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SENIOR FIVE TERM 1

TOPIC 5/6: Trigonometry

Competency: The learner applies trigonometric functions, identities, and equations to solve problems in mathematics and real-life application in navigation, construction, aviation and other fields..

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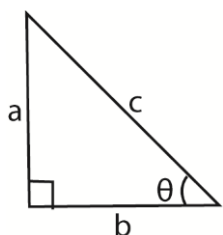
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Trigonometry

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.

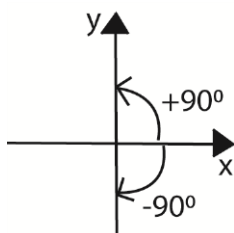
Important to note

(a) For a right angled triangle below

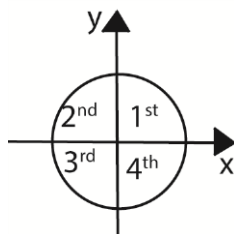


$$\begin{aligned} \bullet \sin \theta &= \frac{a}{c} & \text{cosec} \theta &= \frac{1}{\sin \theta} = \frac{c}{a} \\ \bullet \cos \theta &= \frac{b}{c} & \sec \theta &= \frac{1}{\cos \theta} = \frac{c}{b} \\ \bullet \tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{a}{b} & \cot \theta &= \frac{\cos \theta}{\sin \theta} = \frac{b}{a} \end{aligned}$$

(b) All positive angles are measured anticlockwise from positive x-axis

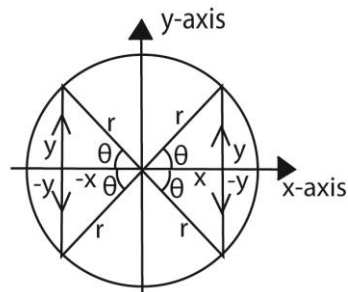


(c) A circle drawn with the centre O, divides the co-ordinate axis into four equal parts called quadrants



The quadrants are also labeled anti-clockwise from the positive x – axis.

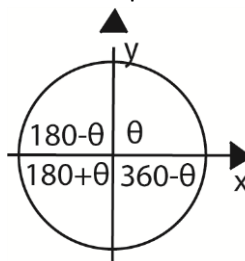
The signs the trigonometric ratios in the quadrants are given below



Ratio	Quadrant			
	1 st	2 nd	3 rd	4 th
cosθ	$\frac{+x}{r}$	$\frac{-x}{r}$	$\frac{-x}{r}$	$\frac{+x}{r}$
sinθ	$\frac{+y}{r}$	$\frac{+y}{r}$	$\frac{-y}{r}$	$\frac{-y}{r}$
tanθ	$\frac{+y}{x}$	$\frac{y}{-x}$	$\frac{y}{x}$	$\frac{-y}{x}$
secθ	$\frac{+r}{x}$	$\frac{-r}{-x}$	$\frac{-r}{x}$	$\frac{+r}{x}$
cosecθ	$\frac{+r}{y}$	$\frac{+r}{y}$	$\frac{-r}{-y}$	$\frac{-r}{y}$
cotθ	$\frac{+x}{y}$	$\frac{-x}{y}$	$\frac{+x}{-y}$	$\frac{-x}{y}$

Note

- If θ is the angle in the 1st quadrat
- In the 2nd quadrat the angle is $(180 - \theta)$
- In the 3rd quadrat the angle is $(180 + \theta)$
- In the 4th quadrat the angle is $(360 - \theta)$



Solving equations

We make use of the quadrants to find the ranges of values within which the angle follows

Example 1

Solve the following equations for $0^\circ \leq \theta \leq 360^\circ$

(i) $3\cos\theta + 2 = 0$

Solution

$$\cos\theta = -\frac{2}{3}$$

But $\cos$ is negative in the 2nd and 3rd quadrants.

Ignoring the negative sign, the angle obtained is referred to as the key or principle angle, i.e. key angle = $\cos^{-1}\frac{2}{3} = 48.2^\circ$ (1d.p)

In the 2nd quadrant, $\theta = 180 - 48.2 = 131.8^\circ$

In the 3rd quadrant, $\theta = 180 + 48.2 = 228.2^\circ$

$$\therefore \{\theta: \theta = 131.8^\circ, 228.2^\circ\}$$

Note that: the key angle is not part of the solution but only a guide.

(ii) $4\cos^2\theta - 1 = 0$

Solution

$$\cos\theta = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$$\text{Key angle, } \theta = \cos^{-1}\frac{1}{2} = 60^\circ$$

When $\cos\theta = \frac{1}{2}$ (positive is 1st and 4th quadrants)

1st quadrant $\theta = 60^\circ$

4th quadrant $\theta = 360 - 60 = 300^\circ$

When $\cos\theta = -\frac{1}{2}$ (positive is 2nd and 3rd quadrants)

3rd quadrant $\theta = 180 - 60 = 120^\circ$

4th quadrant $\theta = 180 + 60 = 240^\circ$

$$\therefore \{\theta: \theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ\}$$

(iii) $\operatorname{cosec}\theta + 2 = 0$

Solution

$$\operatorname{cosec}\theta = -2 \Rightarrow \sin\theta = -\frac{1}{2} \text{ (taking reciprocal)}$$

$$\text{Key angle} = \sin^{-1}\frac{1}{2} = 30^\circ$$

In the 3rd quadrant $\theta = 180 + 30 = 210^\circ$

In the 4th quadrant $\theta = 360 - 30 = 330^\circ$

$$\therefore \{\theta: \theta = 210^\circ, 330^\circ\}$$

(iv) $3\sec^2\theta - 4 = 0$

Solution

$$\sec\theta = \pm\frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\text{Key angle} = \cos^{-1}\frac{\sqrt{3}}{2} = 30^\circ$$

$$\text{For } \cos\theta = \frac{\sqrt{3}}{2}; \theta = 30^\circ, 330^\circ$$

$$\text{For } \cos\theta = -\frac{\sqrt{3}}{2}; \theta = 120^\circ, 210^\circ$$

$$\therefore \{\theta: \theta = 30^\circ, 120^\circ, 210^\circ, 330^\circ\}$$

(d) Definitions of angle

(i) **Acute angle** is an angle between 0° and 90° . It lies in the 1st quadrant

(ii) **Right angle** is an angle = 90°

(iii) **Obtuse angle** is an angle between 90° and 180° . It lies in the 2nd quadrant

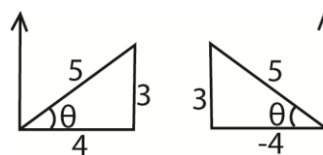
(iv) **Reflex angle** is an angle between 180° and 360° . It lies in the 3rd and 4th quadrant

Example 2

(a) If $\sin\theta = \frac{3}{5}$ and $0^\circ \leq \theta \leq 360^\circ$. Find the possible values of $3\tan\theta - \cot\theta$

Solution

If $\sin\theta = \frac{3}{5}$; θ lies in 1st or 2nd quadrants



In 1st quadrant

$$3\tan\theta - \cot\theta = 3\left(\frac{3}{4}\right) - \left(\frac{4}{3}\right) = \frac{11}{12}$$

In 2nd quadrant

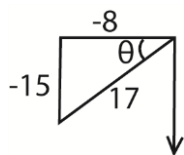
$$3\tan\theta - \cot\theta = 3\left(-\frac{3}{4}\right) - \left(-\frac{4}{3}\right) = -\frac{11}{12}$$

$\therefore$ the possible values are $\pm \frac{11}{12}$

(b) If $\cos\theta = -\frac{8}{17}$ and θ is reflex, find the value of $4\sec^2\theta + \tan\theta$

Solution

If $\cos\theta = -\frac{8}{17}$ and θ is reflex, θ lies in the 3rd quadrant



$$4\sec^2\theta + \tan\theta = 4\left(-\frac{17}{8}\right)^2 + \frac{15}{8} = \frac{319}{16}$$

Example 3

Solve for θ , where $\theta^0 \leq \theta \leq 360^0$

(i) $3\tan^2 3\theta = 1$

Solution

$$\tan 3\theta = \pm \frac{1}{\sqrt{3}}$$

$$\text{taking } \tan 3\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow 3\theta = 30^0, 210^0, 390^0, 570^0, 750^0, 930^0$$

$$\theta = 10^0, 70^0, 130^0, 190^0, 250^0, 310^0$$

$$\text{taking } \tan 3\theta = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow 3\theta = 150^0, 330^0, 510^0, 690^0, 870^0, 1050^0$$

$$\theta = 50^0, 110^0, 170^0, 230^0, 290^0, 350^0$$

$$\therefore \{\theta: \theta = 10^0, 50^0, 70^0, 110^0, 130^0, 170^0, 190^0, 230^0, 250^0, 290^0, 310^0, 350^0\}$$

Note

- If $\theta^0 \leq \theta \leq 360^0$ then $\theta^0 \leq 3\theta \leq 1080^0$
[multiply the interval through by 3]

(ii) $2\cos 2\theta + \sqrt{3} = 0$

Solution

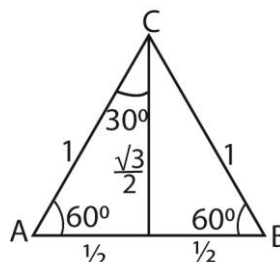
$$\cos 2\theta = -\frac{\sqrt{3}}{2} \text{ and } \theta^0 \leq 2\theta \leq 720^0$$

$$2\theta = 150^0, 210^0, 510^0, 570^0$$

$$\therefore \{\theta: \theta = 75^0, 105^0, 255^0, 285^0\}$$

Set square angles: 30^0 , 45^0 , and 60^0

(i) From equilateral triangle ABC with side equal to 1 unit



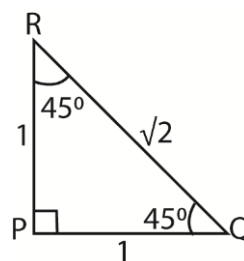
$$\cos 60^0 = \sin 30^0 = \frac{1}{2}$$

$$\cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}$$

$$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}$$

$$\tan 60^0 = \cot 30^0 = \sqrt{3}$$

(ii) From the right angled triangle PQR below



$$\cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}$$

$$\tan 45^0 = 1$$

Example 4

Without using tables or calculators find the value of

(i) $\cos 240^\circ$

Solution

$$\cos 240^\circ = -\cos(240 - 180)^\circ = -\cos 60^\circ = -\frac{1}{2}$$

(ii) $\tan 3990^\circ$

Solution

$$\tan 3990^\circ = \tan [(360 \times 11) + 30]^\circ = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

(iii) $\sin 570^\circ$

Solution

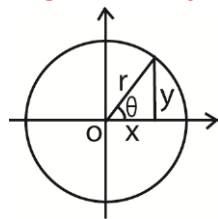
$$\sin 570^\circ = \sin \{(360 \times 1) + 210\}^\circ = -\sin 30^\circ = -\frac{1}{2}$$

(iv) $\sec 225^\circ$

Solution

$$\sec 225^\circ = \sec (225 - 180)^\circ = \sec 45^\circ = -\sqrt{2}$$

The Pythagoras theorem in trigonometry



For any acute angle θ

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

By Pythagoras theorem

$$x^2 + y^2 = r^2$$

Substituting for x and y

$$(r \cos \theta)^2 + (r \sin \theta)^2 = r^2$$

$$r^2(\cos^2 \theta + \sin^2 \theta) = r^2$$

$$\therefore \cos^2 \theta + \sin^2 \theta = 1$$

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$$\text{Now } \tan \theta = \frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \frac{\sin \theta}{\cos \theta}$$

$$\therefore \frac{\sin \theta}{\cos \theta} = \tan \theta$$

Identities

$$\cos^2 \theta + \sin^2 \theta = 1 \dots\dots\dots(i)$$

$$\text{Identity (i)} \div \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta \dots\dots\dots(ii)$$

$$\text{Identity (i)} \div \sin^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta \dots\dots\dots(iii)$$

Example 5

Show that

$$(i) \sin^2 \theta + (1 + \cos \theta)^2 = 2(1 + \cos \theta)$$

Solution

$$\sin^2 \theta + (1 + \cos \theta)^2$$

$$= \sin^2 \theta + 1 + 2 \cos \theta + \cos^2 \theta$$

$$= \sin^2 \theta + \cos^2 \theta + 1 + 2 \cos \theta$$

$$= 1 + 1 + 2 \cos \theta \text{ (Recall that } \sin^2 \theta + \cos^2 \theta = 1) = 2 + 2 \cos \theta = 2(1 + \cos \theta)$$

$$\therefore \sin^2 \theta + (1 + \cos \theta)^2 = 2(1 + \cos \theta)$$

$$(ii) \frac{1 + \sin \theta}{1 + \cos \theta} \cdot \frac{1 + \sec \theta}{1 + \csc \theta} = \tan \theta$$

Solution

$$\frac{1 + \sin \theta}{1 + \cos \theta} \cdot \frac{1 + \sec \theta}{1 + \csc \theta} = \frac{1 + \sin \theta}{1 + \cos \theta} \cdot \frac{1 + \frac{1}{\cos \theta}}{1 + \frac{1}{\sin \theta}}$$

$$= \frac{1 + \sin \theta}{1 + \cos \theta} \cdot \frac{\frac{\cos \theta + 1}{\cos \theta}}{\frac{\sin \theta + 1}{\sin \theta}}$$

$$= \frac{1 + \sin \theta}{1 + \cos \theta} \cdot \frac{\cos \theta + 1}{\cos \theta} \div \frac{\sin \theta + 1}{\sin \theta}$$

$$= \frac{1 + \sin \theta}{1 + \cos \theta} \cdot \frac{\cos \theta + 1}{\cos \theta} \times \frac{\sin \theta}{\sin \theta + 1}$$

$$= \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$$

$$(iii) (\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$$

Solution

$$\begin{aligned} (\tan\theta + \sec\theta)^2 &= \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} \right)^2 = \left(\frac{\sin\theta+1}{\cos\theta} \right)^2 \\ &= \frac{(1+\sin\theta)^2}{\cos^2\theta} = \frac{(1+\sin\theta)^2}{1-\sin^2\theta} \\ &= \frac{(1+\sin\theta)(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = \frac{1+\sin\theta}{1-\sin\theta} \\ \therefore (\tan\theta + \sec\theta)^2 &= \frac{1+\sin\theta}{1-\sin\theta} \end{aligned}$$

Example 6

Solve the following equations for $180^\circ \leq x \leq 180^\circ$

$$(i) 2\cos^2\theta + \sin\theta - 1 = 0$$

Solution

$$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$$

$$2\sin 2\theta - \sin\theta - 1 = 0$$

$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$

$$\text{Either } \sin\theta = 1 \text{ or } \sin\theta = -\frac{1}{2}$$

$$\text{When } \sin\theta = 1; \theta = 90^\circ$$

$$\text{When } \sin\theta = -\frac{1}{2}; \theta = -150^\circ, -30^\circ, 210^\circ, 330^\circ$$

$$[\theta: \theta = -150^\circ, -30^\circ, 90^\circ \text{ for given range}]$$

$$(ii) \cos\theta + \sqrt{3}\sin\theta = 1$$

Solution

1st approach

$$\sqrt{3}\sin\theta = 1 - \cos\theta$$

Squaring both sides

$$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$$

$$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$$

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$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\cos\theta = -\frac{1}{2} \quad \left| \quad \cos\theta = 1 \right.$$

$$\theta = \pm 120^\circ \quad \left| \quad \theta = 0^\circ \right.$$

$$\therefore [\theta: \theta = 0^\circ, \pm 120^\circ]$$

2nd approach

$$\sqrt{3}\sin\theta = 1 - \cos\theta$$

Dividing through by $\cos\theta$

$$\sqrt{3}\tan\theta = \sec\theta - 1$$

Squaring both sides

$$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$$

$$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$$

$$3[\sec^2\theta - 1] = \sec^2\theta - 2\sec\theta + 1$$

$$2\sec^2\theta + 2\sec\theta - 4 = 0$$

$$\sec^2\theta + \sec\theta - 2 = 0$$

$$(\sec\theta + 2)(\sec\theta - 1) = 0$$

$$\sec\theta = -2 \text{ or } \sec\theta = 1$$

$$\cos\theta = -\frac{1}{2} \text{ or } \cos\theta = 1$$

$$\therefore [\theta: \theta = 0^\circ, \pm 120^\circ]$$

3rd approach

$$\sqrt{3}\sin\theta = 1 - \cos\theta$$

Dividing through by $\sin\theta$

$$\sqrt{3} = \operatorname{cosec}\theta - \cot\theta$$

Rearranging

$$\sqrt{3} + \cot\theta = \operatorname{cosec}\theta$$

Squaring both sides

$$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = 1 + \cot^2\theta$$

$$\cot \theta = -\frac{1}{\sqrt{3}}; \Rightarrow \tan \theta = -\sqrt{3}$$

$$\therefore [\theta: \theta = -60^\circ, 120^\circ]$$

Example 7

(a) Given that $7 \tan \theta + \cot \theta = 5 \sec \theta$, derive a quadratic equation for $\sin \theta$. Hence or otherwise, find all values of θ in the interval $0^\circ \leq \theta \leq 180^\circ$ which satisfy the equation, giving your answer to the nearest 0.10 where necessary

Solution

$$7 \tan \theta + \cot \theta = 5 \sec \theta$$

$$7 \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{5}{\cos \theta}$$

$$7 \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} = \frac{5}{\cos \theta}$$

$$7 \sin^2 \theta + \cos^2 \theta = 5 \sin \theta$$

$$7 \sin^2 \theta + (1 - \sin^2 \theta) = 5 \sin \theta$$

$$6 \sin^2 \theta - 5 \sin \theta + 1 = 0$$

$$(3 \sin \theta - 1)(2 \sin \theta - 1) = 0$$

$$\sin \theta = \frac{1}{3} \quad \left| \quad \sin \theta = \frac{1}{2} \right.$$

$$\theta = 19.5^\circ, 160.5^\circ \quad \left| \quad \theta = 30^\circ, 150^\circ \right.$$

$$\therefore [\theta: \theta = 19.5^\circ, 30^\circ, 150^\circ, 160.5^\circ]$$

Example 8

Find the solution of $3 \cot \theta + \operatorname{cosec} \theta = 2$ for $0^\circ \leq \theta \leq 180^\circ$.

