frac{1}{2006}\right) &= \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \dots \times \frac{2006}{2005} \times \frac{2007}{2006} \\ &= \frac{2007}{2} = 1003.5. \end{aligned}$$

- (b)
- Let  $b$  be the capacity of the bottle in ml.
  - Let  $g$  be the capacity of the glass in ml.
  - Let  $j$  be the capacity of the jug in ml.
  - Let  $t$  be the capacity of the tankard in ml.

We are given that

$$\begin{aligned} b + g &= j \\ b &= g + t \\ 3t &= 2j \end{aligned}$$

Therefore,

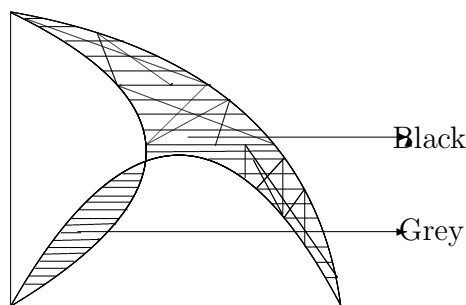
$$\begin{aligned} 3t &= 2j \\ &= 2(b + g) \\ &= 2(g + t) + 2g \\ &= 2g + 2t + 2g = 4g + 2t. \\ \Rightarrow t &= 4g. \end{aligned}$$

Therefore, four glasses would hold one tankard.

14. Let  $x$  represent the number of dogs and  $y$  represent the number of cats. We are given that  $10\%x$  think that are cats and  $90\%y$  think that they are cats. We are also given that  $20\%(x + y)$  think that they are cats. So we solve,

$$\begin{aligned} 10\%x + 90\%y &= 20\%(x + y) \\ \Rightarrow 10x + 90y &= 20x + 20y \\ \Rightarrow x &= 7y. \end{aligned}$$

Altogether the number of animals sum to  $x + y = 8y$  and the percentage of cats is  $\frac{y}{8y} \cdot 100\% = 12.5\%$ .



Let  $OA = OB = r$ . This implies that the area of quadrant  $= \frac{\pi r^2}{4}$  and the area of one semi-circle  $= \frac{\pi r^2}{8}$ . Therefore,  $X + Z = \frac{\pi r^2}{8}$ .

Area inside the quarter-circle but outside one semi circle is  $= \frac{\pi r^2}{4} - \frac{\pi r^2}{8} = \frac{\pi r^2}{8}$  so  $X + T = \frac{\pi r^2}{8}$ . thus  $X + T = X + Z$  and this implies that  $T = Z$  so the areas of the shaded region are equal.

Thus the ratio of the area of the region shaded grey to the area of the region shaded black is 1 : 1.

## NMC 2012 O-LEVEL PAPER 2 SOLUTIONS

## SECTION A

- $\frac{1}{2} \times 2^{2012} = \frac{1}{2} \times 2 \times 2^{2011} = 2^{2011}$ .
- Substituting  $x = 2$  we get  $(2 - 1)f(2) + f\left(\frac{1}{2}\right) = 1$  or

$$f(2) + f\left(\frac{1}{2}\right) = 1. \quad (7)$$

Likewise, substituting  $x = \frac{1}{2}$  gives  $\left(\frac{1}{2} - 1\right)f\left(\frac{1}{2}\right) + f(2) = 1$  or

$$-\frac{1}{2} \cdot f\left(\frac{1}{2}\right) + f(2) = 1. \quad (8)$$

Solving equations (7) and (8) simultaneously (i.e.  $\frac{1}{2} \times (7) + (8)$ ) gives  $\frac{3}{2}f(2) + \frac{3}{2}$  giving  $f(2) = 1$ .

3.

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18} = \frac{60 + 30 + 20 + 15 + 12 + 10}{180} = \frac{147}{180}$$

$$\frac{147}{180} = \frac{120 + 27}{180} = \frac{120}{180} + \frac{27}{180} = \frac{2}{3} + \frac{27}{180}.$$

The fraction that give a sum of  $\frac{27}{180}$  are  $\frac{1}{12}$  and  $\frac{1}{15}$ .

Therefore, the fractions removed are  $\frac{1}{12}$  and  $\frac{1}{15}$ . Their product is  $\frac{1}{12} \times \frac{1}{15} = \frac{1}{180}$ .

- The distance covered by Jim in 4 steps is the same distance covered by John in 3 steps. This implies that the distance covered by Jim in 24 steps ( $4 \times 6$ ) is the same distance covered by John in  $(3 \times 6) = 18$  steps. Since each of John's steps is  $0.5m$ , then Jim covers  $18 \times 0.5 = 9$  metres in 24 steps.
- The sum of the two numbers is  $8 + 2 = 10$ .
- Suppose that  $x$  is the less than 4. This implies that the median is 4. Then,

$$4 = \frac{x + 6 + 4 + 1 + 9}{5} \Rightarrow 20 = x + 20 \Rightarrow x = 0.$$

- Suppose that  $x$  is between 4 and 6. This implies that the median is  $x$ . Then, then

$$x = \frac{x + 6 + 4 + 1 + 9}{5} \Rightarrow 5x = x + 20 \Rightarrow 4x = 20 \Rightarrow x = 5.$$

- Suppose that  $x$  is greater than 6. This implies that the median is 6. Then,

$$6 = \frac{x + 6 + 4 + 1 + 9}{5} \Rightarrow 30 = x + 20 \Rightarrow x = 10.$$

The only possible values of  $x$  are 0, 5 and 10.

7. Let  $x$  be the number of T-shirts that John selects and let  $y$  be the number of T-shirts that Joseph selects. We are given that

$$2x + 7y = 100 \tag{9}$$

$$x - y = 5 \tag{10}$$

(9) minus  $2 \times$  (10) gives  $y = 10$  and  $x = 15$ .

Therefore, they all together select  $x + y = 25$  shirts.

8.  $123456789 \times 98765342 \times 104392831$ .

The last two digits of any product is given by the product of the any two last digits of each number. i.e.  $89 \times 42 \times 31 = 3738 \times 31 = 115878$ . Therefore, the last two digits are 78.

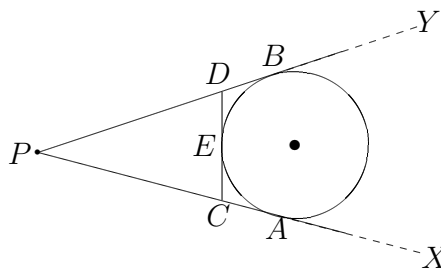
9. The set of all possible 3 positive integers whose product is 36 is

$$\{1, 1, 36\}, \{1, 2, 18\}, \{1, 3, 12\}, \{1, 4, 9\}, \{1, 6, 6\}, \{2, 2, 9\}, \{2, 3, 6\}, \{3, 3, 4\}.$$

The sum of the elements in all sets are different except for  $\{1, 6, 6\}$  and  $\{2, 2, 9\}$ . Since Kitty found out that the sum of integers was not unique, then, it must be one of these sets whose sum is 13.

**NB:** In enumerating the sets, we have excluded sets of the same elements which are arranged differently. For example, sets such  $\{1, 4, 9\}$ ,  $\{4, 9, 1\}$ ,  $\{4, 1, 9\}$ ,  $\{9, 4, 1\}$  and  $\{9, 1, 4\}$  have the same elements and can be taken to be equal and same sets.

- 10.



$\overline{PA} = \overline{PB} = 15$  (Tangent from a common point  $P$ )

Similarly  $\overline{CA} = \overline{CE}$  (tangent from a common point  $C$  and  $\overline{DB} = \overline{DE}$  (Tangent from a common point  $D$ ).

$$\begin{aligned}\text{Perimeter of } \triangle PCD &= \overline{PC} + \overline{CE} + \overline{ED} + \overline{DP} \quad (\overline{CD} = \overline{CE} + \overline{ED}). \\ &= \overline{PC} + \overline{CA} + \overline{DB} + \overline{DP} \\ &= \overline{PA} + \overline{PB} \\ &= 15 + 15 = 30\text{cm}.\end{aligned}$$

## SECTION B

11. Let  $A$  be the amount of alcohol present in the mixture at sometime. Let  $p$  be the fraction of the mixture that Award drinks.

This implies that the amount of alcohol taken by Award is  $Ap$  and the amount of alcohol left in the cup is  $A - Ap = A(1 - p)$ .

At the start, Award drinks  $\frac{1}{5}$  of the water. Liam adds  $\frac{1}{5}$  of the alcohol to the cup. When Award drinks  $\frac{1}{4}$  of the mixture, it implies that the amount of alcohol left is  $\frac{1}{5} \left(1 - \frac{1}{4}\right) = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20}$ .

When Liam replaces with  $\frac{1}{4}$  of alcohol, it implies that the amount of alcohol in the cup is  $\frac{3}{20} + \frac{1}{4} = \frac{3+5}{20} = \frac{8}{20}$ .

When Award drinks  $\frac{1}{3}$  of the mixture, then, the amount of alcohol left is  $\frac{8}{20} \left(1 - \frac{1}{3}\right) = \frac{8}{20} \times \frac{2}{3} = \frac{16}{60}$ .

On the third time Liam replaces with  $\frac{1}{3}$  of the alcohol to fill the cup. This implies that the amount of alcohol in the cup is  $\frac{16}{60} + \frac{1}{3} = \frac{16+20}{60} = \frac{36}{60}$ .

Therefore the percentage of alcohol in the final mixture is  $\frac{36}{60} \times 100\% = 60\%$ .

12. Let  $x$  be the number in the shared corner square. The sum of all the numbers in the squares is equal to the sum of the numbers in the five squares aligned vertically plus the sum of the numbers in the horizontal squares minus  $x$  (because it is repeated). i.e.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 32 + 20 - x \Rightarrow x = 7.$$

Therefore, the number that goes in the shared corner square is 7.

13. It takes David  $56s$  to complete one lap. In  $1s$  he completes  $\frac{1}{56}$  of the lap. Therefore in  $24s$ , David completes  $\frac{24}{56} = \frac{3}{7}$  of the track.

Daniel is running in the opposite direction. He will have to move  $\left(1 - \frac{3}{7}\right) = \frac{4}{7}$  of the track in  $24s$ .

Therefore, to complete one track ( $\frac{7}{7}$ ), Daniel will have to take  $\frac{24 \times 7}{4} = 42s$ .

- 14.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{4} \Rightarrow \frac{1}{y} = \frac{1}{4} - \frac{1}{x} = \frac{x-4}{4x}.$$

Therefore,  $y = \frac{4x}{x-4}$ ,  $x \neq 0$  and  $y \neq 0$ .

- when  $x = \pm 1$ ,  $y = \frac{4}{-3}$  (or  $\frac{-4}{-1-4} = \frac{4}{5}$ ) which is not an integer. Likewise, when  $x = \pm 2$ ,  $y$  turns out not to be an integer. In the same way, when  $x = -3$ , the  $y$ -coordinate turns to be non-integer. The same applies for  $x = 4, x = -5, x = -6, x = \pm 7, x = -8, x = \pm 9, x = \pm 10$ .

- When  $x = 3, y = \frac{4 \times 3}{3-4} = -12$  implying that  $(3, -12)$  is a solution.
- When  $x = -4, y = \frac{4 \times -4}{-4-4} = 2$  implying that  $(-4, 2)$  is a solution.
- When  $x = 6, y = \frac{4 \times 6}{6-4} = 12$  implying that  $(6, 12)$  is a solution.
- When  $x = 8, y = \frac{4 \times 8}{8-4} = 8$  implying that  $(8, 8)$  is a solution.

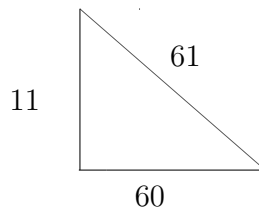
Expected Solutions are  $\{(-12, 3), (-4, 2), (2, -4), (3, -12), (5, 20), (6, 12), (8, 8), (12, 60)$  and  $(20, 5)\}$ .

15. (a)

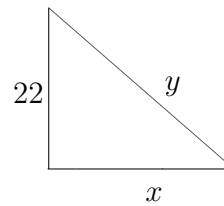
$$(x-a)(x-b)(x-c) \dots (x-z) = (x-a)(x-b)(x-c)(x-d)(x-e) \dots (x-x)(x-y)(x-z) = 0.$$

(b)

Original Triangle



New triangle



The original triangle has perimeter  $11 + 60 + 61 = 132$  units.

Let the other two sides of the new triangle be  $x$  and  $y$  respectively. This implies that  $p = x + y + 22 = 132$  since the perimeter remains the same. Making  $y$  the same subject gives  $y = 110 - x$ .

Using the new triangle, we have

$$\begin{aligned} x^2 + 22^2 &= (110 - x)^2 \\ \Rightarrow x^2 + 484 &= 12100 - 220x + x^2 \\ \Rightarrow -11616 &= -220x \\ \Rightarrow x &= 52.8. \end{aligned}$$

Thus, the length of the hypotenuse of the new triangle is  $110 - x = 110 - 52.8 = 57.2$  units.

**NMC 2010 O-LEVEL PAPER 1 SOLUTIONS  
SECTION A**

1.

$$\begin{aligned} \log \left( \frac{(125)(625)}{25} \right) &= \log (125) (25) \\ &= \log_5 (5^3) (5^2) \\ &= \log_5 5^5 \\ &= 5 \log_5 5 = 5. \end{aligned}$$

2.

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} - ab &= \frac{a^2 + b^2}{ab} - ab \\ &= \frac{a^2 + b^2 - (ab)^2}{ab} \\ &= \frac{a^2 + b^2 - (a - b)^2}{ab} \\ &= \frac{a^2 + b^2 - (a^2 - 2ab + b^2)}{ab} \\ &= \frac{2ab}{ab} = 2. \end{aligned}$$

3. Let  $B$  and  $G$  denote the original number of boys and original number of girls respectively. Then

$$\frac{B}{G - 15} = \frac{2}{1} \quad \text{and} \quad \frac{B - 45}{G - 15} = \frac{1}{5}.$$

The two equations reduce to

$$B = 2(G - 15) \tag{11}$$

$$5B = G + 210. \tag{12}$$

Subtract equation (12) from  $5 \times$  (11) and get  $10G - 150 - G - 210 = 0$  from which we get  $G = 40$ . Therefore, there were 40 girls at the beginning.

4.

$$\text{Cost of one easy payment} = \left(\frac{1}{4} \times 200,000\right) + 5,000 = \text{UGShs } 55,000.$$

$$\text{Cost of three easy payments} = \text{UGShs } 3 \times 55,000 = \text{UGShs } 165,000$$

$$\begin{aligned} \text{Total Cost of advertised product} &= \text{cost of three easy payments} \\ &+ \text{handling charges} \\ &= \text{UGShs } 165,000 + \text{UGShs } 15,000 \\ &= \text{UGShs } 180,000. \end{aligned}$$

$$\begin{aligned} \text{Money saved by buying from the television advertiser} &= \text{UGShs } (200,000 - 180,000) \\ &= \text{UGShs } 20,000. \end{aligned}$$

$$\text{Percentage saved} = \frac{20,000}{200,000} = 10\%.$$

5. Let  $x$  and  $2x$  be the ages in years of walter and his grandfather respectively at the end of 1994.

	Walter	Grandfather	Total
Age at the end of 1994	$x$	$2x$	$3x$
Year of birth	$1994-x$	$1994-2x$	3838

Since the sum of the years in which they were born is equal to 3838, we solve the equation  $1994 - x + 1994 - 2x = 3838$  or  $3988 - 3x = 3838$  from which we get  $x = 50$  years. Therefore, at the end of 2010, Walter will be  $50 + (2010 - 1994) = 66$  years.

6. Let  $m$  be the number of mangoes in Jesse's fridge originally.

	Mangoes at the beginning	Mangoes at the end
Day I	$m$	$m - 0.2m = 0.8m$
Day 2	$0.8m$	$0.8m - 0.2 \times 0.8m = 0.64m$

Therefore,  $0.64m = 32$  implying that  $m = 50$ .

7. Let  $A$  represent numbers between 1 and 100 which are 1 less than a multiple of 5. Such numbers have unit digits of either 4 or 9. So,

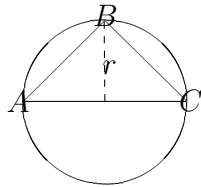
$$\begin{aligned} A &= \{5n - 1 : n = 1, 2, \dots, 20\} \\ &= \{4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64, 69, 74, 79, 84, 89, 94, 99\}. \end{aligned}$$

Let  $B$  represent numbers between 1 and 100 which are 1 greater than a multiple of 4. Then,

$$\begin{aligned} B &= \{4m + 1 : m = 1, 2, \dots, 24.\} \\ &= \{5, 9, 13, \dots, 99\} \\ A \cap B &= \{\text{Those numbers in } A \text{ which are 1 greater than the multiple of 4}\} \\ &= \{9, 29, 49, 69, 89\}. \end{aligned}$$

Let  $C$  be prime numbers between 1 and 100. Then,  $A \cap B \cap C$  are the prime numbers we can get from the set  $A \cap B$ . Therefore,  $A \cap B \cap C = \{29, 89\}$  and their sum is  $29 + 89 = 118$ .

8. Of all triangles that are inscribable in the semi-circle  $ABC$  with the diameter  $AC$  as the base, the one with the greatest area is the one with largest height which is the radius of the circle. Therefore, area of such a triangle is  $A = \frac{1}{2}bh = \frac{1}{2} \times 2r \times r = r^2$ .



9. Let  $R$  be the annual rent. Expenditure: repairs and upkeep equal to  $12.5\%R$  and the Amount Realised is  $R - 12.5\%R - 650,000$ . Since he earns  $5.5\%$  on his investment, it implies

$$\begin{aligned} 5.5\% \times 20,000,000 &= R - 12.5\%R - 650,000 \\ 1,100,000 &= 0.875R - 650,000 \\ 0.875R &= 1,750,000 \Rightarrow R = 2,000,000. \end{aligned}$$

Therefore, monthly rent is  $\frac{2,000,000}{12} = \text{UGShs } 166,666.67$ .

10.

$$\begin{aligned} \text{Area of lune} &= \text{Area of semicircle } APB - \text{Area of segment } AB \\ &= \text{Area of semicircle } APB - (\text{Area of semicircle } APB - \text{Area of triangle } AOB) \\ &= \text{Area of triangle } AOB = \frac{1}{2}r^2. \end{aligned}$$

The ratio of the area of the lune to the area of the triangle is  $1 : 1$ .

11. (a) It takes Okello 9 days to do the work. Assuming that Okello works at a uniform rate, it implies that he does  $\frac{1}{9}$  of the work in one day. Oryem is 50% more efficient than Okello. This means that in one day Oryem does 150% of the work done by Okello, which is  $150\% \times \frac{1}{9} = \frac{1}{6}$ . This it turn implies that Oryem takes 6 days to do the same piece of work if he works at a uniform rate.
- (b) Let  $l$  be the original length and  $w$  be the original width. The original area  $A = lw$ . Suppose that when  $w$  is increased by 10%, the length is decreased by  $d\%$  to maintain the original area. Then

$$\begin{aligned} \text{New Area} &= \text{Original Area} \\ (100 - d)\% \times 110\%w &= lw \\ (100 - d)\% \times 1.1lw &= lw \\ (100 - d)\% \times 1.1 &= 1 \end{aligned}$$

which implies that  $d = 9.1$ .

12. Let the speed of  $M$  be  $r$ . This implies that the speed of  $D$  is  $r - 4$ . We have the following table.

	D	M
Distance in Miles traveled	48	72
Speed in miles per hour	$r - 4$	$r$ .

Since  $D$  and  $M$  travel for the same time, we have,

$$\text{Time taken by } D = \text{Time taken by } M$$

$$\text{OR } \frac{48}{r - 4} = \frac{72}{r}$$

$$\text{OR } 48r = 72(r - 4)$$

and this gives  $r = 12$  miles per hour.

13. The even digits 2, 4, 6, 8 all appear 20 times in between 1 and 100. Their sum is  $20(2 + 4 + 6 + 8) = 400$ .
14. The four sums are

$$7 + a + b + 1 = k \tag{13}$$

$$7 + c + 3 = k \tag{14}$$

$$3 + e + f + 10 = k \tag{15}$$

$$1 + d + 10 = k \tag{16}$$

Adding the four equations gives

$$42 + a + b + c + d + e + f = 4k. \tag{17}$$

since digits 2, 4, 5, 6, 8 and 10 can be distributed among the letters  $a, b, c, d, e$  and  $f$ , we deduce from equation (17) that

$$42 + 2 + 4 + 5 + 6 + 8 + 10 = 4k$$

which implies that  $k = 19$ .

From equation (14), we have that  $7 + c + 3 = k$  which implies that  $c = k - 10 = 19 - 10 = 9$ .

From equation (16), we have that  $1 + d + 10 = k$  which implies that  $d = k - 11 = 19 - 11 = 8$ .

From equation (13), we have that  $7 + a + b + 1 = k$  which implies that  $a + b = k - 8 = 11$ .

Looking at the remaining digits, we can let  $a = 5$  and  $b = 6$ .

From equation (15), we have that  $3 + e + f + 10 = k$  which implies that  $e + f = k - 13 = 6$ .

Looking at the remaining digits, we can let  $e = 2$  and  $f = 4$ .

15. Let  $\overline{DC} = x$ . Since  $\triangle CDE$  is a right angled triangle with  $\angle DCE = 60^\circ$ , we have  $\overline{CE} = \frac{\overline{EC}}{\cos 60} =$

$\frac{2\overline{EC}}{\cos 60} = 2x$  and  $\overline{DE} = \cos 30 \times 2x = \sqrt{3}x$ . So,  $\overline{FE} = \overline{DE} = \sqrt{3}x$ . Since  $\angle FEA = 90^\circ$ , we have  $\overline{AE} = \frac{\overline{FE}}{\tan 60} = \frac{\sqrt{3}x}{\sqrt{3}} = x$ . So,  $\overline{AE} = \overline{DC}$  and  $\overline{AC} = \overline{AE} + \overline{EC} = 3x$ . Therefore,

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2}\overline{BA} \times \overline{BC} \times \sin 60 \\ &= \frac{1}{2} \times 3x \times 3x \times \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{4}x^2 \end{aligned}$$

and

$$\begin{aligned} \text{Area of } \triangle DEF &= \frac{1}{2}\overline{FD} \times \overline{FE} \times \sin 60 \\ &= \frac{1}{2} \times \sqrt{3}x \times \sqrt{3}x \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}x^2. \end{aligned}$$

$$\begin{aligned} \text{Ratio of Area of } \triangle DEF : \text{Area of } \triangle ABC &= \frac{3\sqrt{3}}{4}x^2 : \frac{9\sqrt{3}}{4}x^2 \\ &= 1 : 3. \end{aligned}$$

## NMC 2010 O-LEVEL PAPER 2 SOLUTIONS

## SECTION A

1.

$$\frac{a}{a+b+c+d} = \frac{4}{3}$$

$$\Leftrightarrow 3a = 4(b+c+d)$$

$$\Leftrightarrow d = \frac{3a - 4(b+c)}{4}$$

$$\Leftrightarrow \frac{d}{a} = \frac{3 - 4\frac{(b+c)}{a}}{4}$$

$$\Leftrightarrow \frac{d}{a} = \frac{3 - 4\frac{5}{3}}{4} = -\frac{11}{12}.$$

2. The following table shows the position of Bob and Alice on every turn.

Turn	0th	1st	2nd	3rd	4th	5th	6th
Bob:	12	3	6	9	12	3	6
Alice:	12	5	10	3	8	1	6

The game will take six turns and they will meet at a pointed marked 6.

3.

$$\begin{aligned} 1 + x^2 + x^2 + x^3 + \dots + x^{2010} &= 1 + x^2 + x^2 + x^3 + x^4(1 + x^2 + x^2 + x^3) + \dots \\ &\quad + x^{2007}(1 + x^2 + x^2 + x^3) \\ &= 0 + x^4(0) + \dots + x^{2007}(0) \\ &= 0. \end{aligned}$$

4.  $f(f(f(x))) = f(f(ax+b)) = f(a(ax+b)+b) = a(a(ax+b)+b) + b = a^3x + a^2b + ab + b.$ 

$f(f(f(x))) = 8x + 21$ , it implies that  $a^3 = 8 \Rightarrow a = 2$  and  $a^2b + ab + b = 21 \Rightarrow b = 3$ .  
Therefore,  $a + b = 5$ .

5. Let the George' original wage be  $w$ . After the decrease, his new wage is  $w_1 = 75\%w$ . If this wage bill is increased by  $I\%$ , the new wage bill will be  $w_2 = 75\% \times (100 + I)\%$ . Since the new bill is equal to the first bill, we solve for  $75\% \times (100 + I)\%w = w$  implying that  $I = \frac{100}{3}$ . Therefore, we have to increase the current wage by 33.33% to get the original wage.

6.

$$\begin{aligned}
 (\sqrt{3})^x \cdot (\sqrt{27}) &= \sqrt{3} \\
 \Rightarrow 3^{\frac{x}{2}} \cdot 27^{\frac{x}{2}} &= 3^{\frac{1}{2}} \\
 \Rightarrow (81)^{\frac{x}{2}} &= 3^{\frac{1}{2}} \\
 \Rightarrow (3^4)^{\frac{x}{2}} &= 3^{\frac{1}{2}} \\
 \Rightarrow (3)^{2x} &= 3^{\frac{1}{2}} \\
 \Rightarrow 2x &= \frac{1}{2} \Rightarrow x = \frac{1}{4}.
 \end{aligned}$$

7.

$$\begin{aligned}
 60 &= x^2 + 3xy + 2y^2 \\
 &= x^2 + 2xy + y^2 + xy + y^2 \\
 &= (x + y)^2 + y(x + y) \\
 &= 36 + 6y.
 \end{aligned}$$

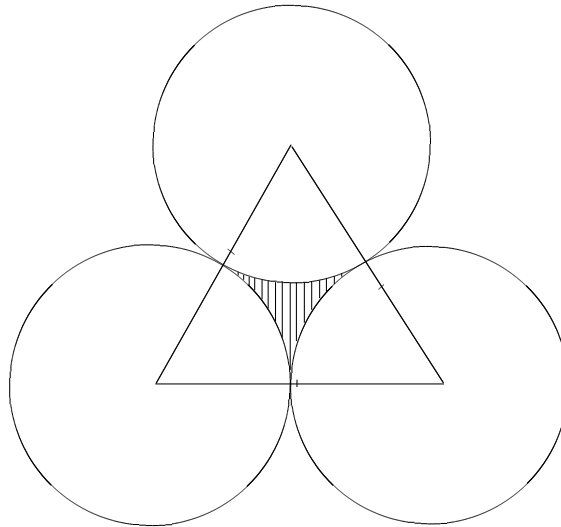
Solving gives  $y = 4$  and  $x = 2$ . Therefore,  $2x + 3y = 16$ .

8.

$$\log_{10} x^2 - \log_{10} 2x = 1 \Rightarrow \log_{10} \frac{x^2}{2x} = 1 \Rightarrow \log_{10} \frac{x}{2} = 1 \Rightarrow \frac{x}{2} = 10 \Rightarrow x = 20.$$

9. Mark's total score on the first 10 quizzes is  $85 \times 10 = 850$ . Mark's total score on 14 quizzes is  $14 \times 90 = 1260$ . Therefore, Mark's average score on the last four questions is  $\frac{1260-850}{4} = 102.5$

10.



Area of shaded region = Area of the equilateral triangle – Area of the three sectors

$$\begin{aligned}
 &= \frac{1}{2} \times 2 \times 2 \times \sin 60 - 3 \left( \frac{60}{360} \times \pi \times 1^2 \right) \\
 &= \sqrt{3} - \frac{\pi}{2}.
 \end{aligned}$$

11. (a) There are 720 such arrangements. To have an arrangement where the lower case letters appear in an increasing alphabetical order, we note that there are
- 2 ways in which the caps and lower case letters alternate.
  - $4! = 24$  ways in which  $abc$  appear as a block.
  - 36 way in which  $ab$  appear as a block, excluding the case  $abc$ .
  - 36 ways in which  $bc$  appear as a block, excluding the case  $abc$ .

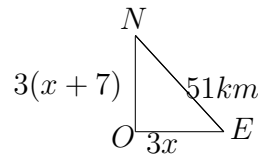
In all, there are 98 ways in which the lower case letters appear in an increasing alphabetical order. The required probability is therefore,  $\frac{98}{720} = 0.136$ .

- (b) Let  $x$  and  $y$  be the length and width of the rectangle. Then,  $x^2 + y^2 = 10$  and  $xy = 3$ . We want to find the perimeter  $p = 2(x + y)$ .

$$\begin{aligned}
 (x + y)^2 &= x^2 + y^2 + 2xy \\
 &= 10 + 2 \times 3 = 16.
 \end{aligned}$$

Therefore,  $x + y = \sqrt{16} = 4$  and the perimeter  $p = 2(x + y) = 8$ .

12. Let Ann's speed be  $x\text{km/hr}$ . This implies that Jason's speed is  $x + 7\text{km/hr}$ . If  $O$  is the camp site, after three hours, Ann is at point  $E$  a distance of  $3x\text{km}$  from point  $O$  and Jason at point  $N$  a distance of  $3(x + 7)\text{km}$  away from point  $O$ . If  $\overline{EN} = 51\text{km}$ , then,



$$\begin{aligned}
 (3x)^2 + (3(x+7))^2 &= 51^2 \\
 \Rightarrow 9x^2 + 9(x^2 + 14x + 49) &= 51^2 \\
 \Rightarrow 18x^2 + 126x + 441 = 2601 \\
 \Rightarrow x^2 + 7x - 120 &= 0 \Rightarrow x = 8 \text{ or } -15.
 \end{aligned}$$

We take  $x = 9$  and disregard  $x = -15$ . The sum of their speed is therefore  $x+x+7 = 23km/hr$ .

13. Let  $x$  denote the time David spends running around the circle. If  $l$  is the length of the circular track, the, David's speed is  $\frac{l}{x}$  and Juma's speed is  $\frac{l}{40}$ . When they first meet, the total distance they have traveled is  $\frac{l}{40} \times 15 + \frac{l}{x} \times 15 = l$  the length of the track. Solving gives  $x = 24$ . the average of their time is  $\frac{24+40}{2} = 32$  seconds.
- 14.

	Rainy morning	Non-rainy mornings
Rainy afternoons	$a$	$b$
Non-rainy afternoons	$c$	$e$

Using the tabulated information, we have,

$$\left. \begin{aligned}
 a + b + e + c = d \\
 a + b + c = 7 \\
 a = 0 \\
 c + e = 5 \\
 b + e = 6
 \end{aligned} \right\} \Rightarrow \left. \begin{aligned}
 b + e + c = d \\
 b + c = 7 \\
 c + e = 5 \\
 b + e = 6
 \end{aligned} \right\} \Rightarrow \left. \begin{aligned}
 b + e + c = d \\
 b + c = 7 \\
 b - e = 2 \\
 b + e = 6
 \end{aligned} \right\}$$

from which we deduce that  $b = 4, e = 2, c = 3$  and  $d = 9$ .

15. Let  $\angle DAE = a^\circ$  and let  $\angle EDC = x$ . Then,

$$\begin{aligned}
 \angle ABC + \angle BCA + 30 + a &= 180 \\
 \Rightarrow 2\angle BCA &= 180 - 30 - a \\
 \Rightarrow \angle BCA &= \frac{180 - 30 - a}{2}
 \end{aligned}$$

$$= 75 - \frac{a}{2}.$$

Similarly,

$$2\angle DEA + a = 180 \Rightarrow \angle DEA = 90 - \frac{a}{2}.$$

Using  $\triangle ADC$ , we have,  $a + 90 - \frac{a}{2} + x + 75 - \frac{a}{2} = 180 \Rightarrow x = 165 - a = 180 - a \Rightarrow x = 15^\circ$ .

