SIMPLIFIED 'A' LEVEL

MATHEMATICS

Statistics, Numerical Methods and Mechanics

2

By

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Contents

STATISTICS

This is a branch of mathematics dealing with collection, presentation, analysis and interpretation of data.

Types of data

(a) Discrete data

Its Information collected by counting and usually takes integral values that do not lie within a given range

(b) Continuous data

Its Information that takes on values within a given range

DISCRETE OR UNGROUPED DATA

(I) Measure of central tendency

These are values of the distribution that tend to locate the central value. They include mean, median and mode

(a) Mean or average

It is denoted by $\bar{x}$ and defined as $\boxed{\bar{x} = \frac{\sum x}{n}}$

Where x is variable given and n total number of the variables

If assumed mean(working mean) A is given then

$$\boxed{\bar{x} = A + \frac{\sum d}{n}} \quad \text{Where } d = x - A$$

If the frequency, f is given then $\boxed{\bar{x} = \frac{\sum fx}{\sum f}}$ Or $\boxed{\bar{x} = A + \frac{\sum fd}{\sum f}}$

(b) Mode

This is the value of the distribution that appears most

(c) Median

This is the middle value of the distribution obtained after the values have been arranged either in ascending or descending order.

median= $\left(\frac{N}{2}\right)^{th}$ *value*

Examples

1. Given the following sets of values

2, 1, 3, 4, 5, 6, 7, 8, 9, 10, 3, 4, 6, 7, 6, 8, 9, 6, 3, 2

(a) Form a frequency table of ungrouped data

(b) Use your table to find the mean and mode

(c) Find the median value

Solution

(a)

x	f	fx	C.f
1	1	1	1
2	2	4	3
3	3	9	6
4	2	8	8
5	1	5	9
6	4	24	13
7	2	14	15
8	2	16	17
9	2	18	19
10	1	10	20
	$\sum f=20$	$\sum fx=109$	

(b) mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{109}{20} = 5.45$

mode= 6 (appear most)

(c) median= $\left(\frac{N}{2}\right)^{th} = \left(\frac{20}{2}\right)^{th} = 10^{th}$ *value* from c.f

Median= 6

2. Given the information below

x	10	11	12	13	14	15	16	17	18
f	4	2	6	3	7	2	1	2	2

Find ;

(a) Mean value (b) Modal value (c) Median

Solution

x	f	fx	C.f
10	4	40	4
11	2	22	6
12	6	72	12
13	3	39	15
14	7	98	22
15	2	30	24
16	1	16	25
17	2	34	27
18	2	36	29
	$\Sigma f=29$	$\Sigma fx=387$	

(b) mean, $\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{387}{29} = 13.34$

Modal value= 14 (appear most)

(c) median= $\left(\frac{N}{2}\right)^{th} = \left(\frac{29}{2}\right)^{th} = 14.5^{th} value$ from c.f

Median= 13

3. The following are marks obtained by 30 students in a mathematics test marked out of 10

Marks (x)	0	1	2	3	4	5	6	7	8	9	10
Frequency (f)	2	2	3	2	3	4	5	2	1	3	3

(a) Find the mean using a working mean of 5

(b) Find the modal value and median

Solution

x	f	d=x-A	F.d	C.f
0	2	-5	-10	2
1	2	-4	-8	4
2	3	-3	-9	7
3	2	-2	-4	9
4	3	-1	-3	12
5	4	0	0	16
6	5	1	5	21
7	2	2	4	23
8	1	3	3	24
9	3	4	12	27
10	3	5	15	30
	$\Sigma f=30$		$\Sigma fd=5$	

(a) $\bar{x} = A + \frac{\Sigma fd}{\Sigma f} = 5 + \frac{5}{30} = 5.167$

(b) Modal value= 6

median= $\left(\frac{N}{2}\right)^{th} = \left(\frac{30}{2}\right)^{th} = 15^{th} value$ from c.f

Median= 5

(ii) Measure of dispersion

These are measures used to find how the observations are spread out from the mean

(a) Range

This is the difference between the larges value and the smallest value.

Examples

Find the range of the following set of values 45, 64, 54, 76, 86

Solution

$$range = 86 - 45 = 41$$

(b) Variance of a population

The variance of x denoted by $var(x)$ is defined as

$$Var(x) = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$$

or

$$Var(x) = \frac{\Sigma x^2}{n} - \bar{x}^2$$

if the frequency is given then;

$$Var(x) = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2$$

or

$$Var(x) = \frac{\Sigma fx^2}{\Sigma f} - \bar{x}^2$$

(c) Standard deviation

$S.d = \sqrt{var(x)}$, $s.d = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$ Or $s.d = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$

Examples

1. Find the variance and standard deviation of the following; 45, 54, 64, 76, 86

Solution

x	x^2
45	2025
54	2916
64	4096
76	5776
86	7396
$\Sigma x=325$	$\Sigma x^2=22209$

$$Var(x) = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$$

$$Var(x) = \frac{22209}{5} - \left(\frac{325}{5}\right)^2 = 216.8$$

$$s.d = \sqrt{Var(x)} = \sqrt{216.8} = 14.72$$

2. The frequency distribution table shows the marks of some students from st Noa girls school

x	55	63	65	66	70	72	75	80	90
f	2	2	3	1	2	2	4	3	1

Calculated the standard deviation

Solution

x	f	fx	fx^2
55	2	110	6050
63	2	126	7938
65	3	195	12675
66	1	66	4356
70	2	140	9800
72	2	144	10368
75	4	300	22500
80	3	240	19200
90	1	90	8100
	Σf=20	Σfx=1411	$\Sigma fx^2=100987$

$$s.d = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

$$s.d = \sqrt{\frac{100987}{20} - \left(\frac{1411}{20}\right)^2}$$

$$s.d = 8.49 marks$$

Using assumed mean to get variance and standard deviation

$$Var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2 \qquad s.d = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \qquad where\ d = x - A$$

Example

1. The frequency distribution table shows the height of some students at Standard high school

Height	154	155	160	164	171	180
Frequency	4	6	8	5	4	3

Determine the variance and standard deviation of the data using a working mean of 160

Solution

x	f	d=x-A	fd	fd^2
154	4	-6	-24	144
155	6	-5	-30	150
160	8	0	0	0
164	5	4	20	80
171	4	11	44	484
180	3	20	60	1200
	Σf=30		Σfd=70	$\Sigma fd^2=2058$

$$Var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

$$Var(x) = \frac{2058}{30} - \left(\frac{70}{30}\right)^2 = 63.156$$

$$s.d = \sqrt{var\ x} = \sqrt{63.156} = 7.95$$

(d) Quartiles

A quartile is a value that divides given values into four equal parts

q_1 is the lower quartile and its defined by:

$q_1 = \left(\frac{1}{4}N\right)^{th} value$ where N is the sum of all the variables

q_3 is the upper quartile and its defined by:

$q_3 = \left(\frac{3}{4}N\right)^{th} value$ where N is the sum of all the variables

(e) Percentiles

A percentile is a value that divides given values into 100 equal parts

P_{10} is the 10th percentile and its defined by:

$P_{10} = \left(\frac{10}{100}N\right)^{th} value$ where N is the sum of all the variables

P_{85} is the 85th percentile and its defined by:

$P_{85} = \left(\frac{85}{100}N\right)^{th} value$ where N is the sum of all the variables

(f) Deciles

A decile is a value that divides given values into 10 equal parts

D_7 is the 7th decile and its defined by;

$D_7 = \left(\frac{7}{10}N\right)^{th}$ *value* where N is the sum of all the variables

Examples

1. The table below shows the marks obtained by 20 students in a test marked out of 20

marks	10	11	12	13	14	15	16	17	18	19	20
number of students	1	2	2	2	2	4	2	1	2	1	1

find;

(a) Mean mark

(b) Standard deviation

(c) 60th percentile

(d) Interquartile range

Solution

x	f	C.f	fx	fx²
10	1	1	10	100
11	2	3	22	242
12	2	5	24	288
13	2	7	26	338
14	2	9	28	392
15	4	13	60	900
16	2	15	32	512
17	1	16	17	289
18	2	18	36	648
19	1	19	19	361
20	1	20	20	400
	Σf=20		Σfx=294	Σfx²=4470

(a) mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{294}{20} = 14.7$

(b) $s.d = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

$s.d = \sqrt{\frac{4470}{20} - (14.7)^2} = 2.722$

(c) 60th percentile= $\left(\frac{60}{100}x20\right)^{th}$

$= 12^{th} value\ from\ c.f$

60th percentile= 15

(d) $q_1 = \left(\frac{1}{4}x20\right)^{th} = 5^{th} value\ from\ c.f$

$q_1 = 12$

$q_3 = \left(\frac{3}{4}x20\right)^{th} = 15^{th} value\ from\ c.f$

$q_3 = 16$

Interquartile range= $16 - 12 = 4$

2. Given the following scores

7 8 6 8 9 10 6 4 5 6 4

4 6 8 7 10 8 6 11 12 8

(a) Form a frequency distribution table of ungrouped data

(b) Find the standard deviation

(c) Calculate the semi interquartile range

(d) Determine the range of 45th and 90th percentile

Solution

x	f	C.f	fx	fx²
4	3	3	12	48
5	1	4	5	25
6	5	9	30	180
7	2	11	14	98
8	5	16	40	320
9	1	17	9	81
10	2	19	20	200
11	1	20	11	121
12	1	21	12	144
	Σf=21		Σfx=153	Σfx²=1217

(a) $s.d = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

$s.d = \sqrt{\frac{1217}{21} - \left(\frac{153}{21}\right)^2} = 2.21$

(b) $q_1 = \left(\frac{1}{4}x21\right)^{th} = 5.25^{th} value\ from\ c.f$

$q_1 = 6$

$q_3 = \left(\frac{3}{4}x21\right)^{th} = 15.25^{th} value\ from\ c.f$

$q_3 = 8$

(c) Semi-Interquartile range= $\frac{8-6}{2} = 1$

(d) 45th percentile= $\left(\frac{45}{100}x21\right)^{th}$

$= 9.45^{th} value\ from\ c.f$

45th percentile= 7

90th percentile= $\left(\frac{90}{100}x21\right)^{th}$

$= 18.9^{th} value\ from\ c.f$

90th percentile= 10

Range = $10 - 7 = 2$

CONTINUOUS OR GROUPED DATA

This is data whose scores or values are said to be continuous and take interval value

Example

class	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79	80 -89
Number of students	4	5	7	3	6	2	1

Draw a frequency table

Solution

class	Frequency	Class width	Class mark, x	Class boundary
20-29	4	10	24.5	19.5-29.5
30-39	5	10	34.5	29.5-39.5
40-49	7	10	44.5	39.5-49.5
50-59	3	10	54.5	49.5-59.5
60-69	6	10	64.5	59.5-69.5
70-79	2	10	74.5	69.5-79.5
80-89	1	10	84.5	79.5-89.5

Terms used

(a) **Class interval (class)**

These constitute the classes of the distribution. In the table above, the classes are $20-29, 30-39, 40-49, 50-59, 60-69, 70-79, 80-89$.

(b) **Class mark (mark)**

This is the mid value of the class interval. It is normally denoted by x. in the above table. Class mark is $24.5, 34.5, 44.5 \dots \dots$

(c) **Class boundary**

These are values that make classes continuous. In the above table the first class boundary is $(20-0.5)-(29+0.5) = 19.5-29.5$. In this case the lower class boundary is 19.5 and upper class boundary is 29.5

For class interval of $2.0-2.9$, the class boundary is $(2.0-0.05)-(2.9+0.05) = 1.95-2.95$

(d) **Class width**

This is the width of each class boundary. It is given by

$$class\ width = upper\ class\ boundary - lower\ class\ boundary$$

Examples

1. The data below shows the heights in centimeters of 70 students

Height (cm)	130-135	135-140	140-145	145-150	150-160	160-170	170-180
Number of students	10	12	8	9	11	15	5

Construct a frequency distribution for the above data.

Solution

Height	Class boundary	Width	x	f
130-135	130-135	5	132.5	10
135-140	135-140	5	137.5	12
140-145	140-145	5	142.5	8
145-150	145-150	5	147.5	9
150-160	150-160	10	155	11
160-170	160-170	10	165	15
170-180	170-180	10	175	5

2. Use the data below to construct a frequency distribution table

Marks	20-<30	30-<40	40-<50	50-<60	60-<70	70-<80	80-<90
Number of students	10	14	9	18	4	3	2

Solution

marks	Class boundary	Width	x	f
20-<30	20-30	10	25	10
30-<40	30-40	10	35	14
40-<50	40-50	10	45	9
50-<60	50-60	10	55	18
60-<70	60-70	10	65	4
70-<80	70-80	10	75	3
80-<90	80-90	10	85	2

3. The table below shows the ages of 35 people

Age	0-	5-	10-	15-	20-	30-	40-
Frequency	4	6	3	5	7	2	8

Draw a frequency table for the data

Solution

Age	Class boundary	Width	x	f
0- 5	0-5	5	2.5	4
5-	5-10	5	7.5	6
10-	10-15	5	12.5	3
15-	15-20	5	17.5	5
20-	20-30	10	25	7
30-	30-40	10	35	2
40-	40-45	5	42.5	8

Note: The last class width is 5 since it's the most common

4. The table below shows the marks of 40 people

marks	-20	-30	-40	-50	-60	-65	-70
Frequency	8	4	7	10	2	2	7

Draw a frequency table for the data

Solution

marks	Class boundary	Width	x	f
-20	10-20	10	15	8
-30	20-30	10	25	4
-40	30-40	10	35	7
-50	40-50	10	45	10
-60	50-60	10	55	2
-65	60-65	5	62.5	2
-70	65-70	5	67.5	7

Note: The first class width is 10 because it's the most common

5. The data below shows the length in centimeter of different phone calls made by Airtel clients

Length (minutes)	<20	<30	<35	<40	<50	<60
Cumulative frequency	4	20	32	42	48	50

Construct a frequency distribution table

Solution

Length	Class boundary	Width	x	f
<20	10-20	10	15	4
<30	20-30	10	25	16
<35	30-35	5	32.5	12
<40	35-40	5	37.5	10
<50	40-50	10	45	6
<60	50-60	10	55	2

Measure of central tendencies

(a) Measure of Mean or average, $\bar{x}$

The mean of grouped data is given by:

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Where f –frequency

x –mid point value

Or $\bar{x} = A + \frac{\sum fd}{\sum f}$

Where $d = x - A$

(b) Median

Median of grouped data is defined by

$$median = L_b + \left(\frac{\frac{\sum f}{2} - c.f_b}{f}\right)C$$

L_b = lower class boundary of the median class

C = class width of the median class

f = frequency of the median class

$c.f_b$ = cumulative frequency before that one of the median class

Examples

The ages of people in the town were as follows

Age	0- <5	<15	<30	<50	<70	<90
Number of people	44	81	105	146	98	47

Calculate;

(i) mean age

(ii) median age

Solution

Age	x	f	fx	C.f	Class boundary
0 - < 5	2.5	44	110	44	0 - < 5
5 - < 15	10	81	810	125	5 - < 15
15 - < 30	22.5	105	2362.5	230	15 - < 30
30 - < 50	40	146 ✓	5840	376 ✓	30 - < 50 → Median class
50 - < 70	60	98	5880	474	50 - < 70
70 - < 90	80	47	3760	521	70 - < 90
		Σf=521	Σfx=18762.5		

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{18762.5}{521} = 36.012 years$$

$$median = L_b + \left(\frac{\frac{\Sigma f}{2} - c.f_b}{f}\right)C$$

$$\frac{\Sigma f}{2} = \frac{521}{2} = 260.5$$

median class boundary is $30 - < 50$, $f = 146$ and $C = 20$

$$median = 30 + \left(\frac{260.5 - 230}{146}\right)20 = 34.178 \text{ years}$$

(c) Mode

Mode of grouped data with equal class width is defined as

$$mode = L_b + \left(\frac{|\Delta_1|}{|\Delta_1| + |\Delta_2|}\right)C$$

L_b = lower class boundary of the modal class

C = class width of the modal class

Δ_1 = modal frequency − frequency before

Δ_1 = modal frequency − frequency after

Example;

The table below shows the weight of 250 students at Kennedy secondary school

Weight (kg)	44.0 − 47.9	48.0 − 51.9	52.0 − 55.9	56.0 − 59.9	60.0 − 63.9	64.0 − 67.9	68.0 − 71.9	72.0 − 75.9
Frequency	3	17	50	45	46	57	23	9

Find;

(i) Average weight | (ii) Median weight | (iii) Modal weight

Solution

Class boundary	x	f	fx	C.f
43.95 − 47.95	45.95	3	137.85	3
47.95 − 51.95	49.95	17	849.15	20
51.95 − 55.95	53.95	50	2697.5	70
55.95 − 59.95	57.95	45	2607.75	115
59.95 − 63.95	61.95	46	2849.7	161
63.95 − 67.95	65.95	57	3759.15	218
67.95 − 71.95	69.95	23	1608.85	241
71.95 - 75.95	73.95	9	665.55	250
		Σf=250	Σfx=15175.5	

(i) $\bar{x} = \frac{\sum fx}{\sum f} = \frac{15175.5}{250} = 60.702kg$

(ii) $median = L_b + \left(\frac{\frac{\Sigma f}{2} - c.f_b}{f}\right)C$

$$\frac{\Sigma f}{2} = \frac{250}{2} = 125$$

median class boundary is $59.95 - 63.95$, $f = 46$ and $C = 4$

$$median = 59.95 + \left(\frac{125 - 115}{46}\right)4 = 60.82 \text{ kg}$$

(iii) Modal class boundary is $63.95 - 67.95$, since 57 is the highest frequency and $C = 4$

$\Delta_1 = 57 - 46 = 11$, $\Delta_2 = 57 - 46 = 34$

$$mode = 63.95 + \left(\frac{11}{11 + 34}\right)4 = 64.93kg$$

Mode of un equal class width

Mode of grouped data with un equal class width is defined as

$$mode = L_b + \left(\frac{|f.d_1|}{|f.d_1| + |f.d_2|}\right)C$$

Modal class is determined from the highest frequency density

$$frequency\ density = \frac{frequency}{class\ width}$$

L_b = lower class boundary of the modal class

C = class width of the modal class

$f.d_1$ = modal frequency density − frequency density before

$f.d_2$ = modal frequency density − frequency density after

Example;

Given the data below

Marks (x)	10-19	20-24	25-34	35-39	40-54	55-64	65-79
Frequency (f)	4	6	7	3	8	6	6

Find the mode

Solution

01F000

Class boundary	Class width	f	Frequency density
9.5 – 19.5	10	4	0.4
19.5 – 24.5	5	6	1.2
24.5 – 34.5	10	7	0.7
34.5 – 39.5	5	3	0.6
39.5 – 54.5	15	8	0.53
54.5 – 64.5	10	6	0.6
64.5 – 79.5	15	6	0.4
		$\Sigma f=40$	

$$mode = L_b + \left(\frac{|f.d_1|}{|f.d_1| + |f.d_2|}\right)c$$

$$mode = 19.5 + \left(\frac{|1.2 - 0.4|}{|1.2 - 0.4| + |1.2 - 0.7|}\right)5$$

$$mode = 22.58$$

Exercise

1. The table below shows the weight (kg) of 150 patients who visited a certain health centre

Weight (kg)	0-9	10-19	20-29	30-39	40-49	50-59	60-69
Frequency (f)	30	16	24	32	28	12	8

Calculate

(i) Mean | (ii) Mode | (iii) Median

An(29.97kg, 36.17kg, 29.99kg)

2. The table below shows the expenditure of examiners in thousands of shillings during A'level marking exercise

10	11	10	12	14	16	20	25
21	22	13	17	18	24	30	32
27	35	40	44	39	50	54	53
44	37	36	39	52	51	57	15
16	19	34	43	26	38	53	40

(a) Form a frequency distribution table with class interval of 5,000 shillings, the lowest class limit being 10,000

(b) Calculate the;

(i) mean expenditure and | (ii) standard deviation

An(30.75, 13.64)

(c) Draw a histogram to represent the above data and use it to determine the mode

3. The information below shows the amount of money in millions given to some districts in Uganda during Kisanjja akuna muchezzo

Amount	25-<30	30-<40	40-<50	50-<60	60-<80
frequency	4	10	32	3	5

(i) Use the above information to draw a histogram

(ii) Use your histogram to estimate the mode

Measure of dispersion

(a) Variance of a population

The variance of x of grouped data denoted by var(x) is defined as

$$Var(x) = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

or

$$Var(x) = \frac{\sum fx^2}{\sum f} - \bar{x}^2$$

or

$$Var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

where $d = x - A$

(b) Standard deviation

$$S.d = \sqrt{var(x)}$$

or

$$s.d = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

or

$$s.d = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

Examples

1. The table below shows the number of crimes committed by students

Number of crimes	5 - < 10	10 - < 20	20 - < 30	30 - < 50	50 - < 100
Number of students	10	15	25	40	25

Calculate the variance and standard deviation for the number of crimes committed

Solution

Number of crimes	x	f	fx	fx²
5-<10	7.5	10	75	562.5
10-<20	15	15	225	3375
20-<30	25	25	625	15625
30-<50	40	40	1600	64000
50-<100	75	25	1875	140625
		Σf=115	Σfx=4400	Σfx²=224187.5

$$Var(x) = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

$$Var(x) = \frac{224187.5}{115} - \left(\frac{4400}{115}\right)^2 = 485.56$$

$$S.d = \sqrt{var(x)} = \sqrt{485.56} = 22.04$$

2. The table below shows the weight of 250 students at St Noa girls' school

Weight (kg)	44.0 – 47.9	48.0 – 51.9	52.0 – 55.9	56.0 – 59.9	60.0 – 63.9	64.0 – 67.9	68.0 – 71.9	72.0 – 75.9
Frequency	3	17	50	45	46	57	23	9

Using assumed mean of 57.95, find;

(a) Average weight | (b) Variance | (c) Standard deviation

Solution

Weight	x	f	d=x-A	fd	fd²
43.95 – 47.95	45.95	3	-12	-36	432
47.95 – 51.95	49.95	17	-8	-136	1088
51.95 – 55.95	53.95	50	-4	-200	800
55.95 – 59.95	57.95	45	0	0	0
59.95 – 63.95	61.95	46	4	184	736
63.95 – 67.95	65.95	57	8	456	3648
67.95 – 71.95	69.95	23	12	276	3312
71.95 -75.95	73.95	9	16	144	2304
		Σf=250		Σfd=688	Σfd²=12320

(a) $\bar{x} = A + \frac{\sum fd}{\sum f} = 57.95 + \frac{6885}{250} = 60.702kg$

(b) $Var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$

$$Var(x) = \frac{12320}{250} - \left(\frac{6885}{250}\right)^2 = 41.71kg$$

$$s.d = \sqrt{var\ x} = \sqrt{41.71} = 6.46kg$$

Exercise

1. Use the frequency distribution below

Time (min)	0 - < 1	1 - < 2	2 - < 3	3 - < 5	5 - < 10
Frequency	10	15	25	40	25

Calculate the variance and standard deviation. **An(4.99, 2.2)**

2. The frequency table shows the height of 80 students to nearest cm

Height (cm)	150 - 154	155 - 159	160 -164	165 - 169	170-174	175-179	180-184	185-189
Number of students	3	7	10	15	25	12	6	2

Calculate the mean and standard deviation. **An(169.625cm, 7.944cm)**

3. The table below shows the amount of money in millions given to some districts in Uganda

Number of crimes	25 - < 30	30 - < 40	40 - < 50	50 - < 60	60 - < 80
Number of students	4	10	4	3	5

Find the mean and standard deviation **An(44.423 millions, 14.73 millions)**

4. A certain factory produces ball bearings in a certain month according to their diameter as shown below

diameter	91 -93	94 -96	97 - 99	100-102	103-105	106-108	109-111
frequency	4	6	34	40	13	3	5

Calculate the mean and standard deviation using assumed mean of 98. **An(1535.2, 39.18)**

Percentiles and Quartiles

(a) Percentiles

A percentile is a value that divides a given distribution into 100 equal parts.

The 60th percentile denoted by P_{60} is defined as

•

$$P_{60} = L_b + \left(\frac{\frac{60}{100}\Sigma f - c.f_b}{f}\right)C$$

L_b = lower class boundary of the 60th class

C = class width of the 60th class
f = frequency of the 60th class
$c.f_b$ = cumulative frequency before that one of the 60th class

(b) Quartiles

A quartile is a value that divides a given distribution into4 equal parts.

The lower quartile denoted by q_1 is defined as

$$q_1 = L_b + \left(\frac{\frac{1}{4}\Sigma f - c.f_b}{f}\right)C$$

L_b = lower class boundary of the q_1 class

C = class width of the q_1 class
f = frequency of the q_1 class
$c.f_b$ = cumulative frequency before that one of the q_1 class

The upper quartile denoted by q_3 is defined as

$$q_3 = L_b + \left(\frac{\frac{3}{4}\Sigma f - c.f_b}{f}\right)C$$

L_b = lower class boundary of the q_3 class

C = class width of the q_3 class
f = frequency of the q_3 class
$c.f_b$ = cumulative frequency before that one of the q_3 class

$$Inter\ quartile\ range = q_3 - q_1$$

$$semi - Inter\ quartile\ range = \frac{q_3 - q_1}{2}$$

Examples

1. The following table shows the marks obtained by 400 students in a physics test marked out of 100

Marks (%)	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90 - 99
Number of students	4	6	2	5	7	8	5	3

Find

(i) Mean
(ii) Standard deviation
(iii) Median and mode
(iv) Semi interquartile range
(v) 40th and 85th percentile range

Solution

Class boundary	x	f	fx	fx²	C.f
19.5 – 29.5	24.5	4	98	2401	4
29.5 – 39.5	34.5	6	207	7142	10
39.5 – 49.5	44.5	2	89	3961	12
49.5 – 59.5	54.5	5	272.5	14851	17
59.5 – 69.5	64.5	7	451.5	29122	24
69.5 – 79.5	74.5	8	596	44402	32
79.5 – 89.5	84.5	5	422.5	35701	37
89.5 – 99.5	94.5	3	283.5	26791	40
		Σf=40	Σfx=2420	Σfx²=164370	

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{2420}{40} = 60.5\%$$

(a) $$s.d = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

$$s.d = \sqrt{\frac{164370}{40} - (60.5)^2} = 21.19\%$$

(b) $$mode = L_b + \left(\frac{|\Delta_1|}{|\Delta_1| + |\Delta_2|}\right)C$$

$$mode = 69.5 + \left(\frac{|8-7|}{|8-7| + |8-5|}\right)10 = 72\%$$

$$median = L_b + \left(\frac{\frac{\Sigma f}{2} - c.f_b}{f}\right)C$$

$$\frac{\Sigma f}{2} = \frac{40}{2} = 20$$

median class boundary is $60 - 69$, $f = 7$ and $C = 10$

$$median = 59.5 + \left(\frac{20 - 17}{7}\right)10 = 63.786\%$$

(c) $$q_1 = L_b + \left(\frac{\frac{1}{4}\Sigma f - c.f_b}{f}\right)C$$

$$\frac{\Sigma f}{4} = \frac{40}{4} = 10$$

q_1 class boundary is $30 - 39$, $f = 6$ and $C = 10$

$$q_1 = 29.5 + \left(\frac{10 - 4}{6}\right)10 = 39.5\%$$

$$q_3 = L_b + \left(\frac{\frac{3}{4}\Sigma f - c.f_b}{f}\right)C$$

$$\frac{3£f}{4} = \frac{3}{4}x40 = 30$$

q_3 *class boundary is* $70 - 79$, $f = 8$ *and* $C = 10$

$$q_3 = 69.5 + \left(\frac{30 - 24}{8}\right)10 = 77\%$$

$$semi - Inter\ quartile\ range = \frac{q_3 - q_1}{2}$$

$$S.I.R = \frac{77 - 39.5}{2} = 18.75\%$$

(d) $P_{40} = L_b + \left(\frac{\frac{40}{100}£f - c.f_b}{f}\right)C$

$$\frac{40}{100}£f = \frac{40}{100}x40 = 16$$

P_{40} *class boundary is* $50 - 59$, $f = 5$ *and* $C = 10$

$$P_{40} = 49.5 + \left(\frac{16 - 12}{5}\right)10 = 57.5\%$$

$$P_{85} = L_b + \left(\frac{\frac{85}{100}£f - c.f_b}{f}\right)C$$

$$\frac{85}{100}£f = \frac{85}{100}x40 = 34$$

P_{85} *class boundary is* $80 - 89$, $f = 5$ *and* $C = 10$

$$P_{85} = 79.5 + \left(\frac{34 - 32}{5}\right)10 = 83.5\%$$

40^{th} and 85^{th} range= $83.5 - 57.5 = 26\%$

2. The height in centimeters of 50 flowers were recorded and gave the following results

												96	100	106	118	101
86	101	114	118	87	92	93	105	102	97	93	101	111	96	100	106	118
107	96	101	104	92	99	107	98	105	113	103	108	92	109	95	100	103
99	106	116	101	105	83	108	92	116	117	102	100	110	88			

(a) Construct a frequency distribution table, using equal class intervals of width of 5cm taking the first interval as 85-89

(b) Use your table to find

(i) Mode and median

(ii) Semi interquartile range

(iii) 70^{th} percentile

Solution

Height (cm)	Class boundary	x	f	C.f
85 - 89	85.5 – 89.5	24.5	4	4
90 - 94	89.5 – 94.5	34.5	6	10
95 - 99	94.5 – 99.5	44.5	7	17
100 - 104	99.5 – 104.5	54.5	13	30
105 - 109	104.5 – 109.5	64.5	10	40
110 - 114	109.5 – 114.5	74.5	5	45
115 - 119	114.5 – 119.5	84.5	5	50
			Σf=50	

(a) $mode = L_b + \left(\frac{|\Delta_1|}{|\Delta_1| + |\Delta_2|}\right)C$

$$mode = 99.5 + \left(\frac{|13 - 7|}{|13 - 7| + |13 - 10|}\right)5$$

mode $= 102.84cm$

$$median = L_b + \left(\frac{\frac{£f}{2} - c.f_b}{f}\right)C$$

$$median = 99.5 + \left(\frac{\frac{50}{2} - 17}{13}\right)5 = 102.58cm$$

(b) $q_1 = L_b + \left(\frac{\frac{1}{4}£f - c.f_b}{f}\right)C$

$$q_1 = 94.5 + \left(\frac{\frac{50}{4} - 10}{7}\right)5 = 96.29cm$$

$$q_3 = L_b + \left(\frac{\frac{3}{4}£f - c.f_b}{f}\right)C$$

$$q_3 = 104.5 + \left(\frac{\frac{3}{4}x50 - 30}{10}\right)5 = 108.25$$

$$S.I.R = \frac{108.25 - 96.29}{2} = 5.98cm$$

(c) $P_{70} = L_b + \left(\frac{\frac{70}{100}£f - c.f_b}{f}\right)C$

$$P_{70} = 104.5 + \left(\frac{\frac{70}{100}x50 - 30}{10}\right)5 = 107cm$$

3. Given the information in the table below

classes	1.0 – 1.9	2.0 – 2.9	3.0 – 3.9	4.0 – 4.9	5.0 – 5.9	6.0 – 6.9	7.0 – 7.9	8.0 – 8.9
Frequency	3	7	8	2	15	5	6	4

find

(a) The mean value and Standard deviation using assumed mean of 4.45

(b) Mode

(c) Median

(d) Interquartile range

(e) 50^{th} percentile

Solution

Class boundary	x	f	d	fd	fd^2	C.f
0.95 - 1.95	1.45	3	-3	-9	27	3
1.95 - 2.95	2.45	7	-2	-14	28	10
2.95 - 3.95	3.45	8	-1	-8	8	18
3.95 - 4.95	4.45	2	0	0	0	20
4.95 - 5.95	5.45	15	1	15	15	35
5.95 - 6.95	6.45	5	2	10	20	40
6.95 - 7.95	7.45	6	3	18	54	46
7.95 - 8.95	8.45	4	4	10	64	50
		$\Sigma f=50$		$\Sigma fd=28$	$\Sigma fd^2=216$	

(i) $\bar{x} = A + \frac{\sum fd}{\sum f} = 4.45 + \frac{28}{50} = 5.01$

(j) $Var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2 = \frac{216}{50} - \left(\frac{28}{50}\right)^2 = 4.0064$

$s.d = \sqrt{var\ x} = \sqrt{4.0064} = 2.002$

(i) $mode = 4.95 + \left(\frac{15-2}{15-2+15-5}\right)1 = 5.52$

(ii) $median = L_b + \left(\frac{\frac{\pounds f}{2} - c.f_b}{f}\right)C$

$median = 4.95 + \left(\frac{\frac{50}{2} - 20}{15}\right)1 = 5.28$

(iii) $q_1 = L_b + \left(\frac{\frac{1}{4}\pounds f - c.f_b}{f}\right)C$

$\frac{\pounds f}{4} = \frac{50}{4} = 12.5$

$q_1 = 2.95 + \left(\frac{12.5 - 10}{8}\right)1 = 2.64$

$q_3 = L_b + \left(\frac{\frac{3}{4}\pounds f - c.f_b}{f}\right)C$

$\frac{3\pounds f}{4} = \frac{3}{4}x50 = 37.5$

$q_3 = 5.95 + \left(\frac{37.5 - 35}{15}\right)1 = 6.45$

$Inter\ quartile\ range = 6.45 - 2.64 = 3.81$

(iv) $P_{50} = L_b + \left(\frac{\frac{70}{100}\pounds f - c.f_b}{f}\right)C$

$\frac{50}{100}\pounds f = \frac{50}{100}x50 = 25$

$P_{50} = 4.95 + \left(\frac{25 - 20}{15}\right)1 = 5.28$

4. Given the information in the table below

class	20 - 29	30 - 34	35 - 44	45 - 64	65 - 74	75 - 84
Frequency	5	5	12	20	10	8

Find ;

(a) Mean value

(b) Standard deviation

(c) Mode

(d) Median

(e) Interquartile range

(f) 90th percentile

Solution

Class boundary	Class width	x	f	F.d	fx	fx^2	C.f
19.5 – 29.5	10	24.5	5	0.5	122.5	3001.25	5
29.5 – 34.5	5	32	5	1	160	5120	10
34.5 – 44.5	10	39.5	12	1.2	474	18723	22
44.5 – 64.5	20	54.5	20	1	1090	59405	42
64.5 – 74.5	10	69.5	10	1	695	48302.5	52
74.5 – 84.5	10	79.5	8	0.8	636	50562	60
			$\Sigma f=60$		$\Sigma fx=3177.5$	$\Sigma fx^2=185113.75$	

(a) $\bar{x} = \frac{\sum fx}{\sum f} = \frac{3177.5}{60} = 52.96$

(b) $s.d = \sqrt{\frac{185113.75}{60} - (52.96)^2} = 16.75$

(c) $mode = L_b + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$

$mode = 34.5 + \left(\frac{1.2 - 1}{1.2 - 1 + 1.2 - 1}\right)10 = 39.5$

(d) $median = L_b + \left(\frac{\frac{\pounds f}{2} - c.f_b}{f}\right)C$

$median = 44.5 + \left(\frac{\frac{60}{2} - 22}{20}\right)20 = 52.5$

(v) $q_1 = L_b + \left(\frac{\frac{1}{4}\pounds f - c.f_b}{f}\right)C$

$q_1 = 34.5 + \left(\frac{\frac{60}{4} - 10}{12}\right)10 = 38.67$

$q_3 = L_b + \left(\frac{\frac{3}{4}\pounds f - c.f_b}{f}\right)C$

$q_3 = 64.5 + \left(\frac{\frac{3}{4}x60 - 42}{10}\right)10 = 67.5$

$Inter\ quartile\ range = 67.5 - 38.67 = 28.83$

(vi) $P_{90} = L_b + \left(\frac{\frac{90}{100}\pounds f - c.f_b}{f}\right)C$

$P_{90} = 74.5 + \left(\frac{\frac{90}{100}x60 - 52}{8}\right)10 = 77$

Exercise

1. The data below gives the weight of some students from a certain school

mass	40 - 44	45 - 49	50 - 54	55 - 59	60 - 64	65 - 69	70 - 74
Frequency	3	30	29	33	13	1	1

Draw an ogive and use it to estimate
   (a) Median mass
   (b) 10 – 90 percentile range
   (c) Interquartile range
   (d) The number of students who weigh above 47kg

An(53.29, 45.83, 15.21, 9.03, 93 students)

2. The table below shows the wages of 40 workers of a certain factory in thousands if shilling per day

2.0	2.1	2.0	2.2	6.4	2.6	3.0	3.5
3.1	3.2	2.3	2.7	2.8	3.4	4.0	3.2
3.7	4.5	5.0	5.4	4.9	6.0	6.4	6.3
5.4	4.7	4.6	4.9	6.2	6.2	6.7	2.5
2.6	2.9	4.4	5.3	4.8	4.8	6.3	5.0

   (a) Form a frequency distribution table with interval of 0.5 thousands shillings starting with 2 thousand shillings
   (b) Calculate the
      (i) Mean wage
      (ii) Standard deviation wage
      (iii) Modal wage
      (iv) Median wage

An(4187.5 /=, 1403/=, 6219.23/=, 4200/=)

Graphs

(a) Grouped data with equal class width

(i) Histogram

This is a graph consisting of vertical bars plotted for class frequencies against class boundary. Histogram is used to determine the mode

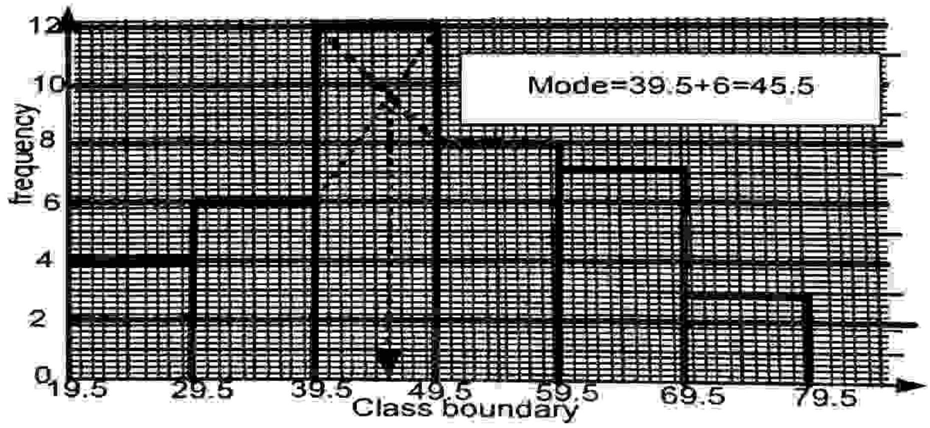
Examples

Given the data below

Marks	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79
Frequency	4	6	12	8	7	3

Draw a histogram and use it to determine the mode

Solution



(ii) Cumulative frequency curve(ogive)

This a curve of cumulative frequency against class boundary the ogive is used to determine median, quartiles, percentiles and deciles

Note The values of cumulative frequency must be plotted against upper class boundary and first value of the lower class boundary must be plotted against cumulative frequency zero.

Examples

Given the data below

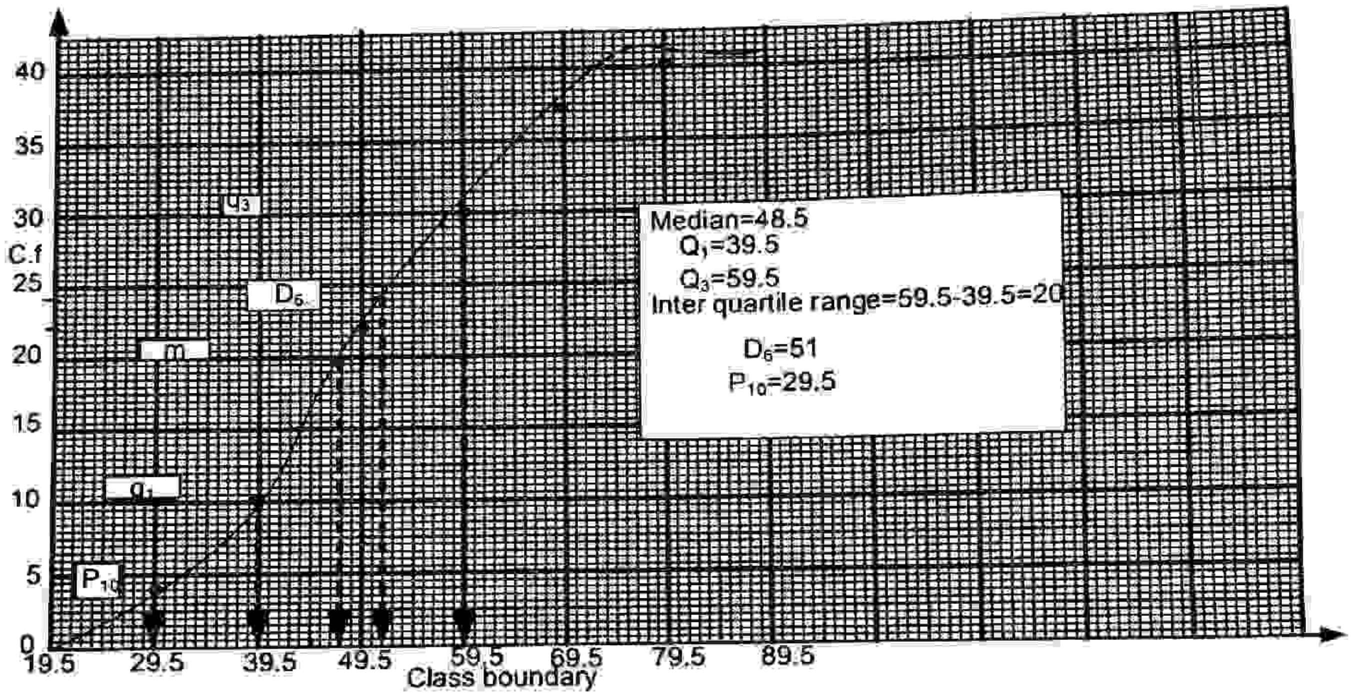
Marks	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79
Frequency	4	6	12	8	7	3

Draw an ogive and use it to determine ;

(a) Median (b) Interquartile range (c) 10th percentile

Solution

Class boundary	19.5 – 29.5	29.5 – 39.5	39.5 – 49.5	49.5 – 59.5	59.5 – 69.5	69.5 – 79.5
C.f	4	10	22	30	37	40



More examples

1. The table below shows the marks obtained by 60 students in a mathematics test of a certain school

Marks	20 - 24	25 - 29	30 - 34	35 - 39	40 - 44	45 - 49	50 - 54	55 - 59	60 - 64
Number of students	10	6	10	15	5	3	4	3	4

(a) Draw a histogram and use it to determine the mode

(b) Draw a cumulative frequency curve and use it to estimate

(i) Median

(ii) Interquartile range

(iii) 60th percentile

(iv) Pass mark if 50 students passed

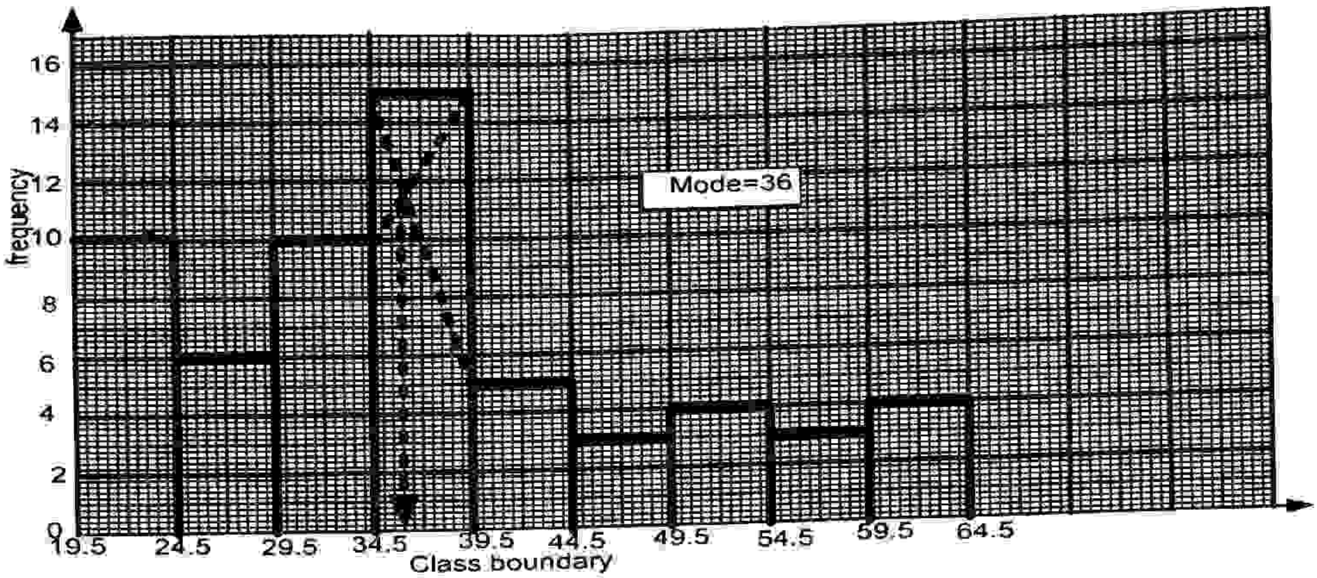
(v) The pass mark if 20 students failed

(vi) The middle range of 40% of the students who did the test

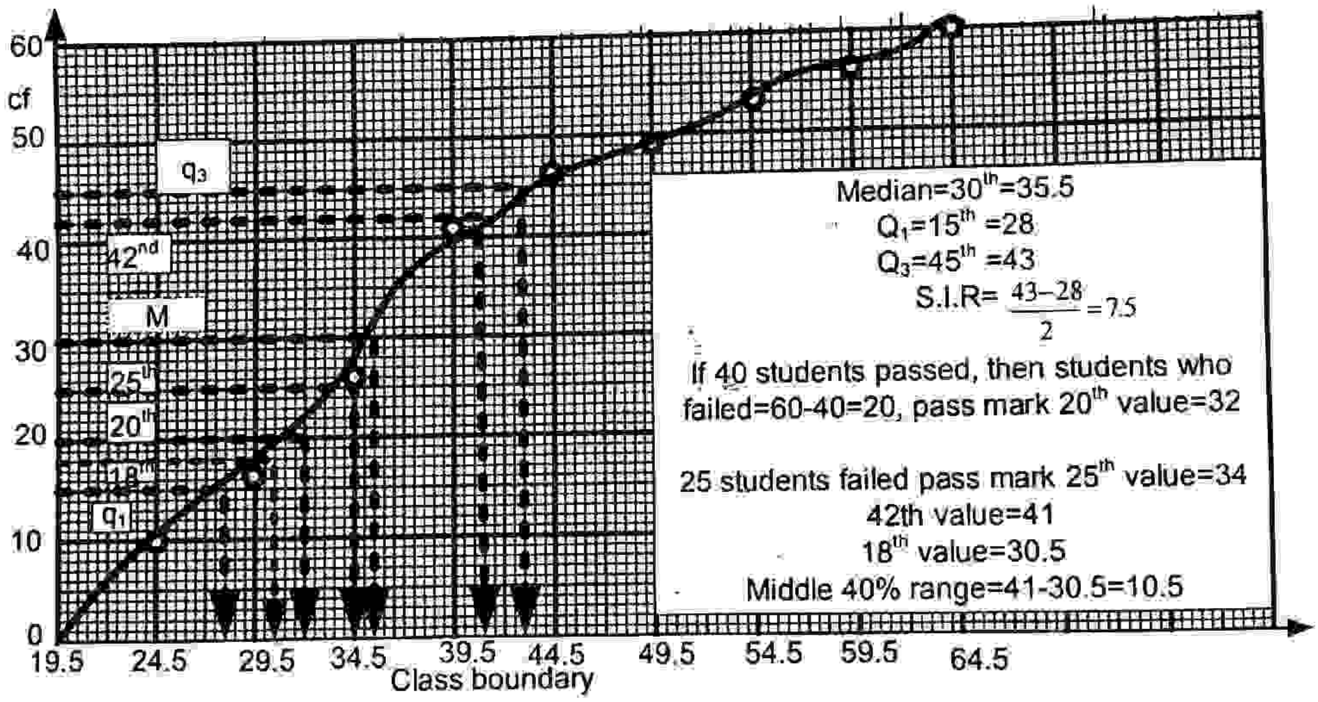
Solution

Class boundary	f	C.f
19.5 – 24.5	10	10
24.5 – 29.5	6	16
29.5 – 34.5	10	26
34.5 – 39.5	15	41
39.5 – 44.5	5	46
44.5 – 49.5	3	49
49.5 – 54.5	4	53
54.5 – 59.5	3	56
59.5 – 64.5	4	60

(a) Histogram



(b) Ogive



middle 40% of students= $\frac{40}{100}x60 = 24$ students

$\therefore$ remainder of the students= $\frac{60-24}{2} = 18$ on either side

we have to get 18th value and $(60-18)$ =42th

2. The table below shows the marks obtained by students in a certain school

Class	30 - < 40	40 - < 50	50 - < 60	60 - < 70	70 - < 80	80 - < 90	90 - < 100
Frequency	4	6	8	12	10	7	3

(a) Plot a histogram and use it to estimate the mode

(b) Plot an ogive and use it to estimate

(i) The median

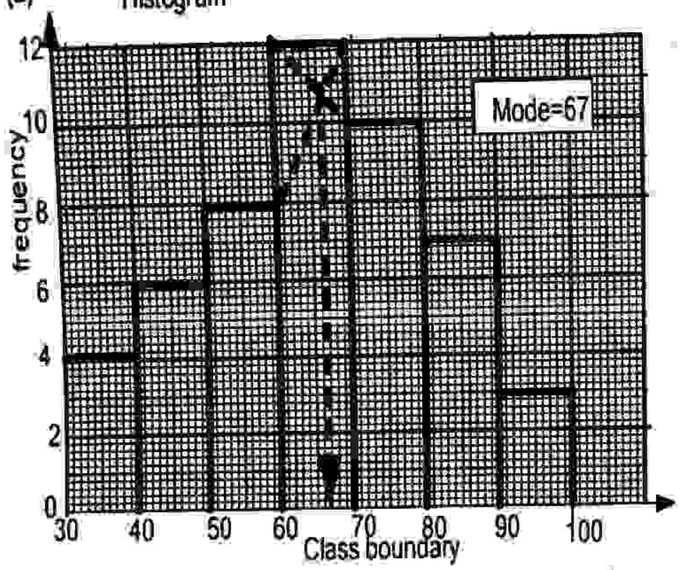
(ii) Interquartile range

(iii) 90th percentile

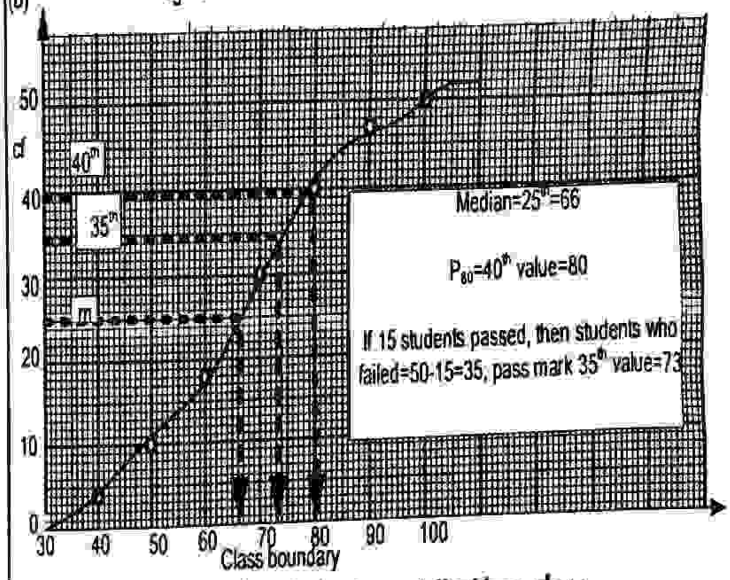
(iv) Pass mark if 15 students passed

Solution

(a) Histogram



(b) ogive



3. The table below shows the time taken by parents to travel to school during visitation day

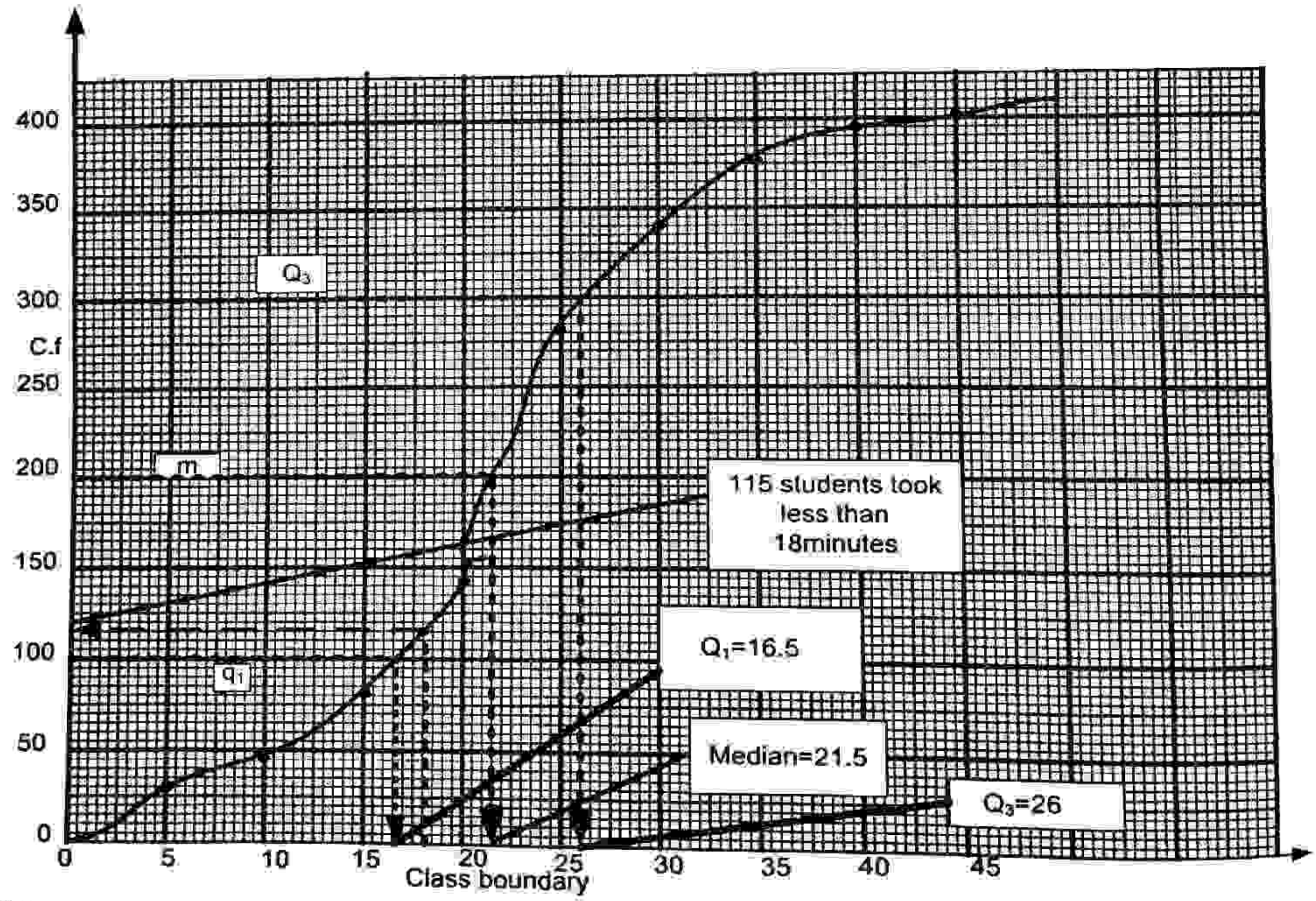
Time taken (minutes)	< 5	< 10	< 15	< 20	<25	< 30	< 35	< 40	< 45
Cumulative frequency	28	45	78	143	280	340	375	395	400

(a) Draw an ogive and use it to find

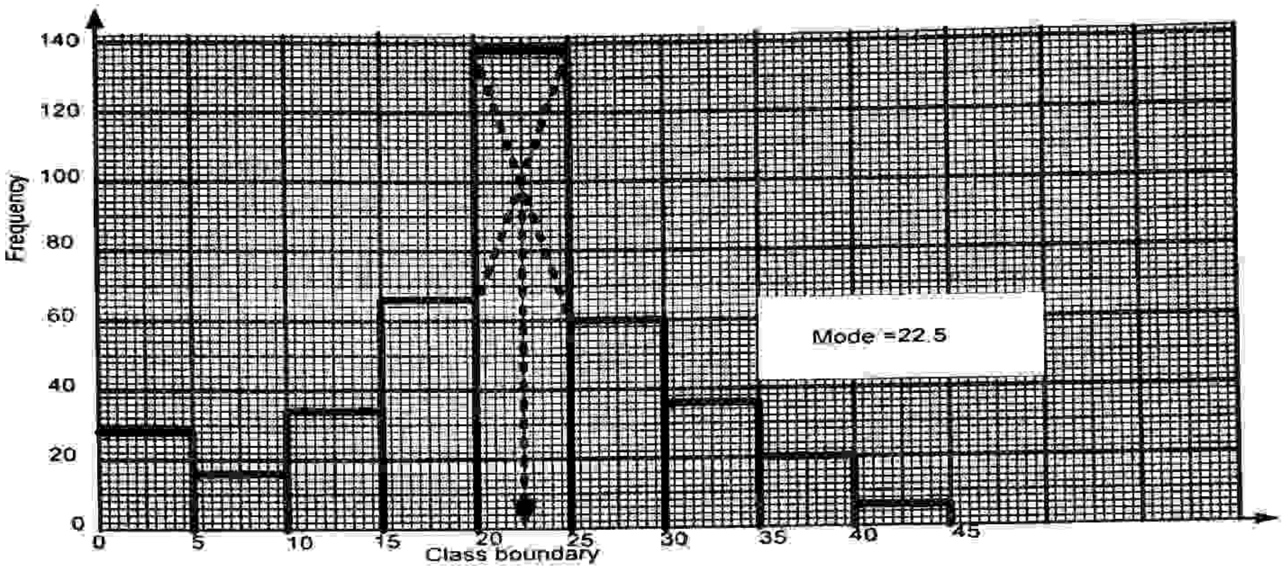
(i) The median time (ii) Interquartile range of the time

(iii) Number of students who spent less than 18 minutes

(b) Draw a histogram and use it to find the modal time



Histogram



GROUPED DATA WITH UN EQUAL CLASS WIDTH

(I) Histogram

This is a graph of frequency density against class boundary

(II) Ogive

This a graph of cumulative frequency against class boundary

Examples

1. The data shows the length in centimeters for different calendars produced by a printing press. A cumulative frequency distribution was formed

Length (cm)	< 20	< 30	< 35	< 40	<50	< 60
Cumulative frequency	4	20	32	42	48	50

(a) Construct a frequency distribution table

(b) Draw a histogram and use it to estimate the modal length. **An(33)**

(c) Find the mean length of the calendar **An(32.1)**

2. The table below shows the ages of employees in a company

Age (years)	< 15	< 20	< 30	< 40	<50	< 60	< 65
Cumulative frequency	0	17	39	69	87	92	98

(a) Construct a cumulative frequency curve

(b) Use the curve to estimate the semi-interquartile range **An(9.5)**

(c) Calculate the mean age of the employees **An(43.26)**

Exercise 1

1. The table shows the number of people in millions in the different age groups in a certain country

Age	Below 10	10 - under 20	20- under 30	30- under 40	40- under 50	50- under 70	70- under 90
Population (million)	2	8	10	14	10	5	1

Calculate the; **Uneb 1988 P1 No.15**

(i) Mean age (ii) Mode (iii) standard deviation

(iv) Draw a histogram to represent the above data **An (i) 35 (ii) 35 (iii) 13.15**

2. The frequency distribution table shows the weight of 100 children measured to the nearest kg

weight	10 - 14	15 - 19	20 - 24	25 - 29	30 - 34	35 - 39	40 - 44	45 - 49
Number of children	5	9	12	18	25	15	10	6

(a) Calculate the mean and standard deviation **Uneb 1989 P1 No.14**

(b) Draw a cumulative frequency curve and use it to estimate

   (i) Median

   (ii) Number of children with weight above 37kg

   An (a) 30.2kg and 8.987kg (i) 31kg (ii) 33

3. The frequency distribution table shows the heights of 80 prisoners measured to the nearest cm

Height	150 - 154	155 - 159	160 - 164	165 - 169	170 - 174	175 - 179	180 - 184	185 - 189
Number of prisoners	3	7	10	15	25	12	6	2

(a) Calculate **Uneb 1990 P1 No.4**

   (i) the mean height

   (ii) standard deviation for the height of the prisoners

(b) Draw a cumulative frequency curve and use it to estimate semi interquartile range

An (a) (i) 169.625cm (ii) 7.955cm (b) 5cm

4. The table below shows the expenditure on examiners in thousands of shillings during the year 1987 ordinary level marking exercise **Uneb 1991 P1 No.14**

10	11	10	12	14	16	20	25
21	22	13	17	18	24	30	32
27	35	40	44	39	50	54	53
44	37	36	39	52	51	57	15
16	19	34	43	26	38	53	40

(a) Form a frequency distribution table with interval of 5000 shillings starting with 10,000 shillings

(b) Draw a histogram to represent the above data and superimpose a frequency polygon

(c) Calculate:

   (i) The mean expenditure

   (ii) Standard deviation **An (i) 30.75 (ii) 13.636**

5. The table shows the weekly earnings of a random sample of workers in a soap factory in Kampala

Weekly earnings	Number of workers
Under 1500	1
1500 and under 2000	4
2000 and under 2500	28
2500 and under 3000	42
3000 and under 3500	33
3500 and under 4000	18
4000 and under 4500	13
4500 and under 5000	9
5000 and above	2

Calculate the; **Uneb 1992 P1 No.14**

   (i) The percentage number of workers earning shs. 3050 and above

   (ii) Mean weekly earnings and standard deviation of the distribution

   (iii) If the wages are increased by 10%, determine the new mean weekly wage and the new standard deviation **An (i) 45% (ii) 3133.333 (iii) 3446.667, 879.30**

6. The table below shows the marks scored in general paper by some students in mock examiners from a certain school **Uneb 1993 No.14**

Marks	31 - 40	41 - 50	51 - 60	61 - 70	71 - 80	81 - 90
Frequency	12	18	14	8	6	2

(a) Draw a histogram and use it to estimate the modal mark. **An(46.5)**

(b) Find the mean, median and standard deviation **An(52.833, 50.5, 13.646)**

7. The table below shows the weight of some fresher's who under went medical examination in a certain university

Weight (kg)	40 - 44	45 - 49	50 - 54	55 - 59	60 - 64	65 - 69	70 - 74	75 - 79	80 - 84
Number of students	3	10	16	10	4	5	4	6	1

(a) Calculate **Uneb 1994 P1 No.14**

   (i) the mode

   (ii) median and mean weight of the students

(b) Draw a cumulative frequency curve and use it to estimate semi interquartile range

An (a) (i) 52kg (ii) 55kg, 57.984kg (b) 16kg

8. The table below shows the amount of cotton (100′s of bales) produced by g Grower's Union over a period of time **Uneb 1995 No.14**

70	41	34	55	45	66	73	77	80	30
50	45	72	30	27	70	55	70	85	70
30	50	60	53	40	45	35	55	20	81
25	51	35	62	60	50	45	35	50	89
53	23	28	65	68	50	65	34	35	76

(a) Form a frequency distribution table starting with 20 – 29 class and using equal class intervals
(b) Draw a cumulative frequency curve for the data and hence estimate the median production
(c) Calculate:
  (i) The mean production
  (ii) Standard deviation of the production **An (b) 5450 bales (c) 5370 bales, 1798 bales**

9. A certain factory produces ball bearings. A sample of the bearings from the factory produced the following results **Uneb 1996 No.15**

Diameter of bearings (mm)	90 - 93	94 - 96	97 - 99	100 - 102	103 - 105	106 - 108
Frequency	4	6	34	40	13	3

(a) Determine the mean and variance of the diameter of the sample bearings. **An(99.83mm, 9.3411mm^2)**
(b) Estimate the median surface area of the bearings produced in the factory **An(31293mm^2)**

10. The frequency distribution table shows the heights of 5.1 students measured to the nearest cm

Height	151 - 153	154 - 156	157 - 159	160 - 162	163 - 165	166 - 168
frequency	2	14	13	13	2	1

(a) Calculate **Uneb 1997 P1 No.11**
  (i) the mean height (ii) standard deviation for the height of the students
(b) Draw a cumulative frequency curve and use it to estimate the median and interquartile range
An (a) (i) 159.133cm (ii) 3.222cm (b) 158cm, 5.2cm

11. The table below shows the ages of people in a certain town **Uneb 1998 P1 No.15**

Age	0 - < 5	5 - < 15	15 - < 30	30 - < 50	50 - < 70	70 - < 90
Number (thousands)	4.4	8.1	10.5	14.6	9.8	4.7

(a) Plot a histogram and use it to estimate the modal age interval
(b) estimate
  (i) average age of the town
  (ii) number of people below the age of 18 years
  (iii) median **An (a) 0–< 5 (b) 36.013 years, 14600 years, 34.178 year**

12. The frequency distribution table shows the weight of 150 patients measured to the nearest kg

weight	0 - 19	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79
Number of patients	30	16	24	32	28	12	8

(a) Calculate the mean and modal weight **Uneb 2001 No.12**
(b) Plot an ogive and use it to estimate
  (i) Median and semi inter quartile range for the weight of the patients
  (ii) Probability that patients weighing between 13kg and 52.5kg is chosen
An (a) 38.833kg and 46.167kg (b) (i) 41.5kg,14.75kg (ii) 0.573

13. The table below shows the cumulative distribution of the ages (in years) of 400 students

Age (years)	< 12	< 13	<14	< 15	<16	< 17	< 18	< 19
Cumulative frequency	0	27	85	215	320	370	395	400

(a) Construct a cumulative frequency curve **Uneb 2002 No.7**
(b) Use the curve to estimate the:
  (i) Median age **An(14.3)** (ii) 20th and 80th percentile range **An(2.1)**

14. The table below shows the time taken by students to solve a mathematics problem **Uneb 2002 No.14**

Time (min)	5 - 9	10 - 14	15 - 19	20 - 24	25 - 29	30 - 34
Frequency	5	14	30	17	11	3

(a) Draw a histogram and use it to estimate the modal time **An(17.3)**
(b) Find the mean time and standard deviation of solving the problem **An(18.5min, 5.9896min)**

15. The frequency distribution table shows the heights of 5.6 students measured to the nearest cm

Height	149 - 152	153 - 156	157 - 160	161 - 164	165 - 168	169 - 172	173 - 176
frequency	5	17	20	25	15	6	2

(a) Calculate **Uneb 2004 No.14**

(i) the mean height

(ii) standard deviation for the height of the students

(b) Draw a cumulative frequency curve and use it to estimate the mean and range of the height of the middle 60% of the candidates **An (a) (i) 160.9cm (ii) 5.5873cm (b) 161.1cm, 10cm**

16. The table below shows the weight of some 5.5 students from a certain school **Uneb 2005 No.15**

Weight (kg)	50 - 53	54 -57	58 - 61	62 - 65	66 - 60	70 - 73	74 - 77	78 - 81
Number of students	3	8	12	18	11	5	2	1

(a) Calculate

(i) Mean

(ii) Standard deviation of the students weight

(b) Draw a cumulative frequency curve and use it to estimate

(i) Median weight

(ii) Number of students who weight between 58.9kg and 66.7kg

An (a) (i) 63.1kg (ii) 6kg, (b) (i) 63.1kg (ii) 29 students

17. The table below shows the weight of some animals from a certain firm **Uneb 2006 No.12**

Weight (kg)	21 - 25	26 -30	31 - 35	36 - 40	41 – 50	51 - 65	66 - 74
Number of students	10	20	15	10	30	45	5

(a) Calculate

(i) Mode

(ii) Standard deviation of the students weight

(b) Draw a cumulative frequency curve and use it to estimate semi interquartile range

An (a) (i) 28.833kg (ii) 11.772kg, (b) 12kg

18. The frequency distribution table shows the amount of money in thousands of shillings that was paid to teachers during a work shop. **Uneb 2008 No.9**

Amount (sh 000's)	110 - 114	115 - 119	120 - 129	130 - 134	135 - 144	145 - 159
frequency	13	20	32	17	16	12

(a) Calculate

(i) the mean amount

(ii) median amount

(b) Draw a histogram and use it to estimate the modal amount

An (a) (i) 126375/= (ii) 128,000/= (b) 118,000/=

19. The table below shows the wages of 40 workers of a certain factory in millions of shilling per annum

1.0	1.1	1.0	1.2	5.4	1.6	2.0	2.5
2.1	2.2	1.3	1.7	1.8	2.4	3.0	2.2
2.7	3.5	4.0	4.4	3.9	5.0	5.4	5.3
4.4	3.7	3.6	3.9	5.2	5.1	5.7	1.5
1.6	1.9	3.4	4.3	2.6	3.8	5.3	4.0

(a) Form a frequency distribution table with interval of 0.5 million shillings starting with 1 million shillings

(b) Calculate **Uneb 2009 No.11**

(i) Mean income

(ii) Standard deviation

(c) Draw a histogram and use it to estimate the modal income

An (b) (i) 3,175,000/= (ii) 1,413,992.374/= (c) 5,200,000/=

20. The table below shows the marks obtained by students in a physics test. **Uneb 2010 No.10**

Marks(%)	25 - 29	30 -34	35 - 39	40 - 44	45 - 49	50 -54	55 - 59	60 - 64	65 - 69	70 -74
Number of students	9	12	10	17	13	25	18	14	8	8

(a) Calculate

(i) Mean

(ii) Standard deviation of the students mark

(b) Draw a histogram and use it to estimate modal mark

An (a) (i) 49.463% (ii) 12.424% (b) 52.5 %

21. The table below shows the marks obtained by 200 students in an examination. **Uneb 2012 No.9**

Marks(%)	10 - 19	20 -29	30 - 39	40 - 49	50 - 59	60 -69	70 - 79	80 - 89
Number of students	18	34	58	42	24	10	6	8

(a) Calculate

(i) Mean mark

(ii) Modal mark

(b) Draw a cumulative frequency curve for the data and use it to estimate the lowest mark for a distinction one if the top 5% of the candidates qualify for the distinction

An (a) (i) 40.2% (ii) 35.5% (b) 75 %

22. The time taken for 55 students to have their lunch to the nearest minute are given in the table below

Time (minutes)	3 - 4	5 - 9	10 - 19	20 -29	30 - 44
Number of children	2	7	16	21	9

(a) Calculate the mean time for the students to have lunch **Uneb 2017 No.12**

(b) Draw a histogram for your data and use it to estimate the modal time for the students to have lunch **An((i) 20.65, (ii) 22minutes)**

23. The frequency distribution below shows the ages of 240 students admitted to a certain University. **Uneb 2018 No.9**

Age (years)	18-<19	19 - <20	20 - <24	24 - <26	26 - <30	30 - <32
Number of students	24	70	76	48	16	6

(a) Calculate the mean age of the students. **An(22.1458years)**

(b) (i) Draw a histogram for the given data.