Solution

$$3 \cot \theta + \operatorname{cosec} \theta = 2$$

$$3 \frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta} = 2$$

$$(3 \cos \theta + 1)^2 = (2 \sin \theta)^2$$

$$9 \cos^2 \theta + 6 \cos \theta + 1 = 4 \sin^2 \theta$$

$$9 \cos^2 \theta + 6 \cos \theta + 1 = 4(1 - \cos^2 \theta)$$

$$13 \cos^2 \theta + 6 \cos \theta - 3 = 0$$

$$\cos \theta = \frac{-6 \pm \sqrt{6^2 + 4 \times 13}}{2 \times 13}$$

$$\cos \theta = 0.3021$$

$$\theta = 72.40$$

$$\cos \theta = 0.7637$$

$$\theta = 40.2$$

$$\therefore [\theta: \theta = 72.4^\circ, 40.2^\circ]$$

Elimination of trigonometric parameter in trigonometry

This involves the use of identities to eliminate the trigonometric values in equation

Example 9

(a) If $x = \tan \theta + \sec \theta$ and $y = \tan \theta - \sec \theta$; show that $xy + 1 = 0$

Solution

$$x + y = \tan \theta$$

$$x - y = 2 \sec \theta$$

$$\sec \theta = \frac{1}{2}(x - y)$$

Using identity: $1 + \tan^2 \theta = \sec^2 \theta$

$$1 + (x + y)^2 = \left[\frac{1}{2}(x - y) \right]^2$$

$$4 + x^2 + 2xy + y^2 = x^2 - 2xy + y^2$$

$$4xy + 4 = 0$$

$$xy + 1 = 0 \text{ as required}$$

(b) $x = 2 + 3 \sin \theta$ and $y = 3 + 2 \cos \theta$ show that $4(x - 2)^2 + (y - 3)^2 = 36$

Solution

$$x = 2 + 3 \sin \theta \Rightarrow \sin \theta = \frac{x-2}{3}$$

$$y = 3 + 2 \cos \theta \Rightarrow \cos \theta = \frac{y-3}{2}$$

Using identity $\sin^2 \theta + \cos^2 \theta = 1$

$$\left(\frac{x-2}{3} \right)^2 + \left(\frac{y-3}{2} \right)^2 = 1$$

$$4(x - 2)^2 + (y - 3)^2 = 36 \text{ as required}$$

(c) $x = 2 \sin \theta$ and $y = \tan \theta$, prove that

$$x = \pm \frac{2y}{\sqrt{1+y^2}}$$

Solution

$$x = 2 \sin \theta; \Rightarrow \operatorname{cosec} \theta = \frac{2}{x}$$

$$y = \tan\theta; \Rightarrow \cot\theta = \frac{1}{y}$$

Using identity: $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$$1 + \left(\frac{1}{y}\right)^2 = \left(\frac{2}{x}\right)^2$$

$$x = \pm \frac{2y}{\sqrt{1+y^2}}$$

Revision exercise 1

1. Solve for θ , where $0^\circ \leq \theta \leq 360^\circ$

(a) $\sec\theta \operatorname{cosec}\theta + 2\sec\theta - 2\operatorname{cosec}\theta - 4 = 0$

[θ : $\theta = 60^\circ, 210^\circ, 300^\circ, 330^\circ$]

(b) $\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0$

[θ : $\theta = 45^\circ, 60^\circ, 225^\circ, 240^\circ$]

2. Show that

(a) $\frac{1 - \cos\theta + \sin\theta}{1 - \cos\theta} = \frac{1 + \cos\theta + \sin\theta}{\sin\theta}$

(b) $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$

(c) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$

(d) $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

(e) $\sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} = \sec\theta + \tan\theta$

3. Solve the following equations for $-180^\circ \leq x \leq 180^\circ$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

[θ : $\theta = -150^\circ, -30^\circ, 90^\circ$]

(ii) $\sin 2\theta + 5\cos 2\theta = 3$

[θ : $\theta = \pm 45^\circ, \pm 135^\circ$]

(iii) $4\cot^2\theta + 24\operatorname{cosec}\theta + 39 = 0$

[θ : $\theta = 16.6^\circ, 23.6^\circ, 156.4^\circ, 163.4^\circ$]

4. Solve each of the following equations in the stated range

(a) $4\cos^2\theta + 2\sin\theta = 4$ $0^\circ \leq \theta \leq 360^\circ$

[θ : $\theta = 0^\circ, 48.6^\circ, 131.4^\circ, 180^\circ, 360^\circ$]

(b) $2\sec^2\theta - 4\tan\theta - 2 =$ $-180^\circ \leq \theta \leq 360^\circ$

[θ : $\theta = -135^\circ, -161.6^\circ, 18.4^\circ, 45^\circ$]

(c) $5\cos^2 3\theta = 3(1 + \sin 3\theta)$, $0^\circ \leq \theta \leq 360^\circ$

[θ : $\theta = 7.9^\circ, 52.1^\circ, 90^\circ, 127.9^\circ, 172.1^\circ$]

5. Solve for θ ; $00 \leq \theta \leq 3600$

(a) $\tan\theta + 3\cot\theta = 4$

[θ : $\theta = 45^\circ, 71.6^\circ, 225^\circ, 251.6^\circ$]

(b) $4\cos\theta - 3\sin\theta = 2$

[θ : $\theta = 29.50, 256.70$]

6. Solve

(a) $\cos\theta + \sqrt{3}\sin\theta = 2$ $0 \leq \theta \leq \pi$

$$\left[\theta = \frac{\pi}{3}\right]$$

(b) $2\cos\theta - \operatorname{cosec}\theta = 0$ $0^\circ \leq \theta \leq 270^\circ$

[θ : $\theta = 45^\circ, 225^\circ$]

(c) $2\sin^2\theta + 3\cos\theta = 0$ $0^\circ \leq \theta \leq 360^\circ$

[θ : $\theta = 240^\circ, 120^\circ$]

(d) $3\sin\theta + 4\cos\theta = 2$ $-180^\circ \leq \theta \leq 180^\circ$

[θ : $\theta = -29.55^\circ, 103.29^\circ$]

(e) $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for

$0^\circ < \theta < 180^\circ$ [θ : $\theta = 38.66^\circ, 116.57^\circ$]

7. Without using a tables or calculator , show that $\tan 15^\circ = 2 - \sqrt{3}$

8. Solve equation

$8\cos^4\theta - 10\cos^2\theta + 3$ for $0^\circ \leq \theta \leq 180^\circ$

[θ : $\theta = 30^\circ, 45^\circ, 135^\circ, 150^\circ$]

9. Eliminate θ from the following equation

(a) $x = a\sec\theta$ and $y = b + c\cos\theta$

[$ac = x(y - b)$]

(b) $x = \sec\theta + \tan\theta$ and $y = \sec\theta - \tan\theta$

[$xy = 1$]

10. Solve the simultaneous equation

$\cos x + 4\sin y = 1$

$4\sec x - 3\operatorname{cosec} y = 5$ for values of x and y between 0° and 360°

[$x = 78.8^\circ, 281.5^\circ$; $y = 11.5^\circ, 168.5^\circ$]

11. Prove each of the following identities

(a) $\sin x \tan x + \cos x = \sec x$

(b) $\operatorname{cosec} x + \tan x \sec x = \operatorname{cosec} x \sec^2 x$

(c) $\operatorname{cosec} x - \sin x = \cot x \cos x$

(d) $(\sin x + \cos x)^2 - 1 = 2\sin x \cos x$

12. Eliminate θ from each of the following pairs of relationships

(a) $x = 3\sin\theta$, $y = \operatorname{cosec}\theta$ [$xy = 3$]

(b) $5x = \sin\theta$, $y = 2\cos\theta$ [$100x^2 + y^2 - 4 = 0$]

(c) $x = 3 + \sin\theta$, $y = \cos\theta$ [$(x-3)^2 + y^2 = 1$]

(d) $x = 2 + \sin\theta$, $\cos\theta = 1 + y$

[$(x-2)^2 + (y+1)^2 = 1$]

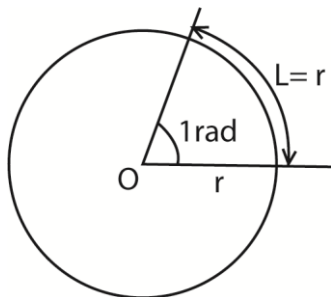
Measuring angles in radians

A radian is defined as an angle subtended at the centre of a circle by an arc that is equal to the

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radius of the circle. One radian is represented by π , where $\pi = \frac{22}{7}$



How to convert between degrees and radians

1 revolution = circumference of a circle

But circumference of a circle subtends an angle 2π at the centre.

$$\Rightarrow 1 \text{ revolution} = 2\pi = 360^\circ$$

$$\pi = 180^\circ$$

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

$$x^\circ = \frac{\pi}{180} \times \text{radians}$$

Example 10

Convert the following angles to radians

(a) 330°

(b) 90°

(c) 30°

Solution

(a) $330^\circ = \frac{\pi}{180} \times 330 = \frac{11\pi}{6} \text{ radians}$

(b) $90^\circ = \frac{\pi}{180} \times 90 = \frac{\pi}{2} \text{ radians}$

(c) $30^\circ = \frac{\pi}{180} \times 30 = \frac{\pi}{6} \text{ radians}$

Converting radians to degrees

$$2\pi \text{ radians} = 360^\circ$$

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$x \text{ radians} = \frac{180^\circ}{\pi} \times$$

Example 11

Convert each of the following radians to degrees

(i) $\frac{\pi}{3}$ radians

(ii) $\frac{2\pi}{5}$ radians

(iii) π radians

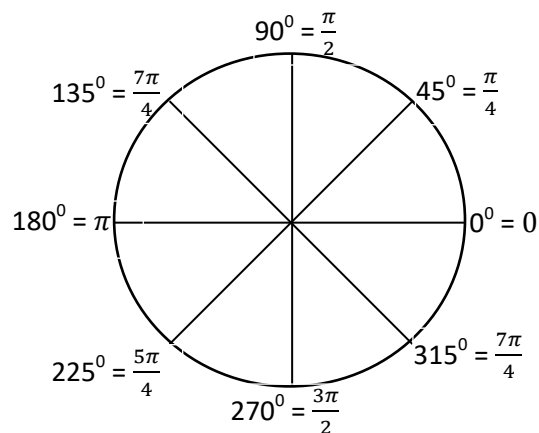
Solution

(i) $\frac{\pi}{3} \text{ radians} = \frac{180^\circ}{\pi} \times \frac{\pi}{3} = 60^\circ$

(ii) $\frac{2\pi}{5} \text{ radians} = \frac{180^\circ}{\pi} \times \frac{2\pi}{5} = 72^\circ$

(iii) $\pi \text{ radians} = \frac{180^\circ}{\pi} \times \pi = 180^\circ$

Some equivalent angles in degrees and radians



Example 12

Find each of the following values

(a) $\sin\left(\frac{2\pi}{3}\right)$

(b) $\cos\left(\frac{4\pi}{3}\right)$

(c) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Convert the angles from radian to degrees

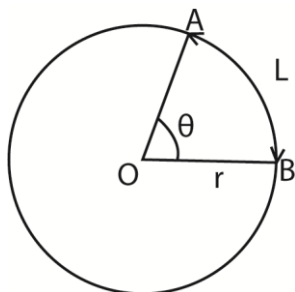
(a) $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2 \times 180}{3}\right) = \sin 120^\circ = \frac{\sqrt{3}}{2}$

(b) $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{4 \times 180}{3}\right) = \cos 240^\circ = -\frac{1}{2}$

$$(d) \tan\left(\frac{7\pi}{4}\right) = \tan\left(\frac{7 \times 180}{4}\right) = \tan 60^\circ = \sqrt{3}$$

Length of an arc

Suppose that the angle subtended by the length L of an arc AB of a circle is θ as shown.



$$\frac{L}{\theta} = \frac{2\pi r}{2\pi}$$

$L = r\theta$ where θ must be in radians

Example 13

Find the length of an arc of a circle of radius 14 if it subtends an angle

- (i) $\frac{\pi}{4}$
- (ii) 150°

Solution

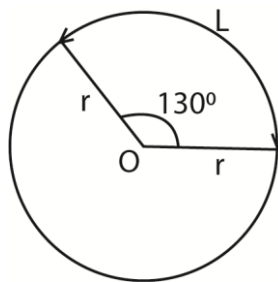
- (i) $L = r\theta = 14 \times \frac{\pi}{4} = 11\text{cm}$
- (ii) Convert degrees to radians
 $150^\circ = \frac{\pi}{180} \times 150 = \frac{5\pi}{6}$ radians
 $L = 14 \times \frac{5\pi}{6} = 36.67\text{cm}$

Example 14

A sector was drawn which had a perimeter of 80cm, and centre angle of 130° . Calculate the radius

Solution

The sides of a sector are composed of an arc, and two more sides which are radii of a circle.



$$2r + L = 80$$

$$L = 80 - 2r$$

Converting 130° to radians

$$130^\circ = \frac{\pi}{180} \times 130 = \frac{13\pi}{18}$$

$$\text{But } L = r\theta$$

$$80 - 2r = \frac{13\pi r}{18}$$

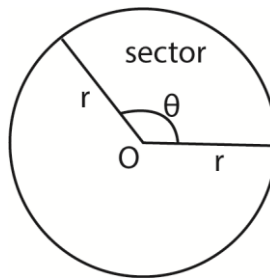
$$2r + \frac{13\pi r}{18} = 80$$

$$\frac{(36+13\pi)r}{18} = 80$$

$$r = 18.74\text{cm}$$

Area of a sector of a circle

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.



The area of a sector of a circle of radius r and central angle θ is given by

$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \left(\frac{\theta}{2}\right)r^2$$

Where θ must be in radians

Example 15

Find the area of a sector with radius 14cm and angle (i) $\frac{\pi}{4}$ (ii) 120°

Solution

$$(i) A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{8}\right) \cdot 14^2 = 77\text{cm}^2$$

(ii) Converting 120° to radians

$$120^\circ = \frac{\pi}{180} \times 120 = \frac{2\pi}{3}$$

$$A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{3}\right) \cdot 14^2 = 205.25\text{cm}^2$$

Solving trigonometric functions whose range is in radians

When the range of the trigonometric function is in radians, the answer should be given in radians

Example 16

Solve the following equations for the ranges indicated

$$(i) \cos\theta + \sqrt{3}\sin\theta = 1 \quad 0 \leq \theta \leq \pi$$

Solution

$$\sqrt{3}\sin\theta = 1 - \cos\theta$$

Squaring both sides

$$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$$

$$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$$

$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\begin{array}{l|l} \cos\theta = -\frac{1}{2} & \cos\theta = 1 \\ \theta = \pm 120^\circ & \theta = 0^\circ \end{array}$$

$$\pm 120^\circ = \pm \frac{\pi}{180} \times 120 = \pm \frac{2\pi}{3} \text{ Radians}$$

$$0^\circ = 0 \text{ radians}$$

$$\therefore \left[\theta: \theta = 0, \pm \frac{2\pi}{3} \right]$$

$$(ii) 2\cos^2\theta + \sin\theta - 1 = 0 \quad 0 \leq \theta \leq \pi$$

Solution

$$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$$

$$2\sin 2\theta - \sin\theta - 1 = 0$$

$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$

$$\text{Either } \sin\theta = 1 \text{ or } \sin\theta = -\frac{1}{2}$$

$$\text{When } \sin\theta = 1; \theta = 90^\circ$$

$$\text{When } \sin\theta = -\frac{1}{2}; \theta = -150^\circ, -30^\circ, 210^\circ, 330^\circ$$

$$[\theta: \theta = \frac{\pi}{180} \times 90 = \frac{\pi}{2} \text{ for given range}]$$

Revision exercise 2

1. Express each of the following in radians

$$(a) 30^\circ \left[\frac{\pi}{6} \right]$$

$$(b) 45^\circ \left[\frac{\pi}{4} \right]$$

$$(c) 120^\circ \left[\frac{2\pi}{3} \right]$$

$$(d) 300^\circ \left[\frac{5\pi}{3} \right]$$

2. Express the following angle in degrees

$$(a) \frac{\pi}{3} \text{ rad } [60^\circ]$$

$$(b) \frac{\pi}{8} \text{ rad } [22.5^\circ]$$

$$(c) 3\pi \text{ rad } [540^\circ]$$

$$(d) 5.2\pi \text{ rad } [936^\circ]$$

3. A sector of the circle of radius 7 cm subtends an angle $\frac{\pi}{3}$ radians at the centre.

Calculate the

$$(a) \text{ Length of the arc } \left[6\frac{2}{3} \text{ cm} \right]$$

$$(b) \text{ Perimeter of the sector } \left[20\frac{2}{3} \text{ cm} \right]$$

$$(c) \text{ Area of the sector } \left[\frac{77}{3} \text{ cm}^2 \right]$$

4. AOB is a sector of a circle, centre O, and is such that OA = OB = 7cm and angle AOB is 300° . Calculate the

$$(a) \text{ Perimeter of sector AOB } \left[17\frac{2}{3} \text{ cm} \right]$$

$$(b) \text{ The area of AOB } \left[\frac{77}{6} \text{ cm}^2 \right]$$

5. Find the value each of the following

$$(a) \sin \pi [0]$$

(b) $\cos 3\pi$ [-1]

(c) $\tan \frac{\pi}{3}$ [$\sqrt{3}$]

6. Solve the following equations for the ranges indicated

(a) $2\sec^2\theta = 3 + \tan\theta$ for $0 \leq \theta \leq 2\pi$

[θ : $\theta = 0.25\pi, 0.85\pi, 1.25\pi, 1.85\pi$]

(b) $2\sin^2\theta \cos\theta + \cos\theta - 1$ for $0 \leq \theta \leq 2\pi$

[θ : $\theta = 0.38\pi, 1.62\pi, 2\pi$]

(c) $2\tan\theta + 4\cot\theta = \operatorname{cosec}\theta$ for $-\pi \leq \theta \leq \pi$

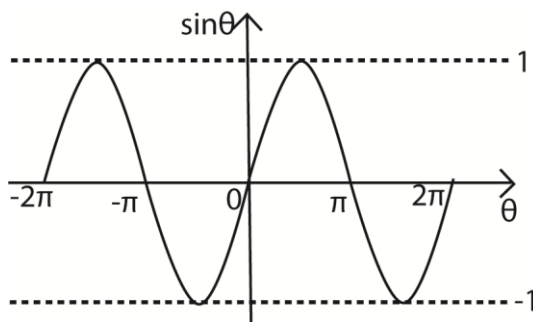
[θ : $\theta = \pm \frac{1}{3}\pi, \pm 0.73\pi$]

Graphs of trigonometric functions

The following are the characteristic of the three major trigonometric functions

The sine function

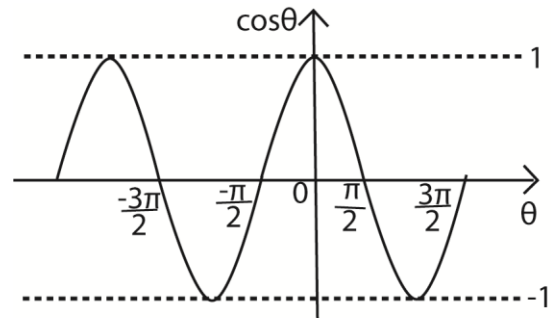
- It is continuous (with no breaks)
- The range $-1 \leq \sin\theta \leq 1$
- The shape of the graph from $\theta = 0$ to $\theta = 2\pi$ is repeated every 2π radians
- This is called a periodic or cyclic function and the width of the repeating pattern that is measured on horizontal axis is called a **period**. The sine wave has a period of 2π , a maximum value of +1 and a minimum value of -1.
- The greatest value of sine wave is called the **amplitude**.



The cosine function

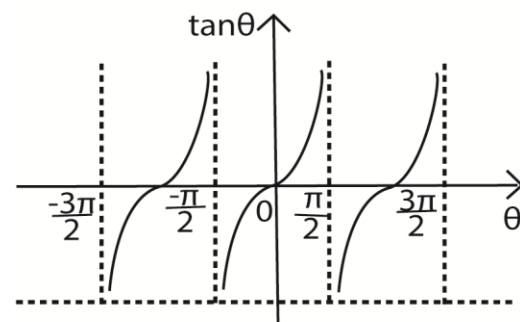
- It is continuous (with no breaks)
- The range $-1 \leq \sin\theta \leq 1$
- Has a period of 2π

- The shape is the same as the sine wave but displaced a distance $\frac{\pi}{2}$ to the left on the horizontal axis. This is called a **phase shift**



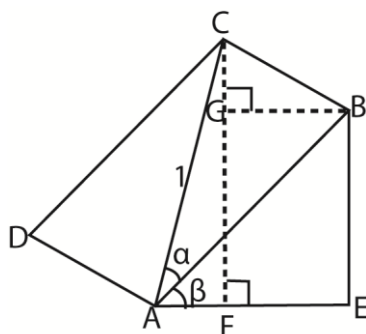
The tan function

- The tan function is found using; $\tan\theta = \frac{\sin\theta}{\cos\theta}$. It follows that $\tan\theta = 0$ when $\sin\theta = 0$; and $\tan\theta$ is undefined when $\cos\theta = 0$
- The graph is continuous, but undefined when $\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
- The range of values for $\tan\theta$ is unlimited
- It has a period π



Compound angles in trigonometry

Consider a cardboard ABCD of unit diagonal that stands on the edge A, making an angle β with the horizontal ground. Let the unit diagonal AC be inclined at an angle α to the side AB (see diagram)



Angles EAB = ABG (Alternative angles)

$$\therefore \text{Angle ABG} = \beta$$

$$\text{Angle [ABG + GBC]} = 90^\circ$$

$$\therefore \text{Angle GBC} = 90 - \beta$$

From triangle BGC,

$$\text{Angle BCG} = 180 - (90 + 90 - \beta)$$

$$\therefore \text{Angle BCG} = \beta$$

From

(1) Triangle ABC:

$$\cos \alpha = \frac{AB}{AC} = \frac{AB}{1}; \Rightarrow AB = \cos \alpha$$

(2) Triangle ABE:

$$\cos \beta = \frac{AE}{AB} = \frac{AE}{\cos \alpha}; \Rightarrow AE = \cos \beta \cos \alpha$$

$$\sin \beta = \frac{BE}{AB} = \frac{BE}{\cos \alpha}; \Rightarrow BE = \cos \alpha \sin \beta$$

(3) Triangle BCG:

$$\cos \beta = \frac{CG}{BC} = \frac{CG}{\sin \alpha}; \Rightarrow CG = \sin \alpha \cos \beta$$

$$\sin \beta = \frac{BG}{BC} = \frac{BG}{\sin \alpha}; \Rightarrow BG = \sin \alpha \sin \beta$$

(4) Triangle ACF:

$$\cos(\alpha + \beta) = \frac{AF}{AC} = \frac{AF - BG}{1} = AE - BG$$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \frac{CF}{AC} = \frac{CG + GF}{1} = CG + GF$$

$$\therefore \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

It follows that

$$(i) \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$(ii) \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

[substituting $-\beta$ for β]

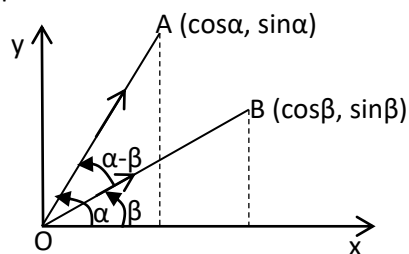
$$(iii) \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$(iv) \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

[substituting $-\beta$ for β]

These can also be derived using vector approach.