**NMC 2015 O-LEVEL PAPER 1 QUESTIONS**

**SECTION A**

- Qn1.** Evaluate  $(\frac{1}{16})a^0 + (\frac{1}{16})^0 - 64^{-\frac{1}{2}} - (32)^{-\frac{4}{5}}$ .
- Qn2.** Simplify the following expression  $\frac{200012}{200013^2 - 200012 \times 200014}$ .
- Qn3.** After finding the average of 125 scores, a student carelessly included the average with the 125 scores and found the average of these 126 numbers. What is the relationship between the true average and the second average obtained?
- Qn4.** Let  $f(2x - 1) = 4x^2 - 10x + 16$ . Find  $f(x)$ .
- Qn5.** A number  $M$  can be written as 47 in base  $x$  and as 74 in base  $y$ . Assuming that both  $x$  and  $y$  are positive integers, find the least possible value of  $x + y$ .
- Qn6.** Evaluate  $x^4$  where  $x = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$ .
- Qn7.** Some chocolates are missing and one of three children ate them. At least one child told the truth and at least one lied. The children made the following statements.  
 (i) Jonh: *Mary did not eat them.* (ii) Mary: *Either John or Tom ate them.*  
 (iii) Tom: *John ate them.*  
 Who actually ate the chocolates?
- Qn8.** Without using a calculator, evaluate  $G(x) = \frac{1}{\sqrt{\frac{3}{2} + \frac{2}{\sqrt{2}}}} + \frac{1}{\sqrt{\frac{3}{2} - \frac{2}{\sqrt{2}}}}$ .
- Qn9.** The 3–digit number  $2a3$  is added to the number 326 to give the 3–digit number  $5b9$ . Given that  $5b9$  is divisible by 9, what is the value of  $a + b$ .
- Qn10.** The number of boys that wrote a mathematics contest in 2014 was 20% higher than in 2013. The number of girls that wrote the contest in 2014 had increased by 50% over the same period, while the total number of contestants had increased by only 30%. What fraction of the students that wrote the contest in 2014 were girls?

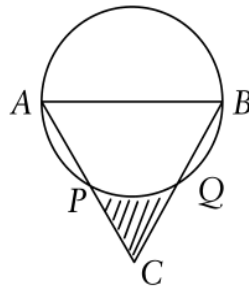
**SECTION B**

- Qn11.** The lengths of the sides of an acute angled triangle are successive integers. Show that the altitude to the second longest side divides it into two segments whose difference in length equals 4.
- Qn12.** A list of five positive integers has mean 12 and range 18. The mode and the median are both 8. Find the largest possible integer in the list?
- Qn13.** (a) Let  $\log_a b = \log_b a$  where  $a \neq b$ ,  $ab > 0$ ,  $a \neq 1$  and  $b \neq 1$ . Find the value of  $ab$ .

- (b) John walked a certain distance at a constant rate. If he had gone  $\frac{1}{2}$  km/hr faster, he would have walked the distance in four-fifths of the time. If he had gone  $\frac{1}{2}$  km/hr slower, he would have been  $2\frac{1}{2}$  hours longer on the road. Find the distance travelled?

**Qn14.** Student lockers at a university are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost shs.200 each. If it costed shs.1,390,600 to label all the lockers, how many lockers are there at the university?

**Qn15.** In the diagram below,  $ABC$  is an equilateral triangle whose base  $AB$  is a diameter of the circle with  $AB = 12$  cm and  $AC, BC$  intersect the circle at points  $P$  and  $Q$  respectively as shown. Find the area of the shaded region.



## NMC 2015 O-LEVEL PAPER 1 SOLUTIONS

### SECTION A

**Qn1.**  $\left(\frac{1}{16}\right) a^0 + \left(\frac{1}{16}\right)^0 - 64^{-\frac{1}{2}} - (32)^{-\frac{4}{5}} = \frac{1}{16} + 1 - \frac{1}{\sqrt{64}} - \frac{1}{(\sqrt[5]{32})^4} = \frac{1}{16} + 1 - \frac{1}{8} - \frac{1}{16} = 1 - \frac{1}{8} = \frac{7}{8}$ .

**Qn2.** Let  $200012 = n$ . Then  $\frac{200012}{200013^2 - 200012 \times 200014} = \frac{n}{(n+1)^2 - n(n+2)} = \frac{n}{n^2 + 2n + 1 - n^2 - 2n} = n = 200012$ .

**Qn3.** True average,  $x = \frac{x_1 + x_2 + x_3 + \dots + x_{125}}{125}$ .

Fake average  $y = \frac{x_1 + x_2 + x_3 + \dots + x_{125} + x}{126} = \frac{125x + x}{126} = \frac{126x}{126} = x$ .

Therefore the true average and the fake average are the same.

**Qn4.** Let  $g(x) = f(2x - 1) = 4x^2 - 10x + 16$ . Then  $g\left(\frac{x+1}{2}\right) = f\left(\frac{2(x+1)}{2} - 1\right) = f(x)$ . Hence

$$\begin{aligned} f(x) &= 4\left(\frac{x+1}{2}\right)^2 - 10\left(\frac{x+1}{2}\right) + 16 \\ &= (x+1)^2 - 5(x+1) + 16 \\ &= x^2 + 2x + 1 - 5x - 5 + 16 \\ &= x^2 - 3x + 12. \end{aligned}$$

Therefore the function  $f(x) = x^2 - 3x + 12$ .

**Qn5.**  $(47)_x = (74)_y$  implies that  $4x + 7 = 7y + 4$ . So,  $7y - 4x = 3 \implies x = \frac{7y-3}{4}$ .

We note that both  $x$  and  $y$  are integers each greater than 7. Suppose  $y = 8$ , then  $x = \frac{56-3}{4} = \frac{53}{4}$  which is not an integer. Suppose  $y = 9$ , then  $x = \frac{63-3}{4} = \frac{60}{4} = 15$ .

The least value of  $x + y = 15 + 9 = 24$ .

**Qn6.** Evaluate  $x^4$  where  $x = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$ . Since  $\sqrt{1} = 1$ , then  $x = \sqrt{1 + \sqrt{2}}$ .

$$x^2 = \left(\sqrt{1 + \sqrt{2}}\right)^2 = 1 + \sqrt{2}. \text{ Hence } x^4 = (x^2)^2 = (1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}.$$

**Qn7.** Suppose John ate the chocolates, then all the three told the truth, a contradiction. If Mary ate the chocolates, then all the three told lies which is also a contradiction. If Tom ate the chocolates, then John and Mary told the truth and Tom lied. Hence Tom ate the chocolates.

**Qn8.**

$$G(x) = \frac{1}{\sqrt{\frac{3}{2} + \frac{2}{\sqrt{2}}}} + \frac{1}{\sqrt{\frac{3}{2} - \frac{2}{\sqrt{2}}}} = \frac{\sqrt{\frac{3}{2} - \frac{2}{\sqrt{2}}} + \sqrt{\frac{3}{2} + \frac{2}{\sqrt{2}}}}{\sqrt{\left(\frac{3}{2} + \frac{2}{\sqrt{2}}\right)\left(\frac{3}{2} - \frac{2}{\sqrt{2}}\right)}}$$

Let  $a = \sqrt{\frac{3}{2} + \frac{2}{\sqrt{2}}}$  and  $b = \sqrt{\frac{3}{2} - \frac{2}{\sqrt{2}}}$ . Then  $(a + b)^2 = a^2 + b^2 + 2ab = \frac{3}{2} - \frac{2}{\sqrt{2}} + \frac{3}{2} + \frac{2}{\sqrt{2}} +$

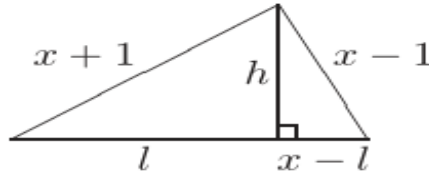
$$2\sqrt{\left(\frac{9}{4} - \frac{4}{2}\right)} = 3 + 2 \times \frac{1}{2} = 4. \text{ Therefore } \frac{\sqrt{\frac{3}{2} - \frac{2}{\sqrt{2}}} + \sqrt{\frac{3}{2} + \frac{2}{\sqrt{2}}}}{\sqrt{\left(\frac{3}{2} + \frac{2}{\sqrt{2}}\right)\left(\frac{3}{2} - \frac{2}{\sqrt{2}}\right)}} = \frac{\sqrt{4}}{0.5} = \frac{2}{0.5} = 4.$$

**Qn9.** Since  $5b9$  is divisible by 9, then the sum  $5 + b + 9 = 14 + b$  is also a multiple of 9. Thus the digit  $b = 4$ . Now,  $2a3 = 5b9 - 326 = 549 - 326 = 223$ . This implies that  $a = 2$ . Hence  $a + b = 4 + 2 = 6$ .

**Qn10.** Let  $x$  and  $y$  be the numbers of boys and girls, respectively, that wrote the contest in 2013. Then the corresponding numbers for the year 2014 are  $1.2x$  and  $1.5y$ . The total number of contestants in 2014 are  $1.3(x + y)$ . We solve  $1.2x + 1.5y = 1.3(x + y)$  to get  $0.2y = 0.1x$  or  $x = 2y$ . The required fraction is  $\frac{1.5y}{1.3(x+y)} = \frac{1.5y}{1.3(3y)} = \frac{1.5}{3.9} = \frac{5}{13}$ .

## SECTION B

**Qn11.** Let the lengths of the triangle be  $x - 1$ ,  $x$  and  $x + 1$  and the length of the perpendicular projection be  $h$  and the length of the longer section of the side of length  $x$  be  $l$  as shown.



Therefore

$$\begin{aligned}
 h^2 &= (x + 1)^2 - l^2 &= (x - 1)^2 - (x - l)^2 \\
 x^2 + 2x + 1 - l^2 &= x^2 - 2x + 1 - x^2 + 2lx - l^2 \\
 4x + x^2 &= 2lx \\
 4 + x &= 2l. \\
 l - (x - l) &= 2l - x = 4.
 \end{aligned}$$

Therefore the altitude to the second longest side divides it into two segments whose difference in length equals 4.

**Qn12.** Let the numbers be  $x_1, x_2, x_3, x_4, x_5$  in ascending order. Clearly  $x_3 = 8$  and  $x_5 = 18 + x_1$ . Therefore the numbers are  $x_1, x_2, 8, x_4, x_1 + 18$ . The mean is 12 which implies that  $x_1 + x_2 + 8 + x_4 + x_1 + 18 = 60$  or  $2x_1 + x_2 + x_4 = 34$ . Since the mode is 8, it is possible that  $x_2 = 8, x_4 = 8$  or both. Suppose  $x_2 = x_4 = 8$ , then  $2x_1 = 18 \implies x_1 = 9$ . This is impossible given that  $x_1$  is the smallest so either is 8 but not both.

Suppose that  $x_4 = 8$ , then  $2x_1 + x_2 = 26$  and  $x_1 \leq x_2 < 8$ . There are no possible pairs  $(x_1, x_2)$  satisfying the two so  $x_2 = 8$ .

In this case  $2x_1 + x_4 = 26$  and  $x_1 \leq 8 < x_4 < x_1 + 18$ . The largest possible value of  $x_1$  is 8 in which the largest possible integer is  $8 + 18 = 26$ .

We only check that  $x_4 = 26 - 16$  does not violate  $8 < x_4 < x_1 = 18$  which is the case.

**Qn13.** (a) Let  $\log_a b = \log_b a$ . This implies that  $b = a^x$  and  $a = b^x$ . Multiplying the two equations, we get  $ab = (ab)^x$ . There are three possibilities.

$$(i) \quad ab = 0 \qquad (ii) \quad x = 1 \text{ making } a = b \qquad (iii) \quad ab = 1.$$

The first two are excluded by the conditions  $a \neq b$  and  $ab > 0$ . We remain with the third possibility that  $ab = 1$ .

(b) All the three walking rates and the corresponding times yield the same distance. Let  $R, T$  denote the first rate and the corresponding time, then we have the following distance expressions.

$$RT = \left(R + \frac{1}{2}\right)\left(\frac{4}{5}\right)T \tag{1}$$

$$RT = \left(R - \frac{1}{2}\right)\left(T + \frac{5}{2}\right) \tag{2}$$

From (1),  $RT - \frac{4}{5}RT = \frac{4}{10}T$ . Hence  $\frac{1}{5}RT = \frac{2}{5}T \implies RT = 2T$ .

From Equation (2),  $RT = (2 - \frac{1}{2})(\frac{RT}{2} - \frac{5}{2}) = \frac{3}{2}(\frac{RT+5}{2}) = \frac{3}{4}(RT + 5)$ .

Thus  $RT = 15$ . The distance travelled is 15km.

**Qn14.** The locker labelling requires  $\frac{1390600}{200} = 6953$ . Lockers 1 through 9 require 9 digits. Lockers 10 through 99 require  $2 \times 90 = 180$  digits. Lockers 100 through 999 require  $3 \times 900 = 2700$  digits. Hence the remaining lockers require  $6953 - 2700 - 180 - 9 = 4064$  digits, and so there must be  $\frac{4064}{4} = 1016$  more lockers each using 4 digits. In all, there are  $1016 + 999 = 2015$  student lockers.

**Qn15.** Let  $N$  be the centre of the circle and then draw the two radii  $NP$  and  $NQ$  so that  $AN = NB = NP = NQ = 6\text{cm}$ .  $\triangle ABC$  is equilateral so  $\angle PAN = 60^\circ$  and hence  $\angle APN = 60^\circ$  since  $AN = NP$ . It follows that  $\angle ANP = 60^\circ$  so,  $\triangle ANP$  is equilateral.

Similarly,  $\angle QNB = 60^\circ$  and therefore  $\angle PNQ = 60^\circ$  as well. Let  $\otimes$  be the area of the figure bounded by the points. Then

$$\begin{aligned}\otimes ABC &= \otimes ANP + \otimes BNQ + \otimes NPQ + \otimes PQC \\ \otimes PQC &= \otimes ABC - (\otimes ANP + \otimes BNQ + \otimes NPQ)\end{aligned}$$

$$\text{But } \otimes ANP = \frac{1}{2} \times a \times b \times \sin \theta = \frac{1}{2} \times 6 \times 6 \times \sin 60^\circ = 9\sqrt{3}.$$

$$\text{Similarly } \otimes BNQ = \frac{1}{2} \times a \times b \times \sin \theta = \frac{1}{2} \times 6 \times 6 \times \sin 60^\circ = 9\sqrt{3}.$$

$$\text{Also } \otimes NPQ = \frac{\theta}{360} \times \pi r^2 = \frac{60}{360} \times \pi \times 6^2 = 6\pi.$$

$$\otimes ABC = \frac{1}{2} \times 12 \times 12 \times \sin 60^\circ = \frac{72\sqrt{3}}{2} = 36\sqrt{3}.$$

$$\text{Hence } \otimes PQC = 36\sqrt{3} - 9\sqrt{3} - 9\sqrt{3} - 6\pi = 18\sqrt{3} - 6\pi = 6(3\sqrt{3} - \pi)\text{cm}^2.$$

## NMC 2015 O-LEVEL PAPER 2 QUESTIONS

### SECTION A

**Qn1.** For what value of  $k$  is the displayed factoring valid?

$$2015^3 + 2015^4 = k(1 + 2015).$$

**Qn2.** When the decimal point of a certain positive decimal number is moved four places to the right, the new number is nine times the reciprocal of the original number. What is the original number?

- Qn3.** A line with slope 2 intersects a line with slope 5 at the point  $(10, 20)$ . What is the distance between the  $x$ -intercepts of these two lines?
- Qn4.** For what value(s) of  $x$  is  $8xy - 12y + 2x - 3 = 0$  true for all values of  $y$ ?
- Qn5.** An employee received a salary raise of 2% last year and 3% this year. After these raises, her current salary is UGX 651,372. What was her salary two years ago (before the first of the two salary raises).
- Qn6.** Alex, Barbra and Charity share Shs1,000,000 among themselves not necessarily getting equal amounts, to be invested in different ways. At the end of one year they have a total of Shs1,500,000. Barbra and Charity have both doubled their money, where as Alex has lost Shs100,000. What was Alex's original portion?
- Qn7.** Let  $g$  be the function such that  $g\left(\frac{x}{3}\right) = x^2 + x + 1$ . Find all values of  $z$  for which  $g(3z) = 7$ .
- Qn8.** Two numbers are such that their difference, their sum and their product are in the ratio  $1 : 7 : 24$  respectively. Find the product of these two numbers.
- Qn9.** Suppose  $A, B$  and  $C$  are three numbers for which  $1001C - 2002A = 4004$  and  $1001B + 3003A = 5005$ . What is the average of the three numbers  $A, B$  and  $C$ ?
- Qn10.** A company sells milk in cylindrical jars. Marketing research suggests that using wider jars will increase sales. Suppose that the diameter of the jars is increased by 25% without altering the volume. By what percentage must the height be decreased?

### SECTION B

- Qn11.** For all real  $x$ ,  $f(x + 1) + f(x) = 1$ . If  $f(9) = 18$ . What is the value of  $f(1) + 2$ ?
- Qn12.** (a) Let  $f$  be the function defined by  $f(x) = ax^2 - \sqrt{2}$  for some positive  $a$ . If  $f(f(\sqrt{2})) = -\sqrt{2}$ , what is the value of  $a$ ?
- (b) When students sat in a dinning hall for lunch each on a separate table, there was one person left. As a result they decided to sit two per table and one table was left free. How many tables were in the dinning hall?
- Qn13.** (a) Country A has  $c\%$  of the world's population and owns  $d\%$  of the world's wealth. Country B has  $e\%$  of the world's population and  $f\%$  of its wealth. Assume that the citizens of A share the wealth of A equally, and assume that those of B share the wealth of B equally. Find the ratio of the wealth of a citizen of A to the wealth of a citizen of B.
- (b) Some boys and girls are having a car wash to raise money for a class trip to China. Initially 40% of the group are girls. Shortly thereafter two girls leave and two boys arrive, and then 30% of the group are girls. How many girls were initially in the group?
- Qn14.** (a) One of the solutions of the equations  $(a - w)(a - x)(a - y)(a - z) = 9$  is  $a = 2$ . If  $w, x, y, z$  are different integers, find the value of  $w + x + y + z$ .
- (b) Two non-zero real numbers  $x$  and  $y$  satisfy the relation  $xy = x - y$ . Find the value of  $\frac{x}{y} + \frac{y}{x} - xy$ .

**Qn15.** Point  $P$  is inside an equilateral triangle  $\triangle ABC$ . Points  $Q, R$  and  $S$  are the feet of the perpendicular from  $P$  to  $\overline{AB}, \overline{BC}$  and  $\overline{CA}$  respectively. Given that  $\overline{PQ} = 1, \overline{PR} = 2$  and  $\overline{PS} = 3$ , what is  $\overline{AB}$ ?

## NMC 2015 O-LEVEL PAPER 2, SOLUTIONS

### SECTION A

**Qn1.**  $2015^3 + 2015^4 = 1015^3(1 + 2015)$ . So, by comparing with  $k(1 + 2015)$  we have  $k = 2015^3$ .

**Qn2.** Let the number be  $y$ . By moving the decimal point four places to the right, it is the same as multiplying the by 10000. Hence  $10000y = 9\left(\frac{1}{y}\right) \implies 10000y^2 = 9$ .

$$\text{Therefore } y = \sqrt{\frac{9}{10000}} = \frac{3}{100} = 0.03.$$

**Qn3.** The equation of a line is given by  $y = mx + c$ . For the line of slope 2 that passes through the point  $(10, 20)$ , its equation is given by  $\frac{y-20}{x-10} = 2 \implies y = 2x$ . Similarly, the equation of

the line of slope 5 that passes through the point  $(10, 20)$ , is  $\frac{y-20}{x-10} = 5 \implies y = 5x - 30$ . The  $x$ -intercepts are obtained by setting  $y = 0$  in the respective equations, hence  $x = 0$  and  $x = 6$ . Therefore the distance between the points is  $6 - 0 = 6$ .

**Qn4.**

$$\begin{aligned} 8xy - 12y + 2x - 3 &= 0 \\ 4y(2x - 3) + 2x - 3 &= 0 \\ (2x - 3)(4y + 1) &= 0 \\ x = 1.5 \text{ or } y &= -0.25. \end{aligned}$$

**Qn5.** Let the original salary be  $x$ . Then  $(1.02x) \times (1.03) = 651372 \implies 1.0586x = 651372$  or  $x = \frac{651372}{1.0586} = 620,000$ .

**Qn6.** Denote the original portions for Alex, Betty, and Charity as  $a, b$ , and  $c$  respectively. Then  $a + b + c = 1000000$  and  $a - 100000 + 2(b + c) = 1500000$ . Substituting  $b + c = 1000000 - a$  in the second equation, we have  $a - 100000 + 2(1000000 - a) = 1500000$ . This yields  $a = \text{Shs. } 400,000$  which is Alex's original portion.

**Qn7.** Let  $x = 9z$ , then  $g\left(\frac{9z}{3}\right) = 81z^2 + 9z + 1$ . This implies that  $81z^2 + 9z + 1 = 7$ . Solving this equation by factorization method, we have  $3(3z + 1)(9z + 2) = 0$ . Thus  $z = -\frac{1}{3}$  or  $z = \frac{2}{9}$ .

**Qn8.** Let the numbers be represented by  $x$  and  $y$ . Then  $x + y = 7(x - y) \implies 8y = 6x$ .

$$\text{Also, } xy = 24(x - y) \implies xy = 24x - 24y = 24x - 18x = 6x \implies x(y - 6) = 0.$$

Since  $x \neq 0$ , it implies  $y = 6$ . Also since  $6x = 8y \implies x = 8$ . Thus  $xy = 48$ .

**Qn9.** Adding the two equations gives  $1001A + 1001B + 1001C = 9009$ . Hence

$$1001(A + B + C) = 9009$$

$$A + B + C = 9$$

$$\frac{A + B + C}{3} = 3.$$

The average of the three numbers  $A$ ,  $B$  and  $C$  is 3.

**Qn10.** Let  $r$ ,  $h$  and  $v$  respectively be the radius, height and volume of the jar that is currently being used. Then the new jar will have the radius of  $1.25r$  and the volume is maintained as  $v$ . If  $H$  is the height of the new jar, then  $v = \pi r^2 h = \pi (1.25r)^2 H \implies \frac{H}{h} = \frac{1}{(1.25)^2} = 0.64$ . Therefore  $H$

is 64% of  $h$ , so the height of a old jar must be reduced by  $(100 - 64)\% = 36\%$

## SECTION B

**Qn11.**

$$f(9) + f(8) = 1 \implies f(8) = 1 - f(9).$$

$$f(8) + f(7) = 1 \implies f(7) = 1 - f(8) = 1 - 1 + f(9) = f(9).$$

$$f(7) + f(6) = 1 \implies f(6) = 1 - f(7) = 1 - f(9).$$

$$f(6) + f(5) = 1 \implies f(5) = 1 - f(6) = 1 - 1 + f(9) = f(9).$$

$$f(5) + f(4) = 1 \implies f(4) = 1 - f(5) = 1 - f(9).$$

$$f(4) + f(3) = 1 \implies f(3) = 1 - f(4) = 1 - 1 + f(9) = f(9).$$

$$f(3) + f(2) = 1 \implies f(2) = 1 - f(3) = 1 - f(9).$$

$$f(2) + f(1) = 1 \implies f(1) = 1 - f(2) = 1 - 1 + f(9) = f(9).$$

Therefore  $f(1) + 2 = f(9) + 2 = 18 + 2 = 20$ .

**Qn12.** (a) Since  $a \neq 0$ , the only  $x$  for which  $f(x) = -\sqrt{2}$  is  $x = 0$ . Since  $f(f(\sqrt{2})) = -\sqrt{2}$ , then

$$f(\sqrt{2}) \text{ must be zero. Finally } f(\sqrt{2}) = 0 \implies a(\sqrt{2})^2 - \sqrt{2} = 0. \text{ Hence } a = \frac{\sqrt{2}}{2}.$$

**ALTERNATIVELY**

$$f(f(\sqrt{2})) = f(2a - \sqrt{2}) = a(2a - \sqrt{2})^2 - \sqrt{2} = -\sqrt{2}.$$

$$\text{Hence } a(2a - \sqrt{2})^2 = 0 \implies 2a - \sqrt{2} = 0 \text{ so that } a = \frac{\sqrt{2}}{2}.$$

(b) Let  $s$  be the number of students and  $t$  the number of tables. Then  $s = t + 1$  and  $s = 2(t - 1)$ . Since the two equations are equal, we have  $t + 1 = 2t - 2 \implies t = 3$ . Thus, there are 3 tables in the dining hall.

- Qn13.** (a) Country A has  $c\%$  of the world's population and owns  $d\%$  of the world's wealth. Let  $P$  be the world's population and  $W$  its wealth. Then  $\frac{Pc}{100}$  citizens of A together own  $\frac{Wd}{100}$  units of wealth. So, each citizen of A owns  $\frac{Wd}{100} \div \frac{Pc}{100} = \frac{Wd}{100} \times \frac{100}{Pc} = \frac{Wd}{Pc}$  units of wealth.

Similarly, each citizen of B owns  $\frac{Wf}{100} \div \frac{Pe}{100} = \frac{Wf}{100} \times \frac{100}{Pe} = \frac{Wf}{Pe}$  units of wealth.

The required ratio is thus  $\frac{Wd}{Pc} \div \frac{Wf}{Pe} = \frac{Wd}{Pc} \times \frac{Pe}{Wf} = \frac{de}{cf}$ .

- (b) Let  $g$  be the number of girls and  $b$  be the number of boys initially in the group. Then  $g = 0.4(g + b)$ .

After the two girls leave and two boys arrive, the size of the entire group is unchanged. So  $g - 2 = 0.3(g + b)$ .

Solving these two equations simultaneously, gives  $g - 2 = 0.3 \times \frac{g}{0.4} = 0.75g$ .

This implies  $(1 - 0.75)g = 2 \implies g = 8$ .

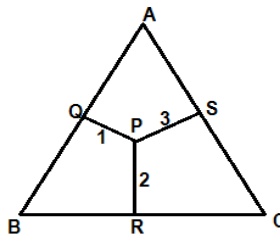
Thus  $b = \frac{0.6g}{0.4} = \frac{0.6 \times 8}{0.4} = \frac{4.8}{0.4} = 12$ . Therefore, initially we had 8 girls in the group.

- Qn14.** (a) Since  $a = 2$  is a solution to the equation, then the product  $(2-w)(2-x)(2-y)(2-z) = 9$ . Since  $w \neq x \neq y \neq z$ , all the four factors on the left side must have different values. i.e  $(2-w)(2-x)(2-y)(2-z) = 3 \times -3 \times 1 \times -1$  not necessarily in the same order. Thus  $2-w+2-x+2-y+2-z = 3-3+1-1 = 0 \implies 8-(w+x+y+z) = 0 \implies w+x+y+z = 8$ .

(b)  $\frac{x}{y} + \frac{y}{x} - xy = \frac{x^2+y^2-(xy)^2}{xy} = \frac{x^2+y^2-(x-y)^2}{xy} = \frac{x^2-y^2-x^2+2xy-y^2}{xy} = \frac{2xy}{xy} = 2$ .

- Qn15.** Let the side length of  $\triangle ABC$  be  $s$ . Then, the area of  $\triangle APB = \frac{1}{2}bh = \frac{1}{2} \times s \times 1 = \frac{1}{2}s$ .

The area of  $\triangle BPC = \frac{1}{2}bh = \frac{1}{2} \times s \times 2 = s$  and the area of  $\triangle CPA = \frac{1}{2}bh = \frac{1}{2} \times s \times 3 = \frac{3s}{2}$ .



The area of  $\triangle ABC$  is the sum of these three areas. Therefore area of  $\triangle ABC = \frac{1}{2} \times s \times \frac{\sqrt{3}}{2} =$

$\frac{\sqrt{3}}{4}s^2 = s + \frac{1}{2}s + \frac{3s}{2} = 3s$ . Hence  $3s = \frac{\sqrt{3}}{4}s^2 \implies (12 - \sqrt{3}s)s = 0$ .

Since  $s \neq 0$ , then  $12 - \sqrt{3}s = 0 \implies s = \frac{12}{\sqrt{3}} = 4\sqrt{3}$ .

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**NMC 2016 0-LEVEL PAPER 1 QUESTIONS**

**SECTION A**

- Qn1.** The reciprocal of  $1 - \frac{1}{x}$  is  $-2$ . What is the value of  $x$ .
- Qn2.** Let  $A + R = 17$ ,  $R + M = 1$ , and  $M + L = 24$ . Compute  $A + L$ .
- Qn3.** Suppose  $(6^{30} + 6^{-30})(6^{30} - 6^{-30}) = 3^x 8^y - 3^{-x} 8^{-y}$ . Find the value of  $x$  and  $y$ .
- Qn4.** Given that  $(24_b)^2 = 642_b$  where  $b > 1$ . Find the value of  $b$ .
- Qn5.** Trickster Rabbit agrees with Foolish Fox to double Fox's money every time Fox crosses the bridge nearby Rabbit's house, as long as Fox pays Shs.400 in toll to Rabbit after each crossing. The payment is made after the doubling, Fox is excited about his good fortune until he discovers that all his money is gone after crossing the bridge three times. How much money did Fox have at the beginning?
- Qn6.** Charles was born in the twentieth century. On his birthday in the year 2014, he noticed that his age was twice the number formed by the rightmost two digits of the year in which he was born. Compute the four-digit year in which Charles was born.
- Qn7.** Given that

$$\frac{a}{b+d+c} = \frac{4}{3} \text{ and } \frac{a}{b+c} = \frac{3}{5}.$$

Find the value of  $\frac{d}{a}$ .

- Qn8.** Peter, Sofia, and Mary got some eggs at a party. Peter had three times as many eggs as Sofia, and Sofia had twice as many eggs as Mary. Peter decides to give some of his eggs to Sofia and Mary so that all three will have the same number of eggs. What fraction of his eggs should Peter give to Sofia?
- Qn9.** Denis, Henry, Jack, Martha, Ronald, and Tom were in a 12-person race with 6 other people. Ronald finished 6 places ahead of Henry. Martha finished 1 place behind Jack. Denis finished 2 places behind Henry. Jack finished 2 places behind Tom. Tom finished 1 place behind Ronald. Martha finished in 6th place. Who finished in 8th place?
- Qn10.** At Mulago Hospital one year, multiple birth statistics were as follows: Sets of twins, triplets, and quadruplets accounted for 1000 of the babies born. There were four times as many sets of triplets as sets of quadruplets, and there was three times as many sets of twins as sets of triplets. How many of these 1000 babies were in sets of triplets?

**SECTION B**

- Qn11.** The sum of the lengths of the three sides of a right angled triangle is 56. The sum of the squares of the lengths of the three sides of the same right angled triangle is 1250. What is the area of the triangle?
- Qn12.** A circle is inscribed in a right angled triangle measuring  $3cm$ ,  $4cm$  and  $5cm$ , Without use of construction method, find the circumference of the circle?
- Qn13.** The four numbers  $a < b < c < d$  can be paired in six different ways. If each pair has a different sum and if the four smallest sums are 1, 2, 3 and 4. hat are all possible values of  $D$ .
- Qn14.**  $ABC$  is a right angled triangle. Points  $D, E$  and  $F$  are on the sides  $\overline{AB}, \overline{AC}$  and  $\overline{BC}$  respectively such that  $\overline{AD} = \overline{AE}$  and  $\overline{CE} = \overline{CF}$ . Find angle  $DEF$ .