(ii) Use the histogram to estimate the modal age. **An(19.58years)**

PRICE INDEX

A price index is a statistical number which represents a change in price with respect to time, environment and other characteristics.

Base year

This is the year or period against which all other years or periods are compared

Current year

This is the year or period for which the index is to be calculated

Types of price indices

- Simple price indices
- Simple Aggregate price indices
- Weighted aggregate price indices / composite price indices
- Average weighted price index/ cost of living index

(a) Simple price indices

Its given by $simple\ price\ index = \frac{P_1}{P_0}x100\%$

Where $P_1 - price\ in\ the\ current\ year\ \ and\ \ P_0 - price\ in\ the\ base\ year$

Examples

1. A loaf of bread costed $sh.1200/=$ in 2008 and $sh.1800/=$ in 2014. Taking 2008 as the base year, find the price relative in 2014.

Solution

$$price\ index = \frac{P_1}{P_0}x100\% = \frac{1800}{1200}x100\% = 150$$

2. In 2016, the price index of a commodity using 2015 as the base was 180. In 2017, the price index using 2016 as the base year was 150. What is the price index in 2017 using 2015 as the base year.

Solution

$$\frac{P_{2016}}{P_{2015}}x100\% = 180$$

$$\frac{P_{2016}}{P_{2015}} = 1.8 \ldots\ldots(i)$$

$$\frac{P_{2017}}{P_{2016}}x100\% = 150$$

$$\frac{P_{2017}}{P_{2016}} = 1.5 \ldots\ldots\ldots(ii)$$

$(i)x(ii)$

$$\frac{P_{2016}}{P_{2015}}x\frac{P_{2017}}{P_{2016}} = 1.8x1.5$$

$$\frac{P_{2017}}{P_{2015}} = 2.7$$

$$\frac{P_{2017}}{P_{2015}}x100\% = 2.7x100 = 270$$

3. The wages of nurses in Uganda in 2010was 350,000/= the wage of the nurse in 2015 was increased by 150,000/=. Using 2010 as the base year calculate the nurses wage index in 2015.

Solution

$$Wage\ index = \frac{W_1}{W_0}x100\% = \frac{500000}{350000}x100\% = 142.9$$

4. The table below shows the price and quantities of selected items consumed in the period 2013 – 2015

	Price (shs) per kg			quantities		
Item	2013	2014	2015	2013	2014	2015
Salt	1000	1500	1900	200	80	70
Beef	5000	8000	10000	300	70	62
Goats Meat	7000	9000	12000	250	100	70
Beans	3000	5000	5500	150	120	80

(a) By using 2013 as the base year, find;
   (i) The price relative of all the items in 2014
   (ii) The simple quantum index of all the items in 2015

(b) By using 2013-2014 as the base year, find the simple price index for all the items in 2015

Solution

(a) (i) $price\ price = \frac{P_{2014}}{P_{2013}} x100\%$

For salt : $price\ index = \frac{1500}{1000} x100\% = 150$

For beef: $price\ index = \frac{8000}{5000} x100\% = 160$

For meat: $price\ index = \frac{9000}{7000} x100\% = 128.57$

For beans: $price\ index = \frac{5000}{3000} x100\% = 166.67$

(i) (ii) $simple\ quantum\ index = \frac{Q_{2014}}{Q_{2013}} x100\%$

For salt : $quantum\ index = \frac{80}{200} x100\% = 40$

For beef: $quantum\ index = \frac{70}{300} x100\% = 23.3$

For meat: $quantum\ index = \frac{100}{250} x100\% = 40$

For beans: $quantum\ index = \frac{120}{150} x100\% = 80$

(b) Since the base year is 2013 – 2014, then we will get the average prices for 2013 and 2014 which will be taken as the base year

$$price\ price = \frac{P_{2015}}{P_{2013-2014}} x100\%$$

For salt : $P.I = 1900 \Big/ \left(\frac{1000+1500}{2}\right) x100\% = 152$

For beef: $P.I = 10000 \Big/ \left(\frac{8000+5000}{2}\right) x100\% = 153.45$

For meat: $P.I = 12000 \Big/ \left(\frac{9000+7000}{2}\right) x100\% = 150$

For beans: $P.I = 5500 \Big/ \left(\frac{5000+3000}{2}\right) x100\% = 137.5$

(b) Simple aggregate price index

Its given by $simple\ aggregate\ price\ index = \frac{\sum P_1}{\sum P_0} x100\%$

Where $P_1 - price\ in\ the\ current\ year$ and $P_0 - price\ in\ the\ base\ year$

Example

1. The table below shows the price of beans and meat per kg in 2000 and 2008

Item	Year 2000	Year 2008
Beans	700	1200
Meat	2500	4500

Using 2000 as the base year, find,
(a) Price relatives of each commodity
(b) Simple aggregate price index

Solution

(a) $price\ index = \frac{P_1}{P_0} x100\%$

For beans : $P.R = \frac{1200}{700} x100\% = 171.43$

For meat: $P.R = \frac{4500}{2500} x100\% = 180$

(b) $S.A.P.I = \frac{\sum P_1}{\sum P_0} x100\%$

$$S.A.P.I = \left(\frac{1200 + 4500}{700 + 2500}\right) x100\% = 178.13$$

2. In 2014 the prices of a shirt, a dress and a pair of shoes were 20,000/=, 35,000/= and 45,000/= respectively. Given that in 2017 the prices were 25,000/= , 50,000/= and y/= respectively. Find y if the aggregate price index was 130.

Solution

$simple\ aggregate\ price\ index = \frac{\sum P_{2017}}{\sum P_{2014}} x100\%$

$$\left(\frac{25000 + 50000 + y}{20000 + 35000 + 45000}\right) x100 = 130$$

$y = 55,000/=$

(c) Weighted aggregate price indices / composite price indices

$$Weighted\ aggregate\ price\ index = \frac{\sum P_1 w}{\sum P_0\, w} x100\%$$

Examples

1. The table below shows the prices(shs) and amounts of item bought for assembling a phone in 2012 and 2015

Item	Price (shs)		quantities
	2012	2015	
transistor	12000	18000	8
resistor	16500	21000	22
capacitor	15000	17000	9
Diode	16000	18000	2
Circuit	20000	25000	1

Calculate the composite price index for the phone taking 2012 as the base year

Solution

$$Weighted\ aggregate\ price\ index = \frac{\sum P_1 w}{\sum P_0 w} x100\%$$

$$Weighted\ aggregate\ price\ index = \frac{18000x8 + 21000x22 + 17000x9 + 18000x2 + 25000x1}{12000x8 + 16500x22 + 15000x9 + 16000x2 + 20000x1} x100\%$$

$$Weighted\ aggregate\ price\ index = 127$$

2. The table below shows the prices(shs) and amounts of item bought weekly by a restaurant in 2005 and 2006

Item	Price (shs)		amount
	2005	2006	
Beans	4000	8000	2
Millet	3500	7000	4
Maize	6000	5000	3
Sorghum	2500	3000	1

Calculate the weighted aggregate price index taking 2005 as the base year

Solution

$$Weighted\ aggregate\ price\ index = \frac{\sum P_1 w}{\sum P_0 w} x100\%$$

$$Weighted\ aggregate\ price\ indices = \frac{8000x2 + 7000x4 + 5000x3 + 3000x1}{4000x2 + 3500x4 + 6000x3 + 2500x1} x100\%$$

$$Weighted\ aggregate\ price\ indices = 145.88$$

3. The table below shows the prices(shs) and amounts of item bought for making a cake in 2005 and 2006

Item	Price (shs)		amount
	2008	2009	
Flour per kg	6000	7800	3
Sugar per kg	5000	4000	1
Milk per litre	1000	1500	2
Eggs per egg	200	300	8

(a) Calculate the weighted aggregate price index taking 2008 as the base year

(b) In 2009, the cost of making a cake was 80,000/=. Using the weighted aggregate price index above, find the cost of the cake in 2008

Solution

(a) $Weighted\ aggregate\ price\ index = \frac{\sum P_1 w}{\sum P_0 w} x100\%$

$$Weighted\ aggregate\ price\ index = \frac{7800x3 + 4000x1 + 1500x2 + 300x8}{6000x3 + 5000x1 + 1000x2 + 200x8} x100\% = 123.3083$$

(b) $\frac{P_1}{P_0} x100\% = 123.3083$

$$\frac{80{,}000}{P_0} x100\% = 123.3083$$

$$P_0 = 64{,}878.033/=$$

(d) Average Weighted price index/ cost of living index

$$Average\ \text{weighted}\ price\ index = \frac{\sum\left(\frac{P_1}{P_0} w\right)}{\sum w} x100\%$$

When the price relative (P.R) is given then: $Average\ \text{weighted}\ price\ index = \frac{\Sigma(\text{P.R} \times w)}{\sum w}$

Examples

1. Calculate the average weighted price for year 2002 taking 2000 as the base for the information below

Item	Price (shs)		amount
	2000	2002	
Food	55000	60000	4
Housing	48000	52000	2
Transport	16000	20000	1

Solution

$$Average\ weighted\ price\ index = \frac{\sum\left(\frac{P_1}{P_0}w\right)}{\sum w}x100\%$$

$$A.W.P.I = \frac{\frac{60000}{55000}x4 + \frac{52000}{48000}x2 + \frac{20000}{16000}x1}{4+2+1}x100\%$$

$$= 111\%$$

2. The table below shows the expenditure (Ug shs) if a student during the first and second terms

Item	Expenditure		amount
	1st term	2nd term	
Clothing	46,500	49,350	5
Pocket money	55,200	37,500	3
Books	80,000	97,500	8

Using the first terms expenditure as the base, find the average weighted price index

Solution

$$A.W.P.I = \frac{\frac{49350}{46500}x5 + \frac{37500}{55200}x3 + \frac{97500}{80000}x8}{5+3+8}x100\%$$

$$= 106.841$$

3. The table below shows the price relatives together with their weights for a certain family

Item	Weight	Price relatives
food	172	120
water	160	124
housing	170	125
Electricity	210	135
Clothing	140	104

Find the cost of living

Solution

$$cost\ of\ living = \frac{\sum(P.R\,x\,w)}{\sum w} = \frac{120x172 + 124x160 + 125x170 + 135x210 + 104x140}{172+160+170+210+140} = 122.82$$

Laspeyre and Pasche aggregate price index

When base year and current year have different quantities, then we use;

$$Laspeyre\ aggregate\ price\ index = \frac{\sum P_1 W_0}{\sum P_0 W_0}x100\%$$

$$Pasche\ aggregate\ price\ index = \frac{\sum P_1 W_1}{\sum P_0 W_1}x100\%$$

Examples

1. The table below shows the items consumed by a certain family in year 2001 and 2002

Item	2001=100		2002	
	Price (shs)	Quantity (kg)	Price(shs)	Quantity (kg)
Rice	2800	20	3200	30
Millet	1500	10	1900	10
Beans	2000	5	2500	70

Using the table above, find

(i) Laspeyre aggregate price index

(ii) Pasche aggregate price index

Solution

$$Laspeyre\ aggregate\ price\ index = \frac{\sum P_1 W_0}{\sum P_0 W_0}x100\% = \frac{3200x20 + 1900x10 + 2500x5}{2800x20 + 1500x10 + 2000x5}x100\%$$

$$= 117.9012$$

$$Pasche\ aggregate\ price\ index = \frac{\sum P_1 W_1}{\sum P_0 W_1}x100\% = \frac{3200x30 + 1900x10 + 2500x70}{2800x30 + 1500x10 + 2000x70}x100\%$$

$$= 121.3389$$

Component	Weight	Price (shs)	
		1998	2005
A	6	35	60
B	5	70	135
C	3	43	105
D	2	180	290
E	1	480	800

Taking 1998 as the base year;
(a) Calculate for 2005 the;
(i) Simple aggregate price index
(ii) Price relatives of each component
(iii) Weighted aggregate price index
(b) Using the price index in (a)(ii) estimate the cost of an engine in 1998 if its cost in 2005 was 1600 US dollars

An((i) 172.03, (ii) 171.4, 192.9, 244.2, 161.1, 166.7, (iii) 182.73, (b) 875.61)

8. The table below shows the prices (in Ug Shs) of some food items in January, June and December together with the corresponding weights.

Item	Weight	Price (shs)		
		Jan	Jun	Dec
Matooke (1 bunch)	4	15,000	13,000	18,000
Meat (1kg)	1	6,500	6,000	7,150
Posho (1kg)	3	2,000	1,800	1,600
Beans (1kg)	2	2,200	2,000	2,800

Taking January as the base month, calculate the;
(i) Simple aggregate price index for June
(ii) Weighted aggregate price index for December

An((i) 88.72, (ii) 116.61)

9. The table below shows the price of beans and meat per kg in 2000 and 2010

Item	Price	
	2000	2010
Beans	3000	5000
Meat	5000	7000

Using 2000 as the base year, find,
(a) Price relatives of each commodity
(b) Simple aggregate price index

An((a) 166.67, 140 (b) 150)

CORRELATIONS AND SCATTER DIAGRAMS

RANK CORRELATIONS

This is the approach used to determine the degree of the relationship between two variables by ranking them.

Rank correlation is determined using

1. Spearman rank correlation coefficient (ρ)
2. Kendall's rank correlation coefficient (τ)

Commenting on the rank correlation coefficient

The table below is used

Correlation coefficient	Interpretation
0 – 0.19	Very low correlation
0.2 – 0.39	Low correlation
0.4 – 0.59	Moderate correlation
0.6 – 0.79	High correlation
0.8 – 1.0	Very high correlation

Note:

The sign associated with the correlation coefficient will be the one responsible for the type of correlation

Eg, -0.84 is a very high negative correlation?

Spearman rank correlation coefficient (ρ)

Its given by $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$

Where d – is the difference between the ranks?

n – Total number of pairs

Examples

1. Given the following score 45, 70, 30, 60, 42, 90, 80, 27. Rank them with the highest score taking rank 1.

Solution

Score	45	70	30	60	42	90	80	27
Rank	5	3	7	4	6	1	2	8

2. Given the following score 45, 50, 70, 45, 50, 48, 80, 50, 65, 90. Rank them with the highest score taking rank 1.

Solution

For case of ties, give positions to the scores and then add the positions and divide by the number of times the score appears

Score	45	50	70	45	50	48	80	50	65	90
Rank	9.5	6	3	9.5	6	8	2	6	4	1

3. Two examiners marked the scripts of 8 candidates. The table shows the marks awarded by each examiner.

Examiner								
x	72	60	56	76	68	52	80	64
y	56	44	60	74	66	38	88	52

Calculate the ranks correlation coefficient and comment on your result

Solution

R_X	R_Y	d^2
3	5	4
6	7	1
7	4	9
2	1	1
4	3	1
8	8	0
1	2	1
5	6	1
		$\sum d^2=18$

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6x18}{8(8^2-1)} = 0.786$$

There is a high positive correlation between x and y

4. The following shows the marks obtained by 10 students in mathematics and physics exams

mathematics	80	80	70	60	65	80	68	90	95	50
Physics	50	45	70	80	70	90	70	80	70	95

Calculate the ranks correlation coefficient and comment on your result

Solution

R_m	R_p	d^2
4	9	25
4	10	36
6	6.5	0.25
9	3.5	30.25
8	6.5	2.25
4	2	4
7	6.5	0.25
2	3.5	2.25
1	6.5	30.25
10	1	81
		$\sum d^2=211.5$

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$\rho = 1 - \frac{6x211.5}{10(10^2-1)} = -0.282$$

There is a very low negative correlation between mathematics and physics

5. The following table gives the order in which six candidates were ranked in two tests x and y

x	E	C	B	E	D	A
y	F	A	D	E	C	C

Calculate the rank correlation coefficient and comment on your results.

R_x	R_y	d^2
5	6	1
3	1	4
2	4	4
6	5	1
4	2.5	2.25
1	2.5	2.25
		$\sum d^2=14.5$

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6x14.5}{6(6^2-1)} = -0.586$$

There is a moderate negative correlation between mathematics and physics

An$(\rho = -0.614)$

Exercise

1. The following shows the marks obtained by 9 students in Biology and Chemistry exams

Biology	60	80	75	85	68	68	90	95	78
Chemistry	70	75	80	78	85	85	90	96	83

Calculate the ranks correlation coefficient and comment on your result **An**$(\rho = 0.57,)$

2. The following shows the marks obtained by 7 candidates in a job interview consisting of oral and aptitude.

Aptitude	G	F	A	D	B	C	E
Oral	D	F	E	B	G	C	A

Calculate the ranks correlation coefficient and comment on your result **An**$(\rho = -0.036,)$

3. The following shows the marks obtained by 10 students in Beginning of term and End of term exams

BOT	15	22	25	28	31	33	36	39	42	48
EOT	30	50	55	30	57	35	60	72	70	75

Calculate the ranks correlation coefficient and comment on your result **An**$(\rho = 0.84,)$

4. The following shows the marks obtained by 10 students in history and geography exams

History	15	20	54	36	40	35	16	36	18	40
Geography	21	16	40	35	16	20	13	20	30	25

Calculate the ranks correlation coefficient and comment on your result **An**$(\rho = 0.38,)$

5. The following shows the grades obtained by 8 students in mock and national exams

Mock	A	B	A	D	O	D	B	F
National	C	B	C	E	E	C	A	F

Calculate the ranks correlation coefficient and comment on your result **An**$(\rho = 0.708,)$

Kendall's rank correlation coefficient (τ)

Its given by $\tau = \frac{2s}{n(n-1)}$

Where s – is the total score of the agreements and disagreements

n – Total number of pairs

Procedure

- Arrange the first row with ranks in ascending order
- Arrange the second row according to the ranks of the first row
- For the first rank in the second row, count to only the right hand side the number of ranks greater than the first rank and take them as positive (agreements). Similarly count to only right hand side the number of ranks less than the first rank and take them as negative (disagreements)
- Add agreements and disagreements to get the total, s
- Repeat the procedure for the remaining ranks in the second row

Examples

1. Two examiners marked the scripts of 8 students and gave the following grades

	A	B	C	D	E	F	G	H
Examiner 1	3	6	7	2	4	8	1	5
Examiner 2	5	7	4	1	3	8	2	6

Calculate kendall's ranks correlation coefficient for the two examiners

Solution

	G	D	A	E	H	B	C	F
Examiner 1	1	2	3	4	5	6	7	8
Examiner 2	2	1	5	3	6	7	4	8
Agreements	6	6	3	4	2	1	1	
Disagreements	-1	0	-2	0	-1	-1	0	
s	5	6	1	4	1	0	1	Σs=18

$$\tau = \frac{2s}{n(n-1)}$$

$$\tau = \frac{2x18}{8(8-1)} = 0.643$$

2. Consultation meeting was held to find out from the local leaders in different areas in kampala whether they are supporting (kwatako) or not supporting (togikwatako) the amendment of article 102 (b) for the removal of age limit. Results were as follows.

	A	B	C	D	E	F	G	H	I
kwatako	9	10	3	32	30	25	17	8	26
Togikwatako	15	46	58	48	92	37	10	80	55

Calculate kendall's ranks correlation coefficient for the data and comment on your result

	D	E	I	F	G	B	A	H	C
Kwatako	1	2	3	4	5	6	7	8	9
togikwatako	5	1	4	7	9	6	8	2	3
Agreements	4	7	4	2	0	1	0	1	
Disagreements	-4	0	-2	-3	-4	-2	-2	0	
s	0	7	2	-1	-4	-1	-2	1	Σs=2

$$\tau = \frac{2s}{n(n-1)}$$

$$\tau = \frac{2x2}{9(9-1)} = 0.056$$

There is a very low positive correlation

3. The table below shows marks obtained by 10 different students in two sets of exams

	A	B	C	D	E	F	G	H	I	J
Set 1	30	25	24	23	35	40	37	30	35	44
Set 2	27	26	25	25	30	43	32	35	42	42

Calculate kendall's ranks correlation coefficient for the data

Solution

	D	C	B	H	A	I	E	G	I	J
SET 1	1	2	3	4	5	6	7	8	9	10
SET 2	4	1	2	7	3	8	5	6	9.5	9.5
Agreements	6	8	7	3	5	2	3	2	0	
Disagreements	-3	0	0	-3	0	-2	0	0	0	
s	3	8	7	0	5	0-1	3	2	0	Σs=28

$$\tau = \frac{2s}{n(n-1)}$$

$$\tau = \frac{2x28}{10(10-1)} = 0.622$$

SIGNIFICANCE OF RANKS CORRELATION COEFFICIENT

- If the $\rho_c > \rho_T$, a significant relation exists
- If the $\rho_c < \rho_T$, no significant relation exists

Where ρ_c − calculated spearman correlation coefficient

ρ_T − table spearman correlation coefficient at either 1% or 5% level

Examples

1. The following shows the marks obtained by 8 students in mathematics and physics exams

mathematics	65	65	70	75	75	80	85	85
Physics	50	55	58	55	65	58	61	65

Calculate the ranks correlation coefficient and comment on the significance of your result at 5% level

Solution

R_M	R_P	d^2
7.5	8	0.25
7.5	6.5	1
6	4.5	2.25
4.5	6.5	4
4.5	1.5	9
3	4.5	2.25
1.5	3	2.25
1.5	1.5	0
		$\Sigma d^2=21$

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$\rho = 1 - \frac{6x21}{8(8^2-1)} = 0.75$$

Since $\rho_c(0.75) > \rho_T(0.71)$, a significant relation exists

2. The following shows the mid term grades and end of term marks obtained by 8 students in mathematics

MOT	A	C	D	B	F	C	O	E
EOT	92	75	63	54	48	45	34	18

Calculate the ranks correlation coefficient and comment on the significance of your result at 5% level

Solution

R_M	R_E	d^2
1	1	0
3.5	2	2.25
5	3	4
2	4	4
7	5	4
3.5	6	6.25
8	7	1
6	8	4
		$\Sigma d^2=25.5$

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

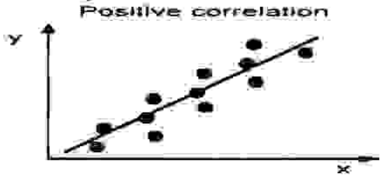
$$\rho = 1 - \frac{6x25.5}{8(8^2-1)} = 0.696$$

Since $\rho_c(0.696) < \rho_T(0.71)$, no significant relation exists

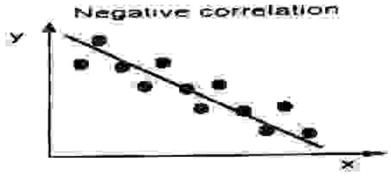
Scatter graphs

It's a graph showing the relation between two variables

Positive correlation



Negative correlation



Examples

1. Given the information in the table below

x	10	15	20	15	30	35	40	45	50	60
y	15	20	35	40	35	50	55	40	55	60

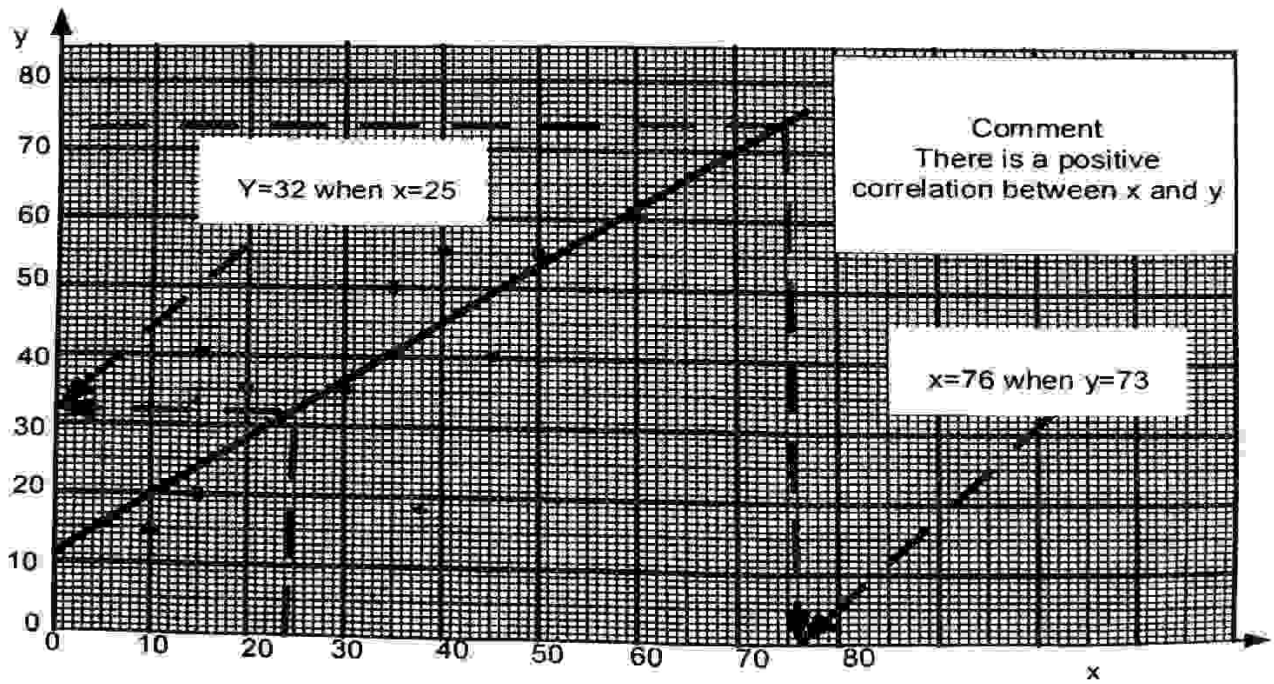
(a) Represent the above information on a scatter diagram and comment on the relationship between x and y

(b) Draw the line of best fit

(c) Use your graph to find

(i) The value of y when $x = 25$ (ii) The value of x when $y = 73$

Solution

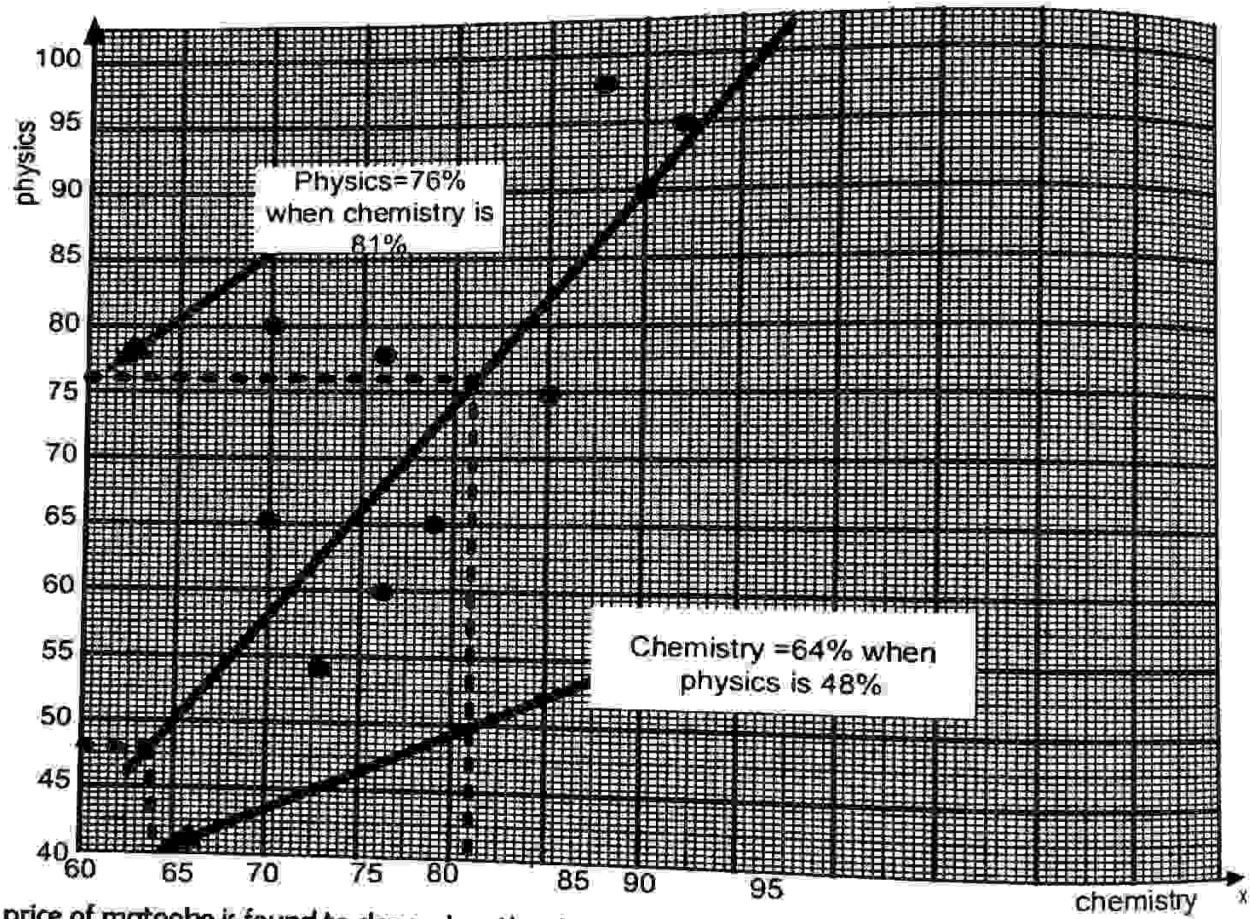


2. The table below shows marks obtained by 10 students in physics and chemistry tests

Physics	80	75	65	90	95	98	78	65	54	60
Chemistry	70	85	70	90	92	88	76	79	73	76

(a) Represent the above information on a scatter diagram
(b) Draw the line of best fit
(c) Use your graph to find
  (i) The mark scored in chemistry if the student got 76% in physics
  (ii) The mark scored in physics if the student got 48% in chemistry

Solution

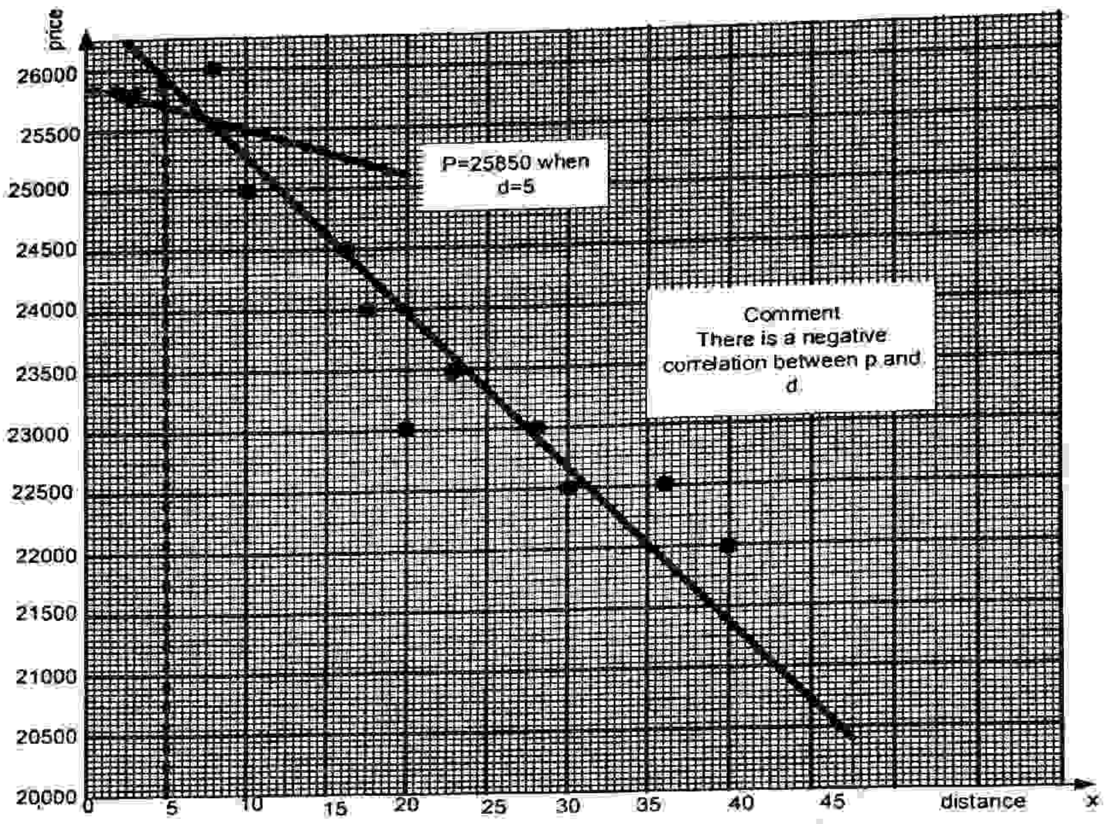


3. The price of matooke is found to depend on the distance the market is away from the nearest town. The table below gives the average price of the matooke from markets around kampala city.

Distance, d (km)	40	8	17	20	24	30	10	28	16	36
Price, P(shs)	22000	26000	24000	23000	23500	22500	25000	23000	24500	22500

  (i) Plot a scatter diagram comment on the relationship between distance and price
  (ii) Draw the line of best fit on your graph
  (iii) Estimate the price of matooke when $d = 5$. **An(155)**

Solution



Exercise 3

1. Eight candidates seeking admission to a university course sat for written and oral test. The scores were shown below.

Written (x)	55	54	35	62	87	53	71	50
Oral (y)	57	60	47	65	83	56	74	63

   (a) Plot the results on a scatter diagram. Comment on the relationship between the written test and oral test
   (b) Draw the line of best fit on your graph and use it to estimate y when $x = 70$ **An(71)**
   (c) Calculate the rank correlation coefficient. Comment on your result **An**($\rho = 0.833$**)**

2. The pair of observations have been made on two random variables X and Y. The ten (x,y) values are. (0,20), (-7,12), (-10,15), (-12,22), (-17,5), (-30,-5), (-32,13), (10,30), (15,40) and (-12,8)
   (a) Plot the results on a scatter diagram **UNEB 1990 No12**
   (b) Draw the line of best fit on your graph
   (c) Estimate the expected value of Y corresponding to $X = -7$ **An(15.7)**
   (d) Calculate the rank correlation coefficient and comment on the significance of the results at 1% significance level. **An**($\rho = 0.894, \tau = 0.778$**)**

Three examiners X, Y and Z each marked the script of ten candidates who sat for a mathematics examination. The table below shows the examiners' ranking of the candidates **UNEB 1991 No12**

Examiner	A	B	C	D	E	F	G	H	I	J
X	8	5	9	2	10	1	7	6	3	4
Y	5	3	6	1	4	7	2	10	8	9
Z	6	3	7	2	5	4	1	10	9	8

Calculate the coefficient of rank correlation of the rankings
   (i) X and Y **An**($\rho = -0.127$**)** (ii) Y and Z **An**($\rho = 0.903$**)**
   (iii) Comment on the significance of each at 5% significant level

3. Three weighing scales from three different shops W, X and Y in a market were used to weigh 10 bags of beans A, B, C.......J and the results in (kg) were given in the table below. One of the scales was known to be in good working condition. **UNEB 1992 No13**

Scales	A	B	C	D	E	F	G	H	I	J
W	65	68	70	63	64	62	73	75	72	78
X	63	68	68	60	65	60	72	73	70	66
Y	63	74	78	75	64	73	79	70	67	79

Determine the rank correlation coefficient for the performance of the scales

(i) W and X **An**$(\rho = 0.8)$ (ii) X and Y **An**$(\rho = 0.185)$

(iii) Which of the three scales. W , X and Y were in good working conditions

4. (a) In many government institutions, officers complain about typing errors. A test was designed to investigate the relationship between typing speed and errors made. Twelve typist A, B,C,.........L were picked at random to type the text. The table below shows the rankings of the typist according to speed and errors made (N.B lowest ranking in errors implies least errors) **UNEB 1994 No14**

Typist	A	B	C	D	E	F	G	H	I	J	K	L
speed	3	4	2	1	8	11	10	6	7	12	5	9
errors	2	6	5	1	10	9	8	3	4	12	7	11

(i) Calculate the coefficient of rank correlation of the rankings **An**$(\rho = 0.8182, \tau = 0.67)$

(ii) Comment on the significance at 1% significance level

(**b**) The cost of travelling a certain distance away from the city centre is found to depend on the route and the distance a given place is away from the centre. The table below gives the average rates of travel charged for distances to be travelled away from the city centre

Distance, S(km)	9	12	14	21	24	30	33	45	46	50
Rate charged, r(shs)	750	1000	1150	1200	1350	1250	1400	1750	1600	2000

(i) Plot the above data on a scatter diagram and draw a line of best fit through the points of the scatter diagram

(ii) Estimate the expected value of r corresponding to $S = 40km$ **An(1601)**

5. (a) In a certain commercial institution, a speed and error typing examinations was administered to 12 randomly selected candidates A, B, C,.........L of the institution. The table below shows their speeds (y) in seconds and the number of errors in their typed scripts (x) **UNEB 1995 No13**

	A	B	C	D	E	F	G	H	I	J	K	L
No. of errors (x)	12	24	20	10	32	30	28	15	18	40	27	35
Speed (y) in seconds	130	136	124	120	153	160	155	142	145	172	140	157

(i) Calculate the coefficient of rank correlation of the rankings **An**$(\rho = 0.84, \tau = 0.7)$

(ii) Comment on your result

(iii) Plot the above data on a scatter diagram and draw a line of best fit through the points of the scatter diagram

6. The following table gives the marks obtained in Calculus, Physics and Statistics by seven students

Calculus	72	50	60	55	35	48	82
Physics	61	55	70	50	30	50	73
Statistics	50	40	62	70	40	40	60

Determine the rank correlation coefficient for the performance of the students in; **UNEB 1996 No16**

(i) Calculus and Physics **An**$(\rho = 0.9)$ (ii) Calculus and Statistics **An**$(\rho = 0.64)$

7. Given the table below **UNEB 1999 No8**

x	80	75	86	60	75	92	86	50	64	75
y	62	58	60	45	68	68	81	48	50	70

Determine the rank correlation coefficient between the variables x and y, comment on your result **An**$(\rho = 0.715)$

8. The table below shows the percentage of sand y, in the soil at different depths x, (in cm)

Soil depth x, (cm)	35	65	55	25	45	75	20	90	51	60
% of sand, y	86	70	84	92	79	68	96	58	86	77

(d) Plot the results on a scatter diagram. Comment on the relationship between the depth of the soil and the percentage of sand in the soil **UNEB 2001 No15**

(e) Draw the line of best fit on your graph and use it to estimate

(i) the percentage of sand in the soil at depth of 31cm **An(92%)**

(ii) depth of the soil with 54% sand **An(96cm)**

(iii) Calculate the rank correlation coefficient. **An**($\rho = -0.949$**)**

9. Eight applicants for a certain job obtained the following marks in aptitude and written tests

Applicant	A	B	C	D	E	F	G	H
Aptitude test	33	45	15	42	45	35	40	48
Written test	57	60	40	75	58	48	54	68

(i) Calculate the coefficient of rank correlation of the applicant's performance in the two tests

(ii) Comment on your result **An**($\rho = 0.78$**) UNEB 2004 No7**

10. The table below shows the marks scored by ten students in mathematics and Fine art tests

	A	B	C	D	E	F	G	H	I	J
Mathematics	40	48	79	26	55	35	37	70	60	40
Fine Art	59	62	68	47	46	39	63	29	55	67

Calculate the coefficient of rank correlation for the students' performance in the two subjects and comment on your result **An**($\rho =$**) UNEB 2005 No7**

11. Below are marks scored by 8 students A, B, C..........H in mathematics, Economics and Geography in the end of term examination **UNEB 2007 No12**

	A	B	C	D	E	F	G	H
Math	52	75	41	60	81	31	65	52
Economics	50	60	35	65	66	45	69	48
Geography	35	40	60	54	63	40	55	72

Determine the rank correlation coefficient for the performance of the students in;

(i) math and Economics **An**($\rho = 0.851$**)** (ii) geography and math **An**($\rho = 0.191$**)**

Comment on the significance of math in the performance of economics and geography. ($\rho = 0.86, \tau = 0.79$ based on 8 observations at 1% level of significance)

12. The heights and masses of ten students are given in the table below **UNEB 2011 No12**

Height (cm)	156	151	152	146	160	157	149	142	158	141
Mass (kg)	62	58	63	58	70	60	55	57	68	56

(a) Plot the data on a scatter diagram.

(b) Draw the line of best fit on your graph and use it to estimate the mass corresponding to a height of 155cm**An(65kg)**

(c) Calculate the rank correlation coefficient for the data. Comment on the significance of the height on masses of students. ($\rho = 0.79, \tau = 0.64$ based on 10 observations at 1% level of significance) **An**($\rho = 0.87, \tau = 0.71$**)**

13. The heights and ages of ten farmers are given in the table below **UNEB 2013 No9**

Height (cm)	156	151	152	160	146	157	149	142	158	140
Age (years)	47	38	44	55	46	49	45	30	45	30

(a) Plot the data on a scatter diagram.

(b) Draw the line of best fit on your graph and use it to estimate

(i) Y when $x = 147$ **An(37)** (ii) X when $y = 43$ **An(151)**

(c) Calculate the rank correlation coefficient for the data. Comment on your results **An**($\rho = 0.752, \tau = 0.6$**)**

14. The table below gives the points awarded to eight schools by three judges J_1, J_2 and J_3 during a music competition. J_1 was the chief judge **UNEB 2015 No12**

J_1	72	50	50	55	35	38	82	72
J_2	60	55	70	50	50	50	73	70
J_3	50	40	62	70	40	48	67	67

(a) Determine the rank correlation between the judgments of

(i) J_1 and J_2 **An**($\rho = 0.744$**)** (ii) J_1 and J_3 **An**($\rho = 0.702$**)**

(b) Who of the two judges had a better correlation with the chief judge?. Give a reason

PROBABILITY THEORY

Probability is the measure of the chance of an event occurring or not occurring

Sample space

A sample space (s) is the set of all possible outcomes of an experiment

Examples

1. When a die is rolled once, $s = \{1, 2, 3, 4, 5, 6\}$
2. When a coin is tossed once, $s = \{H, T\}$

EVENTS

An event (E) is a subset of a sample space

When a coin is tossed twice $s = \{HH, HT, TT, TH\}$

When interested in getting one head, $E = \{HT, TH\}$

Probability of an event

Given an event E over a sample space S

$$P(E) = \frac{n(E)}{n(S)}$$

Examples

1. Find the probability of getting a head when an ordinary coin is tossed once

Solution

$S = \{H, T\}$

$E = \{H\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}$$

2. Find the probability of getting two heads when an ordinary coin is tossed twice

Solution

$S = \{HH, HT, TT, TH\}$

$E = \{HH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$$

3. Find the probability of getting two heads when an ordinary coin is tossed thrice

Solution

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$E = \{HHT, HTH, THH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

4. Find the probability of getting a number less than 3, when an ordinary die is tossed once

Solution

$S = \{1,2,3,4,5,6\}$

$E = \{1,2\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

Intersection of events

For any two events A and B, the probability that A and B occur together is $P(A \cap B)$

Union of events

For any two events A and B, the probability that either A or B or both occur r is $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example

1. The probability that a student passes Economics is $\frac{2}{3}$, the probability that he passes mathematics is $\frac{4}{9}$, if the probability that he passes at least one of them is $\frac{4}{5}$. Find the probability that he passes both subjects

Solution

$P(E \cup M) = P(E) + P(M) - P(E \cap M)$

$$\frac{4}{5} = \frac{2}{3} + \frac{4}{9} - P(E \cap M)$$

$$P(E \cap M) = \frac{14}{45}$$

Complement of events

A^1 Denotes event A does not occur

For events A and B

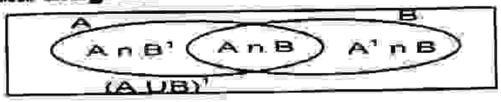
(i) $P(A) + P(A^1) = 1$

(ii) $P(B) + P(B^1) = 1$

(iii) $P(A \cap B) + P(A \cap B)^1 = 1$

(iv) $P(A \cup B) + P(A \cup B)^1 = 1$

Venn diagram



(i) $P(A) = P(A \cap B^1) + P(A \cap B)$

(ii) $P(B) = P(B \cap A^1) + P(A \cap B)$

Contingency table

	B	B'	
A	P(AnB)	P(AnB')	P(A)
A'	P(A'nB)	P(A'nB')	P(A')
	P(B)	P(B')	1

(i) $P(A) = P(A \cap B^1) + P(A \cap B)$

(ii) $P(A^1) = P(A^1 \cap B) + P(A^1 \cap B^1)$

(iii) $P(B) = P(B \cap A^1) + P(A \cap B)$

(iv) $P(B^1) = P(A \cap B^1) + P(A^1 \cap B^1)$

Demergan's rule

(i) $P(A^1 \cap B^1) = P(A \cup B)^1 = 1 - P(A \cup B)$

$P(neither\ A\ nor\ B) = 1 - P(A\ or\ B)$

(ii) $P(A^1 \cup B^1) = P(A \cap B)^1 = 1 - P(A \cap B)$

Types of events

- Undefined events
- Mutually exclusive events
- Independent events
- Exhaustive events

Undefined events

For undefined events, there is no restriction on $P(A \cap B)$

Examples

1. Events A and B are such that $P(A) = \frac{19}{30}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{4}{5}$. Find

(i) $P(A \cap B)$

(ii) $P(A^1 \cap B^1)$

(iii) $P(A^1 \cap B)$

(iv) $P(A^1 \cup B)$

(v) $P(A \cap B^1)$

Solution

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{4}{5} = \frac{19}{30} + \frac{2}{5} - P(A \cap B)$$

$$P(A \cap B) = \frac{7}{30}$$

(ii) $P(A^1 \cap B^1) = P(A \cup B)^1 = 1 - P(A \cup B)$

$$P(A^1 \cap B^1) = 1 - \frac{4}{5} = \frac{1}{5}$$

(iii) $P(A^1 \cap B) = P(B) - P(A \cap B)$

$$P(A^1 \cap B) = \frac{2}{5} - \frac{7}{30} = \frac{1}{6}$$

(iv) $P(A^1 \cup B) = P(A^1) + P(B) - P(A^1 \cap B)$

$$P(A^1 \cup B) = \left(1 - \frac{19}{30}\right) + \frac{2}{5} - \frac{1}{6} = \frac{3}{5}$$

(v) $P(A \cap B^1) = P(A) - P(A \cap B)$

$$P(A \cap B^1) = \frac{19}{30} - \frac{7}{30} = \frac{2}{5}$$

2. Events C and D are such that $P(C) = 0.3$, $P(D) = 0.4$ and $P(C \cap D) = 0.1$. Find

(i) $P(D^1)$

(ii) $P(C \cap D^1)$

(iii) $P(C^1 \cap D)$

(iv) $P(C^1 \cap D^1)$

(v) $P(C^1 \cup D^1)$

Solution

(i) $P(D^1) = 1 - P(D) = 1 - 0.4 = 0.6$

(ii) $P(C \cap D^1) = P(C) - P(C \cap D) = 0.3 - 0.1 = 0.2$

(iii) $P(C^1 \cap D) = P(D) - P(C \cap D) = 0.4 - 0.1 = 0.3$

(iv) $P(C^1 \cap D^1) = P(C^1) - P(C^1 \cap D)$

$= (1 - 0.3) - 0.3 = 0.4$

(v) $P(C^1 \cup D^1) = P(C \cap D)^1 = 1 - P(C \cap D)$

$= 1 - 0.1 = 0.9$

3. Events A and B are such that $P(A) = 0.4, P(A \cap B) = 0.1$ and $P(A \cup B) = 0.9$. Find
   (i) $P(B)$
   (ii) $P(A^1 \cap B)$
   (iii) $P(A \cap B^1)$
   (iv) $P(A \cup B^1)$

Solution

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.9 = 0.4 + P(B) - 0.1$

$P(B) = 0.6$

(ii) $P(A^1 \cap B) = P(B) - P(A \cap B)$

$P(A^1 \cap B) = 0.6 - 0.1 = 0.5$

(iii) $P(A \cap B^1) = P(A) - P(A \cap B)$

$P(A \cap B^1) = 0.4 - 0.1 = 0.3$

(iv) $P(A \cup B^1) = P(A) + P(B^1) - P(A \cap B^1)$

$P(A \cup B^1) = 0.4 + (1 - 0.6) - 0.3 = 0.5$

4. Events A and B are such that $P(A) = 0.7, P(A \cap B) = 0.45$ and $P(A^1 \cap B^1) = 0.18$. find **UNEB 2010 No.10**
   (i) $P(B^1)$
   (ii) $P(A \text{ or } B, \text{but not both } A \text{ and } B)$

Solution

$P(A^1) = P(A^1 \cap B) + P(A^1 \cap B^1)$

$1 - 0.7 = P(A^1 \cap B) + 0.18$

$P(A^1 \cap B) = 0.12$

$P(B) = P(B \cap A^1) + P(A \cap B)$

$1 - P(B^1) = P(B \cap A^1) + P(A \cap B)$

$1 - P(B^1) = 0.12 + 0.45$

$P(B^1) = 0.43$

$P(A \text{ or } B, \text{but not both } A \text{ and } B)$

$= P(A^1 \cap B) + P(A \cap B^1)$

$= 0.12 + P(A) - P(A \cap B)$

$= 0.12 + 0.7 - 0.45 = 0.37$

5. The probability that Anne reads the new vision is 0.75 and the probability that she reads the New vision and not the Daily monitor is 0.65. The probability that she reads neither of the papers is 0.15. find the probability that she reads daily monitor **UNEB 2008 No.1**

Solution

$P(N) = 0.75, P(N \cap D^1) = 0.65, P(N^1 \cap D^1) = 0.15$

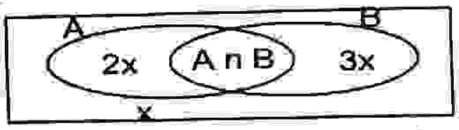
$P(D^1) = P(N \cap D^1) + P(N^1 \cap D^1)$

$1 - P(D) = 0.65 + 0.15$

$P(D) = 0.2$

6. Events A and B are such that $P(A^1 \cap B) = 3x, P(A \cap B^1) = 2x,\ P(A^1 \cap B^1) = x$ and , $P(B) = \frac{4}{7}$. Use a venn diagram to find the value of **UNEB 2011 No.4**
   (i) x
   (ii) $P(A \cap B)$

Solution



$P(A \cup B) = P(A \cap B^1) + P(B)$

$1 - P(A \cup B)^1 = 2x + \frac{4}{7}$

$1 - x = 2x + \frac{4}{7}$

$x = \frac{1}{7}$

$P(A \cap B) = P(B) - P(A^1 \cap B)$

$P(A \cap B) = \frac{4}{7} - 3(\frac{1}{7}) = \frac{1}{7}$

Exercise 4a

1. Events C and D are such that $P(C) = 0.5$, $P(D) = 0.7$ and $P(C \cup D) = 0.8$. Find
   $(i)\ P(C \cap D)$ $(ii) P(C \cap D^1)$
   An $(i) = 0.4, (ii) = 0.1$

2. The probability that two events A and B occur together is $\frac{2}{15}$ and the probability that either or both events occur is $\frac{3}{5}$. Find $(i)\ P(A)$ $(ii)\ P(B)$
   An $P(A) = 0.4, P(B) = \frac{1}{3}$

3. Events A and B are such that $P(A) = 0.36$, $P(B) = 0.25$ and $P(A^1 n B) = 0.24$. Find
(i) $P(A^1)$ (ii) $P(A n B)$ (iii) $P(A uB)$
(iv) $P(A n B^1)$ (v) $P(A^1 u B^1)$

An $(i) = 0.64, (ii) = 0.01, (iii) = 0.6,$
$(iv) = 0.35, (v) = 0.99$

4. Events C and D are such that $P(C) = 0.7$, $P(CnD) = 0.3$ and $P(C u D) = 0.9$. Find
(i) $P(D)$ (ii) $P(C n D^1)$
(iii) $P(C^1 n D)$ (iv) $P(C^1 n D^1)$

An $(i) = 0.5, (ii) = 0.4, (iii) = 0.2, (iv) = 0.1$

5. Events A and B are such that $P(A^1) = \frac{2}{3}$, $P(B) = \frac{1}{2}$ and $P(A n B) = \frac{1}{12}$. Find
(i) $P(A u B)$ (ii) $P(A n B^1)$

An $(i) = 0.75, (ii) = 0.25,$

6. Events A and B are such that $P(A) = \frac{2}{3}$, $P(A n B) = \frac{5}{12}$ and $P(A uB) = \frac{3}{4}$. Find
(i) $P(B)$ (ii) $P(A^1 n B)$

An $(i) = 0.5, (ii) = \frac{1}{12}$,

7. Events A and B are such that $P(A) = P(B)$, $P(A n B) = \frac{1}{10}$ and $P(A uB) = \frac{7}{10}$. Find
(i) $P(A)$ (ii) $P(A^1 u B)$

An $(i) = 0.4, (ii) = 0.7,$

8. Events A and B are such that $P(A) = \frac{7}{12}$, $P(B) = \frac{3}{4}$ and $P(A n B) = \frac{2}{5}$. Find
(i) $P(A u B)$ (ii) $P(A n B^1)$

An $(i) = \frac{14}{15}, (ii) = \frac{11}{60}$,

Mutually exclusive events

Two events A and Bare mutually exclusive if they do not occur together ie. $P(A n B) = 0$

$$\boxed{P(A U B) = P(A) + P(B)}$$

Examples

1. Events A and B are mutually exclusive such that $P(B) = \frac{1}{2}, P(A) = \frac{2}{5}$ find
(i) $P(A U B)$ (ii) $P(A^1 n B)$ (iii) $P(A^1 n B^1)$

Solution

(i) $P(A U B) = P(A) + P(B)$
$P(A U B) = 0.5 + 0.4 = 0.9$

(ii) $P(A^1 n B) = P(B) - P(A n B)$
$P(A^1 n B) = 0.5 - 0 = 0.5$

(iii) $P(A^1 n B^1) = P(A U B)^1 = 1 - P(A U B)$
$= 1 - 0.9 = 0.1$

2. Events A and B are mutually exclusive such that $P(A U B) = \frac{7}{10}, P(A) = \frac{3}{5}$ find
(i) $P(B)$ (ii) $P(A^1 n B)$ (iii) $P(A n B^1)$ (iv) $P(A^1 n B^1)$ (v) $P(A^1 u B^1)$ (vi) $P(A u B^1)$ (vii) $P(A^1 u B)$

Solution

(i) $P(A U B) = P(A) + P(B)$
$0.7 = 0.6 + P(B)$
$P(B) = 0.1$

(ii) $P(A^1 n B) = P(B) - P(A n B)$
$P(A^1 nB) = 0.1 - 0 = 0.1$

(iii) $P(A n B^1) = P(A) - P(A n B)$
$P(A n B^1) = 0.6 - 0 = 0.6$

(iv) $P(A^1 n B^1) = P(A U B)^1 = 1 - P(A U B)$
$= 1 - 0.7 = 0.3$

(v) $P(A^1 U B^1) = P(A n B)^1 = 1 - P(A n B)$
$= 1 - 0 = 1$

(vi) $P(A U B^1) = P(A) + P(B^1) - P(A n B^1)$
$= 0.6 + 0.9 - 0.6 = 0.9$

(vii) $P(A^1 U B) = P(A^1) + P(B) - P(A^1 n B)$
$= 0.4 + 0.1 - 0.1 = 0.4$

3. In an athletics competition in which there are no dead heats, the probability that Kipsiro wins is 0.5, the probability that Bekele wins is 0.2, the probability that Chellimo wins is 0.1. Find the probability that:
(i) Bekele or Kipsiro wins (ii) Neither Kipsiro nor Chellimo wins

Solution

(i) $P(B U K) = P(B) + P(K)$
$P(B U K) = 0.2 + 0.5 = 0.7$

(ii) $P(K^1 n C^1) = P(K U C)^1 = 1 - P(K U C)$
$= 1 - (0.5 + 0.1) = 0.4$

Exercise 4b

1. Events A and B are mutually exclusive such that $P(B) = \frac{2}{3}, P(A) = \frac{1}{2}$, find **Uneb 1991 No.1**
(i) $P(A \cup B)$ (ii) $P(A \cap B^1)$ (iii) $P(A^1 \cap B^1)$
An (i) = 0.9, (ii) = 0.5 (iii) = 0.1
2. Events A and B are mutually exclusive such that $P(B) = \frac{3}{10}, P(A) = \frac{3}{5}$, find
(i) $P(A \cup B)$ (ii) $P(A^1)$ (iii) $P(A^1 \cap B)$
An (i) = 0.9, (ii) = 0.4, (iii) = 0.3
3. Events A and B are mutually exclusive such that $P(B) = 0.4, P(A) = 0.5$, find
(i) $P(A^1 \cup B)$ (ii) $P(B^1)$ (iii) $P(A^1 \cap B^1)$
An (i) = 0.5, (ii) = 0.6, (iii) = 0.1
4. Events A and B are mutually exclusive such that $P(A \cup B) = \frac{8}{10}, P(A) = \frac{2}{5}$, find
(i) $P(B)$ (ii) $P(A^1 \cap B)$ (iii) $P(A \cap B^1)$
(iv) $P(A^1 \cap B^1)$ (v) $P(A \cup B^1)$ (vi) $P(A^1 \cup B)$
An (i) = 0.4, (ii) = 0.4, (iii) = 0.4, (iv) = 0.2, (v) = 0.6, (vi) = 0.6
5. Events A and B are mutually exclusive such that $P(A^1 \cap B) = 0.3, P(A^1 \cup B) = 0.45$ find
(i) $P(B)$ (ii) $P(A)$ (iii) $P(A \cap B^1)$
(iv) $P(A^1 \cap B^1)$ (v) $P(A^1 \cup B^1)$ (vi) $P(A^1 \cup B)$
An (i) = 0.3, (ii) = 0.55, (iii) = 0.55, (iv) = 0.15, (v) = 1, (vi) = 0.45
6. Events A and B are mutually exclusive such that $P(A \cup B) = 0.9, P(A \cup B^1) = 0.6$ find
(i) $P(B)$ (ii) $P(A)$ (iii) $P(A^1 \cup B)$
(iv) $P(A^1 \cap B^1)$ (v) $P(A^1 \cup B^1)$
An (i) = 0.4, (ii) = 0.5, (iii) = 0.5, (iv) = 0.1, (v) = 1,

Independent events

Two events A and B are independent if the occurrence of one does not affect the other

(i) $P(A \cap B) = P(A)x\,P(B)$
(ii) $P(A^1 \cap B) = P(A^1)x\,P(B)$
(iii) $P(A \cap B^1) = P(A)x\,P(B^1)$
(iv) $P(A^1 \cap B^1) = P(A^1)x\,P(B^1)$

Examples

1. Events A and B are independent such that $P(A \cup B) = \frac{8}{10}, P(A) = \frac{1}{2}$ find
(i) $P(B)$
(ii) $P(A^1 \cap B)$
(iii) $P(A \cap B^1)$
(iv) $P(A^1 \cap B^1)$
(v) $P(A \cup B^1)$
(vi) $P(A^1 \cup B)$

Solution

(i) $P(A \cup B) = P(A) + P(B) - P(A)xP(B)$
$0.8 = 0.5 + y - 0.5y$
$y = 0.6 \quad \therefore P(B) = 0.6$
(ii) $P(A^1 \cap B) = P(A^1)x\,P(B)$
$P(A^1 \cap B) = (1 - 0.5)x\,0.6 = 0.3$
(iii) $P(A \cap B^1) = P(A)x\,P(B^1)$
$P(A \cap B^1) = 0.5x\,(1 - 0.6) = 0.2$
(iv) $P(A^1 \cap B^1) = P(A^1)x\,P(B^1)$
$P(A^1 \cap B^1) = 0.5x\,0.4 = 0.2$
(v) $P(A^1 \cup B) = P(A^1) + P(B) - P(A^1)xP(B)$
$= 0.5 + 0.6 - 0.2 = 0.9$
(vi) $P(A \cup B^1) = P(A) + P(B^1) - P(A)xP(B^1)$
$= 0.5 + 0.4 - 0.2 = 0.7$

2. Events A and B are independent such that $P(A \cap B) = \frac{1}{12}, P(A) = \frac{1}{3}$ find
(i) $P(B)$
(ii) $P(A \cup B)$
(iii) Show that A^1 and B are independent

Solution

(i) $P(A \cap B) = P(A)x\,P(B)$
$$\frac{1}{12} = \frac{1}{3}x\,P(B)$$
$$P(B) = \frac{1}{4}$$
(ii) $P(A \cup B) = P(A) + P(B) - P(A)xP(B)$
$$P(A \cup B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = 0.5$$
(iii) $P(A^1 \cap B) = P(B) - P(A \cap B)$
$P(A^1 \cap B) = P(B) - P(A)xP(B)$
$P(A^1 \cap B) = P(B)x[1 - P(A)]$
$P(A^1 \cap B) = P(B)xP(A^1)$

3. Events A and B are independent **UNEB 2009 No.9**
(i) Show that the events A and B^1 are also independent
(ii) Find $P(B)$ given that $P(A) = 0.4$ and $P(A \cup B) = 0.8$

Solution

(i) $P(A \cap B^1) = P(A) - P(A \cap B)$
$P(A \cap B^1) = P(A) - P(A)xP(B)$

$$P(A\cap B^1)=P(A)x[1-P(B)]$$
$$P(A\cap B^1)=P(A)xP(B^1)$$

(ii) $P(A\cup B)=P(A)+P(B)-P(A)xP(B)$

$0.8=0.4+y-0.4y$

$y=0.667 \quad \therefore P(A)=0.667$

4. Events A and B are independent such that $P(A\cup B^1)=\frac{9}{10}$, $P(A)=\frac{2}{5}$ find
   (i) $P(B)$
   (ii) $P(A\cup B)$
   (iii) Show that A^1 and B^1 are independent

Solution

(i) $P(A\cup B^1)=P(A)+P(B^1)-P(A)xP(B^1)$

$0.9=0.4+y-0.4y$

$y=\frac{5}{6} \quad \therefore P(B)=\frac{1}{6}$

(ii) $P(A\cup B)=P(A)+P(B)-P(A)xP(B)$

$P(A\cup B)=0.4+\frac{1}{6}-0.4x\frac{1}{6}=0.5$

(iii) $P(A^1\cap B^1)=1-P(A\cup B)$

$=1-[P(A)+P(B)-P(A)xP(B)]$

$=1-P(A)-P(B)+P(A)xP(B)$

$=P(A^1)-P(B)[1-P(A)]$

$=P(A^1)-P(B)[P(A^1)]$

$=P(A^1)[1-P(B)]$

$=P(A^1)xP(B^1)$

5. The probability of two independent events A and B occurring together $\frac{1}{8}$. The probability that either or both events occur is $\frac{5}{8}$. Find **UNEB 2004 No.2**
   (i) $P(A)$
   (ii) $P(B)$

Solution

$P(A\cup B)=\frac{5}{8}$, $P(A\cap B)=\frac{1}{8}$

$P(A\cap B)=P(A)x\,P(B)$

$\frac{1}{8}=xy$

$x=\frac{1}{8y}\ldots\ldots(1)$

$P(A\cup B)=P(A)+P(B)-P(A)xP(B)$

$\frac{5}{8}=x+y-\frac{1}{8}\ldots\ldots\ldots(2)$

solving simultaneously

$\frac{5}{8}=\frac{1}{8y}+y-\frac{1}{8}$

$8y^2-6y+1=0$

$y=0.5 \text{ or } y=0.25$

$x=\frac{1}{8y}=\frac{1}{8x0.5}=0.25$

$x=\frac{1}{8y}=\frac{1}{8x0.25}=0.5$

$P(A)=0.25, P(B)=0.5$

Or $P(A)=0.5, P(B)=0.25$

6. Abel, Bob and Charles applied for the same job in a certain company. The probability that Abel will take the job is $\frac{3}{4}$, the probability that Bob will take it is $\frac{1}{2}$, while the probability that Charles will take the job is $\frac{2}{3}$, what is the probability that **UNEB 2004 No.9a**
   (i) None of them will take the job
   (ii) One of them will take job

Solution

$$P(\text{None takes})=P(A^1nB^1nC^1)=\frac{1}{4}x\frac{1}{2}x\frac{1}{3}=\frac{1}{24}$$
$$P(\text{one takes})=P(AnB^1nC^1)+P(A^1nBnC^1)+P(A^1nB^1nC)$$
$$=\left(\frac{3}{4}x\frac{1}{2}x\frac{1}{3}\right)+\left(\frac{1}{4}x\frac{1}{2}x\frac{1}{3}\right)+\left(\frac{1}{4}x\frac{1}{2}x\frac{2}{3}\right)=\frac{1}{4}$$

Exercise 4c

1. Events A and B are independent such that $P(A)=0.4, P(B)=0.25$. find
   (i) $P(A\cap B)$ (ii) $P(A\cap B^1)$ (iii) $P(A^1\cap B^1)$
   An (i) $=\frac{1}{10}$ (ii) $=\frac{3}{10}$ (iii) $=0.45$
2. Events A and B are independent such that P(A) = 0.3, P(B) = 0.5. find
   (i) $P(A\cap B)$ (ii) $P(A\cup B)$ (iii) $P(A^1\cap B^1)$
   An (i) $=0.15$ (ii) $=0.65$ (iii) $=0.35$
3. Events A and B are independent such that P(A) = 0.4, P(A U B) = 0.7. find
   (i) $P(B)$ (ii) $P(A\cap B)$ (iii) $P(A^1\cap B)$
   An (i) $=0.5$ (ii) $=0.2$ (iii) $=0.3$
4. Events A and B are independent such that $P(A)=\frac{1}{3}$, $P(B)=\frac{3}{4}$, find the probability that

(i) both A and B occur

(ii) only one occurs **An** $(i) = \frac{1}{4}$ $(ii) = \frac{7}{12}$

5. A mother and her daughter both enter a competition. The probability that the mother wins a prize is $\frac{1}{6}$ and the probability that her daughter wins a prize is $\frac{2}{7}$. Assuming that the two events are independent, find the probability that

(i) Either the mother or the daughter but not both wins a prize

(ii) At least one of them wins a prize **An**

$(i) = \frac{5}{14}$ $(ii) = \frac{17}{42}$

6. Two athletes, Kiprotich and Chebet are attempting to qualify for Olympics games. The probability of Kiprotich qualifying is 0.8, and the probability of both Kiprotich and Chebet qualifying is 0.6. given that the probability of the athletes qualifying are independent events, find the probability that only one of them qualifies **An** $(i) = 0.35$

7. The probability of two independent events A and B occurring together $\frac{1}{10}$. The probability that either or both events occur is $\frac{8}{10}$. Find

$(i)\ P(A)$ $(ii)\ P(B)$

An. $P(A) = 0.77, P(B) = 0.13$

Or $P(A) = 0.13, P(B) = 0.7$

8. The probability that a certain type of computer will break down on the first month of use is 0.1. If the school has two such computers bought at the same time, find the probability that at the end of the first month, just one has broken down. Assume that the performance of the two computers are independent. **An**= 0.18

9. Three teachers enter a marathon race. The respective probabilities of them completing the race are 0.9, 0.7, 0.6. Assuming that their performances are independent, find the probability

(i) They all complete the race

(ii) At least two complete the race **An**

$(i) = 0.378, (iii) = 0.834$

10. The probability that two twins pass an interview are $\frac{1}{3}$ and $\frac{2}{5}$ respectively. Assuming that their performances are independent, find the probability that

(i) They all pass the interview

(ii) Only one passes the interview

An $(i) = \frac{2}{15}, (iii) = \frac{7}{15}$

11. The probability that Angela can solve a certain number is 0.4 and the probability that Jane can solve the same number is 0.5, find the probability that the number will be solve if both students try the number independently **An**= 0.7

12. Three target men take part in a shooting competition, their chances of hitting the target are $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$. Assuming that their performances are independent, find the probability that

(i) Only one will hit the target

(ii) Target will be hit **An** $(i) = \frac{11}{24}, (ii) = \frac{3}{4}$

13. Three football teams Noa, Kitende, and Budo are playing in nationals. The probability that Noa, Kitende and Budo will qualify for the finals is $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{1}{4}$. Find the probability that only two teams will qualify for the finals. **An** $= \frac{5}{12}$

14. Three Athletes Kiprop, Chebet, and Aloysious are competing for a place in Olympics games. The probability that Kiprop, Chebet, and Aloysious will qualify for the Olympics games is $\frac{2}{3}$, $\frac{2}{5}$ and $\frac{5}{6}$. Find the probability that only one athlete will qualify for the Olympics games. **An** $= \frac{23}{90}$

15. The probability that three girls Faith, Jane, and Angella will pass exams is $\frac{2}{3}$, $\frac{2}{5}$ and $\frac{3}{4}$ respectively. Find the probability that;

(i) All the three will fail

(ii) All three will pass (iiii) Only two will pass.