Consider two unit vectors $\underline{OA}$ and $\underline{OB}$ each inclined at angles α and β , respectively to the positive x-axis



$$\text{Using the definition of a vector product: } \underline{OA} \cdot \underline{OB} = |\underline{OA}| \cdot |\underline{OB}| \cos(\alpha - \beta)$$

Since $\underline{OA}$ and $\underline{OB}$ are unit vectors,

$$|\underline{OA}| = |\underline{OB}| = 1$$

$$\therefore \underline{OA} \cdot \underline{OB} = \cos(\alpha - \beta)$$

$$\Rightarrow (\cos \alpha \underline{i} + \sin \alpha \underline{j}) \cdot (\cos \beta \underline{i} + \sin \beta \underline{j}) = \cos(\alpha - \beta)$$

$$\therefore \cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\alpha - \beta)$$

Substituting $90 - \alpha$ for α

$$\begin{aligned} \cos(90 - \alpha) \cos \beta + \sin(90 - \alpha) \sin \beta \\ = \cos(90 - \alpha - \beta) \end{aligned}$$

$$\therefore \sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin(\alpha + \beta)$$

Other expansions can be similar substitutions

$$\begin{aligned} \text{i.e. } \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \end{aligned}$$

Dividing through by $\cos \alpha \cos \beta$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Similarly

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The following is a summary of compound angles

1. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
2. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
3. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
4. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
5. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
6. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example 17

Calculate the value of $\sin 15^\circ$ given that $\sin 45^\circ =$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}, \sin 30^\circ = \frac{1}{2} \text{ and } \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \sin 30^\circ \cos 45^\circ \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = 0.2588\end{aligned}$$

Example 18

Prove that $\tan(45^\circ + A) = \frac{1 + \tan A}{1 - \tan A}$

$$\text{From } \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\begin{aligned}\tan(45^\circ + A) &= \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} \\ &= \frac{1 + \tan A}{1 - \tan A}\end{aligned}$$

Example 19

Acute angles A and B are such that: $\cos A = \frac{1}{2}$, $\sin B = \frac{1}{3}$. Show without using tables or calculator that $\tan(A + B) = \frac{9\sqrt{3} + 8\sqrt{2}}{5}$

Solution

$$\text{Using } \cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{1}{2}\right)^2 + \sin^2 A = 1$$

$$\sin^2 A = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{3}}{2}$$

$$\tan A = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$$

Similarly;

$$\cos^2 B + \left(\frac{1}{3}\right)^2 = 1$$

$$\cos B = \frac{2\sqrt{2}}{3}$$

$$\tan B = \frac{2\sqrt{2}}{3} \div \frac{1}{3} = \frac{1}{2\sqrt{2}}$$

But

$$\begin{aligned}\text{From } \tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{\sqrt{3} + \frac{1}{2\sqrt{2}}}{1 - \sqrt{3} \cdot \frac{1}{2\sqrt{2}}} \\ &= \frac{(2\sqrt{2}\sqrt{3} + 1)(2\sqrt{2} + \sqrt{3})}{(2\sqrt{2} - \sqrt{3})(2\sqrt{2} + \sqrt{3})}\end{aligned}$$

$$\tan(A + B) = \frac{9\sqrt{3} + 8\sqrt{2}}{5}$$

Example 20

Solve $\cos(\theta + 35^\circ) = \sin(\theta + 25^\circ)$ for $0^\circ \leq \theta \leq 360^\circ$

$$\cos\theta\cos 35^\circ - \sin\theta\sin 35^\circ = \sin\theta\cos 25^\circ + \cos\theta\sin 25^\circ$$

Dividing through by $\cos\theta$

$$\cos 35^\circ - \tan\theta\sin 35^\circ = \tan\theta\cos 25^\circ + \sin 25^\circ$$

$$\tan\theta = \frac{\cos 35^\circ - \sin 25^\circ}{\cos 35^\circ + \sin 25^\circ} = \frac{0.3965337825}{1.479884223}$$

$$\theta = 15^\circ, 195^\circ \text{ for } 0^\circ \leq \theta \leq 360^\circ$$

Example 21

(a) Prove that $\frac{2\tan\theta}{1 + \tan^2\theta} = \sin 2\theta$

Solution

$$\begin{aligned}\frac{2\tan\theta}{1 + \tan^2\theta} &= \frac{2\sin\theta}{\cos\theta} \div \left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right) \\ &= \frac{2\sin\theta}{\cos\theta} \div \left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right) \\ &= \frac{2\sin\theta}{\cos\theta} \div \left(\frac{1}{\cos^2\theta}\right)\end{aligned}$$

$$= 2\sin\theta\cos\theta = \sin 2\theta$$

(b) Solve $\sin 2\theta = \cos\theta$ for $0^\circ \leq \theta \leq 90^\circ$

Solution

$$\sin 2\theta = \cos\theta$$

$$2\sin\theta\cos\theta = \cos\theta$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = 30^\circ \text{ for } 0^\circ \leq \theta \leq 90^\circ$$

Example 22

Given that α , β and γ are angles of a triangle, show that $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

Hence find $\tan\gamma$ if $\tan\alpha = 1$ and $\tan\gamma = 2$.

Solution

$$\alpha + \beta + \gamma = 180^\circ \text{ (angle sum of a triangle)}$$

$$\tan(\alpha + \beta + \gamma) = \tan 180^\circ = 0$$

$$\tan[(\alpha + \beta) + \gamma] = 0$$

$$\frac{\tan(\alpha + \beta) + \tan\gamma}{1 - \tan(\alpha + \beta)\tan\gamma} = 0$$

$$\Rightarrow \tan(\alpha + \beta) + \tan\gamma = 0$$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma$$

$$\tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha\tan\beta\tan\gamma$$

$$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

Example 23

In a triangle ABC, prove that $\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C$

$$= \cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$$

$$= \cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$$

Solution

$$\frac{1}{2}(A + B + C) = \frac{1}{2}(180^\circ) = 90^\circ$$

$$\cot\left[\frac{1}{2}(A + B + C)\right] = \cot 90^\circ = 0$$

$$\Rightarrow \frac{1 - \tan\left(\frac{1}{2}A + \frac{1}{2}B\right)\tan\frac{1}{2}C}{\tan\left(\frac{1}{2}A + \frac{1}{2}B\right) + \tan\frac{1}{2}C} = 0$$

$$1 = \tan\left(\frac{1}{2}A + \frac{1}{2}B\right)\tan\frac{1}{2}C$$

$$1 = \left(\frac{\tan\frac{1}{2}A + \tan\frac{1}{2}B}{1 - \tan\frac{1}{2}A\tan\frac{1}{2}B}\right)\tan\frac{1}{2}C$$

$$1 - \tan\frac{1}{2}A\tan\frac{1}{2}B$$

$$= \tan\frac{1}{2}A\tan\frac{1}{2}C + \tan\frac{1}{2}B\tan\frac{1}{2}C$$

$$1 = \tan\frac{1}{2}A\tan\frac{1}{2}B + \tan\frac{1}{2}A\tan\frac{1}{2}C + \tan\frac{1}{2}B\tan\frac{1}{2}C$$

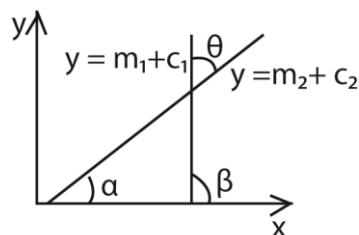
Dividing each side by $\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C$

$$\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C = \cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C$$

Example 24

Prove that the angle θ , between the straight line $y = m_1x + c_1$ and the straight line $y = m_2x + c_2$ is given by $\tan\theta = \frac{m_2 - m_1}{1 + m_2m_1}$

Let the lines be inclines at angles α and β with the x-axis respectively



From the diagram above

$$\theta = \beta - \alpha$$

$$\Rightarrow \tan\theta = \tan(\beta - \alpha)$$

$$= \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2m_1}$$

Revision exercise 3

1. (a) Show that

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$$

(b) If $\sin(\alpha + \beta) = 5\cos(\alpha - \beta)$ show that

$$\tan\alpha = \frac{5 - \tan\beta}{1 + \tan\beta}$$

(c) Without using tables or calculator,
show that $\cos 15^\circ = \sin 75^\circ$

(d) If $\alpha + \beta = 45^\circ$, show that $\tan \alpha = \frac{1 - \tan \beta}{1 + \tan \beta}$

2. Prove that:

$$(i) \frac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)} + 1 = \frac{(1 + \tan \beta)(1 + \cot \alpha)}{\cos \alpha + \tan \beta}$$

$$(ii) \tan \alpha - \tan \beta = \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta}$$

$$(iii) \cot \alpha + \cot \beta = \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}$$

$$(iv) \frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)} = \frac{\tan \alpha - \tan \beta}{\tan \alpha + \tan \beta}$$

$$(v) \frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{\cot \alpha \cot \beta + 1}{\cot \alpha \cot \beta - 1}$$

$$(vi) \cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

3. (a) Determine solution of $\tan 2x + 2 \sin x = 0$
for $0^\circ \leq x \leq 180^\circ$ [x: $x = 0^\circ, 60^\circ, 120^\circ, 180^\circ$]

(vii) Show that in triangle ABC,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

4. Find the values of $\tan \alpha$ for each of the following

$$(a) \sin(\alpha - 30^\circ) = \cos \alpha [\sqrt{3}]$$

$$(b) \sin(\alpha + 45^\circ) = \cos \alpha [\sqrt{2} - 1]$$

$$(c) \cos(\alpha + 60^\circ) = \sin \alpha [2 - \sqrt{3}]$$

$$(d) \sin(\alpha + 60^\circ) = \cos(\alpha - 60^\circ) [1]$$

$$(e) \cos(\alpha + 60^\circ) = 2 \cos(\alpha + 30^\circ) [4 + 3\sqrt{3}]$$

$$(f) \sin(\alpha + 60^\circ) = \cos(45^\circ - \alpha) \left[\frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1} \right]$$

5. Given that

(a) $\tan(\alpha - \beta) = \frac{1}{2}$ and $\tan \alpha = 3$ find the value of $\tan \beta$ [1]

(b) $\tan(\alpha + \beta) = 5$ and $\tan \beta = 2$ find the value of $\tan \alpha \left[\frac{3}{11} \right]$

6. Given that

(a) $\tan(\theta - 45^\circ) = 4$, find the value of θ
 $\left[-\frac{5}{3} \right]$

(b) $\tan(\theta + 60^\circ)$ find the value of $\cot \theta$
 $[8 + 5\sqrt{3}]$

Double angles and half angles in trigonometry

(b) From $\cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta$
 $\Rightarrow \cos 2\theta = \cos^2 \theta - \sin^2 \theta \dots \dots \dots (i)$

Either

$$\cos 2\theta = \cos^2 \theta - 1 + \cos^2 \theta \quad (\cos^2 \theta + \sin^2 \theta = 1)$$

$$\cos 2\theta = 2\cos^2 \theta - 1 \dots \dots \dots (ii)$$

Or

$$\cos 2\theta = 1 - \sin^2 \theta - \sin^2 \theta$$

$$\Rightarrow \cos 2\theta = 1 - 2\sin^2 \theta \dots \dots \dots (iii)$$

It follows that

$$\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \dots \dots \dots (iv)$$

$$\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \dots \dots \dots (iv)$$

The identities imply

$$\cos 6\theta = \cos^2 3\theta - \sin^2 3\theta$$

$$= 2\cos^2 3\theta - 1 = 1 - 2\sin^2 3\theta$$

$$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$$

$$= 2\cos^2 \frac{\theta}{2} - 1 = 1 - 2\sin^2 \frac{\theta}{2}$$

(c) $\sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta$

$$\Rightarrow \sin 2\theta = 2\sin \theta \cos \theta$$

It follows that

$$\sin 6\theta = 2\sin 3\theta \cos 3\theta$$

$$\sin \theta = 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$(d) \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta}$$

$$\Rightarrow \tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta}$$

It follows that

$$\tan \theta = \frac{2\tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$\tan 6\theta = \frac{2\tan 3\theta}{1 - \tan^2 3\theta}$$

Note that in all cases, the angles on the right hand side are half the angles on the left hand side [**half angle formulae**]

Example 25

Show that

(a) $\operatorname{cosec} 2\theta + \cot 2\theta = \cot \theta$

Solution

$$\begin{aligned}\operatorname{cosec} 2\theta + \cot 2\theta &= \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta} \\ &= \frac{1 + \cos 2\theta}{\sin 2\theta} \\ &= \frac{1 + 2\cos^2 \theta - 1}{2\sin \theta \cos \theta} \\ &= \frac{2\cos^2 \theta}{2\sin \theta \cos \theta} = \cot \theta\end{aligned}$$

(b) $\tan 3\theta = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}$

Hence deduce that if $3\theta + \alpha = 45^\circ$, then

$$\tan \alpha = \frac{1 - 3\tan \theta - 3\tan^2 \theta + \tan^3 \theta}{1 + 2\tan \theta - 3\tan^2 \theta - \tan^3 \theta}$$

Solution

$$\begin{aligned}\tan 3\theta &= \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} \\ &= \left\{ \left(\frac{2\tan \theta}{1 - \tan^2 \theta} \right) + \tan \theta \right\} \div \left\{ 1 - \left(\frac{2\tan \theta}{1 - \tan^2 \theta} \right) \tan \theta \right\} \\ &= \frac{2\tan \theta + \tan \theta - \tan^3 \theta}{1 - \tan^2 \theta - 2\tan^2 \theta} = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta} \\ \therefore \tan 3\theta &= \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}\end{aligned}$$

Hence $3\theta + \alpha = 45^\circ \Rightarrow \alpha = 45^\circ - 3\theta$

$\tan \alpha = \tan(45^\circ - 3\theta)$

$$\begin{aligned}&= \frac{\tan 45^\circ - \tan 3\theta}{1 + \tan 45^\circ \tan 3\theta} = \frac{1 - \tan 3\theta}{1 + \tan 3\theta} \\ &= \frac{1 - \left(\frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta} \right)}{1 + \left(\frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta} \right)} \\ &= \frac{1 - 3\tan \theta - 3\tan^2 \theta + \tan^3 \theta}{1 + 2\tan \theta - 3\tan^2 \theta - \tan^3 \theta} \\ \therefore \tan \alpha &= \frac{1 - 3\tan \theta - 3\tan^2 \theta + \tan^3 \theta}{1 + 2\tan \theta - 3\tan^2 \theta - \tan^3 \theta}\end{aligned}$$

Example 26

If $\tan \alpha = \frac{3}{4}$ and α is acute, without using tables or calculator work out the value of

(a) $\tan 2\alpha$

$$\tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha} = \frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{24}{7}$$

(b) $\tan \frac{\alpha}{2}$

similarly $\tan \alpha = \frac{2\tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}} = \frac{3}{4}$

$$\Rightarrow 3\tan^2 \frac{\alpha}{2} + 8\tan \frac{\alpha}{2} - 3 = 0$$

$$(3\tan \frac{\alpha}{2} - 1)(\tan \frac{\alpha}{2} + 3) = 0$$

$$\tan \frac{\alpha}{2} = \frac{1}{3} \text{ or } \tan \frac{\alpha}{2} = -3$$

Since α is acute, $\tan \alpha$ cannot be negative

$$\therefore \tan \frac{\alpha}{2} = \frac{1}{3}$$

Example 27

(a) Show that $\cos 3\alpha = 4\cos^2 \alpha - 3\cos \alpha$. Hence

solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$ for $0^\circ \leq \alpha \leq 180^\circ$

Solution

$$\begin{aligned}\cos 3\alpha &= \cos(2\alpha + \alpha) \\ &= \cos 2\alpha \cos \alpha - \sin 2\alpha \sin \alpha \\ &= (2\cos^2 \alpha - 1)\cos \alpha - 2\sin^2 \alpha \cos \alpha \\ &= (2\cos^2 \alpha - 1)\cos \alpha - 2(1 - \cos^2 \alpha)\cos \alpha \\ &= 2\cos^3 \alpha - \cos \alpha - 2\cos \alpha + 2\cos^3 \alpha \\ &= 4\cos^2 \alpha - 3\cos \alpha\end{aligned}$$

Hence $4x^3 - 3x = \frac{\sqrt{3}}{3}$

i.e. $4\cos^2 \alpha - 3\cos \alpha = \frac{\sqrt{3}}{3}$

$0^\circ \leq \alpha \leq 180^\circ$; $\cos 3\alpha = \frac{\sqrt{3}}{3}$

For the range $0^\circ \leq \alpha \leq 180^\circ$

$$\Rightarrow 0^\circ \leq 3\alpha \leq 540^\circ$$

$$3\alpha = 54.7^\circ, 414.7^\circ$$

$$\alpha = 18.23^\circ, 138.23^\circ \text{ (2d.p.)}$$

$$[\alpha: \alpha = 18.23^\circ, 138.23^\circ]$$

- (b) Given that $t = \tan 22\frac{1}{2}^\circ$, show that $t^2 + 2t - 1 = 0$,

$$\text{Hence show that } \tan 45^\circ = -1 + \sqrt{2}$$

Solution

$$\tan 45^\circ = \frac{2 \tan 22\frac{1}{2}^\circ}{1 - \tan^2 22\frac{1}{2}^\circ}$$

$$1 = \frac{2t}{1-t^2}$$

$$1 - t^2 = 2t$$

$$t^2 + 2t - 1 = 0 \text{ (as required)}$$

solving

$$t = \frac{-2 \pm \sqrt{2^2 - (4 \times 1 \times -1)}}{2 \times 1}$$

$$t = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

Since $22\frac{1}{2}^\circ$ is an acute angle,

$$\tan 22\frac{1}{2}^\circ = -1 + \sqrt{2} \text{ is positive}$$

$$\therefore \tan 22\frac{1}{2}^\circ = -1 + \sqrt{2}$$

Example 28

- (a) Show that $3\sin\theta = 3\sin\theta - 4\sin^3\theta$. Hence solve the equation $\sin 3\theta + \sin\theta = 0$ for $0^\circ \leq \theta \leq 360^\circ$

Solution

$$\begin{aligned} \sin 3\theta &= \sin(2\theta + \theta) \\ &= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \\ &= 2\sin\theta \cos^2\theta + (1 - 2\sin^2\theta)\sin\theta \\ &= 2\sin\theta(1 - \sin^2\theta) + (1 - 2\sin^2\theta)\sin\theta \\ &= 3\sin\theta - 4\sin^3\theta \end{aligned}$$

$$\text{Hence } \sin 3\theta + \sin\theta = 0$$

$$3\sin\theta - 4\sin^3\theta + \sin\theta = 0$$

$$4\sin\theta - 4\sin^3\theta = 0$$

$$4\sin\theta(1 - \sin^2\theta) = 0$$

$$4\sin\theta(1 - \sin\theta)(1 + \sin\theta) = 0$$

$$\sin\theta = 0; \theta = 0^\circ, 180^\circ, 360^\circ$$

$$\sin\theta = 1; \theta = 90^\circ$$

$$\sin\theta = -1; \theta = 270^\circ$$

$$\therefore \theta: \theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$$

- (b) Prove that $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$. Hence solve the equation $\cot 2\theta + 2\cot\theta = 2$ for $0^\circ \leq \theta \leq 360^\circ$

Solution

$$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta \cos\theta}$$

dividing through by $\sin^2\theta$

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$

$$\text{Hence, } \cot 2\theta + 2\cot\theta = 0$$

$$\frac{\cot^2\theta - 1}{2\cot\theta} + 2\cot\theta = 0$$

$$5\cot^2\theta - 4\cot\theta - 1 = 0$$

$$(5\cot\theta + 1)(\cot\theta - 1) = 0$$

$$\cot\theta = -\frac{1}{5} \text{ or } \cot\theta = 0$$

$$\Rightarrow \tan\theta = -5 \text{ or } \tan\theta = 1$$

$$\text{When } \tan\theta = -5; \theta = 101.3^\circ, 281.3^\circ$$

$$\text{When } \tan\theta = 1, \theta = 45^\circ, 225^\circ$$

$$\therefore \theta: \theta = 45^\circ, 101.3^\circ, 225^\circ, 281.3^\circ$$

Revision exercise 4

1. Prove that

$$(a) \sin\alpha \operatorname{cosec}\beta + \cos\alpha \sec\beta = 2\sin(\alpha + \beta) \operatorname{cosec} 2\beta$$

$$(b) \cos^6\theta + \sin^6\theta = 1 - \frac{3}{4}\sin^2 2\theta$$

$$(c) \frac{\sin 3\alpha}{1 + 2\cos 2\alpha} = \sin\alpha \text{ and hence deduce that}$$

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

2. (a) Solve the equation for θ , $0^\circ \leq \theta \leq 360^\circ$
 $\sin^2\theta - 2\sin\theta \cos\theta - 3\cos^2\theta = 0$

$$[\theta: \theta = 71.6^\circ, 135^\circ, 251.6^\circ, 315^\circ]$$

(b) show that $\frac{\cos\theta}{1+\sin\theta} = \cot(\frac{\theta}{2} + 45^\circ)$. Hence or otherwise solve the equation

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1}{2} \text{ for } 0^\circ \leq \theta \leq 360^\circ [\theta = 36.8^\circ]$$