- Qn15.** (a) Prove that if three consecutive integers are multiplied and the middle number is added to the result, the number obtained will always be a cube.
- (b) Brenda's school is on a semester system, with three terms per year. The grade point average(GPA) for a student is the sum of the percentage grades (maximum 100%) in each course divided by the total number of courses. The first term she took five courses, and her GPA was 75%. The second term she took four courses, and her GPA was 80%, in the third term, she took three courses. In one of these courses, her grade was 90%. If her overall GPA for the full year is 80%, what is the lowest possible grade she could have obtained in one of the other two courses she took in the third term?

### NMC 2016 0-LEVEL PAPER 1 SOLUTIONS

#### SECTION A

**Qn1.**

$$\frac{1}{1 - \frac{1}{x}} = -2 \implies \frac{x}{x-1} = -2$$

$$x = -2x + 2 \implies x = \frac{2}{3}.$$

**Qn2.** Adding the three equations gives

$$\begin{aligned} A + R + M + L + R + M &= 17 + 24 + 1 = 42 \\ A + L + 2(R + M) &= 42 \\ A + L &= 42 - 2 = 40. \end{aligned}$$

**Alternatively;** Note that  $A + L = (A + R) + (M + L) - (R + M) = 17 + 24 - 1 = 40$ .

**Qn3.** By using difference of two squares we have  $(6^{30})^2 - (6^{-30})^2 = 3^x 8^y - 3^{-x} 8^{-y}$   
 $6^{60} - 6^{-60} = 3^x 8^y - 3^{-x} 8^{-y}$  This means that  $6^{60} = 3^x 8^y = 3^x 2^{3y}$  or  $2^{60} 3^{60} = 3^x 2^{3y}$ . Clearly  $x = 60$  and  $3y = 60 \implies y = 20$ .

**Qn4.** Expressing in base ten, we have  $(2b + 4)^2 = 6b^2 + 4b + 2$ . Now

$$\begin{aligned} 4b^2 + 16b + 16 &= 6b^2 + 4b + 2 \\ b - 6b - 7 &= 0 \\ (b - 7)(b + 1) &= 0 \\ b &= 7 \text{ or } -1. \end{aligned}$$

Since  $b > 1$ , we conclude that  $b = 7$ .

**Qn5.** Let  $x$  be the amount that the Fox has before crossing for the third time. Then we have that  $2x - 400 = 0 \implies x = 200$ . This means Fox remained with 200 at the end of the second crossing. Then if  $y$  is the amount that remained before the second crossing, then  $2y - 400 = 200 \implies y = 300$ . Then if  $z$  is the initial amount Fox had before the first crossing, then  $2z - 400 = 300 \implies z = 350$ . Therefore Fox initially had Shs. 350.

**Qn6.** Let  $N$  be the number formed by the rightmost two digits of the year in which Charlie was born. Then his current age is  $100 - N + 14 = 114 - N$ . Setting this equal to  $2N$  and solving yields  $N = 38$ , hence the answer is 1938.

**Alternatively**

Let  $N$  be the number formed by the rightmost two digits of the year in which Charlie was born. The number of years from 1900 to 2014 can be thought of as the number of years before Charlie was born plus the number of years since he was born, or  $N$  plus Charlie's age. Thus  $N + 2N = 114$ , which leads to  $N = 38$ , so the answer is 1938.

**Qn7.**  $\frac{a}{b+d+c} = \frac{4}{3}$  and  $\frac{a}{b+c} = \frac{3}{5} \implies b+c = \frac{5a}{3}$ . Substituting this equation into the first we get

$$3a = 4\left(d + \frac{5a}{3}\right) = 4d + \frac{20a}{3}$$

$$\frac{-11a}{3} = 4d \implies \frac{d}{a} = \frac{-11}{12}.$$

**Qn8.** Assign a variable to the number of eggs Mary has, say  $m$ . Then, because we are given that Sofia has twice the number of eggs Mary has, Sofia has  $2m$  eggs, and Peter, having three times the number of eggs as Sofia, has  $6m$  eggs.

For them to all have the same number of eggs, they must each have  $\frac{m+2m+6m}{3} = 3m$  eggs. This means Peter must give  $2m$  eggs to Mary and  $m$  eggs to Sofia, so the answer is  $\frac{m}{6m} = \frac{1}{6}$ .

**Qn9.** Because Martha was 6th, Jack was 5th, so Tom was 3rd. Thus, Ronald was 2nd and the 8th place finisher was Henry.

**Qn10.** We can set up a system of equations where  $a$  is the sets of twins,  $b$  is the sets of triplets, and  $c$  is the sets of quadruplets.

$$\begin{aligned} 2a + 3b + 4c &= 1000 \\ b &= 4c \\ a &= 3b \end{aligned}$$

Solving for  $c$  and  $a$  in the second and third equations and substituting into the first equation yields

$$\begin{aligned} 2(3b) + 3b + 4(0.25b) &= 1000 \\ 6b + 3b + b &= 1000 \\ b &= 100 \end{aligned}$$

Since we are trying to find the number of babies and NOT the number of sets of triplets, the solution  $4c$ . Therefore, we strategically use the second initial equation to realize that  $b = 4c$ , leaving us with the number of babies born as triplets equal to 100.

### SECTION B

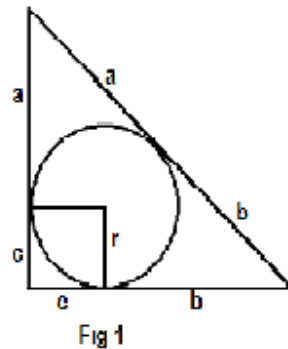
**Qn11.** Let the lengths be  $a, b, c$ . Then we have the following equations.

$$\begin{aligned} a + b + c &= 56 \\ a^2 + b^2 + c^2 &= 1250 \\ a^2 + b^2 &= c^2 \end{aligned}$$

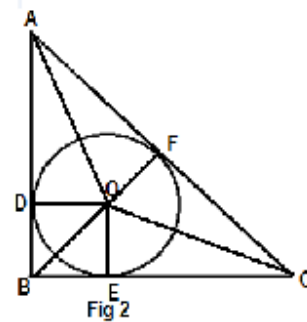
Combining the second and third equations, we get  $2c^2 = 1250 \implies c = 25$ . Hence  $a + b + 25 = 56 \implies a + b = 31$ . This put in the second equation gives  $a^2 + b^2 = 1250 - 625 = 650$ .

From the identity  $(a + b)^2 = a^2 + b^2 + 2ab$  we have  $2ab = (a + b)^2 - (a^2 + b^2) = 31^2 - 25^2 = 336 \implies ab = 168$ . Therefore the area of a triangle  $A = \frac{1}{2}ab = \frac{1}{2} \times 168 = 84$  sq. units.

**Qn12.** Consider Fig 1 and we draw the lines from the incentre to the points of tangency, we know that these meet at right angles so that the square of side  $c$  is formed at the right angle. Hence  $c = r$  where  $r$  is the radius of the circle. But



or



$$\begin{aligned} a + b &= 5 \\ b + c &= 4 \\ a + c &= 3. \end{aligned}$$

Subtracting the third equation from the second equation we get  $b - a = 1$ . This equation added to the first equation gives  $2b = 6 \implies b = 3$ . It follows that  $a = 2$  and  $c = 1$ . The radius of the circle is 1 and the circumference is thus  $2\pi r = 2\pi \times 1 = 2\pi$ .

**Alternatively by considering Fig 2**

We draw the lines  $OA, OB, OC$  from the incentre  $O$  to the points  $A, B, C$  and the perpendiculars  $OD, OE, OF$  from the incentre to the  $D, E, F$  of tangency.

Let the lengths of these perpendiculars be  $r$  which is the radius of the circle.

Area of  $\triangle ABC =$  Area of  $\triangle BOC +$  Area of  $\triangle AOC +$  Area of  $\triangle AOB$ . The lines  $OD, OE, OF$  are the heights of these triangles, hence

$$\frac{1}{2} \times 3 \times 4 = \frac{1}{2} \times 4 \times r + \frac{1}{2} \times 3 \times r + \frac{1}{2} \times 5 \times r$$

Hence  $6 = 2r + 1.5r + 2.5r \implies 6 = 6r \implies r = 1$ .

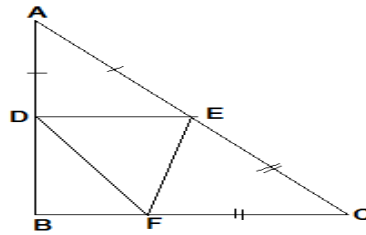
Hence the circumference of the circle is  $2\pi r = 2\pi \times 1 = 2\pi$ .

**Qn13.** The six possible sums are  $a + b, a + c, a + d, b + c, b + d$  and  $c + d$ . Of these the smallest sums are  $a + b = 1$  and  $a + c = 2$ . These two equations combined give  $c = 1 + b$ . The largest possible sums are  $c + d$  and  $b + d$ . The sums in the middle are therefore  $b + c$  and  $a + d$ .

Suppose  $b + c = 3$ . This implies  $2b + 1 = 3 \implies b = 1$ . Hence  $a = 0$ . And  $a + d = 4$  implies  $0 + d = 4 \implies d = 4$ .

Suppose  $b + c = 4$ . This implies  $2b + 1 = 4 \implies b = 1.5$ . Hence  $a = -0.5$  and  $a + d = 3$  implies  $-0.5 + d = 3 \implies d = 3.5$ .

**Qn14.** Let  $\angle DEF = x, \angle AED = p, \angle CEF = q$ . Then  $180^\circ = p + q + x$ .



$$180 = \frac{180 - A}{2} + \frac{180 - C}{2} + x$$

$$180 = 180 + x - \frac{(A + C)}{2}$$

$$180 = 180 + x - 45$$

$$x = 45^\circ$$

- Qn15.** (a) Let  $x$  be the middle integer. Then the other two integers are  $x - 1$  and  $x + 1$ . Therefore  $(x - 1)(x)(x + 1) + x = x[(x - 1)(x + 1) + 1] = x(x^2 - 1 + 1) = x^3$  which is a cube as required.
- (b) Suppose  $x$  and  $y$  are the grades Brenda obtained in the other two courses, then her overall average for the full year was

$$\frac{5(75) + 4(80) + 90 + x + y}{12} = 80$$

$$x + y = 960 - (5(75) + 4(80) + 90) = 175.$$

$$x + y = 175.$$

The highest percentage grade she could obtain in either course is 100%.  
 Suppose  $x = 100\%$ , then  $y = 175 - 100 = 75\%$  which is the minimum possible mark.

### NMC 2016 0-LEVEL PAPER 2 QUESTIONS

#### SECTION A

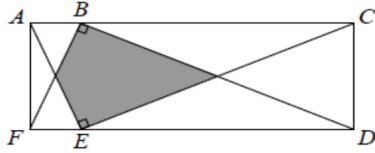
- Qn1.** List all ordered triples  $(a, b, c)$  of positive integers which satisfy  $a + \frac{b}{c} = 11$  and  $b + \frac{a}{c} = 14$ .
- Qn2.** In triangle  $ABC$ ,  $AB = 3$ ,  $BC = 4$ , and  $AC = 6$ . If  $BC$  is extended through  $C$  to  $D$  so that  $CD = BC$ , what is  $AD$ ?
- Qn3.** Ampeirwe's Convenience Centre buys a calculator for  $p$  shillings (where  $p > 0$ ), raises its price by  $n\%$ , then reduces this new price by  $20\%$ . If the final price is  $20\%$  higher than  $p$ , what is the value of  $n$ ?
- Qn4.** A function  $f$  is defined so that if  $n$  is an odd integer, then  $f(n) = n - 1$  and if  $n$  is an even integer, then  $f(n) = n^2 - 1$ . For example, if  $n = 15$ , then  $f(n) = 14$  and if  $n = -6$ , then  $f(n) = 35$ , since  $15$  is an odd integer and  $-6$  is an even integer. Determine all integers  $n$  for which  $f(f(n)) = 3$ .

- Qn5.** A student takes three exams. The second has twice as many questions as the first, and the third has three times as many questions as the first. The student answers exactly 75% of the questions correctly on the first exam, exactly 81% on the second exam, and exactly 85% on the third exam. Out of all the questions on the three exams, what percentage did he answer wrongly?
- Qn6.** The price of a ticket to attend a football match was Shs.10,000 but due to a sponsorship deal by a company to pay up a certain amount of the ticket price for each person attending the match it was reduced. The number of spectators increased by 50%, but the amount of money received only increased by 20%. How much was the new ticket price after reduction?
- Qn7.** A large container, labeled  $R$ , is partially filled with 4 litres of red paint. Another large container, labeled  $W$ , is partially filled with 5 litres of white paint. A small empty can is completely filled with red paint taken from  $R$ , and the contents of the can then emptied into  $W$ . After thorough mixing of the contents of  $W$ , the can is completely filled with some of this mixture from  $W$ , and the contents of the can then emptied into  $R$ . The ratio of red paint to white in  $R$  is now 3 : 1. What is the size of the can, in litres?
- Qn8.** For what non-zero value of the radius,  $r$ , does the volume of a sphere equal its surface area?
- Qn9.** What is the tens digit of the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, and 15?
- Qn10.** Determine all values of  $x$  for which  $(2 \cdot 4^{x^2-3x})^2 = 2^{x-1}$ .

### SECTION B

- Qn11.** My cat keeps to himself most of the time. I only heard him meow, hiss, and buzz on one out of the last 23 days. But I did hear him make at least one of these sounds each day. I heard him hiss on 6 days, buzz on 12 days, and meow on 15 days. On 2 days, I heard him meow and hiss but not buzz, and on 2 days, I heard him buzz and hiss but not meow. On how many days did I hear him meow and buzz but not hiss?
- Qn12.** Nankya and Kintu are driving on a circular track with radius 25 km. Nankya leaves the starting line first, going clockwise around the track at a speed of 80 km/h. Fifteen minutes after Nankya starts, Kintu leaves the same starting line, going counter-clockwise around the track at a speed of 100 km/h. For how many hours will Kintu have been driving when the two of them pass each other for the fourth time?
- Qn13.** In a psychology experiment, an image of a cat or an image of a dog is flashed briefly onto a screen and then Apio is asked to guess whether the image showed a cat or a dog. This process is repeated a large number of times with an equal number of images of cats and images of dogs shown. If Apio is correct 95% of the time when she guesses “dog” and 90% of the time when she guesses “cat”, determine the ratio of the number of times she *guessed* “dog” to the number of times she *guessed* “cat”.

**Qn14.** In the diagram,  $ACDF$  is a rectangle with  $AC = 200$  and  $CD = 50$ . Also,  $\triangle FBD$  and  $\triangle AEC$  are congruent triangles which are right-angled at  $B$  and  $E$ , respectively. What is the area of the shaded region?



**Qn15.** A formula for the sum of the first  $n$  positive integers is  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ .

- (a) What is the sum of the first 200 positive integers,  $1 + 2 + 3 + \dots + 198 + 199 + 200$  ?
- (b) Calculate the sum of the 50 consecutive integers beginning at 151, that is,

$$151 + 152 + 153 + \dots + 198 + 199 + 200.$$

- (c) Starting with the sum of the first 1000 positive integers,  $1 + 2 + 3 + \dots + 999 + 1000$ , every third integer is removed to create the new sum

$$1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \dots + 998 + 1000.$$

Calculate this new sum.

### NMC 2016 O-LEVEL PAPER 2 SOLUTIONS

#### SECTION A

**Qn1.**

$$a + \frac{b}{c} = 11 \implies c = \frac{b}{11 - a} \tag{1}$$

$$b + \frac{a}{c} = 14 \implies c = \frac{a}{14 - b} \tag{2}$$

Equating the two equations (1) and (2)

$$\frac{b}{11 - a} = \frac{a}{14 - b} \quad \text{or} \quad (11 - a)a = (14 - b)b$$

Clearly  $a < 11$  and  $b < 14$  since  $a, b, \& c$  are positive integers.

By taking  $a \in \{1, 2, \dots, 10\}$ , the solution set for the LHS is  $\{10, 18, 24, 30\}$ . Likewise, the solution set for the RHS is  $\{13, 24, 33, 40, 48, 49\}$

The common solution is 24, which happens when  $a = 8$  or  $a = 3$  and  $b = 12$  or  $b = 2$ .

Case 1 For  $a = 3, b = 2$  will make  $c$  in Eqn (2) non integer.

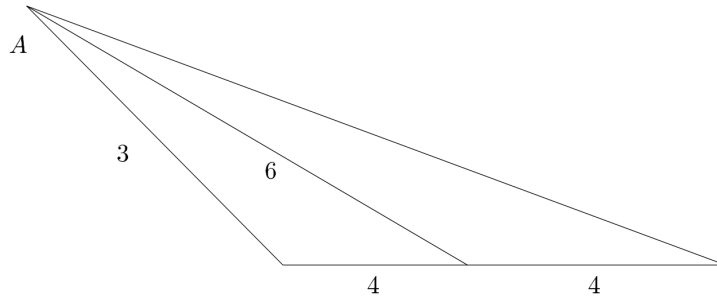
Case 2 For  $a = 3, b = 12$  will make  $c$  in Eqn (2) non integer.

Case 3 For  $a = 8, b = 2$  will make  $c$  in Eqn (2) non integer.

For  $a = 8, b = 12$  gives an integer value of  $c = 4$ . Thus the solution set

$$(a, b, c) = (8, 12, 4)$$

**Qn2.**



By the Cosine rule

$$6^2 = 3^2 + 4^2 - 24 \cos B \Rightarrow \cos B = -\frac{11}{24}$$

Similarly,

$$AD^2 = 3^2 + 8^2 - 24 \cos B$$

$$AD^2 = 3^2 + 8^2 - 24 \left( -\frac{11}{24} \right)$$

$$\begin{aligned} AD^2 &= 73 + 22 = 95 \\ \Leftrightarrow AD &= \sqrt{95} \end{aligned}$$

**Qn3.** When the price of  $p$  shillings is raised by  $n\%$ , the price is multiplied by  $1 + \frac{n}{100}$

When the new price is reduced by  $20\%$ , the new price is multiplied by  $1 - \frac{20}{100} = \frac{80}{100}$  Therefore, after these two price adjustments, the price is

$$p \left( 1 + \frac{n}{100} \right) \left( \frac{80}{100} \right)$$

We are told that this final price is  $20\%$  higher than  $p$ , and so the final price equals

$$p \left( 1 + \frac{20}{100} \right) \text{ or } p \left( \frac{120}{100} \right)$$

In other words,

$$p \left( 1 + \frac{n}{100} \right) \left( \frac{80}{100} \right) = p \left( \frac{120}{100} \right)$$

Simplifying and using the fact that  $p \neq 0$ , we obtain

$$80 \left( 1 + \frac{n}{100} \right) = 120$$

Thus,

$$1 + \frac{n}{100} = \frac{120}{80} = \frac{3}{2} = \frac{150}{100}$$

and so

$$\frac{n}{100} = \frac{50}{100} \quad \text{or} \quad n = 50$$

**Qn4.** *Solution 1*

Let  $m = f(n)$ . The equation  $f(f(n)) = 3$  becomes  $f(m) = 3$ .

Suppose that  $f(m) = 3$  and  $m$  is odd. By definition, we have  $f(m) = m - 1 = 3$  and so  $m = 4$ , which is not odd, so this case cannot happen.

Suppose that  $f(m) = 3$  and  $m$  is even. By definition, we have  $f(m) = m^2 - 1 = 3$  and so  $m^2 = 4$  or  $m = \pm 2$ , each of which is even.

Therefore, if  $f(f(n)) = 3$ , then  $f(n) = 2$  or  $f(n) = -2$ .

Suppose that  $f(n) = 2$  or  $f(n) = -2$  and  $n$  is odd. By definition, we have  $n - 1 = 2$  (giving  $n = 3$ ) or  $n - 1 = -2$  (giving  $n = -1$ ). Each of these resulting values of  $n$  is odd.

Suppose that  $f(n) = 2$  or  $f(n) = -2$  and  $n$  is even. Then  $n^2 - 1 = 2$  or  $n^2 - 1 = -2$  which give  $n^2 = 3$  or  $n^2 = -1$ , neither of which is possible if  $n$  is an integer. Thus, the integers  $n$  for which  $f(f(n)) = 3$  are  $n = 3$  and  $n = -1$ . (We can check by substitution that each of these satisfies the original equation.)

*Solution 2*

We consider the cases of  $n$  even and  $n$  odd separately.

Suppose that  $n$  is even. Then  $n^2$  is even and so  $f(n) = n^2 - 1$  must be odd. Thus,  $f(f(n)) = f(n^2 - 1) = (n^2 - 1) - 1 = n^2 - 2$ , since  $f(m) = m - 1$  when  $m$  is odd.

For  $n$  to be even and  $f(f(n)) = 3$ , we must have  $n^2 - 2 = 3$  or  $n^2 = 5$ . There are no integer solutions to this equation, and so there are no solutions in this case.

Suppose that  $n$  is odd. Then  $f(n) = n - 1$  must be even. Thus,  $f(f(n)) = f(n - 1) = (n - 1)^2 - 1 = n^2 - 2n + 1 - 1 = n^2 - 2n$ . For  $n$  to be odd and  $f(f(n)) = 3$ , we must have  $n^2 - 2n = 3$  or  $n^2 - 2n - 3 = 0$ . Factoring, we obtain  $(n - 3)(n + 1) = 0$  and so  $n = 3$  or  $n = -1$ , both of which are odd.

Thus, the integers  $n$  for which  $f(f(n)) = 3$  are  $n = 3$  and  $n = -1$ . (We can check by substitution that each of these satisfies the original equation.)

**Qn5.** Let the number of equations in the first exam be  $n$ , then the student answered correctly

$$\begin{aligned} & 0.75n + 0.81 \times 2n + 0.85 \times 3n \quad \text{out of } (n + 2n + 3n) \\ \Rightarrow & 0.75n + 0.81 \times 2n + 0.85 \times 3n \quad \text{out of } 6n \end{aligned}$$

Those answered correctly

$$\frac{0.75n + 0.81 \times 2n + 0.85 \times 3n}{6n} = \frac{0.75 + 0.81 \times 2 + 0.85 \times 3}{6} = 0.82 = 82\%$$

Thus 82% answered correctly, and  $100 - 82 = 18\%$  answered wrongly.

**Qn6.** If the new price ticket is  $x$  and the original number of spectators was  $N$ , then

$$1.5 \times Nx = 1.2 \times 10,000 \times N \quad \text{the increase in revenue by 2\%}$$

$$\Rightarrow x = \frac{1.2(10,000N)}{1.5N} = 8,000$$

is the new ticket price.

**Qn7.** Let  $x$  denote the size of the can. After the first transfer, container  $R$  has

$$4 - x \text{ Red}$$

And container  $W$  has

$$x \text{ Red} \quad , \quad 5 \text{ White}$$

After the second transfer,  $R$  has

$$\left( (4 - x) + x \frac{x}{5 + x} \right) \text{ Red} \quad , \quad x \frac{x}{5 + x} \text{ White}$$

Thus using the ratio 1 : 3

$$\begin{aligned} (4 - x)(5 + x) + x^2 &= 3(5x) \\ 20 &= 16x \end{aligned}$$

$$\Rightarrow x = \frac{20}{16} = \frac{5}{4} \text{ litres}$$

**Qn8.** For the volume of a sphere equalling its surface area  $\frac{4}{3}\pi r^3 = 4\pi r^2$ .

$$\frac{4}{3}\pi r^3 - 4\pi r^2 = 0$$

$$4\pi r^2 \left( \frac{1}{3}r - 1 \right) = 0$$

Thus the non zero  $r$  value is  $r = 3$ .

**Qn9.** Among the list 10, 11, 12, 13, 14, 15, the integers 11 and 13 are prime.

Also,  $10 = 2 \times 5$  and  $12 = 2 \times 2 \times 3$  and  $14 = 2 \times 7$  and  $15 = 3 \times 5$ .

For an integer  $N$  to be divisible by each of these six integers,  $N$  must include at least two factors of 2 and one factor each of 3, 5, 7, 11, 13.

Note that

$$22 \times 3 \times 5 \times 7 \times 11 \times 13 = 60060.$$

(This is the least common multiple of 10, 11, 12, 13, 14, 15.)

To find the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, 15, we can find the smallest six-digit positive integer that is a multiple of 60 060.

Note that  $1 \times 60\,060 = 60\,060$  and that  $2 \times 60\,060 = 120\,120$ . Therefore, the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, 15 is 120 120. Therefore, the tens digit of this number is 2.

**Qn10.** Using exponent laws, we obtain the following equivalent equations:

$$\left(2 \cdot 4^{x^2-3x}\right)^2 = 2^{x-1}$$

$$\begin{aligned} 2^2 \cdot 4^{2(x^2-3x)} &= 2^{x-1} \\ 2^2 \cdot (2^2)^{2x^2-6x} &= 2^{x-1} \end{aligned}$$

$$2^2 \cdot 2^{2(2x^2-6x)} = 2^{x-1}$$

$$\frac{2^2 \cdot 2^{4x^2-12x}}{2^{x-1}} = 1$$

$$\begin{aligned} 2^{2+4x^2-12x-(x-1)} &= 1 \\ 2^{4x^2-13x+3} &= 1 \end{aligned}$$

Since  $2^0 = 1$ , this last equation is true exactly when  $4x^2 - 13x + 3 = 0$  or  $(4x - 1)(x - 3) = 0$ . Therefore,  $x = \frac{1}{4}$  or  $x = 3$ . We can check by substitution that each of these values of  $x$  satisfies the original equation.

## SECTION B

**Qn11.** Let M denote the days on which the cat meowed, H the days on which he hissed, and P the days on which he buzzed. Given the data, the following Venn diagram represents the situation, where the expressions inside the circles are the cardinalities of disjoint sections:

We are looking for  $x$ . The labeled sections of the Venn diagram are disjoint sets, so they should add to 23. That is,

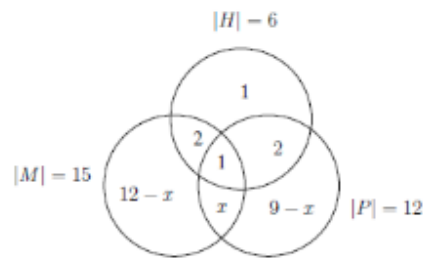
$$23 = 1 + 1 + 2 + 2 + x + (9 - x) + (12 - x) = 27 - x$$

Therefore,  $x = 4$ .

**Qn12.** Since the track is circular with radius 25 km, then its circumference is  $2\pi(25) = 50\pi$  km.

In the 15 minutes that Nankya drives at 80 km/h, he drives a distance of  $\frac{1}{4}(80) = 20$  km (because 15 minutes is one-quarter of an hour).

When Kintu starts driving, she drives in the opposite direction to Nankya. Suppose that Nankya and Kintu



meet for the first time after Kintu has been driving for  $t$  hours. During this time, Kintu drives at 100 km/h, and so drives  $100t$  km. During this time, Nankya drives at 80 km/h, and so drives  $80t$  km.

Since they start  $50\pi - 20$  km apart along the track (the entire circumference minus the 20 km that Nankya drove initially), then the sum of the distances that they travel is  $50\pi - 20$  km. Therefore,  $100t + 80t = 50\pi - 20$  and so  $180t = 50\pi - 20$  or  $t = \frac{5\pi - 2}{18}$ .

Suppose that Nankya and Kintu meet for the next time after an additional  $T$  hours. During this time, Kintu drives  $100T$  km and Nankya drives  $80T$  km. In this case, the sum of the distances that they drive is the complete circumference of the track, or  $50\pi$  km. Thus,  $180T = 50\pi$  or  $T = 5\pi/18$ .

The length of time between the first and second meetings will be the same as the amount of time between the second and third, and between the third and fourth meetings. Therefore, the total time that Kintu has been driving when she and Nankya meet for the fourth time will be

$$t + 3T = \frac{5\pi - 2}{18} + 3 \cdot \frac{5\pi}{18} = \frac{20\pi - 2}{18} = \frac{10\pi - 1}{9} \text{ hours}$$

**Qn13.** Suppose that Opio guesses “cat”  $c$  times and guesses “dog”  $d$  times.

When she guesses “dog”, she is correct 95% of the time.  
 When she guesses “cat”, she is correct 90% of the time.

Thus, when she guesses “cat”, she is shown  $0.9c$  images of cats and so  $c - 0.9c = 0.1c$  images of dogs.

Thus, when she guesses “dog”, she is shown  $0.95d$  images of dogs and so  $d - 0.95d = 0.05d$  images of cats.

(We assume that  $c$  and  $d$  have the property that  $0.9c$  and  $0.95d$  are integers.) Therefore, the total number of images of cats that she is shown is  $0.9c + 0.05d$  and the total number of images of dogs that she is shown is  $0.1c + 0.95d$ .

But the number of images of cats equals the number of images of dogs.  
 Thus,  $0.9c + 0.05d = 0.1c + 0.95d$ , which gives  $0.8c = 0.9d$  or  $\frac{d}{c} = \frac{0.8}{0.9}$

Therefore, the ratio of the number of times that she guessed “dog” to the the number of times that she guessed “cat” is 8 : 9.

**Qn14.** Since  $\triangle FBD$  is congruent to  $\triangle AEC$ , then  $FB = AE$ . Since  $\triangle FAB$  and  $\triangle AFE$  are each right-angled, share a common side  $AF$  and have equal hypotenuses ( $FB = AE$ ), then these triangles are congruent, and so  $AB = FE$ .

Now  $BAFE$  has two right angles at  $A$  and  $F$  (so  $AB$  and  $FE$  are parallel) and has equal sides  $AB = FE$  so must be a rectangle. This means that  $BCDE$  is also a rectangle.

Now the diagonals of a rectangle partition it into four triangles of equal area. (Diagonal  $AE$  of the rectangle

splits the rectangle into two congruent triangles, which have equal area. The diagonals bisect each other, so the four smaller triangles all have equal area.)

Since  $\frac{1}{4}$  of rectangle  $ABEF$  is shaded and  $\frac{1}{4}$  of rectangle  $BCDE$  is shaded, then  $\frac{1}{4}$  of the total area is shaded. (If the area of  $ABEF$  is  $x$  and the area of  $BCDE$  is  $y$ , then the total shaded area is  $\frac{1}{4}x + \frac{1}{4}y$ , which is  $\frac{1}{4}$  of the total area  $x + y$ .)

Since  $AC = 200$  and  $CD = 50$ , then the area of rectangle  $ACDF$  is  $200(50) = 10000$ , so the total shaded area is  $\frac{1}{4}(10000) = 2500$ .

**Qn15.** (a) Using the formula given, the sum of the first 200 positive integers

$$1 + 2 + 3 + \cdots + 198 + 199 + 200 = \frac{200(200 + 1)}{2} = 100(201) = 20100$$

(b) The sum of the first 200 positive integers is equal to the sum of the first 150 positive integers added to the sum of the 50 consecutive integers  $151 + 152 + 153 + \cdots + 198 + 199 + 200$ .