An $(i) = \frac{1}{20}, (ii) = \frac{1}{5}$ $(iii) = \frac{7}{15}$

16. The probability of two independent events A and B occurring together $\frac{2}{15}$. The probability that either or both events occur is $\frac{3}{5}$. Find **UNEB 1998 Nov No.1** $(i)\ P(A)$ $(ii)\ P(B)$

An. $P(A) = \frac{1}{3}, P(B) = \frac{2}{5}$ Or $P(A) = \frac{2}{5}, P(B) = \frac{1}{3}$

17. The interview involves written, oral and practical tests. The probability that an interviewee passes written is 0.8, oral is 0.6 and practical is 0.7. What is the probability that the interviewee will pass; **UNEB 2009 No.13b**

(i) The entire interview

(ii) Exactly two of the interview test

An $(i) = 0.336$ $(ii) = 0.452$

18. The probabilities that three players A, B and C score in a net ball game are $\frac{1}{5}$, $\frac{1}{4}$ and $\frac{1}{3}$ respectively. If the player together in a game, find the probability that **UNEB 2010 No.12a**

(i) Only C scores
(ii) At least one player scores
(iii) Two and only two players score
An =

19. Events A and B are such that $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{2}$. Find $P(A \cup B)$ when A and B are
(i) *independent events*
(ii) *mutually exclusive events*
An $(i) = \frac{3}{5}$ $(ii) = \frac{7}{10}$
UNEB 2012 No.2

Conditional probability
If A and B are two events, then the conditional probability that A occurs given that B has already occurred is $P(A/B)$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Similarly the conditional probability that B occurs given that A has already occurred is $P(B/A)$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Examples

1. Events A and B are such that $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{4}$ and $P(A/B) = \frac{2}{5}$. Find
(i) $P(A \cap B)$
(ii) $P(A \cup B)$
(iii) $P(B/A)$

Solution
(i) $P(A/B) = \frac{P(A \cap B)}{P(B)}$
$P(A \cap B) = 0.4 x 0.25 = 0.1$
(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cup B) = 0.2 + 0.25 - 0.1 = 0.35$
(iii) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.2} = 0.5$

2. Events A and B are such that $P(A) = \frac{4}{7}$, $P(A \cap B^1) = \frac{1}{3}$ and $P(A/B) = \frac{5}{14}$. Find
(i) $P(A \cap B)$
(ii) $P(B)$
(iii) $P(A \cup B)$
(iv) $P(A^1 \cap B^1)$

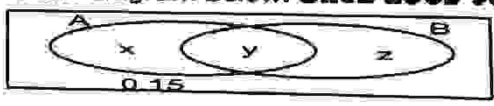
Solution
(i) $P(A \cap B) = P(A) - P(A \cap B^1) = \frac{4}{7} - \frac{1}{3} = \frac{5}{21}$
(ii) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{5}{21}\right)}{\left(\frac{5}{14}\right)} = \frac{2}{3}$
(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cup B) = \frac{4}{7} + \frac{2}{3} - \frac{5}{21} = 1$
(iv) $P(A^1 \cap B^1) = 1 - P(A \cup B) = 1 - 1 = 0$

3. Events A and B are independent. Given that $P(A \cap B^1) = \frac{1}{4}$ and $P\left(A^1/B\right) = \frac{1}{6}$ find **UNEB 2004 No.9**
(i) $P(A)$
(ii) $P(B)$
(iii) $P(A \cap B)$
(iv) $P(A \cup B)^1$

Solution
(i) $P\left(A^1/B\right) = \frac{P(A^1 \cap B)}{P(B)}$
$\frac{1}{6} = \frac{P(A^1) x P(B)}{P(B)}$
$P(A^1) = \frac{1}{6}$, $P(A) = \frac{5}{6}$
(ii) $P(A \cap B^1) = \frac{1}{4}$
$P(A) x P(B^1) = \frac{1}{4}$
$P(B^1) = \frac{3}{10}$, $P(B) = \frac{7}{10}$
(iii) $P(A \cap B) = P(A) - P(A \cap B^1) = \frac{5}{6} - \frac{1}{4} = \frac{7}{12}$
or $P(A \cap B) = P(A) x P(B) = \frac{5}{6} x \frac{7}{10} = \frac{7}{12}$
(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cup B) = \frac{5}{6} + \frac{7}{10} - \frac{7}{12} = \frac{19}{20}$
$P(A \cup B)^1 = 1 - \frac{19}{20} = \frac{1}{20}$

Exercise 4d

1. Events X and Y are such that $P(X^1) = \frac{3}{5}$, $P\left(^{Y}/_{X^1}\right) = \frac{1}{3}$ and $P\left(^{Y^1}/_{X}\right) = \frac{1}{4}$. Find
   (i) $P(Y)$ (ii) $P\left(^{X^1}/_{Y}\right)$
   An $(i) = \frac{1}{2}$ $(ii) = \frac{2}{5}$
2. Events A and B are such that $P(A) = \frac{2}{5}$, $P(^{A}/_{B}) = \frac{1}{2}$ and $P(^{B}/_{A}) = \frac{2}{3}$. Find
   (i) $P(A \, n \, B)$ (ii) $P(B)$ (iii) $P(A \, U \, B)$
   An $(i) = \frac{4}{15}$ $(ii) = \frac{8}{15}$ $(iii) = \frac{2}{3}$
3. Events A and B are such that $P(A) = \frac{1}{3}$, $P(^{B}/_{A}) = \frac{1}{4}$ and $P\left(^{B^1}/_{A^1}\right) = \frac{4}{5}$. Find
   (i) $P\left(^{B^1}/_{A}\right)$ (ii) $P(A \, n \, B)$
   (iii) $P(B)$ (iv) $P(A \, U \, B)$
   An $(i) = \frac{3}{4}$ $(ii) = \frac{1}{12}$ $(iii) = \frac{13}{60}$ $(iv) = \frac{7}{15}$
4. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \, n \, B^1) = \frac{1}{3}$. Find
   (i) $P(A^1 \, U \, B^1)$ (ii) $P\left(^{A^1}/_{B^1}\right)$
   An $(i) = \frac{5}{6}$ $(ii) = \frac{1}{2}$
5. Events A and B are such that $P(A \, n \, B) = \frac{1}{12}$, $P(^{B}/_{A}) = \frac{1}{3}$ and $P(B) = \frac{1}{6}$. Find
   (i) $P(A)$ (ii) $P(^{A}/_{B})$ (iii) $P\left(^{A}/_{B^1}\right)$
   An $(i) = 0.25$ $(ii) = 0.5$ $(iii) = 0.2$
6. Events A and B are such that $P(A \, U \, B) = 0.8$, $P(^{A}/_{B}) = 0.2$ and $P(A^1 \, n \, B) = 0.4$. Find.
   (i) $P(A \, n \, B)$ (ii) $P(B)$ (iii) $P(A)$
   (iv) $P\left(^{A}/_{B^1}\right)$ (v) $P\left(^{A^1}/_{B^1}\right)$
   An $(i) = 0.1$ $(ii) = 0.5$ $(iii) = 0.4$
   $(iv) = 0.6$, $(v) = 0.4$
7. Events A and B are independent. Given that $P(A) = 0.2$ and $P(B) = 0.15$. Find
   (i) $P(A \, nB)$ (ii) $P(^{A}/_{B})$ (iii) $P(A \, UB)$
   An $(i) = 0.03$ $(ii) = 0.2$ $(iii) = 0.32$
8. Events A and B are such that $P(A) = 0.2$, $P(^{A}/_{B}) = 0.4$ and $P(B) = 0.25$. Find
   (i) $P(A \, nB)$ (ii) $P(^{B}/_{A})$ (iii) $P(A \, UB)$
   An $(i) = 0.1$ $(ii) = 0.5$ $(iii) = 0.35$
9. Events A and B are such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(^{A}/_{B}) = \frac{2}{5}$. Find
   (i) $P(A \, n \, B)$ (ii) $P(^{B}/_{A})$
   An $(i) = \frac{1}{10}$ $(ii) = \frac{3}{10}$
10. Events A and B are such that $P(A) = \frac{8}{15}$, $P(B) = \frac{1}{3}$ and $P(^{A}/_{B}) = \frac{1}{5}$. Find the;
    (i) probabilities that both events occur
    (ii) only one of the two events occurs
    (iii) Neither events occur
    An $(i) = \frac{1}{15}$ $(ii) = \frac{11}{15}$ $(iii) = \frac{1}{5}$
11. Events A and B are such that $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{4}$ and $P(^{A}/_{B}) = \frac{2}{3}$. Find
    (i) $P(A \, n \, B)$ (ii) $P(^{B}/_{A})$
    An $(i) = \frac{1}{6}$ $(ii) = \frac{1}{4}$
12. Events A and B are such that $P(B) = \frac{1}{3}$, $P(A) = \frac{1}{2}$ and $P(A \, nB^1) = \frac{1}{3}$. Find.
    (i) $P(A^1 \, U \, B^1)$ (ii) $P\left(^{B^1}/_{A^1}\right)$
    An $(i) = \frac{5}{6}$ $(ii) = \frac{2}{3}$
13. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{8}$ and $P(^{A}/_{B}) = \frac{7}{12}$. Find
    (i) $P(A \, n \, B)$ (ii) $P\left(^{B}/_{A^1}\right)$
    UNEB 2003 No.1 An $(i) = \frac{7}{32}$ $(ii) = \frac{5}{16}$
14. A and B are intersecting sets as shown in the Venn diagram below. **Uneb 2005 9a**



Given that $P(A) = 0.6$, $P\left(^{A^1}/_{B}\right) = \frac{5}{7}$ and $P(AUB) = 0.85$. Find
   (i) the value of x, y and z
   (ii) $P(^{A}/_{B})$ (iii) $P\left(^{A}/_{B^1}\right)$
   An (i) $x = 0.5, y = 0.1, z = 0.25$ $(ii) = \frac{2}{7}$ $(ii) = \frac{10}{13}$
15. Events A and B are independent with A twice as likely to occur as B. if $P(A) = 0.5$, find
    (i) $P(A \, U \, B)$ (ii) $P(^{A \, nB}/_{A})$
    UNEB 2006 No.1 An $(i) = \frac{5}{8}$ $(ii) = \frac{1}{8}$
16. Events A and B are such that $P(A) = \frac{4}{7}$, $P(^{A}/_{B}) = \frac{5}{14}$ and $P(A \, n \, B^1) = \frac{1}{3}$. Find
    i) $P(A \, n \, B)$ (ii) $P(B)$

(iii) $P(^B/_A)$ (iv) $P(A \cup B)$

UNEB 2014 No.9b An $(i) = \frac{5}{21}$ $(ii) = \frac{2}{3}$ $(iii) = \frac{5}{12}$ $(iv) = 1$

17. Two events A and B are such $P(^A/_B) = {^2/_5}$. $P(B) = \frac{1}{4}$ and $P(A) = \frac{1}{5}$. Find: **UNEB 2018** No.2
    (a) $P(A \cap B)$ (b) $P(A \cup B)$
    (b) **An** $(i) = 0.1$ $(ii) = 0.35$

COMBINATIONS

The number of combinations of r objects from n unlike objects is n_{C_r} where

$$n_{C_r} = \frac{n!}{(n-r)!\,r!}$$

Examples

1. Find the number of ways of selecting a football team from 15 players.

   Solution

$$15_{C_{11}} = 1365 \text{ ways}$$

2. A committee of 4 men and 3 women is to be formed from 10 men and 8 women. In how many ways can the committee be formed.

   Solution

$$10_{C_4} x 8_{C_3} = 210x56 = 11760 \text{ ways}$$

3. A group of 9 has to be selected from 10 boys and 8 girls. It can consist of either 5 boys and 4 girls or 4 boys and 5 girls. Find how many different groups can be chosen.

   Solution

$$10_{C_5} x 8_{C_4} + 10_{C_4} x 8_{C_5} = 252x70 + 210x56 = 29400 ways$$

4. A bag contains 5 Pepsi and 4 Mirinda bottle tops. Three bottle tops are picked at random from the bag one after the other without replacement. Find the probability that the bottle tops picked are of the same type. **Uneb 2016 No.8**

   Solution

$n\,(s) = 3 \text{ tops from the } 9 = 9_{C_3} = 84 ways$

$n\,(E) = 3 \text{ pepsi from } 5 + 3 \text{ mirinda from } 4$

$= 5_{C_3} x 4_{C_0} + 5_{C_0} x 4_{C_3}$

$= 10 + 4 = 4 ways$

$P(\text{same type}) = \frac{14}{84} = \frac{1}{6}$

5. In a group of 12 international referees, there are 3 from Africa, 4 from Asia and 5 from Europe. To officiate at a tournament 3 referees are chosen at random from the group find the probability that;
   (i) A referee is chosen from each continent (ii) Exactly 2 referees are chosen from Asia
   (iii) 3 referees are chosen from the same continent

   Solution

$n\,(s) = 3 \text{ refs from the } 12 = 12_{C_3} = 220 ways$

$n\,(E) = 3_{C_1} x 4_{C_1} x 5_{C_1} = 60 ways$

$P(1 \text{ from each}) = \frac{60}{220} = \frac{3}{11}$

(ii) $n\,(E) = 4_{C_2} x 3_{C_1} x 5_{C_0} + 4_{C_2} x 3_{C_0} x 5_{C_1}$

$= 18 + 30 = 48 ways$

$P(2 \text{ from Asia}) = \frac{48}{220} = \frac{12}{55}$

(iii) $n\,(E) = 4_{C_3} x 3_{C_0} x 5_{C_0} + 4_{C_0} x 3_{C_3} x 5_{C_0} + 4_{C_0} x 3_{C_0} x 5_{C_3}$

$= 4 + 1 + 10 = 15 ways$

$P(3 \text{ from same}) = \frac{15}{220} = \frac{3}{44}$

6. Box P contains 4 red and 3 green sweets and box Q contains 7 red and 4 green sweets. A box is randomly selected and 2 sweets are randomly picked from it, one at a time without replacement. If P is twice as likely to be picked as Q, find the probability that both sweets are
   (i) Same colour (ii) of different colours,
   (iii) from P given that they are of different colours.

   Solution

(i) $P(both\ balls\ same\ colour) = \frac{2}{3}\left[\frac{^4C_2 x ^3C_0}{^7C_2}\right] + \frac{2}{3}\left[\frac{^4C_0 x ^3C_2}{^7C_2}\right] + \frac{1}{3}\left[\frac{^7C_2 x ^4C_0}{^{11}C_2}\right] + \frac{1}{3}\left[\frac{^7C_0 x ^4C_2}{^{11}C_2}\right] = 0.4494$

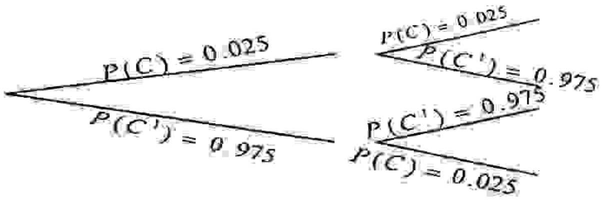
(ii) $P(both\ balls\ different\ colour) = \frac{2}{3}\left[\frac{^4C_1 x ^3C_1}{^7C_2}\right] + \frac{1}{3}\left[\frac{^7C_1 x ^4C_1}{^{11}C_2}\right] = 0.5506$

(iii) $P\left(from\ P/_{different\ colour}\right) = \frac{\frac{2}{3}\left[\frac{^4C_1 x ^3C_1}{^7C_2}\right]}{0.5506} = 0.6919$

PROBABILITY TREE DIAGRAMS

1. A factory makes yoghurts. When an inspector tests a random sample of yoghurts, the probability of any yoghurt being contaminated is 0.025. if a student buys two of the yoghurts made the factory. Find the probability
   (i) Both yogurts are contaminated (ii) Only one is contaminated

Solution

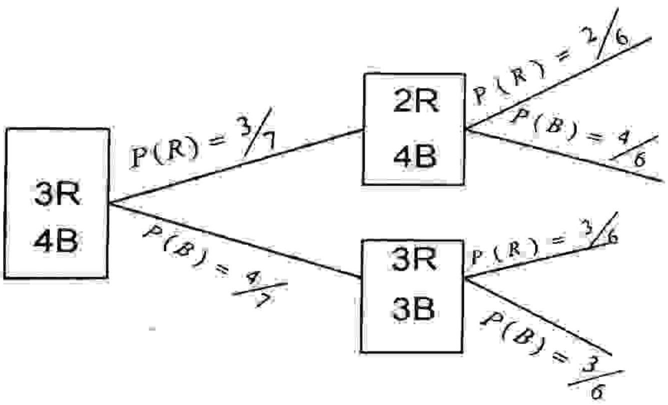


(i) $P(both\ contaminated) = P(CnC)$
$= 0.025x0.025 = 0.00063$

$P(One\ contaminated) = P(CnC^1) + P(C^1nC)$
$= 0.025x0.975 + 0.975x0.025 = 0.0488$

2. A box contains 3 red balls and 4 blue balls. Two balls are randomly drawn one after the other without replacement. Find the probability that
   (i) 1st ball is blue
   (ii) 2nd ball is red
   (iii) 2nd ball is red give that the 1st was blue
   (iv) Both ball are of the same colour
   (v) Different colour

Solution



(i) $P(1^{st}\ ball\ blue) = \frac{4}{7}x\frac{3}{6} + \frac{4}{7}x\frac{3}{6} = 0.5714$

(ii) $P(2^{nd}\ ball\ red) = \frac{4}{7}x\frac{3}{6} + \frac{3}{7}x\frac{2}{6} = 0.4286$

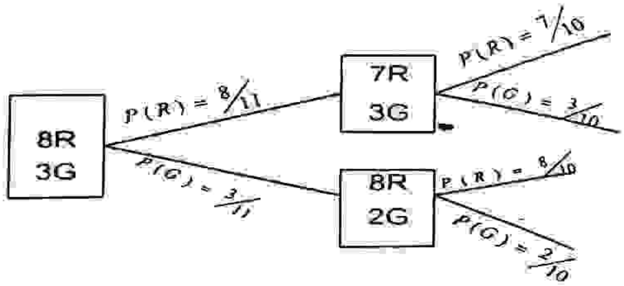
(iii) $P(2^{nd}\ ball\ red/_{1^{st}\ ball\ blue}) = \frac{(\frac{4}{7}x\frac{3}{6})}{(\frac{4}{7}x\frac{3}{6}+\frac{4}{7}x\frac{3}{6})}$
$= 0.5$

(iv) $P(both\ balls\ same\ colour) = \frac{4}{7}x\frac{3}{6} + \frac{3}{7}x\frac{2}{6}$
$= 0.4286$

(v) $P(both\ balls\ different\ colour) = 1 - \frac{3}{7}$
$= 0.5714$

3. A Bag contains 8 red pens and 3 green pens. Two pen are randomly picked one after the other, find the probability of drawing two pens of different colours, if the
   (i) First pen is not replaced (ii) First pen is replaced

Solution



(i) $P(both\ different\ colour) = \frac{8}{11}x\frac{3}{10} + \frac{3}{11}x\frac{8}{10}$
$= \frac{24}{55}$

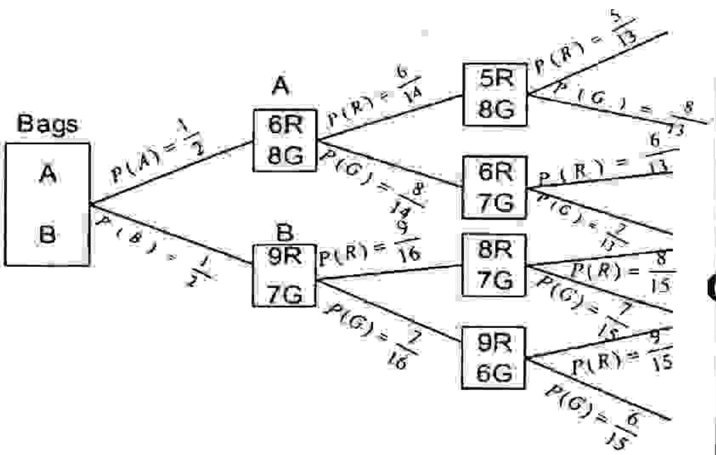
(ii) $P(both\ different\ colour) = \frac{8}{11}x\frac{3}{11} + \frac{3}{11}x\frac{8}{11}$
$= \frac{48}{121}$

4. Box A contains 6 red and 8 green sweets and box B contains 9 red and 7 green sweets. A box is randomly selected and 2 sweets are randomly picked from it, one at a time without replacement. If A is likely to be picked as B, find the probability that both sweets are

(i) Same colour (ii) of different colours,

(iii) from B given that they are of same colours.

Solution



(i) $P(same\ colour) = \frac{1}{2}x\frac{6}{14}x\frac{5}{13} + \frac{1}{2}x\frac{8}{14}x\frac{7}{13} + \frac{1}{2}x\frac{9}{16}x\frac{8}{15} + \frac{1}{2}x\frac{7}{16}x\frac{6}{15}$
$= 0.4738$

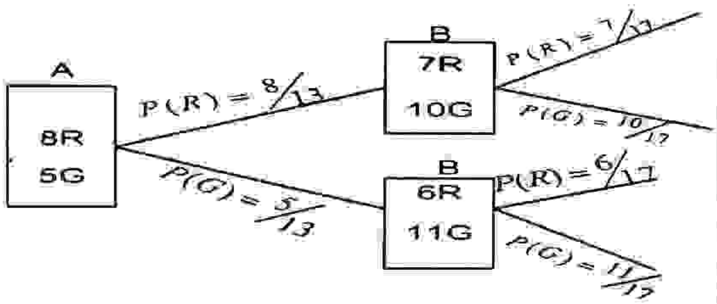
(ii) $P(both\ balls\ different\ colour) = 1 - \frac{3831}{7280}$
$= 0.5262$

(iii) $P\left(^{from\ B}/_{same\ colour}\right) =$

$$= \frac{\frac{1}{2}x\frac{6}{14}x\frac{5}{13} + \frac{1}{2}x\frac{8}{14}x\frac{7}{13}}{(0.4738)} = 0.4987$$

5. Bag A contains 8 red pens and 5 green pens. Bag B contains 6 red and 10 green pens. A pen is randomly picked from bag A and placed in bag B. A pen is then randomly picked form bag B, find the probability that it will be red

Solution



$P(Red\ pen) = \frac{8}{13}x\frac{7}{17} + \frac{5}{13}x\frac{6}{17} = \frac{86}{221}$
$= 0.3891$

Exercise 4e

1. A box contains 15 red balls and 5 black balls. Two balls are randomly drawn one after the other without replacement. Find the probability that;
   (i) Both are red
   (ii) Are of different colour
   (iii) Both are black, given that the second ball is black **An** $(i) = \frac{21}{38}\ (ii) = \frac{15}{38}\ (iii) = \frac{20}{83}$
2. A box contains 4 red balls and 6 black balls. Two balls are randomly drawn one after the other without replacement. Find the probability that;
   (i) Second ball is red, given that the first one ball is red
   (ii) Both balls are red
   (iii) Both balls are of different colours
   An $(i) = \frac{1}{3}\ (ii) = \frac{2}{15}\ (iii) = \frac{8}{15}$
3. A box contains 3 black balls and 5 white balls. Two balls are randomly drawn one after the other without replacement. Find the probability that
   (i) Second ball is white
   (ii) First ball white, given that the second ball is white **An** $(i) = \frac{5}{8}\ (ii) = \frac{4}{7}$
4. A box contains 3 red sweets, 8 blue sweets and 7 green sweets. Three sweets are randomly drawn one after the other without replacement. Find the probability that;
   (i) all sweets are blue (ii) all sweets are red
   (iii) one of each colour
   An $(i) = \frac{7}{102}\ (ii) = \frac{1}{816}\ (iii) = \frac{7}{34}$
5. A box contains 3 red pens and 6 black pens. Three pens are randomly drawn one after the other without replacement. Find the probability that;
   (i) 3 red pens

(ii) Two red pens and one black pen

(iii) More than one black pen

An $(i) = \frac{5}{21}$ $(ii) = \frac{15}{28}$ $(iii) = \frac{19}{84}$

6. A box contains 7 black sweets and 3 white sweets. Three sweets are randomly drawn one after the other with replacement. Find the probability that;
   (i) All three sweets are black
   (ii) A white, black and a white sweet in that order are chosen
   (iii) Two white and one black sweet drawn
   (iv) At least one black sweet drawn

   An $(i) = 0.34$ $(ii) = 0.063$
   $(iii) = 0.19$ $(iv) = 0.97$

7. A die is thrown three times. What is the probability of scoring a two on just one occasion

   An $(i) = \frac{25}{72}$

8. A coin is tossed four times, Find the probability of obtaining less than two heads **An** $(i) = \frac{5}{16}$

9. The probability that I have to wait at the traffic lights on my way to work is 0.25. find the probability that, on two consecutive mornings, I have to wait on at least one morning. **An** $= \frac{7}{16}$

10. The probability that I am late for work is 0.05. Find the probability that, on two consecutive mornings;
    (i) I am late for work twice
    (ii) I am late for work once

    An $(i) = 0.0025$ $(ii) = 0.095$

11. Box A contains 3 red balls and 4 brown balls while box B contains 3 red balls and 2 brown balls. A box is drawn randomly and one ball is randomly drawn from it. Find the probability that the ball;
    (i) The ball is red **An** $(i) = \frac{18}{35}$ $(ii) = \frac{5}{12}$
    (ii) The ball came from box A, given that it is red

12. A bag contains 10 green balls and 6 black balls. Two balls are randomly drawn one after the other without replacement. Find the probability that
    (i) Black given that the first one was green
    (ii) white **UNEB 2000 No.3**

    An $(i) = 0.4$ $(ii) = 0.375$

13. Box P contains 3 red balls and 2 blue balls while box Q contains 2 red balls and 3 blue balls. A box is drawn randomly and two balls are randomly drawn from it, one after the other with out replacement. Find the probability that the balls are of different colour **UNEB 2001 No.10 An** $= \frac{19}{30}$

14. A box contains 4 white balls and 1 black ball. A second box contains 1 white ball and 4 black balls. A ball is drawn at random from the first bag and put into the first bag. Find the probability that a white ball will be picked when a ball is selected from the first bag;

    UNEB 2005 No.9 An $= \frac{7}{10}$

15. (a) A box contains 7 red balls and 6 blue balls. Three balls are selected at random without replacement. Find the probability that;
    (i) They are of the same colour
    (ii) At most two are blue

    (b) Two boxes P and Q contain white and brown cards. P contains 6 white and 4 brown. Q contains 2 white and 3 brown. A box is selected at random and a card is selected. Find the probability that;
    (i) A brown card is selected
    (ii) Box Q is selected given that the card is white **UNEB 2007 No.15**

    An $(i) = 0.1923$ $(ii) = 0.9301, (b)(i) = 0.5, (ii) = 0.4$

16. A box contains two types of balls, red and black. When a ball is picked from the box, the probability that it is red is $^7/_{12}$. Two balls are selected at random from the box without replacement. Find the probability that
    (i) The second ball is black
    (ii) The first ball is red, given that the second one is black

    UNEB 2009 No.13a An $(i) = {}^5/_{11}$ $(ii) = {}^7/_{11}$

17. A bag contains 20 good and 4 bad oranges. If 5 oranges are selected at random without replacement. Find the probability that the 4 are good and the other is bad. **UNEB 2012 No.15a An** =

18. A bag contains 30 white, 20 blue and 20 red balls. Three balls are selected at random without replacement. Find the probability that the first ball is white and the third ball is also white. **UNEB 2014 No.9a An** $= 0.18$

19. A box A contains 4 white and 2 red balls. Box B contains 3 white and 2 red balls. A box is selected at random and two balls are picked one after the other without replacement.
    (i) Find the probability that the two balls picked are red
    (ii) Given that two white balls are picked, what is the probability that they are from box B **UNEB 2015 No.16**

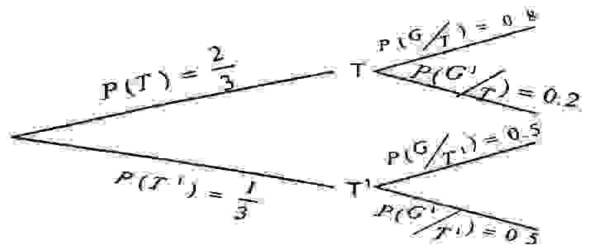
    An $(i) = 0.1333$ $(ii) = 0.3333$

CONDITIONAL PROBABILITY USING A TREE DIAGRAM / BAYE'S RULE

Examples

1. During planting season a farmer treats $\frac{2}{3}$ of his seeds and $\frac{1}{3}$ of the seeds are left untreated. The seeds which are treated have a probability of germinating of 0.8 while the untreated seeds have a probability of germination of 0.5. find the probability that a seed selected at random
   (i) will germinate
   (ii) had been treated, given that it had germinated

Solution

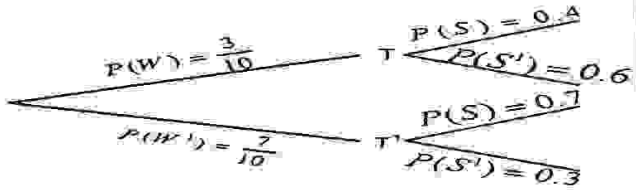


(i) $P(G) = P(TnG) + P(T^1n\,G)$
$= \frac{2}{3}x0.8 + \frac{1}{3}x0.5 = 0.7$

(ii) $P(^T/_G) = \frac{P(TnG)}{P(G)} = \frac{\frac{2}{3}x0.8}{0.7} = 0.762$

2. The probability that a golfer hits the ball on the green if it is windy as he strikes the ball is 0.4 and the corresponding probability if it is not windy as he strikes the ball is 0.7. The probability that the wind blow as he strikes the ball is $\frac{3}{10}$. Find the probability that
   (i) He hits the ball on the green
   (ii) It was not windy, given that he does not hit the ball on to the green

Solution

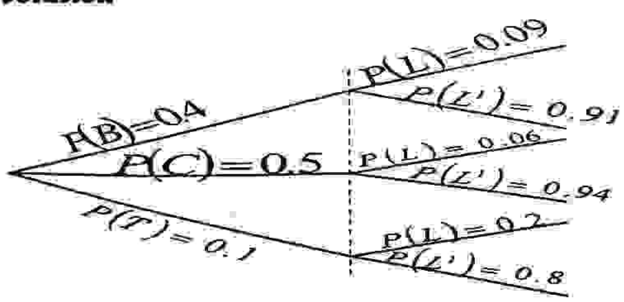


(i) $P(S) = P(WnS) + P(W^1n\,S)$
$= 0.3x0.4 + 0.7x0.7 = 0.61$

(ii) $P\left(^{W^1}/_{S^1}\right) = \frac{P(W^1n\,S^1)}{P(S^1)} = \frac{(\frac{7}{10}x0.3)}{1-0.61} = 0.5385$

3. When students were to go for a geography tour, the school hired three different types of vehicles, buses, coasters and taxis. Of the hiring's 40% were buses, 50% were coasters and 10% were taxis. For bus hired, 9% arrive late, the corresponding percentages for a coaster and a taxi being 6% and 20% respectively. Find the probability that the next vehicle hire
   (i) Will be a bus and will not arrive late
   (ii) Will arrive late
   (iii) Will be a coaster given that it will arrive late

Solution



(i) $P(BnL^1) = 0.4x0.91 = 0.364$

(ii) $P(L) = P(BnL) + P(CnL) + P(TnL)$
$= 0.4x0.09 + 0.5x0.06 + 0.1x0.2$
$= 0.086$

(iii) $P(^C/_L) = \frac{P(CnL)}{P(L)} = \frac{(0.5x0.06)}{0.086} = 0.3488$

Exercise 4f

1. Mo. Salah has an injury. When he is playing, the probability that Liverpool wins is 0.75 but otherwise it is only 0.5. The probability that he will be fit to play this weekend is $\frac{1}{3}$. Determine the probability that Liverpool will win the match. **An**= $\frac{7}{12}$

2. 55% of the teachers at St Noa are male. 30% of the male teachers, teach sciences and 5% of the

female teachers teach sciences. If a teacher is selected at random, what is the probability that;

(i) Teaches sciences
(ii) She is female given that she doesn't teach sciences **An**$(i) = 0.1875 (ii) = 0.5262$

3. The probability that a golfer hits the ball on the green if it is windy as he strikes the ball is 0.4 and the corresponding probability if is not windy as he strikes the ball is 0.7. The probability that the wind blow as he strikes the ball is $\frac{3}{10}$. Find the probability that
   (i) He hits the ball on the green
   (ii) It was not windy, given that he does not hit the ball on to the green **An**$(i) = 0.61 \quad (ii) = 0.5385$

4. In a restaurant, 40% of the customer's order for chicken. If a customer orders chicken, the probability he will take juice is 0.6. If he does not order chicken, the probability that he will take juice is 0.3. find the probability that a customer picked at random will order
   (i) chicken and juice (ii) juice
   An$(i) = 0.24 \ (ii) = 0.42$

5. Phone screens are inspected for defects. The probability that a phone has air bubble is 0.02. If a phone has air bubbles the probability that it is cracked is 0.5 while the probability that a phone free of air bubbles is cracked is 0.05. What is the probability that a phone chosen at random is cracked? **An**= 0.059

6. Whether or not the girls in a certain school go for morning preps depends on whether they are woken up in the morning by preps masters. For 85% of the mornings girls are woken up and the other 15% of the mornings they are not woken up. If the girls are woken up, they go for morning preps 90% of the mornings. If the girls are not woken, they do not go for morning preps 60% of the mornings. On what proportion of mornings do girls go for morning preps. **An**= 0.825

7. In a factory, there are two different machines A and B. Items are produced from A and B with respective probabilities of 0.2 and 0.8. It was established that 5% and 8% produced by A and B respectively are defective. If one item is selected randomly, fi d the probability that
   (i) It is defective
   (ii) It is produced by A given that it is defective. **An**$(i) = 0.074 \ (ii) = 0.1351$

8. Data from electoral commission showed that in the previous election, of all the Kampala parliamentary contestants, 70% were N.R.M, 20% were F.D.C and 10% were independent. 5% of the N.R.M contestants won elections, 95% of the F.D.C contestants won elections and 25% of the independent won elections. If a contestant is chosen at random, find the probability that the person
   (i) won election **An**$(i) = 0.25 \ (ii) = 0.14$
   (ii) An F.D.C won an election

9. A student travels to school by route A or route B. The probability she uses route A is 0.25. The probability that is late to school if she uses route A is $\frac{2}{3}$ and the corresponding probability if she uses route B is $\frac{1}{3}$.
   (i) Find the probability that she will be late to school **An**$(i) = \frac{5}{12} \quad (ii) = \frac{3}{5}$
   (ii) Given that she is late, what is the probability that she used route B

10. A student is to travel to a school for an interview. The probability that he will be in time for the interview when he travels taxi and boda respectively are 0.1 and 0.2 respectively. The probability that he will travel by taxi and boda are 0.6 and 0.4 respectively
   (i) Find the probability that he will be on time
   (ii) Given that he is not time, what is the probability that he traveled by boda
   An$(i) = 0.14 \quad (ii) = 0.372$

11. 64% of the students at A 'level take Arts subjects and 36% take science subjects. The probability of an arts student passing is $\frac{3}{4}$ and a science student passing is $\frac{5}{6}$. Find the probability that a student chosen at random failed exams **An**= 0.22

12. Of the group of students studying A-level in a school, 56% are boys and 44% are girls. The probability that a boy of this group is studying chemistry is $\frac{1}{5}$ and the probability that a girl of this group is studying chemistry is $\frac{1}{11}$;
   (i) Find the probability that a student selected at random from this group is a girl studying chemistry
   (ii) Find the probability that a student selected at random from this group is not studying chemistry
   (iii) Find the probability that a chemistry student selected at random from this group is male
   An$(i) = \frac{1}{25} (ii) = \frac{106}{125} \ (iii) = \frac{14}{19}$

13. When a school wants to buy chalk, the school phones three suppliers A, B or C. of the phone calls to them, 30% are to A, 10% to B and 60% to C. The percentage of occasions when the suppliers deliver chalk after a phone call to them are, 20% for A, 6% for B and 9% for C.

(i) Find the probability that the supplier phoned will not be able to deliver chalk on the day if phoning

(ii) Given that the school phones a supplier and the supplier can deliver chalk that, find the probability that the school phoned supplier B **An**$(i) = 0.88$ $(ii) = 0.05$

14. In a girl's school, it is known that the probability of selecting a girl who is pregnant is 0.02. if the probability of a nurse correctly testing a pregnant girl as being pregnant is 0.7 and the probability of incorrectly testing a girl who is not pregnant as being pregnant is 0.05. What is the probability that?

(i) a girl is tested as being pregnant

(ii) a girl is tested as being pregnant given that she is not pregnant

(iii) a girl is pregnant given that she is tested pregnant

(iv) a girl is not pregnant given that she is tested pregnant. **An**$(i) = 0.063$ $(ii) = 0.05$ $(iii) = 0.2333$ $(iv) = 0.0171$

15. In Kampala city, 30% of the people are F.D.C, 50% are N.R.M and 20% are independent. Records show that in the previous election, 65% of the F.D.C voted, 85% of the N.R.M voted and 50% of the independent voted. A person is randomly selected from the city.

(i) Find the probability that the person voted

(ii) Given that the person didn't vote, determine the probability that he is an F.D.C **An**$(i) = 0.72$ $(ii) = 0.375$

16. Three girls, Annette, Brenda and Sauron pack biscuits in a factory. From the batch allotted to then Annette packs 55%, Brenda 30% and Sauron 15%. The probability that Annette breaks some biscuits in a packet is 0.7 and the respective probabilities for Brenda and Sauron are 0.2 and 0.1. What is the probability that a packet with broken biscuits found by the checker was packed by Annette? **An**$= 0.837$

17. A shop stocks two brands of tooth paste, Colgate tooth paste and Fresh up tooth paste, and of two sizes, large and small. Of the stock 70% is Colgate, and 30% is fresh up. Of the Colgate, 30% are small size and of the fresh up, 40% are small size. Find the probability that;

(i) A tooth paste chosen at random from the stock will be of small size

(ii) Small tooth paste chosen at random from the stock will be of Colgate

An$(i) = 0.33$ $(ii) = \frac{7}{11}$

18. At a bus park, 60% of the buses are of Teso coaches, 25% are Kakise buses and the rest are Y.Y buses. Of the Teso coaches 50% have TVs, while for the Kakise and Y.Y buses only 5% and 1% have TVs respectively. If a bus is selected at random form the park, determine the probability that **UNEB 1999 No.7**

(i) It has a TV

(ii) Kakise bus is selected given that it has a TV **An**$(i) = 0.0315$ $(ii) = 0.0398$

19. On a certain day, fresh fish from lakes, Kyoga, Victoria, Albert and George were supplied to a market in ratio a 30%, 40%, 20% and 10% respectively. Each lake had an estimated ratios of poisoned fish of 2%, 3%, 3% and 1% respectively. If a health inspector picked a fish at random. **UNEB 2002 No.1**

(i) What is the probability that the fish was poisoned?

(ii) Given that the fish was poisoned, what is the probability that it was from lake Albert. **An**$(i) = 0.025$ $(ii) = 0.24$

20. The chance that a person picked from a kampala street is employed is 30 in every 48. The probability that a person is a university graduate is employed is 0.6. Find;. **UNEB 2002 No.4**

(i) the probability that the person picked at random from the street is a university graduate and is employed.

(ii) Number of people that are not university graduates and are employed from a group of 120 people. **An**$(i) = 0.375$ $(ii) = 30$

PROBABILITY DISTRIBUTIONS 1- DISCRETE RANDOM VARIABLE

A probability density function (p.d.f) is discrete if it takes on specific values

Properties of discrete probability density functions

(i) $\sum P(X = x) = 1$ OR $\sum f(x) = 1$

(ii) $P(X = x) \geq 0$

Examples

1. A discrete random variable has a probability function $P(X = x) = \begin{cases} cx^2, & x = 0, 1, 2, 3, 4 \\ 0, & otherwise \end{cases}$

   Find the value of c and draw the graph of $P(X = x)$

   Solution

   $$\sum P(X = x) = 1$$

   $$c(0^2) + c(1)^2 + c(2)^2 + c(3)^2 + c(4)^2 = 1$$

   $$c + 4c + 9c + 16c = 1$$

   $$c = \frac{1}{30}$$

2. A discrete random variable has a probability function

   $$f(x) = \begin{cases} kx, & x = 1, 2, 3, 4 \\ 0, & otherwise \end{cases}$$

   Find the value of k and draw the graph of $f(x)$

   Solution

   $$\sum f(x) = 1$$

   $$k + 2k + 3k + 4k = 1$$

   $$k = \frac{1}{10}$$

3. A random variable X of a discrete probability distribution is given by

   $P(X = 1) = 0.2, \quad P(X = 2) = P(X = 3) = 0.1, \quad P(X = 4) = P(X = 5) = C,$

   Find the value of the constant C and draw the graph of $P(X = x)$

   Solution

   $$\sum P(X = x) = 1$$

   $$0.2 + 0.1 + 0.1 + c + c = 1$$

   $$c = 0.3$$

4. A discrete random variable has a probability function

   $$P(X = x) = \begin{cases} k\left(\frac{2}{3}\right)^x, & x = 0, 1, 2, 3 \ldots\ldots\ldots\ldots \\ 0, & otherwise \end{cases}$$

   Find the value of k

   Solution

   $$k\left(\frac{2}{3}\right)^0 + k\left(\frac{2}{3}\right)^1 + k\left(\frac{2}{3}\right)^2 + k\left(\frac{2}{3}\right)^3 \ldots\ldots\ldots = 1$$

   $$k\left(1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots\ldots\ldots\right) = 1$$

   Sum to infinity $S_\infty = \frac{a}{1-r}$

   $$k\left(\frac{1}{1 - \frac{2}{3}}\right) = 1$$

   $$k = \frac{1}{3}$$

Finding probabilities

1. A discrete random variable Y has a probability distribution

Y	-3	-2	-1	0	1
P(Y=y)	0.1	0.25	0.3	0.15	a

Find

(i) The value of a

(ii) $P(-3 \leq Y < 0)$

(iii) $P(Y > -1)$

(iv) $P(-1 < Y < 1)$

(v) mode

Solution

(i) $\sum P(Y = y) = 1$

$$0.1 + 0.25 + 0.3 + 0.15 + a = 1$$

$a = 0.2$

(ii) $P(-3 \le Y < 0) = P(Y = -3) + P(Y = -2) + P(Y = -1)$
$= 0.1 + 0.25 + 0.3 = 0.65$

(iii) $P(Y > -1) = P(Y = 0) + P(Y = 1)$
$= 0.15 + 0.2 = 0.35$

(iv) $P(-1 < Y < 1) = P(Y = 0) = 0.15$

(v) Mode is the value of y with the highest probability, mode is $= -1$

2. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.15	0.20	0.15	c	0.1

Find

(i) The value of c

(ii) $P(X < 4)$

(iii) $P(X \le 4)$

(iv) $P(2 \le X \le 4)$

(v) $P(X > 2 / X \le 4)$

(vi) Mode

Solution

(i) $\sum P(X = x) = 1$
$0.15 + 0.20 + 0.15 + c + 0.1 = 1$
$c = 0.4$

(ii) $P(X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$
$= 0.15 + 0.20 + 0.15 = 0.5$

(iii) $P(X \le 4) = P(X = 1) + P(X = 2) + (X = 3) + P(X = 4)$
$= 0.15 + 0.20 + 0.15 + 0.4 = 0.9$

(iv) $P(2 \le X \le 4) = P(X = 2) + P(X = 3) + P(X = 4)$
$= 0.2 + 0.15 + 0.4 = 0.75$

(v) $P\left(X > 2 / X \le 4\right) = \frac{P(X>2 \cap X \le 4)}{P(X \le 4)}$
$= \frac{P(X = 3) + P(X = 4)}{P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)}$
$= \frac{0.15 + 0.4}{0.9} = 0.6111$

(vi) Mode is the value of x with the highest probability, mode is $= 4$

3. A discrete random variable X has a probability distribution $f(x) = \begin{cases} kx, & x = 1, 2, 3, 4, 5 \\ 0, & otherwise \end{cases}$

Find

(i) The value of k

(ii) $P(X = 3)$

(iii) $P(X \ge 3)$

(iv) $P(X \le 3)$

(v) $P(1 < X \le 3)$

(vi) $P(X \ge 1 / X < 4)$

Solution

(i) $\sum f(x) = 1$
$k + 2k + 3k + 4k + 5k = 1$
$k = \frac{1}{15}$

(ii) $P(X = 3) = 3k = \frac{3}{15} = \frac{1}{5}$

(iii) $P(X \ge 3) = P(X = 3) + P(X = 4) + (X = 5)$
$= 3k + 4k + 5k = 12k = \frac{4}{5}$

(iv) $P(X \le 3) = P(X = 1) + P(X = 2) + (X = 3)$
$= k + 2k + 3k = 6k = \frac{6}{15}$

(v) $P(1 < X \le 3) = P(X = 2) + P(X = 3)$
$= 2k + 3k = 5k = \frac{1}{3}$

(vi) $P\left(X \ge 2 / X < 4\right) = \frac{P(X \ge 2 \cap X<4)}{P(X<4)}$
$= \frac{P(X = 2) + P(X = 3)}{P(X = 1) + P(X = 2) + P(X = 3)}$
$= \frac{2k + 3k}{k + 2k + 3k} = \frac{5k}{6k} = \frac{5}{6}$

Exercise 5a

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	a	0.05

Find

(i) The value of a

(ii) $P(1 \le X \le 3)$

(iii) $P(X > 2)$

(iv) $P(2 < X < 5)$

(v) Mode

An$(i = 0.1, ii = 0.85, iii = 0.55, iv = 0.5, v = 3)$

2. A random variable X of a discrete p.d.f is given by $P(X = x) = kx, \ x = 12, 13, 14$

$$a = 0.2$$

(ii) $P(-3 \le Y < 0) = P(Y = -3) + P(Y = -2) + P(Y = -1)$
$= 0.1 + 0.25 + 0.3 = 0.65$

(iii) $P(Y > -1) = P(Y = 0) + P(Y = 1)$
$= 0.15 + 0.2 = 0.35$

(iv) $P(-1 < Y < 1) = P(Y = 0) = 0.15$

(v) Mode is the value of y with the highest probability, mode is $= -1$

2. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.15	0.20	0.15	c	0.1

Find
(i) The value of c
(ii) $P(X < 4)$
(iii) $P(X \le 4)$
(iv) $P(2 \le X \le 4)$
(v) $P(X > 2 / X \le 4)$
(vi) Mode

Solution

(i) $\sum P(X = x) = 1$
$0.15 + 0.20 + 0.15 + c + 0.1 = 1$
$c = 0.4$

(ii) $P(X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$
$= 0.15 + 0.20 + 0.15 = 0.5$

(iii) $P(X \le 4) = P(X = 1) + P(X = 2) + (X = 3) + P(X = 4)$
$= 0.15 + 0.20 + 0.15 + 0.4 = 0.9$

(iv) $P(2 \le X \le 4) = P(X = 2) + P(X = 3) + P(X = 4)$
$= 0.2 + 0.15 + 0.4 = 0.75$

(v) $P\left(X > 2 / X \le 4\right) = \frac{P(X>2 \cap X\le4)}{P(X\le4)}$
$= \frac{P(X = 3) + P(X = 4)}{P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)}$
$= \frac{0.15 + 0.4}{0.9} = 0.6111$

(vi) Mode is the value of x with the highest probability, mode is $= 4$

3. A discrete random variable X has a probability distribution $f(x) = \begin{cases} kx, & x = 1,2,3,4,5 \\ 0, & otherwise \end{cases}$

Find
(i) The value of k
(ii) $P(X = 3)$
(iii) $P(X \ge 3)$
(iv) $P(X \le 3)$
(v) $P(1 < X \le 3)$
(vi) $P(X \ge 1 / X < 4)$

Solution

(i) $\sum f(x) = 1$
$k + 2k + 3k + 4k + 5k = 1$
$k = \frac{1}{15}$

(ii) $P(X = 3) = 3k = \frac{3}{15} = \frac{1}{5}$

(iii) $P(X \ge 3) = P(X = 3) + P(X = 4) + (X = 5)$
$= 3k + 4k + 5k = 12k = \frac{4}{5}$

(iv) $P(X \le 3) = P(X = 1) + P(X = 2) + (X = 3)$
$= k + 2k + 3k = 6k = \frac{6}{15}$

(v) $P(1 < X \le 3) = P(X = 2) + P(X = 3)$
$= 2k + 3k = 5k = \frac{1}{3}$

(vi) $P\left(X \ge 2 / X < 4\right) = \frac{P(X\ge2 \cap X<4)}{P(X<4)}$
$= \frac{P(X = 2) + P(X = 3)}{P(X = 1) + P(X = 2) + P(X = 3)}$
$= \frac{2k + 3k}{k + 2k + 3k} = \frac{5k}{6k} = \frac{5}{6}$

Exercise 5a

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	a	0.05

Find
(i) The value of a
(ii) $P(1 \le X \le 3)$
(iii) $P(X > 2)$
(iv) $P(2 < X < 5)$
(v) Mode

An$(i = 0.1, ii = 0.85, iii = 0.55, iv = 0.5, v = 3)$

2. A random variable X of a discrete p.d.f is given by $P(X = x) = kx,\ x = 12, 13, 14$

Write out the probability distribution and find the value of the constant k.

An

x	12	13	14
P(X=x)	12k	13k	14k

$k = \frac{1}{39}$

3. A random variable Y of a discrete probability distribution is given by $P(Y = 8) = 3P(Y = 10)$
$P(Y = 3) = 0.1$, $P(Y = 5) = 0.05$, $P(Y = 6) = 0.45$,
Find $P(Y = 10)$. **An**= 0.1

4. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	k	0.2	0.05

Find
(i) The value of k
(ii) $P(X \geq 4)$
(iii) $P(X < 1)$
(iv) $P(2 \leq X < 4)$

An$(i = \frac{7}{20}, ii = \frac{1}{4}, iii = 0, iv = \frac{13}{20},)$

5. Write out the probability distribution for each of these variables
(a) The number of heads, X obtained when two fair coins are tossed
(b) The number of tails, X obtained when three fair coins are tossed

An (a)

x	0	1	2
P(X=x)	0.25	0.5	0.25

(b)

x	0	1	2	3
P(X=x)	0.125	0.375	0.375	0.125

6. A drawer contains 8 brown socks and 4 blue socks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. This procedure is performed twice more. X is the random variable for the number of brown socks taken. Find the probability distribution for X

An

x	0	1	2	3
P(X=x)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

7. The discrete random variable R has a p.d.f is given by $P(R = r) = c(3 - r)$, $r = 0, 1, 2, 3$
Find
(i) The value of c
(ii) $P(1 \leq R < 3)$

An. $(i = \frac{1}{6}\ \ ii = \frac{1}{2})$

8. A game consists of throwing tennis balls into a bucket from a given distance. The probability that Bridget will get the tennis ball in the bucket is 0.4. A turn consists of 3 attempts
(a) Construct a probability distribution for X, the number of tennis balls that land in the bucket in a turn
(b) Bridget wins a prize if, at the end of her turn, if there are 2 or more tennis balls in the bucket. What is the probability that she does not win a prize

An (a)

x	0	1	2	3
P(X=x)	0.126	0.432	0.288	0.064

(b) =0.648

9. A discrete random variable has a probability function $P(X = x) = \begin{cases} k\left(\frac{4}{5}\right)^x, & x = 0,1,2,3\ \ldots\ldots \\ 0, & otherwise \end{cases}$

Find the value of k. **An** $k = \frac{1}{5}$

10. The number of times a phone freezes in a week follows a discrete random variable with a probability function $P(X = x) = \begin{cases} k\left(\frac{1}{4}\right)^x, & x = 0,1,2,3\ \ldots\ldots\ldots\ldots \\ 0, & otherwise \end{cases}$

Find the probability that a phone freezes not more than twice in a week. **An**= 0.984

11. Lisa plays a game in which she throws 2 dice. If she gets two sixes, she wins 20,000/=, if she gets one six she wins 10,000/=, otherwise she wins nothing. She pays 5,000/= to enter the game. Write down the probability distribution of X, the amount she gains in one turn

An

x	-5000	5000	15000
P(X=x)	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

EXPECTATION OF X, E(X) OR MEAN

The expected value of x is given by

$$E(X) = \sum xP(X = x)$$

Examples

1. A discrete random variable X has a probability distribution

x	-2	-1	0	1	2
P(X=x)	0.3	0.1	0.15	0.4	0.05

Find Expectation, E(x)

Solution

$$E(X) = \sum xP(X = x) = (-2x0.3) + (-1x0.1) + (0x0.15) + (1x0.4) + (2x0.05) = -0.2$$

2. The discrete random variable Y has a probability distribution is given by

$$P(Y = y) = cy, \quad y = 1, 2, 3,$$
$$P(Y = y) = c(8 - y), \quad y = 4, 5, 6, 7$$

Find

(i) The value of c (ii) $mean, \mu$

Solution

y	1	2	3	4	5	6	7
P(Y=y)	c	2c	3c	4c	3c	2c	c

(i) $\sum P(Y = y) = 1$

$$c + 2c + 3c + 4c + 3c + 2c + c = 1$$
$$c = \frac{1}{16}$$

(ii) $E(Y) = \sum yP(Y = y) = (1xc) + (2x2c) + (3x3c) + (4x4c) + (5x3c) + (6x2c) + (7xc)$

$$\mu = 4 = 64c = 64x\frac{1}{16} = 4$$

3. A fair coin is tossed three times, write out the probability distribution for the number of heads, X obtained and hence obtain the expected number of heads

Solution

x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$E(X) = \sum xP(X = x)$

$$= \left(0x\frac{1}{8}\right) + \left(1x\frac{3}{8}\right) + \left(2x\frac{3}{8}\right) + \left(3x\frac{1}{8}\right) = 1.5$$

4. A family plans to have 4 children. Given that X is the number of girls in the family. Find the expected number of girls

Solution

x	0	1	2	3	4
P(X=x)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$E(X) = \sum xP(X = x) = \left(0x\frac{1}{16}\right) + \left(1x\frac{4}{16}\right) + \left(2x\frac{6}{16}\right) + \left(3x\frac{4}{16}\right) + \left(4x\frac{1}{16}\right) = 2$$

5. A box contains 5 red balls and 3 blue balls. Two balls are drawn one at a time without replacement. Find the expected number of blue balls

Solution

x	0	1	2
P(X=x)	$\frac{20}{56}$	$\frac{30}{56}$	$\frac{6}{56}$

$E(X) = \sum xP(X = x)$

$$= \left(0x\frac{20}{56}\right) + \left(1x\frac{30}{56}\right) + \left(2x\frac{6}{56}\right) = 0.75$$

6. At St Noa , movie night follows a discrete random variable. Three girls, Jolly, Cathy and Sheila watch with probabilities of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{5}$ respectively. Write out the probability distribution X, for the number of girls who watch the movie and determine the number of girls who watched.

Solution

$P(X=3) = P(JnCnS) = \frac{1}{4}x\frac{1}{3}x\frac{2}{5} = \frac{1}{30}$

$P(X=2) = P(JnCn\bar{S}) + P(Jn\bar{C}nS) + P(\bar{J}nCnS)$

$= \left(\frac{1}{4}x\frac{1}{3}x\frac{3}{5}\right) + \left(\frac{1}{4}x\frac{2}{3}x\frac{2}{5}\right) + \left(\frac{3}{4}x\frac{1}{3}x\frac{2}{5}\right) = \frac{13}{60}$

$P(X=1) = P(Jn\bar{C}n\bar{S}) + P(\bar{J}nCn\bar{S}) + P(\bar{J}n\bar{C}nS)$

$= \left(\frac{1}{4}x\frac{2}{3}x\frac{3}{5}\right) + \left(\frac{3}{4}x\frac{1}{3}x\frac{3}{5}\right) + \left(\frac{3}{4}x\frac{2}{3}x\frac{2}{5}\right) = \frac{9}{20}$

$P(X=0) = P(\bar{J}n\bar{C}n\bar{S}) = \frac{3}{4}x\frac{2}{3}x\frac{3}{5} = \frac{3}{10}$

x	0	1	2	3
P(X=x)	$\frac{3}{10}$	$\frac{9}{20}$	$\frac{13}{60}$	$\frac{1}{30}$

$E(X) = \sum xP(X=x)$

$= \left(0x\frac{3}{10}\right) + \left(1x\frac{9}{20}\right) + \left(2x\frac{13}{60}\right) + \left(3x\frac{1}{30}\right) \approx 0.$

7. A gambling game consists of tossing 3 coins. A participant is paid 5,000/= if he gets either all head or all tail, otherwise he pays out 3,000/=. What is the participants expected gain per toss

Solution

x	5000	- 3000
P(X=x)	$\frac{2}{8}$	$\frac{6}{8}$

$E(X) = \sum xP(X=x)$

$= \left(5000x\frac{2}{8}\right) + \left(-3000x\frac{6}{8}\right) = -1000/=$

8. Box P contains 4 red and 3 green sweets and box Q contains 5 red and 6 green sweets. A box is randomly selected and 2 sweets are randomly picked from it, one at a time without replacement. If P is twice as likely to be picked as Q, find the probability that both sweets are **UNEB 2011 No15**
   (i) of the same colours,
   (ii) from P given that they are of same colours.
   (iii) expected number of red sweets removed

Exercise 5b

1. A discrete random variable X has a probability distribution

x	0	1	2	3	4
P(X=x)	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$

Find $E(X)$. **An** $E(x) = 2.25$

2. A discrete random variable X has a probability distribution

x	5	6	7	8	9
P(X=x)	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{3}{11}$

Find the mean **An** $\mu = 7$

3. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	k	0.2	0.1

Find
   (i) The value of the constant k
   (ii) Expectation, E(x)

An $k = 0.3, E(x) = 2.9$

4. A discrete random variable X has a probability distribution

x	0	1	2	3
P(X=x)	c	c^2	$c^2 + c$	$3c^2 + 2c$

Find
   (i) The value of the constant c
   (ii) Expectation, E(x)

An $c = 0.2, E(x) = 2.08$

5. Find the expected number of heads when two fair coins are tossed. **An** $E(x) = 1$
6. A family plans to have 3 children. Given that X is the number of boy in the family. Find the expected number of boys **An** $E(x) = 1.5$
7. If X is the random variable for the product of the scores on two tetrahedral dice, where the score is the number on which the die lands, find the expected score for the throw. **An** $E(x) = 6.25$
8. A bag contains 5 black counters and 6 red counters. Two counters are drawn at random, one at a time without replacement. Find the expected number of red counters. **An** $E(x) = \frac{12}{11}$
9. An un biased tetrahedral die is tossed once. If the die lands on the face marked 1, the player has to pay $10,000/=$. If it lands on a face marked with a 2 or a 4, the player wins $5,000/=$ and if it lands on a 3, the player wins $3,000/=$. Find the expected gain in one throw. **An** $E(x) = 750/=$
10. A discrete random variable X can take on values 10 and 20 only. If $E(X) = 16$. Write out the probability distribution for X. **An** $P(X = 10) = 0.4$ and $P(X = 20) = 0.6$.
11. A discrete random variable X can take on values 0, 1, 2 and 3 only. If $E(X) = 1.4$, $P(X \le 2) = 0.9$ and $P(X \le 1) = 0.5$. Find (i) $P(X = 1)$ (ii) $P(X = 0)$

An $(i) = 0.3, (ii) = 0.2$

12. The discrete random variable Y has a probability distribution is given by

$$P(Y = y) = cy, \quad y = 1, 2, 3, 4$$

Find

(i) The value of c, (ii) $mean, \mu$

An $c = 0.1, E(x) = \frac{11}{3}$

13. A discrete random variable has a p.d.f

$$P(X = x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, 3, 4, 5 \\ k, & x = 6 \\ 0, & otherwise \end{cases}$$

Find the:

(i) value of k.

(ii) mode, (iii) mean of X

An $k = \frac{1}{32}, \; mode = 1, \mu = 1\frac{31}{32}$

14. A discrete random variable has a p.d.f

$$P(X = x) = \begin{cases} k\,2^x, & x = 1, 2, 3, 4, 5, 6 \\ 0, & otherwise \end{cases}$$

Find the

(i) value of k, (ii) mean of X

(iii) $P(X < 4 / X > 1)$

An $k = \frac{1}{127}, \; , \mu = 5 \; (iii) = \frac{3}{31}$

Properties of the mean

(i) $E(a) = a$

(ii) $E(ax) = a\,E(x)$

(iii) $E(ax + b) = aE(x) + b$

(iv) $E(ax - b) = aE(x) - b$

Where a and b are constants

Examples

1. A random variable X of a discrete probability distribution is given by

x	1	2	3
P(X=x)	0.1	0.2	0.3

Find

(i) $E(x)$ (ii) $E(5x)$ (iii) $E(4X + 6)$

Solution

(i) $E(X) = \sum xP(X = x)$

$E(X) = (1x0.1) + (2x0.2) + (3x0.3) = 2.2$

(ii) $E(5X) = 5E(X) = 5x2.2 = 11$

(iii) $E(4X + 6) = 4E(X) + 6 = 4x2.2 + 6 = 14.8$

2. A random variable X of a discrete probability distribution is given by

x	-1	0	1	2
P(X=x)	0.25	0.10	0.45	0.20

Find

(i) $P(-1 \le X < 1)$ (ii) $E(X)$ (iii) $E(6X - 2)$

Solution

(i) $P(-1 \le X < 1) = P(X = 1) + P(X = 0)$
$= 0.25 + 0.10 = 0.35$

(ii) $E(X) = \sum xP(X = x)$
$= (-1x0.25) + (0x0.1) + (1x0.45) + (2x0.20)$
$E(X) = 0.6$

(iii) $E(6X - 2) = 6E(X) - 2 = 6x0.6 - 2 = 1.6$

Variance, Var(X)

$$Var(X) = E(X^2) - [E(X)]^2$$

Where $E(X^2) = \sum x^2 P(X = x)$

Examples

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	0.2	0.3	0.1

Find

(i) The mean (ii) The variance

Solution

(i) $E(X) = \sum xP(X = x)$

$= (1x0.1) + (2x0.3) + (3x0.2) + (4x0.3) + (5x0.1)$

$E(X) = 3$

(ii) $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X = x)$

$= (1^2x0.1) + (2^2x0.3) + (3^2x0.2) + (4^2x0.3) + (5^2x0.1)$

$E(X^2) = 10.4$

$Var(X) = 10.4 - [3]^2 = 1.4$

2. The discrete random variable Y has a probability distribution is given by

$P(Y = y) = c|y|, \quad y = -3, -2, -1, 0, 1, 2, 3,$

Find

(i) The value of c (ii) $mean, \mu$ (iii) Standard deviation

Solution

y	-3	-2	-1	0	1	2	3
P(Y=y)	3c	2c	c	0	c	2c	3c

(i) $\sum P(Y = y) = 1$

$3c + 2c + c + 0 + c + 2c + 3c = 1$

$c = \frac{1}{12}$

(ii) $E(X) = \sum xP(X = x)$

$= (-3x3c) + (-2x2c) + (-1xc) + (1xc) + (2x2c) + (3x3c)$

$E(X) = 0$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X = x)$

$= ((-3)^2x3c) + ((-2)^2x2c) + ((-1)^2xc) + ((1)^2$ $+ ((2)^2x2c) + ((2)^2x2c)$

$E(X^2) = 72c = 72x\frac{1}{12} = 6$

$Var(X) = 6 - [0]^2 = 6$

S.D= $\sqrt{6} = 2.45$

3. Two marbles are drawn without replacement from a box containing 3 red marbles and 4 white marbles. The marbles are drawn at random. If X is the random variable for the number of red marbles drawn, find

(i) Expected number of red marbles **(ii)** The standard deviation of X

Solution

x	0	1	2
P(X=x)	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

(i) $E(X) = \sum xP(X = x)$

$= \left(0x\frac{2}{7}\right) + \left(1x\frac{4}{7}\right) + \left(3x\frac{1}{7}\right) = \frac{6}{7}$

(ii) $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X = x)$

$= \left(0^2x\frac{2}{7}\right) + \left(1^2x\frac{4}{7}\right) + \left(2^2x\frac{1}{7}\right) = \frac{8}{7}$

$Var(X) = \frac{8}{7} - \left[\frac{6}{7}\right]^2 = \frac{20}{49}$

4. A vendor stocks 12 copies of a magazine each week and the probability for each possible total number of copies sold is shown below

Number of copies	9	10	11	12
Probability	0.20	0.35	0.30	0.15

(a) Estimate the mean and variance of the number of copies sold

(b) The vendor buys the magazine at 8,500/= and sells at 14,500/=. Any copies not sold are destroyed. Construct a probability distribution table for the vendors weekly profit and hence find the expected weekly profit

Solution

(a) $E(X) = \sum xP(X = x)$

$= (9x0.2) + (10x0.35) + (11x0.3) + (12x0.15)$

$E(X) = 10.4$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X = x)$

$= (9^2x0.2) + (10^2x0.35) + (11^2x0.3) + (12^2x0.15)$

$E(X^2) = 109.1$

$Var(X) = 109.1 - [10.4]^2 = 0.94$

(a) $Profit = S.P - C.P$

9 copies, $Profit$ =9x14500−12x8500 = 28500

10 copies, $Profit$ =10x14500−12x8500= 43000

11 copies, $Profit$ =11x14500−12x8500= 57500

12 copies, $Profit$ =12x14500−12x8500= 72000

y	28500	43000	57500	72000
P(Y=y)	0.20	0.35	0.30	0.35

$E(Y) = \sum yP(Y = y)$

$= (28500x0.2) + (43000x0.35) + (57500x0.3) + (72000x0.15)$

$E(Y) = 48000$

5. The table below shows the number of red and green balls put in three identical boxes A, B and C.

UNEB 2018 No.15

Boxes	A	B	C
Red balls	4	6	3
Green balls	2	7	5

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn",

(a) draw a probability distribution table for X.