3. (a) solve the equation
 $4\cos 2\theta - 2\cos \theta + 3 = 0$ for $0^\circ \leq \theta \leq 360^\circ$
 $[\theta: \theta = 60^\circ, 104.5^\circ, 255.5^\circ, 300^\circ]$
 (b) Solve the equation $\sin \theta + \sin \frac{\theta}{2} = 0$ for
 $0^\circ \leq \theta \leq 360^\circ$
 $[\theta: \theta = -360^\circ, -240^\circ, 0^\circ, 240^\circ, 360^\circ]$
4. (a) Prove that
 $\tan\left(\frac{\pi}{4} + \theta\right) - \tan\left(\frac{\pi}{4} - \theta\right) = 2\tan 2\theta$
 (b) By expressing $2\sin \theta \sin(\theta + \alpha)$ as difference of cosines of two angles or otherwise, where α is constant, find its least value $\left[\frac{-a}{2}\right]$
 (c) Solve for θ in the equation
 $\cos \theta - \cos(\theta + 60^\circ) = 0.4$ for
 $0^\circ \leq \theta \leq 360^\circ$ $[\theta = 126.4^\circ, 353.6^\circ]$
5. (a) Show that $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$.
 Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$
 $[x: x = -0.746, -0.204, 0.959]$
 (b) Find all solutions of the equation
 $5\cos x - 4\sin x = 6$ in the range
 $-180^\circ \leq x \leq 180^\circ$ $[x: x = -59.1^\circ, -18.3^\circ]$
6. (a) Express $\sqrt{\left(\frac{\sin 2\theta - \cos 2\theta - 1}{2 - 2\sin 2\theta}\right)}$ in terms of
 $\tan \theta \left[\frac{1}{\sqrt{\tan \theta - 1}}\right]$
 (b) Find the solution of the equation
 $\sqrt{3}\sin \theta - \cos \theta + 1 = 0$ for $0 \leq \theta \leq 2\pi$
 $\left[\theta: \theta = \frac{4}{3}\pi, 2\pi\right]$
 (c) Factorize $\cos \theta - \cos 3\theta - \cos 7\theta + \cos 9\theta$ in form $A\cos p\theta \sin q\theta \sin r\theta$ where A, p, q and r are constants $[A = -4, p = 5, q = 5, r = 2]$
7. (a) Given that $\sin \alpha + \sin \beta = p$ and $\cos \alpha + \cos \beta = q$ show that
 (i) $\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{p}{q}$
 (ii) $\cos(\alpha + \beta) = \frac{q^2 - p^2}{q^2 + p^2}$
 (b) Solve the simultaneous equation:
 $\cos \alpha + 4\sin \beta = 1$
 $4\sec \alpha - 3\operatorname{cosec} \beta = 5$ $[\theta = 78.5^\circ, 281.5^\circ]$
8. (a) Express $\sin \theta + \sin 3\theta$ in form $p\cos \theta \sin q\theta$ where p and q are constant $[p = 2, q = 2]$
 (b) Find the solution of
 $\cos 7\theta + \cos 5\theta = 2\cos \theta$ for $0^\circ \leq \theta \leq 360^\circ$
 $[0^\circ, 60^\circ, 270^\circ, 360^\circ]$
 (c) Prove that $\frac{\sin A + \sin 4A + \sin 7A}{\cos A + \cos 4A + \cos 7A} = \tan 4A$
9. Eliminate θ from each of the following pairs of expression
 (a) $x + 1 = \cos 2\theta, y = \sin \theta$ $[x + 2y^2 = 0]$
 (b) $x = \cos 2\theta, y = \cos \theta - 1$ $[x = 2y^2 + 4y + 1]$
 (c) $y - 3 = \cos 2\theta, x = 2 - \sin \theta$
 $[2x^2 - 8x + y + 4 = 0]$
10. Solve the following equations for $180^\circ \leq \theta \leq 180^\circ$
 (a) $\sin 2\theta + \sin \theta = 0$ $[\pm 120^\circ, \pm 180^\circ]$
 (b) $\sin 2\theta - 2\cos^2 \theta = 0$ $[-135^\circ, 45^\circ, \pm 90^\circ]$
 (c) $3\cos 2\theta + 2 + \cos \theta = 0$ $[\pm 70.5^\circ, \pm 120^\circ]$
 (d) $\sin 2\theta = \tan \theta$ $[0^\circ, \pm 45^\circ, \pm 135^\circ, \pm 180^\circ]$
11. Solve the following equations for $360^\circ \leq \theta \leq 360^\circ$, giving your answer correct to 1 decimal place
 (a) $\sin \theta = \sin\left(\frac{\theta}{2}\right)$ $[0^\circ, \pm 120^\circ, \pm 360^\circ]$
 (b) $3\cos\left(\frac{\theta}{2}\right) = 2\sin \theta$ $[\pm 180^\circ, 97.2^\circ, 262.8^\circ]$
 (c) $2\sin \theta = \tan\left(\frac{\theta}{2}\right)$ $[0^\circ, \pm 120^\circ, \pm 240^\circ, \pm 360^\circ]$
 (d) $2\cos \theta = 15\cos\left(\frac{\theta}{2}\right) + 2$ $[\pm 209^\circ]$
12. Prove the following identities
 (a) $2\cos^2 \theta - \cos 2\theta = 1$
 (b) $2\operatorname{cosec} 2\theta = \operatorname{cosec} \theta \sec \theta$
 (c) $2\cos^3 \theta + \sin 2\theta \sin \theta = 2\cos \theta$
 (d) $\tan \theta + \cot \theta = 2\operatorname{cosec} 2\theta$
 (e) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$
 (f) $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$
 (g) $\cot \theta - \tan \theta = 2\cot 2\theta$
 (h) $\cot 2\theta + \operatorname{cosec} \theta = \cot \theta$
 (i) $\frac{\cos 2\theta}{\cos \theta + \sin \theta} = \cos \theta - \sin \theta$
 (j) $\frac{\sin 2\theta}{1 - \cos 2\theta} = \cot \theta$
 (k) $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$
 (l) $\tan 3\theta = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}$
 (m) $\tan\left(\frac{\theta}{2}\right) = \frac{\sin \theta}{1 + \cos \theta}$

13. Express $\tan 22\frac{1}{2}^\circ$ in the form $a + b\sqrt{2}$ where a and b are integers [$a = -1, b = \pm 1$]

14. Solve the equation

(i) $4\cos\theta - 2\cos 2\theta = 3$ for $0 \leq \theta \leq \pi \left[\frac{\pi}{3}\right]$

(ii) $\cos 2\theta + \cos 3\theta + \cos \theta = 0$ for $0^\circ \leq \theta \leq 360^\circ$ [$\theta = 45^\circ, 120^\circ, 135^\circ$]

(iii) $\cos\theta + \sin 2\theta = 0$ for $0^\circ \leq \theta \leq 360^\circ$ [$\theta = 90^\circ, 210^\circ, 270^\circ, 330^\circ$]

(iv) $2\sin 2\theta = 3\cos\theta$ for $-180^\circ \leq \theta \leq 180^\circ$ [$\theta = -90^\circ, 48.6^\circ, 90^\circ, 132.4^\circ$]

(v) $\sin\theta - 4\sin 4\theta = \sin 2\theta - \sin 3\theta$ for $-\pi \leq \theta \leq \pi \left[\frac{-\pi}{5}, \frac{-\pi}{2}, \frac{-3\pi}{5}, 0, \frac{\pi}{2}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$ –

Harmonic form in trigonometry

These are trigonometric functions expressed in the form of **$R\cos(x \pm \alpha)$ and $R\sin(x \pm \alpha)$** . They are in two ways

(i) solving equations in the form

$$a\cos\theta + b\sin\theta + c = 0$$

(ii) determining the maximum and minimum values of the function

$$a\cos\theta + b\sin\theta + c = 0 \quad \text{where } a, b \text{ and } c \text{ are constants}$$

A: Solving equations

Example 29

(a) Express $3\cos\theta - 4\sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants

Solution

$$\begin{aligned} \text{Let } 3\cos\theta - 4\sin\theta &= R\cos(\theta + \alpha) \\ &= R(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ &= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \end{aligned}$$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$$R\cos\alpha = 3 \quad \dots\dots\dots (i)$$

$$R\sin\alpha = 4 \quad \dots\dots\dots (ii)$$

$$\text{Eqn (ii)} \div \text{eqn (i)}$$

$$\tan\alpha = \frac{4}{3}; \alpha = 53.1^\circ$$

$$\text{Eqn. (i)}^2 + \text{eqn. (ii)}^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$$

$$R^2[\cos^2\alpha + \sin^2\alpha] = 25$$

$$R^2 = 25$$

$$R = 5$$

$$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^\circ)$$

(b) Solve the equation $3\cos\theta - 4\sin\theta = 5$ for $0^\circ \leq \theta \leq 360^\circ$.

Solution

$$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^\circ)$$

$$\Rightarrow 5\cos(\theta + 53.1^\circ) = 5$$

$$\cos(\theta + 53.1^\circ) = 1$$

$$\theta + 53.1^\circ = 0^\circ, 360^\circ$$

$$\theta = -53.1^\circ, 306.9^\circ$$

$$\text{Hence } \theta = 306.9^\circ$$

Example 30

(a) Express $\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta - \alpha)$

Solution

$$\begin{aligned} \text{Let } \sin\theta - \sqrt{3}\cos\theta &= R\sin(\theta - \alpha) \\ &= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha) \end{aligned}$$

Equating coefficients

$$R\cos\alpha = 1 \quad \dots\dots\dots (i)$$

$$R\sin\alpha = \sqrt{3} \quad \dots\dots\dots (ii)$$

$$\text{Eqn. (ii)} \div \text{eqn. (i)}$$

$$\tan\alpha = \sqrt{3}; \Rightarrow \alpha = 60^\circ$$

$$R^2[\cos^2\alpha + \sin^2\alpha] = 4$$

$$R^2 = 4; R = 2$$

$$\therefore \sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^\circ)$$

(b) Solve the equation

$$\sin\theta - \sqrt{3}\cos\theta + 1 = 0 \text{ for } 0^\circ \leq \theta \leq 360^\circ$$

$$\sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^\circ)$$

$$\Rightarrow 2\sin(\theta - 60^\circ) + 1 = 0$$

$$\sin(\theta - 60^\circ) = -\frac{1}{2}$$

$$\theta - 60^\circ = 210^\circ, 330^\circ$$

$$\theta = 270^\circ, 390^\circ$$

$$\text{Hence } \theta = 270^\circ \text{ for the given range}$$

Example 31

- (a) Express $4\cos\theta - 5\sin\theta$ in the form $A\cos(\theta + \beta)$, where A is constant and β is an acute angle

$$\begin{aligned}\text{Let } 4\cos\theta - 5\sin\theta &= A\cos(\theta + \beta) \\ &= A(\cos\theta\cos\beta - \sin\theta\sin\beta) \\ &= A\cos\theta\cos\beta - R\sin\theta\sin\beta\end{aligned}$$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$$A\cos\beta = 4 \quad \text{..... (i)}$$

$$A\sin\beta = 5 \quad \text{..... (ii)}$$

$$\text{Eqn (ii)} \div \text{eqn (i)}$$

$$\tan\alpha = \frac{5}{4}; \alpha = 51.3^\circ$$

$$\text{Eqn. (i)}^2 + \text{eqn. (ii)}^2$$

$$A^2\cos^2\beta + A^2\sin^2\beta = 4^2 + 5^2 = 41$$

$$A^2[\cos^2\alpha + \sin^2\alpha] = 41$$

$$A^2 = 41$$

$$A = \sqrt{41}$$

$$\therefore 3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^\circ)$$

- (b) Solve the equation $3\cos\theta - 4\sin\theta = 2.2$ for $0^\circ \leq \theta \leq 360^\circ$

Solution

$$3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^\circ)$$

$$\Rightarrow \sqrt{41}\cos(\theta + 51.3^\circ) = 2.2$$

$$\cos(\theta + 51.3^\circ) = \frac{2.2}{\sqrt{41}} = 0.3436$$

$$(\theta + 51.3^\circ) = 69.9^\circ, 290.1^\circ$$

$$\therefore \theta = 18.6^\circ, 238.3^\circ$$

B: Maximum and minimum values

The maximum and minimum values of a circular function may be obtained using three methods

- Express the given function either in the form $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ if possible, where R and α are constants.
- Differentiating the given function with respect to the given function say θ
- Sketching the graphs of the function given and noting their maximum and minimum points.

In this chapter approach I will be considered.

Example 32

Determine the maximum and minimum values of the following, stating the value of θ for which they occur

- (a) $\sqrt{3}\sin\theta + \cos\theta + 7$

$$\begin{aligned}\text{Let } \sqrt{3}\sin\theta + \cos\theta &= R\sin(\theta + \alpha) \\ &= R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)\end{aligned}$$

Equating coefficients

$$R\sin\alpha = 1 \quad \text{..... (i)}$$

$$R\cos\alpha = \sqrt{3} \quad \text{..... (ii)}$$

$$\text{Eqn. (i)} \div \text{eqn. (ii)}$$

$$\tan\alpha = \frac{1}{\sqrt{3}}; \Rightarrow \alpha = 30^\circ$$

$$R^2[\cos^2\alpha + \sin^2\alpha] = [1^2 + (\sqrt{3})^2] = 4$$

$$R^2 = 4; R = 2$$

$$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + 30^\circ)$$

$$\Rightarrow \sqrt{3}\sin\theta + \cos\theta + 7 = 2\sin(\theta + 30^\circ) + 7$$

The minimum value occurs when

$$\sin(\theta + 30^\circ) = -1$$

$$\Rightarrow \text{Minimum value} = 2(-1) + 7 = 5$$

$$\text{Now for } \sin(\theta + 30^\circ) = -1$$

$$\theta + 30^\circ = 270^\circ$$

$$\theta = 240^\circ = \frac{4\pi}{3}$$

$$\text{The minimum value is } \left(\frac{4\pi}{3}, 5\right)$$

And maximum value occurs when

$$\sin(\theta + 30^\circ) = 1$$

$$\Rightarrow \text{Minimum value} = 2(1) + 7 = 9$$

$$\text{Now for } \sin(\theta + 30^\circ) = 1$$

$$\theta + 30^\circ = 90^\circ$$

$$\theta = 60^\circ = \frac{\pi}{3}$$

$$\text{The maximum value is } \left(\frac{\pi}{3}, 9\right)$$

- (b) $5\cos\theta - 12\sin\theta - 13$

Solution

$$\begin{aligned}\text{Let } 5\cos\theta - 12\sin\theta &= R\cos(\theta - \alpha) \\ &= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\end{aligned}$$

$$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$$R\cos\alpha = 5 \dots\dots\dots (i)$$

$$R\sin\alpha = 12 \dots\dots\dots (ii)$$

$$\text{Eqn (ii)} \div \text{eqn (i)}$$

$$\tan\alpha = \frac{12}{5}; \alpha = 67.4^\circ$$

$$\text{Eqn. (i)}^2 + \text{eqn. (ii)}^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 12^2 = 169$$

$$R^2[\cos^2\alpha + \sin^2\alpha] = 169$$

$$R^2 = 169$$

$$R = 13$$

$$\therefore 2\cos\theta - 12\sin\theta = 13\cos(\theta - 67.4^\circ)$$

$$\Rightarrow 5\cos\theta - 12\sin\theta - 13 = 13\cos(\theta - 67.4^\circ) - 13$$

The minimum value occurs when

$$\cos(\theta - 67.4^\circ) = -1$$

$$\Rightarrow \text{Minimum value} = 13(-1) - 13 = -26$$

$$\text{Now for } \cos(\theta - 67.4^\circ) = -1$$

$$\theta - 67.4^\circ = 180^\circ$$

$$\theta = 247.4^\circ$$

The minimum value is $(247.4^\circ, -26)$

And maximum value occurs when

$$\cos(\theta - 67.4^\circ) = 1$$

$$\Rightarrow \text{Minimum value} = 13(1) - 13 = 0$$

$$\text{Now for } \cos(\theta - 67.4^\circ) = 1$$

$$\theta - 67.4^\circ = 0^\circ$$

$$\theta = 67.4^\circ$$

The maximum value is $(67.4^\circ, 0)$

Example 33

- (a) Given that $p = 2\cos\theta + 3\cos 2\theta$ and $q = 2\sin\theta + 3\sin 2\theta$, show that $1 \leq p^2 + q^2 \leq 25$
If $p^2 + q^2 = 19$ and θ is acute, find θ and show that $pq = \frac{-5\sqrt{3}}{4}$

Solution

$$p^2 = 4\cos^2\theta + 12\cos\theta\cos 2\theta + 9\cos^2 2\theta \dots\dots(i)$$

$$q^2 = 4\sin^2\theta + 12\sin\theta\sin 2\theta + 9\sin^2 2\theta \dots\dots(ii)$$

$$\text{Eqn. (i)} + \text{eqn. (ii)}$$

$$p^2 + q^2 = 4 + 12(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta) + 9$$

$$p^2 + q^2 = 13 + 12\cos\theta [\cos(-\theta) = \cos\theta]$$

$$\text{But } -1 \leq \cos\theta \leq 1$$

Multiplying through by 12

$$-12 \leq 12\cos\theta \leq 12$$

Adding 13 throughout

$$1 \leq 12\cos\theta + 12 \leq 25$$

$$\therefore 1 \leq p^2 + q^2 \leq 25 \text{ as required}$$

$$\text{If } p^2 + q^2 = 19, \Rightarrow 13 + 12\cos\theta = 19$$

$$\cos\theta = \frac{1}{2}; \theta = 60^\circ [\theta \text{ is acute}]$$

$$\Rightarrow p = 2\cos 60^\circ + 3\cos 120^\circ = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$q = 2\sin 60^\circ + 3\sin 120^\circ = \sqrt{3} + 3 \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$$

$$\therefore pq = \left(-\frac{1}{2}\right) \left(\frac{5\sqrt{3}}{2}\right) = \frac{-5\sqrt{3}}{4}$$

- (b) Express $f(x) = 5\sin^2\theta - 3\sin\theta\cos\theta + \cos^2\theta$ in the form $p + q\cos(2\theta - \alpha)$

$$\text{Hence show that } \frac{1}{2} \leq f(x) \leq 5\frac{1}{2}$$

Solution

$$\text{Using } \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \text{ and}$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$f(x) = \frac{5}{2}(1 - \cos 2\theta) - 3\sin\theta\cos\theta + \frac{1}{2}(1 + \cos 2\theta)$$

$$= \frac{5}{2} - \frac{5}{2}\cos 2\theta - 3 \cdot \frac{2}{2}\sin\theta\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$= 3 - 2\cos 2\theta - \frac{3}{2}\sin 2\theta$$

$$= 3 - [2\cos\theta + \frac{3}{2}\sin 2\theta]$$

Now:

$$3 - [2\cos\theta + \frac{3}{2}\sin 2\theta] \equiv p + q\cos(2\theta - \alpha)$$

$$= 3 + [q\cos 2\theta\cos\alpha + q\sin 2\theta\sin\alpha]$$

By comparing: $p = 3$, $q\sin\alpha = \frac{3}{2}$ and $q\cos\alpha = 2$

$$\Rightarrow \tan\alpha = \frac{3}{4}; \alpha = 36.9^\circ$$

$$\text{And } q = \sqrt{\left\{\left(\frac{3}{2}\right)^2 + (2)^2\right\}} = \frac{5}{2}$$

$$\Rightarrow 3 - [2\cos\theta + \frac{3}{2}\sin 2\theta] = 3 - \frac{5}{2}\cos(2\theta - 36.9^\circ)$$

$$\text{But } -1 \leq \cos(2\theta - 36.9^\circ) \leq 1$$

$$\text{Multiplying through by } -\frac{5}{2}$$

$$\frac{5}{2} \geq -\frac{5}{2}\cos(2\theta - 36.9^\circ) \geq -\frac{5}{2}$$

$$\text{Adding 3 throughout}$$

$$3 - \frac{5}{2} \leq 3 - \frac{5}{2}\cos(2\theta - 36.9^\circ) \leq 3 + \frac{5}{2}$$

$$\therefore \frac{1}{2} \leq f(x) \leq 5\frac{1}{2}$$

- (c) Find the maximum and minimum points of the function; $f(x) = 3\cos\theta - 4\sin\theta + 20$ for $0^\circ \leq \theta \leq 360^\circ$

Solution

$$\text{Let } 3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$$

$$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$$R\cos\alpha = 3 \dots\dots\dots (i)$$

$$R\sin\alpha = 4 \dots\dots\dots (ii)$$

$$\text{Eqn (ii)} \div \text{eqn (i)}$$

$$\tan\alpha = \frac{4}{3}; \alpha = 53.1^\circ$$

$$\text{Eqn. (i)}^2 + \text{eqn. (ii)}^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$$

$$R^2[\cos^2\alpha + \sin^2\alpha] = 25$$

$$R^2 = 25$$

$$R = 5$$

$$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta - 53.1^\circ)$$

$$\Rightarrow 3\cos\theta - 4\sin\theta + 20 = 5\cos(\theta - 53.1^\circ) + 20$$

The minimum value occurs when

$$\cos(\theta - 53.1^\circ) = -1$$

$$\Rightarrow \text{Minimum value} = 5(-1) + 20 = 15$$

$$\text{Now for } \cos(\theta - 53.1) = -1$$

$$\theta - 53.1^\circ = 180^\circ$$

$$\theta = 126.8^\circ$$

The minimum value is $(126.8^\circ, 15)$

And maximum value occurs when

$$\cos(\theta - 53.1^\circ) = 1$$

$$\Rightarrow \text{Minimum value} = 5(1) + 20 = 25$$

$$\text{Now for } \cos(\theta - 53.1^\circ) = 1$$

$$\theta - 53.1^\circ = 0^\circ, 360^\circ$$

$$\theta = 53.1^\circ, 306.8^\circ$$

The maximum value is $(306.8^\circ, 25)$

Example 34

Find the maximum and minimum points of the following

$$(a) f(\theta) = \frac{1}{3 + \sin\theta - 2\cos\theta}$$

Solution

$$\text{Let } \sin\theta - 2\cos\theta = R\sin(\theta - \alpha)$$

$$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$$R\cos\alpha = 1 \dots\dots\dots (i)$$

$$R\sin\alpha = 2 \dots\dots\dots (ii)$$

$$\text{Eqn (ii)} \div \text{eqn (i)}$$

$$\tan\alpha = 2; \alpha = 63.4^\circ$$

$$\text{Eqn. (i)}^2 + \text{eqn. (ii)}^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2 = 5$$

$$R^2[\cos^2\alpha + \sin^2\alpha] = 5$$

$$R^2 = 5$$

$$R = \sqrt{5}$$

$$\therefore \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63.4^\circ)$$

$$\Rightarrow 3 + \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63.4^\circ) + 3$$

$$\Rightarrow f(\theta) = \frac{1}{3 + \sqrt{5}\sin(\theta - 63.4^\circ)}$$

Note: for a fractional function, a maximum point is obtained when the denominator is minimum and the vice versa for the maximum point