That is,

$$1 + 2 + \cdots + 198 + 199 + 200 = (1 + 2 + \cdots + 148 + 149 + 150) + (151 + 152 + \cdots + 198 + 199 + 200)$$

Therefore, the sum of the 50 consecutive integers  $151 + 152 + 153 + \cdots + 198 + 199 + 200$  is equal to the difference between the sum of the first 200 positive integers and the sum of the first 150 positive integers. From part (15a), the sum of the first 200 positive integers is 20100.

Using the formula, the sum of the first 150 positive integers

$$1 + 2 + 3 + \cdots + 148 + 149 + 150 = \frac{150(150 + 1)}{2} = 10075(151) = 11325$$

Therefore, the sum of the 50 consecutive integers beginning at 151,

$$\begin{aligned} 151 + 152 + \cdots + 199 + 200 &= (1 + 2 + \cdots + 199 + 200) - (1 + 2 + \cdots + 149 + 150) \\ &= 20100 - 11325 \\ &= 8775 \end{aligned}$$

(c) Let the required sum,  $1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \cdots + 998 + 1000$ , be  $S$ .

Let the sum of the first 333 positive multiples of 3,  $3 + 6 + 9 + 12 + \cdots + 996 + 999$ , be  $T$ .

The sum of the first 1000 positive integers is equal to  $S + T$ . That is,

$$1 + 2 + \cdots + 998 + 999 + 1000 = (1 + 2 + 4 + 5 + 7 + \cdots + 998 + 1000) + (3 + 6 + \cdots + 996 + 999)$$

Therefore, the required sum  $S$  is equal to the difference between the sum of the first 1000 positive integers and  $T$ .

Using the formula, the sum of the first 1000 positive integers

$$1 + 2 + 3 + \cdots + 998 + 999 + 1000 = \frac{1000(1000 + 1)}{2} = 500(1001) = 500500$$

We may determine the sum  $T$  by first removing the common factor 3 from each term and then using the formula.

$$\begin{aligned}3 + 6 + 9 + \cdots + 993 + 996 + 999 &= 3(1 + 2 + 3 + \cdots + 331 + 332 + 333) \\ &= 3\left(\frac{3333(333 + 1)}{2}\right) \\ &= 3(333 \times 167) \\ &= 166833\end{aligned}$$

Finally, the required sum  $S$  is  $500500 - 166833 = 333667$ .

NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 1.

SATURDAY 18TH MARCH, 2017

9:00 AM - 12:15 PM

## INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
- (b) *There are **two** sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*
- (c) *Answer scripts of only qualified students should be returned to the contest coordinator, latest **Monday 20th March, 2017**.*
- (d) ***ALL** participants who qualify for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 29th July, 2017 at NOON at Makerere University, Kampala.*
- (e) *National Mathematics Contest 2017 Paper 2 will be done on 17th June, 2017 at various centres.*

## SECTION A : 5 marks each

1. If  $2x = 3y + 5$ . What is the value of  $\frac{4^x}{8^y}$ ?
2. What is the smallest positive integer which is divisible by all 1 digit primes but has no prime digits?
3. Define a function  $f$  on the set of integers by

$$f(n) = \begin{cases} n - 1 & \text{if } n \text{ is even} \\ n^2 - 1 & \text{if } n \text{ is odd.} \end{cases}$$

List all the values of  $n$  for which  $f(f(n)) = 8$ .

4. How many of the two digit numbers between 10 and 99 inclusive have the property that both are perfect squares and do not start with a 4.
5. John did push ups everyday for 7 days. Each day after the first day, he did 5 push ups more than the day before. In total he did 175 push ups. How many push ups did John do on the third day.
6. Find all positive real numbers  $x$  for which

$$\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8.$$

7. A square  $ABCD$  has sides of length  $3\text{cm}$ . Side  $AB$  is extended through  $B$  to  $E$  with  $BE = 1\text{cm}$ . Segment  $DE$  intersects  $BC$  at point  $F$ . What is the area of  $\triangle CDF$ ?



8. A Politician  $A$  lies on Mondays, Tuesdays and Wednesdays and tells the truth on other days of the week. Politician  $B$  lies on Thursdays, Fridays and Saturdays and tells the truth on other days of the week. One day both of them say "Yesterday was one of my lying days". What day is it when they say this if one says the truth and the other one lies?
9. Suppose a given population of a country increased by 20% in 2015 and decreased by 20% in 2016. What is the total percentage change over the two years period?
10. The price of a ticket was UGX10,000 before reduction. When the price was reduced the number of customers increased by 50% but the amount of money collected only increased by 20%. By how much was the ticket price reduced?

**SECTION B : 10 marks each**

11. Georges' car gets 3 more kilometers per litre during highway driving than it does during city driving. On a recent trip he drove 112 kilometers on the highway and 150 kilometers in the city and used exactly 10 litres of petrol. How many kilometers per litre does his car use during city driving?
12. Arrange the numbers from 1 to 9 in the  $3 \times 3$  grid below so that each number occurs once, the product of the entries in the first row is 12. Product of entries in the second row is 112, product of entries in the first column is 216 and that of the second column is 12. Find the value of  $A + B + C$ .

A		
	B	
		C

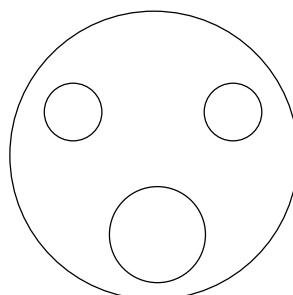
13. If Bonny and Mary leave their houses at the same time, walking directly towards each other, each at their own constant rate, they will meet after 5 minutes. If Bonny leaves 3 minutes later than Mary they meet after Bonny has walked for 3 minutes. How many minutes would it take him to walk all the way from his house to Mary's?
14. Find the ordered triple(s) of real numbers  $(a, b, c)$  which satisfy;

$$a^2 + 1 = 3b + c$$

$$b^2 + 33 = 7c - 3a$$

$$c^2 - 5 = b - 3a$$

15. The radius of each small circle is  $\frac{1}{6}$  of the radius of the large surrounding circle. The radius of the middle sized circle is twice that of either small circle. What fraction of the area of the large circle lies outside the three interior circles?





**SECTION A : 5 marks each**

Qn. 1.

$$\frac{4^x}{8^y} = \frac{2^{2x}}{2^{3y}} = 2^{2x-3y} = 2^5$$

since  $2x - 3y = 5$ .

Qn. 2. The number must be a multiple of  $2 \cdot 3 \cdot 5 \cdot 7 = 210$  that is 210, 420, 630 or 840  $\Rightarrow$  the requested number is 840.

Qn. 3.  $f(n)$  is odd when  $n$  is even and even when  $n$  is odd so,

$$f(f(n)) = \begin{cases} (n-1)^2 - 1 & \text{if } n \text{ is even} \\ n^2 - 2 & \text{if } n \text{ is odd.} \end{cases}$$

Now if  $f(f(n)) = 8 \Rightarrow (n-1)^2 - 1 = 8 \Rightarrow n = -2$  or  $n = 4$ .

Qn. 4. The digits that form two digit perfect squares are 0, 1, 4, 9 so that there are three choices to fill the first position and four choices for the second position thus there are twelve numbers whose two digits are perfect squares.

Excluding those that begin with a digit 4 that is 40, 41, 44 and 49 there are 8 digits that have both digits as perfect squares. [10, 11, 14, 19, 90, 91, 94, 99]

Qn. 5. John did 175 push ups for 7 days (an odd number of days) and on each day he did an equal number of push ups more than the day before (5 more).

The number of push ups that John did on the middle day (Day 4) is equal to the average number of push ups that he completed each day, Mean  $\bar{X} = \frac{175}{7} = 25$  push ups per day. On day 4 he did 25 push ups on the third day.

DAY I - 10, DAY II - 15, DAY III - 20, DAY IV - 25, DAY V - 30, DAY VI - 35, DAY VII - 35

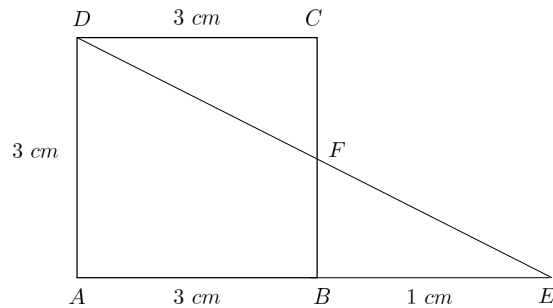
Qn. 6.

$$\begin{aligned} \log_4 x - \log_x 16 &= \frac{7}{6} - \log_x 8. \\ \log_4 x &= \frac{7}{6} + \log_x 16 - \log_x 8 \\ &= \frac{7}{6} + \log_x 2 \\ &= \frac{7}{6} + \frac{\log_4 2}{\log_4 x} \end{aligned}$$

Let  $\log_4 x = y$  so that we have  $y = \frac{7}{6} + \frac{1}{y}$  on solving we get  $y = \frac{3}{2}$  and  $y = -\frac{1}{3}$  thus

$$x = 8 \text{ or } x = \frac{1}{4^{\frac{1}{3}}}$$

Qn. 7. From the diagram By similarity of triangles,  $\frac{BF}{BE} = \frac{3}{4}$  thus  $BF = \frac{3}{4}BE$ . Now  $CF = 3 - \frac{3}{4} = \frac{9}{4}$ .



The area

$$A = \frac{1}{2}b \cdot h = \frac{1}{2} \cdot 3 \cdot \frac{9}{4} = \frac{27}{8} \text{ cm}^2.$$

Qn. 8. The only days that politician *A* would make that statement are Monday and Thursday. The only days that politician *B* would make that statement are Thursday and Sunday  $\Rightarrow$  both made the statement on Thursday if one lies and the other says the truth.

Qn. 9. Let the population of the country be  $N$  before increase then  $N$  would increase to  $1.2N$  and then a decrease of 20% would lead to

$$0.8 \cdot (1.2 \cdot N) = 0.96N.$$

Therefore the total change is  $N - 0.96N = 0.04N$  equivalent to 4%.

Qn. 10. If the new fee is  $x$  and the original number of customers was  $N$  then

$$1.2 \times 10,000 \cdot N = 1.5Nx \Rightarrow x = \frac{12000}{1.5} = 8,000/ =$$

so the ticket was reduced by 2,000/ = .



**SECTION B : 10 marks each**

Qn. 11. Let  $x$  denote the kilometers his car gets during city driving then

$$\frac{112}{x+3} + \frac{150}{x} = 10 \Rightarrow (5x+9)(x-25) = 0 \Rightarrow$$

either  $x = -\frac{9}{5}$  or  $x = 25$ . Therefore the number of kilometers his car gets during city driving is 25km/litre.

Qn. 12. We can fill in the 7, 5 and then the 4 since the last column needs 7 and 5 powers of 2. We can fill in 8 and 2 in the second row then the 6 and 1 in the second column and finally the 3 and 9 in the first column.

3	1	4	$=2^2 \cdot 3$
8	2	7	$=2^4 \cdot 7$
9	6	5	
$2^3 \cdot 3^3$	$2^2 \cdot 3$		

$$A + B + C = 3 + 2 + 5 = 10.$$

Qn. 13. Let  $S_B$  and  $S_M$  denote the speeds of Bonny and Mary respectively. Let  $d$  be the distance between their houses, then

$$5S_B + 5S_M = d = 3S_B + 6S_M \Rightarrow S_M = 2S_B$$

and

$$d = 15S_B$$

therefore Bonny takes 15 minutes to walk all the way to Mary's house.

Qn. 14. Add all the equations to get;

$$a^2 + b^2 + c^2 + 29 = 4b + 8c - 6a$$

Combining common terms we get

$$(a^2 + 6a) + (b^2 - 4b) + (c^2 - 8c) = -29$$

Completing squares we have

$$(a+3)^2 + (b-2)^2 + (c-4)^2 = 0 \Rightarrow a = -3, b = 2 \text{ and } c = 4$$

Qn. 15. Let the radius of the large circle be 1. The sum of the areas of three circles is

$$\pi \left( \frac{1}{36} + \frac{1}{36} + \frac{1}{9} \right) = \frac{\pi}{6}$$

therefore the fraction is  $\frac{5}{6}$  of the total area  $\pi$ .



NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 2.

SATURDAY 17TH JUNE, 2017

9:00 AM - 12:15 PM

**INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:**

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
- (b) *There are **two** sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*
- (c) *Answer scripts of only qualified students should be returned to the contest coordinator, latest **Monday 19th June, 2017.***
- (d) **ALL** participants who qualified and sat for Paper 2 are cordially invited to the certificate and/or prize giving ceremony on Saturday 29th July, 2017 at NOON at Makerere University, Kampala.

**SECTION A : 5 marks each**

- Qn. 1. If the price of sugar increases from 3,000 UGX per kilogram to 5,000 UGX per kilogram. By what percentage will the amount of sugar bought be reduced?
- Qn. 2. When 3430 is divided by the odd number  $n$ , the quotient is a positive prime number and the remainder is 0. What is  $n$ ?
- Qn. 3. Given that,
- $$\frac{1}{x} + \frac{1}{y} = \frac{1}{a}, \quad x + y = b \quad \text{and} \quad x^2 + y^2 = c^2.$$
- Find the possible relationships between  $a$  and  $b$  if  $c = 0$ .
- Qn. 4. What is the ratio of the area of a circle to the area of an inscribed square?
- Qn. 5. Mr. Okello can slash a square kilometre compound in 40 minutes, Chris can do the same job in 30 minutes. How many minutes will it take the two to slash the same compound if they both work at the same rate.
- Qn. 6. A lady drives at  $40\text{km/h}$  while going to visit her uncle. On arrival she realises her uncle had gone on a trip. Immediately she drives back to her home at  $90\text{km/h}$ . What is her average speed for the whole trip?
- Qn. 7. In a triangle  $ABC$ ,  $D$  lies on  $AC$ , and  $E$  lies on  $BC$  with  $DE$  parallel to  $AB$ . If  $AC = 5$ ,  $DC = 4$  and  $BE = 2$  then what is the length of  $EC$ ?



Qn. 8. A litre bottle of a liquid contains 10% alcohol, and another 3 litre bottle full of liquid contains 30% alcohol. They are mixed together. What percent of the mixture is not alcohol?

Qn. 9. Suppose that  $a$  and  $x$  are two positive real numbers for which

$$\log_x a + \log_a x = 3.$$

What is the value of

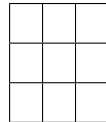
$$(\log_a x)^2 + (\log_x a)^2$$

Qn. 10. Mary paid UGX 480,000 to purchase a certain number of items, but the nice vendor gave her two extra. This decreased the price per item by UGX 1,000. How many items did she receive (including the two extra)?

**SECTION B : 10 marks each**

Qn. 11. In a group of five friends, the sums of the ages of each group of four of them are 125, 129, 131, 137, and 142. What is the age of the youngest of the friends?

Qn. 12. Suppose that the numbers 1, 2, 4, 8, 16, 32, 64, 128, and 256 are placed into the 9 squares in a  $3 \times 3$  grid in such a way that each of the 9 numbers appears exactly once and the product of the numbers appearing in any row or column is the same. What is the value of the product in each row and column?



Qn. 13. In racing over a given distance at uniform speeds, John can beat Isa by 20 meters, Isa can beat David by 10 meters, and John can beat David by 28 meters. How many meters is the distance over which they are racing.

Qn. 14. What is the units digit of  $2^{2017}$ ?

Qn. 15. A circle passes through two adjacent vertices of a square and is tangent to the opposite side of the square. If the side length of the square is 2. What is the radius of the circle?

**SECTION A : 5 marks each**

Qn. 1. Percentage increase in price is

$$\left( \frac{5,000 - 3,000}{3,000} \right) \times 100 = 66.67\%$$

The new price is 1.6667 times the original price  $\Rightarrow$  the amount of sugar that will be bought is

$$\frac{1}{1.6667} = 0.59998 \approx 0.6$$

of a kilogram thus the amount of sugar bought will be reduced by 0.4 that is 40%

Qn. 2.

$$3430 = 343 \times 10 = 7^3 \times 2 \times 5$$

The only way 3430 can be written so as to have an odd number and a prime number is

$$3430 = 2 \times (7^3 \times 5)$$

this implies then  $n = 7^3 \times 5 = 1715$ .

Qn. 3.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a} \Leftrightarrow \frac{x+y}{xy} = \frac{1}{a}$$

since  $x + y = b \Rightarrow ab = xy$  then considering

$$(x+y)^2 = x^2 + 2xy + y^2 \Rightarrow b^2 = c^2 + 2ab$$

then with  $c = 0$  implies

$$b^2 = 2ab \Rightarrow b(b - 2a) = 0 \Rightarrow b = 0$$

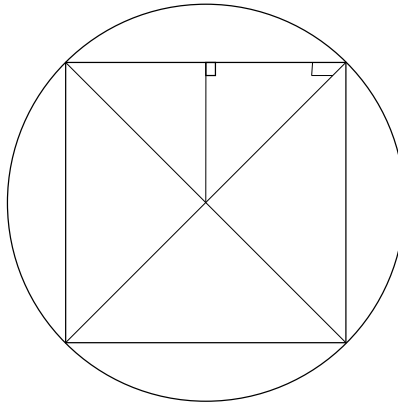
or

$$b = 2a$$

The only possible relationship between  $a$  and  $b$  is

$$a = \frac{b}{2} \text{ or } b = 2a.$$

Qn. 4. Using the figure below



Let  $r$  be the radius of the circle then the inscribed square is composed of four isosceles right angles with both sides equal to  $r$ . The area of the circle is  $\pi r^2$  and the area of the square is  $4 \cdot \left(\frac{1}{2}r^2\right) = 2r^2$ .

The ratio required is  $\pi r^2 : 2r^2 \Rightarrow \pi : 2$ .

Qn. 5. In one minute they can do slash  $\frac{1}{40} + \frac{1}{30}$  of the job. This is equivalent to

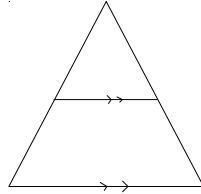
$$\frac{3 + 4}{120} = \frac{7}{120} \Rightarrow$$

the number of minutes to do the job combined is the reciprocal of this rate i.e  $\frac{120}{7}$ .

Qn. 6. Average speed =  $\frac{\text{total distance}}{\text{total time}}$ . Let  $D$  be the distance between her uncle's house and her house. Thus

$$\text{Average speed} = \frac{2D}{\frac{D}{40} + \frac{D}{90}} = \frac{2}{\frac{1}{40} + \frac{1}{90}} = \frac{720}{13} = 55.35 \text{ km/hr}$$

Qn. 7. See figure.



Triangles  $CDE$  and  $CAB$  are similar thus

$$CE : CD = EB : AD \Rightarrow CE : 4 = 2 : 1$$

implying that

$$\frac{CE}{4} = \frac{2}{1} \text{ and } CE = 8cm.$$

Qn. 8. Total amount of non alcohol liquid is  $0.9 \times 1 + 0.7 \times 3 = 0.1 + 2.1 = 3$ . The total amount of liquid is  $1 + 3 = 4 \Rightarrow$  the percentage of non alcohol liquid  $= \frac{3}{4} \times 100 = 75\%$

Qn. 9. Square both sides of

$$\log_x a + \log_a x = 3$$

to get

$$(\log_x a)^2 + 2 \log_a x \cdot \log_x a + (\log_a x)^2 = 9$$

this implies that

$$(\log_x a)^2 + (\log_a x)^2 = 9 - 2 = 7.$$

Qn. 10. Let  $x$  be the number of items that she received then

$$\frac{480,000}{x-2} - \frac{480,000}{x} = 1,000$$

solving gives  $x = 32$ .

**SECTION B : 10 marks each**

Qn. 11. Since there are only five four groups that can be formed then,

$$125 + 129 + 131 + 137 + 142 = 4 \text{ (sum of ages)} = 4 \times 166.$$

The youngest is  $166 - 142 = 24$  years.

Qn. 12. Note that the numbers

$$1, 2, 4, 8, 16, 32, 64, 128, 256$$

are got from the powers of 2 from 0, 1, 2, ... to 8. Since the powers of 2 from 0 to 8 are being placed in the square so that the sum in each row and column is the same then the common sum of the powers is

$$\frac{1 + 2 + 3 + \dots + 8}{3} = 12 \Rightarrow$$

thus the common product is  $2^{12} = 4096$ .

Qn. 13. Let the distance be  $d$  meters and their speeds be  $S_J, S_I$ , and  $S_D$  for John, Isa and David respectively. The following are possible relationships,

$$\frac{S_I}{S_D} = \frac{d}{d-10}; \frac{S_J}{S_D} = \frac{d}{d-28}$$

and

$$\frac{S_J}{S_I} = \frac{d}{d-20}$$

The relation

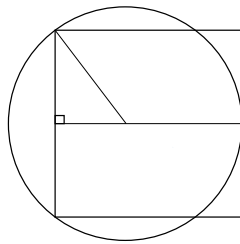
$$\frac{S_J}{S_I} \cdot \frac{S_I}{S_D} = \frac{S_J}{S_D}$$

gives

$$\frac{d}{d-20} \cdot \frac{d}{d-10} = \frac{d}{d-28}.$$

On solving we get  $d = 100$ .

- Qn. 14. The units digits of  $2^1, 2^2, 2^3$ , and  $2^4$  are respectively 2, 4, 8 and 6. Continuing the powers of 2 we have  $2^5 = 32, 2^6 = 64, 2^7 = 128, 2^8 = 256$ . we note therefore that these four digits repeat in the same order all through. The positive integers with increasing powers of 2 so that when the exponent is divisible by 4 units digit is 6. The nearest year to 2017 divisible by 4 is 2016 meaning  $2^{2017}$  is one step into the next cycle whose units digit is 2.
- Qn. 15. Given



From the right triangle in the diagram below we have that;

$$r^2 = 1 + (2 - r)^2$$

showing that  $r = \frac{5}{4}$  units.

NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 1.

SATURDAY 17TH MARCH, 2018

9:00 AM - 12:15 PM

**INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:**

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
- (b) *There are **two** sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*
- (c) *Answer scripts of only qualified students should be returned to the contest coordinator, latest **Monday 19th March, 2018.***
- (d) **ALL** *participants who qualify for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 28th July, 2018 at NOON at Makerere University, Kampala.*
- (e) *National Mathematics Contest 2018 Paper 2 will be done on 23rd June, 2018 at various centres.*

**SECTION A : 5 marks each**

- There are three consecutive positive integers such that the square of the second minus twelve times the first is three less than twice the third. What is the smallest of the three integers?
- In what positive base  $x$  does the equation  $4 \times 12 = 103$  for multiplication of base- $x$  numbers hold?
- What positive integers less than 100 have exactly three distinct factors?
- For what positive integer  $n$  is  $100^2 - n^2 = 9559$ ?
  - Determine one pair  $(x, y)$  of positive integers for which  $x > 1$  and  $y > 1$  and  $xy = 9559$ .
- Let
$$s_n = 1 - 2 + 3 - 4 + 5 - 6 + \dots + (-1)^{n+1}n.$$
Find  $s_1 + s_2 + s_3 + \dots + s_{2018}$ .
- What is the value of
$$(\log_3(\log_2(\log_4(65536))))^{2018}?$$
- In triangle  $ABC$ ,  $D$  lies on  $BC$  so that  $AC = 4$  cm,  $AD = 4$  cm,  $BD = 8$  cm, and  $CD = 1$  cm. What is the value of angle  $ABC$ ?
- A student has four shirts and three trousers. The four shirts are Red, Blue, Green and Yellow, and the three trousers are Red, Blue and Green. If the student picks a shirt and a trouser at random. What is the probability that they are of the same color?



9. In the Zagland, people measure length in two units, zigs and zags. A length of  $2x$  zigs is equal to a length of  $(2x + 6)$  zags and a length of  $(3x + 7)$  zigs is equal to a length of  $(3x + 22)$  zags. what is the value of  $x$ ?
10. The ratio of boys to girls in a class is  $7 : 5$ . If there are 6 more boys than girls, how many girls are in the class?

**SECTION B : 10 marks each**

11. If all people eat the same amount of pizza, and a pizza 16 inches in diameter serves two people, how many inches in diameter should each of two pizzas be in order to serve three people? (Pizzas are circular and are eaten entirely.)
12. How many pairs  $(x, y)$  satisfying  $10 \leq y < x$  and  $x + y \leq 99$  have the property that  $x + y$  and  $x - y$  have the same digits in reverse order? (This allows cases such as  $(44; 36)$  where the sum and difference are 80 and 08.)
13. John and David are running laps around a 500-meter oval track. They started together, but John has pulled ahead because he is 20% faster than David. How many laps will John have run when he
- (a) first passes David?
  - (b) passes David for the second time.
14. A certain sequence of numbers is defined by  $a_n = a_{n-1} + a_{n-2} + a_{n-3}$  for  $n \geq 4$  and  $n$  is a natural number. Suppose  $a_4 = 20$ ,  $a_5 = 36$ , and  $a_9 = 408$ . What is the value of  $a_1$ ?
15. Circle  $B$  passes through the center of circle  $A$  and is tangent to it. Circle  $C$  passes through the center of circle  $B$  and is tangent to it. What fraction of the area of circle  $A$  lies inside circle  $B$  but outside circle  $C$ ?

**SECTION A : 5 marks each**

- Qn. 1. There are three consecutive positive integers such that the square of the second minus twelve times the first is three less than twice the third. What is the smallest of the three integers?

solution

Let the numbers be  $x, x + 1$  and  $x + 2$ . Then  $(x + 1)^2 - 12x = 2(x + 2) - 3 \Rightarrow x(x - 12) = 0$ .  
So the smallest of the three numbers is either 0 or 12.

- Qn. 2. In what positive base  $x$  does the equation  $4 \times 12 = 103$  for multiplication of base- $x$  numbers hold?

solution

Convert every expression to base ten.  $4_x \times 12_x = 103_x$  is equivalent to

$$(4 \times x^0) \times [(1 \times x^1) + (2 \times x^0)] = (1 \times x^2) + (0 \times x^1) + (3 \times x^0)$$

whence  $(x - 5)(x + 1) = 0 \Rightarrow x = 0$  or  $x = 5$  and the positive base therefore is 5.

- Qn. 3. What positive integers less than 100 have exactly three distinct factors?

solution

A number to have three distinct factors has to be a perfect square and they are 4, 9, 25 and 49 with the three distinct factors as  $\{1, 2, 4\}$ ,  $\{1, 3, 9\}$ ,  $\{1, 5, 25\}$  and  $\{1, 7, 49\}$  respectively.

- Qn. 4. (a) For what positive integer  $n$  is  $100^2 - n^2 = 9559$ ?  
(b) Determine one pair  $(x, y)$  of positive integers for which  $x > 1$  and  $y > 1$  and  $xy = 9559$ .

solution

(a)  $100^2 - n^2 = 9559 \Rightarrow n^2 = 100^2 - 9559 = 441$  and therefore  $n = 21$ .

(b) Using the fact in (a) above we have  $100^2 - 21^2 = 9559$ . Using a difference of two squares we get  $100^2 - 21^2 = (100 - 21)(100 + 21)$  thus the numbers are  $x = 121, y = 79$  and  $x = 869, y = 11$  and the order could be interchanged.

- Qn. 5. Let  $s_n = 1 - 2 + 3 - 4 + 5 - 6 + \dots + (-1)^{n+1}n$ . Find  $s_1 + s_2 + s_3 + \dots + s_{2018}$ .

solution

The sums  $s_1 = 1, s_2 = -1, s_3 = 2, s_4 = -2, s_5 = 3, s_6 = -3, \dots$ . Their sums therefore are  $s_1 + s_2 = 0, s_1 + s_2 + s_3 = 2, s_1 + s_2 + s_3 + s_4 = 0, \dots$ . This sum will be zero always for  $n$  even thus  $s_1 + s_2 + \dots + s_{2018} = 0$ .

- Qn. 6. What is the value of

$$(\log_3(\log_2(\log_4(65536))))^{2018}?$$

solution

The value of  $(\log_3(\log_2(\log_4(65536))))^{2018}$  is obtained after evaluating  $\log_4(65536)$ . Since  $4^8 = 65536$  then  $\log_4(65536) = \log_4(4^8) = 8$ . So  $\log_2(8) = 3$  and hence  $(\log_3(3))^{2018} = 1$ .

- Qn. 7. In triangle  $ABC$ ,  $D$  lies on  $BC$  so that  $AC = 4$  cm,  $AD = 4$  cm,  $BD = 8$  cm, and  $CD = 1$  cm. What is the value of angle  $ABC$ ?

solution

From the diagram,  $4^2 = h^2 + (DE)^2 \Rightarrow \frac{63}{4}$ . Now  $(AB)^2 = h^2 + (BE)^2$  But  $BE = 8 + 0.5 = 8.5$  cm

So  $AB = \sqrt{\left(\frac{63}{4} + \frac{17^2}{4}\right)} = \frac{\sqrt{352}}{2}$  cm. From  $\sin \theta = \frac{\sqrt{63}}{2} \div \frac{352}{2} = 0.4231$  so

$$\theta = \sin^{-1}(0.4231) = 25.03^\circ$$

- Qn. 8. A student has four shirts and three trousers. The four shirts are Red, Blue, Green and Yellow, and the three trousers are Red, Blue and Green. If the student picks a shirt and a trouser at random. What is the probability that they are of the same color?

solution

Each of the four shirts has three possible trousers to wear with thus  $4 \times 3 = 12$  possible matches irrespective of color. Matching colors has only three possibilities (Red shirt, Red trouser, Green shirt Green trouser and Blue shirt Blue trouser) thus the probability is  $\frac{3}{12} = \frac{1}{4}$ .

- Qn. 9. In the Zagland, people measure length in two units, zigs and zags. A length of  $2x$  zigs is equal to a length of  $(2x + 6)$  zags and a length of  $(3x + 7)$  zigs is equal to a length of  $(3x + 22)$  zags. what is the value of  $x$ ?

solution

$2x$  zigs =  $(2x + 6)$  zags  $\Rightarrow 1$  zig =  $\frac{x+3}{x}$  zags. Now  $(3x + 7)$  zigs =  $(3x + 7)\frac{x+3}{x} = 3x + 22$  zags therefore  $x = \frac{7}{2}$ .

- Qn. 10. The ratio of boys to girls in a class is  $7 : 5$ . If there are 6 more boys than girls, how many girls are in the class?

solution

Let the number of girls in the class be  $g$ . Then the number of boys in the class is  $g + 6$  so  $\frac{g+6}{g+(g+6)} = \frac{7}{7+5} \Rightarrow g = 15$ .