(b) calculate the mean and variance of X

Solution

$$P(X=0)=\frac{1}{3}\left[\frac{4_{C_2}x3_{C_0}}{7_{C_2}}+\frac{6_{C_2}x7_{C_0}}{13_{C_2}}+\frac{3_{C_2}x5_{C_0}}{8_{C_2}}\right]=0.2332$$

$$P(X=1)=\frac{1}{3}\left[\frac{4_{C_1}x3_{C_1}}{7_{C_2}}+\frac{6_{C_1}x7_{C_1}}{13_{C_2}}+\frac{3_{C_1}x5_{C_1}}{8_{C_2}}\right]=0.5358$$

$$P(X=2)=\frac{1}{3}\left[\frac{4_{C_0}x3_{C_2}}{7_{C_2}}+\frac{6_{C_0}x7_{C_2}}{13_{C_2}}+\frac{3_{C_0}x5_{C_2}}{8_{C_2}}\right]=0.2310$$

x	0	1	2
P(X=x)	0.2332	0.5358	0.2310

(i) $E(X)=\sum xP(X=x)$

$=(0x0.2332)+(1x0.5358)+(2x0.231)$

$E(X)=0.9978$

(ii) $Var(X)=E(X^2)-[E(X)]^2$

$E(X^2)=\sum x^2P(X=x)$

$=(0^2x0.2332)+(1^2x0.5358)+(2^2x0.231)$

$E(X^2)=1.4598$

$Var(X)=0.9978-[1.4598]^2=0.4642$

Properties of the Variance

(i) $Var(a)=0$

(ii) $Var(aX)=a^2\,Var(X)$

(iii) $Var(aX+b)=a^2Var(X)$

(iv) $Var(aX-b)=a^2Var(X)$

Where a and b are constants

Examples

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	0.1	0.05

Find

(i) The mean (*ii*) The variance (*iii*) Var $(3X-2)$

Solution

(iii) $E(X)=\sum xP(X=x)$

$=(1x0.2)+(2x0.25)+(3x0.4)+(4x0.1)+(5x0.05)$

$E(X)=2.55$

(iv) $Var(X)=E(X^2)-[E(X)]^2$

$E(X^2)=\sum x^2P(X=x)$

$=(1^2x0.2)+(2^2x0.25)+(3^2x0.4)+(4^2x0.1)+(5^2x0.05)$

$E(X^2)=7.65$

$Var(X)=7.65-[2.55]^2=1.148$

(v) Var $(3X-2)=3^2Var(X)=9x1.148$

$=10.332$

2. A random variable X of a discrete probability distribution is given by

x	10	20	30
P(X=x)	0.1	0.6	0.3

Find (i) The mean (*ii*) The variance (*iii*)Var $(4X+3)$

Solution

(i) $E(X)=\sum xP(X=x)$

$=(10x0.1)+(20x0.6)+(30x0.3)=22$

(ii) $Var(X)=E(X^2)-[E(X)]^2$

$E(X^2)=\sum x^2P(X=x)$

$=(10^2x0.1)+(20^2x0.6)+(30^2x0.3)=520$

$Var(X)=520-[22]^2=36$

(iii) Var $(4X+3)=4^2Var(X)=16x36$

$=576$

Exercise 5c

1. A random variable X of a discrete probability distribution is given by

x	1	2	3
P(X=x)	0.2	0.3	0.5

Find
(i) $E(X)$ (ii) $E(X^2)$ (iii) $Var\ (X)$
An $E(X) = 2.3, E(X^2) = 5.9, Var(X) = 0.61$

2. A random variable X of a discrete probability distribution is given by

x	-1	0	1	2
P(X=x)	0.25	0.10	0.45	0.20

Find
(i) $P(-1 \leq X < 2)$
(ii) $E(X)$ (iii) $E(2X+3)$
An (i) $= 0.8$, $(ii) = 0.6$, (iii) $= 4.2$

3. A random variable X of a discrete probability distribution is given by: $P(X = 0) = 0.05$, $P(X = 1) = 0.45$, $P(X = 2) = 0.5$
Find;
(i) $E(X)$ (ii) $E(X^2)$ (iii) $Var\ (X)$
An $E(X) = 1.45, E(X^2) = 2.45, Var(X) = 0.348$

4. A random variable X of a discrete probability distribution is given by: $P(X = 1) = 0.1$, $P(X = 2) = 0.2, P(X = 3) = 0.3, P(X = 4)) = 0.4$
Find;
(i) $E(X)$ (ii) $Var(X)$
(iii) $P\left(X = 2/X \geq 2\right)$
An $E(X) = 3,, Var(X) = 1, (iii) = \frac{2}{9}$

5. The discrete random variable Y has a probability distribution is given by
$P(Y = y) = k, \quad y = 1,2,3,4,5,6$
Find;
(i) $mean, \mu$ (ii) $E(3X+4)$
(iii) $E(X^2)$ (iv) Standard deviation
An $\mu = 3.5, (ii) = 15\frac{1}{6}$, (iii) $= 14.5, \sigma = 1.708$

6. The discrete random variable R has a probability distribution is given by
$$P(R = r) = \frac{3r+1}{22}, \quad r = 0,1,2,3$$
Find;
(i) $mean, \mu$ (ii) $E(3R-2)$
(iii) $E(R^2)$
An $\mu = \frac{24}{11}, E(R^2) = \frac{61}{11}, E(3R-2) = \frac{50}{11}$

7. The discrete random variable R has a probability distribution is given by
$$P(R = r) = \begin{cases} \frac{2r+1}{20}, & r = 0,1,2,3 \\ \frac{11-2r}{20}, & r = 4,5 \end{cases}$$
Find
(i) $E(R)$ (ii) $Var(R)$
An , $E(R) = 2.55, Var(R) = 1.45$

8. The discrete random variable X has a probability distribution is given by
$$P(X = x) = \begin{cases} kx, & x = 1,2,3,4,5 \\ k(10-x), & x = 6,7,8,9 \end{cases}$$
Find
(i) Constant k
(ii) $E(X)$ (iii) $Var(X)$
An $k = 0.04, E(X) = 5, Var(X) = 4$

9. The discrete random variable X has a probability distribution is given by
$P(X = x) = kx, \quad x = 1,2,3 \ldots\ldots n$
Where k is a constant
show that $k = \frac{2}{n(n+1)}$
hence find in terms of n, the mean of X.
An$= \frac{1}{3}(2n+1)$

10. A random variable X of a discrete probability distribution is given by:
$P(X = 2) = 0.1, P(X = 4) = 0.3,$
$P(X = 6) = 0.5 \quad P(X = 8) = 0.1$
Find Var (X). **An**$=$ 2.56

11. A random variable X of a discrete probability distribution is given by:
$P(X = 0) = P(X = 1) = 0.1, P(X = 2) = 0.2,$
$P(X = 3) = P(X = 4) = 0.3$
Find Var (X). **An**$=$ 1.64

12. A biased die is tossed once. X is the random variable for the number which appears upper most. The probability distribution of X is as follows.

x	1	2	3	4	5	6
P(X=x)	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$

Find
(i) $E(X)$ (ii) $E(3X-5)$
(iii) $E(X^2)$ (iv) $Var\ (X)$
An $(i) = 3.5$, (ii) $= 5.5$, (iii) $= 14, (iv) = 1.75$

13. A discrete random variable X has a probability distribution

x	-2	-1	0	1	c
P(X=x)	0.1	0.1	0.3	0.4	0.1

Find the value of c when;

(i) $E(X) = 0.3$ (ii) $E(X^2) = 1.8$

An $(i) = 2, (ii) = 3 \text{ or } -3$

14. Find the variance of each of the following probability distribution

(a)

x	-3	-2	0	2	3
P(X=x)	0.3	0.3	0.2	0.1	0.1

(b)

x	1	3	5	7	9
P(X=x)	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$

(c)

x	0	2	5	6
P(X=x)	0.11	0.35	0.46	0.08

An $(a) = 4.2, (b) = 7.33$ $(c) = 3.67$

15. An unfair die is tossed once. X is the random variable for the number which appears upper most. The probability distribution of X is as follows.

x	1	2	3	4	5	6
P(X=x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	y	$\frac{1}{5}$	$\frac{1}{6}$

Find ; (i) Value of y (ii) $E(X)$
(iii) $Var(X)$ $(iv) Var\ (4X)$

An $y = \frac{1}{10}$ $(ii) = 3.5, (iii) = 2\frac{59}{60}$ (iv) $= 47\frac{11}{15}$

16. Two pens are drawn without replacement from a box containing 3 red pens and 4 blue pens. The pens are drawn at random. If X is the random variable for the number of blue pens drawn, find;

(i) $E(X)$ (ii) $E(X^2)$
(iii) $Var(X)$ (iv) $Var\ (3X - 4)$

An $(i) = \frac{8}{7}$, (ii) $= \frac{12}{7}$, $(ii) = \frac{20}{49}$, (iv) $= \frac{180}{49}$

17. If X is the random variable for the sum of the scores on two tetrahedral dice, where the score is the number on which the die lands, find;

(i) $E(X)$ $(ii)\ Var(X)$
(iii) $Var(2X)$ (iv) $Var\ (2X + 5)$

An $(i) = 5, (ii) = 2.5,$ (iii) $= 10,$ (iv) $= 10$

18. A random variable X of a discrete probability distribution is given by

x	10	20	30
P(X=x)	0.1	0.6	0.3

Find Var $(2X + 3)$. **An= 144**

19. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	k	2k	3k	4k	5k

Find

(i) Constant k (ii) Mean
(iii) Standard deviation (iv) $P(2 < X \leq 4)$

An $k = \frac{1}{15}, \sigma = \frac{7}{15}, \mu = 1.247$

20. A discrete random variable has a probability function

$$P(X = x) = \begin{cases} kx, & x = 1, 2, 3, \ldots \ldots \ldots \ldots n \\ 0, & otherwise \end{cases}$$

Where k is a constant and $E(X) = 3$, Find the;

(i) value of k and n.

(ii) variance of X, (iii) $P\left(X = 2/X \geq 2\right)$

An $k = \frac{1}{10}, n = 4,\ Var(X) = 1, (iii) = \frac{2}{9}$

21. A discrete random variable has a probability function

$$f(x) = \begin{cases} k(x^2 + 4), & x = 0, 1, 2, 3, 4 \\ 0, & otherwise \end{cases}$$

Find the

(i) value of k, (ii) $E(X)$
(iii) mode (iv) $P\left(X = 2/X \geq 2\right)$

An $k = \frac{1}{50}, (ii) = 2.8,\ (iii) = 4, (iv) = \frac{7}{10}$

Cumulative distribution function, F(X)

F(x) is given by $F(x) = \sum P(X =)x$

Note $F(+\infty) = 1$ where $+\infty$ is the upper limit

Examples

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	0.1	0.05

Find the cumulative distribution function

Solution

x	1	2	3	4	5
F(x)	0.2	0.45	0.85	0.95	1

2. The random variable X has a cumulative function below

x	-1	0	1	2
F(x)	0.25	0.35	0.80	1

Find the probability distribution function

Solution

x	-1	0	1	2
f(x)	0.25	0.10	0.45	0.20

3. A discrete random variable X has a cumulative distribution

x	1	2	3	4	5
F(x)	0.2	0.32	0.67	0.91	1

Find
(i) Probability distribution function
(ii) $P(X = 3)$
(iii) $P(X > 2)$

Solution

(i)

x	1	2	3	4	5
P(X=x)	0.2	0.12	0.35	0.23	0.1

(ii) P(X=3) = 0.35

(iii) P(X>2) = 0.68

4. The random variable X has a cumulative function below

x	1	2	3	4
F(x)	0.1	0.5	0.80	1

Find the
(i) Mean
(ii) Var(X)
(iii) mode

Solution

x	1	2	3	4
P(X=x)	0.1	0.40	0.30	0.20

(i) $E(X) = \sum xP(X = x)$

$E(X) = (1x0.1) + (2x0.4) + (3x0.3) + (4x0.2)$

$E(X) = 2.6$

(ii) $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X = x)$

$= (1^2x0.1) + (2^2x0.4) + (3^2x0.3) + (4^2x0.2)$

$E(X^2) = 5.92$

$Var(X) = 5.92 - [2.6]^2 = 0.84$

(iii) mode= 2

Exercise: 5d

1. A discrete random variable X has a cumulative distribution

x	0	1	2	3	4
F(x)	0.1	0.3	0.6	0.8	1

Find
(i) $E(X)$
(ii) $Var(X)$ (iii) $Var(6X + 2)$

An $(i) = 15.2, (ii) = 1.56$, (iii) $= 56.16$

2. The random variable X has a cumulative function below

x	1	2	3	4
F(x)	0.13	0.54	0.75	1

Find
(i) $P(X = 2)$ (ii) $P(X > 1)$
(iii) $P(X \geq 3)$ (iv) $P(X < 2)$ (v) $E(X)$

An $(i) = 0.41, (ii) = 0.87$, $(iii) = 0.46$, $(iv) = 0.13, (v) = 2.58$

3. A discrete random variable X has a cumulative distribution

x	3	4	5	6	7
F(x)	0.01	0.23	0.64	0.85	1

Find
(i) Probability distribution function
(ii) $Var(X)$ **An**(0.9724)

4. A discrete random variable has a cumulative probability function. $F(x) = \frac{x^2}{9}$, $x = 1,2,3$

Find the
(i) $F(2)$.
(ii) $P(X = 2)$ (iii) $E(2X - 3)$

An (i) $= \frac{4}{9}$, $(ii) = \frac{1}{3}$, (iii) $= \frac{17}{9}$.

5. A discrete random variable has a cumulative probability function. $F(x) = kx$, $x = 1,2,3$

Find the
(i) $constant\ k$ $(ii)\ P(X < 3)$
(iii) Standard deviation

An $k = \frac{1}{3}, P(X < 3) = \frac{2}{3}, \sigma = 0.816$,

6. A discrete random variable has a cumulative probability function.

$$F(x) = 1 - \left(1 - \frac{x}{4}\right)^x \quad x = 1,2,3,4$$

Find the
(i) $F(3)$. $(ii)\ F(2)$.
(iii) Var(X)

An $F(2) = \frac{63}{64}, F(3) = \frac{3}{4}$, $Var(X) = 0.547$,

BINOMIAL DISTRIBUTION

A binomial distribution is a special type of a discrete random variable in which an experiment gives rise to only two out comes either success or failure

Condition for a binomial distribution

(i) The experiment has a finite (repeated) number of trials, n
(ii) The trials are independent
(iii) The outcome of each trial is either a success or a failure
(iv) The probability, p of a successful outcome is constant for all trials

If a discrete random variable X is the number of **successful** out comes in **n** trials and satisfies the above conditions, then X follows a binomial distribution written as
$X \sim B(n, p)$ or $X \sim Bin(n, p)$

Formula for a binomial distribution

If $X \sim B(n, p)$, the probability of obtaining, r successes in n trials $P(X = r)$ where

$$\boxed{P(X = r) = {}^{n}C_r p^r q^{n-r}} \quad \text{for } r = 0, 1, 2, 3 \ldots\ldots\ldots\ldots n$$

Where $q = 1 - p$

Examples

1. The random variable X is distributed $B(7, 0.2)$. Find:
 (i) $P(X = 3)$ (ii) $P(1 < X \leq 4)$ (iii) $P(X > 1)$

 Solution

 (i) $n = 7, p = 0.2, q = 1 - 0.2 = 0.8$
 $P(X = 3) = {}^{7}C_3\, 0.2^3 0.8^4 = 0.115$
 (ii) $P(1 < X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4)$
 $P(1 < X \leq 4) = {}^{7}C_2\, x0.2^2 x0.8^5 + {}^{7}C_3\, x0.2^3 x0.8^4 + {}^{7}C_4\, x0.2^4 x0.8^3$
 $P(1 < X \leq 4) = 0.275 + 0.115 + 0.029 = 0.419$
 (iii) $P(X > 1) = 1 - P(X \leq 1) = P(X > 1) = 1 - [P(X = 0) + P(X = 1)]$
 $P(X > 1) = 1 - [\,{}^{7}C_0\, x0.2^0 x0.8^7 + {}^{7}C_1\, x0.2^1 x0.8^6]$
 $P(X > 1) = 1 - [0.210 + 0.367] = 0.423$

2. At freedom city supermarket, 60% of the customer's shop on Saturday. Find the probability that in a randomly selected sample of 10 customers
 (i) Exactly 2 shop on Saturday (ii) More than 7 shop on Saturday

 Solution

 (i) $n = 10, p = 0.6, q = 1 - 0.6 = 0.4$
 $P(X = 2) = {}^{10}C_2\, 0.6^2 0.4^8 = 0.011$
 (ii) $P(X > 7) = P(X = 8) + P(X = 9) + P(X = 10)$
 $P(X > 7) = {}^{10}C_8\, x0.6^8 x0.4^2 + {}^{10}C_9\, x0.6^9 x0.4^1 + {}^{10}C_{10}\, x0.6^{10} x0.4^0$
 $P(X > 7) = 0.121 + 0.040 + 0.006 = 0.167$

3. The probability that a marble drawn at random from a box is red is 0.4. If a sample of 6 marbles is taken, find the probability that it will contain;
 (i) No red marble (ii) 5 or 6 red marbles (iii) Less than half red marbles

 Solution

 (i) $n = 6, p = 0.4, q = 1 - 0.4 = 0.6$
 $P(X = 0) = {}^{6}C_0\, 0.4^0 0.6^6 = 0.047$
 (ii) $P(5\ 0r\ 6) = P(X = 5) + P(X = 6)$
 $P(5\ 0r\ 6) = {}^{6}C_5\, 0.4^5 0.6^1 + {}^{6}C_6 0.4^6 0.6^0 = 0.037 + 0.004 = 0.041$
 (iii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$
 $P(X < 3) = {}^{6}C_0\, 0.4^0 0.6^6 + {}^{6}C_1\, 0.4^1 0.6^5 + {}^{6}C_2 0.4^2 0.6^4$

$$= 0.047 + 0.187 + 0.311 = 0.545$$

4. A biased coin is such that the chance of a head appearing uppermost when tossed is twice the chance of a tail appearing uppermost. If the coin is tossed 10 times. Find the probability that
   (i) Exactly 6 heads will appear
   (ii) Between 5 and 8 heads will appear

Solution

(i)
$$P(H) + P(T) = 1$$
$$2x + x = 1$$
$$x = \frac{1}{3}$$
$$n = 10, p = \frac{2}{3}, q = \frac{1}{3}$$
$$P(X = 6) = {}^{10}C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^4 = 0.228$$

(ii)
$$P(5 < X < 8) = P(X = 6) + P(X = 7)$$
$$= {}^{10}C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^4 + {}^{10}C_7 \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^3$$
$$= 0.228 + 0.260 = 0.488$$

5. A box contains a large number of pens. The probability that a pen is faulty is 0.1. How many pens would you need to select to be more than 95% certain of picking at least one faulty pen

Solution

$$n = ?, p = 0.1, q = 0.9$$
$$P(X \geq 1) > 0.95$$
$$P(X \geq 1) = 1 - P(X = 0) > 0.95$$
$$1 - {}^nC_0\, 0.1^0 0.9^n > 0.95$$
$$0.05 > 0.9^n$$

Introducing logs : $log\, 0.05 > n\, log\, 0.9$

$$-1.301 > -0.045n$$
$$\frac{-1.301}{-0.045} < n$$
$$n > 28.9$$

Least value of n is 29

Using cumulative binomial probability table

The tables give values $P(X \geq x)$ for values of n and P

For values of

(i) $P(X \leq x) = 1 - P(X \geq x + 1)$

(ii) $P(X = x) = P(X \geq x) - P(X \geq x + 1)$

Examples

1. The random variable X is distributed $B(5, 0.3)$. Find ;
   (i) $P(X \geq 3)$
   (ii) $P(X > 1)$
   (iii) $P(X \leq 4)$
   (iv) $P(X < 3)$
   (v) $P(X = 2)$

Solution

$$n = 5, P = 0.3$$

(i) $P(X \geq 3) = 0.1631$

(ii) $P(X > 1) = P(X \geq 2) = 0.4718$

(iii) $P(X \leq 4) = 1 - P(X \geq 5) = 1 - 0.0024 = 0.9976$

(iv) $P(X < 3) = P(X \leq 2) = 1 - P(X \geq 3) = 1 - 0.1631 = 0.8369$

(v) $P(X = 2) = P(X \geq 2) - P(X \geq 3) = 0.4718 - 0.1631 = 0.3087$

2. An unbiased coin is tossed 15 times. Find the probability that;
   (i) Exactly eight heads will appear upper most
   (ii) Between 6 and 10 heads will appear
   (iii) Between 6 and 10 heads inclusive will appear

Solution

$$n = 15, P = 0.5$$

(i) $P(X = 8) = P(X \geq 8) - P(X \geq 9) = 0.5000 - 0.3036 = 0.1964$

(ii) $P(6 < X < 10) = P(7 \leq X \leq 9) = P(X \geq 7) - P(X \geq 10) = 0.6964 - 0.1509 = 0.5455$

(iii) $P(6 \leq X \leq 10) = P(X \geq 6) - P(X \geq 11) = 0.8491 - 0.0592 = 0.7899$

3. A student attempts 20 objective questions by guess work. Each question has got four possible alternatives out of which only one is correct. Find the probability that he gets;
   (i) Exactly 9 correct answers
   (ii) At least 12 correct answers

(iii) At most 6 correct answers
(iv) Between 6 and 14 correct answers inclusive
(v) Exactly 7 incorrect answers

Solution

For correct answers, $n = 20$, $P = 0.25$, $q = 0.75$

(i) $P(X = 9) = P(X \geq 9) - P(X \geq 10) = 0.0409 - 0.0139 = 0.027$
(ii) $P(X \geq 12) = 0.0009$
(iii) $P(X \leq 6) = 1 - P(X \geq 7) = 1 - 0.2142 = 0.7858$
(iv) $P(6 \leq X \leq 14) = P(X \geq 6) - P(X \geq 15) = 0.3828 - 0.000 = 0.3828$

For incorrect answers, $n = 20$, $P = 0.75$, $q = 0.25$

(i) $P(X = 7) = 0.002$

Exercise 6a

1. A biased coin is such that a head is three times as likely to occur as a tail. The coin is tossed 5 times. Find the probability that at most two tails occur. **UNEB 2018 No.8 An 0.8965**
2. Tom's chance of passing an examination is $^2/_3$. If he sits for four examinations, calculate the probability that; **UNEB 2014 No.4**
   (i) Only two examinations
   (ii) More than half of the examinations
   An (i) 0.2963 (ii) 0.5926
3. A fair die is rolled 6 times, calculate the probability that; **UNEB 2001 No.10**
   (i) A 2 or 4 appears on the first throw
   (ii) Four 5's will appear in the six throws **An (i)** $\frac{1}{3}$ **(ii) 0.008**
4. The probability that Alex wins a chess game is $\frac{2}{3}$. He plays 8 games, what is the probability that he wins **UNEB 1998 Nov No.7**
   (i) At least 7 games (ii) Exactly 5 games
   An (i) 0.1951 (ii) 0.2731
5. Usain Bolt makes 5 practice runs I the 100m sprint. A run is successful if he runs it in less than 11 seconds. There are 8 chances out of 10 that he is successful. Find the probability that;
   (i) He records at least no success at all
   (ii) He records at least 2 success
   (iii) If he is successful in 5 practice runs, he makes two additional runs. The probability of success in either of the additional runs is 0.7. Determine the probability that Bolt will make 7 successful runs in total. **An (i) 0.0003 (ii) 0.9933 (iii) 0.1606 UNEB 1993 No.13**
6. In a test there are 10 objective questions each with a choice of five possible alternatives out of which only one is correct. If a student guesses each of the answers, find the probability that he gets at least two answers correct **UNEB 1992 No.11 An 0.6242**
7. 30% of the students in the school are day scholars . form a sample of 10 students chosen at random, find the probability that;
   (i) Only 3 are day scholars
   (ii) Less than half are day scholars
   An (i) 0.267 (ii) 0.850
8. The probability that a shopper buys a cake is 0.25. Find the probability that in a random sample of 9 shoppers,
   (i) Exactly 3 buy a cake
   (ii) More than 7 buy a cake
   An (i) 0.234 (ii) 0.0001
9. A bag contains counter books of which 40% are blue and the rest are black. A counter book is taken form the bag, its colour noted then replaced. This is performed 8 times in all. Calculate the probability that;
   (i) Exactly 3 will be blue
   (ii) At least one will be blue
   (iii) More than half will be black
   An (i) 0.279 (ii) 0.983 (iii) 0.594
10. At a certain school, the records taken form admissions office show that the ratio of male to female $S.5$ applicants is 4:6. Basing on this experience, what is the probability that will be more female applicants in a random collection of a dozen applicants. **An 0.6652**
11. The random variable X is B(6,0.42). Find
    (i) $P(X = 6)$ (ii) $P(X = 4)$
    (iii) $P(X \leq 2)$
    An (i) 0.00549 (ii) 0.157 (iii) 0.503
12. During printing of report cards, the computer occasionally makes mistakes. In a batch of 20 cards, 5 are found to have mistakes. A sample of 3 of this batch is taken and examined. What is the probability that;

(i) It contains a mistake
(ii) Exactly one contains a mistake

An (i) 0.5781 (ii) 0.4219

13. An un biased die is thrown seven times. Find the probability of throwing at least 5 sixes. **An 0.002**

14. In a family a couple is equally likely to produce a girl or a boy. Find the probability that in a sample of 5 children there will be more boys than girls. **An 0.5**

15. The probability that it will rain on any given day during examination period is 0.3. calculate the probability that in a given week during examination period, it will rain on;
    (i) Exactly 2 days (ii) At least 2 days
    (ii) At most two days
    (iii) Exactly three days that are consecutive

    An (i) 0.318 (ii) 0.671 (iii) 0.647 (iv) 0.0324

16. A fair coin is tossed 6 times. Find the probability of throwing at least four heads. **An 0.344**

17. The random variable X is $B(4, p)$ and $P(X = 4) = 0.0256$. Find $P(X = 2)$ **An 0.3456**

18. In Agriculture lab, Sylvia plants bean seeds and the probability that they germinate successfully is $\frac{1}{3}$
    (a) She takes 9 seeds. Find the probability that;
        (i) More than five seeds germinate
        (ii) At least three seeds germinate.
    (b) Find the number of seeds that she needs to take in order to be 99% certain that at least one seed germinates

    An (a) (i) 0.0424 (ii) 0.632 (b) 12

19. In a shooting competition, the probability of hitting the target with a single shot is 0.6, if 7 shots are taken find the probability that the target is hit more than twice. **An 0.0105**

20. In mass production of shirts it is found that 5% are defective. Shirts are selected at random and put into packets of 10.
    (a) A packet is selected at random. Find the probability that it contains
        (i) Three defective shirts
        (ii) Less than three defective shirts
    (b) Two packets are selected at random. Find the probability that there are no defective shirts in either packets.

    An (a) (i) 0.0105 (ii) 0.988 (b) 0.358

21. A biased coin is such that it is twice as likely to show a head as a tail. If the coin is tossed 5 times. Find the probability that
    (i) Exactly 3 heads are obtained
    (ii) More than 3 heads are obtained

    An (i) 0.329 (ii) 0.461

22. The probability that a target is hit is 0.3. Find the least number of shots which should be fired if the probability that the target is hit at least once is greater than 0.95. **An 9**

23. 1% of the light bulbs in a box are faulty. Find the largest sample size which can be taken if it is required that the probability that there are no faulty bulbs in the sample is greater than 0.5. **An 68**

24. In a test there are 10 multiples choice questions. Each question has got four possible alternatives out of which only one is correct. If a student guesses each of the answers, find the probability that he gets;
    (i) More than 7 correct answers
    (ii) More than half correct answers **An**

    (i) 0.0004 (ii) 0.0197

25. $X \sim B(n, 0.3)$. Find the least value of n such that $P(X \geq 1) = 0.8$. **An 5**

26. The random variable X is $B(n, 0.6)$. Find the value of n such that $P(X < 1) = 0.0256$. **An 4**

Expectation and variance of a binomial distribution

If $X \sim B(n, p)$ then

$\boxed{E(x) = np}$ and $\boxed{Var\ (x) = npq}$ where $q = 1 - p$

Examples

1. The random variable X is B(4,0.8). Find the mean and the variance

Solution

$E(x) = np$
$E(x) = 4x0.8 = 3.2$

Mean = 3.2
$Var\ (x) = npq$

$Var\ (x) = 4x0.8x0.2 = 0.64$
variance= 0.64

2. In kampala city, it is known that $\frac{3}{5}$ of the voters support F.D.C party, in a sample of 6 voters, what is the expected value and standard deviation of the number of F.D.C's.

Solution

$E(x) = np = 6x0.6 = 3.6$ | $S.d = \sqrt{npq} = \sqrt{6x0.6x0.4} = 1.2$

3. The probability that, it will be a sunny day is 0.4. find the expected number if sunny days in a week and also find the standard deviation

Solution

$E(x) = np = 7x0.4 = 2.8$ | $S.d = \sqrt{npq} = \sqrt{7x0.4x0.6} = 1.3$

4. X is $B(n, p)$ with mean 5 and standard deviation 2. Find the value of n and p.

Solution

$$E(x) = np$$
$$5 = np \ldots\ldots\ldots\ldots\ldots (i)$$
$$S.d = \sqrt{npq}$$
$$2 = \sqrt{npq}$$
$$4 = npq \ldots\ldots\ldots\ldots (ii)$$

(ii) divide with (i)

$$\frac{npq}{np} = \frac{4}{5}$$
$$q = 0.8$$
$$p = 1 - 0.8 = 0.2$$
$$5 = np$$
$$n = \frac{5}{0.2} = 25$$

Mode of the binomial distribution

The mode is the value of X that is **most likely** to occur. The value of X with the highest probability and its close to the mean gives the mode.

Examples

1. The probability that a student is awarded a distinction in the mathematics examination is 0.15. In a randomly selected group of 15 students, what is the most likely number of students awarded a distinction.

Solution

$$E(x) = np$$
$$E(x) = 15x0.15 = 2.25$$
$$P(X = 2) = {}^{15}C_2\,(0.15)^2(0.85)^{13} = 0.286$$
$$P(X = 3) = {}^{15}C_3\,(0.15)^3(0.85)^{12} = 0.218$$

The most likely number of students awarded a distinction are 2

2. In a school, 80% of the students find difficulties in Physics. If a sample of 12 students is chosen.
   (i) What is the most likely number of students who find difficulty in Physics
   (ii) Find the probability that fewer than half find difficulty in Physics.

Solution

(i) $E(x) = np$

$$E(x) = 12x0.8 = 9.6$$
$$P(X = 9) = {}^{12}C_9\,(0.8)^9(0.2)^3 = 0.236$$
$$P(X = 10) = {}^{12}C_{10}\,(0.8)^{10}(0.2)^2 = 0.284$$

The most likely number is 10

(ii) $P(X < 6) = 0.0004$

Exercise 6b

1. 10% of drugs at a certain Pharmacy are expired. A sample of 25 drugs is taken. Find the expected number of expired drugs and the standard deviation. **An 2.5 , 1.5**
2. The probability that a student scores above 60% in a mathematics test is 0.5. In a random sample of 15 students, what is the most likely number of students who will score above 60%. **An 7 and 8**
3. The probability that an apple picked at random from a sack is bad is 0.15.
   (a) Find the standard deviation of the number of bad apples in a sample of 15 apples
   (b) What is the most likely number of bad apples in a sample of 30 apples

   An (a) 1.38 (b) 4
4. The random variable X is $B(n, 0.3)$ and $E(X) = 2.4$. Find n and the standard deviation of X. **An 8 , 1.30**
5. In a group of people, the expected number who wear glasses is 2 and the variance is 1.6. find the probability that;
   (i) A person chosen at random from the group wears glasses

(ii) 6 people in the group wear glasses. **An (i) 0.2 (ii) 0.00551**

6. New vision publishes a governance article in its newspaper each day of the week, except Sunday. A man is able to read on average 8 out of 10 articles.
   (i) Find the expected value and the standard deviation of the number of read articles in a given week
   (ii) What is the probability that he will read at least 5 articles in a given week
   An (i) 4.8 (ii) 0.98

7. The random variable X is $B(10, p)$ where $p < 0.5$. The variance of X is 1.875. Find;
   (i) The value of p (ii) $E(X)$
   (iii) $P(X = 2)$
   An (i) 0.25 (ii) 2.5 (iii) 0.282

8. A die is biased and the probability, p, of throwing a six is known to be less than $\frac{1}{6}$. An experiment consists of recording the number of sixes in 25 throws of the die. The standard deviation of the number of sixes is 1.5. calculate the;
   (i) value of p
   (ii) Probability that exactly three sixes are recorded during a particular experiment.
   An (i) 0.1 (ii) 0.23

9. In a bag there are 6 red pens, 8 blue pens and 6 black pens. An experiment consists of taking a pen at random form the bag, noting its colour and then replacing it in the bag. This procedure is carried out 10 times in all. Find;
   (i) Expected number of red pens drawn
   (ii) Most likely number of black pens drawn
   (iii) Probability that no more than four blue pens are drawn **An (i) 3 (ii) 3 (iii) 0.633**

10. The random variable X is distributed binomially with mean 2 and variance 1.6. find
   (i) The probability that X is less than 6
   (ii) The most likely value of X.
   An (i) 0.994 (ii) 2

11. Each day a bakery delivers the same number of loaves to a certain shop which sells on average 98% of them. Assuming that the number of loaves sold per day has a binomial distribution with standard deviation 7, find the expected number of loaves the shop would expect to sell per day. **An 2500**

12. On average 20% of the bolts produced by a machine are faulty. Samples of 10 bolts are to be selected at random each day. Each bolt will be selected and replaced in the set of bolts which have been produced on that day.
   (i) Find the probability that, in any one sample, two bolts or less will be faulty
   (ii) Calculate the expected value and variance of the number of bolts in a sample which will not be faulty. **An (i) 0.68 (ii) 8, 1.6**

13. An experiment consists of taking 12 shots at a target and counting the number of hits. The expected number of hits was found to be 3. Calculate;
   (i) The probability of hitting the target with a single shot
   (ii) The stand ad deviation of the number of hits in an experiment. **An (i) 0.25 (ii) 1.5**

14. In a certain a family, the probability that they will have a baby boy is 0.6. If there are 5 children in a family determine; **UNEB 1988 No.10**
   (a) The expected number of girls
   (b) The probability that there are at least three girls
   (c) The probability that they are all boys
   An (i) 2 **(ii) 0.317 (iii) 0.0778**

15. The probability of winning a game is 0.8. Ten games are played. What is the; **UNEB 1997** No.7
   (a) Mean number of success and variance
   (b) Probability of at least 8 success in the ten games **An (a) 8, 1.6 (b) 0.6778**

16. In a test there are 10 multiples choice questions. Each question has got four possible alternatives out of which only one is correct. If a student guesses each of the answers, find the;
   (i) Probability that at least four answers are correct **UNEB 1998 mar** No.12
   (ii) Most likely number of correct answers
   An (i) 0.2241 (ii) 2

17. A biased coin is such that a head is twice as likely to occur as a tail. The coin is tossed 15 times. Find the **UNEB 2001 No.**4
   (i) The expected number of heads
   (ii) probability that at most two tails occur.
   An (i) 10 (ii) 0.0793

PROBABILITY DISTRIBUTIONS 2- CONTINUOUS RANDOM VARIABLE

A probability density function (p.d.f) is continuous, if it takes on values between an interval

Properties of a continuous probability density functions

(i) $\int f(x)dx = 1$ | (ii) $f(x) \geq 0$

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \leq x \leq 5 \\ 0, & else\ where \end{cases}$

Find the value of the constant k

Solution

$$\int_0^5 kx\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^5 = 1$$

$$k\left(\frac{5^2}{2} - \frac{0^2}{2}\right) = 1$$

$$k\frac{25}{2} = 1$$

$$k = \frac{2}{25}$$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \leq x \leq 2 \\ 2k(x-1), & 2 \leq x \leq 4 \\ 0, & else\ where \end{cases}$

Find the value of the constant k

Solution

$$\int_0^2 kx\,d + \int_2^4 2k(2x-1)\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[\frac{x^2}{2} - x\right]_2^4 = 1$$

$$k\left(\frac{2^2}{2} - \frac{0^2}{2}\right) + 2k\left\{\left(\frac{4^2}{2} - 4\right) - \left(\frac{2^2}{2} - 2\right)\right\} = 1$$

$$2k + 8k = 1$$

$$k = \frac{1}{10}$$

Sketching f(x)

- Find the initial and final points of f(x)
- Join the initial and final points of f(x) using a line or a curve

Note

- A line is in the form of $y = mx + c$
- A curve has a power of x being 2 and above or fractional power eg $y = x^2$
- A curve with a positive coefficient of x^2 has a minimum turning point while a curve with a negative coefficient x^2 has a maximum turning point

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \leq x \leq 3 \\ 0, & else\ where \end{cases}$

Find the value of the constant k and sketch f(x)

Solution

$$\int_0^3 kx\,dx = 1$$

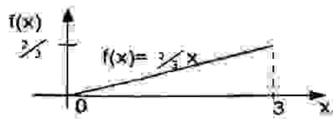
$$k\left[\frac{x^2}{2}\right]_0^3 = 1$$

$$k\left(\frac{3^2}{2} - \frac{0^2}{2}\right) = 1$$

$$k = \frac{2}{9}$$

When $x = 0$, $f(x) = \frac{2}{9}x0 = 0$

When $x = 3$, $f(x) = \frac{2}{9}x3 = \frac{2}{3}$



2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx\ , & 0 \le x \le 3 \\ k(6-x), & 3 \le x \le 6 \\ 0, & else\ where \end{cases}$

Find the value of the constant k and sketch f(x)

Solution

$$\int_0^3 kx\,dx + \int_3^6 k(6-x)dx = 1$$

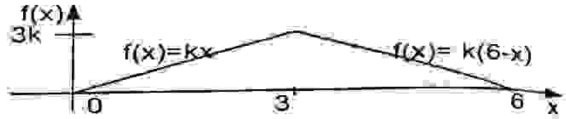
$$k\left[\frac{x^2}{2}\right]_0^3 + k\left[6x - \frac{x^2}{2}\right]_3^6 = 1$$

$$k = \frac{1}{9}$$

When $x = 0,\ f(x) = k(0) = 0$

When $x = 3,\ f(x) = k(3) = 3k$

When $x = 6,\ f(x) = k(6-6) = 0$



3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k\ (x+2)\ , & -2 \le x \le 0 \\ k(2-x), & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find the value of the constant k and sketch f(x)

Solution

$$\int_{-2}^0 k(x+2)\,dx + \int_0^2 k(2-x)dx = 1$$

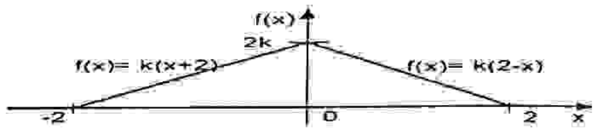
$$k\left[\frac{x^2}{2} + 2x\right]_{-2}^0 + k\left[2x - \frac{x^2}{2}\right]_0^2 = 1$$

$$k = \frac{1}{4}$$

When $x = -2,\ f(x) = k(-2+2) = 0$

When $x = 3,\ f(x) = k(0+2) = 2k$

When $x = 2,\ f(x) = k(2-2) = 0$



Finding probabilities

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 6 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k

(ii) $P(X > 4)$

(iii) $P(X < 3)$

(iv) $P(1 < X < 3)$

(v) $P(X > 2 / X \le 4)$

Solution

(i) $\int_0^6 kx\,dx = 1$

$$k\left[\frac{x^2}{2}\right]_0^6 = 1$$

$$k\left(\frac{6^2}{2} - \frac{0^2}{2}\right) = 1$$

$$k = \frac{1}{18}$$

(i) $P(X > 4) = \frac{1}{18}\int_4^6 x\,dx = \frac{1}{18}\left[\frac{x^2}{2}\right]_4^6$

$$= \frac{1}{18}\left(\frac{6^2}{2} - \frac{4^2}{2}\right) = \frac{5}{9} = 0.5556$$

(ii) $P(X < 3) = \frac{1}{18}\int_0^3 x\,dx = \frac{1}{18}\left[\frac{x^2}{2}\right]_0^3$

$$= \frac{1}{18}\left(\frac{3^2}{2} - \frac{0^2}{2}\right) = \frac{1}{4} = 0.25$$

(iii) $P(1 < X < 3) = \frac{1}{18}\int_1^3 x\,dx = \frac{1}{18}\left[\frac{x^2}{2}\right]_1^3$

$$= \frac{1}{18}\left(\frac{3^2}{2} - \frac{1^2}{2}\right) = \frac{2}{9} = 0.2222$$

(vi) $P(X > 2 / X \le 4) = \frac{P(X > 2 \cap X \le 4)}{P(X \le 4)}$

$$= \frac{P(2 < X \le 4)}{P(X \le 4)} = \frac{\frac{1}{18}\int_2^4 x\,dx)}{\frac{1}{18}\int_0^4 x\,dx}$$

$$= \frac{3}{4}$$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k\left(1 - \frac{x}{10}\right), & 0 \le x \le 10 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(X < 4)$

(iii) $P(2 < X < 6)$

Solution

(i) $\int_0^{10} k\left(1 - \frac{x}{10}\right) dx = 1$

$$k\left[x - \frac{x^2}{20}\right]_0^{10} = 1$$

$$k\left\{\left(10 - \frac{10^2}{20}\right) - \left(0 - \frac{0^2}{20}\right)\right\} = 1$$

$$k = \frac{1}{5}$$

(ii) $P(X < 4) = \frac{1}{5}\int_0^4 \left(1 - \frac{x}{10}\right) dx = \frac{1}{5}\left[x - \frac{x^2}{20}\right]_0^4$

$$= \frac{1}{5}\left\{\left(4 - \frac{4^2}{20}\right) - \left(0 - \frac{0^2}{20}\right)\right\} = 0.64$$

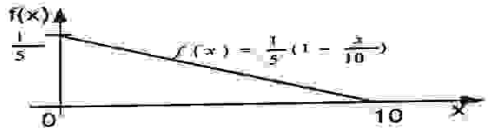
(iii) $P(2 < X < 6) = \frac{1}{5}\int_2^6 \left(1 - \frac{x}{10}\right) dx$

$$= \frac{1}{5}\left[x - \frac{x^2}{20}\right]_2^6 = \frac{1}{5}\left\{\left(6 - \frac{6^2}{20}\right) - \left(2 - \frac{2^2}{20}\right)\right\}$$

$$= 0.48$$

When $x = 0$, $f(x) = \frac{1}{5}x\left(1 - \frac{0}{10}\right) = \frac{1}{5}$

When $x = 10$, $f(x) = \frac{1}{5}x\left(1 - \frac{10}{10}\right) = 0$



3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx(6 - x), & 0 \le x \le 6 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(X \ge 5)$

Solution

(i) $\int_0^6 kx(6 - x)dx = 1$

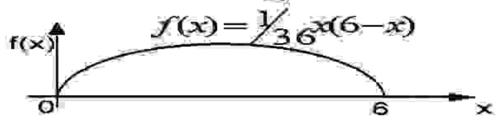
$$k\left[3x^2 - \frac{x^3}{3}\right]_0^6 = 1$$

$$k\left\{\left(3x6^2 - \frac{6^3}{3}\right) - \left(3x0^2 - \frac{0^3}{3}\right)\right\} = 1$$

$$k = \frac{1}{36}$$

When $x = 0$, $f(x) = \frac{1}{36}x0(6 - 0) = 0$

When $x = 6$, $f(x) = \frac{1}{36}x6(6 - 6) = 0$



(ii) $P(X \ge 5) = \frac{1}{36}\int_5^6 x(6 - x)dx = \frac{1}{36}\left[3x^2 - \frac{x^3}{3}\right]_5^6$

$$= \frac{1}{36}\left\{\left(3x6^2 - \frac{6^3}{3}\right) - \left(3x5^2 - \frac{5^3}{3}\right)\right\} = 0.074$$

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2, & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(1 \le X \le 3)$

Solution

(i) $\int_0^4 kx^2 dx = 1$

$$k\left[\frac{x^3}{3}\right]_0^4 = 1$$

$$k\left\{\frac{4^3}{3} - \frac{0^3}{3}\right\} = 1$$

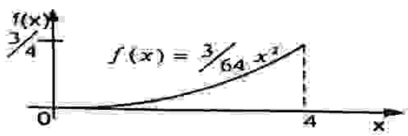
$$k = \frac{3}{64}$$

(ii) $P(1 \le X \le 3) = \frac{1}{36}\int_1^3 x^2 ddx$

$$= \frac{1}{36}\left[\frac{x^3}{3}\right]_1^3 = \frac{1}{36}\left\{\frac{3^3}{3} - \frac{1^3}{3}\right\} = 0.4063$$

When $x = 0$, $f(x) = \frac{3}{64}x0^2 = 0$

When $x = 4$, $f(x) = \frac{3}{64}x4^2 = \frac{3}{4}$



5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x^2+1), & 0 \le x \le 3 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(1 \le X \le 3)$

Solution

(i) $\int_0^3 k(x^2+1)dx = 1$

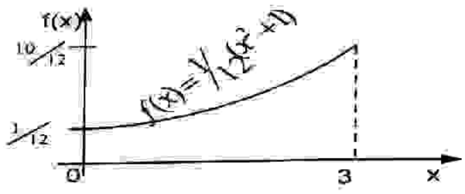
$k = \frac{1}{12}$

(ii) $P(1 \le X \le 3) = \frac{1}{12}\int_1^3 (x^2+1)dx$

$= \frac{1}{12}\left[\frac{x^3}{3} + x\right]_1^3 = 0.8889$

When $x = 0, f(x) = \frac{1}{12}(0^2+1) = \frac{1}{12}$

When $x = 3, f(x) = \frac{1}{12}(3^2+1) = \frac{10}{12}$



6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 0 \le x \le 2 \\ k(2x-3), & 2 \le x \le 3 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(X < 1)$

(iii) $P(X > 2.5)$

(iv) $P\left(0 \le X \le 2 / X \ge 1\right)$

Solution

(i) $\int_0^2 k\ dx + \int_2^3 k(2x-3)\ dx$

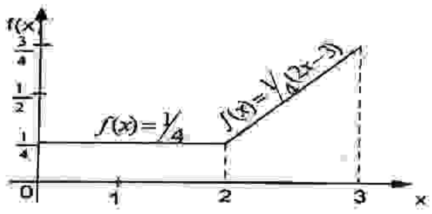
$k[x]_0^2 + k[x^2 - 3x]_2^3 = 1$

$k = \frac{1}{4}$

When $x = 0, f(x) = k = \frac{1}{4}$

When $x = 2, f(x) = k = \frac{1}{4}$

When $x = 3, f(x) = \frac{1}{4}x(2x3 - 3) = 4$



(ii) $P(X < 1) = \frac{1}{4}\int_0^1 dx = \frac{1}{4}[x]_0^1 = \frac{1}{4}$

(iii) $P(X > 2.5) = \frac{1}{4}\int_{2.5}^3 (2x-3)\ dx$

$= \frac{1}{4}[x^2 - 3x]_{2.5}^3 = 0.3125$

(iv) $P\left(0 \le X \le 2 / X \ge 1\right) = \frac{P(0 \le X \le 2 \cap X \ge 1)}{P(X \ge 1)}$

$= \frac{P(1 \le X \le 2)}{P(X \ge 1)} = \frac{\frac{1}{4}\int_1^2 dx}{\frac{1}{4}\int_1^2 dx + \frac{1}{4}\int_2^3 (2x-3)\,dx}$

$= \frac{1}{4} \Big/ \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{1}{3}$

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2)^2, & -2 \le x \le 0 \\ 4k, & 0 \le x \le 4/3 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(-1 < X < 1)$

(iii) $P(X > 1)$

Solution

(i) $\int_{-2}^0 k(x+2)^2 dx + \int_0^2 4k\ dx$

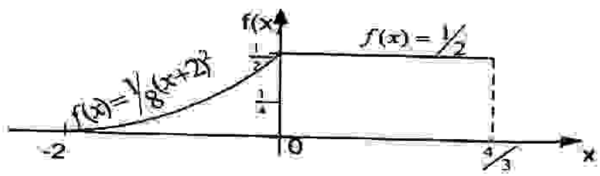
$k\left[\frac{(x+2)^3}{3}\right]_{-2}^0 + 4k[x]_0^2 = 1$

$k = \frac{1}{8}$

When $x = -2, f(x) = k(-2+2)^2 = 0$

When $x = 0, f(x) = k(0+2)^2 = \frac{1}{2}$

When $x = 4/3, f(x) = 4k = \frac{1}{2}$



(ii) $P(-1 < X < 1) = k\int_{-1}^0 (x+2)^2 dx + \int_0^1 4k dx$

$$= k\left[\frac{(x+2)^3}{3}\right]_{-1}^{0} + 4k[x]_0^1 = \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$$

(iii) $P(X > 1) = \int_1^{4/3} 4k dx = \frac{1}{2}[x]_1^{4/3} = \frac{1}{6}$

Finding the constant from a sketch graph

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

(a) sketch f(x) and find the value of constant k

(b) Find ;

(i) $P(X \le 1)$

(ii) $P(1 < X < 2)$

Solution

(a)

When $x = 0, f(x) = kx0 = 0$

When $x = 4, f(x) = kx4 = 4k$

$$\frac{1}{2}x4x4k = 1$$

$$8k = 1$$

$$k = \frac{1}{8}$$

(i) $P(X \le 1) = \frac{1}{8}\int_0^1 x\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_0^1$

$$= \frac{1}{8}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = \frac{1}{16}$$

(ii) $P(1 < X < 2) = \frac{1}{8}\int_1^2 x\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_1^2$

$$= \frac{1}{8}\left(\frac{2^2}{2} - \frac{1^2}{2}\right) = \frac{3}{16}$$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4-x), & 2 \le x \le 4 \\ 0, & else\ where \end{cases}$

(a) sketch f(x) and find the value of the constant k

(b) find;

(i) $P(X < 1)$

(ii) $P(X > 3)$

(iii) $P(1 \le X \le 3)$

(iv) $P(^{X \ge 1}/_{X \le 3})$

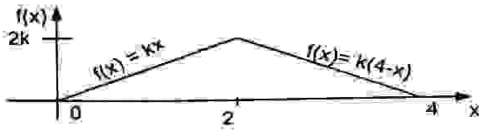
Solution

(a)

When $x = 0, f(x) = kx0 = 0$

When $x = 2, f(x) = kx2 = 2k$

When $x = 4, f(x) = kx(4-4) = 0$



$$\frac{1}{2}x4x2k = 1$$

$$4k = 1$$

$$k = \frac{1}{4}$$

(i) $P(X < 1) = \frac{1}{4}\int_0^1 x\,dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{8}$

(ii) $P(X > 3) = \frac{1}{4}\int_3^4 (4-x)\,dx = \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$

$$= 0.125$$

(iii) $P(1 < X < 3) = \frac{1}{4}\int_1^2 x\,dx + \frac{1}{4}\int_2^3 (4-x)\,dx$

$$= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{4}\left[4 - \frac{x^2}{2}\right]_2^3 = \frac{3}{4}$$

(iv) $P(^{X \ge 1}/_{X \le 3}) = \frac{P(X \ge 1 \cap X \le 3)}{P(X \le 3)}$

$$= \frac{P(1 \le X \le 3)}{P(X \le 3)}$$

$$= \frac{\frac{3}{4}}{\frac{1}{4}\int_0^2 x\,dx + \frac{1}{4}\int_2^3 (4-x)\,dx} = \frac{3}{4}\Big/\frac{7}{8} = \frac{6}{7}$$

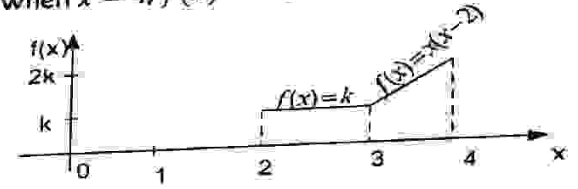
3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 2 \le x \le 3 \\ k(x-2), & 3 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) the value of the constant k and sketch f(x)

(ii) find $P(|X - 2.5| < 0.5)$

Solution

When $x = 2, f(x) = k$
When $x = 3, f(x) = k$
When $x = 4, f(x) = k(4-2) = 2k$



$$(kx1) + \frac{1}{2}x1(k + 2k) = 1$$

$$k = \frac{2}{5}$$

(iii) $P(|X - 2.5| < 0.5) =$
$= P(-0.5 < X - 2.5 < 0.5)$

$$P(2 < X < 3) = \frac{2}{5}\int_2^3 dx = \frac{2}{5}[x]_2^3 = \frac{2}{5}$$

Finding p.d.f from a sketch graph

1. A random variable X of a continuous p.d.f is given by

(a) Find the value of k

(b) Find f(x)

Solution

(a) $Area = 1$

$$\frac{1}{2}x3xk = 1$$

$$k = \frac{2}{3}$$

(b) For interval; $0 \le x \le 1$

$(0,0)$ and $(1, k)$

grad$= \frac{y-0}{x-0} = \frac{\frac{2}{3}-0}{1-0}$

$y = \frac{2}{3}x$

For interval; $1 \le x \le 3$

$(3,0)$ and $(1, k)$

grad$= \frac{y-0}{x-3} = \frac{\frac{2}{3}-0}{1-3}$

$y = -\frac{1}{3}(x-3)$

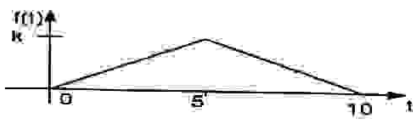
$$f(x) = \begin{cases} \frac{2}{3}x, & 0 \le x \le 1 \\ -\frac{1}{3}(x-3), & 1 \le x \le 3 \\ 0, & else\ where \end{cases}$$

2. A random variable X of a continuous p.d.f is given by

Find ; **Uneb 2012 No.12**

(a) the value of k
(b) Find f(x)
(c) $P\left(X < 1.5 / X > 0.5\right)$
(d) Mean of X

3. The life time, T of a fully charged phone battery is a random variable T of a continuous p.d.f is given by



Find ; **Uneb 2007 No.9**

(a) the value of k
(b) Find f(t)
(c) $P(4 \le X \le 7)$
(d) Mean of T

An; $(a) = \frac{1}{5}, (c) = 0.5, \ (d) = 5$

$$f(t) = \begin{cases} \frac{1}{5}t, & 0 \le t \le 5 \\ \frac{1}{5}(10-t), & 5 \le t \le 10 \\ 0, & else\ where \end{cases}$$

Exercise 7a

1. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}kx^2, & 0\le x\le 2\\ 0, & else\ where\end{cases}$
   (a) Find the value of the constant k
   (b) sketch f(x)
   (c) find (i) $P(X\ge 1)$ (ii) $P(0.5\le X\le 1.5)$
   An. $(a)=\frac{3}{8}$ (c) $(i)=\frac{7}{8}$ $(ii)=\frac{13}{32}$

2. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}k & -2\le x\le 3\\ 0, & else\ where\end{cases}$
   (i) Sketch f(x)
   (ii) Find the value of the constant k
   (iii) find $P(-1.6\le X\le 2.1)$
   An. $(ii)=\frac{1}{5}$ $(iii)=0.74$

3. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}k(4-x), & 1\le x\le 3\\ 0, & else\ where\end{cases}$
   (i) Find the value of the constant k
   (ii) sketch f(x) (iii) find $P(1.2\le X\le 2.4)$
   An. $(i)=\frac{1}{4}$ $(iii)=0.66$

4. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}k(x+2)^2, & 0\le x\le 2\\ 0, & else\ where\end{cases}$
   (a) Find the value of the constant k
   (b) sketch f(x)
   (c) find (i) $P(0\le X\le 1)$ (ii) $P(X\ge 1)$
   An. $(i)=\frac{3}{56}$ $(iii)=\frac{19}{56}$ $(iv)=\frac{37}{56}$

5. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}kx^3, & 0\le x\le c\\ 0, & else\ where\end{cases}$
   Given that $P(X\le 0.5)=\frac{1}{16}$
   (i) Find the value of the constant k and c
   (ii) sketch f(x) **An.** $c=1,\ k=4$

6. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}kx, & 0\le x\le 4\\ 0, & else\ where\end{cases}$
   (i) Find the value of the constant k
   (ii) sketch f(x) (iii) find $P(1\le X\le 2.5)$
   An. $(i)=0.125$ $(iii)=0.328$

7. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}k, & 0\le x\le 2\\ k(2x-3), & 2\le x\le 3\\ 0, & else\ where\end{cases}$
   (a) sketch f(x) and find the value of the constant k
   (b) find;
   (i) $P(X<1)$ (ii) $P(X>2.5)$
   (iii) $P(1\le X\le 2.3)$
   (ii)
   (iii)
   An. $(a)=\frac{1}{4}$ (b) $(i)=\frac{1}{4}$ $(ii)=0.3125$ $(iii)=0.3475$

8. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}a, & 0\le x\le 1.5\\ \frac{a}{2}(2-x), & 1.5\le x\le 2\\ 0, & else\ where\end{cases}$
   Find the **Uneb 2000 No.14 a**
   (i) Value of a (ii) $P(X<1.6)$
   An. $(i)\ a=\frac{16}{25}$ $(ii)=0.9744$

"Expectation or mean of X"

For a continuous random variable with p.d.f, f(x): $E(X)=\int x\,f(x)\,dx$

Examples

1. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}kx^2, & 0\le x\le 3\\ 0, & else\ where\end{cases}$

Find
(i) the value of the constant k and sketch f(x)
(ii) μ, the mean of X
(iii) $P(X\le\mu)$

Solution

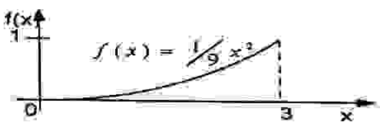
(i) $\int_0^3 kx^2dx=1$

$$k\left[\frac{x^3}{3}\right]_0^3=1$$

$$k=\frac{1}{9}$$

When $x=0$, $f(x)=\frac{1}{9}x0^2=0$

When $x=3$, $f(x)=\frac{1}{9}x3^2=1$



(ii) $E(X)=\frac{1}{9}\int_0^3 x^3dx=\frac{1}{9}\left[\frac{x^4}{4}\right]_0^3=2.25$

$$P(X \le \mu) = \frac{1}{9}\int_0^{2.25} x^2 dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_0^{2.25}$$

$= 0.42$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^3, & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k

(ii) the mean of X

(iii) $P(X \le 1)$

Solution

(ii) $\int_0^2 kx^3 dx = 1$

$$k\left[\frac{x^4}{4}\right]_0^2 = 1$$

$$k\left\{\frac{2^4}{4} - \frac{0^4}{4}\right\} = 1$$

$$k = \frac{1}{4}$$

(ii) $E(X) = \frac{1}{4}\int_0^2 x^4 dx = \frac{1}{4}\left[\frac{x^5}{5}\right]_0^2 = 1.6$

(iii) $P(X \le 1) = \frac{1}{4}\int_0^1 x^3 dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_0^1$

$= 0.0625$

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(4x - x^2), & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k

(ii) $E(X)$

(iii) $P(X \le 1)$

Solution

(i) $\int_0^2 k(4x - x^2)dx = 1$

$$k\left[2x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left\{\left[2(2)^2 - \frac{2^3}{3}\right] - \left[2(0)^2 - \frac{0^3}{3}\right]\right\} = 1$$

$$k = \frac{3}{16}$$

(ii) $E(X) = \frac{3}{16}\int_0^2 x(4x - x^2)dx$

$$= \frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^2 = 0.25$$

(iii) $P(X \le 1) = \frac{3}{16}\int_0^1 (4x - x^2)dx$

$$= \frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^1 = 0.3125$$

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 3x^k, & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k (ii) $E(X)$

(ii) The value of a such that $P(X \le a) = 0.5$

Solution

(i) $\int_0^1 3x^k dx = 1$

$$3\left[\frac{x^{k+1}}{k+1}\right]_0^1 = 1$$

$$3\left\{\frac{1^{k+1}}{k+1} - \frac{0^{k+1}}{k+1}\right\} = 1$$

$$\frac{3}{k+1} = 1$$

$$k = 2$$

(ii) $E(X) = \int_0^1 x(3x^2)dx = 3\left[\frac{x^4}{4}\right]_0^1 = 0.75$

(iii) $P(X \le a) = 3\int_0^1 (x^2)dx$

$0.5 = [x^3]_0^a$

$a^3 = 0.5$

$a = 0.794$

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{4}x, & 0 \le x \le 2 \\ \left(1 - \frac{x}{4}\right), & 2 \le x \le 4 \\ 0, & else\ where \end{cases}$

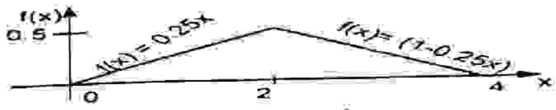
sketch f(x) and find E(X)

Solution

When $x = 0, f(x) = \frac{1}{4}x0 = 0$

When $x = 2, f(x) = \frac{1}{4}x2 = 0.5$

When $x = 4, f(x) = \left(1 - \frac{4}{4}\right) = 0$



$$E(X) = \frac{1}{4}\int_0^2 x^2\,dx + \int_2^4 \left(x - \frac{x^2}{4}\right)dx$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^2}{2} - \frac{x^3}{12}\right]_2^4 = 2$$

(ii) $P(X > 3) = \frac{1}{4}\int_3^4 (4 - x)\,dx$

$$= \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4 = \frac{1}{8} = 0.125$$

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2), & -1 \le x \le 0 \\ 2k(1-x), & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

(i) sketch f(x)

(ii) find the value of k

(iii) $P\left(0 < X < 0.5 / X > 0\right)$

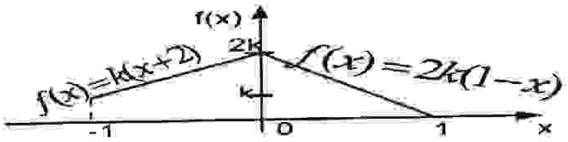
(iv) find E(X) **Uneb 1997 No.10**

Solution

(i) When $x = -1, f(x) = k(-1+2) = k$

When $x = 0, f(x) = k(0+2) = 2k$

When $x = 1, f(x) = 2k(1-1) = 0$



$$\frac{1}{2}x1x(k + 2k) + \frac{1}{2}x1x2k = 1$$

$$k = \frac{2}{5}$$

(iii) $P\left(0 < X < 0.5 / X > 0\right) = \frac{P(0 < X < 0.5)}{P(X > 0)}$

$$= \frac{\frac{4}{5}\int_0^{0.5}(1-x)\,dx}{\frac{4}{5}\int_0^{1}(1-x)\,dx} = \frac{\left[x - \frac{x^2}{2}\right]_0^{0.5}}{\left[x - \frac{x^2}{2}\right]_0^{1}} = \frac{3}{8}\Big/\frac{1}{2} = 0.75$$

(iv) $E(X) = \frac{2}{5}\int_{-1}^{0} x(x+2)\,dx + \frac{4}{5}\int_0^1 x(1-x)\,dx$

$$= \frac{2}{5}\left[\frac{x^3}{3} + x^2\right]_{-1}^{0} + \frac{4}{5}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = -\frac{2}{15}$$

Properties of the mean

(i) $E(a) = a$

(ii) $E(ax) = a\,E(x)$

(iii) $E(ax + b) = aE(x) + b$

(iv) $E(ax - b) = aE(x) - b$

Where a and b are constants

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{20}(x+3), & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) Sketch f(x)

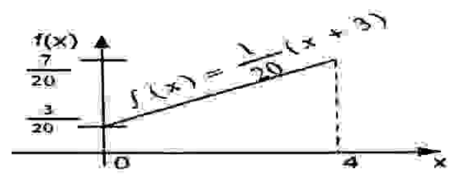
(ii) Find $E(X)$

(iii) Find $E(2X + 5)$

Solution

(i) When $x = 0, f(x) = \frac{1}{20}(0+3) = \frac{3}{20}$

When $x = 4, f(x) = \frac{1}{20}(4+3) = \frac{7}{20}$



$$(ii)\ E(X) = \frac{1}{20}\int_0^4 x(x+3)dx = \frac{1}{20}\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^4$$
$= 2.266$

$(iii)\ \ E(2X+5) = 2x2.266 + 5 = 9.533$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{32}x^2, & 0 \le x \le 2 \\ \frac{3}{32}(6-x), & 2 \le x \le 6 \\ 0, & else\ where \end{cases}$

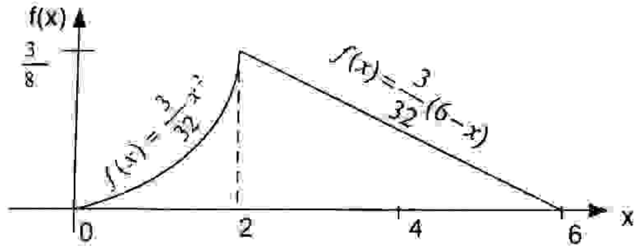
(i) sketch f(x)
(ii) $find\ P(X < 4)$
(iii) find E(X)
(iv) find $E(100x - 20)$

Solution

When $x = 0$, $f(x) = \frac{3}{32}(0)^2 = 0$

When $x = 2$, $f(x) = \frac{3}{32}(2)^2 = \frac{3}{8}$

When $x = 6$, $f(x) = \frac{3}{32}(6-6) = 0$



$$(ii)\ P(X<4) = \frac{3}{32}\int_0^2 x^2 dx + \frac{3}{32}\int_2^4 (6-x)\,dx$$

$$= \frac{3}{32}\left[\frac{x^3}{3}\right]_0^2 + \frac{3}{32}\left[6x - \frac{x^2}{2}\right]_2^4 = \frac{13}{16}$$

$$(iii)\ E(X) = \frac{3}{32}\int_0^2 x^3\,dx + \frac{3}{32}\int_2^6 (6 - x^2)\,dx$$

$$= \frac{3}{32}\left[\frac{x^4}{4}\right]_0^2 + \frac{3}{32}\left[3x^2 - \frac{x^3}{3}\right]_2^6 = 2.875$$

$(iv)\ \ E(100X - 20) = 100x2.875 - 20 = 267.50$

Exercise 7b

1. Sketch f(x) and find E(X) for each of the following continuous random variable

(a) $f(x) = \begin{cases} \frac{3}{4}(x^2+1), & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

(b) $f(x) = \begin{cases} \frac{3}{4}x(2-x), & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

(c) $f(x) = \begin{cases} \frac{1}{18}(6-x), & 0 \le x \le 6 \\ 0, & else\ where \end{cases}$

(d) $f(x) = \begin{cases} kx^3, & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

(e) $f(x) = \begin{cases} \frac{3}{8}, & \frac{2}{3} \le x \le 2 \\ \frac{3}{32}x(4-x), & 2 \le x \le 4 \\ 0, & else\ where \end{cases}$

An. $(a) = \frac{9}{16}$ $\ (b) = 1$ $\ (c) = 2$ $\ (d) = 1.6$ $\ (e) = 2.042$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 0 \le x \le 1 \\ k, & 1 \le x \le 3 \\ k(4-x), & 3 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) sketch f(x)
(ii) find the value of k
(iii) find E(X)

An. $(ii)\ k = \frac{2}{7}$ $\quad (iii) = 2$

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2, & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) Sketch f(x)
(ii) Find $E(X)$
(iii) Find $E(2X + 5)$

An. $(ii) = 3$ $\quad (iii) = 11$

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2(10-x), & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) Find k
(ii) Find $E(X)$
(iii) Find $E(3X - 4)$

An. $(i)\ k = \frac{3}{2500}$ $\quad (ii) = 6$ $\quad (iii) = 14$

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 5 \le x \le 10 \\ 0, & else\ where \end{cases}$

(i) Sketch f(x)
(ii) Find k
(iii) Find $E(X)$
(iv) Find $P(X > 8)$

An. $(ii)\ k = \frac{2}{75}$ $\quad (iii) = \frac{70}{9}$ $\quad (iv) = 0.48$

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k[1 - (x-2)^2], & 1 \le x \le 3 \\ 0, & else\ where \end{cases}$

(i) Find k
(ii) Sketch f(x) (*iii*) Find $E(X)$

An. $(i)\ k = \frac{3}{4}$ $(iii) = 2$

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx(5-x), & 0 \le x \le 5 \\ 0, & else\ where \end{cases}$

(i) Find k
(ii) Sketch f(x) (*iii*) Find $mean\ of\ X$

An. $(i)\ k = \frac{6}{125}$ $(iii) = 2.5$

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(1+cosx), & 0 \le x \le \pi \\ 0, & else\ where \end{cases}$

(i) Find k
(ii) Sketch f(x) (*iii*) Find $mean\ of\ X$

An. $(i)\ k = \frac{1}{\pi}$ $(iii) = 0.9342$

9. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{3}x, & 0 \le x \le 3 \\ k, & 3 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) sketch f(x)
(ii) *find k* (*iii*) find E(X)
(*iv*) find of c such that $P(X > c) = 0.85$

An. $(ii)\ k = \frac{2}{5}$ $(iii) = 2.6$ $(iv)\ c = 1.5$

10. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k\left(x - \frac{1}{a}\right), & 1 \le x \le 3 \\ 0, & else\ where \end{cases}$

Given that $P(X > 1) = 0.8$, Find: **Uneb 2001 No.15**

(i) Values of a and k
(ii) Probability that X lies between 0.5 and 2.5
(iii) $mean\ of\ X$

An. $(i)\ k = \frac{2}{15}, a = -1$ $(ii) = 0.6667, (iii) = 1.8$

11. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2), & -1 \le x \le 0 \\ 2k, & 0 \le x \le 1 \\ \frac{k}{2}(5-x), & 1 \le x \le 3 \\ 0, & else\ where \end{cases}$

(a) Sketch the function f(x) **Uneb 2004 No.11**
(b) Find the value of k and mean of X

An. $(i)\ k = \frac{2}{13}$ $(ii) = \frac{12}{13}$

12. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 2kx, & 0 \le x \le 1 \\ k(3-x), & 1 \le x \le 2 \\ 0, & else\ where \end{cases}$

Uneb 2005 No.11

(a) Sketch the function f(x)
(b) Find the value of k and mean of X

An. $(i)\ k = \frac{2}{5}$ $(ii) = \frac{17}{15}$

13. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha(1 - cosx), & 0 \le x \le \frac{\pi}{2} \\ \alpha\ sinx, & \frac{\pi}{2} \le x \le \pi \\ 0, & else\ where \end{cases}$

Find the **Uneb 2008 No.12**

(i) Value of α (*ii*) Mean , μ
$(iii)\ P\left(\frac{\pi}{3} < X < \frac{3\pi}{4}\right)$

An. $(i)\ \alpha = \frac{2}{\pi}$ $(ii) = 1 + \frac{\pi}{4}$ $(iii) = 0.6982$

14. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k_1x, & 1 \le x \le 3 \\ k_2(4-x), & 3 \le x \le 4 \\ 0, & else\ where \end{cases}$

(a) Show that $k_2 = 3k_1$ **Uneb 2011 No.9**
(b) Find
(i) Value of k_1 and k_2
(ii) Mean , μ

An. $(i)\ k_1 =, k_2 =$ $(ii) =$

15. A random variable Yof a continuous p.d.f is given by $f(x) = \begin{cases} \frac{y+1}{4}, & 1 \le y \le k \\ 0, & else\ where \end{cases}$

Find: **Uneb 2015 No.9**

(i) Values of k
(ii) Expectation of Y
(iii) $P(1 \le Y \le 1.5)$

An. $(i)\ k = 2,$ $(ii) = 1.6667, (iii) = 0.2813$

Variance of X

For a continuous random variable with p.d.f, f(x)

$$Var(X) = E(X^2) - [E(X)]^2 \quad \textbf{Or} \quad Var(X) = E(X^2) - \mu^2$$

Where $E(X^2) = \int x^2 f(x)\, dx$ and μ –Mean

Properties of the Variance

(i) $Var(a) = 0$ (ii) $Var(ax) = a^2\ Var(x)$

(iii) $Var(ax + b) = a^2Var(x)$ (iv) $Var(ax - b) = a^2Var(x)$

Where a and b are constants

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(1-x^2), & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

Find **Uneb 1993 No.10a**

(i) the value of the constant k

(ii) $E(X)$

(iii) Var(X)

Solution

(i) $\int_0^1 k(1-x^2)dx = 1$

$$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$k = 1.5$$

(ii) $E(X) = 1.5\int_0^1 x(1-x^2)dx = 1.5\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$

$$= \frac{3}{8}$$

$$E(X^2) = 1.5\int_0^1 x^2(1-x^2)dx$$

$$= 1.5\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

$$Var(X) = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}x, & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find

(i) $E(X)$ (ii) $Var(X)$ (iii) standard deviation, σ (iv) $Var(3X + 2)$

Solution

(i) $E(X) = \frac{1}{8}\int_0^4 x(x)dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4 = 2.667$

(ii) $E(X^2) = \frac{1}{8}\int_0^4 x^2(x)dx = \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4 = 8$

$Var(X) = 8 - (2.667)^2 = 0.887$

(iii) $\sigma = \sqrt{0.887} = 0.942$

(iv) $Var(3X + 2) = 9x0.887 = 7.983$

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{4}{25}(5-2x), & 0 \le x \le 2.5 \\ 0, & else\ where \end{cases}$

Find

(i) mean, μ (ii) standard deviation, σ

Solution

(i) $E(X) = \frac{4}{25}\int_0^{2.5} x(5-2x)dx$

$$= \frac{4}{25}\left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_0^{2.5} = 0.833$$

(ii) $E(X^2) = \frac{4}{25}\int_0^{2.5} x^2(5-2x)dx$

$$= \frac{4}{25}\left[\frac{5x^3}{3} - \frac{2x^4}{4}\right]_0^{2.5} = 1.041$$

$$Var(X) = 1.041 - (0.833)^2 = 0.347$$

$$\sigma = \sqrt{0.347} = 0.59$$

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{4}(1+x^2), & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