The minimum value occurs when

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$$\sin(\theta - 63.4^\circ) = 1$$

$$\Rightarrow \text{Minimum value} = \frac{1}{3+\sqrt{5}} = 0.31$$

$$\text{Now for } \sin(\theta - 63.4) = 1$$

$$\theta - 63.4^\circ = 90^\circ$$

$$\theta = 153.4^\circ$$

$$\text{The minimum value is } (153.4^\circ, 0.31)$$

And maximum value occurs when

$$\sin(\theta - 63.4^\circ) = -1$$

$$\Rightarrow \text{Maximum value} = \frac{1}{3+\sqrt{5(-1)}} = 1.31$$

$$\text{Now for } \sin(\theta - 63.4^\circ) = -1$$

$$\theta - 63.4^\circ = 270^\circ$$

$$\theta = 333.4^\circ$$

$$\text{The maximum value is } (333.4^\circ, 1.31)$$

$$(b) f(\theta) = \frac{1}{4\sin\theta - 3\cos\theta + 6}$$

Solution

$$\text{Let } 4\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$$

$$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$$R\cos\alpha = 4 \dots\dots\dots (i)$$

$$R\sin\alpha = 3 \dots\dots\dots (ii)$$

$$\text{Eqn (ii)} \div \text{eqn (i)}$$

$$\tan\alpha = 0.75; \alpha = 36.9^\circ$$

$$\text{Eqn. (i)}^2 + \text{eqn. (ii)}^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$$

$$R^2[\cos^2\alpha + \sin^2\alpha] = 25$$

$$R^2 = 25$$

$$R = 5$$

$$\therefore 4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.9^\circ)$$

$$\Rightarrow 4\sin\theta - 3\cos\theta + 6 = 5\sin(\theta - 36.9^\circ) + 6$$

$$\Rightarrow f(\theta) = \frac{1}{5\sin(\theta - 36.9^\circ) + 6}$$

The minimum value occurs when

$$\sin(\theta - 36.9^\circ) = 1$$

$$\Rightarrow \text{Minimum value} = \frac{1}{5(1)+6} = \frac{1}{11}$$

$$\text{Now for } \sin(\theta - 36.4) = 1$$

$$\theta - 36.9^\circ = 90^\circ$$

$$\theta = 126.9^\circ$$

$$\text{The minimum value is } (126.9^\circ, \frac{1}{11})$$

And maximum value occurs when

$$\sin(\theta - 36.9^\circ) = -1$$

$$\Rightarrow \text{Maximum value} = \frac{1}{5(-1)+6} = 1$$

$$\text{Now for } \sin(\theta - 36.9^\circ) = -1$$

$$\theta - 36.9^\circ = 270^\circ$$

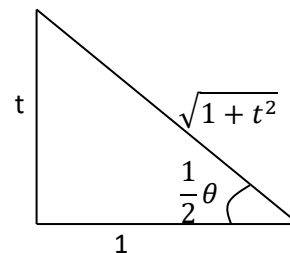
$$\theta = 306.9^\circ$$

$$\text{The maximum value is } (306.9^\circ, 1)$$

The t-formula in trigonometry

Although this form has been tackled indirectly, it is formally stated here

Suppose that $t = \tan\frac{\theta}{2}$, we have



From the triangle above

$$\cos\frac{1}{2}\theta = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin\frac{1}{2}\theta = \frac{t}{\sqrt{1+t^2}}$$

$$\text{But } \cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$$

$$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$$

$$\therefore \cos\theta = \frac{1-t^2}{1+t^2}$$

$$\text{And } \sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$$

$$= 2 \left(\frac{t}{\sqrt{1+t^2}} \right) \left(\frac{1}{\sqrt{1+t^2}} \right)$$

$$\therefore \sin \theta = \frac{2t}{1+t^2}$$

The t- formula is used widely in solving equations and proving trigonometric identities.

These can be extended as follows

(i) For $t = \tan \theta$, $\sin 2\theta = \frac{2t}{1+t^2}$ and $\cos 2\theta = \frac{1-t^2}{1+t^2}$

(ii) For $t = \tan\left(\frac{5x}{4}\right)$, $\sin\left(\frac{5x}{2}\right) = \frac{2t}{1+t^2}$ and

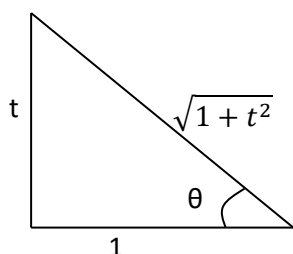
$$\cos\left(\frac{5x}{2}\right) = \frac{1-t^2}{1+t^2}$$

Example 35

Show that if $t = \tan \theta$, then $\sin 2\theta = \frac{2t}{1+t^2}$ and

$2\theta = \frac{1-t^2}{1+t^2}$. Hence solve the equation $\sqrt{3}\cos 2\theta + \sin 2\theta = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

Solution



From the triangle above

$$\cos \theta = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin \theta = \frac{t}{\sqrt{1+t^2}}$$

But $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$= \left(\frac{1}{\sqrt{1+t^2}} \right)^2 - \left(\frac{t}{\sqrt{1+t^2}} \right)^2$$

$$\therefore \cos 2\theta = \frac{1-t^2}{1+t^2}$$

And $\sin 2\theta = 2\sin \theta \cos \theta$

$$= 2 \left(\frac{t}{\sqrt{1+t^2}} \right) \left(\frac{1}{\sqrt{1+t^2}} \right)$$

$$\therefore \sin 2\theta = \frac{2t}{1+t^2}$$

Hence $\sqrt{3}\cos 2\theta + \sin 2\theta = 1$

$$\Rightarrow \sqrt{3} \left(\frac{1-t^2}{1+t^2} \right) + \left(\frac{2t}{1+t^2} \right) = 1$$

$$\sqrt{3} - \sqrt{3}t^2 + 2t = 1 + t^2$$

$$(1 + \sqrt{3})t^2 - 2t + 1 - \sqrt{3} = 0$$

$$t = \frac{2 \pm \sqrt{2^2 - 4(1+\sqrt{3})(1-\sqrt{3})}}{2(1+\sqrt{3})} = \frac{2 \pm \sqrt{12}}{2(1+\sqrt{3})} = \frac{1 \pm \sqrt{3}}{1+\sqrt{3}}$$

$$t = \frac{1+\sqrt{3}}{1+\sqrt{3}} = 1 \text{ or}$$

$$t = \left(\frac{1-\sqrt{3}}{1+\sqrt{3}} \right) \left(\frac{1-\sqrt{3}}{1-\sqrt{3}} \right) = -2 + \sqrt{3}$$

If $\tan \theta = 1$; $\theta = 45^\circ, 225^\circ$

If $\tan \theta = -2 + \sqrt{3}$; $\theta = 165^\circ, 345^\circ$

$\therefore \theta: \theta = 45^\circ, 165^\circ, 225^\circ, 345^\circ$

Example 36

Find all the solutions of the equation $5\cos \theta - 4\sin \theta = 6$ for $-180^\circ \leq \theta \leq 180^\circ$

Solution

Let $t = \tan \frac{\theta}{2}$ then

$$\cos \theta = \frac{1-t^2}{1+t^2}$$

$$\sin \theta = \frac{2t}{1+t^2}$$

$$\Rightarrow 5 \left(\frac{1-t^2}{1+t^2} \right) - 4 \left(\frac{2t}{1+t^2} \right) = 6$$

$$5(1-t^2) - 8t = 6(1+t^2)$$

$$5 - 5t^2 - 8t = 6 + 6t^2$$

$$11t^2 + 8t + 1 = 0$$

$$t = \frac{-8 \pm \sqrt{8^2 - 4 \times 11 \times 1}}{2 \times 11} = \frac{-8 \pm 4.4721}{22}$$

$$t = \frac{-8+4.4721}{22} = -0.1604 \text{ or}$$

$$t = \frac{-8-4.4721}{22} = -0.5669$$

Taking $t = -0.1604$

$$\tan \frac{\theta}{2} = -0.1604; \theta = -18.2^\circ$$

Taking $t = -0.5669$

$$\tan \frac{\theta}{2} = -0.5669; \theta = -59.1^\circ$$

$$\therefore \theta = -59.1^\circ, -18.2^\circ$$

Example 37

Solve the equation

$$3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta) \text{ for } 0^\circ < \theta < 180^\circ$$

Let $t = \tan \theta$

$$3t^2 - 2(1 + t^2) = 2(5 - 3t)$$

$$5t^2 + 6t - 8 = 0$$

$$t = \frac{-6 \pm \sqrt{6^2 - 4(5)(-8)}}{2(5)} = \frac{-6 \pm 14}{10} = -2 \text{ or } \frac{4}{5}$$

$$\text{Taking } t = -2; \theta = \tan^{-1}(-2) = 116.57^\circ$$

$$\text{Taking } t = \frac{4}{5}; \theta = \tan^{-1}\left(\frac{4}{5}\right) = 38.66^\circ$$

$$\text{Hence } \theta = 38.66^\circ, 116.57^\circ$$

Example 38

$$\text{Show that } \tan 4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}, \text{ where } t = \tan\theta.$$

Solution

$$\begin{aligned} \tan 4\theta &= \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} \text{ and } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ &= \frac{2\left(\frac{2t}{1-t^2}\right)}{1 - \left(\frac{2t}{1-t^2}\right)^2} \\ &= \frac{4t(1-t^2)}{t^4-6t^2+1} \end{aligned}$$

Example 39

$$\text{Solve the equation } \cos\theta + \sin\theta + 1 = 0 \text{ for } 0^\circ \leq \theta \leq 360^\circ$$

Solution

$$\cos\theta + \sin\theta + 1$$

$$\text{Let } t = \tan \frac{\theta}{2}$$

$$\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$$

$$1 - t^2 + 2t = 1(1 + t^2)$$

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$$2t + 2 = 0; t = -1$$

$$\therefore \tan \frac{\theta}{2} = -1$$

$$\frac{\theta}{2} = 135^\circ, 315^\circ$$

$$\theta = 270^\circ, 630^\circ$$

$$\text{Hence } \theta = 270^\circ$$

Revision exercise 5

- Solve equation $3\cos\theta + 4\sin\theta = 2$ for $0^\circ \leq \theta \leq 360^\circ$ [119.6°, 346.7°]
- (a) Show that $\cos 4x = \frac{\tan^4 x - 6\tan^2 x + 1}{\tan^4 x + 2\tan^2 x + 1}$
 (b) Show that if $q = \cos 2x + \sin 2x$, then $(1+q)\tan^2 x - 2\tan x + q - 1 = 0$. Deduce that if the roots of the above equation are $\tan x_1$ and $\tan x_2$, the $\tan(x_1 + x_2) = 1$
- Find the values of R and $\tan \alpha$ in each of the following equations
 - $2\cos\theta + 5\sin\theta = R\sin(\theta + \alpha)$ $\left[\sqrt{29}, \frac{2}{5}\right]$
 - $2\cos\theta + 5\sin\theta = R\cos(\theta - \alpha)$ $\left[\sqrt{29}, \frac{5}{2}\right]$
 - $\sqrt{3}\cos\theta + \sin\theta = R\cos(\theta - \alpha)$ $\left[2, \frac{1}{\sqrt{3}}\right]$
 - $5\sin\theta - 12\cos\theta = R\sin(\theta - \alpha)$ $\left[13, \frac{12}{5}\right]$
 - $\cos\theta - \sin\theta = R\cos(\theta + \alpha)$ $[\sqrt{2}, 1]$
- Find the greatest and least values and state the smallest non-negative value of x for which each occurs
 - $12\sin x + 5\cos x$ [13, 67.4°; -13, 247.4°]
 - $2\cos x + \sin x$ [$\sqrt{5}$, 26.6°; $-\sqrt{5}$, 206.6°]
 - $7 + 3\sin x - 4\cos x$ [12, 143.1°; 2, 323.1°]
 - $10 - 2\sin x + \cos x$
 $[10 + \sqrt{5}, 296.6^\circ; 10 - \sqrt{5}, 116.6^\circ]$
 - $\frac{1}{2 + \sin x + \cos x}$ $\left[\frac{2+\sqrt{2}}{2}, 225^\circ; \frac{2-\sqrt{2}}{2}, 45^\circ\right]$
 - $\frac{1}{7 - 2\cos x + \sqrt{5}\sin x}$ $\left[\frac{1}{4}, 311.8^\circ; \frac{1}{10}, 131.8^\circ\right]$
 - $\frac{3}{5\cos x - 12\sin x + 16}$ $\left[1, 112.6^\circ; \frac{3}{29}, 292.6^\circ\right]$
- Solve each of the following equations for $0^\circ \leq x \leq 360^\circ$
 - $\sin x + \sqrt{3}\cos x = 1$ [90°, 330°]

- (b) $4\sin x - 3\cos x = 2$ $[60.4^\circ, 193.3^\circ]$
 (c) $\sin x + \cos x = \frac{1}{\sqrt{2}}$ $[105^\circ, 345^\circ]$
 (d) $5\sin x + 12\cos x = 7$ $[80.0^\circ, 325.2^\circ]$
 (e) $7\sin x - 4\cos x = 3$ $[51.6^\circ, 187.9^\circ]$
 (f) $\cos x - 3\sin x = 2$ $[237.7^\circ, 339.2^\circ]$
 (g) $5\cos x + 2\sin x = 4$ $[63.8^\circ, 339.8^\circ]$
 (h) $9\cos 2x - 4\sin 2x = 6$ $[14.2^\circ, 141.8^\circ, 194.2^\circ, 321.8^\circ]$
 (i) $7\cos x + 6\sin x = 2$ $[118.1^\circ, 323.1^\circ]$
 (j) $9\cos x - 8\sin x = 12$ $[313.6^\circ, 323.1^\circ]$

The factor formulae in trigonometry

The following identities were developed from compound angles

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \dots\dots\dots(i)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \dots\dots\dots(ii)$$

$$\sin(A + B) = \sin A \cos B + \sin B \cos A \dots\dots\dots(iii)$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A \dots\dots\dots(iv)$$

$$\text{eqn. (i) + eqn (ii)}$$

$$\cos(A + B) + \cos(A - B) = 2\cos A \cos B$$

$$\text{eqn. (i) - eqn (ii)}$$

$$\cos(A + B) - \cos(A - B) = -2\cos A \cos B$$

$$\text{eqn. (iii) + eqn (iv)}$$

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B$$

$$\text{eqn. (iii) - eqn (iv)}$$

$$\sin(A + B) - \sin(A - B) = -2\sin B \cos A$$

For simplification, $A + B = \alpha$ and $A - B = \beta$

$$\text{Add: } 2A = \alpha + \beta \text{ i.e. } A = \left(\frac{\alpha + \beta}{2}\right)$$

$$\text{Subtract } 2B = \alpha - \beta \text{ i.e. } B = \left(\frac{\alpha - \beta}{2}\right)$$

Substituting for A and B in the above equation

$$\cos \alpha + \cos \beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha - \cos \beta = -2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

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$$\sin \alpha + \sin \beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin \alpha - \sin \beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

Example 40

Show that if X, Y and Z are angles of a triangle, then

$$(a) \cos X + \cos Y + \cos Z - 1 = 4\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2}$$

solution

$$\text{LHS } \cos X + \cos Y + \cos Z - 1$$

$$= 2\cos \frac{X+Y}{2} \cos \frac{X-Y}{2} + 1 - 2\sin^2 \frac{Z}{2} - 1$$

(to eliminate -1)

$$= 2\cos \frac{180^\circ - Z}{2} \cos \frac{X-Y}{2} - 2\sin^2 \frac{Z}{2}$$

(since $X + Y = 180^\circ - Z$)

$$= 2\sin \frac{Z}{2} \cos \frac{X-Y}{2} - 2\sin^2 \frac{Z}{2}$$

(Since $\cos(90^\circ - A) = \sin A$)

$$= 2\sin \frac{Z}{2} \left[\cos \frac{X-Y}{2} - 2\sin^2 \left\{ \frac{180^\circ - (X+Y)}{2} \right\} \right]$$

$$= 2\sin \frac{Z}{2} \left[\cos \frac{X-Y}{2} - \cos \left\{ \frac{X+Y}{2} \right\} \right]$$

(Since $\sin(90^\circ - A) = \cos A$)

$$= 2\sin \frac{Z}{2} \left[-2\sin \frac{X}{2} \sin \frac{Y}{2} \right]$$

$$= -2\sin \frac{Z}{2} \left[2\sin \frac{X}{2} \sin \frac{Y}{2} \right]$$

(Since $\sin(-A) = -\sin A$)

$$4\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} \text{ as required}$$

$$(b) \sin 3X + \sin 3Y + \sin 3Z =$$

$$- 4\cos \frac{3X}{2} \cos \frac{3Y}{2} \cos \frac{3Z}{2}$$

Solution

$$\text{LHS: } \sin 3X + \sin 3Y + \sin 3Z$$

$$= 2\sin \frac{3(X+Y)}{2} \cos \frac{3(X-Y)}{2} + 2\sin \frac{3Z}{2} \cos \frac{3Z}{2}$$

$$= 2\sin\frac{3(180^\circ - Z)}{2} \cos\frac{3(X-Y)}{2} + 2\sin\frac{3Z}{2} \cos\frac{3Z}{2}$$

$$= -2\cos\frac{3Z}{2} \cos\frac{3(X-Y)}{2} + 2\sin\frac{3Z}{2} \cos\frac{3Z}{2}$$

$$\text{Since } \sin(270^\circ - A) = -\cos A$$

$$= -2\cos\frac{3Z}{2} \left[\cos\frac{3(X-Y)}{2} - \sin\frac{3Z}{2} \right]$$

$$= -2\cos\frac{3Z}{2} \left[\cos\frac{3(X-Y)}{2} - \sin\frac{3\{180^\circ - (X+Y)\}}{2} \right]$$

$$= -2\cos\frac{3Z}{2} \left[\cos\frac{3(X-Y)}{2} - \cos\frac{3(X+Y)}{2} \right]$$

$$= -2\cos\frac{3Z}{2} \left[2\cos\frac{3X}{2} + \cos\frac{-3Y}{2} \right]$$

$$\text{Since } \cos(-A) = \cos A$$

$$= -4\cos\frac{3X}{2} \cos\frac{3Y}{2} \cos\frac{3Z}{2}$$

$$(c) \cos 4X + \cos 4Y + \cos 4Z + 1$$

$$= 4\cos 2X \cos 2Y \cos 2Z$$

Solution

$$\text{LHS: } \cos 4X + \cos 4Y + \cos 4Z + 1$$

$$= 2\cos 2(X+Y) \cos 2(X-Y) + 2\cos^2 2Z - 1 + 1$$

$$= 2\cos 2(180^\circ - Z) \cos 2(X-Y) + 2\cos^2 2Z$$

$$= 2\cos 2Z [\cos 2(X-Y) + \cos 2\{180^\circ - (X+Y)\}]$$

$$= 2\cos 2Z [\cos 2(X-Y) + \cos 2(X+Y)]$$

$$= 2\cos 2Z [2\cos 2X \cos 2Y]$$

$$\text{Since } \cos(-A) = \cos A$$

$$= 4\cos 2Z 2\cos 2X \cos 2Y$$

$$(d) \sin^2 Y + \sin^2 Z = 1 + \cos(Y-Z) \cos X$$

$$\text{LHS: } \sin^2 Y + \sin^2 Z$$

$$= \frac{1}{2}(1 - \cos 2Y) + \frac{1}{2}(1 - \cos 2Z)$$

$$= \frac{1}{2}(2 - \cos 2Y - \cos 2Z)$$

$$= 1 - \frac{1}{2}(\cos 2Y + \cos 2Z)$$

$$= 1 - \cos(180^\circ - X) \cos(Y-Z)$$

$$= 1 + \cos(Y-Z) \cos X$$

Example 41

- (a) Factorize $\cos \theta \cos 3\theta - \cos 7\theta + \cos 9\theta$ and express it in the form $A \cos p\theta \sin q\theta \sin r\theta$ where A, p, q and r are constants