**SECTION B : 10 marks each**

- Qn. 11. If all people eat the same amount of pizza, and a pizza 16 inches in diameter serves two people, how many inches in diameter should each of two pizzas be in order to serve three people? (Pizzas are circular and are eaten entirely.)

solution

Let the diameter of each pizza be  $d$  inches. Then the total amount of pizza in the 2 pizzas is

$$\pi \times \left(\frac{d}{2}\right)^2 \times 2 = \frac{\pi d^2}{2}. \text{ Pizza eaten by one person} = \frac{\pi 8^2}{2} \text{ so pizza eaten by 3 persons is } 3 \times \frac{\pi 8^2}{2}$$

$$\text{therefore } 3 \times \frac{\pi 8^2}{2} = \frac{\pi d^2}{2} \Rightarrow d = 8\sqrt{3}.$$

- Qn. 12. How many pairs  $(x, y)$  satisfying  $10 \leq y < x$  and  $x + y \leq 99$  have the property that  $x + y$  and  $x - y$  have the same digits in reverse order? (This allows cases such as (44; 36) where the sum and difference are 80 and 08.)

solution

Let  $x + y = 10n + m$  and  $x - y = 10m + n$  where  $10m + n$  the largest value that the sum and difference can take on. Adding these equations gives  $2x = 11(n + m)$  meaning  $x$  is a multiple of 11 and possible values are  $x = \{11, 22, 33, 44, 55, 66, 77, 88, 99\}$  subtracting the equations yields  $2y = 9(n - m)$  thus  $y$  is a multiple of 9 whose possible values are  $y = \{9, 18, 27, 36, 45, 54, 63, 70, 81, 90, 99\}$ . With restrictions  $10 \leq y < x$  and  $x + y \leq 99$  then there are **twelve pairs** satisfying thus

$$\{(x, y) = (22, 18), (33, 18), (33, 27), (44, 18), (44, 27), (44, 36), (55, 18), (55, 27), (55, 36), (66, 18), (66, 27), (77, 18)\}$$

- Qn. 13. John and David are running laps around a 500-meter oval track. They started together, but John has pulled ahead because he is 20% faster than David. How many laps will John have run when he
- first passes David?
  - passes David for the second time.

solution

Let the number of laps John will have run when he first passes David be  $l$  and the number of laps David will have run then be  $m$ . Then John will have covered  $500l$  meters and David will have covered  $500m$  meters. Suppose John is running at  $vms^{-1}$  then John is running at  $1.2v ms^{-1}$ . Since at the point John passes David they will both have used the sametime then  $\frac{500l}{1.2v} = \frac{500m}{v} \rightarrow l = 1.2m$  that is  $\frac{6}{5}$ . Since full laps have to be whole numbers,

- David will have done 5 laps and John 6 by the time John first passes David.
- John will pass David the second time when John has done 12 laps and David 10.

- Qn. 14. A certain sequence of numbers is defined by  $a_n = a_{n-1} + a_{n-2} + a_{n-3}$  for  $n \geq 4$  and  $n$  is a natural number. Suppose  $a_4 = 20, a_5 = 36$ , and  $a_9 = 408$ . What is the value of  $a_1$ ?

solution

Write down equations out of the sequence for  $n = 4$  and  $n = 5$  that is  $a_1 + a_2 + a_3 = a_4 = 20$  and  $a_2 + a_3 + a_4 = 36$ . solving the two yields  $a_1 = 4$ .

- Qn. 15. Circle  $B$  passes through the center of circle  $A$  and is tangent to it. Circle  $C$  passes through the center of circle  $B$  and is tangent to it. What fraction of the area of circle  $A$  lies inside circle  $B$  but outside circle  $C$ ?

solution

Fraction of the area required =  $\frac{\text{shaded area}}{\text{area of } A}$ . Let  $r$  be the radius of the circle  $C$  then radius of

circle  $B$  is  $2r$  and radius  $A$  is  $4r$ . Fraction of the area required =  $\frac{\pi(2r)^2 - \pi r^2}{\pi(4r)^2} = \frac{3}{16}$ .

NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 2.

SATURDAY 23RD JUNE, 2018

9:00 AM - 12:15 PM

## INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

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- (b) *There are two sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm).*
- (c) *Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.*
- (d) **Indicate your names, class, school and district on all your answer sheets.**
- (e) *Answer scripts of only qualified students should be returned to the contest coordinator, latest Monday 25th June, 2018.*
- (f) *ALL participants who qualified for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 28th July, 2018 at 8:00am at Makerere University, Kampala.*

## SECTION A : 5 marks each

Qn. 1. Let

$$x \star y = 2x^2 + 2xy.$$

If  $a$  and  $b$  are positive integers such that  $a \star b = 9$  and  $b \star a = 72$ , what is the value of  $\frac{a}{b}$ . Express your answer as a common fraction.

Qn. 2. Tom can paint a fence in 15 hours by himself. Ben can paint the same fence in 10 hours by himself. Tom started painting the fence alone, but then tricked Ben into painting while he left. In total it took 11 hours to paint the fence. How many hours did Tom work?

Qn. 3. The ratio of Grace's weight to Henry's weight is 2 : 3. If the ratio of Henry's weight to Sam's weight is 4 : 5, what is the ratio of Grace's to Sam's?

Qn. 4. What is the value of

$$\frac{1}{8} + \log_2 2\sqrt{2\sqrt{2\sqrt{2}}}$$

Qn. 5. What are the roots of the following equation?

$$\log_2(x + 1) + \log_2(x) = 1.$$

Qn. 6. Suppose that

$$3^{2x+1} = 12.$$

Find the value of  $\left(\frac{1}{81}\right)^x$ .



- Qn. 7. What is the remainder when  $3^{2018}$  is divided by 7?
- Qn. 8. An advertisement reads, "Take 11% off any item which is already discounted 31%." What is the percentage value of the combined discounts?
- Qn. 9. A point  $P$  is inside the square  $ABCD$  with  $AB = 8$ . The distances between  $P$  and the points  $A, B$  and the side  $CD$  are all  $x$ . Find the value of  $x$ .
- Qn. 10. Sarah and Jenny are running towards each other. They are 100 meters apart initially. Sarah runs 2 meter/second faster than Jenny, and they meet in 20 seconds. How fast was Jenny running?

**SECTION B : 10 marks each**

- Qn. 11. A school has two empty rectangular swimming pools, one for adults and the other for children. The volume of the children's pool is 65% of the adult's pool. The staff started supplying water to both pools with the same flow rate. If 80% of the children's pool was filled after  $T$  hours, how much more time do we need in terms of a percent of  $T$  until 95% of the adult's pool is filled?
- Qn. 12. A Laboratory attendant has 100 c.c of a solution that contains 20% acid, the rest being water. Suppose he accidentally adds pure acid to make the solution  $33\frac{1}{3}\%$  acid. How many c.c's of water must he then add to return the solution to 20% acid?
- Qn. 13. The Uganda Mathematics contest team for a school had a smooth trip to the contest venue in the morning, but unfortunately repairs on the route they used meant that some section of the route was closed and a diversion created. This made the journey back to school take twice as long. If the new route to school (including the diversion) was 50% longer than the original trip, and the average speed returning was  $10\text{km/h}$  slower, what was the average speed the team used while going to the contest?
- Qn. 14. A  $3 \times 3$  magic square contains every positive integer from 1 through 9, with one integer per cell, such that the sums of the numbers in each row, each column and each diagonal are the same. When the ordinary magic square shown is completed, what is the sum of all the possible values of  $y$ ?

	$y$	
	5	
		$y + 1$

- Qn. 15. In triangle  $ABC$ ,  $\angle BAC = 30^\circ$  and  $\angle ABC$  is a right angle. The point  $D$  lies on  $AB$  so that  $AD = 2\text{cm}$  and  $\angle CDB = 45^\circ$ . What is the length of  $BC$ ?



NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 2.

SATURDAY 23RD JUNE, 2018

9:00 AM - 12:15 PM

**INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:**

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
- (b) *There are two sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm).*
- (c) *Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.*
- (d) **Indicate your names, class, school and district on all your answer sheets.**
- (e) *Answer scripts of only qualified students should be returned to the contest coordinator, latest Monday 25th June, 2018.*
- (f) *ALL participants who qualified for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 28th July, 2018 at 8:00am at Makerere University, Kampala.*

**SECTION A : 5 marks each**

Qn. 1. Let

$$x \star y = 2x^2 + 2xy.$$

If  $a$  and  $b$  are positive integers such that  $a \star b = 9$  and  $b \star a = 72$ , what is the value of  $\frac{a}{b}$ . Express your answer as a common fraction.

Solution

Since  $a \star b = 2a^2 + 2ab = 9$  and  $b \star a = 2b^2 + 2ab = 72$  then  $\frac{2a(a+b)}{2b(b+a)} = \frac{9}{72} \Rightarrow \frac{a}{b} = \frac{1}{8}$  since  $a + b > 0$ .

Qn. 2. Tom can paint a fence in 15 hours by himself. Ben can paint the same fence in 10 hours by himself. Tom started painting the fence alone, but then tricked Ben into painting while he left. In total it took 11 hours to paint the fence. How many hours did Tom work?

Solution

Let  $x$  be the time Tom worked. Suppose Tom works at a rate  $t$  units per hour and Ben works at a rate of  $b$  units per hour. Let the total work required to paint the fence be  $P$ . Then  $15t = P$  and  $10b = P$ . If the total time taken to paint the fence by both was 11 hours, then  $x \cdot t + (11-x) \cdot b = P$  and therefore  $x \cdot \frac{P}{15} + (11-x) \cdot \frac{P}{10} = P$ . Whence  $x \cdot \frac{1}{15} + (11-x) \cdot \frac{1}{10} = 1 \Rightarrow x = 3$ .



Qn. 3. The ratio of Grace's weight to Henry's weight is 2 : 3. If the ratio of Henry's weight to Sam's weight is 4 : 5, what is the ratio of Grace's to Sam's?

Solution

Let the weight of Grace be  $G$ , of Henry be  $H$  and of Sam be  $S$  then  $G : H = 2 : 3$  and  $H : S = 4 : 5$ . Let the total weight of Grace and Henry be  $W_1$  and the total weight of Henry and Sam be  $W_2$ . Then  $G + H = W_1$  and  $S + H = W_2$ . From the ratios it then implies that  $\frac{3}{5}W_1 + \frac{5}{9}W_2 = W_2$  thus  $W_1 = \frac{20}{27}W_2$ . The required ratio is  $\frac{G}{S} = \frac{\frac{2}{5}W_1}{\frac{5}{9}W_2} = \frac{\frac{2}{5} \cdot \frac{20}{27} \cdot W_2}{\frac{5}{9}W_2} = \frac{8}{15}$ .

Qn. 4. What is the value of

$$\frac{1}{8} + \log_2 2\sqrt{2\sqrt{2\sqrt{2}}}$$

Solution

$\sqrt{2} = 2^{\frac{1}{2}} \Rightarrow 2\sqrt{2} = 2^{\frac{3}{2}}$ , in the same way  $2\sqrt{2\sqrt{2\sqrt{2}}} = 2^{\frac{15}{8}}$  so  $\log_2 2\sqrt{2\sqrt{2\sqrt{2}}} = \log_2 2^{\frac{15}{8}} = \frac{15}{8}$   
 thus  $\frac{1}{8} + \log_2 2\sqrt{2\sqrt{2\sqrt{2}}}$

Qn. 5. What are the roots of the following equation?

$$\log_2(x + 1) + \log_2(x) = 1.$$

Solution

$\log_2(x + 1) + \log_2(x) = 1 \Leftrightarrow \log_2\left(\frac{x(x+1)}{2}\right) = 0 \Rightarrow x^2 + x - 2 = 0 \Leftrightarrow x = -2, 1$ . The only valid solution is  $x = 1$

Qn. 6. Suppose that

$$3^{2x+1} = 12.$$

Find the value of  $\left(\frac{1}{81}\right)^x$ .

Solution

Clearly  $3^{2x+1} = 12 \Leftrightarrow 3^{2x} = 4$ . Since  $81 = 3^4$  then  $\left(\frac{1}{81}\right)^x = \left(\frac{1}{3^4}\right)^x \Rightarrow \left(\frac{1}{81}\right)^x = 3^{-4x} = 4^{-2} = \frac{1}{16}$ .

Qn. 7. What is the remainder when  $3^{2018}$  is divided by 7?

Solution

Since  $3^2 = 2(mod 7)$ ,  $3^3 = 6(mod 7)$ ,  $3^4 = 4(mod 7)$ ,  $3^5 = 5(mod 7)$  and  $3^6 = 1(mod 7)$  and  $2018 = 6 \times 336 + 2$  then  $3^{2018} = 3^{6 \times 336} \times 3^2(mod 7) = 3^2(mod 7) = 2$ .

Qn. 8. An advertisement reads, "Take 11% off any item which is already discounted 31%." What is the percentage value of the combined discounts?

Solution



The price after discount is  $\frac{69}{100}p$  where  $p$  is the initial price. Price after advertisement is  $\frac{89}{100}$  of  $\frac{69}{100}p$  thus  $0.6141p$  thus the combined discount is  $(1 - 0.614) \times 100\% = 38.5\%$ .

- Qn. 9. A point  $P$  is inside the square  $ABCD$  with  $AB = 8$ . The distances between  $P$  and the points  $A, B$  and the side  $CD$  are all  $x$ . Find the value of  $x$ .

Solution

Let  $M$  be the center of the side  $AB$ . Now,  $AM = 4, PM = 8 - x$  and triangle  $AMB$  is a right triangle then  $16 = (8 - x)^2 = x^2$  solving for  $x$  yields  $x = 5$ .

- Qn. 10. Sarah and Jenny are running towards each other. They are 100 meters apart initially. Sarah runs 2 meter/second faster than Jenny, and they meet in 20 seconds. How fast was Jenny running?

Solution

Let the speed of Jenny be  $S$  m/s then that of Sarah will be  $(S + 2)$  m/s. Now for sarah, Distance covered  $D = (S + 2) \times 20$  and for Jenny  $100 - D = 20 \times S$  from where  $S = 1.5$  m/s.

**SECTION B : 10 marks each**

- Qn. 11. A school has two empty rectangular swimming pools, one for adults and the other for children. The volume of the children's pool is 65% of the adult's pool. The staff started supplying water to both pools with the same flow rate. If 80% of the children's pool was filled after  $T$  hours, how much more time do we need in terms of a percent of  $T$  until 95% of the adult's pool is filled?

Solution

When 80% of the children's pool is filled then  $80\% \times 65\%$  of the adults pool will be filled that is 52% in  $T$  hours. To fill the adult pool to 95% will take  $T \times \left(\frac{95}{52}\right) = 1.8269T$  thus it will require an additional 82.69% of  $T$ .

- Qn. 12. A Laboratory attendant has 100 c.c of a solution that contains 20% acid, the rest being water. Suppose he accidentally adds pure acid to make the solution  $33\frac{1}{3}\%$  acid. How many c.c's of water must he then add to return the solution to 20% acid?

Solution

Let  $x$  be the amount of acid added accidentally then  $\frac{20 + x}{100 + x} = \frac{1}{3} \Rightarrow x = 20$  cc. Now, we have 40cc of acid out of 120 cc of solution. Let  $y$  be the number of cc of water required to add to get back back a percentage of 20%. Thus  $\frac{y + 80}{120 + y} = \frac{8}{10} \Rightarrow y = 80$  cc.

- Qn. 13. The Uganda Mathematics contest team for a school had a smooth trip to the contest venue in the morning, but unfortunately repairs on the route they used meant that some section of the route was closed and a diversion created. This made the journey back to school take twice as long. If the new route to school (including the diversion) was 50% longer than the original trip, and the average speed returning was 10km/h slower, what was the average speed the team used while going to the contest?



Solution

Let  $d$  be the distance to the contest and  $t$  be time to travel to the contest. Then the average speed going to the contest is  $\frac{d}{t}$ , which can be denoted as  $v$ . Returning home involves a distance 50% greater than going that is  $(1 + 0.5d) = 1.5d$  and a time twice as long as that of going ( $2t$ ). The average return speed  $= \frac{1.5d}{2t} = \frac{3}{4}v$  whose value is  $10km/hr$  less than  $v$ .  
Therefore  $\frac{3}{4}v = v - 10 \Rightarrow v = 40km/hr$ .

Qn. 14. A  $3 \times 3$  magic square contains every positive integer from 1 through 9, with one integer per cell, such that the sums of the numbers in each row, each column and each diagonal are the same. When the ordinary magic square shown is completed, what is the sum of all the possible values of  $y$ ?

	$y$	
	5	
		$y + 1$

Solution

The sum of the elements in each row, column and diagonal of a  $3 \times 3$  magic square must be  $\frac{1 + 2 + 3 + \dots + 9}{3} = 15$  on average. For the center column to sum to 15 the bottom must be  $10 - y$ . Reasoning in a similar way the magic square is filled as below,

$9 - y$	$y$	6
$y + 2$	5	$8 - y$
4	$10 - y$	$y + 1$

Given that 4, 5, 6 already appear in the table, then  $y \neq 4, 5, 6, y \neq 4, 5, 6$  and  $y + 2 = 4, 5, 6$ . These three equations imply  $y \neq 2, 3, 4, 5, 6$ . Also  $8 - y, 9 - y$  appear in the table and we can only use numbers from 1 to 9 so  $y$  can not be 8 or 9 hence  $y \neq 2, 3, 4, 5, 6, 8, 9$  which leaves only  $y = 1, 7$  whose sum is 8.

Qn. 15. In triangle  $ABC$ ,  $\angle BAC = 30^\circ$  and  $\angle ABC$  is a right angle. The point  $D$  lies on  $AB$  so that  $AD = 2cm$  and  $\angle CDB = 45^\circ$ . What is the length of  $BC$ ?

Solution

From  $\triangle ABC$ ,  $\tan 30 = \frac{x}{x + 2} \Rightarrow x = \frac{2}{\sqrt{3} - 1}$ .

NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 1.

SATURDAY 6TH APRIL, 2019

9:00 AM - 12:15 PM

## INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
- (b) *There are **two** sections and each section carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm). Marks will be awarded for only answers for which a clear and logical layout of the working is provided.*
- (c) *Indicate your names, gender, class, school and district **on all** your answer sheets.*
- (d) *Answer scripts of only registered students should be returned to the contest coordinator, latest **Monday 8th April, 2019**.*
- (e) ***ALL** participants who qualify for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 27th July, 2019 at NOON at Makerere University, Kampala.*
- (f) *National Mathematics Contest 2019 Paper 2 will be done on 29th June, 2019 at various centres.*

## SECTION A : 5 marks each

1. If we mix one litre of lemonade that contains 4% lemon with two litres of lemonade that contains 10% lemon, what is the percentage of lemon in the resulting three litre mixture?
2. If  $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{36290}$ , what is the value of the positive integer  $x$ ?
3. Peter's age is the sum of the ages of his two brothers. Six years ago, his age was the product of the ages of his two brothers. How old is Peter and his brothers?
4. Seera's score on a mathematics test was recorded incorrectly by the teacher. Her real score was exactly four times the score that the teacher recorded. When the teacher corrected her mistake, the average score of the class went up by 2. There are 24 students in Seera's class (including Seera). What was Seera's real score on the mathematics test?
5. Suppose you begin counting the numbers 1, 2, 3, 4, 5 starting with 1, then 2, 3, 4, 5 and back to 1 as in 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, ... What is the 2019<sup>th</sup> number you count?
6. John and Brenda play a game where they toss a coin that is equally likely to come up heads or tails. They take turns flipping the coin, with John going first. The first person to flip tails wins assuming that the result of John's toss is revealed after Brenda's toss, determine the probability that John wins the game.
7. An employer reduced the salary of his employees by 20%. The employees complained. After a month he increased the new salary by 20%. One of the employees still complained. What was the reason for the complaint?

8. What is the value of  $x$  in  $5^x - 5^{x-2} = 120\sqrt{5}$ ?
9. The triangle  $ABC$  has edge lengths  $BC = 20$ ,  $CA = 21$ , and  $AB = 13$ . What is its height  $h$  shown in the Figure 1?
10. Alice is driving to Bob's house, intending to arrive at a certain time. If she drives at 60 km/h she will arrive 5 minutes late. If she drives at 90 km/h she will arrive 5 minutes early. Determine the speed at which she should drive to arrive exactly on time.

**SECTION B : 10 marks each**

11. Okello and Mugisha decide to race each other on an 800 metre track. Since Mugisha is faster than Okello, the two decided to give Okello a head start. In the first race, Okello was given a 20 metre head start, however, Mugisha still won and finished 2 seconds earlier than Okello. In the second race, Okello was given a 38 metre head start, and this time Okello won and finished 1 second ahead of Mugisha. Assuming both Okello and Mugisha ran at uniform speeds in both races, determine the speeds (in metres per second) of both runners.
12. In a sequence, each term after the first is the sum of squares of the digits of the previous term. For example, if the first term is 42 then the next term is  $4^2 + 2^2 = 20$ . The next term after 20 is then  $2^2 + 0^2 = 4$ , followed by  $4^2 = 16$ , which is then followed by  $1^2 + 6^2 = 37$ , and so on, giving the sequence 42, 20, 4, 16, 37, and so on.
  - (a) If the first term is 44, what is the 2019<sup>th</sup> term?
  - (b) If the first term is 25, what is the 2019<sup>th</sup> term?
13. The number 102564 has the property that if the last digit is moved to the front, the resulting number, namely 410256, is 4 times larger than the original number:  $410256 = 4 \times 102564$ . Find a six-digit number whose last digit is 9 and becomes 4 times larger when we move this 9 to the front.
14. A  $3 \times 3$  magic square consists of nine different numbers placed in a grid in such a way that the sum of the numbers in each row, each column, and the two main diagonals is the same. The numbers 5, 10, 15, 20, 25, 30, 35, 40, and 45 are used to form a magic square. Some of these numbers have been placed, as shown. Find the possible value(s) of  $X$  and  $Y$ .

20	45	$a$
		$X$
		$Y$

15. The Figure 2 below shows an 8 by 9 rectangle cut into three pieces by two parallel slanted lines  $AE$  and  $BD$ . The three pieces all have the same area. How far apart are the slanted lines?

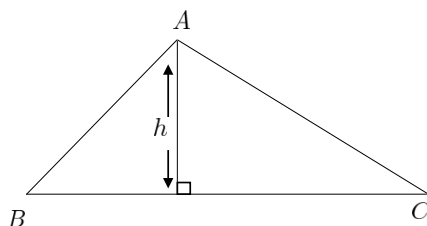


Figure 1: Qn 9 figure

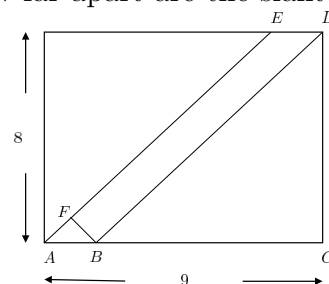


Figure 2: Qn 15 figure

NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 1 SOLUTIONS.

SATURDAY 6TH APRIL, 2019

9:00 AM - 12:15 PM

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- (f) *National Mathematics Contest 2019 Paper 2 will be done on 29th June, 2019 at various centres.*

## SECTION A : 5 marks each

1. Let the quantity of Lemon in 1 litre be 4 units. Then the quantity of lemon units in 2 litres is 20. So the total number of Lemon units in the 3 litres is 24. Hence the percentage of lemon in 3 litre mixture is  $\frac{24}{3}\% = 8\%$ .
2.  $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{36290} \Rightarrow x(x+1) = 36290$  therefore  $x = 190$ .
3. Let  $a$  and  $b$  be the ages of Peter's two brothers six years ago. Where we assume that  $a \geq b$ . Then Peter's age six years ago was  $ab$ , so his age now is  $ab + 6$ . But his brothers' ages now must be  $a + 6$  and  $b + 6$  so Peter's age now is  $a + 6 + b + 6 = a + b + 12$ . This implies that  $ab + 6 = a + b + 12 \Rightarrow ab - a - b = 6$ . If we add both sides by 1 and factor the resulting equation gives,  $(a - 1)(b - 1) = 7$ . The only integer solution to this equation that satisfy  $a \geq b > 0$  is when  $a - 1 = 7$  and  $b - 1 = 1$  thus  $a = 8$  and  $b = 2$ . These were the ages of the brothers six years ago so the brothers are now 14 and 8 respectively  $\Rightarrow$  Peter's age is 22 years.
4. Since the average score of the class of 24 students went up by 2 when the teacher corrected Seera's mistake, the total score of all students must have gone up by  $2 \times 24 = 48$  points. Thus 48 points must represent the difference between the originally recorded score and four times that score that is 48 must equal to exactly three times the originally recorded score. So the originally recorded score must have been  $\frac{48}{3} = 16$ . So Seera's real score must have been  $16 \times 4 = 64$ .

5. The following pattern appears while counting:

1, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, ...

We notice that 1 appears at the first, ninth, 17th, 25th, ... position, so 1 also appears at the 2017th position  $\Rightarrow$  there is a 2 at 2018th and 3 appears at 2019th.

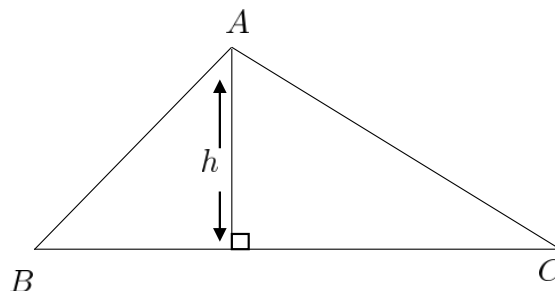
6. Let  $T$  denote tails and  $H$  denote heads. Suppose John rolls/tosses the coin but the result of his roll is revealed only after Brenda has rolled. The four equally likely outcomes in the sample space are  $TT, TH, HT$  and  $HH$ . If the fourth outcome occurs, then the game is a draw, so they play again. For John to win the first two events have to occur out of the three possible events thus  $\frac{2}{3}$  is the required probability.

7. Let  $x$  be the original salary, decreased by 20%  $\Rightarrow 0.8x$  of the salary is paid and this is increased by 20% to  $0.8 \times 1.2 \times x = 0.96x \Rightarrow 96\%$  of the original salary is got meaning it is still less than the original so the employees should continue to complain.

8.

$$5^x - 5^{x-2} = 120\sqrt{5} \Leftrightarrow 5^x \left(1 - \frac{1}{25}\right) = 24 \times 5^{\frac{3}{2}} \Leftrightarrow 5^x \left(\frac{24}{25}\right) = 24 \times 5^{\frac{3}{2}} \Leftrightarrow x = \frac{7}{2}.$$

9. From the triangle  $ABD$



we have

$$x^2 + h^2 = 169 \tag{1.1}$$

and from triangle  $ACD$  we have

$$21^2 = (20 - x)^2 + h^2 \tag{1.2}$$

Solving (1.1) and (1.2) gives  $x = 3.2$ . Thus from (1.1) we have  $h = \sqrt{13^2 - x^2} = 12.6$

10. Let  $D$  and  $V$  be distance and velocity, respectively. Let  $T$  denote the exact time and let  $x$  denote the velocity corresponding to that time. Noting that 5 minutes is  $\frac{1}{12}$  of an hour. For the late journey we have

$$\frac{D}{60} = T + \frac{1}{12}, \tag{1.3}$$

For the early Journey we have

$$\frac{D}{90} = T - \frac{1}{12} \tag{1.4}$$



and for the exact journey we have

$$\frac{D}{x} = T \Rightarrow D = Tx \tag{1.5}$$

From (1.3) and (1.4) we get  $x = 72km/hr$ .

**SECTION B : 10 marks each**

11. Suppose Mugisha ran 800 m in  $t$  seconds. Then Okello ran 780 m in  $t + 2$  seconds and 762 m in  $t - 1$  seconds. Since Okello ran at uniform speed in both races we have

$$\frac{780}{t + 2} = \frac{762}{t - 1} \Rightarrow t = 128sec.$$

Therefore Okello runs at  $\frac{780}{130} = 6m/s$  and Mugisha runs at  $\frac{800}{128} = 6.25m/s$ .

12. (a) If the first term is 44, what is the 2019<sup>th</sup> term? The following are the first few terms of the sequence:

$$\begin{aligned} &44, \\ &4^2 + 4^2 = 32, \\ &3^2 + 2^2 = 13, \\ &1^2 + 3^2 = 10, \\ &1^2 + 0^2 = 1, \\ &1^2 = 1, \\ &1^2 = 1, \\ &\vdots \end{aligned}$$

This implies that the sequence stabilises at 1 and the 2019th term is 1.

- (b) If the first term is 25, what is the 2019<sup>th</sup> term? The following are the first few terms of the sequence:

$$\begin{aligned} &25, \\ &2^2 + 5^2 = 29, \\ &2^2 + 9^2 = 85, \\ &8^2 + 5^2 = 89, \\ &8^2 + 9^2 = 64 + 81 = 145, \\ &\vdots \end{aligned}$$

This sequence simplifies to

$$25, 29, 85, 145, 42, 20, 4, 16, 37, 58, 89, 145, 42, 20, 4, 16, 37, 58, 89, \dots$$

So the 8th, 16th, 24th, ..., 2016th term is 4 thus the 2017th term is 16, 2018th is 37 and the 2019th term is 58.

13. We need to find a number  $a, b, c, d, e$  so that,

$$\begin{array}{cccccc} a & b & c & d & e & 9 \\ & & & & \times & 4 \\ \hline 9 & a & b & c & d & e \end{array}$$

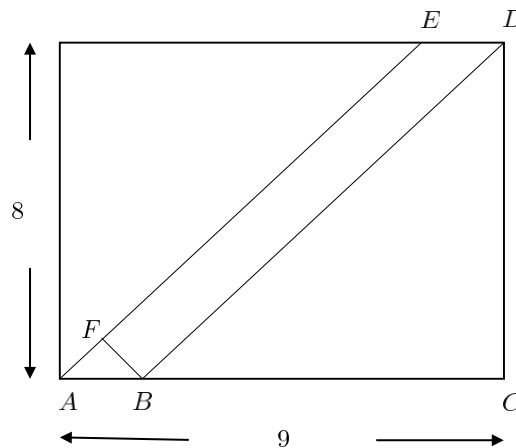
This implies that  $e = 6, d = 7, c = 0, b = 3$  and  $a = 2$  thus the six digit number whose last digit is 9 and becomes 4 times larger when we move 9 to the front is 230769.

14. For

20	45	$a$
		$X$
		$Y$

The sum of all numbers is  $5 + 10 + 15 + 20 + \dots + 45 = 225$ . The sum of three rows and three columns is the same that is sum of three rows and/or three columns is  $\frac{225}{3} = 75$ . The number in the right upper corner is  $75 - (20 + 45) = 10 = a$ . Thus  $75 - (X + Y) = 10$ , possible values of  $X$  and  $Y$  are  $(35, 30)$  or  $(30, 35)$ .

15. Given the figure,



Since the area of the rectangle is  $8 \times 9 = 72$ , the area of each of the three pieces must be  $\frac{72}{3} = 24$ . In particular the area of right triangle  $BCD$  must be 24. Since  $CD = 8$  and the area of  $\triangle BCD$  is

$$\frac{1}{2}BC \times CD = BC \times 4 \Rightarrow BC = \frac{24}{4} = 6.$$

By Pythagoras Theorem  $BD = \sqrt{6^2 + 8^2} = 10$ . Since the area of the parallelogram  $ABDE = 24$  and is given by  $24 = BD \times BF$  then  $BF = \frac{24}{10} = 2.4$ .

NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 2.