Find

(i) mean, μ (ii) standard deviation, σ (iii) $P(|X - \mu| < \sigma)$

Solution

(i) $\mu = \frac{3}{4}\int_0^1 x(1+x^2)dx$

$$= \frac{3}{4}\left[\frac{x^2}{2} + \frac{x^4}{4}\right]_0^1 = 0.5625$$

(ii) $E(X^2) = \frac{3}{4}\int_0^1 x^2(1+x^2)dx$

$$= \frac{3}{4}\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1 = 0.4$$

$$Var(X) = 0.4 - (0.5625)^2 = 0.0835$$

$$\sigma = \sqrt{0.0835} = 0.289$$

(iii) $P(|X - \mu| < \sigma) = P(|X - 0.5625| < 0.289)$

$$= P(0.2735 < X < 0.8515)$$

$$= \frac{3}{4}\int_{0.2735}^{0.8515} (1 + x^2)dx$$

$$= \frac{3}{4}\left[x + \frac{x^3}{3}\right]_{0.2735}^{0.8515} = 0583$$

Exercise 7c

1. Find E(X) and standard deviation for each of the following continuous random variables

(i) $f(x) = \begin{cases} \frac{3}{8}x^2, & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

(ii) $f(x) = \begin{cases} \frac{1}{5}, & -2 \le x \le 3 \\ 0, & else\ where \end{cases}$

(iii) $f(x) = \begin{cases} k(4-x), & 1 \le x \le 3 \\ 0, & else\ where \end{cases}$

(iv) $f(x) = \begin{cases} k(x+2)^2, & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

(v) $f(x) = \begin{cases} kx^3, & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

(vi) $f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 2 \\ \frac{1}{4}(2x-3), & 2 \le x \le 3 \\ 0, & else\ where \end{cases}$

(vii) $f(x) = \begin{cases} \frac{1}{8}(x+2)^2, & -2 \le x \le 0 \\ \frac{1}{2}, & 0 \le x \le \frac{4}{3} \\ 0, & else\ where \end{cases}$

An. (a) = 1.5, 0.387 (b) = 0.5, 1.44 (c) = 1.833, 0.553 (d) = 1.214, 0.545 (e) = 0.8, 0.163, (f) = 1.792, 0.912 (g) = 0.278, 0.672

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2, & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find

(i) The value of k and sketch f(x)

(ii) $E(X)$ and $Var(X)$

(iii) $P(1 < X < 2)$

An. (i) $k = \frac{3}{64}$ (ii) $= 3, \frac{3}{5}$ (iii) $= \frac{7}{64}$

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find

(i) Constant k (ii) $E(X)$

(iii) $Var(X)$ (iv) $P(0.75 < X < 1.5)$

(v) mode

An. (i) $k = 1$ (ii) $= 1$ (iii) $= \frac{1}{6}$ (iv) $= \frac{19}{32}$ (v) $= 1$

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{27}x^2, & 0 \le x \le 3 \\ \frac{1}{3}, & 3 \le x \le 5 \\ 0, & else\ where \end{cases}$

(i) Sketch f(x)

(ii) Find $E(X)$ (iiii) standard deviation

An. (ii) $= 3.417$ (iii) $= 1.008$

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{x(4-x)}, & 1 \le x \le 3 \\ 0, & else\ where \end{cases}$

(i) Show that $k = \frac{2}{ln 3}$

(ii) Find $E(X)$ and $Var(X)$

An. (ii) $= 2$, $4 - \frac{4}{ln3}$

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(ax - x^2), & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

(i) Show that $k = \frac{3}{6a-8}$

(ii) Given that $E(X) = 1$, find the value of a and k

(iii) For the above values of a and k, find $Var(X)$

An. (ii) $a = 2$, $k = 0.75$ (ii) $= 0.2$

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 12(x^2 - x^3), & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

Find mean and standard deviation

An. $\mu = 0.6, \sigma = 0.2$

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{\beta}, & 0 \le x \le \beta \\ 0, & else\ where \end{cases}$

(i) Show that $k = 1$

(ii) Find the mean and standard deviation, in terms of β

An. (ii) $\mu = \frac{\beta}{2}$, $\sigma = \frac{\beta}{2\sqrt{3}}$

9. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}(x+1), & 2 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find

(i) $E(X)$ (ii) $Var(X)$

(iii) $P(2.5 < X < 3)$

An. $(i) = \frac{37}{12}$ $(ii) = \frac{47}{144}$ $(iii) = 0.234$

10. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(1-x)^2, & 2 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find

(i) Constant k (ii) Mean

(iii) standard deviation

An. (i) $k = \frac{3}{26}$ $(ii) = \frac{1}{4}$ $(iii) = 0.194$

11. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4-x), & 2 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find

(i) Constant k (ii) $E(X)$

(iii) $Var(X)$ (iv) $P(X < 1)$

(v) $P(2 < X < 3)$ (vi) $P(1 < X < 3)$

An. (i) $k = \frac{1}{4}$ $(ii) = 2$ $(iii) = \frac{2}{3}$

$(iv) = \frac{1}{8}$ $(v) = \frac{3}{8}$ $(vi) = \frac{3}{4}$

12. A man leaves at a point which is 20 minutes walk from the taxi stage. Taxis arrive at the stage punctually. If the p.d.f for getting a taxi is given by $f(x) = \begin{cases} \frac{1}{20}, & 0 \le x \le 20 \\ 0, & else\ where \end{cases}$

Find **Uneb 1999 No 10b**

(i) Expected time it takes to wait for a taxi

(ii) Variance of the time it takes to wait for a taxi

An. $(i) = 0.05$ mins $(ii) = \frac{100}{3}$ mins

Mode

This is the value of X for which f(x) is maximum in the f(x) given range of X

(i) The mode is obtained from $\frac{d}{dx}f(x) = 0$

The maximum value is confirmed if $\frac{d^2}{dx^2}f(x) = negative$

(ii) When a sketch of f(x) is drawn, the value of X for which f(x) is maximum gives the mode

Note: For any line the mode can only be determine from a sketch of f(x)

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(2+x)(4-x), & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find

(i) The value of k and sketch f(x)

(ii) mode

Solution

(i) $\int_0^4 k(2+x)(4-x)dx = 1$

$$\int_0^4 k(8+2x-x^2)dx = 1$$

$$k\left[8x + x^2 - \frac{x^3}{3}\right]_0^4 = 1$$

$$k = \frac{3}{80}$$

(ii) $\frac{d}{dx}f(x) = 0$

$$\frac{d}{dx}\frac{3}{80}(8+2x-x^2) = 0$$

$$\frac{3}{80}(2-2x) = 0$$

$$x = 1$$

Mode = 1

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{108}x(6-x)^2, & 0 \le x \le 6 \\ 0, & else\ where \end{cases}$

Find

(i) Mean (ii) Standard deviation (iii) mode

Solution

(i) $\mu = \int_0^6 \frac{1}{108}x^2(6-x)^2dx$

$$= \int_0^6 \frac{1}{108}(36x^2 - 12x^3 + x^4)dx$$

$$=\frac{1}{108}\left[12x^3-3x^4+\frac{x^5}{5}\right]_0^6=2.4$$

(ii) $E(X^2)=\int_0^6 \frac{1}{108}x^3(6-x)^2dx$

$$=\int_0^6 \frac{1}{108}(36x^3-12x^4+x^5)dx$$

$$=\frac{1}{108}\left[9x^4-\frac{12x^5}{5}+\frac{x^{56}}{6}\right]_0^6$$

$$=7.2$$

$$\sigma=\sqrt{7.2-2.4^2}=1.2$$

(iii) $\frac{d}{dx}f(x)=0$

$$\frac{d}{dx}\frac{1}{108}(36x-12x^2+x^3)=0$$

$$\frac{1}{108}(36-24x+3x^2)=0$$

$$(6-x)(2-x)=0$$

$x=2$ **or** $x=6$

Mode $=2$

3. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}ksinx, & 0\le x\le\pi\\ 0, & else\ where\end{cases}$

Find

(i) Value of k

(ii) $P(X\ge\frac{\pi}{3})$

(iii) Mean

(iv) Var(X)

(v) *mode*

Solution

(i) $\int_0^\pi ksinx\,dx=1$

$$k[-cosx]_0^\pi=1$$

$$k(-cos\pi--cos0)=1$$

$$k=\frac{1}{2}$$

(ii) $P\left(X\ge\frac{\pi}{3}\right)=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}sinx\,dx=\frac{1}{2}[-cosx]_{\frac{\pi}{3}}^{\pi}$

$$=\frac{1}{2}\left(-cos\pi--cos\frac{\pi}{3}\right)=\frac{3}{4}$$

(iii) $E(X)=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}xsinx\,dx$

sign	derivative	integral
+	x	sinx
-	1	-cosx
+	0	-sinx

$$=\frac{1}{2}[-xcosx+sinx]_0^\pi=\frac{\pi}{2}$$

(iv) $E(X^2)=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}x^2sinx\,dx$

sign	derivative	integral
+	x^2	sinx
-	2x	-cosx
+	2	-sinx
-	0	cosx

$$=\frac{1}{2}[-x^2cosx+2xsinx+2cosx]_0^\pi=\frac{\pi^2-4}{2}$$

$$Var(X)=\frac{\pi^2-4}{2}-\left(\frac{\pi}{2}\right)^2=\frac{\pi^2-8}{4}$$

(v) $\frac{d}{dx}f(x)=0$

$$\frac{d}{dx}\frac{1}{2}(sinx)=0$$

$$\frac{1}{2}(cosx)=0$$

$$x=cos^{-1}0$$

$x=90°$

Mode $=\frac{\pi}{2}$

4. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}kx, & 0\le x\le 6\\ 0, & else\ where\end{cases}$

(a) sketch f(x)

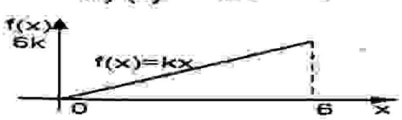
(b) find the value of constant k

(c) mode

Solution

(a) When $x=0$, $f(x)=kx0=0$

When $x=6$, $f(x)=kx6=6k$



$$\frac{1}{2}x6x6k=1$$

$$k=\frac{1}{18}$$

Mode $=6$

5. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}10kx^2, & 0\le x\le 0.6\\ 9k(1-x), & 0.6\le x\le 1.0\\ 0, & else\ where\end{cases}$

Find

(i) The value of k and Sketch f(x)

(ii) Most likely value of X

(iii) *Expected valu of X*

(iv) $P(X>0.8)$

(v) P(0.4 < X < 0.8)

Solution

(i) $\int_0^{0.6} 10kx^2 + \int_{0.6}^{1} 9k(1-x)dx = 1$

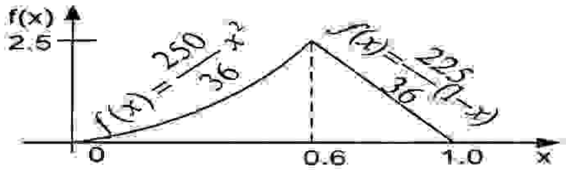
$$10k\left[\frac{x^3}{3}\right]_0^{0.6} + 9k\left[x - \frac{x^2}{2}\right]_0^{1.0} = 1$$

$$k = \frac{25}{36}$$

When $x = 0$, $f(x) = \frac{250}{36}(o)^2 = 0$

When $x = 0.6$, $f(x) = \frac{250}{36}(0.6)^2 = 2.5$

When $x = 1.0$, $f(x) = \frac{225}{36}(1-1) = 0$



(ii) Most likely value of X= 0.6

(ii) **(iii)** $E(X) = \frac{25}{36}\int_0^{0.6} 10x^3\,dx + \frac{25}{36}\int_{0.6}^{1} 9x(1-x)\,dx$

$$= \frac{250}{36}\left[\frac{x^4}{4}\right]_0^{0.6} + \frac{225}{36}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0.6}^{1} = 0.591$$

(iii) **(iv)** $P(X > 0.8) = \frac{25}{36}\int_{0.8}^{1} 9(1-x)\,dx$

$$= \frac{225}{36}\left[x - \frac{x^2}{2}\right]_{0.8}^{1} = \mathbf{0.125}$$

(v) $P(0.4 < X < 0.8) = \frac{25}{36}\int_{0.4}^{0.6} 10x^2\,dx + \frac{25}{36}\int_{0.6}^{0.8} 9(1-x)\,dx$

$$= \frac{250}{36}\left[\frac{x^3}{3}\right]_{0.4}^{0.6} + \frac{225}{36}\left[x - \frac{x^2}{2}\right]_{0.6}^{0.8} = \mathbf{0.727}$$

Median

This is the value of X for which : $\int_a^m f(x)dx = 0.5$

Where m is the median and a is the lower limit

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}x, & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

Find the median

Solution

$$\int_0^m \frac{1}{8}x\,dx = 0.5$$

$$\left[\frac{1}{16}x^2\right]_0^m = 0.5$$

$$\frac{1}{16}m^2 = 0.5$$

$$m = \pm 2.828$$

Median is 2.828 since it lies in the range

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{5}(x+2), & -1 \le x \le 0 \\ \frac{4}{5}(1-x), & 0 \le x \le 10 \\ 0, & else\ where \end{cases}$

Find the median

Solution

We need to first integrate the first interval if its ≥ 0.5. if its less, then the median lies in the second interval

$$\int_{-1}^{0} \frac{2}{5}(x+2)\,dx = \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{-1}^{0} = 0.6$$

Then the median lies in the interval $-1 \le x \le 0$

$$\int_{-1}^{m} \frac{2}{5}(x+2)\,dx = 0.5$$

$$\frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{-1}^{m} = 0.5$$

$$m = -0.129 \text{ or } m = -3.871$$

Median is -0.129 since it lies in the range

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{3}(x+1), & -1 \le x \le 0 \\ \frac{1}{3}(2-x), & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find the median

Solution

We need to first integrate the first interval if its ≥ 0.5.
if its less, then the median lies in the second interval

$$\int_{-1}^{0} \frac{2}{3}(x+1)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + x\right]_{-1}^{0} = \frac{1}{3}$$

Then the median lies in the interval $0 \le x \le 2$

$$\frac{1}{3} + \int_{0}^{m} \frac{1}{3}(2-x)\,dx = 0.5$$

$$\frac{1}{3}\left[2x - \frac{x^2}{2}\right]_0^m = \frac{1}{6}$$

$$2m - \frac{m^2}{2} = \frac{1}{2}$$

$$m^2 - 4m + 1 = 0$$

$$m = 0.268\ \ or\ m = 3.732$$

Median is 0.268 since it lies in the range

Exercise 7d

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k\,x(4-x^2), & 0 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k (*ii*) Median of X
(*iii*) mean of X (*iv*) Standard deviation

An. $(i)\ k = 0.25$ $(ii) = 2.613$
$(iii) = 1.067$ $(iv) = 0.442$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find

(i) Constant k
(ii) Median of X (*iii*) mode

An. $(i)\ k = 1$ $(ii) = 1$ $(iii) = 1$

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} cx, & 0 \le x \le 1 \\ c(2-x), & 1 \le x \le 2 \\ 0, & else\ where \end{cases}$

(i) Sketch f(x) (*ii*) Find Constant *c*
(*iii*) Median m (*iv*) *mode*

Uneb 1992 No.12

An. $(ii)\ c = 1$ $(iii) = 1$ $(iv) = 1$

4. The continuous random variable X has a probability density function

$f(x) = \begin{cases} kx(4-x^2) & 0 \le x \le 2 \\ 0 & else\ where \end{cases}$

Find **Uneb 1998 No.11**

(i) Constant k (*ii*) Median
(*iii*) Mean (*iv*) Standard deviation

An $(i) k = \frac{1}{4}$ $(ii)\ median = 2.6131$
$(iii)\ E(x) = 1.0667,$ $(iv)\delta = 0.4422$

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha, & 2 \le x \le 3 \\ \alpha(x-2), & 3 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) Sketch f(x) (*ii*) Find Constant α
(*iii*) Median m (*iv*) $P(2.5 < X < 3.5)$

Uneb 2006 No.15

An. $(ii)\ \alpha = \frac{2}{5}$ $(iii) = 3.225$ $(iv) = 0.65$

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \beta, & 0 \le x \le 2 \\ \beta(3-x), & 2 \le x \le 3 \\ 0, & else\ where \end{cases}$

Find the

(i) Value of β (*ii*) Mean, μ
(*iiii*) Median m (*iv*) Standard deviation, σ
(*v*) $P(X < \mu - \sigma)$

An. $(i)\ \beta = \frac{2}{5}$ $(ii) = \frac{19}{15}$ $(iii) = \frac{5}{4}$
$(iv) = 0.75$ $(v) = 0.207$

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le k \\ 0, & else\ where \end{cases}$

Find the

(i) Sketch f(x) (*ii*) Value of *k*
(*iii*) Mean, μ (*iv*) Median of X

An. $(ii)\ k = \frac{7}{3}$ $(iii) = \frac{49}{36}$ $(iv) = \frac{4}{3}$

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k, & 1 \le x \le 2 \\ 0, & else\ where \end{cases}$

Find the

(i) Sketch f(x) (*ii*) Value of *k*

(iii) Mean, μ (iv) Median of X
(v) $P(|X - m| > 0.5)$

An. (ii) $k = \frac{2}{3}$ (iii) $= \frac{49}{36}$

(iv) $= 1.25$ (v) $= \frac{17}{48}$

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 2k(x+1), & -1 \le x \le 0 \\ k(2-x), & 0 \le x \le 2 \\ 0, & \text{else where} \end{cases}$

(i) Find Sketch f(x) and find the value of k
(ii) E(X) and $Var(X)$
(iii) mode

An. (i) $k = \frac{1}{3}$ (ii) $= \frac{1}{3}$, $= \frac{5}{18}$ (iii) $= 0$

CUMULATIVE DISTRIBUTION FUNCTION, F(x)

The cumulative distribution function F(x) is defined by $F(x) = \int_a^x f(x)dx$

Steps in finding F(X)

- For each interval, integrate its function from a lower to x with respect to x
- Substitute the upper limit in the integral and carry it forward to the next interval
- Continue the process until when the last upper limit has been substituted to a 1

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{6}(x+1), & 1 \le x \le 3 \\ 0, & \text{else where} \end{cases}$

Find F(x)

Solution

$$F(x) = \int_1^x \frac{2}{15}(x+1)dx$$

$$= \frac{1}{6}\left[\frac{x^2}{2} + x\right]_1^x$$

$$= \frac{1}{6}\left\{\left(\frac{x^2}{2} + x\right) - \left(\frac{1^2}{2} + 1\right)\right\}$$

$$F(x) = \frac{1}{6}\left(\frac{x^2}{2} + x - \frac{3}{2}\right)$$

$$F(3) = \frac{1}{6}\left(\frac{3^2}{2} + 3 - \frac{3}{2}\right) = 1$$

$$F(x) = \begin{cases} 0, & x \le 1 \\ \frac{1}{6}\left(\frac{x^2}{2} + x - \frac{3}{2}\right), & 1 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{26}(1-x)^2, & 2 \le x \le 4 \\ 0, & \text{else where} \end{cases}$

Find F(x)

Solution

$$F(x) = \int_2^x \frac{3}{26}(1-x)^2 dx$$

$$= \int_2^x \frac{3}{26}(1 - 2x + x^2)dx$$

$$= \frac{3}{26}\left[x - x^2 + \frac{x^3}{3}\right]_2^x$$

$$= \frac{3}{26}\left\{\left(x - x^2 + \frac{x^3}{3}\right) - \left(2 - 2^2 + \frac{2^3}{3}\right)\right\}$$

$$F(x) = \frac{3}{26}\left(x - x^2 + \frac{x^3}{3} - \frac{2}{3}\right)$$

$$F(4) = \frac{3}{26}\left(4 - 4^2 + \frac{4^3}{3} - \frac{2}{3}\right) = 1$$

$$F(x) = \begin{cases} 0, & x \le 2 \\ \frac{3}{26}\left(x - x^2 + \frac{x^3}{3} - \frac{2}{3}\right), & 2 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$$

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ (2-x), & 1 \le x \le 2 \\ 0, & \text{else where} \end{cases}$

Find F(x)

Solution

For $0 \le x \le 1$; $F(x) = \int_0^x x\,dx$

$$= \left[\frac{x^2}{2}\right]_0^x$$

$$= \left\{\frac{x^2}{2} - \frac{0^2}{2}\right\}$$

$$F(x) = \frac{x^2}{2}$$

$$F(1) = \frac{1^2}{2} = \frac{1}{2}$$

For $1 \le x \le 2$; $F(x) = \frac{1}{2} + \int_1^x (2-x)dx$

$$= \frac{1}{2} + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left\{\left(2x - \frac{x^2}{2}\right) - \left(2x1 - \frac{1^2}{2}\right)\right\}$$

$$F(x) = \left(2x - \frac{x^2}{2}\right) - 1$$

$$F(2) = \left(2x2 - \frac{2^2}{2}\right) - 1 = 1$$

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{2}, & 0 \le x \le 1 \\ \left(2x - \frac{x^2}{2}\right) - 1, & 1 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{5}, & 0 \le x \le 2 \\ \frac{2}{5}(3-x), & 2 \le x \le 3 \\ 0, & \text{else where} \end{cases}$

Find F(x)

Solution

For $0 \le x \le 2$; $F(x) = \int_0^x \frac{2}{5}\,dx$

$$= \frac{2}{5}\,[x]_0^x$$

$$= \frac{2}{5}\,\{x - 0\}$$

$$F(x) = \frac{2}{5}x$$

$$F(2) = \frac{2}{5}(2) = \frac{4}{5}$$

For $2 \le x \le 3$; $F(x) = \frac{4}{5} + \int_2^x \frac{2}{5}(3-x)dx$

$$= \frac{4}{5} + \frac{2}{5}\left[3x - \frac{x^2}{2}\right]_2^x$$

$$= \frac{4}{5} + \frac{2}{5}\left\{\left(3x - \frac{x^2}{2}\right) - \left(3x2 - \frac{2^2}{2}\right)\right\}$$

$$F(x) = \frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \frac{4}{5}$$

$$F(3) = \frac{2}{5}\left(3x3 - \frac{3^2}{2}\right) - \frac{4}{5} = 1$$

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{2}{5}x, & 0 \le x \le 2 \\ \frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \frac{4}{5}, & 2 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

Finding the median, quartiles and probabilities from F(x)

The median is the value of m for which $F(m) - F(a) = 0.5$

The lower quartile is the value of q_1 for which $F(q_1) - F(a) = 0.25$

The upper quartile is the value of q_3 for which $F(q_3) - F(a) = 0.75$

Where a is lower limit

Example

1. The continuous random variable X has a cumulative distribution function given below

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{16} & 0 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$$

Find;

(i) $P(0.3 \le x \le 1.8)$

(ii) Median m

(iii) Interquartile range

(iv) Sketch F(x)

Solution

(i) $P(0.3 \le x \le 1.8) = F(1.8) - F(0.3)$

$$= \frac{1.8^2}{16} - \frac{0.3^2}{16} = 0.2025 - 0.0056 = 0.197$$

(ii) Median m, $F(m) - F(0) = 0.5$

$$\frac{m^2}{16} - \frac{0^2}{16} = 0.5$$

$$m^2 = 8$$

$$m = \pm 2.828$$

$$median = 2.828$$

(iii) $F(q_1) - F(0) = 0.25$

$$\frac{{q_1}^2}{16} - \frac{0^2}{16} = 0.25$$

$${q_1}^2 = 4$$

$$q_1 = \pm 2$$

$$q_1 = 2$$

$$F(q_3) - F(0) = 0.75$$

$$\frac{{q_3}^2}{16} - \frac{0^2}{16} = 0.75$$

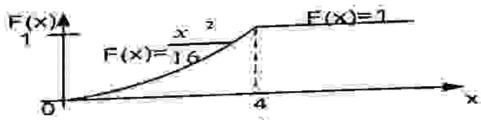
$${q_3}^2 = 12$$

$$q_3 = \pm 3.464$$

$$q_3 = 3.464$$

Interquartile range= $3.464 - 2 = 1.5$

(iv)



2. The continuous random variable X has a c.d.f $F(x) = \begin{cases} 0, & x \le 0 \\ 2x - x^2 & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$

Find;

(i) $P(x \le 0.5)$ | (ii) Median m | (iii) Interquartile range

Solution

(i) $P(x \le 0.5) = F(0.5) - F(0)$

$$= (2x0.5 - 0.5^2) - (2x0 - 0^2) = 0.75$$

(ii) Median m, $F(m) - F(0) = 0.5$

$$(2xm - m^2) = 0.5$$

$$m^2 - 2m + 0.5 = 0$$

$$m = 1.71$$

or $m = 0.293$

$$median = 0.293$$

(iii) $F(q_1) - F(0) = 0.25$

$$(2xq_1 - {q_1}^2) = 0.25$$

$${q_1}^2 - 2q_1 + 0.25 = 0$$

$$q_1 = 0.134$$

Or $q_1 = 1.866$

Lower quartile is 0.134

$$F(q_3) - F(0) = 0.75$$

$$(2xq_3 - {q_3}^2) = 0.75$$

$${q_3}^2 - 2q_3 + 0.75 = 0$$

$$q_3 = 0.5$$

Or $q_3 = 1.5$

Upper quartile is 0.5

Interquartile range= $0.5 - 0.134 = 0.366$

3. The cumulative distribution function is given below $F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{6} & 0 \le x \le 2 \\ -\frac{x^2}{3} + 2x - 2 & 2 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$

Find;

(i) $P(1 \le x \le 2.5)$ | (ii) Median m

Solution

(i) $P(1 \le x \le 2.5) = F(2.5) - F(1)$

$$= -\frac{(2.5)^2}{3} + 2x2.5 - 2 \quad - \frac{1^2}{6}$$

$$= \frac{11}{12} - \frac{1}{6} = 0.75$$

(ii) Median m,

$$P(0 \le x \le 2) = F(2) - F(0)$$

$$= \frac{2^2}{6} - \frac{0^2}{6} = \frac{2}{3}$$

Since $\frac{2}{3} > 0.5$ then median lies between $0 \le x \le 2$

$F(m) - F(0) = 0.5$

$$\frac{m^2}{6} = 0.5$$

$$m^2 = 3$$

$$m = \pm 1.73$$

$$mediam = 1.73$$

Exercise 7e

1. The random variable X has a probability density function $f(x) = \begin{cases} \frac{3}{8}x^2 & x \leq 2 \\ 0 & else\ where \end{cases}$

Find
(i) Cumulative mass function, F(x) and sketch F(x)
(ii) The median, m

An (i) $F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{8}x^3 & 0 \leq x \leq 2 \\ 1 & x \geq 0 \end{cases}$

$(ii)\ m = 1.59$

3. The random variable X has a probability density function $f(x) = \begin{cases} \frac{1}{4}(4-x) & 1 \leq x \leq 3 \\ 0 & else\ where \end{cases}$

Find
(i) Cumulative mass function, F(x) and sketch F(x)
(ii) $P(1.5 \leq x \leq 2)$ $(iiii)$ The median, m

An (i) $F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{8}(8x - x^2 - 7) & 1 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$

$(ii)\ P(1.5 \leq x \leq 2) = \frac{9}{32}$

4. The random variable X has a p.d.f
$f(x) = \begin{cases} k & 1 \leq x \leq 6 \\ 0 & else\ where \end{cases}$

Find
(i) Value of the constant k
(ii) Cumulative mass function, F(x) and sketch F(x)
(iii) Interquartile range

An (I) $k = \frac{1}{5}$

$(ii)\ F(x) = \begin{cases} 0 & x \leq 1 \\ \frac{1}{5}(x-1) & 1 \leq x \leq 6 \\ 1 & x \geq 6 \end{cases}$

(iii) 2.5

5. The random variable X has a probability density function $f(x) = \begin{cases} \frac{1}{4} & 0 \leq x \leq 2 \\ \frac{1}{4}(2x-3) & 2 \leq x \leq 3 \\ 0 & else\ where \end{cases}$

Find
(i) Cumulative mass function, F(x) and sketch F(x)
(ii) Median, m

An $F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{4} & 0 \leq x \leq 2 \\ \frac{1}{4}(x^2 - 3x + 4) & 2 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$

$m = 2$

6. The random variable X has a cumulative distribution function $F(x) = \begin{cases} 0 & x < 0 \\ x^4 & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$

Find
(i) $P(0.3 \leq x \leq 0.6)$ (ii) The median, m
(iii) The value of a such that $P(X > a) = 0.4$

An (0.1215, 0.841, 0.880)

7. The continuous random variable X has a p.d.f
$f(x) = \begin{cases} \frac{1}{3} & 0 \leq x \leq 3 \\ 0 & else\ where \end{cases}$

Find
(i) E(X) (ii) Var(X)
(iii) Cumulative mass function, F(x) and sketch F(x)
(iv) $P(x \geq 1.8)$ (v) $P(1.1 \leq x \leq 1.7)$

An $(i) = 1.5, (ii) = 0.75\ (iv) = 0.4, (v) = 0.2$

$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{3}x & 0 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$

8. The continuous random variable X has a probability density function
$f(x) = \begin{cases} kx^2 & 1 \leq x \leq 2 \\ 0 & else\ where \end{cases}$

Find
(i) Constant k and Standard deviation, σ
(ii) Cumulative mass function, F(x)
(iii) Median, m

An $(i)k = \frac{3}{7}, \delta = 0.272, (iii) = 1.65$

$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{7}(x^3 - 1) & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$

9. The continuous random variable X has a probability density function
$f(x) = \begin{cases} k(4 - x^2) & 0 \leq x \leq 2 \\ 0 & else\ where \end{cases}$

Find
(ii) Constant k
(iii) E(x) and Var(X)
(iv) Cumulative mass function, F(x)
(v) Median, m $(v)\ P(0.69 \leq x \leq 0.7)$

An $(i)k = \frac{3}{16}$ $(ii)\ E(x) = \frac{3}{4},\ var(x) = \frac{19}{80}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{4}x - \frac{1}{16}x^3 & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

$(iv) m = 0.695$ $(v) = 0.007$

10. The continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{x}{3} - \frac{2}{3} & 2 \le x \le 3 \\ \alpha & 3 \le x \le 5 \\ 2 - \beta x & 5 \le x \le 6 \\ 0 & otherwise \end{cases}$$

Find
- (i) Constant α and β
- (ii) Cumulative mass function, F(x)
- (iii) $P(2 \le x \le 3.5)$ (iv) $P(x \ge 5.5)$

An $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{3}$

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{x^2}{6} - \frac{2x}{3} + \frac{2}{3} & 2 \le x \le 3 \\ \frac{x}{3} - \frac{5}{6} & 3 \le x \le 5 \\ 2x - \frac{x^2}{6} - 5 & 5 \le x \le 6 \\ 1 & x \ge 6 \end{cases}$$

$(iii) = \frac{1}{3}$ $(iv) = \frac{1}{24}$

11. The continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{1+x}{6} & 1 \le x \le 3 \\ 0 & otherwise \end{cases}$$

- (i) Sketch f(x)
- (ii) Find the mean of X
- (iii) Determine Cumulative mass function, F(x) and sketch F(x)
- (iv) Find m such that $P(X \le m) = 0.5$

An $(ii)\ mean = \frac{19}{9}$ $(iv) m = 2.16$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{5}x + \frac{1}{12}x^2 - \frac{1}{4} & 1 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

12. A factory is supplied with flour at the beginning of each week. The weekly demand, X thousand tones, for flour from this factory is a continuous random variable having a probability density function $f(x) = \begin{cases} k(1-x)^4 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$

Find
- (i) Constant K
- (ii) the mean of X (iii) Variance of x

An $(i)\ k = 5,$ $(ii) = \frac{1}{6}$ $(iii) = \frac{5}{252}$

13. A continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{1}{4} & 0 \le x \le 1 \\ \frac{x^3}{5} & 1 \le x \le 2 \\ 0 & else\ where \end{cases}$$

Find
- (i) Cumulative mass function, F(x) and sketch F(x)
- (ii) Median, m (iii) Interquartile range

An $$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4}x & 0 \le x \le 1 \\ \frac{1}{5} + \frac{x^4}{20} & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

$(ii) = 1.565$ $(iiii) = 0.821$

14. The continuous random variable X has a probability density function

$$f(x) = \begin{cases} k(x+3) & -3 \le x \le 3 \\ 0 & else\ where \end{cases}$$

- (a) Show that $k = \frac{1}{18}$
- (b) Find
  - (i) E(x) and Var(X)
  - (ii) The lower quartile
- (c) Given that $E(ax+b) = 0$ and $var\ (ax+b) = 1$, find the value of the constants a and b where $a > 0$

An $E(x) = 1, Var(x) = 2,$ $q_1 = 0,$ $a = b = \frac{1}{\sqrt{2}}$

15. A continuous random variable X has a probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 8 \\ 8k & 8 \le x \le 9 \\ 0 & else\ where \end{cases}$$

- (i) Sketch f(x)
- (ii) Show that $k = 0.025$
- (iii) Find F(x) (iv) Calculate $P(X > 6)$

An $$F(x) = \begin{cases} 0 & x < 0 \\ 0.0125x^2 & 0 \le x \le 8 \\ 0.2x - 0.8 & 8 \le x \le 9 \\ 1 & x \ge 9 \end{cases}$$

$(iv) = 0.55$

16. The continuous random variable X has a probability density function

$$f(x) = \begin{cases} ax - bx^2 & 0 \le x \le 2 \\ 0 & else\ where \end{cases}$$

If $E(x) = 1$, find
- (i) The value of the constants a and b
- (ii) Var(X) (iii) F(x)

An $a = 1.5, b = 0.75,\quad Varx = 0.2$

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.75x^2 - 0.25x^3 & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

17. The continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{k}{x} & 1 \le x \le 9 \\ 0 & else\ where \end{cases}$$

Find

(i) The value of the constant k and median value

(ii) Mean and Variance of X

(iii) F(x) and sketch F(x)

(iv) The median, m

An $k = 0.455, m = 3$

$\mu = 3.64,\quad Varx = 4.95$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{In9}\, Inx & 1 \le x \le 9 \\ 1 & x \ge 9 \end{cases}$$

18. The continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{20}{5^5}\, w^3\,(5 - w) & 0 \le x \le 5 \\ 0 & else\ where \end{cases}$$

Find

(i) F(x) and sketch F(x)

(ii) $P(2 < W < 4)$

(iii) Mean and Variance of X

(iv) The mode

An $(ii) = 0.650, (iii)\mu = 3.33,$

$Varx = 0.794\ (iv) = 3.5$

$$F(x) = \begin{cases} 0 & w < 0 \\ \frac{w^4}{5^5}\,(25 - w) & 0 \le w \le 5 \\ 1 & w \ge 5 \end{cases}$$

19. A continuous random variable X has a probability density function

$$f(x) = \begin{cases} kx\,(3 - x) & 0 \le x \le 2 \\ k(4 - x) & 2 \le x \le 4 \\ 0 & else\ where \end{cases}$$

Find **Uneb 1994 No.11**

(i) Value of k and Mean

(ii) F(x), cumulative distribution function

(iii) $P(1.5 \le X \le 3)$

$$\textbf{An}\ F(x) = \begin{cases} 0 & x \le 0 \\ \frac{3}{16}\left(\frac{3}{2}x^2 - \frac{x^3}{3}\right) & 0 \le x \le 2 \\ \frac{3}{4}x - \frac{3}{32}x^2 - \frac{1}{2} & 2 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$$

$$(i)k = \frac{3}{16}, \mu = 1.75, (iii) = \frac{11}{16}$$

20. A continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{2}{3a}\,(x + a) & -a \le x \le 0 \\ \frac{1}{3}(2a - x) & 0 \le x \le 2a \\ 0 & else\ where \end{cases}$$

Find **Uneb 1995 No.12**

(i) Value of a $\quad (ii)$ Median of X

$(iii)\ P(X \le 1.5 / X > 0)$

(iv) F(x), cumulative distribution function

$$\textbf{An}\ F(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{3}(x^2 + 2x + 1) & -1 \le x \le 0 \\ \frac{1}{6}(2 + 4x - x^2) & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

$(i) = 1,\quad (ii) = 0.2679,\quad (iii) = 0.9375$

21. A continuous random variable X has a probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k(4 - x^2) & 1 \le x \le 2 \\ 0 & else\ where \end{cases}$$

Find **Uneb 1998 No.13**

(i) Value of k

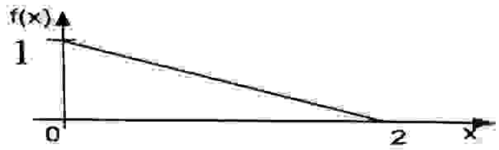
(ii) E(X) and var (X)

(iii) F(x), cumulative distribution function

$$\textbf{An}\ F(x) = \begin{cases} 0 & x \le 0 \\ \frac{3}{13}x^2 & 0 \le x \le 1 \\ \frac{1}{13}(24x - 2x^3 - 19) & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

$(i) = \frac{6}{13}, (ii) = 1.1923,\quad (iii) = 0.1399$

22. The probability density function f(x) of a random variable X takes on the form shown in the diagram below



Find **Uneb 2000 No.14b**

(i) Expression for f(x)

(ii) F(x), cumulative distribution function

(iii) Mean and variance of X

An $(iii)\mu = \frac{2}{3}, var(X) = \frac{2}{9}$

Finding f(x) from F(x)

f(x) can be obtained from ; $f(x) = \frac{d}{dx}F(x)$

Example

1. The continuous random variable X has a c.d.f $F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^3}{27} & 0 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$

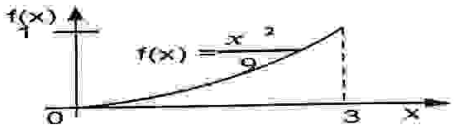
Find the probability density function, f(x) and sketch f(x)

Solution

$$f(x) = \frac{d}{dx}F(x)$$

$$f(x) = \frac{d}{dx}\left(\frac{x^3}{27}\right) = \frac{3x^2}{27} = \frac{x^2}{9}$$

$$f(x) = \begin{cases} \frac{x^2}{9}, & 0 \le x \le 3 \\ 0 & else\ where \end{cases}$$



2. The continuous random variable X has a c.d.f $F(x) = \begin{cases} 0, & x \le 0 \\ kx^3 & 0 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$

Find the
(i) Value of the constant k
(ii) probability density function, f(x)

Solution

(i) $F(4) - F(0) = 1$

$$k(4^3 - 0^3) = 1$$

$$k = \frac{1}{64}$$

(ii) $f(x) = \frac{d}{dx}F(x)$

$$f(x) = \frac{d}{dx}\left(\frac{x^3}{64}\right) = \frac{3x^2}{64}$$

$$f(x) = \begin{cases} \frac{3x^2}{64}, & 0 \le x \le 4 \\ 0 & else\ where \end{cases}$$

3. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le -2 \\ k\left(4x - \frac{1}{3}x^3 + \frac{16}{3}\right) & -2 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

Find the
(i) Value of the constant k
(ii) $P(-1 \le x \le 1.3)$
(iii) probability density function, f(x)
(iv) Variance of x

Solution

(i) $F(2) - F(-2) = 1$

$$k\left\{\left(4x2 - \frac{1}{3}(2)^3 + \frac{16}{3}\right) - \left(4x - 2 - \frac{1}{3}(-2)^3 + \frac{16}{3}\right)\right\} = 1$$

$$\left(\frac{32}{3} - 0\right) = 1$$

$$k = \frac{3}{32}$$

(ii) $P(-1 \le x \le 1.3) = F(1.3) - F(-1)$

$$= \frac{3}{32}\left\{\left(4x1.3 - \frac{1}{3}(1.3)^3 + \frac{16}{3}\right) - \left(4x - 1 - \frac{1}{3}(-1)^3 + \frac{16}{3}\right)\right\} = \frac{3}{32}\left(9.801 - \frac{5}{3}\right) \doteq 0.7626$$

(iii) $f(x) = \frac{d}{dx}F(x)$

$$f(x) = \frac{3}{32}\frac{d}{dx}\left(4x - \frac{1}{3}x^3 + \frac{16}{3}\right) = \frac{3}{32}(4 - x^2)$$

$$f(x) = \begin{cases} \frac{3}{32}(4-x^2), & -2 \le x \le 2 \\ 0 & else\ where \end{cases}$$

(iv) $E(x) = \int_{-2}^{2} \frac{3}{32} x(4-x^2)\,dx$

$$E(x) = \frac{3}{32}\left[2x^2 - \frac{x^4}{4}\right]_{-2}^{2} = \frac{3}{32}\left\{\left(2(2)^2 - \frac{2^4}{4}\right) - \left(2(-2)^2 - \frac{(-2)^4}{4}\right)\right\} = 0$$

$$Var\ x = E(x^2) - [E(x)]^2$$

$$= \int_{-2}^{2} \frac{3}{32}x^2(4-x^2)\,dx - 0^2 = \frac{3}{32}\left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_{-2}^{2} = \frac{3}{32}\left\{\left(\frac{4}{3}(2)^3 - \frac{2^5}{5}\right) - \left(\frac{4}{3}(-2)^3 - \frac{(-2)^5}{5}\right)\right\} = \frac{4}{5}$$

4. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le 0 \\ 2x - 2x^2 & 0 \le x \le 0.25 \\ a + x & 0.25 \le x \le 0.5 \\ b + 2x^2 - x & 0.5 \le x \le 0.75 \\ 1, & x \ge 0.75 \end{cases}$$

Find the
- (i) Value of the constant a and b
- (ii) probability density function, f(x)
- (iii) Mean of x

Solution

(i) For $0 \le x \le 0.25$, $F(x) = 2x - 2x^2$

$F(0.25) = 2x0.25 - 2x(0.25)^2 = 0.375$

For $0.25 \le x \le 0.5$, $F(x) = a + x$

$F(0.25) = a + 0.25 = 0.375$

$\mathbf{a = 0.125}$

For $0.5 \le x \le 0.75$, $F(x) = b + 2x^2 - x$

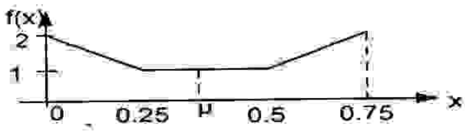
$F(0.75) = b + 2(0.75)^2 - 0.75 = 1$

$\mathbf{b = 0.625}$

$b = 0.625.$

(ii)

$$f(x) = \begin{cases} 2 - 4x & 0 \le x \le 0.25 \\ 1 & 0.25 \le x \le 0.5 \\ 4x - 1 & 0.5 \le x \le 0.75 \\ 0 & else\ where \end{cases}$$



$$\mu = \frac{0.5 + 0.25}{2} = 0.375$$

Exercise 7f

1. The continuous random variable X has a cumulative distribution function F(x)

$$\text{where } F(x) = \begin{cases} 0, & x \le 2 \\ 0.25x - 0.5 & 2 \le x \le 6 \\ 1, & x \ge 6 \end{cases}$$

Find the
- (i) Probability density function, f(x)
- (ii) Sketch f(x)
- (iii) E(x) (iv) Interquartile range

An (iii) $\mu = 4$ (iv) $= 2$

(i) $f(x) = \begin{cases} \frac{1}{4} & 2 \le x \le 6 \\ 0 & else\ where \end{cases}$

2. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le 0 \\ x^3 & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$$

Find the
(i) The median (ii) Mean

An (i) $m = 0.794$ (ii) $\mu = 0.75$

3. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le 0 \\ x - kx^2 & 0 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

Find the
- (i) Value of k
- (ii) Probability density function, f(x)
- (iii) Median of x (ii) Variance of X

An (i) $k = 0.25$ (iii) $m = 0.586$, (iv) $= \frac{2}{9}$

(ii) $f(x) = \begin{cases} 1 - 0.5x & 0 \le x \le 2 \\ 0 & else\ where \end{cases}$

4. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{2x}{3} & 0 \le x \le 1 \\ \frac{x}{3} + k & 1 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

Find the

(i) Value of k

(ii) Probability density function, f(x) and sketch it

(iii) Mean μ (iv) Standard deviation σ

(v) $P(|X-\mu| < \sigma)$

An (i) $k = \frac{1}{3}$ (iii) $\mu = \frac{5}{6}$, $(iv) = 0.5528$ $(v) =$ 0.608

$$f(x) = \begin{cases} \frac{2}{3} & 0 \le x \le 1 \\ \frac{1}{3} & 1 \le x \le 2 \\ 0 & else\ where \end{cases}$$

5. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le 1 \\ \frac{(x-1)^2}{12} & 1 \le x \le 3 \\ \frac{(14x - x^2 - 25)}{24} & 3 \le x \le 7 \\ 1, & x \ge 7 \end{cases}$$

Find the

(i) Probability density function, f(x) and sketch it

(ii) Mean of x (iii) Variance of X

(iv) Median of X (v) $P(2.8 \le x \le 5.2)$

An (ii) $\mu = \frac{11}{3}$ (iii) $= \frac{14}{9}$, (iv) $m = 3.45$, $(v) = 0.595$

$$f(x) = \begin{cases} \frac{1}{6}(x-1) & 1 \le x \le 3 \\ \frac{1}{12}(7-x) & 3 \le x \le 7 \\ 0 & else\ where \end{cases}$$

6. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le 1 \\ \frac{x^2-1}{2} - x & 1 \le x \le 2 \\ 3x - \frac{x^2}{2} & 2 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

Find the **Uneb 2003 NO.10**

(i) Probability density function, f(x) and sketch it

(ii) $P(1.2 < X < 2.4)$

(iii) Mean of x

An (ii) $= 0.8$ (iii) $\mu = 2$,

7. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \le 0 \\ \alpha x & 0 \le x \le 1 \\ \frac{x}{3} + \beta & 1 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

Find the

(i) Value of α and β

(ii) Probability density function, f(x) and sketch it

(iii) Mean of X and variance of X

(iv) $P(X < 1.5 / X > 1)$

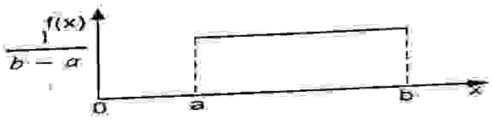
An (i) $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$ (iii) $\mu = \frac{5}{6}$, $Var(X) = \frac{19}{36}$ $(iv) = 0.4998$

UNIFORM OR RECTANGULAR DISTRIBUTION

A continuous random variable X is said to be uniformly distributed over the interval a and b, if the p.d.f is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & else\ where \end{cases}$$

Graph of f(x)



Examples

1. X is uniformly distributed between 6 and 9.
   (i) Write the probability density function
   (ii) Find $P(7.2 \le X \le 8.4)$

Solution

(i) $f(x) = \begin{cases} \frac{1}{9-6} & 6 \le x \le 9 \\ 0 & else\ where \end{cases}$

(ii) $P(7.2 \le X \le 8.4) = \int_{7.2}^{8.4} \frac{1}{3} dx$

$= \frac{1}{3}[x]_{7.2}^{8.4} = \frac{1}{3}(8.4 - 7.2)$

$P(7.2 \le X \le 8.4) = 0.4$

2. X is uniformly distributed between 0 and $\frac{\pi}{2}$.
   (i) Write the probability density function
   (ii) Find $P\left(\frac{\pi}{3} \le X \le \frac{\pi}{2}\right)$

Solution

(i) $f(x) = \begin{cases} \frac{1}{\frac{\pi}{2}-0} & 0 \le x \le \frac{\pi}{2} \\ 0 & else\ where \end{cases}$

(ii) $P\left(\frac{\pi}{3} \le X \le \frac{\pi}{2}\right) = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\pi} dx = \frac{2}{\pi}[x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{2}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{3}$

Expectation of X, E(x)

$$E(x) = \int_a^b x f(x)\, dx$$

$$\int_a^b x\left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{2(b-a)}[x^2]_a^b$$

$$= \frac{1}{2(b-a)}(b^2 - a^2) = \frac{(b-a)(b+a)}{2(b-a)}$$

$$E(x) = \frac{(b+a)}{2}$$

Variance of X, var(x)

$$var(x) = \int_a^b x^2 f(x)\, dx - [E(x)]^2$$

$$\int_a^b x^2\left(\frac{1}{b-a}\right) dx - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{1}{3(b-a)}[x^3]_a^b - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{1}{3(b-a)}(b^3 - a^3) - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{(b^2+ab+a^2)}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12}$$

$$Var(x) = \frac{(b-a)^2}{12}$$

Examples

1. X is a rectangular distribution between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
   (i) Write the probability density function
   (ii) Sketch f(x)
   (iii) Find the mean and variance

Solution

(i) $f(x) = \begin{cases} \frac{1}{\frac{\pi}{2}-(-\frac{\pi}{2})} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & else\ where \end{cases}$

(ii) $E(x) = \frac{(b+a)}{2} = \frac{(\frac{\pi}{2}+-\frac{\pi}{2})}{2} = 0 Var(x) = \frac{(b-a)^2}{12} = \frac{(\frac{\pi}{2}--\frac{\pi}{2})^2}{12} = \frac{\pi^2}{12}$

2. A random variable X is uniformly distributed over the interval $-3 \le x \le -1$.
   (i) Find $P(-2 \le X \le -1.5)$
   (i) Find the mean and standard deviation

Solution

(i) $P(-2 \le X \le -1.5) = \int_{-2}^{-1.5} \frac{1}{2} dx$
$= \frac{1}{2}[x]_{-2}^{-1.5} = \frac{1}{2}(-1.5 - -2) = \frac{1}{4}$

(i) $E(x) = \frac{(b+a)}{2} = \frac{(-1+-3)}{2} = -2$

(ii) $Var(x) = \frac{(b-a)^2}{12} = \frac{(-1--3)^2}{12} = \frac{1}{3}$

Exercise 7g

1. X follows a uniform distribution with probability density function
$f(x) = \begin{cases} k & 3 \le x \le 6 \\ 0 & else\ where \end{cases}$
Find
(i) K (ii) E(x) (iii) Var(x)
(iv) $P(X > 5)$ **An (i)** $\frac{1}{3}$ **(ii) 4.5**
(iii) 0.75 **(iv)** $\frac{1}{3}$

2. X is distributed uniformly over $-5 \le x \le -2$.
Find
(a) $P(-4.3 \le X \le -2.8)$ (b) $E(x)$
(c) Standard deviation of X **An (i)** 0.5
(ii) -3.5 **(iii) 0.866**

3. The continuous random variable has a probability density function
$f(x) = \begin{cases} \frac{1}{4} & 1 \le x \le k \\ 0 & else\ where \end{cases}$
Find
(i) K (ii) $P(2.1 \le X \le 3.4)$
(iii) E(x) (iv) Var(x)
An (I) 5 **(II) 0.325** **(iii) 3** **(iv)** $1\frac{1}{3}$

4. The continuous random variable has a probability density function
$f(y) = \begin{cases} \frac{1}{5} & 32 \le y \le 37 \\ 0 & else\ where \end{cases}$
Find the probability that y lies within one standard deviation of the mean **An 0.577**

5. The continuous random variable X has a cumulative distribution function
$F(x) = \begin{cases} 0 & x < 2 \\ \frac{x-2}{5} & 2 \le x \le 7 \\ 1 & x > 7 \end{cases}$
Find E(x) and Var(x) **An 4.5** $2\frac{1}{12}$

6. The continuous random variable X is uniformly distributed in the interval $a \le x \le b$. The lower quartile is 5 and the upper quartile is 9. Find;
   (i) Value of a and b
   (ii) $P(6 \le X \le 7)$
   (iii) Cumulative distribution function F(x)
An (i) $a = 3,\ b = 11$ **(ii) 0.125**
$F(x) = \begin{cases} 0 & x < 3 \\ \frac{x-3}{8} & 3 \le x \le 11 \\ 1 & x > 11 \end{cases}$

7. During rush hours, it was observed that the number of vehicles departing for Entebbe from Kampala old taxi park take a random variable X with a rectangular distribution over the interval (X_1, X_2). If in one hour, the expected number of vehicles leaving the stage is 12 and variance of 3, find the;
   (i) Value of X_1 and X_2
   (ii) Probability that at least 11 vehicles leave the stage
Uneb 2002 NO.11 An (i) X_1 =, X_2 = **(ii)**

8. The number of patients visiting a certain hospital is uniformly distributed between 150 and 210 ; **Uneb 2014 No.1**
   (i) Write down the probability density function (p.d.f) of the number of patients
   (ii) $P(170 < X < 194)$ **An (II) 0.4**

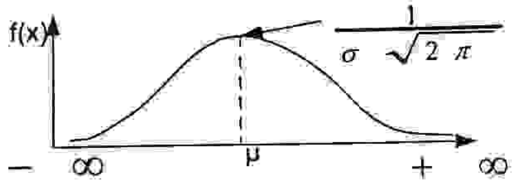
NORMAL DISTRIBUTION

A normal variable is a continuous variable which follows a normal distribution with mean μ and variance, $\sigma^2 a1$

$$\boxed{X \sim N(\mu, \sigma^2)}$$

Its p.d.f is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$

Sketch of $f(x)$ gives a normal curve



Properties of the curve

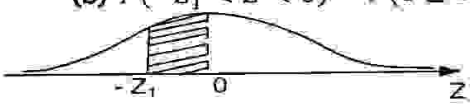
- It is bell shaped
- It is symmetrical about μ
- It extends from $-\infty$ to ∞
- The maximum value of f(x) is $\frac{1}{\sigma\sqrt{2\pi}}$
- The total are under the curve is 1

How to read the cumulative normal distribution table

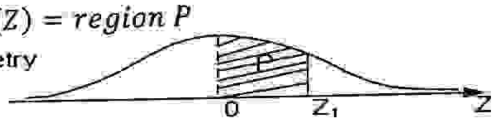
(i) Between 0 and any Z value

(a) $P(0 \le Z \le Z_1) = \Phi(Z) = region\ P$

(b) $P(-Z_1 < Z < 0) = P(0 \le Z \le Z_1) = \Phi(Z) = region\ P$



by symmetry

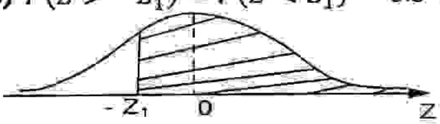


The cumulative normal table reads the region P which is given as $P(0 \le Z \le Z_1)$

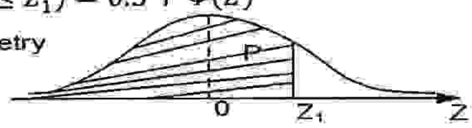
(ii) Less than any positive Z value

(a) $P(Z < Z_1) = 0.5 + P(0 \le Z \le Z_1) = 0.5 + \Phi(Z)$

(b) $P(Z > -Z_1) = P(Z < Z_1) = 0.5 + P(0 \le Z \le Z_1) = 0.5 + \Phi(Z)$



by symmetry

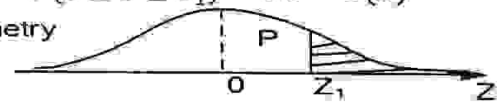


(iii) Greater than any positive Z value

(a) $P(Z > Z_1) = 0.5 - P(0 \le Z \le Z_1) = 0.5 - \Phi(Z)$

(b) $P(Z < -Z_1) = P(Z > Z_1) = 0.5 - P(0 \le Z \le Z_1) = 0.5 - \Phi(Z)$

by symmetry



Examples

1. Find

(i) $P(Z < 0.16)$ (iii) $P(Z < 0.345)$ (v) $P(Z > 1.5)$

(ii) $P(Z < 2)$ (iv) $P(Z > 0.85)$ (vi) $P(Z > 1.36)$

Solution

(i) $P(Z < 0.16) = 0.5 + \Phi(0.16) = 0.5 + 0.0636 = 0.5636$

(ii) $P(Z < 2) = 0.5 + \Phi(2) = 0.5 + 0.4772 = 0.9772$

(iii) $P(Z < 0.345) = 0.5 + \Phi(0.345) = 0.5 + 0.1331 + 0.0019 = 0.6350$

(iv) $P(Z > 0.85) = 0.5 - \Phi(0.85) = 0.5 - 0.3023 = 0.1977$

(v) $P(Z > 1.5) = 0.5 - \Phi(1.5) = 0.5 - 0.4332 = 0.0668$

2. Find

(i) $P(Z < -0.25)$ **(iii)** $P(Z > -2)$ **(v)** $P(Z < -1.377)$

(ii) $P(Z < -2)$ **(iv)** $P(Z > -1.377)$ **(vi)** $P(Z > 1.377)$

Solution

(i) $P(Z < -0.25) = P(Z > 0.25) = 0.5 - \Phi(0.25) = 0.5 - 0.0987 = 0.4013$

(ii) $P(Z < -2) = P(Z > 2) = 0.5 - \Phi(2) = 0.5 - 0.4772 = 0.0228$

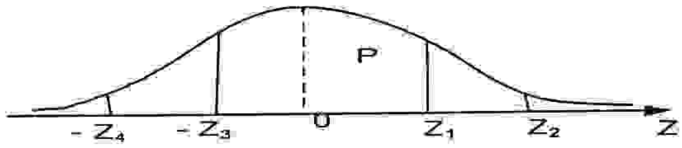
(iii) $P(Z > -2) = P(Z < 2) = 0.5 + \Phi(2) = 0.5 + 0.4772 = 0.9772$

(iv) $P(Z > -1.377) = P(Z < 1.377) = 0.5 + \Phi(1.377) = 0.5 + 0.4147 + 0.0011 = 0.9158$

(v) $P(Z < -1.377) = P(Z > 1.377) = 0.5 - \Phi(1.377) = 0.5 - (0.4147 + 0.0011) = 0.0842$

(vi) $P(Z > 1.377) = 0.5 - \Phi(1.377) = 0.5 - (0.4147 + 0.0011) = 0.0842$

Other important results



(i) Between two Z values on the same side of the mean

(a) $P(Z_1 < Z < Z_2) = P(0 < Z < Z_2) - P(0 < Z < Z_1)$

(b) $P(-Z_4 < Z < -Z_3) = P(0 < Z < Z_4) - P(0 < Z < Z_2)$

(ii) Between two Z values on the opposite side of the mean

(a) $P(-Z_3 < Z < Z_1) = P(0 < Z < Z_3) + P(0 < Z < Z_1)$

(b) $P(|Z| < Z_1) = P(-Z_1 < Z < Z_1) = 2xP(0 < Z < Z_1)$

(c) $P(|Z| > Z_1) = 1 - P(|Z| < Z_1) = 1 - 2xP(0 < Z < Z_1)$

Examples

Find

(i) $P(1.5 < Z < 1.88)$
(ii) $P(0.345 < Z < 1.751)$
(iii) $P(-2.5 < Z < 1)$
(iv) $P(-2.696 < Z < 1.865)$
(v) $P(-2 < Z < -1)$
(vi) $P(-1.4 < Z < -0.6)$
(vii) $P(|Z| < 1.75)$
(viii) $P(|Z| < 1.433)$
(ix) $P(|Z| > 1.433)$

Solution

(i) $P(1.5 < Z < 1.88) = \Phi(1.88) - \Phi(1.5) = 0.4699 - 0.4332 = 0.0367$

(ii) $P(0.345 < Z < 1.751) = \Phi(1.751) - \Phi(0.345) = 0.4600 - 0.1350 = 0.325$

(iii) $P(-2.5 < Z < 1) = \Phi(1) + \Phi(2.5) = 0.3413 + 0.4938 = 0.8351$

(iv) $P(-2.696 < Z < 1.865) = \Phi(1.865) + \Phi(2.696) = 0.469 + 0.4964 = 0.9654$

(v) $P(-2 < Z < -1) = \Phi(2) - \Phi(1) = 0.4772 - 0.3413 = 0.1359$

(vi) $P(-1.4 < Z < -0.6) = \Phi(1.4) - \Phi(0.6) = 0.4192 - 0.2257 = 0.1935$

(vii) $P(|Z| < 1.75) = P(-1.75 < Z < 1.75) = 2x\Phi(1.75) = 2x0.4625 = 0.925$

(viii) $P(|Z| < 1.433) = P(-1.433 < Z < 1.433) = 2x\Phi(1.433) = 2x0.424 = 0.848$

(ix) $P(|Z| > 1.433) = 1 - P(|Z| < 1.433) = 1 - 2x\Phi(1.433) = 1 - 2x0.424 = 0.152$

Standardizing a normal variable X

$X \sim N(\mu, \sigma^2)$ can be standardized using the equation below and read from a cumulative normal table

$$Z = \frac{X - \mu}{\sigma}$$

Examples

1. Given that the random variable X is $X \sim N(300, 25)$. Find;

(i) $P(X > 305)$
(ii) $P(X < 291)$
(iii) $P(X < 312)$
(iv) $P(X > 286)$

Solution

(i) $P(X > 305) = P\left(Z > \frac{305-300}{5}\right) = P(Z > 1)$
$= 0.5 - P(0 < Z < 1)$
$= 0.5 - 0.3413 = 0.1587$

(ii) $P(X < 291) = P\left(Z < \frac{291-300}{5}\right)$

$= P(Z < -1.8)$

$= P(Z > 1.8) = 0.5 - P(0 < Z < 1.8)$

$= 0.5 - 0.4641 = 0.0359$

(iii) $P(X < 312) = P\left(Z < \frac{312-300}{5}\right) = P(Z < 2.4)$

$= 0.5 + P(0 < Z < 2.4)$

$= 0.5 + 0.4918 = 0.9918$

(iv) $P(X > 286) = P\left(Z > \frac{286-300}{5}\right)$

$= P(Z > -2.8)$

$P(Z < 2.8) = 0.5 + P(0 < Z < 2.8)$

$= 0.5 + 0.4974 = 0.9974$

2. Given that the random variable X is $X \sim N(10,4)$. Find;
   (i) $P(X < 7)$
   (ii) $P(X > 12)$
   (iii) $P(7 < X < 12)$
   (iv) $P(9 < X < 11)$

Solution

(i) $P(X < 7) = P\left(Z < \frac{7-10}{2}\right)$

$= P(Z < -1.5) = P(Z > 1.5)$

$= 0.5 - P(0 < Z < 1.5)$

$= 0.5 - 0.4332 = 0.0668$

(ii) $P(X > 12) = P\left(Z > \frac{12-10}{2}\right) = P(Z > 1)$

$= 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413 = 0.1587$

(iii) $P(7 < X < 12) = P\left(\frac{7-10}{2} < Z < \frac{12-10}{2}\right)$

$= P(-1.5 < Z < 1)$

$= P(0 < Z < 1.5) + P(0 < Z < 1)$

$= 0.4332 + 0.3413 = 0.7745$

(iv) $P(9 < X < 11) = P\left(\frac{9-10}{2} < Z < \frac{11-10}{2}\right)$

$= P(-0.5 < Z < 0.5)$

$= 2xP(0 < Z < 0.5) = 2x\, 0.1915 = 0.3830$

3. Given that the random variable X is $X \sim N(50,8)$. Find;
   (i) $P(48 < X < 54)$
   (ii) $P(52 < X < 55)$
   (iii) $P(46 < X < 49)$
   (iv) $P(|X - 50| < \sqrt{8})$

Solution

(i) $P(48 < X < 54) = P\left(\frac{48-50}{\sqrt{8}} < Z < \frac{54-50}{\sqrt{8}}\right)$

$= P(-0.707 < Z < 1.414)$

$= P(0 < Z < 1.414) + P(0 < Z < 0.707)$

$= 0.4213 + 0.2601 = 0.6814$

(ii) $P(52 < X < 55) = P\left(\frac{52-50}{\sqrt{8}} < Z < \frac{52-50}{\sqrt{8}}\right)$

$= P(0.707 < Z < 1.768)$

$= P(0 < Z < 1.768) - P(0 < Z < 0.707)$

$= 0.4615 - 0.2601 = 0.2014$

(iii) $P(46 < X < 49) = P\left(\frac{46-50}{\sqrt{8}} < Z < \frac{49-54}{\sqrt{8}}\right)$

$= P(-1.414 < Z < -0.354)$

$= P(0 < Z < 1.414) - P(0 < Z < 0.354)$

$= 0.4213 - 0.1383 = 0.283$

(iv) $P(|X - 50| < \sqrt{8}) = P(-\sqrt{8} < X - 50 < \sqrt{8})$

$= P(-\sqrt{8} + 50 < X < \sqrt{8} + 50)$

$= P\left(\frac{-\sqrt{8} + 50 - 50}{\sqrt{8}} < Z < \frac{\sqrt{8} + 50 - 50}{\sqrt{8}}\right)$

$= P(-1 < Z < 1)$

$= 2xP(0 < Z < 1) = 2x0.3413 = 0.6826$

4. A random variable X is normally distributed with mean 65 and variance 100, find the probability that X assumes a value between 50 and 70

Solution

$P(50 < X < 70) = P\left(\frac{50-65}{10} < Z < \frac{70-65}{10}\right)$

$= P(-1.5 < Z < 2.5)$

$= P(0 < Z < 1.5) + P(0 < Z < 2.5)$

$= 0.4332 + 0.4938 = 0.927$

5. Lengths of metal strips produced by a machine are normally distributed with mean length of 150cm and standard deviation of 10cm. find the probability that the length of a randomly selected strip is
   (i) Shorter than 165cm
   (ii) Within 5cm of the mean

Solution

(i) $P(X < 165) = P\left(Z < \frac{165-150}{10}\right) = P(Z < 1.5)$

$= 0.5 + P(0 < Z < 1.5)$

$= 0.5 + 0.4332 = 0.9332$

(ii) $P(150 - 5 < X < 150 + 5)$

$= P(145 < X < 155)$

$= P\left(\frac{145 - 150}{10} < Z < \frac{155 - 150}{10}\right)$

$= P(-0.5 < Z < 0.5)$

$= 2xP(0 < Z < 0.5) = 2x0.1915 = 0.383$

6. In end of year exams, the marks are normally distributed with a mean 50 and standard deviation 5. If a mark of 45 is required to pass the exam, what percentage of the students failed the exam.