Solution

$$f(\theta) = \cos 9\theta + \cos \theta - (\cos 7\theta + \cos 3\theta)$$

$$= 2\cos 5\theta \cos 4\theta - 2\cos 5\theta \cos 2\theta$$

$$= 2\cos 5\theta (\cos 4\theta - \cos 2\theta)$$

$$= -4\cos 5\theta (-\sin 3\theta \sin \theta)$$

$$= -4\cos 5\theta \sin 3\theta \sin \theta$$

$$\Rightarrow A = -4, p = 5, q = 3, r = 1$$

- (b) Given that

$$p = \sin \alpha + \sin \beta$$

$$q = \cos \alpha + \cos \beta. \text{ Show that}$$

$$\frac{2pq}{p^2 + q^2} = \sin(\alpha + \beta)$$

Solution

$$\frac{2pq}{p^2 + q^2} = \frac{2(\sin \alpha + \sin \beta)(\cos \alpha + \cos \beta)}{\sin^2 \alpha + 2\sin \alpha \sin \beta + \sin^2 \beta + \cos^2 \alpha + 2\cos \alpha \cos \beta + \cos^2 \beta}$$

$$= \frac{2 \left[2\sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \right] \left[2\cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \right]}{2 + 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)}$$

$$= \frac{2 \left[2\sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \right] \left[2\cos^2 \frac{\alpha - \beta}{2} \right]}{2 + 2\cos(\alpha - \beta)}$$

$$= \frac{2[\sin(\alpha + \beta)][1 + \cos(\alpha - \beta)]}{2[1 + \cos(\alpha - \beta)]}$$

$$= \sin(\alpha + \beta)$$

Example 42

$$\text{Solve } 5\cos^2 3\theta = 3(1 + \sin 3\theta) \text{ for } 0^\circ \leq \theta \leq 90^\circ.$$

Solution

$$5\cos^2 3\theta = 3(1 + \sin 3\theta)$$

$$5(1 - \sin^2 3\theta) = 3(1 + \sin 3\theta)$$

$$5 - 5\sin^2 3\theta = 3 + 3\sin 3\theta$$

$$5\sin^2 3\theta + 3\sin 3\theta - 0 = 0$$

$$(\sin 3\theta + 1)(5\sin 3\theta - 2) = 0$$

$$\sin 3\theta + 1 = 0$$

$$3\theta = \sin^{-1}(-1) = -90^\circ, 270^\circ$$

Example 43

(a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for

$$0 \leq \theta \leq 180^\circ$$

$$\cos 2x = 4\cos^2 x - 2\sin^2 x$$

$$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$$

$$3\cos^2 x - \sin^2 x = 0$$

$$4\cos^2 x - 1 = 0$$

$$(2\cos x + 1)(2\cos x - 1) = 0$$

Either

$$2\cos x + 1 = 0$$

$$\cos x = -\frac{1}{2}$$

$$x = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$$

Or

$$2\cos x - 1 = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

$$\therefore x(60^\circ, 120^\circ)$$

Alternatively

$$\cos 2x = 4\cos^2 x - 2\sin^2 x$$

$$= \frac{4}{2}(1 + \cos 2x) - \frac{2}{2}(1 - \cos 2x)$$

$$= 2 + 2\cos 2x - 1 + \cos 2x$$

$$2\cos 2x + 1 = 0$$

$$\cos 2x = -\frac{1}{2}$$

$$2x = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ, 240^\circ$$

$$x = 60^\circ, 120^\circ$$

Alternatively

$$\cos 2x = 4\cos^2 x - 2\sin^2 x$$

$$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$$

$$3\cos^2 x - \sin^2 x = 0$$

$$\sin^2 x = 3\cos^2 x$$

$$\tan^2 x = 3$$

$$\tan x = \pm\sqrt{3}$$

Either

$$\tan x = \sqrt{3}$$

$$x = \tan^{-1} \sqrt{3} = 60^\circ$$

Or

$$\tan x = -\sqrt{3}$$

$$x = \tan^{-1} -\sqrt{3} = 120^\circ$$

$$\text{Hence } x = 60^\circ, 120^\circ$$

Alternatively

$$\cos 2x = 4\cos^2 x - 2\sin^2 x$$

$$1 - 2\sin^2 x = 4(1 - \sin^2 x) - 2\sin^2 x$$

$$1 = 4 - 4\sin^2 x$$

$$4\sin^2 x = 3$$

$$\sin^2 x = \frac{3}{4}$$

$$\sin x = \pm \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$x = 60^\circ, 120^\circ$$

Alternatively

$$\cos 2x = 4\cos^2 x - 2\sin^2 x$$

$$1 - 2\sin^2 x = 4\cos^2 x - 2\sin^2 x$$

$$4\cos^2 x = 1$$

$$\cos x = \pm \sqrt{\frac{1}{4}} = \pm \frac{1}{2}$$

$$x = 60^\circ, 120^\circ$$

(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan$

$$x = \left(\frac{p+1}{p-1}\right)\tan\alpha.$$

Hence solve the equation

$$\sin(x + \alpha) = p\sin(x - \alpha) \text{ for } p = 2 \text{ and } \alpha = 20^\circ.$$

$$\sin x \cos \alpha + \cos x \sin \alpha = p(\sin x \cos \alpha - \cos x \sin \alpha)$$

$$\cos x \sin \alpha (p + 1) = \sin x \cos \alpha (p - 1)$$

$$\cos x \sin \alpha \left(\frac{p+1}{p-1}\right) = \sin x \cos \alpha$$

$$\frac{\cos x \sin \alpha}{\sin x \cos \alpha} \left(\frac{p+1}{p-1}\right) = \frac{\sin x \cos \alpha}{\sin x \cos \alpha}$$

$$\tan x = \left(\frac{p+1}{p-1}\right) \tan \alpha$$

$$\text{For } \sin(x + 20^\circ) = 2\sin(x - 20^\circ)$$

$$\tan x = \frac{2+1}{2-1} \tan 20^\circ = 3 \tan 20^\circ$$

$$x = \tan^{-1}(3 \tan 20^\circ) = 47.52^\circ$$

Example 44

Prove that in any triangle ABC, $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2 - b^2}{c^2}$

Solution

$$\begin{aligned} \frac{a^2 - b^2}{c^2} &= \frac{(2R\sin A)^2 - (2R\sin B)^2}{(2R\sin C)^2} \\ &= \frac{4R^2(\sin^2 A - \sin^2 B)}{4R^2\sin^2 C} \\ &= \frac{(\sin A + \sin B)(\sin A - \sin B)}{\sin^2[180^\circ - (A+B)]} \\ &= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \cdot 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{\sin^2(A+B)} \\ &= \frac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)} \\ &= \frac{\sin(A-B)}{\sin(A+B)} \end{aligned}$$

Inverse trigonometric functions

Note that

- (a) If $\theta = \cos^{-1}\left(\frac{1}{2}\right)$ then $\cos\theta = \frac{1}{2}$
- (b) $\tan^{-1}(\tan\alpha) = \tan(\tan^{-1}\alpha) = \alpha$
- (c) $\cos^{-1}[\cos(x+y)] = \cos[\cos^{-1}(x+y)] = x+y$
- (d) $\sin(\sin^{-1}\theta) = \sin^{-1}(\sin\theta)$

To avoid errors test the values

Example 45

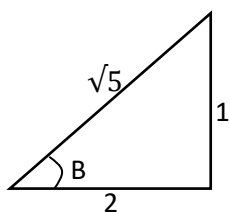
Show that

(a) $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

Solution

$A = \tan^{-1}\frac{1}{3}$ and $B = \sin^{-1}\frac{1}{\sqrt{5}}$

$\Rightarrow \tan A = \frac{1}{3}$ and $B = \frac{1}{\sqrt{5}}$



$\Rightarrow \tan B = \frac{1}{2}$

LHS = $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = A + B$

$$\begin{aligned} &= \tan^{-1}[\tan(A+B)] \\ &= \tan^{-1}\left(\frac{\frac{1}{3} + \frac{1}{2}}{1 - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right) \\ &= \tan^{-1}\frac{3+2}{6-1} \\ &= \tan^{-1}\frac{5}{5} = \tan^{-1}1 = \frac{\pi}{4} \end{aligned}$$

(b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Solution

Let $A = \tan^{-1}\frac{1}{3}$ and $B = \tan^{-1}\frac{1}{7}$

$\Rightarrow \tan A = \frac{1}{3}$ and $\tan B = \frac{1}{7}$

LHS: $\tan^{-1}\tan(2A+B)$ but $\tan 2A = \frac{2\frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2} = \frac{3}{4}$

$$\begin{aligned} \therefore \tan^{-1}\tan(2A+B) &= \frac{\frac{3}{4} + \frac{1}{7}}{1 - \left(\frac{3}{4}\right)\left(\frac{1}{7}\right)} \\ &= \tan^{-1}\left(\frac{21+4}{28-3}\right) \\ &= \tan^{-1}1 \\ &= \frac{\pi}{4} \end{aligned}$$

(c) $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Solution

Let $\theta = \cos^{-1}x$; $\Rightarrow x = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\sin x = \frac{\pi}{2} - \theta$

$\therefore \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Example 46

Solve the equations

(a) $\tan^{-1}(2\theta + 1) + \tan^{-1}(2\theta - 1) = \tan^{-1}(2)$

Solution

Let $A = \tan^{-1}(2\theta + 1)$ and $B = \tan^{-1}(2\theta - 1)$

$\Rightarrow \tan A = 2\theta + 1$ and $\tan B = 2\theta - 1$

$$\therefore A + B = \tan^{-1} 2 \text{ or } \tan(A + B) = 2$$

$$\frac{2\theta + 1 + 2\theta - 1}{1 - (2\theta + 1)(2\theta - 1)} = 2$$

$$4\theta = 2(1 - 4\theta^2 - 1)$$

$$2\theta^2 + \theta - 1 = 0$$

$$(2\theta - 1)(\theta + 1) = 0$$

$$\theta = \frac{1}{2} \text{ or } \theta = -1$$

$$(b) \tan^{-1}(1 + \theta) + \tan^{-1}(1 - \theta) = 32$$

$$\text{Let } A = \tan^{-1}(1 + \theta) \text{ and } B = \tan^{-1}(1 - \theta)$$

$$\Rightarrow \tan A = 1 + \theta \text{ and } \tan B = 1 - \theta$$

$$\therefore A + B = 32 \text{ or } \tan(A + B) = \tan 32$$

Introducing tangents

$$\frac{1 + \theta + 1 - \theta}{1 - (1 + \theta)(1 - \theta)} = \tan 32$$

$$\theta^2 \tan 32 = 2$$

$$\theta = \sqrt{2 \cot 32} = \pm 1.789$$

Example 47

$$\text{If } x = \tan^{-1} \alpha \text{ and } y = \tan^{-1} \beta;$$

$$\text{Show that } x + y = \tan^{-1} \left(\frac{\alpha + \beta}{1 - \alpha\beta} \right)$$

Solution

$$\tan x = \alpha; \quad \tan y = \beta$$

$$(x + y) = \tan^{-1}(\tan(x + y))$$

$$= \tan^{-1} \left(\frac{\alpha + \beta}{1 - \alpha\beta} \right)$$

Example 48

Solve the equation

$$\tan^{-1} \left(\frac{1}{x-1} \right) + \tan^{-1}(x+1) = \tan^{-1}(-2)$$

Solution

$$\text{Let } A = \tan^{-1} \left(\frac{1}{x-1} \right) \text{ and } B = \tan^{-1}(x+1)$$

$$\Rightarrow A + B = \tan^{-1}(-2)$$

$$\begin{aligned} \frac{\frac{1}{x-1} + (x+y)}{1 - \left(\frac{1}{x-1} \right)(x+y)} &= -2 \\ \frac{1+x^2-1}{x-1-x-1} &= -2 \\ \therefore x^2 &= 4; x = \pm 2 \end{aligned}$$

Example 50

Without using tables or calculators determine the values of $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8}$.

Solution

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8}$$

$$= \frac{\frac{1}{2} + \frac{1}{5}}{1 - \left(\frac{1}{2} \right) \left(\frac{1}{5} \right)} + \tan^{-1} \frac{1}{8}$$

$$= \tan^{-1} \frac{7}{9} + \tan^{-1} \frac{1}{8}$$

$$= \frac{\frac{7}{9} + \frac{1}{8}}{1 - \left(\frac{7}{9} \right) \left(\frac{1}{8} \right)} = \tan^{-1} \left(\frac{65}{65} \right) = \frac{\pi}{4}$$

Example 51

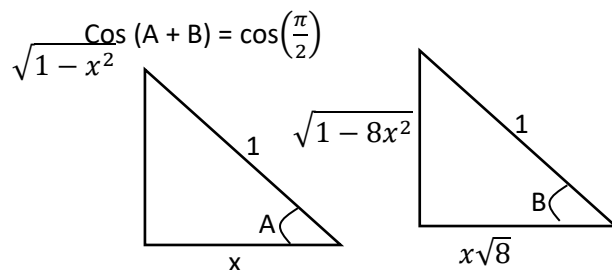
Solve equations

$$(a) \cos^{-1} x + \cos^{-1} x\sqrt{8} = \frac{\pi}{2}$$

Solution

$$\text{Let } A = \cos^{-1} x \text{ and } B = \cos^{-1} x\sqrt{8}$$

$$A + B = \frac{\pi}{2}$$



$$x(x\sqrt{8}) - (\sqrt{1-x^2})(\sqrt{1-8x^2}) = 0$$

$$x(x\sqrt{8}) = (\sqrt{1-x^2})(\sqrt{1-8x^2})$$

$$8x^4 = (1-x^2)(1-8x^2)$$

$$1 - 9x^2 = 0$$

$$(1 - 3x)(1 + 3x) = 0$$

$$\text{Either } x = \frac{1}{3} \text{ or } x = -\frac{1}{3}$$

We discard the negative value, so the root is $x = \frac{1}{3}$

$$(b) 2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}(x\sqrt{2}) = \frac{\pi}{2}$$

Solution

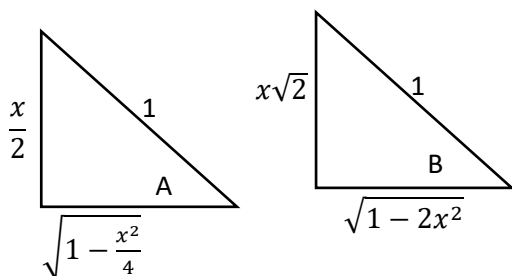
$$\text{Let } A = \sin^{-1}\left(\frac{x}{2}\right) \text{ and } B = \sin^{-1}(x\sqrt{2})$$

$$2A + B = \frac{\pi}{2}$$

$$2A = \frac{\pi}{2} - B$$

$$\sin(2A) = \sin\left(\frac{\pi}{2} - B\right)$$

$$2\sin A \cos A = \cos B$$



$$2\left(\frac{x}{2}\right) \cdot \sqrt{1 - \frac{x^2}{4}} = \sqrt{1 - 2x^2}$$

$$x \cdot \sqrt{\frac{4 - x^2}{4}} = \sqrt{1 - 2x^2}$$

$$\frac{x}{2} \cdot \sqrt{4 - x^2} = \sqrt{1 - 2x^2}$$

$$\frac{x^2}{4} \cdot (4 - x^2) = (1 - 2x^2)$$

$$x^4 - 12x^2 + 4 = 0$$

$$x^2 = \frac{12 \pm \sqrt{144 - 4(4)(1)}}{2(1)}$$

$$x^2 = \frac{12 \pm \sqrt{128}}{2} = 6 \pm 4\sqrt{2}$$

$$x = \sqrt{6 \pm 4\sqrt{2}}$$

After testing for $x = \sqrt{6 + 4\sqrt{2}}$ and for $x = \sqrt{6 - 4\sqrt{2}}$, the value that satisfies the equation is $x = \sqrt{6 - 4\sqrt{2}} = 0.5858$

Hence the value of $x = 0.5858$

Revision exercise 6

1. If $p = \sin \alpha + \sin \beta$ and $q = \cos \alpha + \cos \beta$ show that $\frac{p}{q} = \tan \frac{\alpha + \beta}{2}$

2. (a) Prove that:

$$(i) (\sin 2x - \sin x)(1 + 2\cos x) = \sin 3x$$

$$(ii) \cos 4\theta = \frac{\tan^4 \theta - 6\tan^2 \theta + 1}{\tan^4 \theta + 2\tan^2 \theta + 1}$$

$$(iii) \frac{\sin x + 2\sin 2x + \sin 3x}{\sin x + 2\sin x + \sin 3x} = \tan^2 \frac{x}{2}$$

3. Solve the equation for $0^\circ \leq x \leq 180^\circ$:

$$(a) \sin x + \sin 3x + \sin 5x + \sin 7x = 0$$

$$[x: x = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ]$$

$$(b) \sin 7x + \sin x + \sin 5x + \sin 3x = 0$$

$$[x: x = 60^\circ, 180^\circ]$$

$$(c) \sin x + \sin 4x = 0$$

$$[x: x = 0^\circ, 60^\circ, 72^\circ, 144^\circ, 180^\circ]$$

$$(d) \cos(x + 10^\circ) - \cos(x + 30^\circ) = 0 \quad [70^\circ]$$

$$(e) \cos 5x - \sin 2x = \cos x$$

$$[x: x = 0^\circ, 70^\circ, 90^\circ, 110^\circ, 180^\circ]$$

$$(f) \sin 2x + \sin 10x + \cos 4x = 0$$

$$[x: x = 22.5^\circ, 35^\circ, 55^\circ, 67.5^\circ, 95^\circ, 112.5^\circ, 115^\circ, 157.5^\circ, 175^\circ]$$

4. Show that

$$(a) \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$(b) 2\sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

(c) the positive value that satisfies the equation $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$ is $\frac{1}{6}$

$$(d) \tan^{-1}(-x) = -\tan^{-1} x$$

$$(e) \cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$$

5. Prove that

$$(a) \frac{\sin A - \sin B}{\sin A + \sin B} = \tan\left(\frac{A-B}{2}\right) \cot\left(\frac{A+B}{2}\right)$$

$$(b) \sin 3x + \sin x = 4\sin x \cos^2 x$$

$$(c) \frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \tan 2x$$

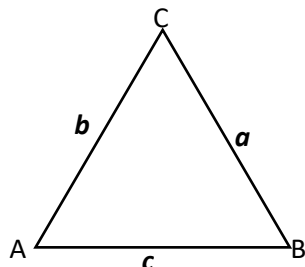
$$(d) \sin(A + B) - \sin(A - B) = 2\cos A \sin B$$

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- (e) $\frac{\sin 5x + \sin x}{\sin 4x + \sin 2x} = 2\cos x - \sec x$
 (f) $\cos 3x + \cos x = 4\cos^2 x - 2\cos x$

Solution to triangles in trigonometry

In a triangle ABC



- (a) Six elements are considered: three angles and three sides

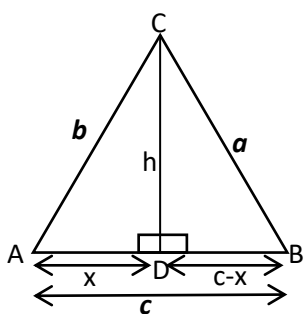
Capital letters denote angles and **small bold and italics letters** sides

- (b) The opposite side of angle A is a, of angle B is b and of angle C is c.
 (c) The angle sum of a triangle is two right angles i.e. $A + B + C = 180^\circ$
 (d) The sides are independent except that the sum of the two sides of the triangle should be equal to or greater than the third side

How to deal with triangles

The cosine rule

- (a) Given an acute angle A



From triangle

$$\text{ACD: } x^2 + h^2 = b^2 \dots\dots\dots (i)$$

$$\text{BCD: } (c - x)^2 + h^2 = a^2$$

$$c^2 - 2cx + x^2 + h^2 = a^2 \dots\dots\dots (ii)$$

Substituting eqn. (i) into eqn. (ii)

$$c^2 - 2cx + b^2 = a^2$$

But

$$x = b\cos A$$

$$\Rightarrow b^2 + c^2 - 2b\cos A = a^2$$

$$a^2 = b^2 + c^2 - 2b\cos A \quad (1)$$

Similarly;

$$b^2 = a^2 + c^2 - 2ac\cos B \quad (2)$$

$$c^2 = a^2 + b^2 - 2ab\cos C \quad (3)$$

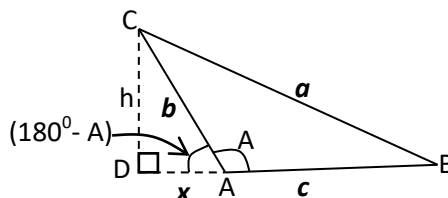
It follows that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

- (b) Given an obtuse angle A



In triangle ABC, A is an obtuse angle and CD is the altitude.

From triangle

$$\text{ACD: } x^2 + h^2 = b^2 \dots\dots\dots (i)$$

$$\text{BCD: } (c - x)^2 + h^2 = a^2$$

$$c^2 - 2cx + x^2 + h^2 = a^2 \dots\dots\dots (ii)$$

Substituting eqn. (i) into eqn. (ii)

$$c^2 - 2cx + b^2 = a^2$$

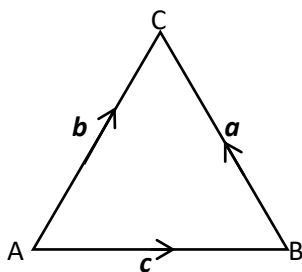
But

$$x = b\cos (180^\circ - A) = -b\cos A$$

From triangle ACD

$$a^2 = b^2 + c^2 - 2b\cos A \text{ as before}$$

The cosine rule can be derived using the vector approach.