SATURDAY 29TH JUNE, 2019

9:00 AM - 12:15 PM

## INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
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- (f) *ALL participants who have sat for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 27th July, 2019 at 8:00am at Makerere University, Kampala.*

## SECTION A : 5 marks each

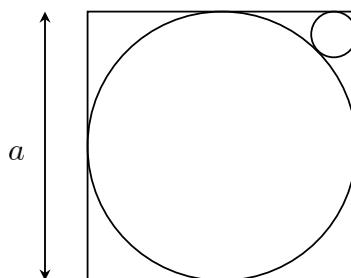
- Qn. 1. What is the value of  $x$  in the equation;  $3 \cdot 16^x + 36^x - 2 \cdot 81^x = 0$ . [The dot ( $\cdot$ ) represents multiplication].
- Qn. 2. Let  $x + \frac{1}{x} = 10$ . Find the value of  $\frac{x^3}{970} + \frac{1}{970x^3}$ .
- Qn. 3. Given that  $B_1 = 1!, B_2 = 1! + 2!, B_3 = 1! + 2! + 3!, B_4 = 1! + 2! + 3! + 4!, \dots$ . What is the ones digit of the term  $B_{2019}$ ?
- Qn. 4. Given that the function  $f(x)$  satisfies  $f(1) = 4$  and  $f(x) \cdot f(y) = f(x - y) + f(x + y)$ . Find  $f(5)$ .
- Qn. 5. In a sequence  $\{a_n\}$  for  $n \geq 4$  and  $n \in \mathbb{N}$  where  $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ , the terms are related by  $a_n = a_{n-1} + a_{n-3}$ . Given that  $a_6 = 30$ , what is the value of  $a_8 - a_1$ ?
- Qn. 6. After John has used 20% of his pocket money and Joy has used 40% of her pocket money, John has twice as much pocket money left as Joy. Find John's original amount of pocket money as a percentage of Joy's original amount of pocket money.
- Qn. 7. Suppose a circle is inscribed in a square each of whose sides is one unit in length. Let  $A_1$  be the area of the square minus the area of the circle. Next a second square is inscribed inside the first circle and a second circle is inscribed inside the second square. Let  $A_2$  be the area of the second square minus the area of the second circle. Then a third square is inscribed inside the second circle and a third circle inside the third square. Let  $A_3$  be the area of the third square minus the area of the third circle. If this process is repeated until  $A_{10}$  is found, what is  $A_1 + A_2 + A_3 + A_4 + A_5$ ? [Hint: Leave  $\pi$  in your answer.]



- Qn. 8. Suppose Juliet can do a job in 4 hours and Janet can do the same job in 5 hours. If they work together for 2 hours and then Juliet quits, how long will it take Janet to finish the job?
- Qn. 9.  $A$  is a two-digit whole number that does not contain zero as a digit.  $B$  is a three-digit whole number, and  $A\%$  of  $B$  is 400. Find all possible values of  $A$  and  $B$ .
- Qn. 10. Suppose two bicycle riders are headed towards each other, the first is moving at  $10\text{mph}$  and the second at  $15\text{mph}$ . At the instant the bicyclists are 30 miles apart, a bird flying at  $20\text{mph}$  and headed in the same direction as the first rider, passes the first rider. When the bird reaches the second rider, it instantaneously changes direction and heads back toward the first rider. When the bird reaches the first rider it again instantaneously changes direction and heads toward the second rider. If the bird continues to go back and forth at  $20\text{mph}$ , how far will it travel from the time the bikers are 30 miles apart until they reach each other?

**SECTION B : 10 marks each**

- Qn. 11. Find the numbers  $m$  and  $n$  such that  $\sqrt{m} - \sqrt{n} = 2\sqrt{2 - \sqrt{3}}$ .
- Qn. 12. Three men, Peter, John, and James, had to go on a journey of forty kilometres. Peter could walk one kilometre an hour, John could walk two kilometres an hour, and James could ride his bicycle at eight kilometres an hour. They start at the same time. James rode with Peter a certain distance, and dropped him to walk the remainder, rode back to meet John on the way and carried him to their destination, where they all arrived at the same time. How long did the journey take? Assume each person went at a uniform rate throughout and the bicycle could only carry one extra person at a time.
- Qn. 13. For  $x \notin \left\{0, -\sqrt{2}, \frac{-1}{\sqrt{2}}\right\}$ . Define  $f(x) = \frac{-1}{x + \sqrt{2}}$ . Using the notation:  
 $f^{(1)} = f(x)$ ,  $f^{(2)} = f(f(x))$ ,  $f^{(3)} = f(f(f(x)))$   
and in general  $f^{(n+1)} = f(f^{(n)}(x))$ . Find and simplify  $f^{(2019)}(x)$ .
- Qn. 14. Three gamblers went into a betting shop. At the start of betting they had money in ratios  $6 : 5 : 4$  and at the end, their money was in ratios  $7 : 6 : 5$ . One of the gamblers won 30,000 UGX. How much money did he have at the end?
- Qn. 15. If a circle of radius  $\frac{a}{2}$  is inscribed in a square of side  $a$ , what is the radius of a circle  $Q$  that can be inscribed in a corner of the square tangent to two sides of the square and to the circle inscribed in the square?



NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 2 SOLUTIONS.

SATURDAY 29TH JUNE, 2019

9:00 AM - 12:15 PM

## INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
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- (f) *ALL participants who have sat for Paper 2 are cordially invited to the certificate and prize giving ceremony on Saturday 27th July, 2019 at 8:00am at Makerere University, Kampala.*

## SECTION A : 5 marks each

Qn. 1. What is the value of  $x$  in the equation;  $3 \cdot 16^x + 36^x - 2 \cdot 81^x = 0$ . [The dot ( $\cdot$ ) represents multiplication].

**Solution :**

$$3 \cdot 16^x + 36^x - 2 \cdot 81^x = 0 \Leftrightarrow \frac{3 \cdot 16^x}{36^x} + \frac{36^x}{36^x} - \frac{2 \cdot 81^x}{36^x} = \frac{0}{36^x} \Leftrightarrow 3 \cdot \left(\frac{4}{9}\right)^x + 1 - 2 \cdot \left(\frac{9}{4}\right)^x = 0.$$

Let  $u = \left(\frac{4}{9}\right)^x$  to have  $3u^2 + u - 2 = 0$  whose solution set is  $\left\{-1, \frac{2}{3}\right\}$ . Since  $u > 0$  then  $u = \frac{2}{3}$  is the only value that satisfies the equation. Therefore  $x = \frac{1}{2}$ . ■



Qn. 2. Let  $x + \frac{1}{x} = 10$ . Find the value of  $\frac{x^3}{970} + \frac{1}{970x^3}$ .

**Solution :** Consider  $\left(x + \frac{1}{x}\right)^3 = 10^3$ . On expanding we get

$$\left(x^3 + \frac{1}{x^3}\right)^3 = 1000 - 3\left(x + \frac{1}{x}\right) = 970.$$

*This means that*

$$\frac{x^3}{970} + \frac{1}{970x^3} = \frac{1}{970} \left(x^3 + \frac{1}{x^3}\right) = \frac{970}{970} = 1.$$

■



Qn. 3. Given that  $B_1 = 1!$ ,  $B_2 = 1! + 2!$ ,  $B_3 = 1! + 2! + 3!$ ,  $B_4 = 1! + 2! + 3! + 4!$ ,  $\dots$ . What is the ones digit of the term  $B_{2019}$ ?

**Solution :**

$$B_1 = 1! = 1,$$

$$B_2 = 1! + 2! = 3,$$

$$B_3 = 1! + 2! + 3! = 9,$$

$$B_4 = 1! + 2! + 3! + 4! = 33,$$

$$B_5 = 1! + 2! + 3! + 4! + 5! = 153,$$

$$B_6 = 1! + 2! + 3! + 4! + 5! + 6! = 873, \dots$$

*We note that the ones digit of  $n!$  is zero  $\Rightarrow n \geq 4$  the ones digit of  $B_n$  is 3 thus the ones digit of  $B_{2019}$  is 3. ■*



Qn. 4. Given that the function  $f(x)$  satisfies  $f(1) = 4$  and  $f(x) \cdot f(y) = f(x - y) + f(x + y)$ . Find  $f(5)$ .

**Solution :** *Substitute different values of  $x$  in the function relation given,*

$$f(1) \cdot f(0) = f(1 + 0) + f(1 - 0) \Leftrightarrow f(0) = 2.$$

*For*

$$f(1) \cdot f(1) = f(1 + 1) + f(1 - 1) \Leftrightarrow f(2) = 14.$$

*Also*

$$f(2) \cdot f(1) = f(2 + 1) + f(2 - 1) \Leftrightarrow f(3) = 52.$$

*Now consider,*

$$f(3) \cdot f(2) = f(5) + f(1) \Leftrightarrow f(5) = 724.$$

■



Qn. 5. In a sequence  $\{a_n\}$  for  $n \geq 4$  and  $n \in \mathbb{N}$  where  $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ , the terms are related by  $a_n = a_{n-1} + a_{n-3}$ . Given that  $a_6 = 30$ , what is the value of  $a_8 - a_1$ ?

**Solution :**

$$a_4 = a_3 + a_1$$

$$a_5 = a_4 + a_2 = a_2 + a_3 + a_1$$

$$a_6 = a_5 + a_3 = a_2 + 2a_3 + a_1$$

$$a_7 = a_6 + a_4 = a_2 + 3a_3 + 2a_1$$

$$a_8 = a_7 + a_5 = 2(a_2 + 2a_3 + a_1) + 3a_1 \Leftrightarrow a_8 = 2 \times 30 + a_1 \Rightarrow a_8 - a_1 = 60$$

■



- Qn. 6. After John has used 20% of his pocket money and Joy has used 40% of her pocket money, John has twice as much pocket money left as Joy. Find John's original amount of pocket money as a percentage of Joy's original amount of pocket money.

**Solution :** Let  $x$  be the original amount of pocket money John had and  $y$  be the original amount that Joy had. Given this then

$$0.8x = 2(0.6y) \Leftrightarrow y = \frac{2}{3}x.$$

The required percentage is

$$\left(\frac{x}{y}\right) \times 100\% = \left(\frac{x}{\frac{2x}{3}}\right) \times 100\% = 150\%.$$

■



Qn. 7. Suppose a circle is inscribed in a square each of whose sides is one unit in length. Let  $A_1$  be the area of the square minus the area of the circle. Next a second square is inscribed inside the first circle and a second circle is inscribed inside the second square. Let  $A_2$  be the area of the second square minus the area of the second circle. Then a third square is inscribed inside the second circle and a third circle inside the third square. Let  $A_3$  be the area of the third square minus the area of the third circle. If this process is repeated until  $A_{10}$  is found, what is  $A_1 + A_2 + A_3 + A_4 + A_5$ ? [Hint: Leave  $\pi$  in your answer.]

**Solution :** The area of the square is  $1 \text{ units}^2$  and that of the circle is  $\pi r^2 = \frac{\pi}{4} \Rightarrow$

$A_1 = 1 - \frac{\pi}{4}$ . The side of the second square is  $\frac{1}{\sqrt{2}}$  and the radius of the second circle is

$$\frac{1}{2\sqrt{2}} \Rightarrow A_2 = \left(\frac{1}{\sqrt{2}}\right)^2 - \pi \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{2} \left(1 - \frac{\pi}{4}\right).$$

The diagonal of the third square is twice the radius of the second circle that is  $2 \cdot \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$  so the length of each of its sides is  $\frac{1}{2}$  and the radius of the third circle is

$$\frac{1}{4} \Rightarrow A_3 = \left(\frac{1}{2}\right)^2 - \pi \left(\frac{1}{4}\right)^2 = \frac{1}{4} \left(1 - \frac{\pi}{4}\right).$$

Continuing in this way we get

$$A_1 + A_2 + A_3 + A_4 + A_5 = \left(1 - \frac{\pi}{4}\right) \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right) = \frac{31}{64}(4 - \pi)$$

■



- Qn. 8. Suppose Juliet can do a job in 4 hours and Janet can do the same job in 5 hours. If they work together for 2 hours and then Juliet quits, how long will it take Janet to finish the job?

**Solution :** *Juliet can do the job in 4 hours so her rate is  $\frac{1}{4}$  of the job per hour.*

*Janet's rate is  $\frac{1}{5}$  of the job per hour. If they work together then their combined rate is  $\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$ . Using the fact that work done is rate times time we have that,  $\frac{9}{20} \times 2 = \frac{9}{10}$  of the job will be done in the first 2 hours remaining  $\frac{1}{10}$  of the job.*

*This means that Janet will finish the job in  $t$  hours thus,  $\frac{1}{5} \times t = \frac{1}{10} \Rightarrow t = 0.5$  of an hour. Thus the work will be done in  $2\frac{1}{2}$  hours. ■*



Qn. 9.  $A$  is a two-digit whole number that does not contain zero as a digit.  $B$  is a three-digit whole number, and  $A\%$  of  $B$  is 400. Find all possible values of  $A$  and  $B$ .

**Solution :**  $\frac{A}{100}B = 400 \Rightarrow AB = 40000 = 2^6 \times 5^4$ . Since  $A$  is a two digit number that does not contain 0 so  $A$  can not have both 2 and 5 as a factor. Also  $A > 40$  otherwise  $B \geq 1000$  which is more than a three digit number thus the only possibility is  $A = 2^6$  and  $B = 5^4$  thus  $A = 64$  and  $B = 625$ . ■



- Qn. 10. Suppose two bicycle riders are headed towards each other, the first is moving at  $10\text{mph}$  and the second at  $15\text{mph}$ . At the instant the bicyclists are 30 miles apart, a bird flying at  $20\text{mph}$  and headed in the same direction as the first rider, passes the first rider. When the bird reaches the second rider, it instantaneously changes direction and heads back toward the first rider. When the bird reaches the first rider it again instantaneously changes direction and heads toward the second rider. If the bird continues to go back and forth at  $20\text{mph}$ , how far will it travel from the time the bikers are 30 miles apart until they reach each other?

**Solution :** *Since one rider is riding at  $10\text{mph}$  and the other at  $15\text{mph}$ , the gap between is being closed at a rate of  $25\text{mph}$ . Since  $\text{Time} = \frac{\text{distance}}{\text{speed}} = \frac{30}{25} = \frac{6}{5}$  hours, so the bird will fly  $20\text{mph}$  for  $\frac{6}{5}$  hours covering a distance  $20 \times \frac{6}{5} = 24$  miles.*

■

**SECTION B : 10 marks each**

Qn. 11. Find the numbers  $m$  and  $n$  such that  $\sqrt{m} - \sqrt{n} = 2\sqrt{2 - \sqrt{3}}$ .

**Solution :** Squaring both sides of  $\sqrt{m} - \sqrt{n} = 2\sqrt{2 - \sqrt{3}}$  gives,

$$m - 2\sqrt{mn} + n = 8 - 4\sqrt{3}$$

equating similar terms gives  $m + n = 8$  and  $mn = 12$  whose solution is  $m = 6$  and  $n = 2$  since  $m > n$  as  $2\sqrt{2 - \sqrt{3}} > 0$ . ■



Qn. 12. Three men, Peter, John, and James, had to go on a journey of forty kilometres. Peter could walk one kilometre an hour, John could walk two kilometres an hour, and James could ride his bicycle at eight kilometres an hour. They start at the same time. James rode with Peter a certain distance, and dropped him to walk the remainder, rode back to meet John on the way and carried him to their destination, where they all arrived at the same time. How long did the journey take? Assume each person went at a uniform rate throughout and the bicycle could only carry one extra person at a time.

**Solution :** Let  $x$  be the distance Peter rode with James and  $y$  be the distance John walked. Using the formula  $\text{Time} = \frac{\text{distance}}{\text{speed}}$  on the times taken by the three to arrive, we have,

Peter:  $\frac{x}{8} + \frac{40-x}{1}$  where  $\frac{x}{8}$  is when riding with James and  $40-x$  when walking.

John:  $\frac{y}{2} + \frac{40-y}{8}$  where  $\frac{y}{2}$  is when walking and  $\frac{40-y}{8}$  when riding with James.

James:  $\frac{x}{8} + \frac{x-y}{8} + \frac{40-y}{8}$  where  $\frac{x}{8}$  riding with Peter,  $\frac{x-y}{8}$  when riding back alone and  $\frac{40-y}{8}$  is when riding with John.

Given that they all arrived at the same time then

$$\frac{x}{8} + \frac{40-x}{1} = \frac{y}{2} + \frac{40-y}{8} \Leftrightarrow 7x + 3y = 280$$

and

$$\frac{y}{2} + \frac{40-y}{8} = \frac{x}{8} + \frac{x-y}{8} + \frac{40-y}{8} \Leftrightarrow 2x - 5y = 0.$$

Now solving  $7x + 3y = 280$  and  $2x - 5y = 0$  gives the values of  $x$  and  $y$  as  $x = \frac{1400}{41}$

and  $y = \frac{560}{41}$  Substituting back into the equations give the time as  $10\frac{5}{41}$  hours. ■



Qn. 13. For  $x \notin \left\{0, -\sqrt{2}, \frac{-1}{\sqrt{2}}\right\}$ . Define  $f(x) = \frac{-1}{x + \sqrt{2}}$ . Using the notation:

$$f^{(1)} = f(x), f^{(2)} = f(f(x)), f^{(3)} = f(f(f(x)))$$

and in general  $f^{(n+1)} = f(f^{(n)}(x))$ . Find and simplify  $f^{(2019)}(x)$ .

**Solution :** Since  $f(x) = \frac{-1}{x + \sqrt{2}}$  then

$$f^{(2)}(x) = \frac{-(x + \sqrt{2})}{1 + \sqrt{2}x}, f^{(3)}(x) = \frac{-(1 + \sqrt{2}x)}{x}, f^{(4)}(x) = x$$

implying that  $f^{(5)}(x) = f(x)$ ,  $f^{(6)}(x) = f^{(2)}(x)$ , in general  $f^{(n)}(x) = f^{(m)}(x)$  when  $n \equiv m \pmod{4}$ . Now

$$2019 \equiv 3 \pmod{4} \Rightarrow f^{(2019)}(x) = f^{(3)}(x) = \frac{-(1 + \sqrt{2}x)}{x}.$$

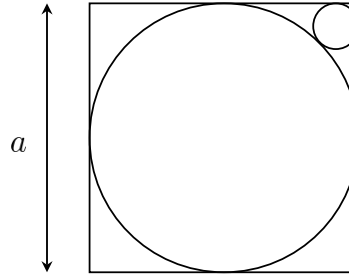
■



- Qn. 14. Three gamblers went into a betting shop. At the start of betting they had money in ratios  $6 : 5 : 4$  and at the end, their money was in ratios  $7 : 6 : 5$ . One of the gamblers won 30,000 UGX. How much money did he have at the end?

**Solution :** *Let us assume that the total money involved in the process is  $270x$  where 270 is the product of the sum of the ratios before and after betting. At the start the gamblers had  $108x, 90x$  and  $72x$ , at the end they have  $105x, 90x$  and  $75x$ , So the last player is the only one who won and by  $30,000 = 3x$ . The last player therefore had  $75x = 750,000$ . ■*

- Qn. 15. If a circle of radius  $\frac{a}{2}$  is inscribed in a square of side  $a$ , what is the radius of a circle  $Q$  that can be inscribed in a corner of the square tangent to two sides of the square and to the circle inscribed in the square?



**Solution :** Let  $O$  be the center of the inscribed circle of radius  $\frac{a}{2}$  and  $P$  the vertex of the square where the smaller circle is inscribed. Let  $r$  be the radius of the smaller circle with center  $Q$ . The length of the line segment  $OP$  is half the length of a diagonal of the square of side  $a$  that is  $\frac{\sqrt{2}a}{2}$ . The length of  $OP$  is also equal to  $\frac{a}{2}$  plus  $r$  plus the length of the line segment  $QP$ . Now construct a square of side  $2r$  with  $P$  being one of its vertices and circumscribing the smaller circle of radius  $r$ . The length of line segment  $QP$  is half the length of the diagonal of the square of radius  $r$ . Therefore the length of  $QP$  is

$$\sqrt{2}r \Rightarrow \frac{\sqrt{2}a}{2} = \frac{a}{2} + r + \sqrt{2}r.$$

Making  $r$  the subject here gives  $r = \left( \frac{3 - 2\sqrt{2}}{2} \right) a$ . ■

NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER.

SATURDAY 28TH MAY, 2022

9:00 AM - 12:15 PM

## INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
- (b) *There are **two** sections and each carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm).*
- (c) *Indicate your names, birth date, gender, class, school and district **on all** your answer sheets.*
- (d) *Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.*
- (e) *Answer scripts of only registered students should be returned to the contest coordinator, latest Monday 30th May, 2022.*
- (f) *ALL participants who score a pass mark are cordially invited to the certificate and prize giving ceremony on Saturday 23rd July, 2022 at 8:00am at Makerere University, Kampala.*

## SECTION A : 5 marks each

1. The sequence  $a_1, a_2, a_3, \dots$  starts with  $a_1 = 60$ . To work out  $a_{n+1}$  for  $n \geq 1$  you add 1 to the digit sum of  $a_n$  and square the result e.g

$$a_2 = (6 + 0 + 1)^2 = 49.$$

Find the value of the term  $a_{2022}$ .

2. If

$$\log_x 64 + \log_{y^2} 16 = \frac{5}{3}$$

and

$$\log_y 64 + \log_{x^2} 16 = 1$$

for all positive numbers  $x$  and  $y$ , what is the value of  $\log_2(xy)$ ?

3. A function  $f$  satisfies  $f(xy) = \frac{f(x)}{y^2}$  for all positive real numbers  $x$  and  $y$ . Given that  $f(25) = 80$ , Find the value of  $f(2020)$ .
4. Given that  $xy > 0$  and that  $x - y = xy$ . Find the value of

$$\frac{120}{x} - \frac{120}{y} + 2022.$$

5. John invested in a business and lost 20% during the first year of the business. The following year he made a gain of 25%. What is his net gain over the two years?



6. If  $f(\sin x) = \sin 3x$ . Evaluate  $f(\cos x)$ .
7. Find the length of a side of a square if a circle that passes through one corner of the square and is tangent internally to two sides of the square has radius 10.
8. Suppose a bag contains six letters of the word "GOOGOO". If you take one letter out of the bag at a time and line them up from left to right. What is the probability that you will spell the word "GOOGOO"?
9. What is the  $2022^{nd}$  term in the sequence?

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \dots$$

10. Suppose it takes  $h$  minutes to fill a tank using inlet  $A$  and  $c$  minutes to fill the same tank using inlet  $B$ . Starting with an empty tank, inlet  $A$  is turned on and after one minute inlet  $B$  is also turned on. How long will it take to fill the tank. [Give your answer in terms of  $h$  and  $c$ ]

**SECTION B : 10 marks each**

11. Opio and Peter ran a race, during which they both ran at constant speeds. Opio began the race  $30m$  ahead of Peter. After 3 minutes, Peter was 18 m ahead of Opio. Peter won the race exactly 7 minutes after it began. How far from the finish line was Opio when Peter won?
12. Given that the equation  $\left(\frac{1}{4}\right)^x + \left(\frac{1}{2}\right)^{x-1} + b = 0$  has a positive solution. What is the interval in which the real number  $b$  should be located?
13. A box contains colored balls, each marked with a positive integer. Half of the balls are blue and the others are gold. One fifth of the balls are marked with prime numbers. Four dozen are marked with numbers that are not prime. One third of the blue balls are marked with prime numbers. How many;
- (a) of the gold balls are marked with numbers that are not prime?
- (b) balls are not blue or are marked with numbers that are not prime or are both not blue and marked with a number that is not prime?
14. If the following pattern continues. Name the row on which the number 289 will appear. [Explain your answer]

$$\begin{array}{cccc} & & & 1 \\ & & & & 3 & & 5 \\ & & & & & & & 11 \\ & & & & & & & & 19 \\ 13 & & 15 & & 17 & & 19 \end{array}$$

15. A goat is tied to the corner of a building whose base is  $8ft$  by  $10ft$  with a twelve feet rope. How many square feet of grass can the goat reach to eat. [Ignore the dimensions of the goat].



NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER.

SATURDAY 28TH MAY, 2022

9:00 AM - 12:15 PM

**INSTRUCTIONS TO CANDIDATES AND SUPERVISORS:**

- (a) *This competition is conducted on the assumption that proper security is maintained. UMS reserves the right, should there be evidence or suspicion of any malpractice, to reject scripts from a particular school/institution.*
- (b) *There are **two** sections and each carries 50 marks. Attempt all questions in exactly three hours fifteen minutes (9 : 00am to 12 : 15pm).*
- (c) *Indicate your names, birth date, gender, class, school and district **on all** your answer sheets.*
- (d) *Marks will be awarded for only answers for which a clear and logical layout of the working is provided. A correct solution, poorly presented, will not earn full marks.*
- (e) *Answer scripts of only registered students should be returned to the contest coordinator, latest Monday 30th May, 2022.*
- (f) *ALL participants who score a pass mark are cordially invited to the certificate and prize giving ceremony on Saturday 23rd July, 2022 at 8:00am at Makerere University, Kampala.*

**SECTION A : 5 marks each**

1. **Solution :**

$$\begin{aligned} a_2 &= (6 + 0 + 1)^2 = 49 \\ a_3 &= (4 + 9 + 1)^2 = (14)^2 = 196 \\ a_4 &= (1 + 9 + 6 + 1)^2 = 289 \\ a_5 &= (2 + 8 + 9 + 1)^2 = 400 \\ a_6 &= (4 + 0 + 0 + 1)^2 = 25 \\ a_7 &= (2 + 5 + 1)^2 = 64 \\ a_8 &= (6 + 4 + 1)^2 = 121 \\ a_9 &= (1 + 2 + 1 + 1)^2 = 25 \end{aligned}$$

*The process will repeat after three steps. This means that*

$$a_6 = a_9 = a_{12} = 25.$$

*Since 2022 is a multiple of 3 then  $a_{2022} = 25$ .* ■

2. **Solution :** *Convert the logarithms to base two.*

$$\frac{\log_2 64}{\log_2 x} + \frac{\log_2 16}{\log_2 x^2} = \frac{5}{3}$$

$$\frac{\log_2 64}{\log_2 y} + \frac{\log_2 16}{\log_2 y^2} = 1$$

On reducing and evaluating the logarithms we get,

$$\frac{6}{\log_2 x} + \frac{2}{\log_2 y} = \frac{5}{3}$$

$$\frac{6}{\log_2 y} + \frac{2}{\log_2 x} = 1$$

Letting

$$\log_2 x = a$$

$$\log_2 y = b$$

and solving the above equation we have,  $a = 4$ ,  $b = 12$  and therefore

$$\log_2(xy) = \log_2 x + \log_2 y = 16.$$

■

3. **Solution :**

$$f(xy) = \frac{f(x)}{y^2}$$

$$f(25) = 80$$

Therefore

$$xy = 25$$

Substituting  $x = \frac{25}{y}$  into  $\frac{f(x)}{y^2} = 80$  we get

$$f(x) = \frac{80 \times 625}{x^2}$$

For  $f(2020)$  we have  $xy = 2020 \Rightarrow x = \frac{2020}{y}$  and  $y = \frac{2020}{x}$  and

$$f(2020) = \frac{f(x)}{y^2} = \frac{80 \times 65}{x^2} \times \frac{1}{\frac{(2020)^2}{x^2}} = \frac{500}{40804} = \frac{125}{10201} = 0.012253.$$

■

4. **Solution :**

$$120 \left( \frac{1}{x} - \frac{1}{y} \right) + 2022 = 120 \left( \frac{-(y-x)}{xy} \right) + 2022 = -120 + 2022 = 1902.$$

■

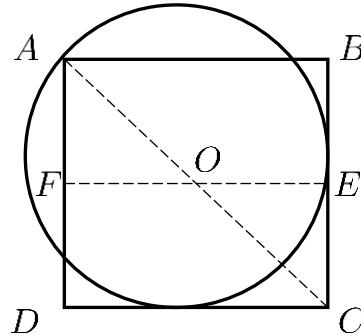
5. **Solution :** Let  $x$  be the original investment into the business in 2020, After the first year (2020) the amount invested would be  $0.8x$ , after the second year the investment will be increased by 25% thus become  $1.25 \times 0.8x = x$  implying that the net gain is zero. ■

6. **Solution :** Recall  $\cos 30^\circ = \sin 60^\circ$  so

$$f(\cos 30^\circ) = f(\sin 60^\circ) = \sin(60^\circ \times 3) = \sin 180^\circ = 0.$$

■

7. **Solution :** Let  $ABCD$  be the square with  $A$  on the circle as shown.



Let  $O$  be the center of the circle, let  $BC$  be tangent to the circle at  $E$ , Draw the diagonal  $AC$  passing through  $O$ . Draw  $OE$ , extend  $EO$  intersecting  $AD$  at  $F$ , we have  $OE = OA = 10$ .  $AFO$  is an isosceles right angle therefore

$$OF = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

Therefore,

$$\begin{aligned} EF &= 10 + OF \\ &= 10 + 5\sqrt{2} \end{aligned}$$

■

8. **Solution :** There are  $6!$  ways of lining up letters in the word “GOOGOO” without regard of which letters are where. There are:

- 2 ways of choosing the first letter as  $G$ ,
- 4 ways of choosing the second letter as “ $O$ ”,
- 3 ways of choosing the 3rd letter as “ $O$ ” and
- 1 way of choosing the fourth letter as  $G$  since one  $G$  is already taken for beginning,
- 2 ways of choosing an “ $O$ ” since there are only two left and
- 1 way for the last letter “ $O$ ”.

Therefore

$$P(\text{GOOGOO}) = \frac{2 \times 4 \times 3 \times 1 \times 2 \times 1}{6!} = \frac{1}{15}$$

■

9. **Solution :** Given the sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \dots$$



We note that the first three terms have the numerator increasing from 1 to 2 and back to 1 while denominator increases to 2, the next three terms have the numerator decrease from 3 to 1 while denominator increase to 3, the next four terms have the numerator decrease from 4 to 1 while denominator increase to 4, the next five terms have the numerator decrease from 5 to 1 while denominator increase to 5, and so on.

We note that the number of terms will be in the format 3, 3, 4, 5, 6, 7, 8, 9, ... Now given that  $3 + 3 = 1 + 2 + 3$  then the number of terms of the sequence where the numerator of the terms of  $n$  decreasing from  $n$  to 1 while denominator increasing from 1 to  $n$  will be of the form

$$3 = 1 + 2, 3, 4, 5, 6, 7, 8, 9, \dots$$

therefore the number  $n$  closest to 2022 such that  $\frac{n(n+1)}{2} = 2022$  is 63 that is

$$\frac{63(64)}{2} = 2016$$

and the next terms will be

$$\frac{64}{1}, \frac{63}{2}, \frac{62}{3}, \frac{61}{4}, \frac{60}{5}, \frac{59}{6}, \dots$$

Thus the 2022<sup>nd</sup> term is

$$\frac{59}{6}$$

■

10. **Solution :** After 1 minute, there is only  $\left(1 - \frac{1}{h}\right)$  of the total space in the tank left to be filled.

Let  $t$  be the total time it takes to fill the tank when both inlets are open, then

$$\frac{1}{h} + \frac{1}{c} = \frac{1}{t} \Rightarrow t = \frac{hc}{h+c}$$

The total time it takes for the tank to fill is

$$\begin{aligned} 1 \text{ minute} + \left(1 - \frac{1}{h}\right) \left(\frac{hc}{h+c}\right) &= 1 + \left(1 - \frac{1}{h}\right) \left(\frac{hc}{h+c}\right) \\ &= \frac{h+hc}{h+c} \\ &= \frac{h(1+c)}{h+c} \end{aligned}$$

■

11. **Solution :** Over the 3 min of the race, Peter ran 48 m further than Opio. Why?

We note that at 0 mins; Opio was at 30m mark. If Opio ran  $d$  m in the 3 minutes then he will be at  $(d + 30)m$  mark after 3 minutes. Since Peter is 18m ahead of Opio after 3 mins then Peter is at

$$(d + 30 + 18) = d + 48$$

Therefore, In 3 minutes Peter ran  $(d + 48)m$  which is 48m further than Opio.

Since each runs at a constant speed then Peter runs  $\frac{48}{3} = 16\text{m/min}$  faster than Opio.

As Peter finishes the race after 7 minutes then Peter runs for another 4 mins.