Solution

$P(X<45)=P\left(Z<\frac{45-50}{5}\right)=P(Z<-1.0)$
$=P(Z>1.0)$
$=0.5-P(0<Z<1.0)$

$P(X<45)=0.5-0.3413=01587$
% of students who failed= $01587x100\%$
$=15.87\%$

7. A bakery supplies bread to the shop every day. The taken to deliver bread to the shop is normally distributed with mean 12 minutes and a standard deviation of 2 minutes. Estimate the number of days during the year when he takes.
(i) Longer than 17 minutes (ii) Less than 10 minutes (iii) Between 9 and 13 minutes

Solution

(i) $P(X>17)=P\left(Z>\frac{17-12}{2}\right)=P(Z>2.5)$
$=0.5-P(0<Z<2.5)$
$=0.5-0.4938=0.0062$
Number if days= $0.0062x365=2\ days$

(ii) $P(X<10)=P\left(Z<\frac{10-12}{2}\right)$
$=P(Z<-1)=P(Z>1)$
$=0.5-P(0<Z<1)$
$=0.5-0.3413=0.1587$
Number if days= $0.1587x365=58\ days$

(iii) $P(9<X<13)=P\left(\frac{9-12}{2}<Z<\frac{13-12}{2}\right)$
$=P(-1.5<Z<0.5)$
$=P(0<Z<1.5)+P(0<Z<0.5)$
$=0.4332+0.1915=0.6247$
Number if days= $0.6247x365=228days$

Exercise 8a

1. Given that the random variable X is $X\sim N(2,2.89)$. Find; $P(X<0)$ **Uneb 2011 No.7**
2. Given that the random variable X is $X\sim N(300,25)$. Find:
   (i) $P(X>308)$ (ii) $P(X>311.5)$
   (iii) $P(X<294)$ (iv) $P(X<290.5)$
   (v) $P(X>302)$ (vi) $P(X>312)$
   An (i) 0.0548 (ii) 0.0107 (iii) 0.8849 (iv) 0.9713 (v) 0.6554 (vi) 0.9918
3. If $X\sim N(50,20)$. Find;
   (i) $P(X>60.3)$ (ii) $P(X<47.3)$
   (iii) $P(X>48.9)$ (iv) $P(X>53.5)$
   (v) $P(X<59.8)$ (vi) $P(X<62.3)$
   An (i) 0.0106 (ii) 0.273 (iii) 0.5972 (iv) 0.2168 (v) 0.9857 (vi) 0.99702
4. If $X\sim N(-8,12)$. Find:
   (i) $P(X<-9.8)$ (ii) $P(X>0)$
   (iii) $P(X<-3.4)$ (iv) $P(X>-5.7)$
   (v) $P(X<-10.8)$ (vi) $P(X>-1.6)$
   An (i) 0.3015 (ii) 0.0105 (iii) 0.9079 (iv) 0.2533 (v) 0.2097 (vi) 0.0323
5. If $X\sim N(a,a^2)$. Find:
   (i) $P(X<0)$ (ii) $P(X>0)$
   (iii) $P(X<0.5a)$ (iv) $P(X>1.5a)$
   An (i) 0.1587 (ii) 0.8413 (iii) 0.6915 (vi) 0.3085
6. If $X\sim N(100,80)$. Find;-
   (i) $P(85<X<112)$ (ii) $P(105<X<115)$
   (iii) $P(85<X<92)$ (iv) $P(|X-100|<\sqrt{80})$
   An (i) 0.8634 (ii) 0.2413 (iii) 0.1388 (vi) 0.6826
7. If $X\sim N(84,12)$. Find;
   (i) $P(80<X<89)$
   (ii) $P(X<79\ or\ X>92)$
   (iii) $P(76<X<82)$
   (iv) $P(|X-84|>2.9)$
   (v) $P(87<X<93)$
   An (i) 0.8014 (ii) 0.085 (iii) 0.2714 (vi) 0.4028 (v) 0.18862
8. The masses of packages from a particular machine are normally distributed with a mean of 200g and a standard deviation of 2g, find the probability that a randomly selected package from the machine weighs
   (i) Less than 197g
   (ii) More than 200.5g
   (iii) Between 198.5g and 199.5g.
   An (i) 0.0668 (ii) 0.4013 (iii) 0.1747
9. The heights of boys at a school follow a normal distribution with mean 150.3cm and variance 25cm. find the probability that a boy picked at random form the group has a height;
   (i) Less than 153cm
   (ii) More than 158cm
   (iii) Between 150cm and 158cm
   (iv) More than 10cm difference from the mean height
   An (i) 0.7054 (ii) 0.0618 (iii) 0.4621 (iv) 0.00456
10. The masses of a certain type of cabbage are normally distributed with mean of 1000g and a standard deviation of 0.15kg. In a batch of 800 cabbages, estimate how many have a mass between 750g and 1290g. **An 740**

11. The number of hours of life of a phone battery is normally distributed with a mean of 150 hours and a standard deviation of 12 hours. In a quality control test, two batteries are chosen at random from a batch. If both batteries have a life less than 120 hours, the batch is rejected. Find the probability that the batch is rejected.
An 0.00003844

12. Cartons of milk from quality supermarket are advertised as containing 1 litre, but in fact the volume of the content is normally distributed with a mean of 1012ml and a standard deviation of 5ml.
   (i) Find the probability that a randomly chosen carton contains more than 1010ml.
   (ii) In a batch of 1000 cartons, estimate the number of cartons that contain less than the advertised volume of milk. **An (i) 0.6554 (ii) 8**

13. A random variable X is such that $X \sim N(-5, 9)$. Find the probability that;
   (i) A randomly chosen item from the population will have a positive value
   (ii) Out of 10 items chosen randomly, exactly 4 will have a positive value.
An (i) 0.0478 (ii) 0.000817

14. The life of a laptop is normally distributed with a mean 2000 hours and a standard deviation of 120 hours. Estimate the probability that the life of such a laptop will be.
   (i) Greater than 2150 hours
   (ii) Greater than 1910 hours
   (iii) Within a range 1850 hours to 2090 hours
An (i) 0.1056 (ii) 0.7734 (iii) 0.6678

15. The weight of sim-sim paste supplied to a shop have a normal distribution with mean 1.5kg and standard deviation 0.6kg. The paste has three sizes.

$$Size\ 1, under\ 0.9kg.$$
$$Size\ 2, from\ 0.9kg\ to\ 2.4kg,$$
$$Size\ 3, over\ 2.4kg$$

   (i) Find the proportion of paste in the three sizes
   (ii) The prices of paste are 1,600/= for size 1, 4,000/= for size 2 and 6,000/= for size 3. Find the expected total cost of 100 sim-sim paste chosen at random from those supplied
An (i) 0.159, 0.775, 0.067 (ii) 375,640/=

16. A random variable X is such that $X \sim N(8, 25)$. Find
   (i) $P(|X-8| < 6.2)$
   (ii) 3 random observations of X are made, find the probability that exactly 2 observations will lie in the interval defined by $|X-8| < 6.2$.
An (i) 0.785 (ii) 0.397

17. The manufacture of a new model car states that, when the car is travelling at 100km/h, the petrol consumption has a mean of 32.4 miles per litre with standard deviation of 1.4 miles per litre. Assuming a normal distribution, find the probability that a randomly chosen car of that model will have a petrol consumption greater than 30 miles per litre when travelling at 100 km/h. **An 0.957**

18. A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg. Given that the weight of the bags are normally distributed, find the **Uneb 1989 No.14**
   (i) Probability that the weight of any bag of maize randomly selected lies between 41.0 and $42.5kg$
   (ii) Percentage of bags whose weight exceeds 43kg
   (iii) Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg
An (i)=0.2029, (ii)=6.68%, (iii)=113

19. A sugar factory sells sugar in bags of mean weight $50kg$ and standard deviation $2.5kg$. Given that the weight of the bags are normally distributed, find the **Uneb 2000 No.15**
   (i) Probability that the weight of any bag of sugar randomly selected lies between 51.5 and 53kg
   (ii) Percentage of bags whose weight exceeds 54kg
   (iii) Number of bags that will be rejected out of 1000 bags purchased for weighing below 45.0kg
An (i)=0.1592, (ii)=5.48%, (iii)=23

20. In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. Given that the weight of the students are normally distributed, find the **Uneb 2005 No.13**
   (i) Probability that the weight of any student randomly selected is 52.8 kg or less
   (ii) Number of students who weigh over 75kg
   (iii) Weight of the middle 56% of the students
An (i)=0.4013, (ii)=1 (iii) $49.251 < X < 59.750$

How to obtain Z-values from a given probability

If you are interested in finding the Z-value whose probabilities are given, it's important to note here that the Z-value may be positive or negative.

Sign	Probability	Z-value
$<$	<0.5	-
$>$	>0.5	-
$<$	>0.5	+
$>$	<0.5	+

Examples

1. $P(Z < Z_1) = 0.25$, find Z_1

Solution

$P(Z < Z_1) = 0.5 - 0.25 = 0.25$

$Z_1 = -0.674$(negative since $0.25 < 0.5$) read directly from a critical table

2. $P(Z < Z_1) = 0.0968$, find Z_1

Solution

$P(Z < Z_1) = 0.5 - 0.0968 = 0.4032$

$Z_1 = -1.3$(negative since $0.0968 < 0.5$) read directly from a critical table

3. $P(Z < Z_1) = 0.05$, find Z_1

Solution

$P(Z < Z_1) = 0.5 - 0.05 = 0.45$

$Z_1 = -1.645$(negative since $0.05 < 0.5$) read directly from a critical table

Or using a cumulative normal distribution

$P(0 < Z < Z_1) = 0.45$

From the table 0.45 lies between 0.4495 and 0.4505. Since the extra information to the right hand side is **add.** So we consider the smallest value ie 0.4495 but 0.4495 corresponds to 1.64. To get the next.

$$\begin{array}{r} 0.4500 \\ -\ 0.4495 \\ \hline 0.0005 \end{array}$$

So we look for 0.0005 in the add column which gives 5

$\therefore 1.64 + 0.005 = -1.645$

4. $P(Z < a) = 0.787$, find a

Solution

$P(Z < a) = 0.787 - 0.5 = 0.287$

using a cumulative normal distribution

$P(0 < Z < Z_1) = 0.287$

From the table 0.287 lies between 0.2852 and 0.2881. Since the extra information to the right hand side is **add.** So we consider the smallest value ie 0.2852 but 0.2852 corresponds to 0.79.

To get the next.

$$\begin{array}{r} 0.2870 \\ -\ 0.2852 \\ \hline 0.0018 \end{array}$$

So we look for 0.0018 in the add column which gives 6

$\therefore a = 0.79 + 0.006 = -1.645$

5. $P(Z < b) = 0.01$, find b

Solution

$P(Z < b) = 0.5 - 0.01 = 0.49$

$b = -2.326$ (negative since $0.01 < 0.5$)

6. $P(Z > b) = 0.01$, find b

Solution

$P(Z > b) = 0.5 - 0.01 = 0.49$

$b = 2.326$ read directly from a critical table

7. $P(Z > a) = 0.812$, find a

Solution

$P(Z > a) = 0.812 - 0.5 = 0.312,$

$a = -0.885$ read directly from a critical table

Further examples

1. If $X \sim N(100,36)$ and $P(X < a) = 0.8907$, find the value of a.

Solution

$$P(X < a) = 0.8907$$
$$P\left(Z < \frac{a-100}{6}\right) = 0.8907 - 0.5 = 0.3907$$
$$1.23 = \frac{a-100}{6}$$
$$a = 107.38$$

2. If $X \sim N(24,9)$ and $P(X > a) = 0.974$, find the value of a.

Solution

$$P(X > a) = 0.974$$
$$P\left(Z > \frac{a-24}{3}\right) = 0.974 - 0.5 = 0.474$$
$$-1.943 = \frac{a-24}{3}$$
$$a = 18.171$$

3. If $X \sim N(70,25)$ and $P(|X - 70| < a) = 0.8$, find the value of a and hence the limit within which the central 80% of the distribution lies.

Solution

$$P(|X - 70| < a) = 0.8$$
$$P(-a < X - 70 < a) = 0.8$$
$$P(-a + 70 < X < a + 70) = 0.8$$
$$P\left(\frac{-a+70-70}{5} < Z < \frac{a+70-70}{5}\right) = 0.8$$
$$P\left(\frac{-a}{5} < Z < \frac{a}{5}\right) = 0.8$$
$$2xP\left(0 < Z < \frac{a}{5}\right) = 0.8$$
$$P\left(0 < Z < \frac{a}{5}\right) = 0.4$$

From table $z = 1.282$
$$1.282 = \frac{a}{5}$$
$$a = 6.41$$

But $P(-a + 70 < X < a + 70) = 0.8$
$$P(63.59 < X < 76.41) = 0.8$$

Central 80% of the distribution lies between 63.59 and 76.41

4. The height of flowers in a farm is normally distributed with mean 169cm and standard deviation 9cm. If X stands for the height of flowers in cm. find X values for
   (a) $P(X < a) = 0.8$
   (b) $P(X > b) = 0.6$

Solution

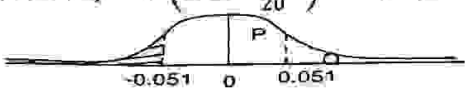
(a) $P(X < a) = 0.8$
$$P\left(Z < \frac{a-169}{9}\right) = 0.8 - 0.5 = 0.3$$
$$0.842 = \frac{a-169}{9}$$
$$a = 176.38$$

(b) $P(X > b) = 0.6$
$$P\left(Z > \frac{b-169}{9}\right) = 0.6 - 0.5 = 0.1$$
$$-0.253 = \frac{b-169}{9}$$
$$b = 166.72$$

5. The marks of 500 students in a mock examination for which the pass mark was 50%. Their marks are normally distributed with mean 45 marks and standard and deviation 20 marks.
   (a) Given that the pass mark is 41, estimate the number of candidates who passed the examination
   (b) If 5% of the candidates obtain a distinction by scoring X marks or more, estimate the value of X
   (c) Estimate the interquartile range of the distribution

Solution

(i) $P(X \geq 41) = P\left(Z \geq \frac{41-45}{20}\right) = P(Z \geq -0.2)$



$$P(Z \geq -0.2) = 0.5 + P(0 < Z < 0.2)$$
$$= 0.5 + 0.0793 = 0.5793$$

Number of candidates who passed
$$= 0.5793x500 = 290$$

(ii) $P(X > X_0) = 0.05$
$$P\left(Z > \frac{X_0-45}{20}\right) = 0.5 - 0.05 = 0.45$$
$$1.645 = \frac{X_0 - 45}{20}$$
$$X_0 = 78$$

(iii) interquartile range $= q_3 - q_1$
$$P\left(0 < Z < \frac{q_3-45}{20}\right) = 0.25$$
$$0.674 = \frac{q_3-45}{20}$$
$$q_3 = 58.48$$
$$P\left(\frac{q_1-45}{20} < Z < 0\right) = 0.25$$
$$-0.674 = \frac{q_1-45}{20}$$
$$q_1 = 31.52$$

interquartile range $= q_3 - q_1$
$$= 58.48 - 31.52 = 26.96$$

6. (a) In a certain athletics competition, points are awarded according to level of performance. The average grade was 82 points with a standard deviation of 5 points. All competitors whose grades ranged between 88 to 94 points received certificates. If the grades are normally distributed and 8 competitors received certificates. How many participants took part in the competition

(b) If certificates were to be awarded to only those having between 90 and 94 points. What proportion of the participants would acquire certificates

Solution

(i) $P(88 < X < 94) = P\left(\frac{88-82}{5} < Z < \frac{94-82}{5}\right) = \frac{8}{n}$

$= P(1.2 < Z < 2.4)$

$= P(0 < Z < 2.4) - P(0 < Z < 1.2))$

$= 0.4918 - 0.3849 = 0.1069$

$0.1069 = \frac{8}{n}$

$n = 74.84$

Hence 74 participants took part

(ii) $P(90 < X < 94) = P\left(\frac{90-82}{5} < Z < \frac{94-82}{5}\right) = \frac{8}{n}$

$= P(1.6 < Z < 2.4)$

$= P(0 < Z < 2.4) - P(0 < Z < 1.6))$

$= 0.4918 - 0.4452$

$P(1.6 < Z < 2.4) = 0.0466$

$= 0.0466x100 = 4.66\%$

7. A random variables X follows a normal distribution with mean 40 and standard deviation 6. Determine.

(i) The area below $X = 34$

(ii) Area above $X = 34$

(iii) The area between $X = 43$ and $X = 52$

(iv) The X-value that has 45% of the area below it

(v) The X-value that has 13% of the are above it

Solution

(i) $P(X < 34)) = P\left(Z < \frac{34-40}{6}\right) = P(Z < -1))$

$= 0.5 - P(0 < Z < 1))$

$= 0.5 - 0.3413 = 0.1587$

(ii) $P(X > 34)) = P\left(Z > \frac{34-40}{6}\right) = P(Z > -1))$

$= 0.5 + P(0 < Z < 1))$

$= 0.5 + 0.3413 = 0.8413$

(iii) $P(43 < X < 51) = P\left(\frac{43-40}{6} < Z < \frac{51-40}{6}\right)$

$= P(0.5 < Z < 2)$

$= P(0 < Z < 2) - P(0 < Z < 0.5))$

$= 0.4772 - 0.1915 = 0.2857$

(iv) $P(X < X_0) = 0.45$

$P\left(Z < \frac{X_0 - 40}{6}\right) = 0.5 - 0.45 = 0.05$

$-0.126 = \frac{X_0 - 40}{6}$

$X_0 = 39.244$

(v) $P(X > X_1) = 0.13$

$P\left(Z > \frac{X_1-40}{6}\right) = 0.5 - 0.13 = 0.37$

$0.392 = \frac{X_1 - 40}{6}$

$X_1 = 42.352$

8. The period of a certain machine approximately follows a normal distribution with mean of five years and standard deviation of one year. Given that the manufacturer of this machine replaces the machine that fails under guarantee, determine the;

(i) Length of the guarantee required so that not more than 2% of the machine that fail are replaced

(ii) Proportion of the machine that would be replaced if the guarantee period was four years

Solution

(i) $P(X < X_0) = 0.02$

$P\left(Z < \frac{X_0 - 5}{1}\right) = 0.5 - 0.02 = 0.48$

$-2.054 = \frac{X_0 - 5}{1}$

$X_0 = 2.946$

Guarantee period is 2.946 years

(ii) $P(X < 4) = P\left(Z < \frac{4-5}{1}\right) = P(Z < -1))$

$P(Z < -1)) = 0.5 - P(0 < Z < 1))$

$= 0.5 - 0.3413 = 0.1587$

$P(X < 4) = 15.87\%$

Exercise 8b

1. Find the values of a in the following

(i)	$P(Z < a)) = 0.506$	(iii)	$P(Z < a)) = 0.0296$	(v)	$P(Z > a)) = 0.713$
(ii)	$P(Z < a)) = 0.787$	(iv)	$P(Z < a)) = 0.325$	(vi)	$P(Z > a)) = 0.154$

(vii) $P(|Z| < a)) = 0.6$
(viii) $P(Z < a)) = 0.9738$
(ix) $P(Z < a)) = 0.2435$
(x) $P(Z > a)) = 0.82$
(xi) $P(Z > a)) = 0.2351$
(xii) $P(|Z| < a)) = 0.6372$
(xiii) $P(|Z| > a)) = 0.097$
(xiv) $P(|Z| < a)) = 0.5$
(xv) $P(|Z| > a)) = 0.0404$

An (i) 0.015 (ii) 0.796 (iii) -1.887(iv) -0.454 (v) -0.562 (vi) 1.019 (vii) 0.842 (viii) 1.94 (ix) -0.695 (x) -0.915 (xi) 0.722 (xii) 0.91 (xiii) 1.66 (xiv) 0.674 (xv) 2.05

2. Find the value of a if
   (i) $P(Z < a) = 0.9693$
   (ii) $P(Z > a) = 0.3802$
   (iii) $P(Z > a) = 0.7367$
   (iv) $P(Z < a) = 0.0793$
   (v) $P(|Z| < a) = 0.9$

   An $[(i)\ a = 1.87,\ (ii)\ a = 0.305,\ (iii)\ a = -0.633, (iv)\ a = -1.41, (v)\ a = 1.645]$

3. If $X \sim N(60,25)$, find a if
   (i) $P(X > a) = 0.2324$ (ii) $P(X > a) = 0.0702$
   (iii) $P(X > a) = 0.837$ (iv) $P(X > a) = 0.7461$
   An (i) 63.66 (ii) 67.37 (iii) 55.09(iv) 56.69

4. If $X \sim N(45,16)$, find a if
   (i) $P(X < a) = 0.0317$ (ii) $P(X < a) = 0.895$
   (iii) $P(X < a) = 0.0456$ (iv) $P(X < a) = 0.996$
   An (i) 37.57 (ii) 50.01 (iii) 38.24 (iv) 55.61

3. If $X \sim N(400,64)$, find a if
   (i) $P(|X - 400| < a)) = 0.75$
   (ii) $P(|X - 400| < a)) = 0.98$
   (iii) $P(|X - 400| < a)) = 0.95$
   (iv) $P(|X - 400| < a)) = 0.975$
   (v) The limit within which the central 95% of the distribution lies
   (vi) Interquartile range of the distribution

   An (i) 9.2 (ii) 18.61 (iii) 15.68 (vi) 17.92
   (v) $384.32 < X < 415.68$ **(vi)** $(394.61, 405.39)$

5. If $X \sim N(80,36)$, find a if $P(|X - 80| < a)) = 0.9$ and hence find the limits within which the central 90% of the distribution lies **An 9.87**, $70.13 < X < 89.87$

6. Bags of flour packed by a particular machine have masses which are normally distributed with mean 500g and standard deviation 20g. 2% of the bags are rejected for being underweight and 1% of the bags are rejected for being overweight. Between what range of values should the mass of a bag of flour lie if it is to be accepted. **An** $(458.92, 546.52)$

7. The masses of mangoes sold at a market are normally distributed with mean mass 600g and standard deviation 20g.
   (i) If a mango is chosen at random, find the probability that its mass lies between 570g and 610g
   (ii) Find the mass exceeded by 7% of the mangoes
   (iii) In one day, 1000 mages are sold. Estimate how many weigh less than 545g

   An (i) 0.6247 (ii) 629.52g (iii) 3

8. The length of metal strips are normally distributed with a mean of 120cm and standard deviation of 10cm.
   (a) Find the probability that a strip selected at random has a length
      (i) Greater than 105cm
      (ii) Within 5cm of the mean
   (b) Strips that are shorter than Lcm are rejected. Estimate the value of L, if 9% or all strips are rejected
   (c) In a sample of 500 strips, estimate the number having a length over 126cm

   An a) (i) 0.9332 (ii) 0.383 (b) 106.6 (c) 137

9. The number of shirts sold in a week by a boutique are normally distributed with a mean of 2080 and standard deviation of 50. Estimate
   (i) The probability that in a given week fewer than 2000 shirts are sold
   (ii) The number of weeks in a year that between 2060 and 2130 shirts are sold
   (iii) Interquartile range of the distribution
   (iv) The least number n of shirts such that the probability that more than n are sold in a given week is less than 0.02

   An (i) 0.0548 (ii) 26 (iii) 67.4 (iv) 2183

10. Batteries for a transistor radio have a mean life under normal usage of 160 hours, with a standard deviation of 30 hours. Assuming the battery life follows a normal distribution
   (i) Find the percentage of batteries which have a life between 150 hours and 180 hours
   (ii) Calculate the range, symmetrical about the mean, within which 75% of the battery lives lie
   (iii) If the radio takes four of these batteries and requires all of them to be working, find the probability that the radio will run for at least 135 hours

   An (i) 37.8% (ii) $(125.5, 194.5)$ (iii) 0.405

11. The length of type A rod is normally distributed with mean of 15cm and a standard deviation of 0.1cm. the length of another type B rod is also

normally distributed with mean of 20cm and standard deviation 0.16cm. For type A rod to be acceptable, its length must be between 14.8cm and 15.2cm and type B rod, the length must be between 19.8cm and 20.2cm

(i) What proportion of type A rod is of acceptable length?

(ii) What is the probability that one of them is of acceptable length?

An (i) 95.44%, (ii) 0.7528, 0.2375

12. The life time of a bulb approximately follows a normal distribution with mean of 800 hours and standard deviation of 80 hours. Given that the manufacturer guarantees to replaces the bulbs that blow below 660 hours. **UNEB 1993 No. 12**

(i) What percentage of the bulbs will he have to replace under the guarantee?

(ii) The manufacture is only willing to replace a maximum of 1% of the bulbs. What should be the guaranteed life time of the bulbs

(iii) Instead of reducing the guaranteed life time as in (ii), the mean life time was increased by superior technology. What should be the new mean so that only 1% are replaced if the guaranteed life time remains at 660 hours but the standard deviation is reduced to 70 hours

An (i) 4.01%, (ii)613.92 hours, (iii) 822 hours

13. The marks of 1000 students in an examination were normally distributed with mean 55 marks and standard and deviation 8 marks.

(i) If a mark of 71 or more is required for A-pass, estimate the number of A -passes awarded

(ii) If 15% of the candidates failed, estimate the minimum mark required for a pass

(iii) Calculate the probability that two candidates chosen at random both passed the examination

An (i) 23 (ii)47, (iii) 0.7225

14. The burning life time of a bulb approximately follows a normal distribution with mean of 1300 hours and standard deviation of 125 hours.

(i) What the probability that the bulb selected at random will burn for more than 1500 hours

(ii) Given that the manufacturer guarantees to replace any bulb that burns for less than 1050 hours, what percentage of the bulbs will have to be replaced

(iii) If two bulbs are installed at the same time, what is the probability that both will burn less than 1400 hours but more than 1200 hours

An (i) 0.0548, (ii)2.28, (iii) 0.3320

15. The volume of a soft drink bottles by a certain company is approximately normally distributed with mean 300ml and standard deviation 2ml. determine the probability that in a sample of 10 bottles at least two contain less than 297.4ml. **UNEB 1988 No. 12 An 0.2515**

16. The marks in an examination were found to be normally distributed with mean 53.9 and standard deviation 16.5. 20% of the candidates who sat this examination failed. Find the pass mark for this examination **UNEB 2015 No.7**
An 40.007

Finding the values of μ OR σ OR BOTH

1. If $X \sim N(100, \sigma^2)$ and $P(X < 106) = 0.8849$. find the value of the standard deviation,σ

Solution

$$P(X < 106) = 0.8849$$
$$P\left(Z < \frac{106-100}{\sigma}\right) = 0.8849 - 0.5 = 0.3849$$
$$\frac{106-100}{\sigma} = 1.2$$
$$\sigma = 5$$

2. The length of a certain item follows a normal distribution with mean μcm and standard deviation 6cm. it is known that 4.78% of the items have length greater than 82cm. find the value if the mean μ

Solution

$$P(X > 82) = 0.0478$$
$$P\left(Z > \frac{82-\mu}{6}\right) = 0.5 - 0.0478 = 0.4522$$
$$\frac{82-\mu}{6} = 1.667$$
$$\mu = 72cm$$

3. The masses of boxes of oranges are normally distributed such that 30% of them are greater than 4.00kg and 20% are greater than 4.53kg. estimate the mean and standard deviation of the masses

Solution

$$P(X > 4) = 0.3$$

$$P\left(Z > \frac{4-\mu}{\sigma}\right) = 0.5 - 0.3 = 0.2$$

$$\frac{4-\mu}{\sigma} = 0.524$$

$$4 = \mu + 0.524\sigma \ldots\ldots\ldots\ldots (i)$$

$$P(X > 4.53) = 0.2$$

$$P\left(Z > \frac{4.53-\mu}{\sigma}\right) = 0.5 - 0.2 = 0.3$$

$$\frac{4.53-\mu}{\sigma} = 0.842$$

$$4.53 = \mu + 0.842\sigma \ldots\ldots\ldots\ldots (ii)$$

$(ii)-(i)$: $0.53 = 0.318\sigma$

$\sigma = 1.67$kg

Also $4 = \mu + 0.524x1.67$

$$\mu = 3.13kg$$

4. The speed of cars passing a certain Entebbe high way can be taken to be normally distributed. 95% of the car ear travelling at less than 85m/s and 10% are traveling at less 55m/s
   (i) Find the average speed of the cars passing through the high way
   (ii) Find the proportion of the cars that travel at more than 70m/s

Solution

(i) $P(X < 85) = 0.95$

$$P\left(Z < \frac{85-\mu}{\sigma}\right) = 0.95 - 0.5 = 0.45$$

$$\frac{85-\mu}{\sigma} = 1.645$$

$$85 = \mu + 1.645\sigma \ldots\ldots\ldots\ldots (i)$$

$$P(X < 55) = 0.1$$

$$P\left(Z < \frac{55-\mu}{\sigma}\right) = 0.5 - 0.1 = 0.4$$

$$\frac{55-\mu}{\sigma} = -1.282$$

$$55 = \mu - 1.282\sigma \ldots\ldots\ldots\ldots (ii)$$

$(i)-(ii)$: $30 = 2.927\sigma$

$\sigma = 10.25$m/s

Also $85 = \mu + 1.645x10.25$

$$\mu = 68.14m/s$$

(ii) $P(X > 70) = P\left(Z > \frac{70-68.14}{1024}\right) = P(Z > 0.182)$

$$= 0.5 - 0.0722 = 0.4278$$

5. The masses of articles produced in a particular shop are normally distributed with mean μ and standard deviation σ. 5% of the articles have masses greater than 85g and 10% have masses less than 25g.
   (i) find the value of μ and σ
   (ii) Find the symmetrical limit, about the mean, within which 75% of the masses lie

Solution

(i) $P(X > 85) = 0.05$

$$P\left(Z < \frac{85-\mu}{\sigma}\right) = 0.5 - 0.05 = 0.45$$

$$\frac{85-\mu}{\sigma} = 1.645$$

$$85 = \mu + 1.645\sigma \ldots\ldots\ldots\ldots (i)$$

$$P(X < 25) = 0.1$$

$$P\left(Z < \frac{25-\mu}{\sigma}\right) = 0.5 - 0.1 = 0.4$$

$$\frac{25-\mu}{\sigma} = -1.282$$

$$25 = \mu - 1.282\sigma \ldots\ldots\ldots\ldots (ii)$$

$(i)-(ii)$: $60 = 2.927\sigma$

$\sigma = 20.5$g

Also $85 = \mu + 1.645x20.5$

$$\mu = 51.3g$$

(ii) $P(|X - 51.3| < a) = 0.75$

$$P(-a + 51.3 < X < a + 51.3) = 0.75$$

$$P\left(\frac{-a + 51.3 - 51.3}{20.5} < Z < \frac{a + 51.3 - 51.3}{20.5}\right) = 0.75$$

$$P\left(\frac{-a}{20.5} < Z < \frac{a}{20.5}\right) = 0.75$$

$$2xP\left(0 < Z < \frac{a}{20.5}\right) = 0.75$$

$$P\left(0 < Z < \frac{a}{20.5}\right) = 0.375$$

$$\frac{a}{20.5} = 1.15$$

$$a = 23.575$$

Lower limit$= -23.575 + 51.3 = 27.73$

upper limit$= 23.575 + 51.3 = 74.88$

6. A total population of 700 students sat a mock examination for which the pass mark was 50%. Their marks were normally distributed. 28 students scored below 40% while 35 students scored above 60%.
   (a) Find the mean mark and standard deviation of the students **UNEB 1995 No. 13**
   (b) What is the probability that a students chosen at random passed the exam
   (c) Suppose the pass mark is lowered by 2%, how many more students will pass

Solution

(i) $P(X < 40) = \frac{28}{700} = 0.04$

$$P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.5 - 0.04 = 0.46$$

$$-1.751 = \frac{40-\mu}{\sigma}$$

$$\mu - 1.75\sigma = 40 \ldots\ldots\ldots\ldots\ldots (i)$$

$P(X > 60) = \frac{35}{700} = 0.05$

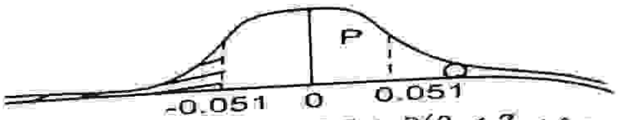
$P\left(Z > \frac{60-\mu}{\sigma}\right) = 0.5 - 0.05 = 0.45$

$$1.645 = \frac{60-\mu}{\sigma}$$

$$\mu + 1.645\sigma = 60 \ldots\ldots\ldots\ldots\ldots (ii)$$

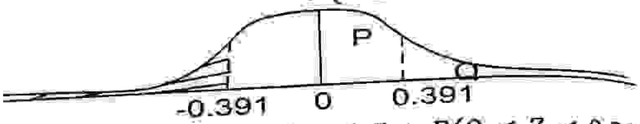
$\mu = 50.3\,,\ \sigma = 5.889$

(ii) $P(X \geq 50) = P\left(Z \geq \frac{50-50.3}{5.889}\right)$
$= P(Z \geq -0.051)$



$P(Z \geq -0.051) = 0.5 + P(0 < Z < 0.051)$
$= 0.5 + 0.0203 = 0.5203$

(iii) $P(X \geq 48) = P\left(Z \geq \frac{48-50.3}{5.889}\right)$
$= P(Z \geq -0.391)$



$P(Z \geq -0.391) = 0.5 + P(0 < Z < 0.391)$
$= 0.5 + 0.1521 = 0.6521$

More proportion $= 0.6521 - 0.5203 = 0.1318$

More students $= 0.1318x700 = 92$

Exercise 8c

1. $X \sim N(45, \sigma^2)$ and $P(X > 51) = 0.288$. Find σ **An 10.7**
2. $X \sim N(21, \sigma^2)$ and $P(X < 27) = 0.9332$. Find σ **An 4**
3. $X \sim N(\mu, 25)$ and $P(X < 27.5) = 0.3085$. Find μ **An 30**
4. $X \sim N(\mu, 12)$ and $P(X > 32) = 0.8438$. Find μ **An 35.5**
5. $X \sim N(\mu, \sigma^2)$ and $P(X > 80) = 0.0113$, $P(X > 30) = 0.9713$. Find σ and μ **An** 52.73, 11.96
6. $X \sim N(\mu, \sigma^2)$ and $P(X > 102) = 0.42$, $P(X < 97) = 0.25$ Find σ and μ **An** 100.8, 5.71
7. $X \sim N(\mu, \sigma^2)$ and $P(X < 57.84) = 0.90$, $P(X < 50) = 0.5$ Find σ and μ **An 50, 6.12**
8. $X \sim N(\mu, \sigma^2)$ and $P(X < 35) = 0.20$, $P(35 < X < 45) = 0.65$ Find σ and μ **An 39.5, 5.32**
9. length of rods produced in a workshop follow a normal distribution with mean μ and variance 4. 10% of the rods are less than 17.4cm long. Find the probability that a rod chosen at random will be between 18cm and 23cm long. **An 0.7725**
10. The length of a stick follows a normal distribution. 10% are of length 250cm or more while 55% have a length over 240cm. find the probability that a stick, picked at random is less than 235cm long. **An 0.203**
11. (a) The diameter of bolts produced by a particular machine follows a normal distribution with mean 1.34cm ad standard deviation 0.04cm. A bolt is rejected if its diameter is less than 1.24cm or more than 1.40cm. Find the percentage of bolts that are accepted.
    (b) The setting of the machine is altered so that the mean diameter changes but the standard deviation remains the same. With the new settings, 3% of the bolts are rejected because they are too large in diameter. Find the;
    (i) new mean diameter of the bolts produced by the machine
    (ii) percentage of bolts which are rejected because they are too small in diameter.
    An (a) 92.7% (b) (i) 1.32 **(ii) 1.7%**
12. A certain make of car tyres can be safely used for 25000km on average before it is replaced. The makers guarantee to pay compensation to anyone whose tyre does not last for 22000km. They expect 7.5% of all the tyres sold to qualify for compensation. If the distance, X travelled before a tyre is replaced has a normal distribution.
    (i) Find the standard deviation of X
    (ii) Estimate the number of tyres per 1000 which will not have been replaced when they have covered 26500km
    An (i) 2080 (ii) 236
13. A cutting machine produces steel rods which must not exceed 100cm in length. The mean length of a large batch of rods taken from the machine is found to be 99.80cm and the standard deviation of these length is 0.15cm.
    (i) If the length of the rods are normally distributed, find the percentage of rods which are too long
    (ii) The position of the cut can be adjusted without altering the standard deviation of

the length, find how small the mean length should be if no more than 2% of the rods are to be rejected for being longer than 100cm.

(iii) If the mean length is maintained at 99.8cm, find how much the standard deviation must be reduced if no more than 4% of the rods are to be rejected for being longer than 100cm.

An (i) 9.1%, (ii) 99.69 (iii) 0.4mm

14. The continuous random variable X is normally distributed with mean μ and standard deviation σ. If $P(X < 53) = 0.04$ and $P(X < 65) = 0.97$, find the interquartile range. **An 4.46**

15. Tea sold in packages marked 750g. the masses are normally distributed with mean 760g and standard deviation σ. What is the maximum value of σ, if less than 1% of the packages are underweight. **An 4.299**

16. In an examination 30% of the candidates fail and 10% achieve distinctions. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of the candidates were normally distributed, estimate the mean mark and standard deviation. **An** μ =**104.31,** $\sigma = 38.76$

17. At st Noah junior, the heights of students are normally distributed. 10% are over 1.8 meters and 20% are below 1.6 meters
    (a) Find the mean height, μ and standard deviation, σ of the students
    (b) Find the inter quartile range **An (a)** $\mu = 1.68$, **σ =0.09, (b) 0.13**

18. Observation of a very large number of cars at a certain point on a motor way established that the speeds are normally distributed. 90% of the cars have speeds less than $77.7km/hr$ ands only 5% of cars have speeds less $than\ 63.1km/hr$. Find the mean speed, μ and standard deviation, σ. **An** $\mu = 71.305$, $\sigma = 4.988$

19. The diameter of a sample of oranges to the nearest cm were **Uneb 1998 march No.11**

Diameter (cm)	8	9	10	11	12	13	14
Frequency	9	15	21	32	19	13	11

    (i) Calculate the mean and diameter
    (ii) Assuming the distribution is normal, find the minimum diameter if the smallest 10% of the oranges are rejected for being too small
    An (i) $\bar{x} = 11, \sigma = 1.6633$ (ii) 8.703

20. A sample of 100 apples is taken from a load. The apples have the following distribution of size

Diameter (cm)	6	7	8	9	10
Frequency	11	21	38	17	13

Assuming that the distribution is approximately normal with mean μ and standard deviation σ
    (i) Determine μ and σ
    (ii) Find the range of size of apples for packing, if 5% are to be rejected as too small and 5% are to be rejected as too large **An** (i) $\mu = 8, \sigma = 1.16$ (ii) $(6.10, 9.90)$

21. The volumes of soda in bottles are normally distributed with a mean of 333ml. Given that 20% of the bottles contain more than 340ml, find the;
    (i) standard deviation of the volume of drink in a bottle
    (ii) Percentage of bottles that contain less than 330ml. **An (i) 8.31, (ii) 35.9%**

22. The heights of 500 students are normally distributed with a standard deviation of 0.080m. if the height of 129 of the students are greater than the mean height but less than 1.806m, find the mean height. **An 1.75**

23. The masses of boxes of apples are normally distributed such that 20% of them are greater than 5.08kg and 15% are greater than 5.62kg. find the mean and standard deviation of the masses. **An 2.74, 2.78**

24. The masses of packets of sugar are normally distributed. If 5% of the packets have mass greater than 510g and 2% have mass greater than 515g. Find the mean and standard deviation of this distribution **An 490g, 12.2g**

25. On a particular day, 50% of the teachers had arrived at work by 8.30am, and 10% had not arrived by 8.55am. assuming the arrival is normally distributed,
    (i) Find the standard deviation of the arrival time in minutes
    (ii) 80 teachers are selected at random. Find the expectation of the number of these teachers that arrived between 8.30am and 8.55am **An (i) 19.50 (ii) 32**

26. Sugar packed in 500g packets is observed to be approximately normally distributed with standard deviation of 4g. If only 2% of the packets contained less than 500g of sugar. Find the mean weight of the sugar in the packets. **Uneb 2008 No.4 An 508.216g**

27. Sixty students sat for a mathematics contest whose pass mark was 40 marks. There scores in the contest were approximately normally distributed, 9 students cored less than 20 marks while 3 scored more than 70 marks. Find the;

(i) Mean scored and standard deviation of the contest
(ii) Probability that a student chosen at random passed the contest

Uneb 2008 No.15

An $\mu = 39.32, \sigma = 18.65, 0.4856$

28. The marks in an examination were normally distributed with mean μ and standard deviation σ. 10% of the candidates had more than 75 marks and 20% had less than 40 marks. Find the values of μ and σ. **Uneb 2014 No.6 An 53.87, 16.48**

29. The number of cows owned by residents in village is assumed to be normally distributed. 15% of the residents have less than 60 cows. 5%of the residents have over 90 cows. **Uneb 2017 No.15**
    (a) Determine the value of the mean and standard deviation of the cows
    (b) If there are 200 residents, find how many have more than 80 cows

    An $\mu = 71.5926, \sigma = 11.1899$, 45

30. A random variable X has a normal distribution when $P(X > 9) = 0.9192$ and $P(X < 11) = 0.7580$. Find: **Uneb 2018 No.12**
    (a) the values of the mean and standard deviation.
    (b) $P(X > 10)$

    An $\mu = 10.3333, \sigma = 0.9524$, 0.6386

NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

Under certain conditions, the normal distribution can be used as an approximation to the binomial distribution.

Condition

(i) The number of trials of the binomial experiment should be large is $n > 20$

(ii) The probability of success not to small or too large ie P constant and very close to 0.5

$$X \sim N(np, npq)$$

The Z value is obtained from $Z = \frac{X \pm 0.5 - \mu}{\sigma} = \frac{X \pm 0.5 - np}{\sqrt{npq}}$

Where ± 0.5 is used to make the binomial distribution continuous

Note; 0.5 must be subtracted from the minimum value and added to the maximum value

(i) $P(X \geq x_1) = P\left(Z \geq \frac{(X-0.5)-np}{\sqrt{npq}}\right)$

(ii) $P(X \leq x_1) = P\left(Z \leq \frac{(X+0.5)-np}{\sqrt{npq}}\right)$

(iii) $P(x_1 \leq X \leq x_2) = P\left(\frac{(X-0.5)-np}{\sqrt{npq}} \leq Z \leq \frac{(X+0.5)-np}{\sqrt{npq}}\right)$

Examples

1. Among spectators watching a football match, 80% were the home team supporters while 20% were the visiting team supporter. If 2500 of the spectators are selected randomly, what is the probability that there are at least 541 visitors in the sample. **Uneb 2006 No. 11**

Solution

$$P(X \geq 541) = P\left(Z \geq \frac{(541-0.5) - 2500x0.2}{\sqrt{2500x0.2x0.8}}\right)$$
$$= P(Z \geq 2.025)$$

$$P(Z \geq 2.025) = 0.5 - P(0 < Z < 2.025)$$
$$P(Z \geq 2.025) = 0.5 - 0.47 = 0.0215$$

2. During elections of head a girl in a school, 40% of the students supported Gillian. 150 students were randomly selected. Find the probability that more than 55 supported Gillian.

Solution

$$P(X > 55) = P(X \geq 56)$$
$$= P\left(Z \geq \frac{55.5 - 150x0.4}{\sqrt{150x0.4x0.6}}\right) = P(Z \geq -0.75)$$

$$P(Z \leq 0.75) = 0.5 + P(0 < Z < 0.75)$$
$$= 0.5 + 0.2734 = 0.7734$$

3. In a box containing different pens, the probability that a pen is red is 0.35. find the probability that in a random sample of 400 pens from the box.
   (i) Less than 120 are red pens
   (ii) More than 160 are red pens
   (iii) Between 120 and 150 inclusive are red pens

Solution

$n = 400, p = 0.35, q = 0.65$

Mean $\mu = np = 400x0.35 = 140$

$\sigma = \sqrt{npq} = \sqrt{400x0.35x0.65} = \sqrt{91}$

(i) $P(X < 120) = P(X \leq 119)$

$$= P\left(Z \leq \frac{119.5 - 140}{\sqrt{91}}\right) = P(Z \leq -2.149)$$
$$= P(Z \geq 2.149) = 0.5 - P(0 \leq Z \leq 2.149)$$
$$= 0.5 - 0.4821 = 0.0158$$

(ii) $P(X > 160) = P(X \geq 161)$

$$= P\left(Z \geq \frac{160.5 - 140}{\sqrt{91}}\right) = P(Z \geq 2.149)$$
$$= 0.5 - P(0 \leq Z \leq 2.149)$$
$$= 0.5 - 0.4821 = 0.0158$$

(iii) $P(120 \leq X \leq 150)$

$$= P\left(\frac{119.5 - 140}{\sqrt{91}} \leq Z \leq \frac{150.5 - 140}{\sqrt{91}}\right)$$
$$= P(-2.149 \leq Z \leq 1.101)$$
$$= P(0 \leq Z \leq 2.149) + P(0 \leq Z \leq 1.101)$$
$$= 0.4821 + 0.3645 = 0.8466$$

4. If an unbiased coin is tossed 100 times, what is the probability that;
   (i) There will be more than 60 heads
   (ii) There will be less than 43 heads
   (iii) There will be between 45 heads and 55 heads inclusive

Solution

$n = 100, p = 0.5, q = 0.5$
Mean $\mu = np = 100x0.5 = 50$
$\sigma = \sqrt{npq} = \sqrt{100x0.5x0.5} = 5$

(i) $P(X > 60) = P(X \geq 61)$
$= P\left(Z \geq \frac{(61 - 0.5) - 50}{5}\right) = P(Z \geq 2.1)$
$P(Z \geq 2.1) = 0.5 - P(0 < Z < 2.1)$
$P(Z \geq 2.1) = 0.5 - 0.4821 = 0.0179$

(ii) $P(X < 43) = P(X \leq 42)$
$= P\left(Z \leq \frac{(42 + 0.5) - 50}{5}\right) = P(Z \leq -1.5)$
$P(Z \leq -1.5) = 0.5 - P(0 < Z < 1.5)$
$= 0.5 - 0.4332 = 0.0668$

(iii) $P(45 \leq X \leq 55)$
$= P\left(\frac{44.5 - 50}{5} \leq Z \leq \frac{55.5 - 50}{5}\right)$
$P(-1.1 \leq Z \leq 1.1) = 2xP(0 \leq Z \leq 1.1)$
$= 2x0.3643 = 0.7286$

5. It is known that 72% of NTV viewers watch news at 9pm. What is the probability that in a sample of 500 viewers chosen at random
   (i) More than 350 watch news
   (ii) Fewer than 340 watch news
   (iii) Exactly 350 watch news

Solution

(i) $P(X > 350) = P(X \geq 351)$
$= P\left(Z \geq \frac{(351 - 0.5) - 500x0.72}{\sqrt{500x0.72x0.28}}\right)$
$= P\left(Z \geq \frac{350.5 - 360}{10.04}\right) = P(Z \geq -0.946)$
$P(Z \geq -0.946) = 0.5 + P(0 < Z < 0.946)$
$P(Z \geq -0.946) = 0.5 - 0.328 = 0.8280$

(ii) $P(X < 340) = P(X \leq 339)$
$= P\left(Z \leq \frac{(339 - 0.5) - 360}{10.04}\right)$
$= P(Z \leq -2.042)$
$P(Z \leq -2.042) = 0.5 - P(0 < Z < 2.042)$
$P(Z \leq -2.042) = 0.5 - 0.4794 = 0.0206$

(iii) $P(X = 350) = P(349.5 \leq X \leq 350.5)$
$= P\left(\frac{349.5 - 360}{10.04} \leq Z \leq \frac{350.5 - 360}{10.04}\right)$
$= P(-1.046 \leq Z \leq -0.946)$
$P(-1.046 \leq Z \leq -0.946) = 0.3522 - 0.3280$
$P(X = 350) = 0.0242$

6. A pair of balanced dice, each numbered from 1 to 6 is tossed 150 times. Determine the probability that a sum of seven appears at least 26 time

Solution

P(sum of 7)= $p = \frac{6}{36} = \frac{1}{6}$

$P(X \geq 26) = P\left(Z \geq \frac{(26 - 0.5) - 150x\frac{1}{6}}{\sqrt{150x\frac{1}{6}x\frac{5}{6}}}\right)$
$= P\left(Z \geq \frac{25.5 - 25}{4.56}\right) = P(Z \geq 0.11)$
$P(Z \geq 0.11) = 0.5 - P(0 < Z < 0.11)$
$P(Z \geq 0.11) = 0.5 - 0.0438 = 0.4562$

7. Two players play a game in which each of them tosses a balanced coin. The game ends in a draw if both get the same result. Determine the probability that in 100 trials, the game ends in a draw.
   (i) At least 53 times
   (ii) At most 53 times

Solution

P(draw)= $p = \frac{2}{4} = \frac{1}{2}$

$P(X \geq 53) = P\left(Z \geq \frac{(53 - 0.5) - 100x0.5}{\sqrt{100x0.5x0.5}}\right)$
$= P\left(Z \geq \frac{52.5 - 50}{5}\right) = P(Z \geq 0.5)$
$P(Z \geq 0.5) = 0.5 - P(0 < Z < 0.5)$
$P(Z \geq 0.5) = 0.5 - 0.1915 = 0.3085$

$P(X \leq 53) = P\left(Z \leq \frac{53.5 - 50}{5}\right) = P(Z \leq 0.7)$
$P(Z \leq 0.7) = 0.5 + P(0 < Z < 0.7)$
$P(Z \leq 0.7) = 0.5 + 0.2580 = 0.7580$

8. In a certain book, the number of words per page follows a normal distribution with mean 800 words and standard deviation 40 words. Three pages are chosen at random, what is the probability that
   (i) None of them has between 830 and 845 words
   (ii) At let two pages have between 830 and 845 words

Solution

(i) $P(830 \leq X \leq 845) = P\left(\frac{830-800}{40} \leq Z \leq \frac{845-800}{40}\right) = P(0.75 \leq Z \leq 1.25)$

$$P(0.75 \le Z \le 1.25) = 0.3522 - 0.3280 = 0.0962$$

$$P(X = 0) = \binom{3}{0}(0.0962)^0(0.9038)^3 = 0.7383$$

(ii) $P(X \ge 2) = P(X = 2) + P(X = 3)$

$$= \binom{3}{2}(0.0962)^2(0.9038)^1 + \binom{3}{3}(0.0962)^3(0.9038)^0 = 0.02509 + 0.00089 = 0.02598$$

Exercise 8d

1. The random variable $X \sim B(200, 0.7)$. Find
   (i) $P(X \ge 130)$ (ii) $P(136 \le X < 148)$
   (iii) $P(X < 142)$ (iv) $P(X = 152)$
   An((i)0.9474, (ii) 0.6325, (iii) 0.5914, (iv) 0.0111)
2. An ordinary unbiased die is thrown 120 times. Find the probability of obtaining at least 24 sixes **An(0.1958)**
3. A pair of dice is tossed 144 times and the sum of the outcomes recorded. Find the probability that a sum of 7 occurs at least 26 times **An(0.3688)**
4. Records form Uganda police traffic department shows that on weekend nights, one out of every ten drivers on the road is drunk. A random sample of four hundred drivers is checked on a weekend. Find the probability that the number of drunk drivers is at least 35 but less than 47. **Uneb 1990 No.14 An(0.6807)**
5. Records from Mulago hospital show that, on average 80% of patients have malaria. If a sample of 200 patients is randomly selected, find the probability that;
   (i) More than 170 patients have malaria
   (ii) At least 155 patients have malaria
   An((i)0.0138, (ii) 0.8345)
6. In a school 45% of the boys are circumcised. Find the probability that, in a group of 200 boys more than 97 are circumcised. **An(0.1432)**
7. 10% of phones imported to Uganda are I-phones, a random sample of 1000 phones is taken. Find the probability that;
   (i) Less than 80 are I-phones
   (ii) Between 90 and 115 inclusive are I-phones
   (iii) More than 120 or more are I-phones
   An((i)0.0154, (ii) 0.8145, (iii) 0.02)
8. During Christmas season, the probability that a message is sent on phone successfully is 0.85
   (i) When 8 messages are sent, find the probability that at least 7 are successfully sent
   (ii) When 50 messages are sent, find the probability that at least 45 are successfully sent. **An((i)0.657, (ii) 0.2142)**
9. One-fifth of Acholi have Nodding syndrome. Find the probability that the number of Acholi people with the syndrome is;
   (i) More than 20 in a random sample of 100 people
   (ii) Exactly 20 in a random sample of 100 people
   (iii) More than 200 in a random sample of 1000 people
   An((i)0.4502, (ii) 0.0996, (iii) 0.484)
10. If a fair die is thrown 300 times, what is the probability that
    (i) There will be more than 60 sixes
    (ii) There will be fewer than 45 sixes
    An((i)0.0519, (ii) 0.1971)
11. A coin is biased such that head is twice as likely to occurs as a tail. The coin us tossed 120 times. Find the probability that there will be;
    (i) Between 42 and 51 tails inclusive
    (ii) 48 tails or less
    (iii) Less than 34 tails
    (iv) At least 72 and at most 90 heads
    An((i)0.3729, (ii) 0.9501, (iii) 0.1039, (iv) 0.9290)
12. 400 students sat a test which consists of 80 true or false questions. None of the candidates knows any of the answers and so guesses
    (i) If the pass mark is 38, how many of the candidates would be expected to pass
    (ii) What should the new pass mark, if it is decided that only 115 candidates pass
    An((i)285, (ii) 45)
13. A lorry of potatoes has on average one rotten potato in six. A green grocer tests a random sample of 100 potatoes and decides to turn away the lorry if he finds more than 18 rotten potatoes in the sample. Find the probability that he accepts the consignment. **An(0.6886)**
14. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected. **Uneb 2000 No.6 An(0.0558)**

15. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. Determine the probability that a sum of seven appears;
    (i) Exactly 40 times
    (ii) Between 25 and 35 inclusive times
    Uneb 2002 No.10 (i) 0.0108, (ii) 0.7286
16. On average 20% of all the eggs sold in supplied have cracks. Find the probability that in a sample of 900 eggs supplied by the farm will have more than 200 cracked eggs. **Uneb 2004 No.5** **An(0.0439)**
17. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that in a sample of 300 boiled eggs will have more than 50 cracked eggs. **Uneb 2005 No.4** **An(0.215)**
18. A die is tossed 40 times and the probability of getting a six on any one toss is 0.122, estimate the probability of getting between 6 to 10 sixes. **Uneb 2007 No.1 An(0.114)**
19. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each question correct. A student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68 or more. Calculate
    (i) The probability that he fails the examination
    (ii) The probability that he obtains a distinction **Uneb 2012 No.15b**
    An((i)0.1308, (ii) 0.0629)
20. A research station supplies three varieties of seeds S_1, S_2 and S_3 in the ratio of $4:2:1$. The probabilities of germination of S_1, S_2 and S_3 are 50%, 60% and 80% respectively.
    (i) Find the probability that a selected seed will germinate
    (ii) Given that 150 seeds are selected at random, find the probability that less 90 seeds will germinate. **Uneb 2013 No.16**

DISTRIBUTION OF SAMPLE MEAN OF A NORMAL DISTRIBUTION POPULATION

If a random X of sample of size n taken from a normal distribution with mean μ and variance σ^2, then distribution of the sample mean $\bar{x}$ is also said to be normally distributed with mean μ and variance $\frac{\sigma^2}{n}$ such that $\bar{x} \approx \left(\mu, \frac{\sigma^2}{n}\right)$

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Example

1. At a certain school, the masses of students are normally distributed with mean 70kg and standard deviation 5kg. If 4 students are randomly selected, find the probability that their mean mass is less 65kg.

Solution

$$P(\bar{X} < 65) = P\left(Z < \frac{65 - 70}{\frac{5}{\sqrt{4}}}\right) = P(Z < -2)$$

$$P(Z > 2) = 0.5 - P(0 < Z < 2)$$
$$P(\bar{X} < 65) = 0.5 - 0.4772 = 0.0228$$

2. A random sample of size 15 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample is less than 58

Solution

$$P(\bar{X} < 58) = P\left(Z < \frac{58 - 60}{\frac{4}{\sqrt{15}}}\right)$$

$$= P(Z < -1.936)$$
$$P(Z > 1.936) = 0.5 - P(0 < Z < 1.936)$$
$$P(\bar{X} < 58) = 0.5 - 0.4736 = 0.0264$$

3. The height of students are normally distributed with mean 164cm and standard deviation $7.2cm$. calculate the probability that the mean height of a sample of 36 students will be between 162cm and $166cm$.

Solution

$$P(162 < \bar{X} < 166) = P\left(\frac{162-164}{\frac{7.2}{\sqrt{36}}} < Z < \frac{166-164}{\frac{7.2}{\sqrt{36}}}\right)$$
$$= P(-1.667 < Z < 1.667)$$

$$= 2xP(0 < Z < 1.667)$$
$$= 2x0.4522 = 0.9044$$

4. The height of a certain plant follows a normal distribution with mean 21cm and standard deviation $\sqrt{90}cm$. A random sample of 10 plants is taken and the mean height calculate. Find the probability that this sample mean lies between 18cm and 27cm

Solution

$$P(18 < \bar{X} < 27) = P\left(\frac{18-21}{\frac{\sqrt{90}}{\sqrt{10}}} < Z < \frac{27-21}{\frac{\sqrt{90}}{\sqrt{10}}}\right)$$
$$= P(-1 < Z < 2)$$

$$= P(0 < Z < 1) + P(0 < Z < 2)$$
$$= 0.3413 + 0.4772 = 0.8185$$

5. A large number of random sample of size n is taken from a distribution X where $X \sim N(74, 36)$ and the sample mean $\bar{X}$ for each random sample is calculated. If $P(\bar{X} > 72) = 0.854$, find the value of n

Solution

$$P(\bar{X} > 72) = P\left(Z > \frac{72 - 74}{\frac{6}{\sqrt{n}}}\right) = 0.854$$

$$P\left(Z > \frac{-\sqrt{n}}{3}\right) = 0.854$$

From table $Z = -1.054$

$$\frac{-\sqrt{n}}{3} = -1.054$$
$$n = 10$$

6. The distribution of a random variable X is $X \sim N(25, 340)$ and the sample mean $\bar{X}$ for each random sample is calculated. If $P(\bar{X} > 28) = 0.005$, find the value of n

Solution

$$P(\bar{X} > 28) = P\left(Z > \frac{28-25}{\frac{\sqrt{340}}{\sqrt{n}}}\right) = 0.005$$

$$P\left(Z > \frac{3\sqrt{n}}{\sqrt{340}}\right) = 0.005$$

From table $Z = 2.576$

$$\frac{3\sqrt{n}}{\sqrt{340}} = 2.576$$

$$n = 250$$

Exercise 8e

1 If $X \sim N(200, 80)$ and a random sample of size 5 is taken from the distribution, find the probability that the sample mean,

(i) Is greater than 207

(ii) Lies between 201 and 209

An((i) 0.0401, (ii)0.3891,)

2 If $X \sim N(200, 10)$ and a random sample of size 10 is taken from the distribution, find the probability that the sample mean lies out side the range 198 to 205. **An(0.3206,)**

3 If $X \sim N(50, 12)$ and a random sample of size 12 is taken from the distribution, find the probability that the sample mean,

(i) Is less than 48.5

(ii) Is less than 52.3

(iii) Lies between 50.7 and 51.7

An((i) 0.0668, (ii)0.9893, (iii) 0.1974)

4 Biscuits are produced with weight (Wg) where W is $N(10,4)$ and are packed at random into boxes consisting of 25 biscuits. Find the probability that;

(i) A biscuit chosen at random weighs between 9.25g and 10.7g

(ii) The content of a box weighs between 245g and 255g

(iii) The average weight of the biscuits in the box lies between 9.7g and 10.3g

An((i) 0.2924, (ii)0.0796, (iii) 0.5468)

5 A normal distribution has a mean of 40 and a standard deviation of 4. If 25 items are drawn at random, find the probability that their mean is;

(i) 41.4 or more

(ii) Between 38.7 and 40.7

(iii) Less than 39.5

An((i) 0.0401, (ii)0.7571, (iii) 0.2660)

7. A random sample of size 25 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample is

(i) less than 58

(ii) Greater than 58

(iii) between 58 and 62

An(0.0062, 0.9938, 0.9876)

8. At st Noah junior, the marks of the pupils can be modeled by a normal distribution with mean 70% and standard deviation 5%. If four pupils are chosen at random. Find the probability that the mean mark is

(i) less than 65%

(ii) Greater than 65%

(iii) Greater than 75%

(iv) between 72% and 75%

An(0.9772, 0.0228, 0.0228, 0.1891)

9. The volume of soda in bottles are normally distributed with mean of 758ml and a standard deviation of 12ml. a random sample of 10 bottles is taken and the mean volume found. Find the probability that the sample mean is less than 750ml. **An(0.0176)**

10. The height of cassava plants are normally distributed with mean of 2m and a standard deviation of 40cm. A random sample of 50 cassava plants is taken and the mean height found. Find the probability that the sample mean lies between 195cm and 205cm. **An(0.6234)**

11. In an examination, marks are normally distributed with mean 64.5 and variance 64. The mean mark in a random sample of 100 scripts is denoted by $\bar{X}$. Find

(i) $P(\bar{X} > 65.5)$

(ii) $P(63.8 < \bar{X} < 64.5)$

An((i) 0.1056, (ii)0.3092)

12. The marks of an examination were normally distributed. 20% of the students scored below 40 marks while 10% of the students scored above 75 marks

(i) Find the mean mark and standard deviation of the students

(ii) If 25 students were chosen at random from those who sat for the examination, what is the probability that their average mark exceeds 60

(iii) If a sample of 8 students were chosen, find the probability that not more than 3 scored between 45 and 65 marks

An((a) μ =53.87, σ =16.473, (b) =0.0313,

$(c) = 0.5419)$

13. The life time of batteries produced by a certain factory s normally distributed. Out of 10000 batteries selected at random, 668 have life time less than 130 hours and 228 have life time more than 200 hours.
    (i) Find the mean and standard deviation of the battery life time
    (ii) Find the percentage of the batteries with life time between 150 and 180 hours
    (iii) If the sample of 25 batteries is selected at random, find the probability that the mean of the life time exceeds 165 hours

    An((a) $\mu = 160, \sigma = 20, (b) = 53.28\%,$ $(c) = 0.1056)$

14. If a large number of samples, size n are taken from a population which follows a normal distribution with mean 74 and standard deviation.
    (i) Find n if the probability that the sample mean exceeds 75 is 0.282
    (ii) Find n if the probability that the sample mean exceeds 70.4 is 0.00135

    An((i) 12, (ii)25,)

15. A normal distribution has a mean of 30 and a variance of 5. Find the probability that:
    (i) The average of 10 observations exceeds 30.5
    (ii) The average of 40 observations exceeds 30.5
    (iii) The average of 100 observations exceeds 30.5
    (iv) Find n such that the probability that he average of n observations exceed 30.5 is less than 1% **An((i) 0.2399, (ii)0.0787, (iii) 0.0127 (iv)** $n > 108)$

16. The random variable is such that $X \sim N(\mu, 4)$. A random sample, size n is taken from the population. Find the least n such that $P(|\bar{X} - \mu| < 0.5) = 0.95$ **An(62)**

17. Boxes made in a factory have weights which are normally distributed with a mean of $4.5kg$ and a standard deviation of $2.0kg$. If a sample of 16 boxes is drawn at random, find the probability that their mean is; **Uneb 1998 nov No. 14**
    (i) Between 4.6 and 4.7 kg
    (ii) Between 4.3 and 4.7 kg

    An((i) 0.0761, (ii)0.3108)

18. The masses of soap powder in a certain packet are normally distributed with mean 842g and variance 225 g^2. Find the probability that a random sample of 120 packets has sample mean with mass. **Uneb 2009 No. 15**
    (i) Between 844 g and 846 g
    (ii) Less than 843g

    An((i) 0.0702, (ii)0.7673)

ESTIMATION OF POPULATION PARAMETERS

Statistical estimation is used to describe the unknown characteristics of the population (population parameter) by using sample characteristic.

A sample is a representation of a population parameter such as population mean, μ and population variance σ^2.

Types of parameter estimation

- Point estimates
- Interval estimates

(a) Point estimates

(i) The unbiased estimate of the population mean, μ is

$\bar{x} = \frac{\sum x}{n}$ or $\bar{x} = \frac{\sum fx}{\sum f}$ Where $\bar{x}$ is sample mean

(ii) The unbiased estimate of the population variance, σ^2 is $\hat{\sigma}^2$ where

$\hat{\sigma}^2 = \frac{n}{n-1}s^2$ Where s^2 is sample variance

OR $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$ or $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

Examples

1. Find the best un biased estimate of the mean μ and variance σ^2 of the population from which each of the following sample is drawn

(i) 46, 48, 50, 45, 53, 50, 48, 51

(ii)

x	20	21	22	23	24	25
f	4	14	17	26	20	9

(ii) $\sum x = 100, \sum x^2 = 1028, n = 10$

(iii) $\sum x = 120, \sum x^2 = 2102, n = 8$ **An(15, 43.14)**

(iv) $\sum x = 330, \sum x^2 = 23700, n = 34$ **An(9.71, 621.12)**

(v) $\sum x = 738, \sum x^2 = 16526, n = 50$ **An(14.76, 114.96)**

Solution

x	f	fx	fx²
45	1	45	2025
46	1	46	2116
48	2	96	4608
50	2	100	5000
51	1	51	2601
53	1	53	2809
	Σf=8	Σfx=391	Σfx²=19159

Unbiased estimate for the mean

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{391}{8} = 48.875$$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$$

$$\hat{\sigma}^2 = \frac{8}{8-1}\left[\frac{19159}{8} - \left(\frac{391}{8}\right)^2\right] = 6.982$$

x	f	fx	fx²
20	4	80	1600
21	14	294	6174
22	17	374	8228
23	26	598	13754
24	20	480	11520
25	9	225	5625
	Σf=90	Σfx=2051	Σfx²=46901

Unbiased estimate for the mean

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{2051}{90} = 22.789$$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$$

$$\hat{\sigma}^2 = \frac{8}{8-1}\left[\frac{46901}{2051} - \left(\frac{2051}{90}\right)^2\right]$$

$$\hat{\sigma}^2 = 1.809$$

(a) Unbiased estimate for the mean

$$\bar{x} = \frac{\sum x}{n} = \frac{100}{10} = 10$$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

$$\hat{\sigma}^2 = \frac{10}{10-1}\left[\frac{1028}{10} - \left(\frac{100}{10}\right)^2\right] = 3.11$$

2. The fuel consumption of a new car model was being tested. In one trials 8 cars chosen at random were driven under identical condition and the distance $x\ km$ covered on one litre of petrol was

recorded. The following results were obtained, $\sum x = 152.98, \sum x^2 = 2927.1$. Calculate the unbiased estimate of the mean and variance of the distance covered by the car

Solution

Unbiased estimate for the mean

$$\bar{x} = \frac{\sum x}{n} = \frac{152.98}{8} = 19.1225$$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

$$\hat{\sigma}^2 = \frac{8}{8-1}\left[\frac{2927.1}{8} - \left(\frac{152.98}{8}\right)^2\right] = 0.25$$

(b) Interval estimate

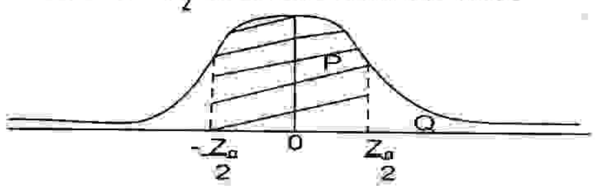
Here we are interested in obtaining the interval over which the true population lies (confidence interval)

Confidence intervals

The unbiased estimate of the population mean, μ is $\bar{x}$

$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ Where n is sample size

Z is the area under the normal curve leaving an area of $\frac{\alpha}{2}$ on either side of the curve



$$P\left(-Z_{\alpha/2} < Z < Z_{\alpha/2}\right) = \alpha/2$$

$$P\left(-Z_{\alpha/2} < \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} < Z_{\alpha/2}\right) = \alpha/2$$

$$P\left(\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = \alpha/2$$

Confidence intervals $\left[\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$

Confidence limits $\left(\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$

Or $\mu = \bar{x} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

(i) Confidence interval for population mean, μ

- of a normal or non-normal population
- with known population variance σ^2 or standard deviation σ
- Using any sample size, n large or small

The confidence interval is obtained from

$$\mu = \bar{x} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \qquad \text{Where } \bar{x} \text{ is sample mean}$$

Examples

1. The length a bar of a metal is normally distributed with mean of 115cm and a standard deviation of 3cm. find the 95% confidence limits for the mean length of the bar

Solution

$$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$$

$$Z_{\alpha/2} = 1.96$$

$$\mu = \bar{x} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\mu = 115 \pm 1.96x\frac{3}{1}$$

Lower limit = 109.12

Upper limit= 120.88

2. The mass of vitamin E in a capsule is normally distributed with standard deviation 0.042mg. a random sample of 5 capsules was taken and the mean mass if vitamin E was found to be 5.12mg. calculate a symmetric 95% confidence interval for the population mean mass.

Solution

$$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$$

$$Z_{\alpha/2} = 1.96$$

$$\mu = \bar{x} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\mu = 5.12 \pm 1.96x\frac{0.042}{\sqrt{5}}$$

Lower limit = 5.08

Upper limit= 5.16

3. It is known that an examination paper is marked in such on way that the standard deviation of the marks is 15.1. In a certain school, 80 candidates take the examination and they have an average mark of 57.4. find

    (i) 95% and

(ii) 99% confidence limits for the mean mark in the exam

Solution

(a) $\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha_{/2} = 1.96$

$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

$\mu = 57.4 \pm 1.96x\frac{15.1}{\sqrt{80}}$

$(54.091, 60.709)$

(iii) $\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$

$Z\alpha_{/2} = 2.575$

$\mu = 57.4 \pm 2.575\frac{15.1}{\sqrt{80}}$

Lower limit = 53.053

Upper limit= 61.746

4. After a particular rainy night, 12 worms were picked and their length in cm were measured; 9.5, 9.5, 11.2, 10.6, 9.9, 11.1, 10.9, 9.8, 10.1, 10.2, 10.9, 11.0.
Assuming that this sample came from a normal population with standard deviation 2, find the 95% confidence interval for the mean length of all the worms.

Solution

$$\bar{x} = \frac{\sum x}{n} = \frac{9.5 + 9.5 + 11.2 + 10.6 + 9.9 + 11.1 + 10.9 + 9.8 + 10.1 + 10.2 + 10.9 + 11.1}{12} = 10.39$$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha_{/2} = 1.96$

$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

$\mu = 10.39 \pm 1.96x\frac{2}{\sqrt{36}}$

$[9.26\text{cm}, 11.52\text{cm}]$

5. In a certain school, 10 students sat for an examination .The student marks were 29, 92, 38, 87, 32, 46, 71, 75, 59, 51 . Assuming the sample was taken from a population where the scores follow a normal distribution with variance 210.
   (i) Calculate the sample mean mark
   (ii) 95% confidence limits for mean an mark in the exam

Solution

$$\bar{x} = \frac{\sum x}{n} = \frac{29 + 92 + 38 + 87 + 87 + 75 + 32 + 46 + 71 + 59 + 51}{10} = 58$$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha_{/2} = 1.96$

$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

$\mu = 58 \pm 1.96x\frac{\sqrt{210}}{\sqrt{10}}$

Lower limit = 49.018

Upper limit= 66.982

6. The height of students are normally distributed with mean, μ and the standard deviation, σ. On the basis of the results obtained from a random sample of 100 students from the school, the 95% confidence interval of the mean was calculated and found to be $(177.22cm, 179.18cm)$. Calculate;
   (i) The value of the sample mean | (ii) The value of standard deviation
   (iii) 90% confidence interval of the mean, μ

Solution

(i) $\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha_{/2} = 1.96$

$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

$177.22 = \bar{x} - 1.96\frac{\sigma}{\sqrt{100}}$.........(i)

$179.18 = \bar{x} + 1.96\frac{\sigma}{\sqrt{100}}$.........(ii)

adding

$2\bar{x} = 356.4$

$\bar{x} = 178.2$

(ii) $177.22 = 178.2 - 1.96\frac{\sigma}{\sqrt{100}}$

$\sigma = 5$

(iii) $\frac{\alpha}{2} = \frac{0.90}{2} = 0.45$

$Z\alpha_{/2} = 1.645$

$\mu = 1782 \pm 1.645x\frac{5}{\sqrt{100}}$

$[177.38\text{cm}, 179.02\text{cm}]$

7. A plant produces steel sheets whose weights are known to be normally distributed with a standard deviation of 2.4kg. A random sample of 36 sheets had a mean weight of 31.4kg.
   (i) find the 99% confidence limits for the population mean.

(ii) Find the width of the 99% confidence limit

Solution

(i) $\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$

$Z\alpha_{/2} = 2.575$

$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

$\mu = 31.4 \pm 2.575x\frac{2.4}{\sqrt{36}}$

$(30.37, 32.43)$

(ii) Width $= 32.43 - 30.37$

$= 2.06kg$

Or $width = 2xZ\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

$= 2x2.575x\frac{2.4}{\sqrt{36}}$

$= 2.06kg$

8. The marks scored by students are normally distributed with mean μ and standard deviation 1.3. It is required to have a 95% confidence interval for μ with width less than2. Find the least number of students that be sampled to achieve this.