Given a triangle above with $BC = a$, $AC = c$ and $AB = b$

$$BC = BA + AC = AC - AB$$

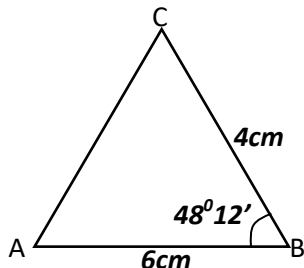
$$a = b - c$$

$$\begin{aligned} \Rightarrow a \cdot a &= (b - c)(b - c) \\ &= b \cdot b - 2b \cdot c + c \cdot c \\ &= b \cdot b + c \cdot c - 2b \cdot c \\ \therefore a^2 &= b^2 + c^2 - 2bccosA \\ \text{since } b \cdot c &= |bc|cosA \end{aligned}$$

Example 52

Solve the triangle in which $AB = 6\text{cm}$, $BC = 4\text{cm}$ and angle $ACB = 48^\circ 12'$

Solution



$$\begin{aligned} \text{Using: } b^2 &= a^2 + c^2 - 2accosB \\ &= 6^2 + 4^2 - 2(6)(4)cos48.2^\circ \end{aligned}$$

$$1^\circ (\text{degree}) = 60' (\text{minutes})$$

$$b = 4.47\text{cm}$$

$$\text{Using: } cosA = \frac{b^2 + c^2 - a^2}{2bc} = \frac{20.0 + 36 - 16}{2(4.47)(6)}$$

$$A = 41.8^\circ$$

$$\text{But } A + B + C = 180^\circ$$

$$41.8^\circ + 48.2^\circ + C = 180^\circ$$

$$C = 90^\circ$$

$$\therefore AC = 4.47\text{cm, angles } BAC = 41.8^\circ \text{ and } ACB = 90^\circ$$

Example 53

In a triangle ABC, prove that

$$\begin{aligned} \text{(a) } a^2 &= (b - c)^2 + 4bccsin^2\left(\frac{A}{2}\right) \text{ hence that } a \\ &= (b - c)sec\alpha \text{ where } tan\alpha = \frac{\sqrt{bc}sin\left(\frac{A}{2}\right)}{b - c} \\ \text{From } cosA &= 1 - 2sin^2\left(\frac{A}{2}\right) \\ \text{Substituting for } cosA &\text{ into the cosine formula } a^2 = b^2 + c^2 - 2bccosA \\ a^2 &= b^2 + c^2 - 2bc[1 - 2sin^2\left(\frac{A}{2}\right)] \\ a^2 &= b^2 + c^2 - 2bc + 4sin^2\left(\frac{A}{2}\right) \\ a^2 &= (b - c)^2 + 4bccsin^2\left(\frac{A}{2}\right) \end{aligned}$$

Hence, substituting for $sin^2\left(\frac{A}{2}\right)$ into $tan\alpha$ expression we get

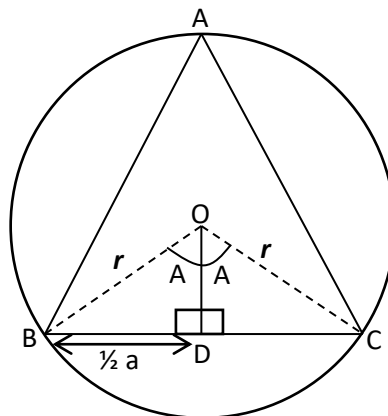
$$a^2 = (b - c)^2 + (b - c)^2tan^2\alpha$$

$$a^2 = (b - c)^2(1 + tan^2\alpha)$$

$$a^2 = (b - c)^2sec^2\alpha$$

$$a = (b - c)sec\alpha$$

The Sine Rule



The figure shows a circle with center O and radius r circumscribing triangle ABC

Angle BOC = 2A [angle subtended by the same arc at the centre of the circle is twice the angle formed at any point on the circumference]

Triangle BOC is isosceles

OD bisects angle BOC and side BC

$$\therefore BD = \frac{1}{2}a$$

From triangle BOD

$$\sin A = \frac{a}{2r} \text{ i.e. } \frac{a}{\sin A} = 2r$$

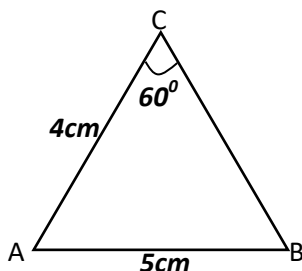
if instead we consider triangles AOC and AOB, we obtain $\frac{b}{\sin B} = 2r$ and $\frac{c}{\sin C} = 2r$

$$\text{In general: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 54

Solve the triangle in which AB = 5cm, AC = 4cm and angle ACB = 60°

Solution



Using sine rule

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow B = \sin^{-1} \left(\frac{b \sin C}{c} \right)$$

$$B = \sin^{-1} \left(\frac{4}{5} \sin 60^\circ \right) = 43.9^\circ$$

$$\text{From } A + B + C = 180^\circ$$

$$A = (180 - 60 - 43.9)^\circ = 76.1^\circ$$

$$\text{Similarly } a = \frac{b \sin A}{\sin B} = \frac{4 \sin 76.1^\circ}{\sin 43.9^\circ} = 5.6 \text{ cm}$$

$$\therefore \overline{AB} = 5.6 \text{ cm}, \hat{BAC} = 76.1^\circ, \hat{ABC} = 43.9^\circ$$

Example 55

Prove that in any triangle

$$\frac{a^2 - b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$$

Solution

From sine rule formula;

$$a = 2r \sin A, b = 2r \sin B, c = 2r \sin C$$

By substitution

$$\frac{a^2 - b^2}{c^2} = \frac{(2r \sin A)^2 - (2r \sin B)^2}{(2r \sin C)^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$\text{But } A + B + C = 180^\circ$$

$$C = 180^\circ - (A + B)$$

$$\sin C = \sin[180^\circ - (A + B)] = \sin(A + B)$$

By substitution

$$\frac{a^2 - b^2}{c^2} = \frac{\sin^2 A - \sin^2 B}{\sin(A+B)} = \frac{(\sin A + \sin B)(\sin A - \sin B)}{\sin(A+B)}$$

$$= \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) \cdot 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{\sin(A+B)}$$

$$= \frac{2 \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A-B)}{\sin(A+B)} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\text{Hence } \frac{a^2 - b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$$

Example 56

Prove that in any triangle ABC,

$$\sin \frac{1}{2}(B - C) = \frac{b - c}{a} \cos \frac{1}{2}A$$

Solution

From sine rule formula;

$$a = 2r \sin A, b = 2r \sin B, c = 2r \sin C$$

By substitution

$$\frac{b - c}{a} = \frac{2r \sin B - 2r \sin C}{2r \sin A} = \frac{\sin B - \sin C}{\sin A}$$

$$\text{But } A + B + C = 180^\circ$$

$$A = 180^\circ - (B + C)$$

$$\sin A = \sin[180^\circ - (B + C)] = \sin(B + C)$$

By substitution

$$\begin{aligned} \frac{b-c}{a} &= \frac{\sin B - \sin C}{\sin A} = \frac{\sin B - \sin C}{\sin(B+C)} \\ &= \frac{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B+C)}{2\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(B+C)} \\ &= \frac{\sin\frac{1}{2}(B+C)}{\cos\frac{1}{2}(B+C)} \end{aligned}$$

$$\text{From } A + B + C = 180^\circ$$

$$B + C = 180^\circ - A$$

$$\frac{1}{2}(B + C) = \left(90^\circ - \frac{1}{2}A\right)$$

$$\sin\frac{1}{2}(B + C) = \sin\left(90^\circ - \frac{1}{2}A\right) = \cos\frac{1}{2}A$$

By substitution

$$\frac{b-c}{a} = \frac{\sin\frac{1}{2}(B+C)}{\cos\frac{1}{2}A}$$

$$\therefore \sin\frac{1}{2}(B - C) = \frac{b-c}{a} \cos\frac{1}{2}A$$

The Tangent Rule

It states that in a triangle ABC

$$\tan\frac{1}{2}(A - B) = \left(\frac{a-b}{a+b}\right) \cot\frac{1}{2}C$$

$$\tan\frac{1}{2}(C - A) = \left(\frac{c-a}{c+a}\right) \cot\frac{1}{2}B$$

$$\tan\frac{1}{2}(b - c) = \left(\frac{b-c}{b+c}\right) \cot\frac{1}{2}A$$

Proof

$$\text{From } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a = 2r\sin A, b = 2r\sin B, c = 2r\sin C$$

$$\begin{aligned} \frac{a-b}{a+b} &= \frac{2r\sin A - 2r\sin B}{2r\sin A + 2r\sin B} = \frac{\sin A - \sin B}{\sin A + \sin B} \\ &= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} \end{aligned}$$

$$= \frac{2\cos(90 - \frac{1}{2}C)\sin\frac{1}{2}(A-B)}{2\sin(90 - \frac{1}{2}C)\cos\frac{1}{2}(A-B)}$$

$$= \frac{\cos(90 - \frac{1}{2}C)\tan\frac{1}{2}(A-B)}{\sin\frac{1}{2}(90 - C)}$$

$$= \frac{\sin\frac{1}{2}C \tan\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$$

$$\frac{a-b}{a+b} = \tan\frac{1}{2}C \tan\frac{1}{2}(A - B)$$

$$\therefore \tan\frac{1}{2}(A - B) = \left(\frac{a-b}{a+b}\right) \cot\frac{1}{2}C$$

Example 56

Show that in a triangle PQR

$$\tan\frac{1}{2}(Q - C) = \left(\frac{q-r}{q+r}\right) \cot\frac{1}{2}P$$

Hence solve the triangle in which $q = 15.32$, $r = 28.6$ and $P = 39^\circ 52'$

Solution

$$\text{From } \frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R}$$

$$p = 2r\sin P, q = 2r\sin Q, r = 2r\sin R$$

$$\frac{q-r}{q+r} = \frac{2r\sin Q - 2r\sin R}{2r\sin Q + 2r\sin R} = \frac{\sin Q - \sin R}{\sin Q + \sin R}$$

$$= \frac{2\cos\frac{1}{2}(Q+R)\sin\frac{1}{2}(Q-R)}{2\sin\frac{1}{2}(Q+R)\cos\frac{1}{2}(Q-R)}$$

$$= \frac{2\cos(90 - \frac{1}{2}P)\sin\frac{1}{2}(Q-R)}{2\sin(90 - \frac{1}{2}P)\cos\frac{1}{2}(Q-R)}$$

$$= \frac{\cos(90 - \frac{1}{2}P)\tan\frac{1}{2}(Q-R)}{\sin\frac{1}{2}(90 - P)}$$

$$= \frac{\sin\frac{1}{2}P \tan\frac{1}{2}(Q-R)}{\cos\frac{1}{2}P}$$

$$\frac{q-r}{q+r} = \tan\frac{1}{2}P \tan\frac{1}{2}(Q - R)$$

$$\therefore \tan\frac{1}{2}(Q - R) = \left(\frac{q-r}{q+r}\right) \cot\frac{1}{2}P$$

Hence

$$\tan \frac{1}{2}(Q - R) = \frac{15.32 - 28.6}{15.32 + 29.6} \cot 39^\circ 52'$$

$$= -0.3621$$

$$\frac{1}{2}(Q - R) = -19.9^\circ \text{ i.e. } Q - R = -39.9^\circ$$

$$\text{But } P + Q + R = 180$$

$$Q + R = 180 - 39.9 = 140.1^\circ$$

$$\text{Solving } Q = 50.15^\circ \text{ and } R = 89.95^\circ$$

$$\text{Now } p = \frac{q \sin P}{\sin Q} = \frac{15.32 \sin \left[39 + \frac{52}{60} \right]^\circ}{\sin 50.15} = 12.79$$

$$\therefore p = 12.79, Q = 50.15^\circ, R = 89.95^\circ$$

Example 57

$$\text{Show that } \frac{a+b-c}{a+b+c} = \tan \frac{1}{2} A \tan \frac{1}{2} B$$

Solution

$$\begin{aligned} \text{LHS} &= \frac{a+b-c}{a+b+c} \\ &= \frac{2r \sin A + 2r \sin B - 2r \sin C}{2r \sin A + 2r \sin B + 2r \sin C} \\ &= \frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} \\ &= \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) - 2 \sin \frac{1}{2} C \cos \frac{1}{2} C}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) + 2 \sin \frac{1}{2} C \cos \frac{1}{2} C} \\ &= \frac{2 \sin \left(90 - \frac{1}{2} C \right) \cos \frac{1}{2}(A-B) - 2 \sin \frac{1}{2} C \cos \frac{1}{2} C}{2 \sin \left(90 - \frac{1}{2} C \right) \cos \frac{1}{2}(A-B) + 2 \sin \frac{1}{2} C \cos \frac{1}{2} C} \\ &= \frac{2 \cos \frac{1}{2} C \cos \frac{1}{2}(A-B) - 2 \sin \frac{1}{2} C \cos \frac{1}{2} C}{2 \cos \frac{1}{2} C \cos \frac{1}{2}(A-B) + 2 \sin \frac{1}{2} C \cos \frac{1}{2} C} \\ &= \frac{\cos \frac{1}{2}(A-B) - \sin \frac{1}{2} C}{\cos \frac{1}{2}(A-B) + \sin \frac{1}{2} C} \\ &= \frac{\cos \frac{1}{2}(A-B) - \sin \left(90 - \frac{1}{2}(A+B) \right)}{\cos \frac{1}{2}(A-B) + \sin \left(90 - \frac{1}{2}(A+B) \right)} \\ &= \frac{\cos \frac{1}{2}(A-B) - \cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B) + \cos \frac{1}{2}(A+B)} \\ &= \frac{-2 \sin \frac{1}{2} A \sin \left(-\frac{1}{2} B \right)}{\cos \frac{1}{2} A + \cos \frac{1}{2} B} \\ &= \tan \frac{1}{2} A \tan \frac{1}{2} B \end{aligned}$$

Expressions for $\sin A$, $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$ in terms of the sides of the triangle

(a) $\sin A$

From the identity

$$\sin^2 A = 1 - \cos^2 A = (1 - \cos A)(1 + \cos A)$$

$$= \left(1 - \frac{b^2 + c^2 - a^2}{2bc} \right) \left(1 + \frac{b^2 + c^2 - a^2}{2bc} \right)$$

$$= \left(\frac{2bc - b^2 + c^2 - a^2}{2bc} \right) \left(\frac{2bc + b^2 + c^2 - a^2}{2bc} \right)$$

$$= \frac{[a^2 - (b-c)^2][(b+c)^2 - a^2]}{4b^2 c^2}$$

$$\therefore \sin^2 A = \frac{(a+c-b)(a+b-c)(b+c-a)(b+c+a)}{4b^2 c^2}$$

$$\text{Let } s = \frac{1}{2} [\text{perimeter of triangle}]$$

$$= \frac{1}{2} [a + b + c]$$

$$2s = [a + b + c]$$

$$a + b = 2s - c; \text{ i.e. } a + b - c = 2s - c - c = 2(s - c)$$

$$a + c = 2s - b; \text{ i.e. } a + c - b = 2s - b - b = 2(s - b)$$

$$b + c = 2s - a; \text{ i.e. } b + c - a = 2s - a - a = 2(s - a)$$

$$\therefore \sin^2 A = \frac{2(s-b) \cdot 2(s-c) \cdot 2(s-a) \cdot 2s}{4b^2 c^2}$$

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Similarly, } \sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}$$

(b) $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$

$$\text{From } \sin^2 \frac{1}{2} A = \frac{1}{2} (1 - \cos A)$$

$$= \frac{1}{2} \left(1 - \frac{b^2 + c^2 - a^2}{2bc} \right)$$

$$= \left(\frac{2bc - b^2 + c^2 - a^2}{4bc} \right)$$

$$= \left(\frac{a^2 - (b-c)^2}{4bc} \right)$$

$$= \left(\frac{(a+c-b)(a+b-c)}{4bc} \right)$$

$$= \left(\frac{2(s-b) \cdot 2(s-c)}{4bc} \right)$$

$$= \left(\frac{(s-b)(s-c)}{bc} \right)$$

$$\therefore \sin \frac{1}{2} A = \sqrt{\left(\frac{(s-b)(s-c)}{bc} \right)}$$

Similarly;

$$\sin \frac{1}{2} B = \sqrt{\left(\frac{(s-b)(s-c)}{ac} \right)}$$

$$\sin \frac{1}{2} C = \sqrt{\left(\frac{(s-b)(s-c)}{ab} \right)}$$

Also;

$$\cos^2 \frac{1}{2} A = \frac{1}{2} (1 + \cos A)$$

$$= \left(\frac{2bc + b^2 + c^2 - a^2}{4bc} \right)$$

$$= \left(\frac{(b+c)^2 - a^2}{4bc} \right)$$

$$= \left(\frac{(b+c-a)(a+b+c)}{4bc} \right)$$

$$= \left(\frac{2(s-a) \cdot 2s}{4bc} \right)$$

$$= \left(\frac{s(s-a)}{bc} \right)$$

$$\therefore \cos \frac{1}{2} A = \sqrt{\left(\frac{s(s-a)}{bc} \right)}$$

Similarly;

$$\cos \frac{1}{2} B = \sqrt{\left(\frac{s(s-b)}{ac} \right)}$$

$$\cos \frac{1}{2} C = \sqrt{\left(\frac{s(s-c)}{ab} \right)}$$

The expression for $\tan \frac{1}{2} A$ can be deduced as follows

$$\tan \frac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

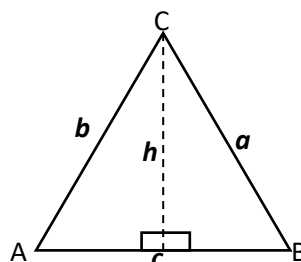
Similarly;

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$$\tan \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\tan \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Area of a triangle in trigonometry



$$\text{Area, } \Delta = \frac{1}{2} (\text{base})(\text{perpendicular height})$$

$$= \frac{1}{2} ch$$

$$= \frac{1}{2} cb \sin A$$

Substituting for

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta = \frac{1}{2} bc \times \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

This is a convenient form given the three sides of a triangle. The formula is called Hero's formula from the first mathematician who suggested it.

Example 58

The area of a triangle is 336m^2 . The sum of the three sides is 84m and one side is 28m . Calculate the length of the remaining two sides

Solution

$$\text{Given } \Delta = 336, a + b + c = 84 \text{ and } a = 28$$

$$s = \frac{1}{2} (a + b + c) = \frac{1}{2} (84) = 42$$

$$28 + b + c = 84$$

$$b + c = 56, \text{ or } c = 56 - b$$

$$\text{But } \Delta^2 = s(s-a)(s-b)(s-c)$$

$$336^2 = 42(42-28)(42-b)(42-56+b)$$

$$b^2 - 56b + 780 = 0$$

$$b = \frac{56 \pm \sqrt{56^2 - 4 \times 1 \times 780}}{2 \times 1}$$

$$b = 30 \text{ or } 26$$

substituting for $c = 56 - b$

$$c = 26 \text{ or } 30$$

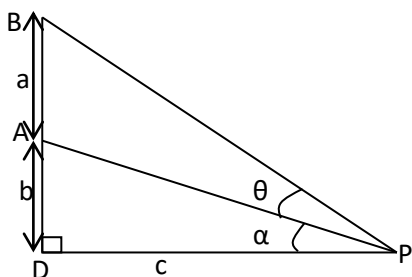
$\therefore$ the remaining sides are 30m and 26m

Applications of trigonometry in finding distances and bearings

Example 59

A vertical pole BAD stands with its base D on a horizontal plane where $BA = a$ and $AD = b$. A point P is situated on the horizontal plane at a distance C from D and the angle $APB = \theta$. Prove that $\theta = \tan^{-1} \left(\frac{ac}{b^2 + ab + c^2} \right)$

Solution



Let angle $APD = \alpha$

$$\text{For triangle APD: } \tan \alpha = \frac{b}{c}$$

$$\text{For triangle DPB: } \tan(\theta + \alpha) = \frac{a+b}{c}$$

$$\Rightarrow \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha} = \frac{a+b}{c}$$

Substituting for $\tan \alpha$

$$\Rightarrow \frac{\tan \theta + \frac{b}{c}}{1 - \left(\frac{b}{c}\right) \tan \theta} = \frac{a+b}{c}$$

$$c^2 \tan \theta + bc = ac + bc - ab \tan \theta - b^2 \tan \theta$$

$$(b^2 + ab + c^2) \tan \theta = ac$$

$$\tan \theta = \frac{ac}{b^2 + ab + c^2}$$

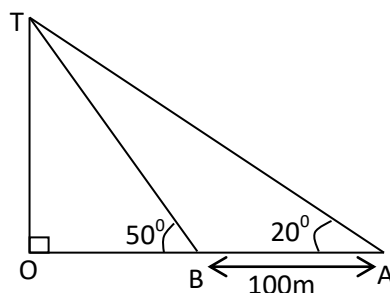
$$\therefore \theta = \tan^{-1} \left(\frac{ac}{b^2 + ab + c^2} \right)$$

Example 60

The angle of the top of a vertical tower from a point A is 20° and from another point B is 50° . Given that A and B lie on the same horizontal plane in the same direction where $AB = 100\text{m}$. Find the height of the tower

Solution

Let OT be the height of the tower



$$\angle A\hat{T}B = 50^\circ - 20^\circ = 30^\circ$$

Using sine rule

$$\frac{TB}{\sin 20^\circ} = \frac{100}{\sin 30^\circ}$$

$$TB = \frac{100 \sin 20^\circ}{\sin 30^\circ}$$

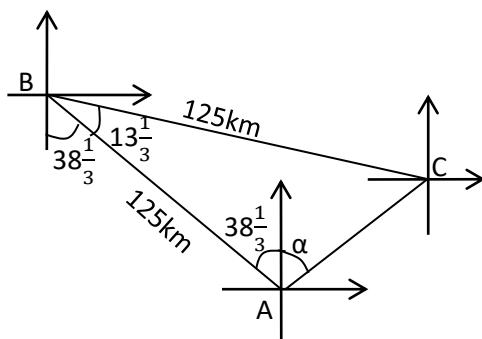
$$\text{But } OT = TB \sin 50^\circ$$

$$OT = \frac{100 \sin 20^\circ \sin 50^\circ}{\sin 30^\circ} = 26.2\text{m}$$

Example 61

From a point A, a pilot flies in the direction $N38^\circ 20'W$ to point B 125km from A. He then flies in the direction $S50^\circ 40'E$ for 125km. He wishes to return to A from this point. How far and in what direction must he fly.