Over the 4 mins, he runs  $4 \times 16 = 64\text{ m}$  further than Opio. After 3 mins Peter runs 64 meters further than Opio. So Opio is 64 m from the finish line. ■

12. **Solution :**

$$\left(\frac{1}{4}\right)^x + \left(\frac{1}{2}\right)^{x-1} + b = 0$$

$$\left(\left(\frac{1}{2}\right)^x\right)^2 + 2\left(\frac{1}{2}\right)^x + b = 0$$

Let

$$y = \left(\frac{1}{2}\right)^x$$

such that

$$y^2 + 2y + b = 0$$

to have

$$y = \left(\frac{1}{2}\right)^x = -1 \pm \sqrt{1-b}$$

$$x = \frac{\ln(\sqrt{1-b}-1)}{-\ln 2}$$

We need  $\ln(\sqrt{1-b}-1) < 0$  for a positive solution.

$$0 < \sqrt{1-b}-1 < 1 \Rightarrow 1 < \sqrt{1-b} < 2 \Rightarrow -3 < b < 0$$

■

13. **Solution :** Four dozen is  $1 - \frac{1}{5} = \frac{4}{5}$  of the total no of balls. The total number of balls is

$$48 \div \frac{4}{5} = 60$$

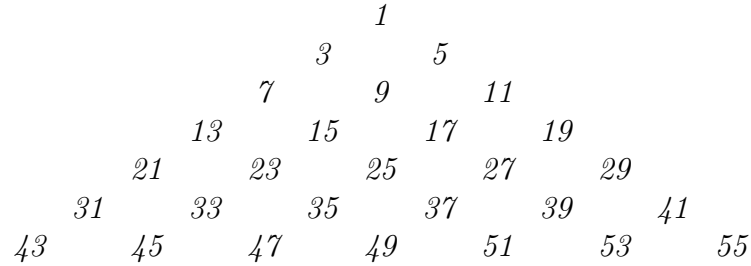
- There are  $60 - 48 = 12$  balls marked with prime numbers and 48 balls marked with non prime numbers.
- There are 30 blue balls and 30 gold balls
- Further there are  $\frac{1}{3} \times 30 = 10$  blue balls marked with prime numbers.
- There are  $12 - 10 = 2$  gold balls marked with prime numbers
- Therefore there are 28 gold balls marked with non prime number

	Blue	Gold	Total
Prime	10	2	12
Non-prime	20	28	48
Total	30	30	60

$$60 - 10 = 50$$

■

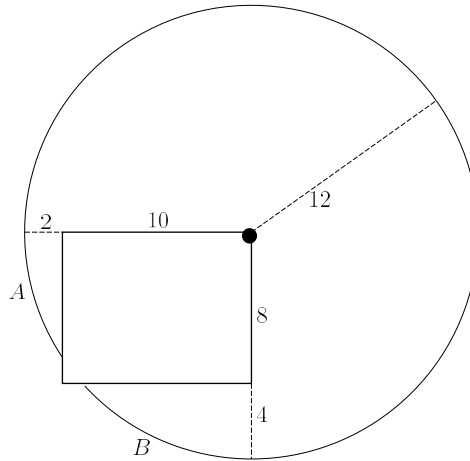
14. **Solution :**



The number 289 =  $(17)^2 \Rightarrow$  it appears in the 17<sup>th</sup> row and in the 9<sup>th</sup> element since the 1<sup>st</sup> row has 1, 3<sup>rd</sup> row has  $3^2 = 9$ , 5<sup>th</sup> row has  $5^2 = 25$  in the middle, ...

■

15. **Solution :**



Total area that the goat reach is;

- $\frac{3}{4}$  of the circle whose radius is 12 ft; i.e

$$\frac{3}{4}(\pi r^2) = \frac{3}{4}(144)\pi$$

and

- $\frac{1}{4}$  of the circular part

$$A = \frac{1}{4}(\pi \cdot 2^2) = \pi$$

and

- $\frac{1}{4}$  of the circular part

$$B = \frac{1}{4}(\pi \cdot 4^2) = 4\pi$$

Total area reached is

$$\frac{3}{4}(144)\pi + \pi + 4\pi = 113\pi.$$

■

**QUESTIONS AND SOLUTIONS****SECTION A: 5 marks each**

1. A goat is tied to one corner of a square plot of side 15m by a rope 7m long. Find how much area the goat can graze on the plot.

**Solution**

The goat can graze on a quarter of the area of a circle centered at the corner of the plot where the goat is tied, and has radius equal to the length of the rope. Hence, the required area,

$$A = \frac{90}{360}\pi r^2 = \frac{1}{4} \times \frac{22}{7} \times 7 \times 7 = \frac{77}{2} = 38.5\text{cm}^2.$$

2. James loves leftovers. On Thursday, he ate  $\frac{1}{4}$  of the Pizza leftover; on Friday,  $\frac{1}{3}$  of what remained then; on Saturday,  $\frac{1}{2}$  of what remained then; and on Sunday he finished it off. On what day did he eat more than on any other day?

**Solution**

Let  $J$  be the amount of Pizza leftover before Thursday's dinner. James eats  $\frac{1}{4}J$  that day and  $\frac{3}{4}J$  remains. On Friday, he eats  $\frac{1}{3}$  of  $\frac{3}{4}J$  which is  $\frac{1}{4}J$  again, and  $\frac{3}{4}J - \frac{1}{4}J = \frac{1}{2}J$  remains. On Saturday, he eats  $\frac{1}{2}$  of  $\frac{1}{2}J$  which is  $\frac{1}{4}J$  again. Here,  $\frac{1}{2}J - \frac{1}{4}J = \frac{1}{4}J$  remains, which he ate on Sunday. Clearly, James eats the same amount each day.

3. Alice and her brother John are walking together at the same pace. Alice stops to talk to Laura while John continues on at the same pace. After conversing with Laura, Alice continues on at the same pace as before. She later passes John, who stopped to sit on a bench. How long has John been sitting on the bench when Alice passes by?

**Solution**

Alice completes two tasks (talking to Laura, then walking to the bench) in the same time that her brother John completes two tasks (walking to the bench, then sitting on it). Since the walking pace is the same for both, the times spent walking to the bench are the same. Hence, John has been sitting on the bench for the same amount of time Alice talked to Laura.

4. In the sum shown, each letter represents a digit from 1 to 9, inclusive. What is the value of  $P + Q + R$  where

$$QR + PPP + PPP = 2023.$$

**Solution**

If the value of  $P$  is less than 9, then  $QR + PPP + PPP$  is at most  $99 + 888 + 888 = 1875$ . Since the given sum is 2023, then the value of  $P$  cannot be less than 9 and thus must equal 9. When  $P = 9$ , we get  $QR + 999 + 999 = 2023$ , and so  $QR = 2023 - 999 - 999 = 25$ . Therefore  $Q = 2, R = 5$ , and so  $P + Q + R = 9 + 2 + 5 = 16$ .



5. Ellena’s birthday is on a Wednesday and Bruce’s birthday is 121 days after Ellena’s. On what day of the week is Bruce’s birthday?

**Solution**

Since Ellena’s birthday is on a Wednesday, then any exact number of weeks after Ellena’s birthday will also be a Wednesday. Therefore, exactly 17 weeks after Ellena’s birthday is also a Wednesday. Since there are 7 days in each week, then  $7 \times 17 = 119$  days after Ellena’s birthday is a Wednesday. Since 119 days after Ellena’s birthday is a Wednesday, then 121 days after Ellena’s birthday is a Friday (since 2 days after Wednesday is Friday). Therefore, Bruce’s birthday is on a Friday.

6. When 15 is added to a set of 10 distinct natural numbers, the median changes from 6 to 8. Find the median of the new set if 7 replaces 15.

**Solution**

Originally, the median is the average of the 5th and 6th numbers. And the 6th number must be 8 (because it is the new median when 15 is added). Now replacing 15 with 7 puts 7 in the middle location (the 6th position). Hence the median of the new set is 7.

7. The sum of the ages of a father and his two sons who are twins is 48. Ten years hence, his age will be twice the sum of the sons’ ages. Find the present age of their father?

**Solution**

Let the age of the father be  $m$  years.

Let the age of one of his children be  $n$  years and age of other child be  $p$  years. Since they are twins, it means  $n = p$ .

From the given information,  $m + 2n = 48$  .....(1)

After 10 years,

$(m + 10) = 2[(n + 10) + (p + 10)] = 2[2n + 20] = 4n + 40$ . .....(2)

Substituting equation (2) into equation (1), gives

$$4n + 30 + 2n = 48 \implies 6n = 18 \implies n = 3.$$

The age of the father is  $48 - 6 = 42$  years.

8. In a row of children, David is 7th from the left and Vianney is 9th from the right. When they interchange among themselves, David becomes 18th from the left. What will be Vianney’s present position from the right?

**Solution**

**Before interchange,**

David is 7th from the left and Vianney is 9th from the right. ....(1)

**After interchange,**

David is 18th from the left and 9th from the right. ....(2)

Vianney is 7th from the left. ....(3)

From (2) we get that there are total of  $(18 + 9 - 1) = 26$  children in the given row. This together with (3) implies that the after interchange, Vianney is  $(26 - 7 + 1) = 20$ th from the right.



9. In a garden, there are 14 horizontal lines and 16 vertical lines of avocado trees. The distance between any two trees is 3 metres and a distance of 2 metres is left from all sides of the boundary of the garden. Find the area of the whole garden?

**Solution**

For columns, there are 15 gaps between the two corner avocado trees  $(15 \times 3)$  metres and 2 metres on each side is left. Therefore length is  $(45 + 4)m = 49m$ .

For rows, there are 13 gaps between the two corner avocado trees  $(13 \times 3)$  metres and 2 metres on each side is left. Therefore width is  $(39 + 4)m = 43m$ .

The area of the garden is length  $\times$  width  $= 49 \times 43 = 2107m^2$ .

10. There are 100 questions in a 2 hour examination. Among these questions are 40 mathematics questions. It is suggested that thrice (three times) as much time be spent on each mathematics problem as for each other question. How many minutes should be spent on mathematics problems?

**Solution**

Let  $y$  be the total time spent on each mathematics problem. Now, for 40 maths questions, the time taken will be  $40y$ . For remaining questions, the time taken will be  $60 \times \frac{y}{3} = 20y$ .

Total time taken to do the examination = 2hrs =  $60 \times 2 = 120$  minutes.

But then  $40y + 20y = 120$  implying that  $60y = 120 \implies y = 2$ .

Therefore, the time taken to solve 40 mathematics problems =  $40 \times 2 = 80$  minutes.

**SECTION B:**

11. The heights of 4 athletes on a team are 135 cm, 160 cm, 170 cm, and 175 cm. Kiplimo joins the team. On the new team of 5 athletes, the modal height of the players is equal to the median height which is equal to the mean (average) height. How tall is Kiplimo?

**Solution**

Since the heights of the 4 athletes on the team are all different, then if Kiplimo's height is different from each of these, there is no single mode height. Therefore, Kiplimo's height must be equal to the height of one of the 4 athletes on the team for there to be a single mode. If Kiplimo's height is 135 cm, then the median height of the 5 athletes is 160 cm which is not possible, since the median does not equal the mode. Similarly, if Kiplimo's height is 175 cm, then the median height of the 5 athletes is 170cm which is not possible.

Therefore, Kiplimo's height must equal 160 cm or 170 cm, since in either case the median height of the 5 athletes will equal Kiplimo's height, which is the mode.

If Kiplimo's height is 170 cm, then the mean height of the 5 athletes is

$$\frac{135 + 160 + 170 + 170 + 175}{5} = 162cm.$$

If Kiplimo's height is 160 cm, then the mean height of the 5 athletes is

$$\frac{135 + 160 + 160 + 170 + 175}{5} = 160cm.$$

When Kiplimo's height is 160 cm, the heights of the 5 athletes (measured in cm) are: 135, 160, 160, 170, and 175. In this case, each of the mode, median and mean height of the 5 athletes equals 160 cm.

12. Seven distinct positive integers will be placed in the seven boxes of the “H”-shaped Figure 1 below. The integers must be placed so that the three integers in the left vertical column, those in the right vertical column, and those in the one horizontal row all have the same sum. The integers  $k$  and  $n$  are each between 4 and 18, inclusive. The figure contains seven different integers and the three sums are equal. Determine all possible values of  $k$ .

7		$n + 6$
10	$n$	18
$k$		4

**Solution**

Since the three sums are equal, we have  $7 + 10 + k = 10 + n + 18 = (n + 6) + 18 + 4$  which gives  $k + 17 = n + 28 = n + 28$ , which means that  $k = n + 11$ . Since  $k$  is between 4 and 18, inclusive, and  $k = n + 11$ , then we must have  $n \leq 7$ . Since  $n$  is between 4 and 18, inclusive, then  $n \geq 4$  as well. Putting this together,  $n$  can equal 4, 5, 6, or 7, which give values for  $k$  of 15, 16, 17, or 18, respectively, and values of  $n + 6$  of 10, 11, 12, or 13, respectively. Since the figure already contains 10 and 18, we cannot have  $k = 18$  or  $n + 6 = 10$ , which means that  $n \neq 7$  and  $n \neq 4$ .

This means that the possible values of  $k$  are 16 and 17.

13. Starting on the 22nd floor of their apartment building, Hassan goes up the stairs and Henry goes up by an elevator. When Hassan starts going up, Henry waits for 2 minutes for the elevator and Hassan goes up from each floor to the next floor in 15 seconds. The elevator goes up from each floor to the next floor in 3 seconds. Hassan and Henry arrive on the  $n$ th floor at exactly the same time. What is the value of  $n$ ?

**Solution**

From the 22nd floor to the  $n$ th floor, Hassan and Henry each go up  $n - 22$  floors. Since Hassan goes from each floor to the next in 15 seconds, this takes his  $15 \times (n - 22)$  seconds. Meanwhile, Henry waits for 2 minutes implying 120 seconds. Since Henry goes from each floor to the next in 3 seconds, it takes a total of  $120 + 3 \times (n - 22)$  seconds to reach the  $n$ th floor. Since their travel times are equal, then

$$15 \times (n - 22) = 120 + 3 \times (n - 22)$$

$$15 \times (n - 22) - 3 \times (n - 22) = 120$$

$$12 \times (n - 22) = 120$$

$$n - 22 = 10$$

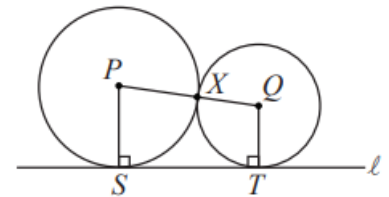
$$\text{and so } n = 32.$$

14. The integers 1 to 32 are spaced evenly and in order around the outside of a circle. Straight lines that pass through the centre of the circle join these numbers in pairs. Which number is paired with 12?

**Solution**

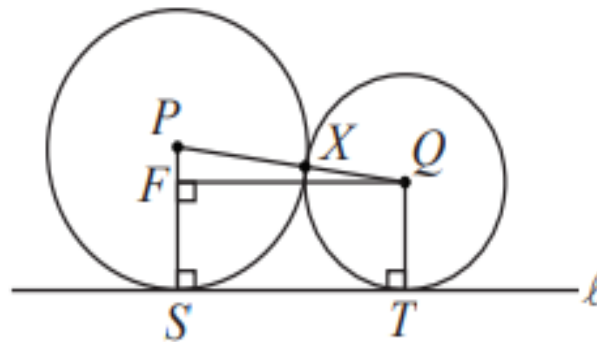
The integers 1 to 32 are spaced evenly and in order around the outside of a circle. Consider drawing a first straight line that passes through the centre of the circle and joins any one pair of these 32 numbers. This leaves  $32 - 2 = 30$  numbers still to be paired. Since this first line passes through the centre of the circle, it divides the circle in half. In terms of the remaining 30 unpaired numbers, this means that 15 of these numbers lie on each side of the line drawn between the first pair. Let the number that is paired with 12 be  $n$ . If we draw the line through the centre joining 12 and  $n$ , then there are 15 numbers that lie between 12 and  $n$  (moving in either direction, clockwise or counter clockwise). Beginning at 12 and moving in the direction of the 13, the 15 numbers that lie between 12 and  $n$  are the numbers 13, 14, 15, ..., 26, 27. Therefore, the next number after 27 is the number  $n$  that is paired with 12. The number paired with 12 is 28.

16. In Figure 2, circles with centres  $P$  and  $Q$  touch a line  $\ell$  (that is, are tangent to  $\ell$ ) at points  $S$  and  $T$ , respectively, and touch each other at point  $X$ . Because the circles touch at  $X$ , line segment  $PQ$  passes through  $X$  and so  $PQ = PX + XQ$ . Because the circles touch  $\ell$  at  $S$  and  $T$ ,  $PS$  and  $QT$  are perpendicular to  $\ell$ .
15. If the circle with centre  $P$  has a radius of 25 and the circle with centre  $Q$  has a radius of 16, determine the area of trapezoid  $PQTS$ .



**Solution**

Draw a perpendicular from  $Q$  to  $F$  on  $PS$  as shown in Figure 2 below. Quadrilateral  $FQTS$  has



three right angles and so it must be a rectangle. Therefore,  $FS = QT$ . Since  $QT$  is a radius, then  $FS = QT = 16$ . Since  $PS$  is a radius, then  $PS = 25$ . Therefore,  $PF = PS - FS = 25 - 16 = 9$ . We note that  $PQ = PX + XQ = 25 + 16 = 41$  (they are both radii). By the Pythagorean Theorem in  $\triangle PFQ$ , which is right-angled at  $F$ ,

$$FQ = \sqrt{PQ^2 - PF^2} = \sqrt{41^2 - 9^2} = \sqrt{1681 - 81} = 40.$$

Since  $FQTS$  is a rectangle,  $ST = FQ = 40$ . Finally, the area of trapezoid  $PQTS$  equals

$$\frac{1}{2}(PS + QT) \times ST = \frac{1}{2}(25 + 16) \times 40 = 41 \times 20 = 820.$$



**QUESTIONS AND SOLUTIONS**

**SECTION A: 5 marks each**

1. Suppose  $x + \frac{2}{y} = 2$  and  $y + \frac{2}{x} = 4$ . What is the value of  $\frac{1}{2} + \frac{x}{y}$ , for  $x, y \neq 0$ ?

**Solution**

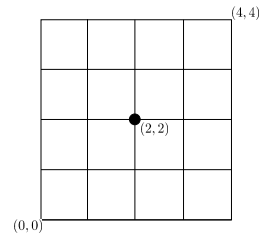
From  $x + \frac{2}{y} = 2$  and  $y + \frac{2}{x} = 4 \Leftrightarrow xy = 2y - 2 = 4x - 2 \Leftrightarrow \frac{x}{y} = \frac{1}{2}$  thus  $\frac{x}{y} + \frac{1}{2} = 1$ .

2. John started reading 16 books, one at a time. The first book took him one day to read, the second book took 2 days to read, the third took him three days to read, and so on. John finished reading the first book on a Monday and the second book on a Wednesday. On what day of the week did he finish reading the 16th book?

**Solution**

The number of days spent reading the 16 books is  $1 + 2 + 3 + 4 + 5 + \dots + 16 = 136$  days. The 16<sup>th</sup> day was read on 135<sup>th</sup> day after the 1<sup>st</sup> day (Monday) thus  $135 \pmod{7} = 2 \Rightarrow$  the day was Wednesday, denote remainders 0, 1, 2 by 0– Monday, 1– Tuesday and 2– Wednesday.

3. A student moves from the point  $(0, 0)$  to the point  $(4, 4)$  in exactly eight steps each of distance 1 in either the positive  $x$  or  $y$  direction. What is the number of possible paths the student can follow so as he/she passes through the point  $(2, 2)$ ?



**Solution**

There are two possible paths to leave  $(0, 0)$ , There are two possible paths to leave  $(1, 0)$ , There are two possible paths to leave  $(0, 1)$ , There are two possible paths to leave  $(1, 1)$ . Thus  $2 \times 2 \times 2 \times 2 = 2^4 = 16$  paths passing through  $(2, 2)$  plus two ways namely  $(0, 0) \rightarrow (1, 0) \rightarrow (2, 0) \rightarrow (2, 1) \rightarrow (2, 2)$  and  $(0, 0) \rightarrow (0, 1) \rightarrow (0, 2) \rightarrow (1, 2) \rightarrow (2, 2)$  in total 18 paths to  $(4, 4)$  passing through  $(2, 2)$ .

4. In a supermarket there is a five day sales promotion where prices on all items are discounted by  $\frac{1}{3}$  of the price on the first day. Beginning on the second day and each day after they have additional 10% off the previous day's price. What is the overall percentage discount that the buyer enjoys if he/she buys an item on the fourth day of the promotion?

**Solution**

The first day you pay  $\frac{2}{3}x$ , if  $x$  is the original price

The second day you pay  $0.9 \times \frac{2}{3}x$

The third day you pay  $0.9 \times 0.9 \times \frac{2}{3}x$

The fourth day you pay  $0.9 \times 0.9 \times 0.9 \times \frac{2}{3}x = 0.486x$

Thus the overall percentage discount the buyer enjoys if he/she buys an item on the fourth day is 51.4%.



5. For every positive integer  $n$ , the Collatz function  $f(n)$  is defined by;

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \in \mathbb{E}(\text{set of even numbers}) \\ 3n + 1, & \text{if } n \in \mathbb{O}(\text{set of odd numbers}) \end{cases}$$

If  $f^k(n) = f \circ f \circ f \circ \dots \circ f(n)$ , composed  $k$ -times in succession. Find the value of  $f^{2023}(40)$ .

**Solution**

$f(40) = 20, f^2(40) = 10, f^3(40) = 5, f^4(40) = 16, f^5(40) = 8, f^6(40) = 4,$   
 $f^7(40) = 2, f^8(40) = 1, f^9(40) = 4, f^{10}(40) = 2, f^{11}(40) = 1, f^{12}(40) = 4$  and so on, the value of  $f^k(40) = 4$  for  $k > 8$  keeps repeating itself, now  $f^{(8+3k)}(40) = 1 \forall k \in \mathbb{N}, k = 0, 1, 2, 3, 4, \dots$  so  $2023 = 8 + 2015 = 8 + 671 \times 3 + 2$  so  $f^{2023}(40) = f^{(8+671 \times 3 + 2)}(40) = 2$ .

6. What is the last digit of  $3^{2023}$ ?

**Solution**

Consider the following trial digits for a few powers,  $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, 3^6 = 729, 3^7 = 2187, 3^8 = 6561, 3^9 = 19683, 3^{10} = 59049 \dots$  There is a pattern in the last digit of 3, 9, 7, 1 and repeats itself in the same order.  $3^{4k-3}$  has the last digit as 3 for  $k = 1, 2, 3, \dots$ , for  $2023 = (4 \times 506) - 3 + 2 \Rightarrow 3^{2023}$  will have the last digit as 7.

7. A day is said to be prime if both the month and the day are prime. For example March 13th is a prime day since March (3) is prime and 13 is also a prime number. How many prime days are there in 2023?

**Solution**

The prime months are February (2), March (3), May (5), July (7), and November (11), the prime days are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31. Recall 2023 has 28 days in February, therefore Feb has 9 prime days, each of the months March, May and July has 11 prime days, November has 10 prime days (no 31st)  $\Rightarrow$  a total of 52 prime days in 2023.

8. Among Juliet, David, Annet, Violet and Alice, there is exactly one spy. Each person makes one statement, the spy and exactly one other person will tell the truth. The statements are given in this order,

- Juliet: Violet is not a spy
- David: Violet is going to lie
- Annet: The spy is either David or Violet
- Violet: The spy has already made a statement
- Alice: I am the spy.
- Who is the spy?

**Solution**

Juliet is the spy.

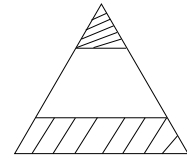
9. At the beginning of the first day, a box contains 1 black ball, 1 gold ball, and no other balls. At the end of each day, for each gold ball in the box, 2 black balls and 1 gold ball are added to the box; this means that at the end of the first day, there are 5 balls in the box. If no balls are removed from the box, how many balls are in the box at the end of the seventh day?

**Solution**

End of day	No. of Black balls	No. of Gold balls
1	$1 + 2$	$1 + 1$
2	$1 + 2 + 4$	$1 + 1 + 1 + 1$
3	$7 + (4 \times 2)$	$4 + 4$
4	$15 + (8 \times 2)$	$8 + 8$
5	$31 + (16 \times 2)$	$16 + 16$
6	$63 + (32 \times 2)$	$32 + 32$
7	$127 + (64 \times 2)$	$64 + 64 = 128$

The total number of balls is  $128 + 255 = 383$  balls.

10. Consider a cake in the shape of an equilateral triangle with height 3 cm. Cut it into three strips of equal height 1 cm as shown in the figure below. Find the total combined area of the top and the bottom strips of the triangular cake.



**Solution**

The shaded triangle on top may be rotated and fit together with the shaded trapezoid on the bottom to form a parallelogram. The height of the parallelogram is 1 cm, we find the base of the parallelogram; the adjacent  $= \frac{3}{\sqrt{3}} = \sqrt{3}$  the base  $= 2 \times \text{adjacent} \Rightarrow \text{base} = 2\sqrt{3}$ , therefore the area of the parallelogram is  $bh = 2\sqrt{3} \times 1 = 2\sqrt{3}$  sq. units.

**SECTION B : 10 marks each**

11. Let  $x$  and  $y$  be positive numbers satisfying  $2 + \log_2 x = 3 + \log_3 y = \log_6(x + y)$ . Find the value of  $\frac{1}{x} + \frac{1}{y}$ .

**Solution**

From  $2 + \log_2 x = 3 + \log_3 y = \log_6(x + y)$ , Let  $a = 2 + \log_2 x = 3 + \log_3 y = \log_6(x + y)$  then  $x = 2^{a-2}, y = 3^{a-3}, x + y = 6^a$  thus

$$\frac{1}{x} + \frac{1}{y} = \frac{x + y}{xy} = \frac{6^a}{2^{a-2} \times 3^{a-3}} = 2^2 \times 3^3 = 108.$$

12. The ratio of paint to water in three separate solutions of paint and water is 1 : 1, 1 : 2 and 1 : 3. If equal quantities of each solution are combined together and mixed what will be the ratio of water to paint in the resulting solution?

**Solution**

Assume that we take a bucket of each of the solution then solution 1 will contain  $\frac{1}{2}$  water and  $\frac{1}{2}$  paint, solution 2 will contain  $\frac{2}{3}$  water and  $\frac{1}{3}$  paint and solution 3 will contain  $\frac{3}{4}$  water and  $\frac{1}{4}$  paint, now we have, total fraction of paint as  $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$  bucket of paint or  $\frac{13}{12}$  paint. For water we have  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$  buckets or  $\frac{23}{12}$  thus the required ratio is 23 : 13.

13. The distance between the dormitory and the class is  $1\text{KM}$ . You leave the dormitory at the same time each day, walk at a steady speed of  $3\text{ KM}$  per hour and arrive just as the class starts. Today you were distracted by the good weather and walked the first half kilometre at a speed of  $2\text{ KM/hr}$ . At what speed must you run(walk) the last half of the kilometre in order to arrive just as the class starts?

**Solution**

Recall Distance = speed  $\times$  time. Initially you take  $t = \frac{2}{3}$  hours = 20 minutes from dormitory to class. Today it took you 15 minutes to cover first half KM i.e Distance = speed  $\times$  time thus  $\frac{1}{2} = 2 \times \text{time} \Rightarrow t = 15$  minutes. You therefore have 5 minutes to reach on time from  $\frac{1}{2}\text{ KM} = \text{speed} \times \text{time}$  where  $t = 5\text{mins} = \frac{1}{12}\text{hr}$  thus speed =  $6\text{ km/hr}$ .

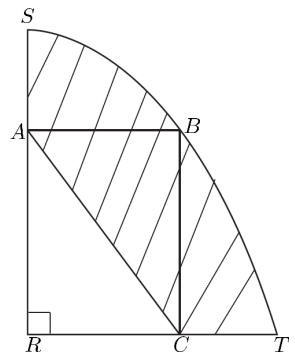
14. What is the value of  $\log_3 4 \times \log_4 5 \times \log_5 6 \times \dots \times \log_{80} 81$ ?

**Solution**

Convert all logarithms into base 10 that is,

$$\log_3 4 \times \log_4 5 \times \log_5 6 \times \dots \times \log_{80} 81 = \frac{\log_{10} 4}{\log_{10} 3} \times \frac{\log_{10} 5}{\log_{10} 4} \times \frac{\log_{10} 6}{\log_{10} 5} \times \dots \times \frac{\log_{10} 81}{\log_{10} 80} = \frac{\log_{10} 81}{\log_{10} 3} = 4.$$

15. In the figure on the right, arc  $SBT$  is one quarter of a circle with centre  $R$  and radius  $r$ . The length plus width of the rectangle  $ABCR$  is 8 units and the perimeter of the shaded region is  $10 + 3\pi$ . Find the value of  $r$ .



**Solution**

Let  $x = \overline{RC}$ ,  $y = \overline{AR} \Rightarrow x+y = 8$ , Now  $x^2+y^2 = r^2$  since the rectangle  $ABCR$  has one vertex in the circle. The perimeter of the shaded area is  $AS+SBT+TC+AC = (r-y)+\frac{\pi r}{2}+(r-x)+\sqrt{x^2 + y^2} = 3r + \frac{\pi r}{2} - 8$  thus  $18 + 3\pi = (3 + \frac{\pi}{2})r \Rightarrow r = 6$  units.



NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 1 QUESTIONS AND SOLUTIONS.

**SECTION A: 5 marks each**

1. Find the value of  $y$  if  $x + y = 8$  and  $\log_2(xy) = 2$ .

**Solution**

From  $x + y = 8 \rightarrow y = 8 - x$  thus  $\log_2(x(8 - x)) = 2 \Leftrightarrow x^2 - 8x + 4 = 0$  whose solutions are  $x = 4 \pm 2\sqrt{3}$  thus  $y = 8 - (4 \pm 2\sqrt{3}) = 4 \pm 2\sqrt{3}$ .

2. Let the terms  $a_1, a_2, a_3, \dots$  be a sequence such that first term  $a_1 = 2023$  and each succeeding term in the sequence is obtained as the sum of the cubes of the digits of the previous term. What is the value of the term  $a_{2024}$ ?

**Solution**

$a_1 = 2023, a_2 = 2^3 + 0^3 + 2^3 + 3^3 = 43, a_3 = 4^3 + 3^3 = 91, a_4 = 9^3 + 1^3 = 730, a_5 = 7^3 + 3^3 + 0^3 = 370, a_6 = 3^3 + 7^3 + 0^3 = 370$ . The terms in the sequence become constant as  $n \rightarrow \infty$  for  $n \geq 5$  thus  $a_{2024} = 370$ .

3. Solve for  $x$  in the equation  $6^{\frac{1}{x}} + 3 \cdot 9^{\frac{1}{x}} = 2 \cdot 4^{\frac{1}{x}}$ .

**Solution**

Re-write  $6^{\frac{1}{x}}$  as  $2^{\frac{1}{x}} \cdot 3^{\frac{1}{x}}$ ,  $9^{\frac{1}{x}}$  as  $3^{\frac{2}{x}}$  and  $4^{\frac{1}{x}}$  as  $2^{\frac{2}{x}}$  substituting these into the main equation we get,  $2^{\frac{1}{x}} \cdot 3^{\frac{1}{x}} + 3 \cdot 3^{\frac{2}{x}} = 2 \cdot 2^{\frac{2}{x}}$  dividing through by  $3^{\frac{1}{x}}$  and then 2 we get,  $1 + 3 \cdot \left(\frac{3}{2}\right)^{\frac{1}{x}} = 2 \cdot \left(\frac{2}{3}\right)^{\frac{1}{x}}$ , let  $y = \left(\frac{2}{3}\right)^{\frac{1}{x}}$  then the equation becomes  $1 + 3y^{-1} = 2y \Leftrightarrow 2y^2 - y - 3 = 0$  whose solution is either  $y = \frac{3}{2} \Rightarrow x = -1$  or  $y = -1 \Rightarrow \left(\frac{2}{3}\right)^{\frac{1}{x}} = -1$  which has no solution. Thus the only solution is  $x = -1$ .