Solution

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha_{/2} = 1.96$

$width = 2xZ\alpha_{/2}\frac{\sigma}{\sqrt{n}} < 2$

$2x1.96x\frac{1.3}{\sqrt{n}} < 2$

$2x1.96x\frac{1.3}{2} < \sqrt{n}$

$6.49 < n$

$n > 6.49$

Least number should be 7

(ii) Confidence interval for population mean, μ

- of a normal or non -normal population
- with un known population variance σ^2 or standard deviation σ
- Using a large sample size (n≥ 30)

If the population variance, σ^2 is not given or unknown, then the confidence interval is obtained from

$$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\hat{\sigma}}{\sqrt{n}}$$

Where $\hat{\sigma}^2 = \frac{n}{n-1}s^2$ $s^2 - sample\ variance$

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

Examples

1. The fuel consumption of a new car model was being tested. In one trials 50 cars chosen at random were driven under identical condition and the distance $x\ km$ covered on one litre of petrol was recorded. The following results were obtained, $\sum x = 525, \sum x^2 = 5625$. Calculate the 95% confidence interval for the mean petrol consumption, in km per litre of cars of this type.

Solution

Unbiased estimate for the mean

$\bar{x} = \frac{\sum x}{n} = \frac{525}{50} = 10.5$

The unbiased estimate of the population variance,

$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$

$\hat{\sigma}^2 = \frac{50}{50-1}\left[\frac{5625}{50} - (10.5)^2\right]$

$\hat{\sigma} = 1.515$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha_{/2} = 1.96$

$\mu = 10.5 \pm 1.96x\frac{1.515}{\sqrt{50}}$

$[10.08km/litre, 10.92km/litre]$

2. The height x cm of each man in a random sample of 200 men living in mbale district was measured. The following results were obtained $\sum x = 35000, \sum x^2 = 62000000$
   (a) Calculate the unbiased estimate of the mean and variance of the heights of men living in mbale
   (b) Determine the 90% confidence interval for the mean height of men living in mbale

Solution

(a) Unbiased estimate for the mean

$\bar{x} = \frac{\sum x}{n} = \frac{3500}{200} = 175$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

$$\hat{\sigma}^2 = \frac{200}{200-1}\left[\frac{62000000}{200} - (175)^2\right]$$

$$\hat{\sigma}^2 = 376.884$$

(b) $\frac{\alpha}{2} = \frac{0.90}{2} = 0.45$

$$Z_{\alpha/2} = 1.645$$

$$\mu = 175 \pm 1.645x\frac{\sqrt{376.884}}{\sqrt{200}}$$

$[172.742\text{cm}, 177.258\text{cm}]$

3. A random sample of 100 observations from a normal population with mean, μ gave the following results $\sum x = 8200, \sum x^2 = 686000$
   (a) Calculate the unbiased estimate of the mean and variance of the heights of men living in mbale
   (b) Determine the 98% confidence interval for the mean
   (c) Determine the 99% confidence interval for the mean

Solution

(a) Unbiased estimate for the mean

$$\bar{x} = \frac{\sum x}{n} = \frac{8200}{100} = 82$$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

$$\hat{\sigma} = \sqrt{\frac{100}{100-1}\left[\frac{686000}{100} - (82)^2\right]}$$

$$\hat{\sigma} = 11.72$$

(b) $\frac{\alpha}{2} = \frac{0.98}{2} = 0.49$

$$Z_{\alpha/2} = 2.326$$

$$\mu = 82 \pm 2.326x\frac{11.72}{\sqrt{100}}$$

$[79.274cm, 84.726cm]$

(c) $[78.981cm, 85.102cm]$

4. The mean and standard deviation of a random sample of size 100 is 900 and 60 respectively. Given that the population is normally distributed, find a 96% confidence interval of the population mean.

Solution

$$\hat{\sigma} = \sqrt{\left(\frac{100}{100-1}\right)x60^2} = 60.302$$

$$\frac{\alpha}{2} = \frac{0.96}{2} = 0.48$$

$$Z_{\alpha/2} = 2.054$$

$$\mu = \bar{x} \pm Z_{\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}$$

$$\mu = 900 \pm 2.054x\frac{60.302}{\sqrt{100}}$$

$[887.614, 912.386]$

Exercise 8f

1. The concentration, in mg per litre of a trace element in 7 randomly chosen samples of water from a spring were.
   240.8, 237.3, 236.7, 236.6, 234.2, 233.9, 232.5.
   Determine the unbiased estimate of the mean and the variance of the concentration of the trace element per litre of water from the spring. **An(236, 7.58)**
2. Cartons of orange are filled by a machine. A sample of 10 cartons selected at random from the production contained the following quantities (in mm) 201.2, 205.0, 209.1, 202.3, 204.6, 206.4, 210.1, 201.9, 203.7, 207.3
   Calculate unbiased estimate of the mean and variance of the population from which the sample was taken. **An(205.16, 9.223)**
3. A factory produces cans of meat whose masses are normally distributed with standard deviation 18g. A random sample of 25 cans is found to have a mean of 458g. Find the 99% confidence interval for the population mean mass of a can of meat produced at the factory. **An(448.7, 467.3)**
4. A tennis ball is known to have a height of bounce which is normally distributed with standard deviation 2cm. A sample of 60 tennis balls is tested and the mean height of bounce of the sample is 140cm. Find
   (i) 95%
   (ii) 98% confidence interval for the mean height of bounce of the tennis ball
   An(139.5, 140.51) (139.4, 140.6)
5. A random sample of 100 is taken from a population. The sample is found to have a mean of 76.0 and standard deviation 12.0. find

(i) 90% (ii) 97%

(ii) 99% confidence interval for the mean of the population **An(74.02, 77.98) (73.38, 78.62) (72.89,79.11)**

6. 150 bags of flour of a particular brand are weighed and the mean mass is found to be 748g with standard deviation 3.6g. find
   (i) 90% (ii) 95%
   (ii) 98% confidence interval for the mean mass of bags of flour of this brand

An(747.51, 748.49) (747.42, 748.58) (747.31,748.69)

7. A random sample of 100 readings taken from a normal population gave the following data: $\bar{x} = 82, \sum x^2 = 686800$. Find
   (i) 98%
   (ii) 99% confidence interval for the population mean **An(79.19, 84.81) (78.89, 85.11)**

8. 80 people were asked to measure their pulse rates when they woke up in the morning. The mean was 69 beats and the standard deviation 4 beats. Find
   (i) 95% (ii) 99%
   (ii) 97% confidence interval for the population mean **An(68.12, 69.88) (67.84, 70.16) (68.0, 70.0)**

9. The 95% confidence interval for the mean length of a particular brand of light bulb is $[1023.3h, 1101.7h]$. This interval is based on results from a sample of 36 light bulbs. Find the 99% confidence interval for the mean length of life for this brand of light bulb, assuming that the length of life is normally distributed.
An(1011,1114)

10. A random sample of 6 items taken from a normal population with variance $4.5cm^2$ gave the following data:
$12.9cm, 13.2cm, 14.6cm, 12.6cm, 11.3cm, 10.1cm$
   (i) Find the 95% confidence interval for the population mean
   (ii) What is the width of this confidence interval **An(10.75, 14.15) (3.4)**

11. An efficiency expert wishes to determine the mean time taken to drill a number of holes in a metal sheet. Determine how large a random sample is needed so that the expert can be 95% certain that the sample mean will differ from the true mean time by less than 15 seconds. Assume that it is known from previous studies that the population standard deviation is 40 seconds. **An(28)**

12. A random sample of 60 loaves is taken from a population whose mean masses are normally distributed with mean μ and standard deviation 10g
   (i) Calculate the width of 95% confidence interval for μ based on the sample
   (ii) Find the confidence level of a 95% confidence interval having the same width as before but based on a random sample of 40 loaves **An((i) 5.06g, (ii)89%)**

13. The distribution of measurements of masses of a random sample of bags packed in a factory is shown below.

Mass (kg)	72.5	77.5	82.5	87.5	92.5	97.5	102.5	107.5
Frequency	6	18	32	57	102	51	25	9

   (i) Find the mean and standard deviation of the masses
   (ii) Find the 95% confidence limits **An((ii)90.5,92.2)**

14. The age , X in years of 250 mothers when their first child was born is given below

x	18 -	20 -	22 -	24 -	26 -	28 -	30 -	32 -	34 -	36 -	38 -
Number of mothers	14	36	42	57	48	26	17	7	2	0	1

   (i) Estimate the mean and standard deviation of X
   (ii) If 250 mothers are randomly selected form a large population of mothers, find the 95% confidence limits for the mean age of the total population **An((i)25.3, 3.6 (ii) (24.9, 25.8))**

15. The life time of 200 electrical components were recorded and classified in the frequency distribution

Life time	0 -	100 -	200 -	300 -	400 -	500 -	600 -	700 -	800 -	900 -	1000 -
frequency	80	48	30	18	10	5	4	3	2	0	0

   (i) Estimate the mean and standard deviation of the distribution
   (ii) find the 90% confidence limits for the population mean using a suitable normal approximation for the distribution of the sample mean **An(ii)(173.5, 214.5)**

NUMERICAL METHODS, ERRORS AND FLOW CHARTS

ERRORS

An error is the difference between exact and approximate value

Types of errors

(a) Rounding errors

These are errors that arise as a result of simply approximating the true value of different numbers

Example

Round of the following number to the given number of decimal places or significant figures

(i) 3.896234 to 4 dp. **An(3.8962)**

(ii) $\frac{2}{3}$ to 3 dp. **An(0.667)**

(iii) 0.00639 to 4dp. **An(0.0064)**

(iv) 5.00257 to 3s.f. **An(5.00)**

(v) 1.0457162 to 3s.f. **An(1.05)**

(vi) 0.00652673 to 4s.f. **An(0.006527)**

(vii) 7.00214 to 4 s.f . **An(7.002)**

(viii) 5415678 to 3 s.f. **An(5420000)**

(b) Truncation errors

These occur when an infinite number is terminated at some point

Examples

Truncate the following numbers to the given number of decimal places (d.p) or significant figures, s.f

(i) 4.56172 to 2dp.**An(4.56)** (ii) $\frac{2}{3}$ to 3dp.**An(0.666)** (iii) 1.345618 to 4s.f. **An(1.345)**

Common terms used

(a) Error or absolute error

If x represents an approximate value to X and Δx is the error in approximation

$$|error| = |exact\ value - approximate\ value|$$

$$|\Delta x| = |X - x|$$

Examples

Round off 32.52632 to 2dp and determine the absolute error

Solution

$X = 32.53$ $x = 32$

$$|\Delta x| = |X - x|$$

$$|\Delta x| = |32.53 - 32|$$

$$|\Delta x| = |-0.53| = 0.53$$

(b) Relative error

$$Relative\ error = \frac{absolute\ error}{exact\ value} = \frac{|\Delta x|}{X} = \frac{|X - x|}{X}$$

(c) Percentage error or percentage relative error

$$percentage\ Relative\ error = \frac{absolute\ error}{exact\ value} x100\% = \frac{|\Delta x|}{X} x100\% = \frac{|X - x|}{X} x100\%$$

Example

Find the percentage error in rounding $\sqrt{3}$ to 2dp

Solution

$X = \sqrt{3} = 1.732050808$ $x = 1.73$

$$percentage\ \ error = \frac{|X - x|}{X} x100\%$$

$$\%\ error = \frac{|1.732050808 - 1.73|}{1.732050808} x100\%$$

$$percentage\ \ error = 0.118\%$$

(d) **Maximum possible error in an approximated number**

This depends on the number of dp the number is rounded to

If the number is rounded off to n dp, then the maximum possible error in that number is $= 0.5x10^{-n}$

Examples

1. If a student weighs 50kg. Find the range where his weight lies

Solution

$n = 0dp, e = 0.5x10^{-0} = 0.5$

$minimum\ value = 50 - 0.5 = 49.5kg$

$Maximum\ value = 50 + 0.5 = 50.5kg$

$Range\ of\ value = (49.5, 50.5)$

2. If x is given to stated level of accuracy state the lower and upper bounds of x

(a) 6.45 (b) 0.278 (c) 4.0

Solution

(a) $n = 2dp, e = 0.5x10^{-2} = 0.005$

$lower\ bound = 6.45 - 0.005 = 6.445$

$upper\ bound = 6.45 + 0.005 = 6.455$

(b) $n = 3dp, e = 0.5x10^{-3} = 0.0005$

$lower\ bound = 0.278 - 0.0005 = 0.2775$

$upper\ bound = 0.278 + 0.0005 = 0.2785$

(c) $n = 1dp, e = 0.5x10^{-1} = 0.05$

$lower\ bound = 4.0 - 0.05 = 3.95$

$upper\ bound = 4.0 + 0.05 = 4.05$

3. A value of $w = 150.58m$ was obtained when measuring the width of the football pitch. Given that the relative error in this value was 0.07%,find the limit within which the value of w lies

Solution

$$\%\ Relative\ error = \frac{|\Delta w|}{w} x100\%$$

$$0.07 = \frac{|\Delta w|}{150.58} x100\%$$

$$|\Delta w| = 0.105$$

$lower\ limit = 150.58 - 0.105 = 150.475$

$upper\ limit = 150.58 + 0.105 = 150.685$

Absolute error in an operation

When the minimum and maximum value is known then;

$$Absolute\ error\ = \frac{1}{2}[\mathbf{maximum\ value - minimum\ value}]$$

(I) Absolute error in addition

Given two number a and b with errors Δa and Δb

$(a + b)_{max} = a_{max} + b_{max} = (a + \Delta a) + (b + \Delta b)$ $(a + b)_{min} = a_{min} + b_{min} = (a - \Delta a) + (b - \Delta b)$

(II) Absolute error in subtraction

Given two number a and b with errors Δa and Δb

$(a + b)_{max} = a_{max} - b_{min} = (a + \Delta a) - (b - \Delta b)$ $(a + b)_{min} = a_{min} - b_{max} = (a - \Delta a) - (b + \Delta b)$

Examples

1. Given that $a = 2.453$, $b = 6.79$. find the limits and hence the absolute error of

(i) $a + b$ (ii) $a - b$

Solution

$a = 2.453, \Delta a = 0.0005, b = 6.79, \Delta b = 0.005$

(I) $(a + b)_{maximum} = a_{max} + b_{max}$

$(a + b)_{max} = (2.453 + 0.0005) + (6.79 + 0.005)$

$= 9.2485$

$(a + b)_{miniimum} = a_{min} + b_{min}$

$(a + b)_{min} = (2.453 - 0.0005) + (6.79 - 0.005)$

$= 9.2375$

$lower\ limit = 9.2375$

$upper\ limit = 9.2485$

$Absolute\ error\ = \frac{1}{2}[9.2485 - 9.2375] = 0.0055$

(ii) $(a - b)_{maximum} = a_{max} - b_{min}$

$(a - b)_{max} = (2.453 + 0.0005) - (6.79 - 0.005)$

$= -4.3315$

$(a - b)_{miniimum} = a_{min} - b_{max}$

$(a - b)_{min} = (2.453 - 0.0005) - (6.79 + 0.005)$

$= -4.3425$

$lower\ limit = -4.3425$

$upper\ limit = -4.3315$

$Absolute\ error\ = \frac{1}{2}[-4.3315 - -4.3425]$

$= 0.0055$

(iii) Absolute error in multiplication

Given two number a and b with errors Δa and Δb

$(ab)_{maximum} = a_{max}b_{max} = (a + \Delta a)(b + \Delta b)$ | $(ab)_{minimum} = a_{min}b_{min} = (a - \Delta a)(b - \Delta b)$

Examples

1. Given that $a = 4.617$, $b = 3.65$. find the:
   (i) the maximum possible error in a and b
   (ii) the absolute error ab

Solution

(i) $a = 4.617, \Delta a = 0.0005, b = 3.65, \Delta b = 0.005$

(ii) $(ab)_{maximum} = a_{max}b_{max}$

$(ab)_{maxi} = (4.617 + 0.0005)(3.65 + 0.005)$

$= 16.87696$

$(ab)_{minimum} = a_{min}b_{min}$

$(ab)_{min} = (4.617 - 0.0005)(3.65 - 0.005)$

$= 16.82853$

$Absolute\ error = \frac{1}{2}[\boldsymbol{max - min}]$

$Absolute\ error = \frac{1}{2}[16.87696 - 16.82853]$

$= 0.02422$

2. Given that $a = 4.617$, $b = -3.65$. find the:
   (i) the limits of values where ab lies
   (ii) the interval of values where ab lies
   (iii) the absolute error in ab

Solution

$a = 4.617, \Delta a = 0.0005\ b = -3.65, \Delta b = 0.005$

(i) $(ab)_{maximum} = a_{max}b_{max}$

$(ab)_{max} = (4.617 + 0.0005)(-3.65 + 0.005)$

$= -16.83079$

$(ab)_{minimum} = a_{min}b_{min}$

$(ab)_{min} = (4.617 - 0.0005)(-3.65 - 0.005)$

$= -16.87331$

$lower\ limit = -16.87331$

$upper\ limit = -16.83079$

Interval $of\ values = [-16.87331, -16.83079]$

$Absolute\ error = \frac{1}{2}[-16.83079 - -16.87331]$

$= 0.02126$

(iv) Absolute error in division

Given two number a and b with errors Δa and Δb

$\left(\frac{a}{b}\right)_{maximum} = \frac{a_{max}}{b_{min}} = \frac{(a + \Delta a)}{(b - \Delta b)}$ | $\left(\frac{a}{b}\right)_{minimum} = \frac{a_{min}}{b_{max}} = \frac{(a - \Delta a)}{(b + \Delta b)}$

Examples

1. Given that $a = 1.26$, $b = 0.435$. find the absolute error of
   (i) Find the maximum possible errors in a and b
   (ii) Range of values where $\frac{a}{b}$ lies
   (iii) Absolute error in $\frac{a}{b}$

Solution

(i) $\Delta a = 0.005$

$\Delta b = 0.0005$

(ii) $\left(\frac{a}{b}\right)_{maximum} = \frac{a_{max}}{b_{min}} = \frac{(1.26 + 0.005)}{(0.435 - 0.0005)}$

$= 2.91139$

$\left(\frac{a}{b}\right)_{minimum} = \frac{a_{min}}{b_{max}} = \frac{(1.26 - 0.005)}{(0.435 + 0.0005)}$

$= 2.88175$

$Range\ of\ values = (2.88175, 2.91139)$

(iii) $Absolute\ error = \frac{1}{2}[2.91139 - 2.88175]$

$= 0.01482$

2. The numbers 2.6754, 4.8006, 15.175 and 0.92 have been rounded off to the given number of dp. Find the range of values within which the exact value of $2.6754\left(4.8006 - \frac{15.175}{0.92}\right)$ can be expected to lie

Solution

Maximum value $= 2.67545\left(4.80065 - \frac{15.1745}{0.925}\right) = -31.529$

$minimum\ value = 2.675445\left(4.80055 - \frac{15.1755}{0.925}\right) = -31.045$

$Range\ of\ values = (-31.529, -31.045)$

3. Obtain the interval of values within which the exact value of $\frac{15.36+27.1-1.672}{2.36x1.043}$ lies

Solution

Maximum value= $\frac{15.365+27.15-1.6715}{2.355x1.0425} = 16.920$

Minimum value = $\frac{15.355+27.05-1.6725}{2.365x1.0435} = 16.505$

$interval\ of\ values = [16.505, 16.920]$

4. Given that $N = \frac{12.4}{4.20} - \frac{10.80}{6.124}$
   (a) Write the possible error in each of the values given
   (b) Estimate the range of values within which N lies. Hence estimate the absolute error in N

   Solution $Range\ of\ values = (1.1725, 1.2052),\ Absolute\ error = 0.0164$

Other examples

1. The dimensions of a rectangle are $7.4cm$ and $6.25cm$.
   (i) State the maximum possible error in each dimension
   (ii) Find the range which the area of the rectangle lies

   Solution

   (i) $l = 7.4cm, \Delta l = 0.05$
   $w = 6.25cm, \Delta w = 0.005$
   (ii) $A = lw$

   $A_{max} = 7.45x6.255 = 46.5998$
   $A_{min} = 7.35x6.245 = 45.9008$
   $Range\ of\ values = [45.900, 46.5998]$

2. The numbers $a = 24.57$, $b = 12.49$ and $c = 7.2$ are calculated with percentage errors of 5, 3 and 1 respectively. Find the limit to two decimal places within which the exact value of the expression $ab - \frac{b}{c}$ lies.

Solution

$\Delta a = \frac{5x24.57}{100} = 1.229$ $\quad \Delta b = \frac{3x12.49}{100} = 0.375$ $\quad \Delta c = \frac{1x7.2}{100} = 0.072$

$$lower\ limit = (24.57 - 1.229)(12.49 - 0.375) - \left(\frac{12.49 + 0.375}{7.2 - 0.072}\right) = 282.97$$

$$Upper\ limit = (24.57 + 1.229)(12.49 + 0.375) - \left(\frac{12.49 - 0.375}{7.2 + 0.072}\right) = 330.24$$

3. A mobile money business man makes an annual profit of 8 million with a margin error of $\pm 5\%$ and an annual loss of 2 million with a margin error of $\pm 3\%$
   (i) Find the range of values corresponding to his gross income
   (ii) If his annual income tax is 1.5 million, what percentage of the gross income goes to tax, giving your answer as a range of values.

   Solution

(i) Max profit= $8x105\% = 8.4m$
Min profit= $8x95\% = 7.6m$
Max loss= $2x103\% = 2.06m$
Min loss= $2x97\% = 1.94m$
Max income= $max\ profit - min\ loss$
Max income= $8.4 - 1.94 = 6.46m$

Min income= $min\ profit - max\ loss$
Min income= $7.6 - 2.06 = 5.54m$
$Range\ of\ income = [5.54, 6.46]$

(ii) Lower limit = $\frac{1.5}{6.46}x100 = 23.22\%$
Upper limit = $\frac{1.5}{5.54}x100 = 27.08\%$
$Range\ of\ values = [23.22\%, 27.08\%]$

Exercise 9a

1. Two numbers A and B have maximum possible errors e_a and e_b respectively. **UNEB 2018 No.6**
   (a) Write an expression for the maximum possible errors in their sum.
   (b) If $A = 2.03$ and $B = 1.547$, find the maximum possible error in $A + B$. **An (0.0035)**
2. Given that $y = \frac{1}{x} + x$ and $x = 2.4$ correct to one decimal places, find the limits within which

y lies. **UNEB 2017 No.6 An (lower limit=2.8755, upper limit=2.7582)**

3. The numbers $A = 6.341$ and $B = 2.6$ have been ronded to the given number of decimal places.
   (a) Find the maximum possible error in AB
   (b) Determine the interval within which $\frac{A^2}{B}$ can be expected to lie. Give your answer correct to 3 decimal places

4. (i) Given the numbers $x = 2.678$ and $y = 0.8765$ measured to the nearest number of d.p indicated
   a) State the maximum possible error in x and y
   b) Determine the maximum possible error in xy
   c) Find the limit within which the product xy lies, correct to 4 dp

   An($(a)\ \Delta x = 0.0005, \Delta y = 0.00005,$ $(b) 0.000572, (c)(2.3467, 2.3478)$**)**

   (ii) The radius of a circle is measured as $5.34m$ to the nearest cm .calculate the lower bound of its area, correct to 3 significant figures. **An(89.4m²)**

5. (a) Determine the maximum absolute error in $\frac{\sqrt{z}}{x^2y^2}$ given that $x = 2.4, y = 5.4$ and $z = 1.8$ all numbers rounded.
   (b) The variable v, r and h are related by the formula. $V = \frac{r^2}{3h} + 5$
   In an experiment, the value of r and h were found to be 4.1 and 40.8 respectively .calculate the lower and upper limits of v correct to 3 dp
   An((a)0.000123 (b) (5.134,5.141))

6. (a) A company had a capital of $shs. 500 million$, the profit in certain year was $shs\ 25.8 million$ in section A of the company and $shs\ 14.56\ million$ in $section\ B$ of the company .There was possible error of 5% in $section\ A$ and an 8% error in section. Find the maximum and minimum values of the total profit of the section as a percentage of the capital.

   (b) Given that $a = 42.326, b = 27.26$ and $c = -12.93$ are rounded off to the given d.p. Find its range within which the exact value of the expression , $\frac{A}{B+C}$ lies where. A, B and C are rounded off to a, b and c **An((max=8.56% , min=7.58%(b) (-2.95137,2.95576))**

7. (a) When $x = 0.8$ $e^x = 2.2255$ and $e^{-x} = 0.4493$ correct to 4d.p
   (i) Round off the value of e^x and e^{-x} to 2 d.p
   (ii)Truncate the values of e^x and e^{-x} to 2d.p
   (iii) If the maximum possible error in the value of $e^x$and e^{-x} is $\pm$ 0.0005 what are the corresponding maximum and minimum values of the quotient $\frac{e^x}{e^{-x}}$. Give your answer to 3 d.p **An((i)2.23, 0.45, (ii) 2.22, 0.44, (iii)4.954, 4.953)**

   (b) Given $x = 3, y = 12, z = 6$ all to the nearest integer, find the maximum values of
   (i) $\frac{x+y}{z}$ (ii) $\frac{xy}{z}$ (iii) $\frac{x}{y} - \frac{y}{z}$

   An((i)0.8696, (ii)7.9545, (iii)-1.4649)

8. If the error in each of the values of e^x and e^{-x} is $\pm$ 0.0005. Find the maximum and minimum values of the quotient $\frac{e^x}{e^{-x}}$ When $x = 0.04$, give your answer to 3 d.p **An(1.084 and 1.082)**

9. The numbers $a = 26.23, b = 13.18$ and $c = 5.1$ are calculated with percentage errors of 4, 3 and 2 respectively. Find the limit to two decimal places within which the exact value of the expression $ab - \frac{b}{c}$ lies. **An(319.21, 367.87)**

10. The length, width and height of water tank all rounded off to 3.65m, 2.14 and 2.5m respectively. Determine in m^3 the least and greatest amount of water the tank can contain. **An(19.066,19.992)**

11. (a) Given that the values $x = 4, y = 6$ and $z = 8$ each has been approximate to the nearest integer. Find the maximum and minimum values of
    (i) $\frac{y}{x}$ (ii) $\frac{z-x}{y}$ (iii) $(x + y)z$

    An((i)1.85714, 1.22222, (ii)0.909091,0.46154 (iii)93.5, 67.5)

    (b) Given $x = 2.2255, y = 0.449$ correct to the given no of d.p.state the maximum possible error in the value of x and y hence determine, the
    (i) Absolute error
    (ii) Limit within which the value of quotient $\frac{x}{y}$ lies giving your ans to 2d.p.

    An($(i)\ \Delta x = 0.00005, \Delta y = 0.0005, 0.01, (ii)(4.95, 4.96)$**)**

12. (a) Given that $x = 2.5, y = 14.2, z = 8.1$ all the values given correct to one d.p ,find the maximum value of
    (i) $\frac{x+y}{z^2}$ (ii) $\frac{x-y}{z}$ (iii) $\frac{1}{x} + \frac{1}{y} - \frac{1}{z}$
    Correct to 3 d.p **An((i)0.259, (ii)-1.478 (iii)0.358)**

(b) Given the number $x = 2.678$ and $y = 0.8765$, measured to the nearest number of d.p indicated

(i) state the maximum possible error in x and y

(ii) Determine the absolute error in xy

iii) Find the limit within which the product xy lies correct to 4 d.p

An$((i)\ \Delta x = 0.0005, \Delta y = 0.00005,$ $(ii) 0.000572, (iii)(2.3473 \pm 0.000572))$

13. Given the numbers $a = 23.037, b = 8.4658$ measured to the nearest number of d.p indicated.

(i) Determine the absolute error in $\frac{a}{b}$

(ii) Find the limits within which $\frac{a}{b}$ lies, correct to 4 d.p

An$((i)\ 0.00007513, (ii)(2.7211, 2.7213))$

14. A soda company makes an annual profit of 1080 million with a margin error of $\pm 10\%$ and an annual loss of 560 million with a margin error of $\pm 5\%$

(i) Find the range of values corresponding to his gross income

(ii) If his annual income tax is 75 million, what percentage of the gross income goes to tax, giving your answer as a range of values.

An$((i)$ 384 to 656 million,(ii)11.4% to 19.5%)

ERROR PROPAGATION

Triangular inequality

It states that $|a \pm b| \leq |\Delta a| + |\Delta b|$

(i) Addition

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

$$|e_{x+y}| = |[(x + \Delta x) + (y + \Delta y)] - (x + y)|$$

$$|e_{x+y}| = |\Delta x + \Delta y|$$

$$|e_{x+y}| \leq |\Delta x| + |\Delta y|$$

$$e_{max} = |\Delta x| + |\Delta y|$$

$$R.E_{max} = \left|\frac{\Delta x}{x + y}\right| + \left|\frac{\Delta y}{x + y}\right|$$

Alternatively

$$absolute\ error = \frac{1}{2}|max - min|$$

$$= \frac{1}{2}([(x+\Delta x)+(y+\Delta y)]-[(x-\Delta x)+(y-\Delta y)])$$

$$|e_{x+y}| = |\Delta x + \Delta y|$$

$$|e_{x+y}| \leq |\Delta x| + |\Delta y|$$

$$e_{max} = |\Delta x| + |\Delta y|$$

$$R.E_{max} = \left|\frac{\Delta x}{x + y}\right| + \left|\frac{\Delta y}{x + y}\right|$$

Examples

1. Given numbers $x = 7.824$ and $y = 2.36$ rounded to the given number of decimal places. Find the limit within which $(x + y)$lies

Solution

$\Delta x = 0.0005, \Delta y = 0.005$

$e_{x+y} = \Delta x + \Delta y = 0.0005 + 0.005 = 0.0055$

Working value= $x + y = 7.824 + 2.36 = 10.184$

$upper\ limit = 10.184 + 0.0055 = 10.1895$

$lower\ limit = 10.184 - 0.0055 = 10.1785$

Alternatively

$(x + y)_{max} = 7.8245 + 2.365 = 10.1895$

$(x + y)_{min} = 7.8235 + 2.355 = 10.1785$

2. If $x = 4.95$ and $y = 2.2$ are each rounded off to the given number of decimal places. Calculate;

(i) The percentage error in $x + y$

(ii) the limit within which $(x + y)$ is expected to lies. Give your answer to two decimal places

Solution

(i) $\Delta x = 0.005, \Delta y = 0.05$

$$\%\ error = \left[\left|\frac{\Delta x}{x + y}\right| + \left|\frac{\Delta y}{x + y}\right|\right] x100\%$$

$$\%\ error = \left[\left|\frac{0.005}{4.95 + 2.2}\right| + \left|\frac{0.05}{4.95 + 2.2}\right|\right] x100\%$$

$$\%\ error = 0.769$$

Alternatively

Working value= $x + y = 4.95 + 2.2 = 7.15$

$e_{x+y} = \Delta x + \Delta y = 0.005 + 0.05 = 0.055$

$$\%\ error = \frac{0.055}{7.15} x100\% = 0.769$$

(ii) $upper\ limit = 7.15 + 0.055 = 7.21$

$lower\ limit = 7.15 - 0.055 = 7.10$

Alternatively

$(x + y)_{max} = 4.955 + 2.25 = 7.21$

$(x + y)_{min} = 4.945 + 2.15 = 7.10$

(ii) Subtraction

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

$$e_{x-y} = |[(x+\Delta x) - (y+\Delta y)] - (x-y)|$$
$$|e_{x-y}| = |\Delta x - \Delta y|$$
$$|e_{x-y}| \leq |\Delta x| + |\Delta y|$$
$$e_{max} = |\Delta x| + |\Delta y|$$
$$R.E_{max} = \left|\frac{\Delta x}{x-y}\right| + \left|\frac{\Delta y}{x-y}\right|$$

Alternatively

$$absolute\ error = \frac{1}{2}|max - min|$$
$$= \tfrac{1}{2}([(x+\Delta x)-(y-\Delta y)]-[(x-\Delta x)-(y+\Delta y)])$$
$$|e_{x-y}| = |\Delta x + \Delta y|$$
$$|e_{x-y}| \leq |\Delta x| + |\Delta y|$$
$$e_{max} = |\Delta x| + |\Delta y|$$
$$R.E_{max} = \left|\frac{\Delta x}{x-y}\right| + \left|\frac{\Delta y}{x-y}\right|$$

Examples

1. Given numbers $x = 6.375$ and $y = 4.46$ rounded off to the given number of decimal places. Find the limit within which $(x-y)$lies

Solution

$\Delta x = 0.0005, \Delta y = 0.005$

$|e_{x-y}| = |\Delta x| + |\Delta y| = |0.0005| + |0.005| = 0.0055$

Working value= $x - y = 6.375 - 4.46 = 1.915$

$upper\ limit = 1.915 + 0.0055 = 1.9205$

$lower\ limit = 1.915 - 0.0055 = 1.9095$

Alternatively

$(x-y)_{max} = 6.3755 - 4.455 = 1.9205$

$(x-y)_{min} = 6.3745 - 4.465 = 1.9095$

2. If $x = 1.563$ and $y = 9.87$ are each rounded off to the given number of decimal places. Calculate;
   (i) The percentage error in $x - y$
   (ii) the limit within which $(x-y)$ is expected to lies. Give your answer to three decimal places

Solution

(i) $\Delta x = 0.0005, \Delta y = 0.005$

$$\%\ error = \left[\left|\frac{\Delta x}{x-y}\right| + \left|\frac{\Delta y}{x-y}\right|\right] x100\%$$
$$\%\ error = \left[\left|\frac{0.0005}{1.563-9.87}\right| + \left|\frac{0.005}{1.563-9.87}\right|\right] x100\%$$
$$\%\ error = 0.0662$$

Alternatively

Working value= $x - y = 1.563 - 9.87 = -8.307$

$|e_{x-y}| = |\Delta x| + |\Delta y| = |0.0005| + |0.005| = 0.0055$

$$\%\ error = \frac{0.0055}{|-8.307|} x100\% = 0.0662$$

(ii) $upper\ limit = -8.307 + 0.0055 = -8.302$

$lower\ limit = -8.307 - 0.0055 = -8.313$

Alternatively

$(x-y)_{max} = 1.5635 - 9.865 = -8.302$

$(x-y)_{min} = 1.5625 - 9.875 = -8.313$

(iii) Multiplication

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

$$|e_{xy}| = |[(x+\Delta x)(y+\Delta y)] - (xy)|$$
$$|e_{xy}| = |xy + y\Delta x + x\Delta y + \Delta x\Delta y - xy|$$

Since Δx and Δy are very small, then $\Delta x\Delta y \approx 0$

$$|e_{xy}| = |y\Delta x + x\Delta y|$$
$$|e_{xy}| \leq |y\Delta x| + |x\Delta y|$$
$$e_{max} = |y\Delta x| + |x\Delta y|$$
$$R.E_{max} = \left|\frac{y\Delta x}{xy}\right| + \left|\frac{x\Delta y}{xy}\right|$$
$$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$$

Alternatively

$$absolute\ error = \frac{1}{2}|max - min|$$
$$= \frac{1}{2}([(x+\Delta x)(y+\Delta y)] - [(x-\Delta x)(y-\Delta y)])$$
$$|e_{xy}| = |y\Delta x + x\Delta y|$$
$$|e_{xy}| \leq |y\Delta x| + |x\Delta y|$$
$$e_{max} = |y\Delta x| + |x\Delta y|$$
$$R.E_{max} = \left|\frac{y\Delta x}{xy}\right| + \left|\frac{x\Delta y}{xy}\right|$$
$$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$$

Examples

1. Given numbers $x = 6.375$ and $y = 4.46$ rounded off to the given number of decimal places. Find the limit within which (xy)lies

Solution

$\Delta x = 0.0005, \Delta y = 0.005$ $\quad |e_{xy}| = |y\Delta x| + |x\Delta y|$

$|e_{xy}| = |4.46x0.0005| + |6.375x0.005| = 0.0341$

Working value= $xy = 6.375x4.46 = 28.4325$

$upper\ limit = 28.4325 + 0.0341 = 28.4666$

$lower\ limit = 28.4325 - 0.0341 = 28.3984$

Alternatively

$(xy)_{max} = 6.3755x4.465 = 28.4666$

$(xy)_{min} = 6.3745x4.455 = 28.3984$

2. If $x = 1.563$ and $y = 9.87$ are each rounded off to the given number of decimal places. Calculate;
   (i) The percentage error in xy
   (ii) the limit within which (xy) is expected to lies. Give your answer to three decimal places

Solution

(i) $\Delta x = 0.0005, \Delta y = 0.005$

$$\%\ error = \left[\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|\right]x100\%$$

$$\%\ error = \left[\left|\frac{0.0005}{1.563}\right| + \left|\frac{0.005}{9.87}\right|\right]x100\%$$

$$\%\ error = 0.0826$$

Alternatively

Working value= $x = 1.563x9.87 = 15.4268$

$$|e_{xy}| = |y\Delta x| + |x\Delta y|$$

$$|e_{xy}| = |9.87x0.0005| + |1.563x0.005| = 0.0128$$

$$\%\ error = \frac{0.0128}{15.4268}x100\% = 0.0826$$

(ii) $upper\ limit = 15.4268 + 0.0128 = 15.440$

$lower\ limit = 15.4268 - 0.0128 = 15.414$

Alternatively

$(xy)_{max} = 1.5635x9.875 = 15.440$

$(xy)_{min} = 1.5625x9.865 = 15.414$

(iv) Division

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

$$\left|e_{x/y}\right| = \left|\frac{(x+\Delta x)}{(y+\Delta y)} - \left(\frac{x}{y}\right)\right|$$

$$\left|e_{x/y}\right| = \left|\frac{xy + y\Delta x - x\Delta y - xy}{y^2 + y\Delta y}\right|$$

$$\left|e_{x/y}\right| = \left|\frac{y\Delta x - x\Delta y}{y^2\left(1+\frac{\Delta y}{y}\right)}\right|$$

Since Δx and Δy are very small, then $\frac{\Delta y}{y} \approx 0$

$$\left|e_{x/y}\right| = \left|\frac{y\Delta x - x\Delta y}{y^2}\right|$$

$$\left|e_{x/y}\right| \leq \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$e_{max} = \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$R.E_{max} = \frac{|y\Delta x| + |x\Delta y|}{|y^2|} \div \frac{x}{y}$$

$$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$$

Alternatively

$$absolute\ error = \frac{1}{2}|max - min|$$

$$= \frac{1}{2}\left|\frac{(x+\Delta x)}{(y-\Delta y)} - \frac{(x-\Delta x)}{(y+\Delta y)}\right|$$

$$e_{x/y} = \left|\frac{x\Delta y + y\Delta x}{y^2 - \Delta y^2}\right|$$

Since Δx and Δy are very small, then $\Delta y^2 \approx 0$

$$\left|e_{x/y}\right| = \left|\frac{y\Delta x + \Delta y}{y^2}\right|$$

$$\left|e_{x/y}\right| \leq \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$e_{max} = \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$R.E_{max} = \frac{|y\Delta x| + |x\Delta y|}{|y^2|} \div \frac{x}{y}$$

$$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$$

Examples

1. Given numbers $x = 5.794$ and $y = 0.28$ rounded off to the given number of decimal places. Find the limit within which $\left(\frac{x}{y}\right)$ lies

Solution

$\Delta x = 0.0005, \Delta y = 0.005$

$$\left|e_{x/y}\right| = \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$\left|e_{x/y}\right| = \frac{|0.28x0.0005| + |5.794x0.005|}{|0.28^2|} = 0.3713$$

Working value= $x/y = 5.794/0.28 = 20.6929$

$upper\ limit = 20.6929 + 0.3713 = 21.0642$

$lower\ limit = 20.6929 - 0.3713 = 20.3198$

Alternatively

$(x/y)_{max} = 5.7945/0.275 = 21.0709$

$(x/y)_{min} = 5.7935/0.285 = 20.3281$

2. If $x = 7.37$ and $y = 2.00$ are each rounded off to the given number of decimal places. Calculate;
   (i) The percentage error in x/y
   (ii) the limit within which (x/y) is expected to lies. Give your answer to three decimal places

Solution

(i) $\Delta x = 0.005, \Delta y = 0.005$

$$\%\ error = \left[\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|\right]x100\%$$

$$\%\ error = \left[\left|\frac{0.005}{7.37}\right| + \left|\frac{0.005}{2.00}\right|\right]x100\% = 0.318$$

Alternatively

$$\left|e_{x/y}\right| = \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$\left|e_{x/y}\right| = \frac{|2.00x0.005| + |7.37x0.005|}{|2.00^2|} = 0.0117$$

Working value= $x/y = 7.37/2.00 = 3.685$

$$\%\ error = \frac{0.0117}{3.685}x100\% = 0.318$$

(ii) $upper\ limit = 3.685 + 0.0117 = 3.697$

$lower\ limit = 3.685 - 0.0117 = 3.673$

Alternatively

$(x/y)_{max} = 7.375/1.995 = 3.697$

$(x/y)_{min} = 7.365/2.005 = 3.673$

3. (a) The numbers A, B and C are approximated by a, b and c with errors e_1, e_2 and e_3 respectively. Show that the maximum possible relative error in taking the approximation of $\frac{A}{B+C}$ as $\frac{a}{b+c}$ is $\left|\frac{e_1}{a}\right| + \left|\frac{e_2}{b+c}\right| + \left|\frac{e_3}{b+c}\right|$
   (b) Given that $a = 40.235$, $b = 14.15$ and $c = 2.45$ are rounded off to the given decimal places. Find the range within which the given range if the approximation $\frac{A}{B+C}$ lies

Solution

(a) $b + c = d$ with an error of e_d such that

$e_d = e_2 + e_3$

$$e = \left(\frac{a + e_1}{d + e_d}\right) - \frac{a}{d}$$

$$e = \frac{ad + de_1 - ae_d - ad}{d^2 + de_d} = \frac{de_1 - ae_d}{d^2\left(1 + \frac{e_d}{d}\right)}$$

Since e_1 and e_d are very small, then $\frac{e_d}{d} \approx 0$

$$|e| = \left|\frac{de_1 - ae_d}{d^2}\right|$$

$$|e| \leq \frac{|de_1| + |ae_d|}{|d^2|}$$

$$e_{max} = \frac{|de_1| + |ae_d|}{|d^2|}$$

$$R.E_{max} = \frac{|de_1| + |ae_d|}{|d^2|} \div \frac{a}{d} = \left|\frac{e_1}{a}\right| + \left|\frac{e_d}{d}\right|$$

(b) $R.E_{max} = \left|\frac{e_1}{a}\right| + \left|\frac{e_2}{b+c}\right| + \left|\frac{e_3}{b+c}\right|$

$$e_{max} = \left[\left|\frac{e_1}{a}\right| + \left|\frac{e_2}{b+c}\right| + \left|\frac{e_3}{b+c}\right|\right]x\frac{a}{b+c}$$

$$e_{max} = \left[\left|\frac{0.0005}{40.235}\right| + \left|\frac{0.005}{14.15 + 2.45}\right| + \left|\frac{0.005}{14.15 + 2.45}\right|\right]x\frac{40.235}{14.15 + 2.45}$$

$e_{max} = 0.00149$

Range= $\frac{40.235}{14.15+2.45} \pm 0.00149$

$[2.4223, 2.4252]$

ERROR IN FUNCTIONS

Given a function $f(x)$ with a maximum possible error Δx.

Absolute error, $|e| = |\Delta x|f^1(x)$

Maximum possible relative error, $R.E = \frac{|\Delta x|f^1(x)}{f(x)}$

Example

1. Find the absolute error and maximum relative error in each of the functions
(i) $y = x^4$ (ii) $y = x^{\frac{3}{2}}$ (iii) $y = sinx$

Solution

(i) $|e| = |\Delta x|f^1(x) = 4x^3|\Delta x|$

$R.E = \frac{4x^3|\Delta x|}{x^4} = \frac{4|\Delta x|}{x}$

(ii) $|e| = |\Delta x|f^1(x) = \frac{3}{2}x^{\frac{1}{2}}|\Delta x|$

$R.E = \frac{\frac{3}{2}x^{\frac{1}{2}}|\Delta x|}{x^{\frac{3}{2}}} = \frac{3}{2}\frac{|\Delta x|}{x}$

(iii) $|e| = |\Delta x|f^1(x) = cosx\,|\Delta x|$

$R.E = \frac{cosx\,|\Delta x|}{sinx} = |\Delta x||cotx|$

2. Given that the error in measuring an angle is $0.4°$. Find the maximum possible error and relative error in $tanx$ if $x = 60°$

Solution

$|e| = |\Delta x|f^1(x) = (1 + tan^2x)\,|\Delta x|$

$|e| = (1 + tan^2 60)\left|\frac{0.4}{180}\pi\right| = 0.0280$

$R.E = \frac{0.0280}{tan60} = 0.0162$

Error in a function that has more than one variable

Given a function $f(x, y)$ with a maximum possible error Δx and Δy respectively.

Absolute error, $|e| = |\Delta x|f^1(x) + |\Delta y|f^1(y)$

Maximum possible relative error, $R.E = \frac{|\Delta x|f^1(x)+|\Delta y|f^1(y)}{f(x,y)}$

Examples

1. Two decimal numbers X and Y are recorded off to give x and y with errors Δx and Δy respectively. Show that the maximum relative error recorded in x^4y is given by $4\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Solution

$|e| = |\Delta x f^1(x) + \Delta y f^1(y)|$

$|e| = |\Delta x 4x^3 y + \Delta y x^4|$

$|e| \le 4|x^3y||\Delta x| + |x^4||\Delta y|$

$|e_{max}| = 4|x^3y||\Delta x| + |x^4||\Delta y|$

$R.E = \frac{4|x^3y||\Delta x| + |x^4||\Delta y|}{x^4y}$

$R.E = 4\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

2. Two decimal numbers x and y are recorded off to give X and Y with errors Δx and Δy respectively. Show that the maximum relative error recorded is approximating $x^{\frac{1}{2}}y^3$ by $X^{\frac{1}{2}}Y^3$ is given by
$\frac{1}{2}\left|\frac{\Delta x}{X}\right| + 3\left|\frac{\Delta y}{Y}\right|$

Solution

$|e| = |\Delta x f^1(x) + \Delta y f^1(y)|$

$|e| = \left|\Delta x \frac{1}{2}X^{-\frac{1}{2}}Y^3 + \Delta y 3X^{\frac{1}{2}}Y^2\right|$

$|e| \le \frac{1}{2}\left|\frac{1}{2}X^{-\frac{1}{2}}Y^3\right||\Delta x| + 3\left|X^{\frac{1}{2}}Y^2\right||\Delta y|$

$|e_{max}| = \frac{1}{2}\left|X^{-\frac{1}{2}}Y^3\right||\Delta x| + 3\left|X^{\frac{1}{2}}Y^2\right||\Delta y|$

$R.E = \frac{\frac{1}{2}\left|X^{-\frac{1}{2}}Y^3\right||\Delta x| + 3\left|X^{\frac{1}{2}}Y^2\right||\Delta y|}{X^{\frac{1}{2}}Y^3}$

$R.E = \frac{1}{2}\left|\frac{\Delta x}{X}\right| + 3\left|\frac{\Delta y}{Y}\right|$

3. The volume of a cylinder is given by $v = \pi r^2 h$. The radius r and height h of the cylinder are measured with corresponding errors Δr and Δh respectively. Find the absolute error and range of values within which v lies if $r = 4.6cm$ and height $h = 15.8cm$

Solution

$|e| = |\Delta r|f^1(r) + |\Delta h|f^1(h)$

$|e| = 2\pi rh|\Delta r| + \pi r^2|\Delta h|$

$|e| = 2\pi x4.6x15.8|0.05| + \pi x4.6^2|0.05|$

$|e| = 26.157$

Working value= $\pi r^2 h$

Working value= $\pi x4.6^2x15.8 = 1050.322$

Range = 1050.322 ± 26.157

Exercise 9b

1. The numbers x and y approximated by X and Y with errors of Δx and Δy respectively.
   (a) Show that the maximum relative error in XY is $\left|\frac{\Delta x}{X}\right| + \left|\frac{\Delta y}{Y}\right|$
   (b) If $x = 4.95$ and $y = 2.013$ are each rounded off to the given number of decimal places, calculate the
   (i) Percentage error in xy
   (ii) Limits within which xy is expected to lie. Give your answer to three decimal places.
2. The numbers X and Y are approximated with possible errors of Δx and Δy respectively.
   (a) Show that the maximum absolute error in the quotient $\frac{X}{Y}$ is given by $\frac{|Y||\Delta x|+|X||\Delta y|}{Y^2}$
   (b) Given that $x = 2.68$ and $y = 0.9$ are rounded to the given number of decimal places. Find the interval within which the exact value of $\frac{X}{Y}$ is expected to lie. **An** $[2.8158, 3.1588]$
3. The numbers x and y are approximated with possible errors of Δx and Δy respectively.
   (a) Show that the maximum absolute error in the quotient $\frac{x}{y}$ is given by $\frac{|y||\Delta x|+|x||\Delta y|}{y^2}$
   (b) Find the interval within which the exact value of $\frac{2.58}{3.4}$ is expected to lie.
4. The numbers x and y are approximated with possible errors of e_x and e_y respectively.
   (a) Show that the maximum absolute error in the quotient $\frac{x}{y}$ is given by $\left|\frac{x}{y}\right|\left\{\left|\frac{e_x}{x}\right| + \left|\frac{e_y}{y}\right|\right\}$
   (b) A car covers a distance of $75.6km$ with an average speed of $36.5kmh^{-1}$, where the quantities are rounded off to 1 decimal point, find the range within which the time the car takes in travel lies.
5. (a) Two decimal numbers x and y are recorded off to give X and Y with errors E_1 and E_2 respectively. Show that the maximum relative error recorded is approximately x^2y by X^2Y is given by $2\left|\frac{E_1}{X}\right| + \left|\frac{E_2}{Y}\right|$
   (b) Given that Y and Z are measured with possible errors Δy and Δz respectively. Show that the relative error in the product YZ is $\frac{Z|\Delta y|+Y|\Delta z|}{YZ}$
6. (a) Two positive real numbers N_1 and N_2 are rounded off to give n_1 and n_2 respectively. Determine the maximum relative error in using n_1n_2 for N_1N_2. State any assumptions made.
   (b) If $N_1 = 2.765$, $N_2 = 0.72$, determine the range within which the exact values of:
   (i) $N_1N_2(N_1 - N_2)$ (ii) $\frac{N_2-N_1}{N_1N_2}$
   are expected to lie. Give your answers to three decimal places **ANS** $(i)\ (4.03, 4.111)$, $(ii)\ (-1.037, -1.017)$
7. (a) The numbers X and Y were estimated with maximum possible errors of Δx and Δy respectively. Show that the percentage relative error in XY is $\left[\left|\frac{\Delta x}{X}\right| + \left|\frac{\Delta y}{Y}\right|\right]x100$
   (b) Obtain the range of values with in which the exact value of 3.551×2.71635 lies
   ANS $(9.6444, 9.6471)$
8. (a). Two numbers A and B are approximated by a and b with errors x and y respectively. Show that the maximum percentage error in approximating $\frac{A}{B}$ is given by $\left[\left|\frac{x}{a}\right| + \left|\frac{y}{b}\right|\right]x100\%$. Hence find the maximum percentage error in $\frac{3.25^2}{4.562}$
   (b) A cylindrical pipe has a radius of 2.5cm measured to the nearest unit. If the relative absolute error made in calculating its volume is 0.125, find the absolute relative error made in measuring its height
   ANS $((a)\ 0.32, (b) 0.000211)$
9. The volume of a cone is given by $v = \frac{1}{3}\pi r^3 h$. The radius r and height h of the cone are measured with corresponding errors Δr and Δh respectively show that the maximum possible relative error in the volume is $3\left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta h}{h}\right|$
10. Given that $A = |x||y|sin\theta$
    (a) Deduce that the maximum possible relative error in A is given by $\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right| + cot\theta|\Delta\theta|$ where $\Delta x, \Delta y$ and $\Delta\theta$ are small numbers compared to x, y and θ respectively
    (b) find the error made in the area, if x and y are measured with errors of ± 0.05 and angle with an error of $\pm 0.5^0$ given that $x = 2.5cm$, $y = 3.4cm$ and $\theta = 30^o$
    ANS (0.212)

Miscellaneous exercises 1

1. A value of $P = 673.16$ was obtained in a certain experiment. Given that the relative error in the measurement of this value is 0.01%, find the limits within which the value of P is expected to lie.
2. The relative error obtained in determining the value of T=873.16 is 0.02%, find
   (i) The error in the measurement of this value
   (ii) The value within which T lies
3. Given that $T = 29.7°C$ and $M = 48kg$
   (i) Write down the possible errors in each value
   (ii) Write down the ranges within which each value lie
4. Given two parcels, with A having a mass of 2kg and B having a mass of 6kg, each mass being rounded off to the given decimal places. Find the interval within which the total mass lies
5. A student measured the length and the breadth of a rectangular sheet of iron as 3.6m and 2.3m respectively.
   (i) Write down the maximum possible error in each measurement
   (ii) Find the limits within which the area of the sheet lies
6. Given that $M = \frac{4.05+2.023}{5.67-4.9312}$
   (i) Write down the maximum possible error in each of the values above
   (ii) Find the range of values within M lies
   (iii) Estimate the maximum percentage error in M
7. Find the interval within which 4^x is expected to lie, if the measured of $x = 2.4$
8. Given $A = 3, B = 12, C = 16$ all to the nearest integer, find the maximum values of:
   (i) $\frac{A+C}{B}$ (ii) $\frac{AB}{C}$ (iii) $\frac{A}{B} - \frac{B}{C}$
9. Three adjacent blocks of sugar cane with areas 400 acres, 600acres, 900 acres to the nearest fifty acres. What
   (i) The possible error in each measurement
   (ii) The greatest total area of the blocks
   (iii) The least possible area of the three blocks
10. Find the range of values within which $\frac{0.38}{4.28} + \frac{030}{2.14}$ lies, if the numbers are rounded off to the given number of decimal place.
11. Given that $X = 2.7654, Y = 3.80$, state the maximum possible error in X and Y, hence find the interval within which the following are expected to lie
    (i) $X + Y$ (ii) $X - Y$ (iii) $\frac{X}{Y}$
    (iv) $\frac{X+Y}{XY}$ (v) $X\left(Y - \frac{X}{Y}\right)$
12. The numbers X and Y are measured with possible errors of ΔX and ΔY respectively.
    (a) Show that the maximum relative error in XY is $\frac{|\Delta X|Y+|\Delta Y|X}{XY}$
    (b) If $X = 6.42$ and $Y = 29.3$ are each rounded off to the given number of decimal places, calculate the
    (i) Write down the maximum possible error in X and Y
    (ii) Find the interval in which the product XY lies.
13. Find the range of values within which $\frac{1.362(7.54-13.2)}{4.7}$ lies, if the numbers are rounded off to the given number of decimal place.
14. Find the maximum absolute error in the expression $\frac{\sqrt{A}}{B^2C^3}$ given that $A = 2.8, B = 6.4$ and $C = 3.4$ all rounded off
15. Given that $Z = |x||y|sin\theta$
    (a) Deduce that the maximum possible relative error in Z is given by $\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right| + cot\theta|\Delta\theta|$ where $\Delta x, \Delta y$ and $\Delta\theta$ are small numbers compared to x, y and θ respectively
    (b) find the maximum percentage relative error in Z, given that $x = 5.5cm, y = 6.8cm$ and $\theta = 45°$
16. Given that the area of a triangle whose adjacent sides are of size a and b, and the angle between the sides is θ $A = |a||b|sin\theta$ given that $a = 2.5cm, b = 3.4cm$ and $\theta = 30°$
    (a) Write down the possible errors in each measurement
    (b) Error made in the area
    (c) Value within which the area is expected to lie,
17. Find the range of values within which the exact value of $2.6954\left(4.6006 - \frac{16.175}{0.82}\right)$ lies, if the numbers are rounded off to the given number of decimal place.
18. Given that $A = 2.5, B = 1.71, C = 16.01$, state the maximum possible error in A, B and C, hence find the limits within which the following are expected to lie
    (i) $\frac{A+C}{B}$ (ii) $\frac{AB}{C}$
    Give your answer to 2 decimal places

19 The numbers x and y are approximated by A and B with possible errors of e_1 and e_2 respectively.

(a) Show that the absolute relative error in the quotient $\frac{x}{y}$ is given by $\left|\frac{e_1}{A}\right| + \left|\frac{e_2}{B}\right|$

(b) If $A = 4.67$ and $B = 1.813$ are each rounded off to the given number of decimal places, calculate the

(i) Write down the maximum possible error in A and B

(ii) Find the absolute relative error in the quotient $\frac{4.67}{1.813}$

(iii) Find the limit within which the quotient $\frac{4.67}{1.813}$ lies

20. Given that $A = 3.3366, B = 0.559$, state the maximum possible error in A and B, hence find

(i) Absolute error in the quotient $\frac{A}{B}$

(ii) the limits within which $\frac{A}{B}$ is expected to lie

Give your answer to 3 decimal places

21. The numbers M and N are approximated with possible errors of e_1 and e_2 respectively.

(a) Show that the maximum absolute error in the quotient MN is given by $\frac{|e_1|N+|e_2|M}{MN}$

(b) Given that $M = 6.43, N = 37.2$, write down the maximum possible error in M and N. Hence find the interval in which

(i) Product MN lies

(ii) The quotient $\frac{M}{N}$ lies

Solutions

1. An $P_{lower} = 673.09, P_{lupper} = 673.23$

2. **An**$(|e| = 0.175, [873.985, 873.335]$

3. An$((i)|\Delta T| = 0.05, |\Delta M| = 0.5,$
$(ii)[29.65, 29.75], [47.5, 48.5]$

4. An$((i)[7.0, 9.0]$, **5. An**$((i)|\Delta l| = 0.05, |\Delta w| = 0.05$
$(ii)\ A_{lower} = 7.99, A_{lupper} = 8.58$

6. An$(\ (ii)[8.16, 8.28]\ (iii) = 0.73\%$

7. **An** $[25.93, 29.79], or\ [25.99, 29.86]$

8. An$(\ (i) = 1.739,\ (ii) = 2.823, (iii) = -0.393$

9. An$(\ (i) = 50\ acres,\ (ii) = 2050, (iii) = 1750$

10. An $[0225, 0.233]$,

11. An$(|\Delta X| = 0.00005, |\Delta Y| = 0.005, (i)[6.560, 6.570]$
$(ii)[-1.0397, -1.0296](iii)[0.727, 0.729]$
$(iv)[0.623, 0.626](v)[8.479, 8.514]$

12. An$(|\Delta X| = 0.005, |\Delta Y| = 0.05, (i)[6.560, 6.570]$,

13. An$[-1.675, -1.607]$, 14. **An** $7.15x10^{-5}$

15. An 2.25% **16. An**$((i)|\Delta a| = 0.05, |\Delta b| = 0.05,$
$(ii) = 0.211, (iii)\ limit = 4.0382, 4.4618$

17. An$[-41.097, -40.443]$,

18. An $(i) limits = 10.76, 10.89, (ii)\ limits = 0.26, 0.2$

19. An$((i)|\Delta A| = 0.005, |\Delta B| = 0.0005,$
$(ii) = 0.00347, (iii)\ limit = 2.572, 2.579$

20. An$(|\Delta A| = 0.00005, |\Delta B| = 0.0005,$
$(i) = 0.053, (ii)\ limit = 5.916, 6.022$

22. **An**$(|\Delta M| = 0.005, |\Delta N| = 0.05,$

23. $(i) = [238.69, 239.70], (ii) = [0.172, 0.173]$

LINEAR INTERPOLATION AND EXTRAPOLATION

(a) Linear interpolation

This deals with computation of values that lie within given values

Examples

1. The table below shows the values of a function f(x)

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Find the value of

(i) $f(1.88)$ (ii) X corresponding to $f(x) = 0.4$

Solution

(i)

1.8	1.88	2.0
0.532	y	0.484

$$\frac{y-0.532}{1.88-1.8} = \frac{0.484-0.532}{2.0-1.8}$$

$$y = 0.513$$

(ii)

2.2	X_0	2.4
0.436	0.4	0.384

$$\frac{X_0-2.2}{0.4-0.436} = \frac{2.4-2.2}{0.384-0.436}$$

$$X_0 = 2.34$$

2. Given the table below

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Using linear interpolation find

(i) $f(x)$ when $x = 10.15$ (ii) $f^{-1}(2.02)$

Solution

(i)

10	10.15	11
2.42	y	2.18

$$\frac{y-2.42}{10.15-10} = \frac{2.18-2.42}{11-10}$$

$$y = 2.384$$

(ii)

11	X_0	12
2.18	2.02	1.92

$$\frac{X_0-11}{2.02-2.18} = \frac{12-11}{1.92-2.18}$$

$$X_0 = 11.62$$

3. Given the table below

x^0	40.0^0	40.4^0	40.8^0	50.4^0
$\sin x^0$	0.6428	0.6481	0.6534	0.7705

Find

(i) $\sin 40.5°$ (ii) $\sin^{-1} 0.6445$

Solution

(i)

40.4^0	40.5^0	40.8^0
0.6481	y	0.6534

$$\frac{y - 0.6481}{40.5 - 40.4} = \frac{0.6534 - 0.6481}{40.8 - 40.4}$$

$$y = 0.6494$$

(ii)

40.0^0	X_0	40.4^0
0.6428	0.6445	0.6481

$$\frac{X_0 - 40}{0.6445 - 0.6428} = \frac{40.4 - 40}{0.6481 - 0.6428}$$

$$X_0 = 40.13$$

(b) Linear extrapolation

This deals with computation of values that lie outside given values

Given the table below

x	2.2	2.6	3.1
x^3	10.648	17.576	29.791

Find 3.4^3

Solution

2.6	3.1	3.4
17.576	29.791	y

$$\frac{y - 29.791}{3.4 - 3.1} = \frac{29.791 - 17.576}{3.1 - 2.6}$$

$$y = 37.12$$

Exercise 10a

1. The table below is an extract from the table of $\cos x$. **Uneb 1991 No.2 b**

	0'	10'	20'	30'	40'	50'
80^0	0.1736	0.1708	0.1679	0.1650	0.1622	0.1593

Use linear interpolation to determine

(i) $\cos 80°36'$ **(ii)** $\cos^{-1}(0.1685)$. **An 0.1633,** $80°18'$

2. The table below shows variation of temperature with time in a certain experiment **Uneb 1999 No.5**

Time, (s)	0	120	240	360	480	600
Temperature, (°C)	100	80	75	65	56	48

Use linear interpolation to determine

(i) Value of $°C$ corresponding to $400s$

(ii) Time at which the temperature is $77°C$. **An** $62°C$, $192s$

3. The table below shows the values of a function ln(x) for given values of x **Uneb 2000 No.2**

X	1.4	1.5	1.6	1.7
lnx	0.3365	0.4055	0.4700	0.5306

Using linear interpolation or extrapolation, find

(i) $ln(1.66)$

(ii) The value of x corresponding to $ln(x) = 0.400$. **An** 0.5064, 1.492

4. The table below shows variation of temperature with time in a certain experiment **Uneb 2001 No.2**

Time, T(s)	0	10	15	20	30
Temperature, θ°C	80	70.2	65.8	61.9	54.2

Use linear interpolation to determine

(i) Value of $\theta°C$ corresponding to $T = 18s$

(ii) Time, T at which the temperature is $\theta = 60°C$. **An** $63.5°C$, $22.5s$

5. Given the table below **Uneb 2002 No.3**

x	-1.0	-0.5	-1.4
y	-1.0	-2.2	-3.7

Using linear interpolation or extrapolation to find

(i) y when $x = 0.5$ **An (-4.6),** (ii) x when $y = -4.5$ **An (0.458)**

6. In an examination, scaling is done such that candidate A who had originally scored 35% gets 505 and candidate B with 40% gets 65%, determine the original mark for candidate C whose new mark is 80% **Uneb 2004 No.3 An (45%)**

7. The table below is an extract of $log_{10} x$ **Uneb 2004 No.3**

x	80.00	80.20	80.50	80.80
$log_{10}x$	1.9031	1.9042	1.9058	1.9074

Using linear interpolation find

(i) $log_{10} 80.759$

(ii) The number whose logarithm is 1.90388 **An ((i) 1.9072, (ii) 80.14)**

8. The table below is an extract from the table of $sec x$. **Uneb 2005 No.2**

$X = 60^0$	0'	12'	24'	36'	48'
secx	2.0000	2.0122	2.0245	2.0371	2.0498

Use linear interpolation to determine

(ii) $sec\ 60°15$

(iii) angle whose secant is 2.0436 **An 2.0307,** $60°42'$

9. The table below shows the values of a function f(x) for given values of x **Uneb 2007 No.11a**

x	2	3	4	5
f(x)	3.88	5.11	8.14	11.94

Using linear interpolation or extrapolation, find

(i) $f(2.15)$

(ii) The value of x corresponding to $f(x) = 10.72$ **An 4.06,** 4.68

10. The table below shows the distance in kilometers a truck moves with a given amount of fuel in litres (l)

Distance (km)	20	28	33	42
Fuel (l)	10	13	21	24

Using linear interpolation or extrapolation, find **Uneb 2009 No.3**

(i) How far the truck can move on 27.5 l of fuel

(ii) The amount of fuel required to cover 29.8 km **An 52.5km,** $15.88l$

11. The table below shows the values of a continuous function f with respect to t **Uneb 2010 No.6**

t	0	0.3	0.6	1.2	1.6
f(t)	2.72	3.00	3.32	4.06	4.95

Using linear interpolation or extrapolation, find

(i) $f(t)$ when $t = 0.9$

(ii) The value of t corresponding to $f(t) = 4.48$. **An 3.69,** 1.48

12. The table below shows the delivery charges by a courier company. **Uneb 2011 No.3**

Mass (gm)	200	400	600
Charge(shs)	700	1200	3000

'Using linear interpolation or extrapolation, find

(i) the delivery charge of a parcel weighing 352gm

(ii) Mass of a parcel whose delivery charge is shs 3,300. **An 1080,** $633.33kg$

13. The table below shows the cost y shillings for hiring a motor cycle for a distance x kilometers.

Distance (x km)	10	20	30	40
Cost (shs.y)	2800	3600	4400	5200

Using linear interpolation or extrapolation, find **Uneb 2012 No.7**

(i) the cost of hiring the motorcycle for a distance of 45km

(ii) Distance mukasa travelled if he paid shs 4000 **An 5600,** $25km$

14. The table below shows the values of a function f(x) **Uneb 2013 No.2**

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Find the value of

(i) $f(2.08)$ (ii) X when to $f(x) = 0.5$ **An 0.465, 1.9**

15. Given the table below **Uneb 2014 No.6**

x	0	10	20	30
y	6.6	2.9	-0.1	-2.9

Using linear interpolation find

(i) y when $x = 16$ (ii) x when $y = -1$ **An ((i) 1.1, (ii) 23.2)**

16. The table below shows the values of a function f(x) for given values of x **Uneb 2015 No.6**

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Using linear interpolation or extrapolation, find

(i) $f(10.4)$

(ii) The value of x corresponding to $f(x) = 1.46$ **An ((i) 2.324, (ii) 13.769)**

17. The table below shows the values of a function f(x) for given values of x **Uneb 2016 No.6**

x	0.4	0.6	0.8
f(x)	-0.9613	-0.5108	-0.2231

Use linear interpolation to determine $f^{-1}(-0.4308)$ correct to 2 decimal places. **An 0.66**

18. The table below shows how T varies with S. **Uneb 2018 No.3**

T	-2.9	-0.1	2.9	3.1
S	30	20	12	9

Use linear interpolation/extrapolation to estimate the value of

(a) T when $S = 26$ (b) S when $T = 3.4$

An ((i) -1.75, (ii) 4.5)

Miscellaneous exercises 2

1. The table below shows the values of a function f(x)

x	1	2	3
f(x)	2	8	11

Find the value of

(i) $f(1.15)$ (iii) X when $f(x) = 6.4$

(ii) X corresponding to $f(x) = 9$ (iv) f(x) when $x = 4$

2. The table below shows the distance in cm travelled by a spider on the ceiling in four seconds of its motion

t(s)	0	1	2	3	4
d(cm)	0	5	38	68	104

Use linear interpolation to estimate

(i) Distance travelled when $t = 2.3$

(ii) The time when the distance travelled is $100cm$

3. The distance between Kajjansi and Kampala town is 20km.Seguuku, Zaana and Kibuye are 8km, 12km and 16km respectively from Kajjansi and the taxi charges are also respectively 500/=, 800/=, 1000/= and 1500/=. Nakimboowa is going to Visit her cousin brother Opio living 11km from Kajjansi

(i) Find how much she will be charged in this taxi

(ii) Suppose she had only 850/= and the taxi left her at a distance worth the money, find how far from Kampala town the taxi leaves her

4. The table below shows the values of two variables x and y

x	29	-0.1	-2.9
y	12	20	30

Use linear interpolation to find the value of

(i) x when $y = 16$ (ii) y when $x = -1$

5. The table below shows the distance, r(m) of a particle along a straight line after time intervals of 3 seconds

t(s)	0	3	6	9	12	15	18
r(m)	8.6	7.5	4.6	3.1	2.8	2.3	1.8

Use linear interpolation or extrapolation to estimate

(i) Distance travelled when $t = 11.4s$ (iii) r when $t = 21s$

(ii) The time when $r = 6.5m$ (iv) t when $r = 1.0m$

6. The table below shows the values of P and Q

P	0	8	12	20
Q	9.2	6.0	4.4	1.5

Use linear interpolation to find,

(i) Q when $P = 15$ (ii) P when $Q = 5.0$

7. The table below shows the values of;

x	6	10	15	20
t	13	25	39	56

Use linear interpolation to find,

(i) t when $x = 12$ (ii) x when $y = 48$

8. The table below shows the values of;

θ	0'	6'	12'	18'	24'	30'
sin10°	0.1736	0.1754	0.1771	0.1788	0.1805	0.1822

Use linear interpolation to find,

(i) $\sin 10°16^1$ (ii) $\sin^{-1} 0.1747$

9. The table below shows the values of a function f(x)

x	10	20	30	40
f(x)	0.1708	0.1679	0.1650	0.1622

Use linear interpolation the value of;

(i) $f(36)$ (ii) X when $f(x) = 0.1685$

10. If $f(0.120) = 1.7652$, $f(0.125) = 1.7666$, find $f(0.123)$

11. A body freely moving from rest is acted on by some variable force F(N) as shown in the table below

Distance (m)	0	4	10	15	20	25	31
Force, F(N)	5	8.0	11	12	13.6	10.5	5

Find;

(i) The force when the body has travelled a distance of 25m

(ii) The distance when a force $F = 12.8N$ is acting on the body

(iii) The force when the boy has travelled a distance of 34.7m

12. The table shows values of x and f(x)

x	0.8	0.9	1.0	1.1	1.2	1.3
f(x)	0.28	0.260	0.241	0.218	0.192	0.172

Find;

(i) The value of $f(1.07)$ (iii) The value of $f(1.4)$

(ii) The value of x when $f(x) = 0.231$

13. Given the table below

T	1	2	3	4	5	6
D	5	17	34	57	85	105

Find;

(i) The value of D when $T = 4.6$ (ii) The value of T when $D = 70$

(iii) The value of D when $T = 7$

14. Given the table below

T	0	120	240	360	480	600
θ	100	80	75	65	56	48

Find;

(i) The value of θ when $T = 3704.6$ (iii) The value of θ when $T = 720$

(ii) The value of T when $\theta = 70.2$ (iv) The value of T when $\theta = 38$

15. The table below shows the temperature (°C) at different times for a certain oven

Time (min)	0	10	20	30	40	50	60
Temp (°C)	40	80	100	130	180	200	300

Find;

(i) Time when temperature will be at 53°C

(ii) Temperature after 54 minutes (iii) Expected temperature after 70 minutes

16. The charges of sending parcels by a certain distributing company depend on the weights of the parcels. For the parcels of weight 500g, 1kg, 1.5kg, 2kg, 3kg and 5kg, the charges are 1000/=, 2000/=, 3500/=, 4000/= respectively. Estimate
    (i) What the distributor would charge for a parcel of weight 27kg
    (ii) If the sender pays 6200/= what is the weight of his parcel
17. Show that the linear extrapolation formula for approximating a value f(c), can be given by $f(c) = f(b) + \left(\frac{c-b}{b-a}\right)(f(b) - f(a))$
18. The diameter, d(mm) of an egg produced by a hen of a certain farm depends on the mass (gm) of the layer' mash ratio it is fed as shown below

Food ratio, m(gm)	200	290	330	410	440	500
Diameter, d(mm)	30.2	34.2	36.2	40.1	41.0	46.2

Assuming the egg to be spherical;
    (i) The optimum amount of the food the hen should be given if it is to produce an egg of average diameter of 38.2gm
    (ii) The radius of an egg if the food ratio supplied is 540gm
19. The table below shows the values of;

x	5	10	15
t	13	24.1	38.7

Use linear interpolation or extrapolation to find,
    (i) t when $x = 8$
    (ii) x when $t = 44$
20. The table below shows the variation of temperature (°C) with time in a certain laboratory experiment

Time (s)	0	120	240	360	480	600
Temp (°C)	100	80	75	65	56	48

Use linear interpolation or extrapolation to find;
    (i) Time when temperature will be at 76°C
    (ii) Temperature after 400 s
    (iii) Expected temperature after 620s
    (iv) Time when temperature will be at 40°C

Solutions

1. **An**$(i) = 2.9, (ii) = 2.33 (iii) = 1.73\ (iv) = 14$
2. **An**$(i) = 47, (ii) = 3.89$
3. **An**$(i) = 725/=\ \ (ii) = 7km$
4. **An**$(i)x = 1.4\ (ii)y = 23.2$
5. **An** $(iii) = 1.3m\ \ (iv) = 22.8s$
6. **An**$(i)Q = 3.3125\ (ii)P = 10.5$
7. **An**$(i)t = 30.6\ (ii)X = 6.33$
8. **An**$(i)Y = 30.6\ (ii)X = 6.33$
9. **An**$(i) = 0.1782\ (ii)y = 10°4^1$
10. **An**$(i) = 0.1633\ (ii)X = 17.93$
11. **An**$= 1.7657$
12. **An**$(i) = 12.36N, (ii) = 17.5 (iii) = 1.61N$
13. **An**$(i) = 0.2249, (ii) = 1.043 (iii) = 0.152$
14. **An(i)** $D = 73.8, (ii)T = 4.46 (iii) = 125$
15. **An** $(i)Q = 64.25, (ii)T = 297.6\ (iii)Q = 40\ \ (iv)\ T = 750$
16. **An** $(i)D = 3.25min, (ii) = 240°C\ (iii) = 400°C$
17. **An** $(i) = 3875/=, (ii) = 5.93kg$
18. **An** $(i) = 371.01gm\ (ii) = 24.46mm$
19. **An** $(i) = 19.66\ (ii) = 16.82$
20. **An**$(i) = 216s, (ii) = 62°C\ (iii) = 46.67°C\ \ (iv) = 720s$

TRAPEZIUM RULE

It is used for estimating an integral area under a curve of a continuous function over a given interval $[a, b]$

If $y = f(x)$

$$A = \int_a^b y\,dx$$

Using several strips between $x = a$ and $x = b$ of equal width, trapezium rule can be used to determine the area

$$A \approx \frac{1}{2}h[(first + last\ ordinate) + 2(sum\ of\ remaining\ ordinates)]$$

Where $h = \frac{b-a}{subinterval}$

Note:

(i) Sub-interval, sub-division and strips are the same

(ii) Sub-interval =(ordinates−1)

Examples

1. Use trapezium rule with 4 sub-intervals to estimate to 2 decimal places $\int_{0.2}^{1.0} \frac{2x+1}{x^2+x} dx$

Solution UNEB 2014 No.3

$$d = \frac{1-0.2}{4} = 0.2$$

x	$\frac{2x+1}{x^2+x}$	
0.2	5.833	
0.4		3.214
0.6		2.292
0.8		1.806
1.0	1.500	
sum	7.333	7.312

$$\int_{0.2}^{1.0} \frac{2x+1}{x^2+x} dx \approx \frac{1}{2}x0.2[7.333 + 2(7.312)]$$

$$\approx 2.20$$

2. Use trapezium rule with 4 sub-intervals to estimate to 3 decimal places $\int_0^{\frac{\pi}{2}} \cos x\, dx$

Solution

x	f(x) =cosx	
0	1.0000	
Π/8		0.9239
2Π/8		0.7071
3Π/8		0.3827
Π/2	0.0000	
sum	1.0000	2.0137

$$d = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

$$\int_0^{\frac{\pi}{2}} \cos x\, dx \approx \frac{1}{2} x \frac{\pi}{8}[1 + 2(2.01137)]$$

$$\approx 0.987$$

3. Use trapezium rule with 7 ordinates to estimate $\int_0^3 \frac{1}{1+x} dx$ correct to 3 dp **UNEB 2006 No.8**

Solution

x	$\frac{1}{1+x}$	
0	1.0000	
0.5		0.6667
1.0		0.5000
1.5		0.4000
2.0		0.3333
2.5		0.2857
3.0	0.2500	
Sum	1.2500	2.1857

$$h = \frac{3-0}{7-1} = 0.5$$

$$\int_0^3 \frac{1}{1+x} \approx \frac{1}{2}x0.5[1.25 + 2(2.1857)]$$

$$\approx 1.405$$

4. (a) use trapezium rule to estimate the integral value of $\int_2^3 \frac{x}{1+x^2} dx$ using five sub-intervals correct to 3dp.