Solution



From the diagram

Let $\hat{BAC} = \hat{BCA} = \theta$

$$\Rightarrow 2\theta + 13\frac{1}{3} = 180^\circ$$

$$\theta = 83\frac{1}{3}$$

$$\text{But } 38\frac{1}{3} + \alpha = \theta$$

$$38\frac{1}{3} + \alpha = 83\frac{1}{3}$$

$$\alpha = 45^\circ$$

From the sine rule

$$\frac{AC}{\sin 13\frac{1}{3}} = \frac{125}{\sin 83\frac{1}{3}}$$

$$AC = 29\text{km}$$

$\therefore$ he has to fly 29km in the direction $S45^\circ W$

Example 62

(a) Prove that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \sin B + \sin A \cos B}$$

Dividing numerator and denominator on the R.H.S by $\cos A \cos B$

$$\tan(A - B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \sin B}{\cos A \cos B} + \frac{\sin A \cos B}{\cos A \cos B}}$$

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$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Hence show that $\frac{1 - \tan 15^\circ}{1 + \tan 15^\circ} = \frac{1}{\sqrt{3}}$

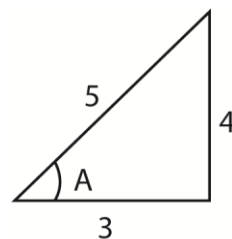
$$\begin{aligned} \frac{1 - \tan 15^\circ}{1 + \tan 15^\circ} &= \frac{\tan 45^\circ - \tan 15^\circ}{1 + \tan 45^\circ \tan 15^\circ} \\ &= \tan(45^\circ - 15^\circ) \tan 30^\circ = \frac{1}{\sqrt{3}} \end{aligned}$$

(b) Given that $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$ where A and B are acute, find the values of

(i) $\tan(A + B)$

(ii) $\operatorname{cosec}(A + B)$

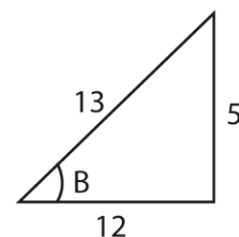
Solution



$$\cos A = \frac{3}{5}$$

$$\sin A = \frac{4}{5}$$

$$\tan A = \frac{4}{3}$$



$$\cos B = \frac{12}{13}$$

$$\sin B = \frac{5}{13}$$

$$\tan B = \frac{5}{12}$$

$$\begin{aligned} \text{(i) } \tan(A + B) &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \sin B - \sin A \cos B} \\ &= \frac{\frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13}}{\frac{3}{5} \cdot \frac{5}{13} - \frac{4}{5} \cdot \frac{12}{13}} = 3.9375 \end{aligned}$$

$$\begin{aligned} \text{(ii) } \operatorname{cosec}(A + B) &= \frac{1}{\sin(A + B)} \\ &= \frac{1}{\sin A \cos B + \cos A \sin B} \\ &= \frac{1}{\frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13}} \\ &= 1.0317 \end{aligned}$$

Example 63

Express $\cos(\theta + 30^\circ) - \cos(\theta + 48^\circ)$ in the form $R \sin P \sin Q$, where R is constant.

Hence solve the equation

$$\cos(\theta + 30^\circ) - \cos(\theta + 48^\circ) = 0.2$$

Solution

$$\begin{aligned}\cos(\theta + 30^\circ) - \cos(\theta + 48^\circ) \\ = -2\sin\left(\frac{\theta + 30^\circ + \theta + 48^\circ}{2}\right) \sin\left(\frac{\theta + 30^\circ - \theta - 48^\circ}{2}\right) \\ = -2\sin(\theta + 39^\circ)\sin(-9^\circ)\end{aligned}$$

$$\cos(\theta + 30^\circ) - \cos(\theta + 48^\circ) = 0.$$

$$\Rightarrow -2\sin(\theta + 39^\circ)\sin(-9^\circ) = 0.2$$

$$\sin(\theta + 39^\circ) = 0.63925$$

$$\theta + 39^\circ = 39.74^\circ$$

$$\theta = 0.74^\circ$$

Example 64

Express $7\cos 2\theta + 6\sin 2\theta$ in form $R\cos(2\theta - \alpha)$, where R is a constant and α is an acute angle.

$$7\cos 2\theta + 6\sin 2\theta \equiv R\cos(2\theta - \alpha)$$

$$7\cos 2\theta + 6\sin 2\theta \equiv R\cos 2\theta \cos \alpha +$$

$$R\sin 2\theta \sin \alpha$$

Comparing both sides

$$R\cos \alpha = 7 \dots\dots\dots (i)$$

$$R\sin \alpha = 6 \dots\dots\dots (ii)$$

(i)2 + (ii)2 gives

$$R = \sqrt{7^2 + 6^2} = \sqrt{85}$$

From equation (i)

$$\sqrt{85}\cos \alpha = 7$$

$$\alpha = \cos^{-1}\left(\frac{7}{\sqrt{85}}\right) = 40.6^\circ$$

Hence solve $7\cos 2\theta + 6\sin 2\theta = 5$ for $0^\circ \leq \theta \leq 180^\circ$. (07marks)

$$\therefore 7\cos 2\theta + 6\sin 2\theta = \sqrt{85} \cos(2\theta - 40.6^\circ) = 5$$

$$2\theta - 40.6 = \cos^{-1}\left(\frac{5}{\sqrt{85}}\right) = 57.16^\circ, 302.84^\circ$$

$$\theta = 48.88^\circ, 171.72^\circ$$

Revision exercise 7

1. Solve the triangles

(a) $a = 17\text{m}$, $b = 21.42\text{m}$, $B = 51^\circ 34'$

$$[A = 38.44^\circ, C = 90^\circ, c = 27.34\text{m}]$$

(b) $b = 107.2\text{m}$, $c = 76.69\text{m}$, $B = 102^\circ 25'$

$$[A = 33.26^\circ, C = 44.32^\circ, a = 60.21\text{m}]$$

(c) $a = 7\text{m}$, $b = 3.59\text{m}$, $C = 47^\circ$

$$[A = 103^\circ 2', B = 29^\circ 52', c = 5.25\text{m}]$$

(d) $A = 60^\circ$, $b = 8\text{m}$, $C = 15$

$$[a = 13, B = 32.2^\circ, C = 87.8^\circ]$$

2. Show that for all values of x

$$\cos x + \cos\left(x + \frac{2\pi}{3}\right) + \cos\left(x + \frac{3\pi}{3}\right) = 0$$

3. (a) Simplify $\frac{\sin 3\theta}{\sin \alpha} - \frac{\cos 3\theta}{\cos \alpha} \left[\frac{2\sin(3\theta - \alpha)}{\sin 2\alpha} \right]$

(b) Express $5\sin\theta + 12\cos\theta$ in the form $r\sin(\theta + \alpha)$ where r and α are constant.

Hence determine the minimum value of

$$5\sin\theta + 12\cos\theta + 7. [r = 13, \alpha = 67.4^\circ, -6]$$

(c) Given that $\tan\theta = \frac{3}{4}$, where θ is acute,

find values of $\tan 2\theta$ and $\tan \frac{\theta}{2}$

$$[\tan 2\theta = \frac{24}{7} \text{ and } \tan \frac{\theta}{2} = \frac{1}{3}]$$

4. (a) Show that $2 \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

(b) Find x given that

$$\tan^{-1}(1+x) + \tan^{-1}(1-x) = 32$$

$$[x = \pm 1.789]$$

(c) Given that $\sin \alpha + \sin \beta = p$ and

$$\cos \alpha + \cos \beta = q$$

$$\text{Show that } \sin(\alpha + \beta) = \frac{2pq}{p^2 + q^2}$$

5. (a) By expressing $2\sin\theta\sin(\theta + \alpha)$ as a difference of cosines of two angles or otherwise, where α is constant, find the least value [minimum value = $\cos \alpha - 1$. It occurs when $\theta = \frac{\alpha}{2}$]

(b) Solve for x in the equation

$$\cos x - \cos(x + 60^\circ) = 0.4 \text{ for}$$

$$0^\circ \leq x \leq 360^\circ [x = 126.4^\circ, 353.6^\circ]$$

6. (a) Prove that in any triangle ABC

$$\frac{b^2 - c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$$

- (b) Show that for any isosceles triangle ABC with AB = c the base, is given by $\Delta = \frac{1}{2}c\sqrt{s(s-c)}$ where s is the perimeter of the triangle
Given that $\Delta = \sqrt{3}$ and $s = 4$, determine the sides of the triangle [1, 3.5, 3.5]
7. Given that $\tan^{-1} \alpha = x$ and $\tan^{-1} \beta = y$, by expressing α and β as tangents ratio of x and y and manipulating the ratios show that $x + y = \tan^{-1} \left(\frac{\alpha + \beta}{1 - \alpha\beta} \right)$
Hence or otherwise
(i) Solve for x in
$$\tan^{-1} \left(\frac{1}{x-1} \right) + \tan(x+1) = \tan(-2)$$

[x = ± 2]
(ii) Without using tables of calculators determine the value of
$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8} \left[\frac{\pi}{4} \right]$$
8. (a) Prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ where ABC has all angles acute and R is the radius of the circumcircle.
(b) From the top of a vertical cliff 10m high, the angle of depression of ship A is 10° and ship B is 15° . The Bearings of A and B from the cliff are 162° and 202.5° respectively. Find the bearing of B from A [301.5 $^\circ$]
9. (a) Prove that $(\sin 2x - \sin x)(1 + 2\cos x) = \sin 3x$
(b) A vertical pole BAO stands with its base O on a horizontal plane, where BA = c and AO = b, a point P is situated on horizontal plane at a distance x from O and angle APB = θ
Prove that $\tan \theta = \frac{cx}{x^2 + b^2 + bc}$
As P takes different positions on the horizontal plane, find the value of x for which θ is greatest. [18 $^\circ$ 26', when x = b = c]
10. (a) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.
(b) Find all the solutions to $2\sin^2 x = 1$ for $00^\circ \leq x \leq 360^\circ$.
[x = $10^\circ, 50^\circ, 130^\circ, 170^\circ, 250^\circ, 290^\circ$]
11. Solve $\cos x + \sqrt{3}\sin x = 2$ for $0^\circ \leq x \leq 360^\circ$
[x = 60°]
12. From the top of a tower 12.6m high, the angles of depression of ship A and B are 12° and 18° respectively. the bearing of ship A and ship B from the tower are 148° and 209.5° respectively
Calculate
(i) How far the ships are from each other [53.14m]
(ii) The bearing of ship A from ship B [108.1 $^\circ$]
13. (a) Solve $\sin 3x + \frac{1}{2} = 2\cos^2 x$ for $0^\circ \leq x \leq 360^\circ$
[x = $30^\circ, 60^\circ, 120^\circ, 150^\circ, 240^\circ, 300^\circ$]
(b) Given that in any triangle ABC, $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot\left(\frac{A}{2}\right)$ solve the triangle with two sides 5 and 7 and the included angle 45° .
[A = 45° , B = 89.4° , C = 45.6°]
14. (a) Solve $\cot^2 x = 5(\cos x + 1)$ for $0^\circ \leq x \leq 360^\circ$ [9.6 $^\circ, 170.4^\circ, 270^\circ$]
(b) Use $\tan \frac{\theta}{2} = t$ to solve $5\sec \theta - 2\sin \theta = 2$ for $0^\circ \leq x \leq 360^\circ$ [46.4 $^\circ, 270^\circ$]
15. Given that $\sin 2x = \cos 3x$, find the values of $\sin \theta$, $0 \leq x \leq \pi$ [0.309 3dp]
16. (a) Show that $\tan\left(\frac{A+B}{2}\right) - \tan\left(\frac{A-B}{2}\right) = \frac{2\sin B}{\cos A + \cos B}$
(b) Find in radians the solution of the equation $\cos \theta + \sin 2\theta = \cos 3\theta$ for $0 \leq \theta \leq \pi$ $\left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right]$
17. (a) Show that $\cot A + \tan 2A = \cot A \sec 2A$
(b) Show that $\tan 3x = \frac{3t-t^3}{1-3t^2}$, where $t = \tan x$. Hence or otherwise show that $\tan^{-1} \left(\frac{\pi}{12} \right) = 2 - \sqrt{3}$
18. (a) Find all the values θ , $00^\circ \leq \theta \leq 360^\circ$, which satisfies the equation $\sin^2 \theta - \sin 2\theta - 3\cos^2 \theta = 0$ [$\theta = 135^\circ, 315^\circ$]

- (b) Show that $\frac{\cos x}{1+\sin x} = \cot\left(\frac{x}{2} + 45^\circ\right)$.
Hence or otherwise solve $\frac{\cos x}{1+\sin x} = \frac{1}{2}$,
 $0^\circ \leq x \leq 360^\circ$ [$x = 36.8^\circ$]
19. (a) Given that X, Y and Z are angles of a triangle XYZ. Prove that
 $\tan\left(\frac{X-Y}{2}\right) = \frac{x-y}{x+y} \cot \frac{Z}{2}$.
Hence solve the triangle if $x = 9\text{cm}$,
 $y = 5.7\text{cm}$ and $Z = 57^\circ$. [$z = 7.6\text{cm}$, $X = 84.4^\circ$]
(b) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to solve
the equation $3\cos\theta - 5\sin\theta = -1$ for
 $0^\circ \leq \theta \leq 360^\circ$ [40.84° , 201.1°]
20. Prove that
 $\tan\left(\frac{\pi}{4} + \theta\right) - \tan\left(\frac{\pi}{4} - \theta\right) = 2\tan 2\theta$
21. (a) Solve the equation $3\cos x + 4\sin x = 2$ for
 $0^\circ \leq x \leq 360^\circ$ [$x = 119.5^\circ$, 346.7°]
(b) If A, B, C are angles of a triangle. Show
that
 $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B$
22. (a) Solve $2\sin 2\theta = 3$ for $-180^\circ \leq x \leq 180^\circ$
[-90° , 48.6° , 90° , 131.4°]
(b) Solve $\sin x - \sin 4x = \sin 2x - \sin 3x$ for
 $\pi \leq x \leq \pi$
 $\left[-\frac{\pi}{5}, -\frac{\pi}{2}, -\frac{3\pi}{5}, 0, \frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}\right]$
23. Without using tables or calculator, show
that $\tan 150 = 2 - \sqrt{3}$
24. (a) Solve the equation $\cos x + \cos 2x = 1$ for
 $0^\circ \leq x \leq 360^\circ$ [$x = 38.67^\circ$, 321.33°]
(b) (i) Prove that $\frac{\cos A + \cos B}{\sin A + \sin B} = \cot \frac{A+B}{2}$
(ii) $\frac{\cos A + \cos B}{\sin A + \sin B} = \tan \frac{C}{2}$ where A, B and C
are angles of a triangle
25. Given that $\sin(\theta - 45^\circ) = 3\cos(\theta + 45^\circ)$ show
that $\tan \theta = 1$. Hence find θ if $0^\circ \leq \theta \leq 360^\circ$.
[45° , 225°]
26. (a) Use the factor formula to show
that $\frac{\sin(A+2B) + \sin A}{\cos(A+2B) + \cos A} = \tan(A+B)$
(b) Express $y = 8\cos x + 6\sin x$ in the form
 $R\cos(x - \alpha)$ where R is positive and α is
acute
Hence find the maximum and minimum
values of $\frac{1}{8\cos x + 6\sin x + 15}$ [0.2, 0.04]
27. Express $\sin x + \cos x$ in the form $R\cos(x - \alpha)$.
Hence, find the greatest value of
 $\sin x + \cos x - 1$. [0.4142]
28. (a) Solve $\cos x + \cos 3x = \cos 2x$, $0 \leq x \leq 360^\circ$
[$x = 45^\circ$, 60° , 135° , 225° , 300° , 315°]
(b) Show that $\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1+\sin\theta}{\cos\theta}$
29. Show that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{7}{9}$
30. (a) Solve $3\sin x + 4\cos x = 2$ for
 $180^\circ \leq x \leq 180^\circ$. [-29.55° , 103.29°]
(b) Show that in any triangle ABC
 $\frac{a^2 - b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$
31. (a) Prove that $\frac{2\tan x}{1+\tan^2 x} = \sin 2x$
(b) Solve $\sin 2x = \cos x$; $0^\circ \leq x \leq 90^\circ$
[$x = 30^\circ$, 90°]
32. (a) Solve the equation
 $8\cos^4 x - 10\cos^2 x + 3 = 0$; $0^\circ \leq x \leq 180^\circ$
[30° , 45° , 135° , 150°]
(b) Prove that $\cos 4A - \cos 4B - \cos 4C =$
 $4\sin 2B \sin 2C \cos 2A - 1$ given that A, B and
C are angles of a triangle
33. Given that $\cos 2A - \cos 2B = -p$ and
 $\sin 2A - \sin 2B = q$, prove that
 $\sec(A+B) = \frac{1}{q} \sqrt{p^2 + q^2}$
34. Solve
(a) $4\sin^2 \theta - 12\sin 2\theta + 35\cos^2 \theta = 0$; for
 $0^\circ \leq \theta \leq 90^\circ$ [74.0°]
(b) $3\cos \theta - 2\sin \theta = 2$, for $0^\circ \leq \theta \leq 360^\circ$
[$\theta = 22.62^\circ$, 270.00°]
35. Solve the equation $\sin 2\theta + \cos 2\theta \cos 4\theta =$
 $\cos 4\theta \cos 6\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$. [$\theta = 0, \frac{3\pi}{16}$]
36. (a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$
for $0 \leq \theta \leq 180^\circ$ [$\theta = 60^\circ$, 120°]
(b) Show that if $\sin(x + \alpha) = p \sin(x - \alpha)$ then
 $\tan x = \left(\frac{p+1}{p-1}\right) \tan \alpha$. Hence solve the
equation $\sin(x + \alpha) = p \sin(x - \alpha)$ for $p = 2$
and $\alpha = 20^\circ$. [$x = 47.52^\circ$]
37. Solve the equation
 $3\tan^2 \theta + 2\sec^2 \theta = 2(5 - 3\tan \theta)$ for

$$0^\circ < \theta < 180^\circ \quad [\theta = 38.66^\circ, 116.57^\circ]$$

38. (a) Show that $\tan 4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}$, where

$$t = \tan \theta$$

(b) Solve the equation

$$\sin x + \sin 5x = \sin 2x + \sin 4x \text{ for}$$

$$0^\circ < x < 90^\circ \quad [x = 60^\circ]$$

39. Solve $2\cos 2\theta - 5\cos \theta = 4$ for

$$0^\circ \leq \theta \leq 360^\circ \quad [\theta = 138.59^\circ, 221.41^\circ]$$

Applications of trigonometry

1. Navigation and GPS

Application: Calculating distances and directions between locations.

Example: Pilots and sailors use trigonometry to determine their course and position using angles and coordinates.

2. Architecture and Construction

Application: Designing buildings, bridges, and tunnels.

Example: Architects use trigonometric ratios to calculate structural angles and ensure stability in designs.

3. Astronomy

Application: Measuring distances between celestial bodies.

Example: Astronomers use trigonometry to calculate the distance from Earth to stars and planets using parallax methods.

4. Engineering

Application: Analyzing forces, motion, and wave patterns.

Example: Civil engineers use trigonometry to calculate load distributions and angles in mechanical systems.

5. Geography and Surveying

Application: Mapping terrain and measuring land.

Example: Surveyors use trigonometry to determine land elevations and distances between points without direct measurement.

6. Computer Graphics and Game Development

Application: Rendering realistic movement and angles.

Example: Trigonometric functions help animate characters, simulate physics, and create 3D environments.

7. Medical Imaging

Application: Interpreting scans and modeling body structures.

Example: Techniques like CT scans and ultrasounds rely on trigonometric calculations to reconstruct internal images.

8. Music and Sound Engineering

Application: Modeling sound waves and frequencies.

Example: Trigonometric functions describe waveforms used in synthesizers and audio processing.

Thank you Dr. Bbosa Science