4. For all values of  $x$ , the function  $f(x)$  satisfies  $f(x+1) + f(x-1) = f(x)$ . If  $f(0) = 5$  and  $f(1) = 7$ , then find the value of  $f(2022) + f(2023) + f(2024)$ .

**Solution**

$f(x+1) = f(x) - f(x-1) \Rightarrow f(x+2) = f(x+1) - f(x)$  and  $f(x+1) = f(x) - f(x-1)$ , from these we have  $f(x+2) = -f(x-1) \rightarrow f(x+3) = -f(x)$  and  $f(x+6) = -f(x+3)$  but  $f(x+3) = -f(x)$  thus  $f(x+6) = f(x)$  which implies that the function is periodic with period 6; the following is true about the functional values of  $f(x)$ ,  $f(2022) = f(0) = 5$ ,  $f(2023) = f(1) = 7$ ,  $f(2024) = f(2) = 2$ , since  $f(2) = f(1) - f(0) = 7 - 5 = 2$  Thus  $f(2022) + f(2023) + f(2024) = 5 + 7 + 2 = 14$ .



5. There are 120 different 5-digit numbers that use each of the digits 2, 4, 5, 7, and 8 exactly once. If listed in numerical order from smallest to largest, what number is in the 97<sup>th</sup> position of the list?

### Solution

There are 24 numbers that start with each digit so the first 96 numbers consist of all the numbers which start with 2, 4, 5 and 7 therefore the 97<sup>th</sup> number in the list is the smallest which starts with 8 that is 82, 457.

6. Suppose that increasing the bus fare by 10% results in a 5% decrease in the number of passengers. To achieve an 8% increase in revenue, what percentage increase in the fare should be implemented? Find the required fare increase percentage assuming that the fare increase does not exceed 50%.

### Solution

Let  $c_0$  and  $n$  represent the original cost and the number of passengers respectively, the cost and the number of passengers after the increase are  $c_0 \left(1 + \frac{x}{100}\right)$  and  $n \left(1 - \frac{x}{200}\right)$ . The revenue becomes  $c_0 \left(1 + \frac{x}{100}\right) \cdot n \left(1 - \frac{x}{200}\right) = nc_0 \left(1 + \frac{x}{100}\right)$  solving gives  $x^2 - 100x + 1600 = 0 \Leftrightarrow x = 20\%$  or  $x = 80\%$  thus  $x = 20\%$  as  $x \leq 50$ .

7. A worker can finish a job in 3 days with his present supply of a certain type of machine. However, with 3 more of these machines the job could be done in 2 days. If the machines all work at the same rate, how many days would it take to do the job with one machine?

### Solution

Let  $x$  be the number of machines that can finish the job in 3 days then  $3x = 2(x + 3)$  is the number of days that one machine would take to finish the job thus  $3 \times 6 = 18$  days.

8. Four suspects of a crime made the following statements to the police:

Ryan said "Bruce did it."

Andrew said "I didn't do it."

Bruce said "Anne did it."

Anne said: "Bruce lied when he said I did it."

Determine who committed the crime if exactly one of the four statements is

- (a) true
- (b) false.

### Solution

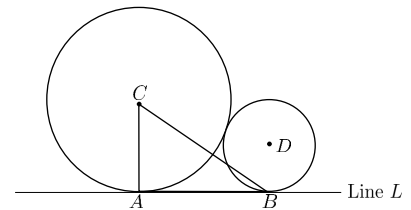
Both answers in (a) and (b) are obtained by using that each person is the criminal and seeing how many of the statements are true or false in each case. Only Andrew in (a) satisfy the information given. and in (b), it is Bruce.

9. Let  $A$  and  $B$  be two cars initially positioned at opposite points along a straight Road. When they both travel at their original constant speeds, it takes 30 minutes for them to meet each other. If car  $A$  doubles its speed while  $B$  maintains the original speed, they meet in 25 minutes. Determine the time in minutes it will take for them to meet if car  $B$  doubles its speed while  $A$  retains the original speed.

**Solution**

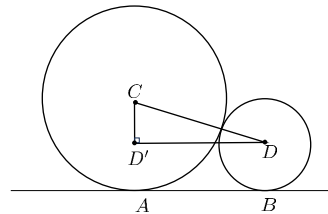
Let the distance between  $A$  and  $B$  be  $L$  then let  $s_1$  and  $s_2$  be the original speed of  $A$  and  $B$  respectively from the condition we have  $(s_1 + s_2)30 = L$  and  $(2s_1 + s_2)25 = L$  solving gives  $s_1 = L(\frac{1}{25} - \frac{1}{30})$  and  $s_2 = L(\frac{2}{30} - \frac{1}{25})$ , we want to find  $t$  such that  $(s_1 + 2s_2)t = L$  thus  $t = \frac{100}{6} = \frac{50}{3}$  minutes.

10. The larger circle is of radius  $7\text{cm}$  centered at  $C$  is tangent to the smaller circle of radius  $1\text{cm}$  centered at  $D$ . If the circles are tangent to the line  $L$  at  $A$  and  $B$ . What is the area of the triangle  $ABC$ ?



**Solution**

Apply the pythagoras Theorem to the triangle  $CD'D$  as in the diagram below



to get  $(CD)^2 = (CD')^2 + (D'D)^2 \Rightarrow CD = \sqrt{(7 - 1)^2 + (7 + 1)^2}$  and  $CD = 10\text{cm}$ , so the area of triangle  $ABC$  is  $\frac{1}{2} \times 8 \times 7 = 28\text{cm}$ .

**SECTION B: 10 marks each**

11. The average age of three siblings, Alice, Bob and John is  $x$ . It will become  $2x$  when John gets as old as Bob is now. It will become  $1.5x$  when Bob gets as old as Alice is now. It will become 30 when John gets as old as Alice is now. Find the current age of John.

**Solution**

Let  $a$  represent the age of Alice now,  $b$  represent the age of Bob now and  $J$  represent the age of John now, thus  $\frac{a+b+J}{3} = x \Leftrightarrow a+b+J = 3x$ . After  $n$  years John will be  $J+n = b$  thus new average is  $\frac{a+b+J}{3} + n = 2x = x+x \Leftrightarrow x = n = b - J$ . After  $k$  years  $\frac{a+b+J}{3} + k = 1.5x = 0.5x + x$  so  $a - b = 0.5x$ . Now after  $y$  years average is 30 thus  $\frac{a+b+J}{3} + y = 30$  from the facts that  $y = a - J, x = n = b - J$  and  $a - b = 0.5x$  we have  $1.5x = a - J$  thus  $x + 1.5x = 2.5x = 30$  thus  $x = 12$ . so the ages are  $J, J + 12, J + 18$  with average 12 thus  $J = 2, 14$  and  $20$  years meaning John is 2 years now.

12. Four sparrows, Pip, Pap, Pep and Pop found a dish of seeds that was free of dirt said Pip: “In turns each will take two grains, Plus a third of what remains and I’ll go first, then Pep, then Pop, With Pap the last. And then we stop.” But Pap cried out, “It isn’t fair. Mine’s two seeds less than half Pep’s share.” But Pip was boss; his word was law So little Pap got nothing more. Poor Pap, his share was rather small! How many seeds were there in all?

**Solution**

Let  $N$  be the number of seeds in the dish, the table below shows the amount of seeds each bird takes,

Pip	Remaining	Pep	Remaining
$\frac{1}{3}(N - 2) + 2$ $= \frac{N+4}{3}$	$N - \left(\frac{N+4}{3}\right)$ $= \frac{2N-4}{3}$	$\frac{1}{3}\left(\frac{2N-4}{3}\right) + 2$ $= \frac{2N+14}{9}$	$\frac{2N-4}{3} - \frac{2N+14}{9}$ $= \frac{4N-20}{9}$

Pop	Remaining	Pap
$\frac{1}{3}\left(\frac{4N-20}{9} - 2\right) + 2$ $= \frac{4N+16}{27}$	$\frac{4N-20}{9} - \frac{4N+16}{27}$ $= \frac{8N-76}{27}$	$\frac{1}{3}\left(\frac{8N-76}{27} - 2\right) + 2$ $= \frac{8N+32}{81}$

$$\Leftrightarrow \frac{8N+32}{81} = \frac{1}{2} \left(\frac{2N+8}{9}\right) - 2 \Leftrightarrow N = 158 \text{ seeds.}$$

13. All of John’s steps with his left foot are 0.5 metres. His right foot steps are the same, except that every third one is twice as long. He starts with his left foot and alternates, except that after every third left foot step, he takes a step back with his left foot. How many steps, forward and backward, does he take to travel the 19 metres from the bus stop to the Ministry of Silly Walks?

**Solution**

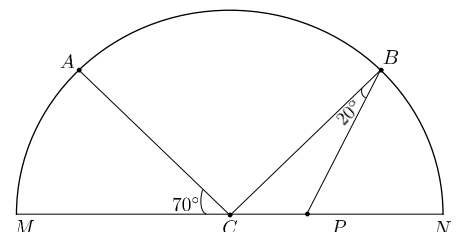
John repeats the sequence  $LRLRLBS$  where  $L$  and  $R$  be the ordinary left and right steps,  $B$ —Backward left step and  $S$  is a  $1m$  right step. This is the 7 steps and the distance covered is  $5 \times (0.5) - 0.5 + 1 = 3$ . Doing this 6 times will take 42 steps and cover  $8m$ . John then takes an additional  $L$  and an additional  $R$  to bring him to the destination thus 44 steps are required.

14. A 5-digit number is said to be “elegant” if (i) its digits are from 1 to 9, (ii) it contains each digit only once, and (iii) it contains the digits 2, 4 and 8, in that order, but not necessarily consecutively. (For instance, 24385 is elegant). How many elegant numbers are there?

**Solution**

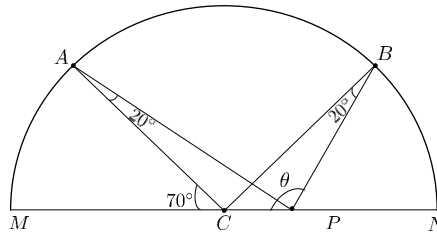
We must select 3 positions for the 2, 4 and 8 (already chosen order). There are 10 ways to do it namely  $248AB, 24A8B, 24AB8$  etc of the remaining 2 positions we have 6 choices for the first digit (anything other than 0, 2, 4, 8) and 5 for the second (not 0, 2, 4, 8 or the digit just selected) the total number is  $10 \times 6 \times 5 = 300$ .

15. Two distinct points  $A$  and  $B$  are on a semicircle with diameter  $MN$  and center  $C$ . The point  $P$  is on  $\overline{CN}$  and  $\angle CBP = 20^\circ$  as indicated in the diagram. What is the value of  $\angle BPN$  if  $\angle ACM = 70^\circ$ .



**Solution**

Consider the figure below



Observe that,  $\angle CPA = 50^\circ$ . Let  $R$  be the radius of the half circle and let  $\theta = \angle CPB$  and to  $\triangle CPB$  we find that  $\frac{\sin 50^\circ}{R} = \frac{\sin 20^\circ}{R}$  and  $\frac{\sin \theta}{R} = \frac{\sin 20^\circ}{CP} \Leftrightarrow \frac{\sin \theta}{R} = \frac{\sin 50^\circ}{R} \Leftrightarrow \sin \theta = \sin 50^\circ \Rightarrow \theta = 180^\circ - 50^\circ = 130^\circ \Rightarrow \angle BPN = 180^\circ - 130^\circ = 50^\circ$

**END**



NATIONAL MATHEMATICS CONTEST  
O-LEVEL PAPER 2 SOLUTIONS.

SATURDAY 29TH JUNE, 2024

SECTION A: 5 marks each

1. Let  $x, y, z$  and  $w$  be integers satisfying  $x \log_{10}(2) + y \log_{10}(3) + z \log_{10}(5) + w \log_{10}(7) = 2024$ . Use the above information to find the value of  $4x + 3y + 2z + w$ .

**Solution:**

Exponentiating the expression  $x \log_{10}(2) + y \log_{10}(3) + z \log_{10}(5) + w \log_{10}(7) = 2024$  yields  $2^x 3^y 5^z 7^w = 10^{2024} = 2^{2024} 3^0 5^{2024} 7^0$ . By the fundamental theorem of arithmetic, (compare terms on both sides)  $x = z = 2024$  and  $y = w = 0$ , hence  $4x + 3y + 2z + w = 12144$ .

2. Find the number of arrangements of 4, 5, 6, 8, 9, 10, 14 into a single row so that every pair of the adjacent numbers in the arrangement has a common factor greater than 1.

**Solution:**

Let  $X = \{4, 5, 6, 8, 9, 10, 14\}$ . The arrangement must either start with 5, 10 and end with 6, 9 or start with 9, 6 and end with 10, 5 since  $\gcd(5, x) = 1$  for every  $x \in X$  except when  $x = 10$ , and  $\gcd(9, x) = 1$  for every  $x \in X$  except when  $x = 6$ .

Since the greatest common factor of  $\{4, 6, 8, 10, 14\}$  is 2, then every permutation of 4, 8, 14 between the 6 and 10 gives a possible arrangement, thus there are  $2 \times 3! = 12$  ways and these are;

5, 10, 14, 8, 4, 6, 9	5, 10, 14, 4, 8, 6, 9
5, 10, 8, 14, 4, 6, 9	5, 10, 8, 4, 14, 6, 9
5, 10, 4, 8, 14, 6, 9	5, 10, 4, 14, 8, 6, 9
9, 6, 14, 8, 4, 10, 5	9, 6, 14, 4, 8, 10, 5
9, 6, 8, 14, 4, 10, 5	9, 6, 8, 4, 14, 10, 5
9, 6, 4, 8, 14, 10, 5	9, 6, 4, 14, 8, 10, 5

3. Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ ,  $x \mapsto f(x)$  be a function such that  $f(x + 1) = \frac{3}{1 + \frac{1}{f(x)}}$ . Determine  $f(1)$  if

$$f(7) = \frac{4374}{2191}.$$

**Solution:**

From  $f(x + 1) = \frac{3}{1 + \frac{1}{f(x)}}$ , we see that  $f(x) \neq 0$  and  $f(x) = \frac{f(x + 1)}{3 - f(x + 1)}$ . In particular,

$$f(7) = \frac{4374}{2191} \text{ implies that } f(6) = \frac{1458}{733}, f(5) = \frac{486}{247}, f(4) = \frac{162}{85}, f(3) = \frac{54}{31}, f(2) = \frac{18}{13}, \text{ hence}$$

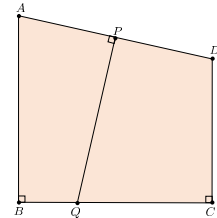
$$f(1) = \frac{6}{7}. \text{ Algorithmically, } f(6) = \frac{\frac{4374}{2191}}{3 - \frac{4374}{2191}} = \frac{\frac{4374}{2191}}{\frac{3}{2191}} = \frac{4374}{3} = \frac{1458}{1}.$$

4. Two friends, Anzoa and Origa play a game in which Anzoa rolls two ordinary dice while Origa rolls one. What is the probability that Origa's die roll exceeds the total of Anzoa's?

**Solution:**

The set of Origa's possible outcomes is  $\{x \in \mathbb{Z} : 1 \leq x \leq 6\}$  while that of Anzoa is  $\{(x, y) \in \mathbb{Z}^2 : 1 \leq x, y \leq 6\}$ . Furthermore, the sum of Anzoa's output belongs to the set  $\{z \in \mathbb{Z} : 2 \leq z \leq 12\}$ . Let  $X$  be the random variable for Origa's output and  $Z$  be the random variable for the sum of Anzoa's output. Depending on whether Origa rolls 1, 2, 3, 4, 5, or 6, the probability that Anzoa's total is lower is respectively  $0, 0, \frac{1}{36}, \frac{3}{36}, \frac{6}{36}, \frac{10}{36}$ . Since these outputs of Anzoa's and Origa's rolls are independent, it follows from the multiplicative rule that, the overall probability that Origa's die roll exceeds the total of Anzoa's is  $\sum_{i=1}^6 \Pr(\{X = i\} \cap \{Z \geq i\}) = \sum_{i=1}^6 \Pr(\{X = i\}) \cdot \Pr(\{Z < i\}) = \frac{1}{6} \left( \frac{1}{36} + \frac{3}{36} + \frac{6}{36} + \frac{10}{36} \right) = \frac{5}{54}$ .

5. In the Figure on the right,  $\vec{AB}$  is perpendicular to  $\vec{BC}$ ,  $\vec{DC}$  is perpendicular to  $\vec{BC}$ , and  $\vec{PQ}$  is the perpendicular bisector of  $\vec{AD}$ . If  $AB = 9$ ,  $BC = 8$ , and  $DC = 7$ , find the area of the quadrilateral APQB.



**Solution:**

Shift the origin of the  $xy$ -plane to be at B. Since P is the midpoint of  $\vec{AD}$ , its coordinates are  $(4, 8)$ . Let  $(a, b)$  be the coordinates of Q, the intersection of the perpendicular bisector of  $\vec{AD}$  and the line segment  $\vec{BC}$ . Since along  $\vec{AD}$ , is along the  $x$ -axis, we have  $b = 0$ . By gradient considerations, we have  $\frac{0 - 8}{a - 4} \cdot \frac{7 - 9}{8 - 0} = -1$ , hence  $a = 2$ . Therefore, the area enclosed by the quadrilateral APQB is  $\frac{1}{2} \cdot 8(2 + 4) + \frac{1}{2} \cdot 1 \cdot 4 = 26$  square units.

6. There are 2024 antelopes and a certain number of lions living across nine game parks in Uganda. In each park, the number of antelopes is eleven times the total number of lions in all the other parks. Find the number of lions that live in the nine game parks of Uganda.

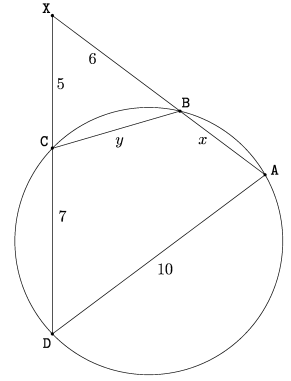
**Solution:**

Let  $X_1, \dots, X_9$  be the number of antelopes in each of the nine parks and let  $x_1, \dots, x_9$  be the number of lions in each of the nine parks so that a park contains  $X$  antelopes and  $x$  lions and so on. Therefore,  $X_1 = 11(x_2 + x_3 + \dots + x_9)$  and  $X_2 = 11(x_1 + x_3 + \dots + x_9)$  and so on. Therefore,  $X_1 + \dots + X_9 = 8 \cdot 11(x_1 + \dots + x_9)$  and hence, since total number of antelopes is 2024, we have  $2024 = 88(x_1 + \dots + x_9)$ . Therefore  $(x_1 + \dots + x_9) = \frac{2024}{88} = 23$  and hence the total number of lions in the nine parks is 23.

7. A cyclic quadrilateral has all four vertices on the circumference of a circle. Find the area enclosed by the cyclic quadrilateral ABCD with sides  $x, y, 7, 10$  shown in the figure on the right. (In 598 - 670 AD, Brahmagupta showed the area  $A$ , enclosed by a cyclic quadrilateral whose edges have lengths  $a, b, c, d$  to be

$$A := \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

$$\text{where } s = \frac{a+b+c+d}{2}.$$



**Solution:**

From the Figure, triangles XDA and XBC are similar, hence,  $\frac{7+5}{6} = \frac{10}{y} = \frac{6+x}{5}$ , thence  $x = 4$  and  $y = 5$ . Let  $a = 4, b = 5, c = 7$  and  $d = 10$  and  $A$  be the plane area enclosed by a cyclic quadrilateral with edge lengths  $a, b, c, d$ . From this, we obtain  $s = \frac{a+b+c+d}{2} = 13$  hence  $A = \sqrt{(s-a)(s-b)(s-c)(s-d)} = 36$  square units by Brahmagupta's formula.

8. Suppose  $a, b$ , and  $c$  are positive integers with  $a < b < c$  such that  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ . What is  $a+b+c$ ?

**Solution:**

First note that we must have  $\frac{1}{a} < 1$ , so  $a > 1$ . Since  $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$ , we must also have  $\frac{1}{a} > \frac{1}{3}$ ; so  $a < 3$ . Thus,  $a = 2$ . Now  $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$  where  $2 < b < c$ . Similar to before,  $\frac{1}{b} > \frac{1}{4}$ , so  $b < 4$ . Thus,  $b = 3$ . With  $a = 2$  and  $b = 3$  we have  $\frac{1}{2} + \frac{1}{3} + \frac{1}{c} = 1$ , which is satisfied when  $c = 6$ . To conclude,

$$a + b + c = 2 + 3 + 6 = 11.$$

9. A house is in the shape of a triangle, perimeter  $P$  metres and area  $A$  square metres. The garden consists of all the land within 5 metres of the house. How much land the garden and the house together occupy?

**Solution:**

The garden consists of 3 rectangular pieces and three sectors of a circle. The rectangular pieces all have width 5 metres and the total length is  $P$  metres. Their combined area is therefore  $5P$  square metres.

At a corner, with interior angle  $x$ , the angle within the sector is  $360^\circ - 180^\circ - x = 180^\circ - x$ . The sum of the angles in all three sectors is  $3(180^\circ) - (\text{sum of interior angles}) = 360^\circ$ .

Therefore, the sectors fit together to form a circle of radius 5, and the combined area is  $25\pi$ . The total area of house and garden is  $A + 25\pi + 5p$  square metres.



10. Find the sum of all the even integers from 546 to 854 inclusive.

**Solution:**

$$(546 + 854) + (548 + 852) + (550 + 850) + \dots + (696 + 704) + (698 + 702) + 700$$

$$1400 + 1400 + 1400 + \dots + 1400 + 1400 + 700$$

Since  $698 - 546 = 152$  and we have only even numbers we have  $76 + 1 = 77$  times 1400 are added together.

$$(77 \times 1400) + 700 = 108500$$

**SECTION B: 10 marks each**

11. A Mersenne prime is any prime number of the form  $2^\ell - 1$ , where  $\ell \in \mathbb{Z}_{\geq 2}$ , for example, 3 and 7. Show that, if an integer  $n$  is the sum of two different powers of 2 and also the sum of two different Mersenne primes, then  $n$  is the sum of two different square numbers. For example,  $10 = 2^3 + 2 = (2^3 - 1) + (2^2 - 1) = 7 + 3 = 3^2 + 1^2$ . **Note:**  $\mathbb{Z}_{\geq 2}$  is a set of positive integers greater or equal to 2.

**Solution:**

Suppose  $n$  is the sum of two different powers of 2 and also the sum of two different Mersenne primes, i.e.,  $n = 2^x + 2^y$  for some  $x, y \in \mathbb{Z}_{\geq 0}$  with  $x \neq y$  and  $n = (2^w - 1) + (2^z - 1)$  where  $(2^w - 1)$  and  $(2^z - 1)$  are primes with  $w \neq z$ . Without loss of generality, suppose  $x > y$  and  $w > z$ . Since all Mersenne primes are odd, it follows that  $n$  is even hence  $y \geq 1$ . Furthermore,  $2^x + 2^y = n = 2^w + 2^z - 2$  implies that  $2^{x-1} + 2^{y-1} = 2^{w-1} + 2^{z-1} - 1$ . Since  $z \geq 2$ , it follows that  $2^{w-1} + 2^{z-1} - 1$  is odd hence  $y = 1$  in which case we find that  $2^{x-2} = 2^{w-2} + 2^{z-2} - 1$ . Now, if  $z > 2$ , then there are no positive integer solutions. However, if  $z = 2$ , then  $x = w$  (which must be an odd prime number since  $w > z = 2$ ) and so  $n = 2^x + 2$ . In this case, we have  $x = 2r + 1$  for some  $r \in \mathbb{Z}_{\geq 1}$  and  $n = 2^{2r+1} + 2 = (2^r + 1)^2 + (2^r - 1)^2$ .

12. Let ABCD be a cyclic quadrilateral such that the line segments  $\overrightarrow{CD}$  and  $\overrightarrow{BA}$  are extended to meet at E. Furthermore, suppose that the (straight) line through D which is tangent to the circle ADE meets the (straight) line  $\overrightarrow{CB}$  at F. Prove that the triangle CDF is isosceles.

**Solution:**

There are two possible diagrams based on the description in the question, though one is given in Figure 1. In the cyclic quadrilateral ABCD, we have,  $\angle DCB + \angle DAB = \pi$  (opposite interior angles of a cyclic quadrilateral are supplementary) and  $\angle EAD + \angle DAB = \pi$ , (angles on straightline are supplementary). Therefore,  $\angle EAD = \angle DCF$ , (or by the alternate segment theorem). Using the alternate segment theorem with circle ADE and tangent at D, we deduce that  $\angle EAD = \angle FDC$  (vertically opposite angles), hence  $\angle FDC = \angle FCD$ . Using the properties of isosceles triangles, we can conclude that triangle CDF is isosceles.

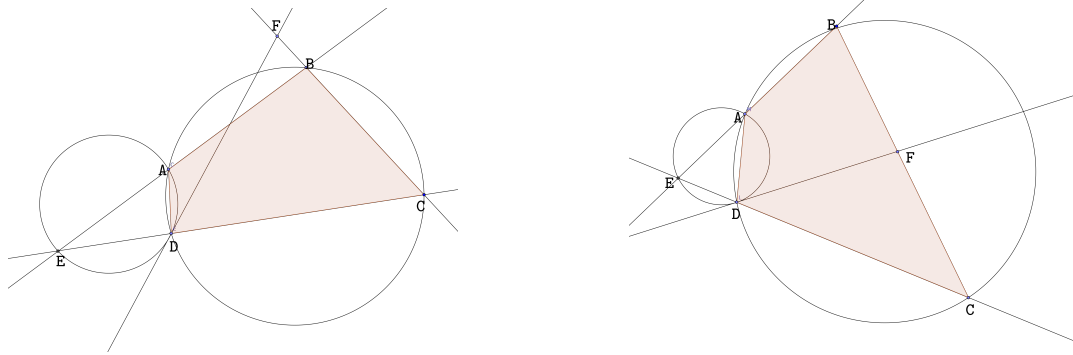
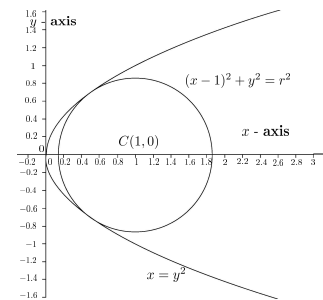


Figure 1: Sketch diagram of the first configuration.

13. A parabola  $x = y^2$  intersects (in at least one point) the circle with center  $C(1,0)$  and radius  $r > 0$  (has equation  $(x - 1)^2 + y^2 = r^2$ ). Find the least value of  $r$  for which this is true.



**Solution:**

Let  $(\alpha, \beta)$  be the point of intersection of the parabola  $x = y^2$  and the circle centered at  $(1,0)$  of radius  $r$ . It is clear that,  $\alpha = \beta^2$  and  $r = \sqrt{(\alpha - 1)^2 + (\beta - 0)^2} = \sqrt{(\beta^2 - 1)^2 + \beta^2}$ . By completion of squares,  $r = \sqrt{(\beta^2 - 1)^2 + \beta^2} = \sqrt{\beta^4 - \beta^2 + 1} = \sqrt{\left(\beta^2 - \frac{1}{2}\right)^2 + \frac{3}{4}} \geq \frac{\sqrt{3}}{2}$ .

14. If  $x, y,$  and  $z$  are positive real numbers satisfying  $x^2 + y^2 = 9, y^2 + yz + z^2 = 16$  and  $x^2 + \sqrt{3}xz + z^2 = 25$ , what is the value of  $2xy + xz + \sqrt{3}yz$ ?

**Solution:**

We need to find the value of  $2xy + xz + \sqrt{3}yz$

$$x^2 + y^2 = 9 \tag{1.1}$$

$$y^2 + yz + z^2 = 16 \tag{1.2}$$

$$x^2 + \sqrt{3}xz + z^2 = 25 \tag{1.3}$$

from  $(x + y)^2 = x^2 + 2xy + y^2$ , we have  $(x + y)^2 - 2yx = x^2 + y^2$  substitute into Eqn (1.1) to have  $(x + y)^2 - 2xy = 9$  that imply

$$2xy = (x + y)^2 - 9 \tag{1.4}$$

from  $(y + z)^2 = y^2 + 2yz + z^2$ , we have  $(y + z)^2 - yz = y^2 + yz + z^2$ , substitute into Eqn (1.2) to have  $(y + z)^2 - yz = 16 \Rightarrow yz = (y + z)^2 - 16$

$$\sqrt{3}yz = \sqrt{3}(y + z)^2 - 16\sqrt{3} \tag{1.5}$$



from  $(x + z)^2 = x^2 + 2xz + z^2$ , we have

$$(x + z)^2 - 2xz = x^2 + z^2 \quad (1.6)$$

adding equations (1.1) to (1.2) we get,

$$x^2 + 2y^2 + yz + z^2 = 25 \quad (1.7)$$

Equations (1.4) = (1.3) so that  $x^2 + \sqrt{3}xz + z^2 = x^2 + 2y^2 + yz + z^2$  to have

$$\sqrt{3}xz = 2y^2 + yz \quad (1.8)$$

Add equations (1.6) to (1.8) to get,

$$x^2 + \sqrt{3}xz + z^2 = (x+z)^2 - 2xz + 2y^2 + yz \text{ (substitute into (1.3)) to have } 25 = (x+z)^2 - 2xz + 2y^2 + yz.$$

Therefore,

$$xz = \frac{1}{2}((x+z)^2 + 2y^2 + yz - 25) \quad (1.9)$$

Adding (1.4), (1.5) and (1.9), we get

$$2xy + xz + \sqrt{3}yz = (x+y)^2 - 9 + \sqrt{3}(y+z)^2 - 16\sqrt{3} + \frac{1}{2}((x+z)^2 + 2y^2 + yz - 25)$$

15. Peter and John played a game. Peter wrote on a blackboard all integers from 1 to 18 and offered John to choose 8 different integers from this list. To win the game John had to choose 8 integers such that among them the difference between any two is either less than 7 or greater than 11. Can John win the game? Justify your answer.

### Solution:

Assume that such integers  $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$  are chosen and listed in the increasing order. Since all requirements are about differences between integers, we can shift them and assume that  $x_1 = 1$ . Since the difference between any two integers is either less than 7 or greater than 11, the integers 8, 9, 10, 11, 12 cannot be in the list. Consider the following pairs of the remaining integers: (2, 13), (3, 14), (4, 15), (5, 16), (6, 17), (7, 18). The difference between two integers, forming the same pair, is 11. Thus, only one integer from each pair can be represented in the list. So these 6 pairs can produce only six integers in the list. With  $x_1 = 1$  it makes 7. Therefore, 8 integers cannot be chosen.

END