(b) (i) find the exact value of $\int_2^3 \frac{x}{1+x^2} dx$

(ii) find the error in your estimation

(iii) Suggest how the error may be reduced

Solution

x	$\frac{x}{1+x^2}$	
2	0.40000	
2.2		0.37671
2.4		0.35503
2.6		0.33505
2.8		0.31674
3.0	0.30000	
Sum	0.70000	1.38353

(a) $h = \frac{3-2}{5} = 0.2$

$$\int_2^3 \frac{x}{1+x^2} dx \approx \frac{1}{2}x0.2[0.7 + 2(1.38353)]$$

$$\approx 0.3467$$

(b) $\int_2^3 \frac{x}{1+x^2} dx = \left[\frac{1}{2} In(1+x^2)\right]_2^3 = \frac{1}{2}(In10 - In5)$

$$= 0.3466$$

$error = |exact\ value - approximate\ value|$

$error = |0.3466 - 0.3467| = 0.0001$

(v) Error is reduced by increasing the number of sub intervals

5. (a) use trapezium rule to estimate the integral value of $\int_0^1 x^2e^x\,dx$ using five sub-intervals correct to 3dp.

(b) (i) find the exact value of $\int_0^1 x^2e^x\,dx$

(ii) find the error in your estimation.

Solution

(a) $h = \frac{1-0}{5} = 0.2$

x	x^2e^x	
0	0	
0.2		0.0489
0.4		0.2387
0.6		0.6560
0.8		1.4243
1.0	2.7183	
Sum	2.7183	2.3679

$$\int_0^1 x^2e^x\,dx \approx \frac{1}{2}x0.2[2.7183 + 2(2.3679)]$$

$$\approx 0.74541 \approx 0.745$$

(b) $\int_0^1 x^2e^x\,dx = [x^2e^x - 2xe^x + 2e^x]_0^1$

$= (1^2e^1 - 2x1e^1 + 2e^1) - (0^2e^0 - 2x0e^0 + 2e^0)$

$= 0.718$

$error = |exact\ value - approximate\ value|$

$error = |0.718 - 0.745| = 0.027$

$R.E = \frac{0.027}{0.718} = 0.038$

6. Use trapezium rule with six strips to estimate $\int_0^{\pi} xsinx\,dx$ correct to 2dp. Determine the percentage relative error in your estimation. **An(3.07,3.14,2%) UNEB 1999 No.9b**

Exercise 10b

1. Use trapezium rule to estimate the approximate value of $\int_0^1 \frac{1}{1+x^2}dx$ using 6 ordinates correct to 3 decimal places **An[0.784] UNEB 2000 No.5**

2. Use trapezium rule with six strips to estimate $\int_2^4 \frac{10}{2x+1}dx$ correct to 4dp. Determine the percentage error in your estimation and suggest how this may be reduced. **An(2.9418, 0.098%) UNEB 2001 No.9b**

3. (a) (i) Use the trapezium rule to estimate the area of $y = 5^{2x}$ between the x-axis, $x = 0$, and Using five sub intervals. Give your answer correct to 3dp. **UNEB 2004 No.12**

(ii) Find the exact value of $\int_0^1 5^{2x}\,dx$

(iii) Find the percentage error in the calculations in (a) i) and (a) ii) above

(b) Show the equation $e^x + x - 4 = 0$ has a real root between 1 and 1.2. Use the NRm to find the root of the equation correct to 3 significant figures **Ans[a)i)=7.712, ii)7.4560, iii)3.43%, b=1.07]**

4. **(I)** use the trapezium rule to estimate the area of $y = 3^x$ between the x-axis, $x = 1$ and $x = 2$ using five strips . give your answer correct to 4s.f **UNEB 2005 No.10**

(II) find the exact values of $\int_1^2 3^x\,dx$

(iii) Find the percentage error in calculation (i) and (ii) above

An((I)=5.483, (II)=5.461, (III)=0.403%)

5. (a) Show that the equation $f(x) = x^3 + 3x - 9$ has a root between $x = 1$ and $x = 2$. Using the newton raphson formula once, estimate the root of the equation rounded off to 2 s.f.

(b) Use the trapezium rule with 7ordinates to find the value of $\int_0^{\pi} \sqrt{(1 + sinx)}dx$ correct to 2dp **UNEB 2007 No.8 An [a=1.6, b=3.98]**

6. Use trapezium rule with 6 ordinates to evaluate $\int_0^1 e^{-x^2}dx$ correct to 2 decimal places **UNEB 2008 No.5 An[0.74]**

7. Use trapezium rule with 6 ordinates to estimate $\int_1^2 \frac{ln\,x}{x}dx$ correct to 3 decimal places **UNEB 2009 No.6 An [0.237]**

8. Use trapezium rule with 5 strips to approximate $\int_0^1 \frac{1}{1+x}dx$ correct to 1 decimal places **UNEB 2010 No.3 An [0.7]**

9. (a) use trapezium rule to estimate the integral value of $\int_0^{\pi/3} tanx\,dx$ using five sub-intervals correct to 3dp. **An(0.704)**

(b) (i) find the exact value of $\int_0^{\pi/3} tanx\,dx$ **An(0.693)**

(ii) find the percentage error in your estimation and suggest how to reduce the error **UNEB 2011 No.11 An(1.587%)**

10. Use trapezium rule with 4 sub-intervals to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+sinx}dx$ correct to 3 decimal places **UNEB 2012 No.3 An(1.013)**

11. Use trapezium rule with 6 ordinates to evaluate $\int_0^2 \frac{1}{1+x^2}\,dx$ correct to 3 decimal places **UNEB 2013 No.5 An(1.105)**

12. Use trapezium rule with 5 sub-intervals to evaluate $\int_2^4 \frac{5}{1+x}\,dx$ correct to 3 decimal places **UNEB 2015 No.2 An(2.559)**

13. A student Used trapezium rule with 5 sub-intervals to evaluate $\int_2^3 \frac{x}{x^2-3}\,dx$ correct to 3 decimal places. Determine **UNEB 2017 No.11**
    (a) The value the student obtained
    (b) The actual value of the integral
    (c) The error the student made in the estimation and suggest how the student can reduce the error **An(0.917, 0.896, 0.021)**

14. (a) Use the trapezium rule with 6-ordinates to estimate the value of $\int_0^{\pi/2}(x+\sin x)\,dx$, correct to three decimal places. **An(2.225) UNEB 2018 No.11**

    (b) (i) Evaluate $\int_0^{\pi/2}(x+\sin x)\,dx$, correct to three decimal places. **An(2.234)**

    (ii) Calculate the error in your estimation in (a) above. **An(0.009)**

    (iii) Suggest how the error may be reduced.

15. Use trapezium rule to estimate $\int_1^{1.4}(x+\tan x)\,dx$ using five ordinate **An(1.6617)**

16. Using five subintervals, find the value of $\int_0^1(x^2e^x)\,dx$ correct to 2dp and hence find the percentage error in your answer **An(0.75, 4.17%)**

17. Use the trapezium rule to estimate the area of $y=e^{-2x}$ between the x-axis, $x=1$ and $x=2$. Using six ordinates , Give your answer correct to 2 significant figures **An (0.0593)**

18. Use trapezium rule to evaluate
    (a) $\int_2^{8/3} x^2(x-1)\,dx$ use five sub intervals
    (b) $\int_2^3 \frac{dx}{x(x^2+x)^{1/2}}$ use five ordinates

    Give your answers to 2 decimal places. **An (a)=4.98, (b)=0.14**

19. (a) Use trapezium rule with seven ordinates to estimate $\int_0^{\pi} x\sin x\,dx$ correct to 2dp.
    (b) Determine the percentage error in your estimation and suggest how this may be reduced. **An(b)=2.23%**

20. Use trapezium rule with eight ordinates to estimate $\int_1^8 \log_{10} e^{3x}\,dx$ correct to 3dp. **An=41.038**

LOCATION OF REAL ROOTS

The range where the root of an equation lies can be located using the following methods

(i) Graphical method (ii) Sign change method

(I) Sign change

Examples

1. Show that the equation $x^3 - 6x^2 + 9x + 2 = 0$ has a root between -1 and 0

Solution

$$f(x) = x^3 - 6x^2 + 9x + 2$$
$$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2 = -14$$
$$f(0) = (0)^3 - 6(0)^2 + 9(0) + 2 = 2$$

Since there is a sign change, then the root lies between -1 and 0

2. Show that the equation $x^3 - 3x - 12$ has a root between 2 and 3.

Solution

$$f(x) = x^3 - 3x - 12$$
$$f(2) = 2^3 - 3x2 - 12 = -10$$
$$f(3) = 3^3 - 3x3 - 12 = 2$$

Since there is a sign change, then the root lies between 2 and 3

3. Show that the equation $x^2 = In(4 - x)$ has a root between 1 and 2

Solution

$$f(x) = x^2 - In(4 - x)$$
$$f(1) = 1^2 - In(4 - 1) = -0.099$$
$$f(2) = 2^2 - In(4 - 2) = 3.307$$

Since there is a sign change, then the root lies between 1 and 2

(II) Graphical method

One or more graphs can be drawn to locate the root

(a) Single graph method

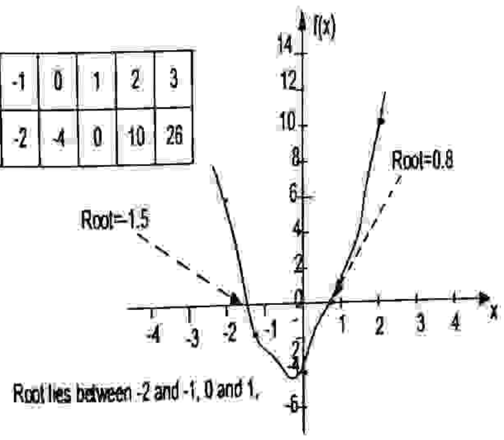
When one graph is drawn, then the root lies between the two points where the curve crosses the x-axis

Examples

1. Using suitable graphs locate the interval over which the root of the equation $3x^2 + x - 4 = 0$ lie

Solution

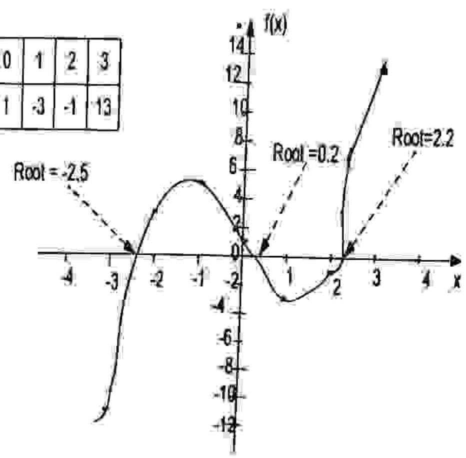
x	-3	-2	-1	0	1	2	3
f(x)	20	6	-2	-4	0	10	26



2. Show graphically that there is one positive real root of the equation $x^3 - 5x + 1 = 0$

Solution

x	-3	-2	-1	0	1	2	3
f(x)	-11	3	5	1	-3	-1	13



(b) Double graph method

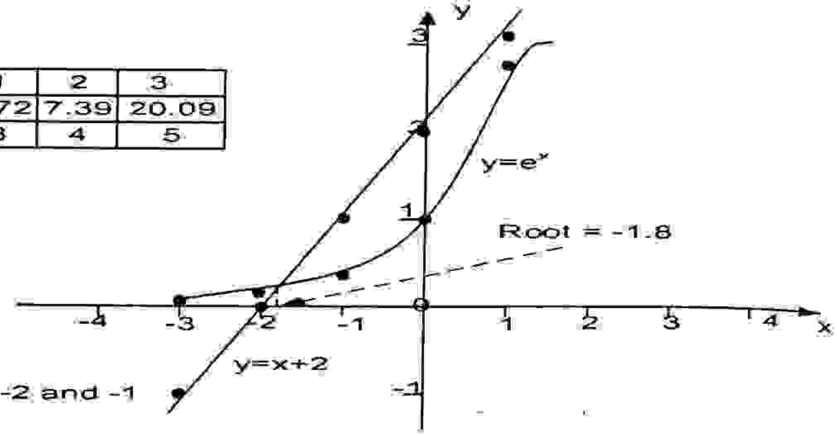
When two graph are drawn, then the root lies between the two points where the two curves meet

Examples

1. Use a graphical method to show that the equation $e^x - x - 2 = 0$ has only one real root by drawing two graphs of $y = e^x$ and $y = x + 2$

Solution

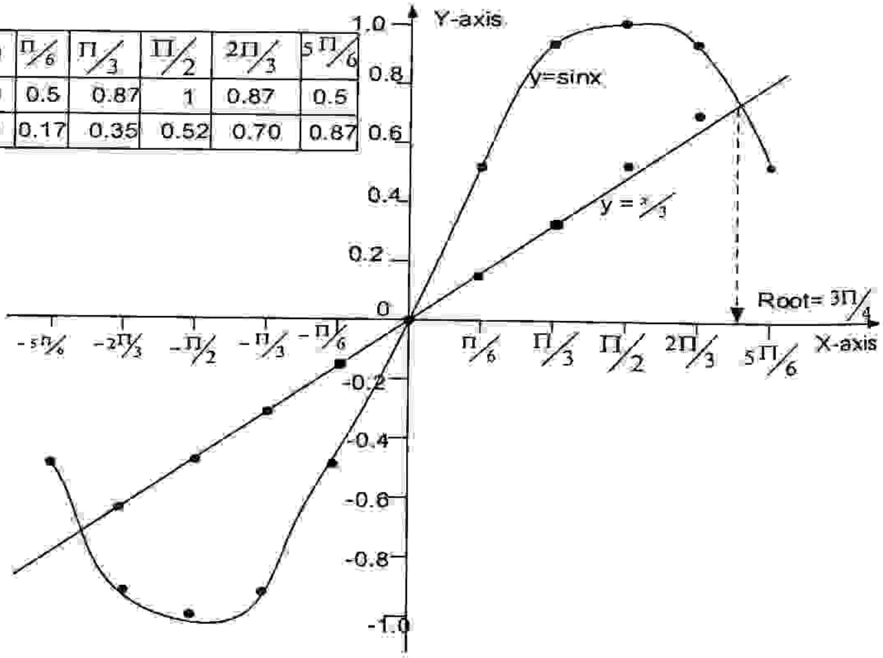
x	-3	-2	-1	0	1	2	3
e^x	0.05	0.14	0.37	1	2.72	7.39	20.09
X+2	-1	0	1	2	3	4	5



2. Given the equation $y = \sin x - \frac{x}{3}$, show by plotting two suitable graphs on the same axes that the positive root lies between $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$

Solution

x	$-5\pi/6$	$-2\pi/3$	$-\pi/2$	$-\pi/3$	$-\pi/6$	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$
sinx	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1	0.87	0.5
$x/3$	-0.87	-0.70	-0.52	-0.35	-0.17	0	0.17	0.35	0.52	0.70	0.87



Exercise 10c

1. By sketching the graphs of 2x and $\tan x$ show that the equation $2x = \tan x$ has only one root between $x = 1.1$ and 1.2. Use linear interpolation to find the value of the root correct to 2dp **UNEB 1995 No.1b An=1.17,**

2. Show graphically that the equation $x + \log_e x = 0.5$ has only one real root that lies between 0.5 and 1 **UNEB 1999 No.3**

3. Show graphically that the positive real root of the equation $2x^2 + 3x - 3 = 0$, lies between 0 and 1. **An**= 0.7

4. On the same axes, draw graphs of $y = 3 - 3x$ and $y = 2x^2$ to show that the root of the equation $2x^2 + 3x - 3 = 0$ lies between -3 and -2. **An**= -2.2

5. Show graphically that the positive real root of the equation $x^3 - 3x - 1 = 0$, lies between 1 and 2 **An**= 1.6

6. On the same axes, draw graphs of $y = 3x - 1$ and $y = x^3$ to show that the root of the equation $x^3 - 3x - 1 = 0$ lies between 0 and 1. **An**= 0.35

7. Using suitable graphs and plotting them on the same axes, find the real root of the equation $e^{2x} sinx - 1 = 0$, in the interval $x = 0.1$ and $x = 0.8$ **An**= 0.44

8. Show graphically that equation $e^{-x} = x$ has only one real root between 0.5 and 1.0. **An**= 0.56

9. Show graphically that equation $e^x = -2x + 2$ has only one real root between 0 and 1.0.

10. On the same axes, draw graphs of $y = 9x - 4$ and $y = x^3$ to show that the root of the equation $x^3 - 9x + 4 = 0$ lies between 2.5 and 3

11. Show that the positive real root of the equation $4 + 5x^2 - x^3 = 0$,lies between 5 and 6

12. On the same axes, draw graphs of $y = x + 1$ and $y = \tan x$ to show that the root of the equation $\tan x - x - 1 = 0$lies between 1 and 1.5

13. Using suitable graphs and plotting them on the same axes, find the real root of the equation $5e^x = 4x + 6$, in the interval $x = -2$ and $x = -1$

14. On the same axes, draw graphs of $y = 2x + 1$ and $y = \log_e(x + 2)$ to show that the root of the equation $\log_e(x + 2) - 2x - 1 = 0$lies between -1 and 0

15. Using suitable graphs and plotting them on the same axes, find the real root of the equation $9\log_{10} x = 2(x - 1)$, in the interval $x = 3$ and $x = 4$

16. On the same axes, draw graphs of $y = 2x$ and $y = \tan x$ to show that the root of the equation $2x = \tan x$ lies between 1.1 and 1.2. Hence use linear interpolation to find the value of the root correct to 2 decimal places

17. Show graphically that equation $\log_e(x) + x = \frac{1}{2}$ has only one real root between 0.5 and 1.0. Hence use linear interpolation to find the value of the root correct to 2 decimal places

METHOD OF SOLVING FOR ROOTS

The following methods can be used

(a) Interpolation

Examples

1. Show that the equation $x^4 - 12x^2 + 12 = 0$ has a root between 1 and 2. Hence use linear interpolation to get the first approximation of the root

Solution

$$f(x) = x^4 - 12x^2 + 12$$
$$f(1) = 1^4 - 12(1)^2 + 12 = 1$$
$$f(2) = 2^4 - 12(2)^2 + 12 = -20$$

Since there is a sign change, then the root lies between 1 and 2

x	1	x_0	2
f(x)	1	0	-20

$$\frac{x_0 - 1}{0 - 1} = \frac{2 - 1}{-20 - 1}$$
$$x_0 = 1.05$$

2. Show that the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between 1 and 2. Hence use linear interpolation twice to get the approximation of the root of the root

Solution

$$f(x) = 2x - 3\cos\left(\frac{x}{2}\right)$$
$$f(1) = 2x1 - 3\cos\left(\frac{1}{2}\right) = -0.633$$
$$f(2) = 2x2 - 3\cos\left(\frac{2}{2}\right) = 2.379$$

Since there is a sign change, then the root lies between 1 and 2

x	1	x_0	2
f(x)	-0.633	0	2.379

$$\frac{x_0 - 1}{0 - -0.633} = \frac{2 - 1}{2.379 - -0.633}$$
$$x_0 = 1.2102$$

$$f(1.2102) = 2x1.2102 - 3\cos\left(\frac{1.2102}{2}\right)$$
$$= -0.047$$

x	1.2102	x_0	2
f(x)	-0.047	0	2.379

$$\frac{x_0 - 1.2102}{0 - -0.047} = \frac{2 - 1.2102}{2.379 - -0.047}$$
$$x_0 = 1.226$$

3. Shows that the equation $3x^2 + x - 5 = 0$ has a real root between $x = 1$ and $x = 2$. Hence use linear interpretation to calculate the root to 2 dp

Solution

Solution

$$f(x) = 3x^2 + x - 5$$
$$f(1) = 3x1^2 + 1 - 5 = -1$$
$$f(2) = 3x2^2 + 2 - 5 = 9$$

Since there is a sign change, then the root lies between 1 and 2

x	1	x_0	2
f(x)	-1	0	9

$$\frac{x_0 - 1}{0 - -1} = \frac{2 - 1}{9 - -1}$$

$$x_0 = 1.1$$
$$f(1.1) = 3x1.1^2 + 1.1 - 5 = -0.27$$

x	1.1	x_0	2
f(x)	-0.27	0	9

$$\frac{x_0 - 1.1}{0 - -0.27} = \frac{2 - 1.1}{9 - -0.27}$$
$$x_0 = 1.13$$

Assignment

1. Show that the equation $e^x - 2x + 1 = 0$ has a root between $x = 1$ and $x = 1.5$. Hence use linear interpolation to obtain an approximation of the root.
2. Show that the positive real root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8. Hence use linear interpolation to determine the root to three decimal places.
3. Given that $f(x) = 3xe^x - 1 = 0$. Use linear interpolation to obtain the root of $f(x)$ lying between $x = 0.2$ and $x = 0.3$ correct to 3dp. **An(2.58)**
4. Show that one of the roots of the equation $x^2 = 3x - 1$ lie between 2 and 3. By using linear interpolation , find the root to two decimal places **An(2.62)**

(b) General iterative method

This involves generating equation by splitting the original equation into several equations by making x the subject

Examples

1. Given $x^2 + 4x - 2 = 0$. Find the possible equations for estimating the roots

Solution

Let x_{n+1} be a better approximation

x_n be next approximation

$$x_{n+1} = \frac{2}{x_n} - 4 \qquad x_{n+1} = \sqrt{(2 - 4x_n)} \qquad x_{n+1} = \frac{2 - x_n^2}{4}$$

2. Given $f(x) = x^3 - 3x - 12 = 0$. Generate equations inform of $x_{n+1} = g(x_n)$ that can be used to solve the equation $f(x) = 0$

Solution

Let x_{n+1} be a better approximation

x_n be next approximation

$$x_{n+1} = \frac{x_n^3 - 12}{3}$$
$$x_{n+1} = \sqrt[3]{(3x_n + 12)} = \frac{12}{x_n^2 - 3}$$
$$x_{n+1} = \sqrt{\left(3 + \frac{12}{x_n}\right)} = \frac{3x_n + 12}{x_n^2}$$

Test for convergence

From the several iterative equations obtained, the equation whose $|f^1(x_n)| < 1$ is the one which converges and gives the correct root

Examples

1. Given the two iterative formulas

(i) $x_{n+1} = \frac{x_n^{\ 3}-1}{5}$ **(ii)** $x_{n+1} = \sqrt{\left(5+\frac{1}{x_n}\right)}$

Using $x_0 = 2$ deduce a more suitable formula for solving the equation. Hence find the root correct to 2dp

Solution

$$x_{n+1} = \frac{x_n^{\ 3}-1}{5}$$

$$f(x_n) = \frac{x_n^{\ 3}-1}{5}, \quad f^1(x_n) = \frac{3x_n^{\ 2}}{5}$$

$$f^1(2) = \frac{3(2)^2}{5} = 2.4$$

Since $|f^1(2)| = 2.4 > 1$, then it will not converge

$$x_{n+1} = \sqrt{\left(5+\frac{1}{x_n}\right)}$$

$$f(x_n) = \sqrt{\left(5+\frac{1}{x_n}\right)} \quad f^1(x_n) = -\frac{1}{2}x_n^{\ -2}\left(5+\frac{1}{x_n}\right)$$

$$f^1(2) = -\frac{1}{2}(2)^{-2}\left(5+\frac{1}{2}\right) = -0.0533$$

Since $|f^1(2)| = |-0.0533| < 1$, then it will converge so this equation gives the root

$$x_{n+1} = \sqrt{\left(5+\frac{1}{x_n}\right)} \quad |e| = 0.005 \quad x_0 = 2$$

$$x_1 = \sqrt{\left(5+\frac{1}{2}\right)} = 2.3452$$

$$x_2 = \sqrt{\left(5+\frac{1}{2.3452}\right)} = 2.3295$$

$$x_3 = \sqrt{\left(5+\frac{1}{2.3295}\right)} = 2.3301$$

$|e| = |2.3301 - 2.3295| = 0.0006 < 0.005$

Hence root is 2.33

2. Show that the iterative formula for solving the equation is $x^3 = x + 1$ is $x_{n+1} = \sqrt{\left(1+\frac{1}{x_n}\right)}$ starting with $x_0 = 1$ find the solution of the equations to 3 s.f

Solution

$$x_{n+1} = \sqrt{\left(1+\frac{1}{x_n}\right)} \quad |e| = 0.005 \quad x_0 = 1$$

$$x_1 = \sqrt{\left(1+\frac{1}{1}\right)} = 1.41421$$

$$x_2 = \sqrt{\left(1+\frac{1}{1.41421}\right)} = 1.30656$$

$$x_3 = \sqrt{\left(1+\frac{1}{1.30656}\right)} = 1.32869$$

$$x_4 = \sqrt{\left(1+\frac{1}{1.32869}\right)} = 1.32389$$

$|e| = |1.32389 - 1.32869| = 0.0048 < 0.005$

Hence root is 1.32

Exercise 10d

1. Given the following iterative formulas

   (i) $x_{n+1} = 5 - \frac{3}{x_n}$

   (ii) $x_{n+1} = \frac{1}{5}(x_n^{\ 2}+3)$

   Taking $x_0 = 5$ deduce a more suitable iterative formula for solving the equation. Deduce also the possible original equation

2. Show that the iterative formula for solving the equation $x^2 - 5x + 2 = 0$ can be written in two ways as $x_{n+1} = 5 - \frac{2}{x_n}$ **or** $x_{n+1} = \frac{x_n^{\ 2}+2}{5}$ Starting with $x_0 = 4$, deduce the more suitable formula for solving for the equation and hence find he root correct to 2dp.
   An(4.56)

3. Show that the iterative formula for solving the equation $x^3 - x - 1 = 0$ is $x_{n+1} = \sqrt{\left(1+\frac{1}{x_n}\right)}$ hence starting $x_0 = 1$ find the root of the equation correct to 3s.f **UNEB 1993 No2b**
   An(1.32)

4. (a) Show that the iterative formula for solving the equation $2x^2 - 6x - 3 = 0$ is $x_{n+1} = \frac{2x_n^{\ 2}+3}{4x_n+6}$

   (b) Show that the positive root for

$2x^2 - 6x - 3 = 0$ lies between 3 and 4. Find the root correct to 2 decimal places **UNEB 1996 No14 An(3.436)**

5. Given two iterative formulae 1 and 2 (shown below) for calculating the positive root of the quadratic equation $f(x) = 0$.

   Formula 1: $x_{n+1} = {}^1/_2\,(x_n{}^2 - 1)$

   Formula 2: $x_{n+1} = {}^1/_2\left(\frac{x_n{}^2+1}{x_n-1}\right)$

   Taking $x_0 = 2.5$, use each formula thrice to two decimal places to decide which is the more suitable formula. Give a reason for your answer **UNEB 2001 No11a**

6. **(a) (i)** Show that the equation $e^x - 2x - 1 = 0$ has a root between $x = 1$ and $x = 1.5$
   (ii) Use linear interpolation to obtain an approximation for the root
   (b) (i) Solve the equation in (a)(i), using each formula below twice. Take the approximation in (a) (ii) as the initial value.

   Formula 1: $x_{n+1} = {}^1/_2\,(e^{x_n} + 1)$

   Formula 2: $x_{n+1} = \frac{e^{x_n}(x_n-1)+1}{e^{x_n}-2}$

   (ii) Deduce with a reason which of the two formula is appropriate for solving the given equation in (a)(i). Hence write down a better approximate root, correct to 2dp. **UNEB 2012 No11 An(1.18, 1.26)**

7. If b is the first approximation to the root of the equation $x^2 = a$, show that the second approximation to the root is given by $\frac{b+\frac{a}{b}}{2}$.
   Hence taking $b = 4$, estimate $\sqrt{17}$ correct to 3dp **UNEB 2001 No.11 b An [4.123]**

8. An iterative formula for solving the equation $f(x) = 0$ **, is given by** $x_{n+1} = {}^1/_3\left(\frac{2x_n{}^3+12}{x_n{}^2}\right)$
   (i) Taking $x_0 = 2$, deduce the root of the equation to three decimal places
   (ii) Find the equation which is solved by this iterative formula **An**= 2.289

9. Show that the iterative formula for finding the fifth root of a number N is given $\frac{1}{5}\left(4x_n + \frac{N}{x_n{}^4}\right)$
   Hence use $x_0 = 2$ to find $\sqrt[5]{50}$ correct to 2 decimal places **An**= 2.19

(c) Newton Raphson Method

Its given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} \quad n = 0, 1, 2 \ldots\ldots$$

Examples

1. Use newton raphson method to find the root of the equation $x^3 + x - 1 = 0$ using $x_0 = 0.5$ as the initial approximation, correct your answer to 2 decimal places.

Solution

$f(x) = x^3 + x - 1, \quad f^1(x) = 2x + 1$

$$x_{n+1} = x_n - \left(\frac{x_n{}^3 + x_n - 1}{2x_n + 1}\right)$$

$$x_{n+1} = \frac{x_n(2x_n + 1) - (x_n{}^3 + x_n - 1)}{2x_n + 1}$$

$$x_{n+1} = \frac{x_n{}^3 + 1}{2x_n + 1}$$

$x_0 = 0.5, \quad |e| = 0.005$

$$x_1 = \frac{0.5^3 + 1}{2x0.5 + 1} = 0.563$$

$$x_2 = \frac{0.5625^3 + 1}{2x0.5625 + 1} = 0.554$$

$$x_3 = \frac{0.5543^3 + 1}{2x0.5543 + 1} = 0.555$$

$|0.555 - 0.554| = 0.001 < 0.005$

Root is 0.56

2. Show that the equation $5x - 3\,cos2x = 0$ has a root between 0 and 1. Hence use Newtons Raphson method to find the root of the equation correct to 2 decimal places

Solution

$f(x) = 5x - 3\,cos2x, \quad f^1(x) = 5 + 6\,sin2x$

$$x_{n+1} = x_n - \left(\frac{5x_n - 3\,cos2x_n}{5 + 6\,sin2x_n}\right)$$

$$x_{n+1} = \frac{x_n(5 + 6\,sin2x_n) - (5x_n - 3\,cos2x_n)}{5 + 6\,sin2x_n}$$

$$x_{n+1} = \frac{6\,sin2x_n + 3\,cos2x_n}{5 + 6\,sin2x_n}$$

$x_0 = 0.6, \quad |e| = 0.005$

$$x_1 = \frac{6\,sin2(0.6) + 3\,cos2(0.6)}{5 + 6\,sin2(0.6)} = 0.631$$

$$x_2 = \frac{6\,sin2(0.631) + 3\,cos2(0.631)}{5 + 6\,sin2(0.631)} = 0.618$$

$$x_3 = \frac{6\,sin2(0.6) + 3\,cos2(0.6)}{5 + 6\,sin2(0.6)} = 0.623$$

$$x_4 = \frac{6\,sin2(0.6) + 3\,cos2(0.6)}{5 + 6\,sin2(0.6)} = 0.622$$

$|0.623 - 0.622| = 0.001 < 0.005$

Root is 0.62

3. Use newton raphson iterative formula to show that the cube root of a number N is given $\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$

Hence taking $x_0 = 2.5$ determine $\sqrt[3]{10}$ correct to 3 decimal places

Solution

$x = N^{1/3}$

$x^3 = N$

$x^3 - N = 0$

$f(x) = x^3 - N, \quad f^1(x) = 3x^2$

$$x_{n+1} = x_n - \left(\frac{x_n^3 - N}{3x_n^2}\right) = \frac{x_n(3x_n^2) - (x_n^3 - N)}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$$

$x_0 = 2.5, N = 10, |e| = 0.0005$

$$x_1 = \frac{1}{3}\left(2x2.5 + \frac{10}{2.5^2}\right) = 2.2$$

$$x_2 = \frac{1}{3}\left(2x2.5 + \frac{10}{2.5^2}\right) = 2.1554$$

$$x_3 = \frac{1}{3}\left(2x2.5 + \frac{10}{2.5^2}\right) = 2.1544$$

$$x_4 = \frac{1}{3}\left(2x2.5 + \frac{10}{2.5^2}\right) = 2.1544$$

$|2.1544 - 2.1554| = 0.000 < 0.005$

Root is 2.154

EXERCISE 10e

1. Derive the simplest iterative formula for the NRM for the root of the equation $e^{3x} = 3$, using your formula with $x_0 = \frac{1}{3}$, find the root correct to 4dp **An($x_3 = 0.3662$)**
2. Using the iterative formula NRM, show that the reciprocal of a number N is $x_n(2 - Nx_n)$. **UNEB 1991 No.1 a**
3. Use newton raphson iterative formula to show that the cube root of a number N is given $\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$ Hence use the iterative formula to find $\sqrt[3]{96}$ correct to 3 decimal places **UNEB 1992 No.1 An(4.579**
4. Show that the equation $3x^3 + x - 5 = 0$ has a real root between $x = 1$ and $x = 2$ **UNEB 1993 No.1**
   (i) Using linear interpolation, find the first approximation for this root to 2dp
   (ii) using NRM twice find the value of this root correct to 2dp. **Ans [I)=1.045, II)=1.09]**
5. Show graphically that there is one positive real root of the equation $xe^{-x} - 2x + 5 = 0$
   Using Newton raphson method, find this root correct to 1dp **UNEB 1994 No.1 Ans [2.6]**
6. Using the iterative formula for NRM, show that the fourth root of the number N is $\frac{1}{4}\left(3x_n + \frac{N}{x_n^3}\right)$.
   Hence show that $(45.7)^{1/4} = 2.600$ (3dp) **UNEB 1997 No.12**
7. On the same axes, draw graphs of $y = x^3$ and $y = 2x + 5$ Using NRM twice, find the psotiive root of the equation $x^3 - 2x - 5 = 0$ correct to 2 decimal place **UNEB 1998 march No.6 Ans [2.09]**
8. (a) Show that the Newton Raphson's formula for finding the smallest positive root of the equation $3tan\,x + x = 0$ is $\frac{6x_n - 3sin2\,x_n}{6+2\,cos2\,x_n}$
   (b) By sketching the graphs of $y = tan\,x$, $y = {}^{-x}/_3$ or otherwise, find the first approximation to the required root and use it to find the actual root correct to 3dp **UNEB 1998 march No.10 An [2.456]**
9. (a) Show that the root of the equation $f(x) = e^x + x^3 - 4x = 0$ has a root between 1 and 2
   (b) Use the Newtown- Raphson method to find the root of the equation in (a) correct to 2 decimal places **UNEB 1998 Nov No.12 An [1.12]**
10. (i) Show that the iterative formula for approximating the root of $f(x) = 0$ by NRM for the $xe^x + 5x - 10 = 0$ is $x_{n+1} = \frac{x_n^2 e^{x_n} + 10}{x_n e^{x_n} + e^{x_n} + 5}$

(ii) Show that the root of the equation in (i) above lies between 1 and 2. Hence find the root of the equation correct to 2 dp **UNEB 1999 No.12 An [1.20]**

11. (a) Use a graphical method to find a first approximation to the real root of $x^3 + 2x - 2 = 0$
   (b) Use the Newtown- Raphson method to find the root of the equation in (a) correct to 2 decimal places **UNEB 2002 No.13 An [0.77]**

12. (a) Show that the equation $x = ln(8 - x)$ has a root between 1 and 2
   (b) Use the Newtown- Raphson method to find the root of the equation in (a) correct to 3 decimal places **UNEB 2003 No.9 An [1.82]**

13. (a) Use a graphical method to find a first approximation to the real root of $x^3 - 3x + 4 = 0$
   (b) Use the Newtown- Raphson method to find the root of the equation in (a) correct to 2 decimal places **UNEB 2004 No.12 An [-2, -2.20]**

14. Show graphically that equation $e^x + x - 4 = 0$ has only one real root. Use NRM to find the approximation of the equation correct to 3dp **UNEB 2005 No12 An 1.07**

15. Show that the NRM for approximating the K^{th} root of the number N is given by

$$x_{n+1} = \frac{1}{k}\left((k-1)x_n + \frac{N}{{x_n}^{k-1}}\right)$$. Hence use your formula to find the positive square root of 67 correct to 4s.f **UNEB 2006 No.14 An(8.185)**

16. (a) Show that the equation $x^3 + 3x - 9 = 0$ has a root between 1 and 2.
   (b) Use the Newton Raphson method once to calculate the root of the equation in (a) correct to 2 decimal places **UNEB 2007 No6 An=1.6**

17. **(a)** Show graphically that there is only one positive real root of the equation $e^x - 2x - 1 = 0$, between 1 and 2
   (b) Use the Newton Raphson method to calculate the root of the equation in (a) correct to 2 decimal places. **UNEB 2008 No14 An 1.26**

18. (a) Show that the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between 1 and 2.
   (b) Use the Newton Raphson method to calculate the root of the equation in (a) correct to 2 decimal places **UNEB 2009 No14 An=1.23**

19. (a) If a is the first approximation to the root of the equation $x^5 - b = 0$, show that the second approximation is given by $\frac{4a + \frac{b}{a^4}}{5}$
   (b) Show that the positive real root of the equation $x^5 - 17 = 0$, lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3dp **UNEB 2010 No9 An=1.762**

20. (a) (i) On the same axes, draw graphs of $y = x^2$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$ at interval of $\frac{\pi}{8}$
   (ii) Use your graphs, to find 1 decimal place an approximate root of the equation $x^2 - \cos x = 0$
   (b) Use the Newton Raphson method to calculate the root of the equation $x^2 - \cos x = 0$, taking the approximate root in (a) as an initial approximation. Correct your answer to 3 decimal places **UNEB 2011 No14 An(a(ii) 0.8, (b) 0.824)**

21. (a) Show that the Newton Raphson formula for approximating the root of the equation $x^3 - 6x^2 + 9x + 2 = 0$ is given by
$$x_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$$
   (b) Use the formula above, with an initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to two decimal places. **UNEB 2016 No An(-0.20)**

22. (a) On the same axes, draw graphs of $y = x$ and $y = 4\sin x$ to show that the root of the equation $x - 4\sin x = 0$ lies between 2 and 3
   (b) Use the Newton Raphson method to calculate the root of the equation $x - 4\sin x = 0$, taking the approximate root in (a) as an initial approximation. Correct your answer to 3 decimal places **UNEB 2017 No14 An(2.475)**

23. (a) Draw on the same axes the graphs of the curves $y = 2 - e^{-x}$ and $y = \sqrt{x}$ for $2 \le x \le 5$.
   (b) Determine from your graph the interval within which the root of the equation $e^{-x} + \sqrt{x} - 2 = 0$ lies. Hence, use Newton – Raphson's method to find the root equation correct to 3 decimal places. **UNEB 2018 No14 An(3.921)**

24. Show graphically that equation $e^x + x - 8 = 0$ has only one real root. Use NRM to find the approximation of $x = ln(x - 8)$ correct to 3dp **a=1.821**

25. Draw using the same axes, the graphs of $y = x^2$ and $y = \sin 2x$ for $0 \leq x \leq \pi/2$ from your graph obtain to one decimal place an approximation of the non-zero root of the equation $x^2 - \sin 2x = 0$. Using NRM, calculate to 2dp a more suitable approximation. **An=0.97**

26. Given the equation $ln(1+2x) - x = 0$
   (a) Show that the root of the equation above lies between 1 and 1.5
   (b) Use Newton raphson iterative formula two times to estimate the root of the equation, correct to two decimal places. **An 1.26**

27. (a) An iterative formula for solving an equation is given by $X_{n+1} = \sqrt[3]{(3x_n + 3)}$ $n = 0, 1, 2$
   (i) Find the equation whose root is being sought
   (ii) Show that the equation has one root and using an appropriate starting value, find the root correct to 2dp
   (b) By drawing a graph, show that the root of the equation $2\tan x = 3x$ lies between $\pi/6$ and $\pi/3$. Hence by using NRM, find the root of the equation to 2dp **Ans [a=2.0, b=0.97]**

28. (a) Find an approximation value of $\int_0^{0.5}(1 - x^2)dx$, using the trapezoidal rule with intervals of 0.1. Show by integration that the magnitude of the error in the approximation is less than 0.001
   (b) Show that the equation $x = \cos x - 3$ has a root between -4 and -3. Use an iterative method to calculate this root correct to 3dp
   Ans[a=0.4778,0.4783. b=-3.794]

Miscellaneous exercise: 3

1. (a) Show that the equation $2x^2 - 6x - 3 = 0$ has a real root between $x = 3$ and $x = 4$
   (*b*) Using linear interpolation, find the first approximation for this root to 2dp
   (*c*) Show that the Newton Raphson formula for approximating the root of the equation $2x^2 - 6x - 3 = 0$ is given by $x_{n+1} = \frac{2x_n^2+3}{4x_n-6}$
   (*d*) Use the formula above, with an initial approximation of $x_0 = 3$, to find the root of the given equation correct to 2dp

2. (a) Show that the equation $f(x) = 3xe^x - 1 = 0$ has a real root between 0.2 and 0.3
   (*b*) Using linear interpolation, find the first approximation for this root to 3dp
   (*c*) Show that the Newton Raphson formula for approximating the root of the equation $3xe^x - 1 = 0$ is given by $x_{n+1} = \frac{3x_n^2e^{x_n}+1}{3e^{x_n}(x_n+1)}$
   (*d*) Use the formula above, to find the root of the given equation correct to 3 decimal places.

3. (a) Show that the equation $\sin x - \left(\frac{x}{2}\right) = 0$ has a real root between 1 and 2
   (*b*) Show that the Newton Raphson formula for approximating the root of the equation $\sin x - \left(\frac{x}{2}\right) = 0$ is given by $x_{n+1} = \frac{2(x_n\cos x_n - \sin x_n)}{2\cos x_n - 1}$
   (*c*) Use the formula above, to find the root of the given equation correct to 2 decimal places.

4. (a) Show that the equation $2x^2 + 3x - 4 = 0$ has a real root between $x = 0.2$ and $x = 1.0$
   (*b*) Show that the Newton Raphson formula for approximating the root of the equation $2x^2 + 3x - 4 = 0$ is given by $x_{n+1} = \frac{2x_n^2+4}{4x_n+3}$
   (*c*) Use the formula above, to find the root of the given equation correct to two decimal places.

5. (a) Show that the equation $x^3 - 5x - 40 = 0$ has a real root between $x = 3$ and $x = 4$
   (*b*) Show that the Newton Raphson formula for approximating the root of the equation $x^3 - 5x = 40$ is given by $x_{n+1} = \frac{2x_n^3+40}{3x_n^2-5}$
   (*c*) Use the formula above, to find the root of the given equation correct to two decimal places.

6. (a) Show graphically that the root of the equation $e^x\sin x = 1$ has a root between 0.2 and 1.2
   (b) Use the Newtown- Raphson method to find the root of the equation in (a) correct to 2 decimal places

7. (a) On the same axes, draw graphs of $y = x - 0.5$ and $y = \ln x$ to show that the root of the equation $x + \ln x = 0.5$ lies between 0 and 1
   (b) Use the Newton Raphson method to calculate the root of the equation $x + \ln x = 0.5$, taking the approximate root in (a) as an initial approximation. Correct your answer to 3 decimal places

8. (a) On the same axes, draw graphs of $y = x$ and $y = e^{-x}$ to show that the root of the equation $e^{-x} - x = 0$ lies between 0.1 and 0.9
   (b) Use the Newton Raphson method to calculate the root of the equation $e^x - x = 0$, taking the approximate root in (a) as an initial approximation. Correct your answer to 3 decimal places

9. (a) On the same axes, draw graphs of $y = x^2$ and $y = 2x + 1$ to show that the root of the equation $x^2 - 2x - 1 = 0$ lies between 2 and 3
(b) Use the Newton Raphson method to calculate the root of the equation $x^2 - 2x - 1 = 0$, taking the approximate root in (a) as an initial approximation. Correct your answer to 2 decimal places

10. (a) Show graphically that the root of the equation $sinx - cosx = 0$ has a root between 0 and 1
(b) Use the Newtown- Raphson method to find the root of the equation in (a) correct to 2 decimal places

11. (a) Show the equation $e^x + x - 4 = 0$ has a real root between 1 and 1.2.
(b) Use the NRM to find the root of the equation correct to 3 significant figures

12. (a) Show that the equation $f(x) = x^3 + 3x - 9$ has a root between $x = 1$ and $x = 2$.
(b) Using the newton Raphson formula, to estimate the root of the equation rounded off to 2 s.f.

13. (a) Show that the equation $e^x simx = 1$ has a positive root between $x = 0$ and $x = 1$.
(b) Using the newton Raphson formula once, estimate the root of the equation rounded off to 2 d.p.

14. (a) Show that the equation $e^x + x^3 - 4x = 0$ has a real root between 1.0 and 1.5
(b) Show that the Newton Raphson formula for approximating the root of the equation $e^x + x^3 - 4x = 0$ is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+2x^3{}_n}{e^{x_n}+3x^2{}_n-4}$
(c) Use the formula above, to find the root of the given equation correct to 2 dp.

15. (a) Show that the equation $x + 3tanx = 0$ has a real root between 2 and 3
(b) Show that the Newton Raphson formula for approximating the root of the equation $x + 3tanx = 0$ given by
$$x_{n+1} = \frac{3}{2}\left(\frac{2x_n - \sin 2x_n}{3 + \cos^2 x_n}\right)$$
(c) Use the formula above, to find the root of the given equation correct to 2 decimal places.

16. (a) Show that if x is the square root of 7, then there is only one value of x between 2 and 3
(b) Show that the Newton Raphson formula for approximating the square root of 7 is given by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$
(c) Use the formula above, to find the root, correct to 2 decimal places

17. (a) On the same axes, draw graphs of $y = x^2$ and $y = sin2x$ to show that the root of the equation $x^2 = sin2x$ for $0 \le x \le {}^{\pi}/_2$
(b) Use the Newton Raphson method to calculate the root of the equation $x^2 = sin2x$, taking the approximate root in (a) as an initial approximation. Correct your answer to 2 decimal places

18. (a) On the same axes, draw graphs of $y = e^{-x}$ and $y = sinx$ to show that the root of the equation $e^{-x}\ sinx = 0$ lies between 0 and 1.0
(b) Use the Newton Raphson method to calculate the root of the equation $e^{-x}\ sinx = 0$, taking the approximate root in (a) as an initial approximation. Correct your answer to 2 decimal places

19. (a) Show that the equation $e^x + x = 10$ has a positive real root between $x = 2$ and $x = 2.5$
(b) Show that the Newton Raphson formula for approximating the root of the equation $e^x + x = 10$ is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+10}{e^{x_n}+1}$
(c) Use the formula above, to find the root of the given equation correct to 2 decimal places.

Solutions

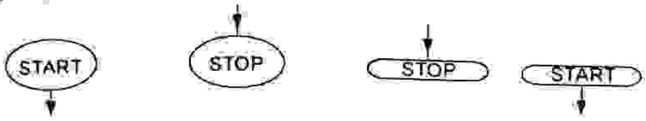
1. **An**$(d) = 3.44$
2. **An**$(b) = 0.258, (c) = 0.258$
3. **An**$(c) = 1.90$
4. **An**$(c) = 0.85$
5. **An**$(c) = 3.90$
6. **An**$(b) = 0.58$
7. **An**$(b) = 0.766$
8. **An**$(b) = 0.567$
9. **An**$(b) = 2.41$
10. **An**$(b) = 0.79$
11. b= 1.07
12. An= 1.6
13. An= 0.58
14. An= 1.12
15. An= 2.14
16. An= 2.65
17. An= 0.97

FLOW CHARTS

A flow chart is a diagram comprising of systematics steps followed in order to solve a problem.

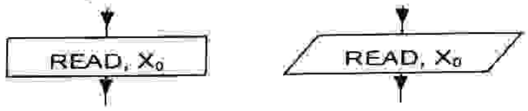
Shapes used

1. START/STOP

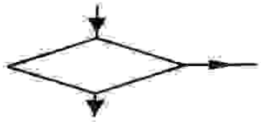


2. OPERATION/ ASSIGNMENT

This indicates that the new number N is obtained by adding one to the previous N



3. Decision box



Notes

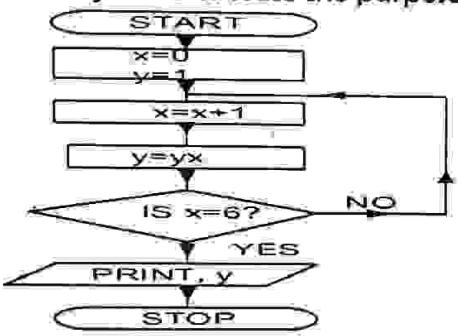
All other shapes can be interchanged except for the decision box

DRY RUN OR TRACE

This is the method of predicting the outcome of a given flow chart using a table

Examples

1. Perform a dry run and state the purpose of the flow chart. **UNEB 2006 No.10a**



SOLUTION

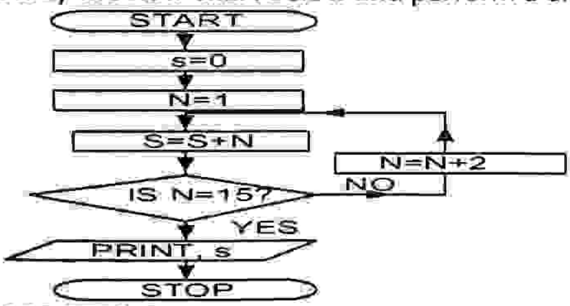
Dry run

x	y
0	1
1	1
2	2
3	6
4	24
5	120
6	720

Purpose is to compute and print 6!

Relationship is $y = x!$

2. Study the flow chart below and perform a dry run of a flow chart

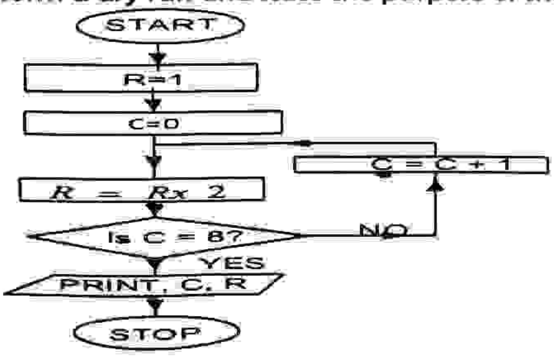


SOLUTION

N	s	Is N =15?
1	1	No
3	4	No
5	9	No
7	16	No
9	25	No
11	36	No
13	49	No
15	64	YES

Purpose is to compute and print the first 8 square number

3. Perform a dry run and state the purpose of the flow chart

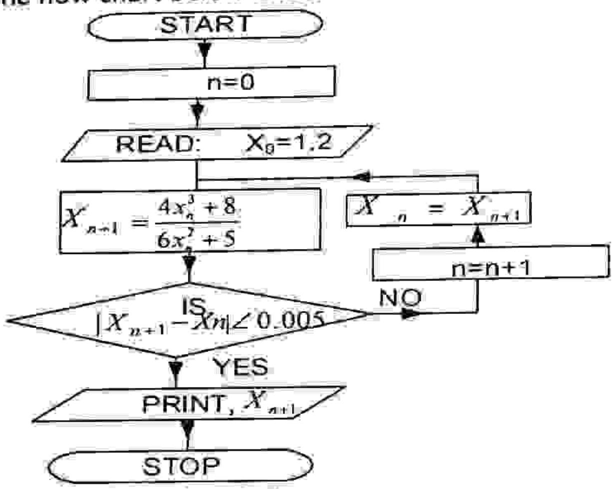


SOLUTION

C	R	Is C =8?
0	1	No
1	2	No
2	4	No
3	8	No
4	16	No
5	32	No
6	64	No
7	128	No
8	256	YES

Purpose is to compute and print 2^8

4. The flow chart below is used to read and print the root of the equation $2x^3 + 5x - 8 = 0$



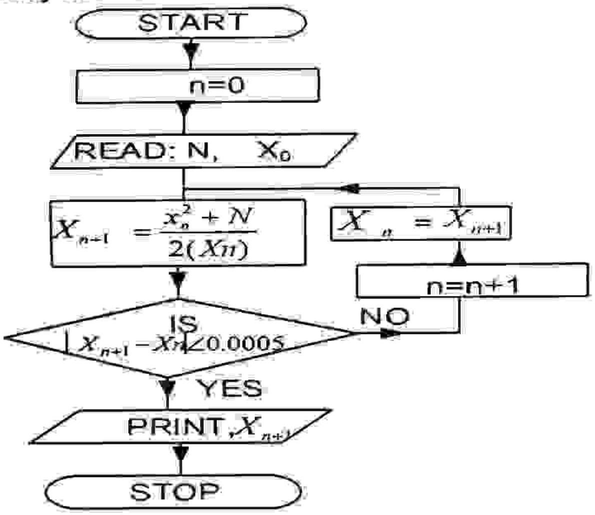
Carry out a dry run of the flow chart and obtain the root of with an error of 0.005

Solution

n	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	1.2	1.0933	0.1067
1	1.0933	1.0867	0.0066
2	1.0867	1.0866	0.001

Root is = 1.087

5. Study the flow chart below



(i) Carry out a dry run of the flow chart, taking $N = 20$, $x_0 = 4.0$ and obtain the root of correct to 3 dp

(ii) State the purpose

Solution

n	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	4.0	4.5	0.5
1	4.5	4.4722	0.0278
2	4.4722	4.4721	0.0001

Root is = 4.472

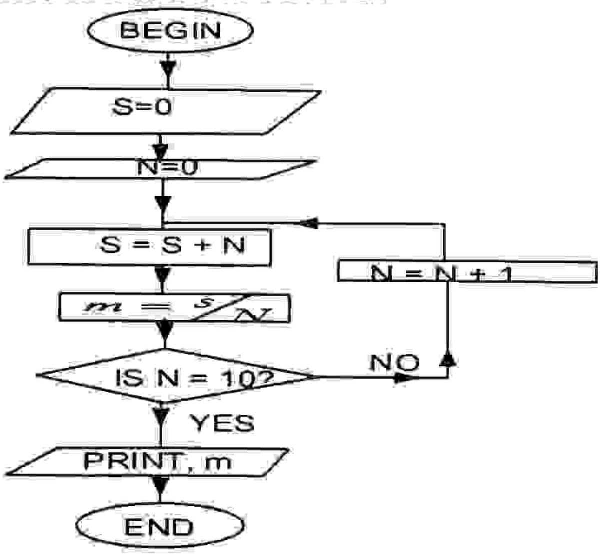
(ii) to print the square root of a number N

Constructing flow charts

1. Draw a flow char that reads and prints the mean of the first ten counting numbers

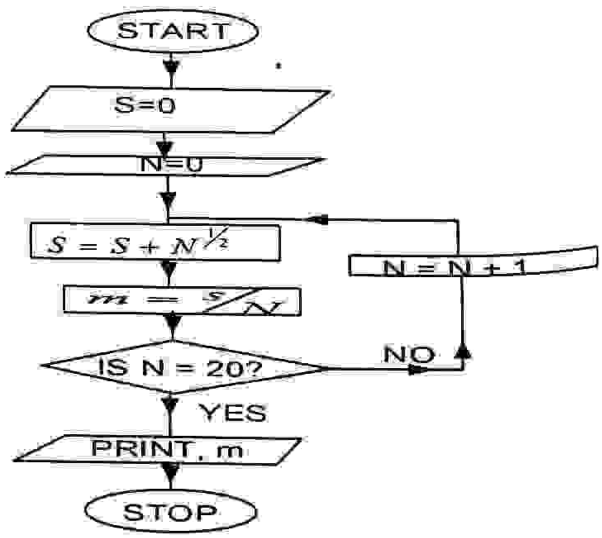
Solution

Let s be sum and m the mean



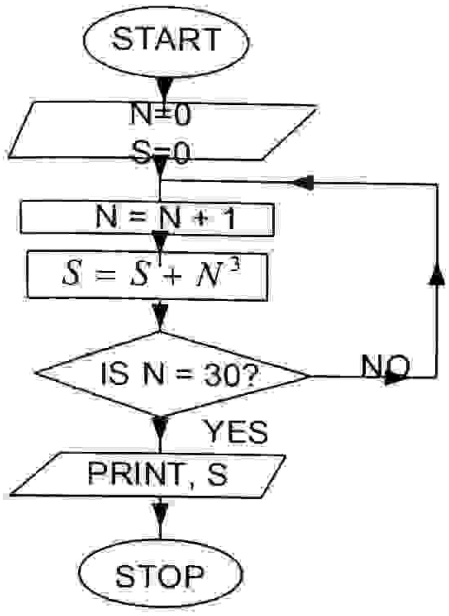
2. Draw a flow chart for computing and printing the mean of the square roots of the first 20 natural numbers

Solution



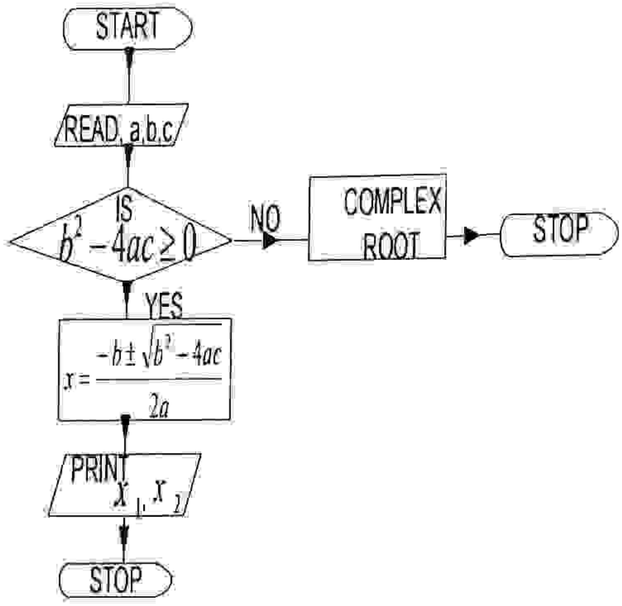
3. Draw a flow chart that computes and prints the sum of the cubes of the first 30 natural numbers

Solution



4. Draw a flow chart that computes the root of the equation $ax^2 + bx + c = 0$

Solution



Newton Raphson and Flow charts

1. (a) Show that the iterative formula base on Newton Raphson's method for approximating the sixth root of a number N is given by $x_{n+1} = \frac{1}{6}\left(5x_n + \frac{N}{x_n^5}\right)$

(b) Draw a flow chart that;

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes and prints the root to three decimal places

(c) Taking $N = 60$, $x_0 = 1.9$, perform a dry run for the flow chart, give your root correct to three decimal places

Solution

(a) $x = N^{1/6} \quad \therefore x^6 = N$

$$x^6 - N = 0$$

$$f(x) = x^6 - N, \quad f^1(x) = 6x^5$$

$$x_{n+1} = x_n - \left(\frac{x_n{}^6 - N}{6x_n{}^5}\right)$$

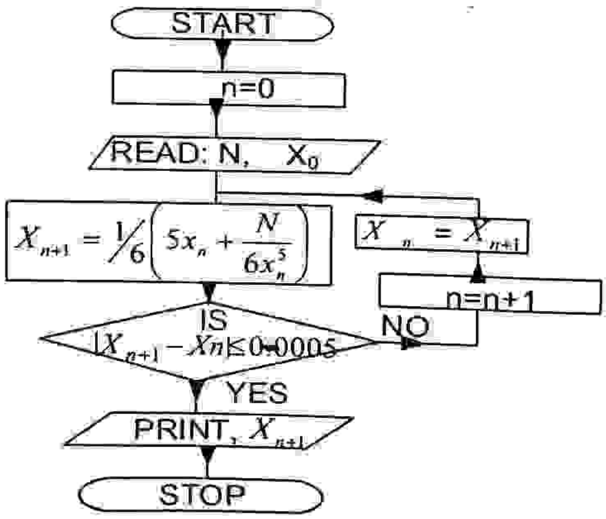
$$x_{n+1} = \frac{x_n(6x_n{}^5) - (x_n{}^6 - N)}{6x_n{}^5}$$

$$x_{n+1} = \frac{5x_n{}^6 + N}{6x_n{}^5} = \frac{1}{6}\left(5x_n + \frac{N}{x_n^5}\right)$$

Dry run

n	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	1.9	1.9872	0.0872
1	1.9872	1.9787	0.0085
2	1.9787	1.9786	0.0001

Root is = 1.979



2. (a) Show that the iterative formula base on Newton Raphson's method for approximating the fourth root of a number N is given by $x_{n+1} = {}^3/_4\left(x_n + \frac{N}{3x_n^3}\right)$

(b) Draw a flow chart that:

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes and prints the root after four iterations

(c) Taking $N = 39.0$, $x_0 = 2.0$, perform a dry run for the flow chart, give your root correct to three decimal places

Solution

(a)

$$x = N^{1/4} \qquad \therefore x^4 = N$$

$$x^4 - N = 0$$

$$f(x) = x^4 - N, \quad f^1(x) = 4x^3$$

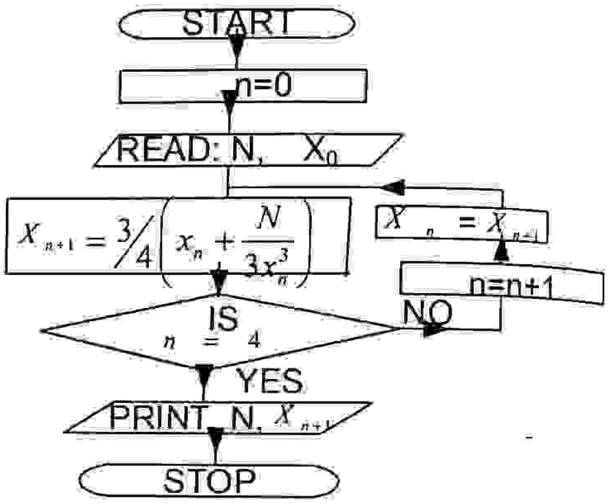
$$x_{n+1} = x_n - \left(\frac{x_n{}^4 - N}{4x_n{}^3}\right) = \frac{x_n(4x_n{}^3) - (x_n{}^4 - N)}{4x_n{}^3}$$

$$= \frac{3x_n{}^4 + N}{4x_n{}^3} = {}^3/_4\left(x_n + \frac{N}{3x_n^3}\right)$$

Dry run

n	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	2.0	2.71875	0.71875
1	2.71875	2.52424	0.19451
2	2.52424	2.49938	0.02486
3	2.49938	2.49899	0.00039

Root =2.499



3. (a) Show that the iterative formula based on Newton Raphson's method for finding the natural logarithm of a number N is given by $x_{n+1} = \frac{e^{x_n}(x_n - 1) + N}{e^{x_n}}$, $n = 0, 1, 2 \ldots\ldots\ldots\ldots$

(b) Draw a flow chart that;

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes and prints the natural logarithm after four iterations and gives the natural logarithm to three decimal places

(c) Taking, $N = 10, x_0 = 2$, perform a dry run for the flow chart, give your root correct to three decimal places

Solution

(a) $x = \text{In N} \quad \therefore e^x = N \quad \therefore e^x - N = 0$

$$f(x) = e^x - N, \quad f^1(x) = e^x$$

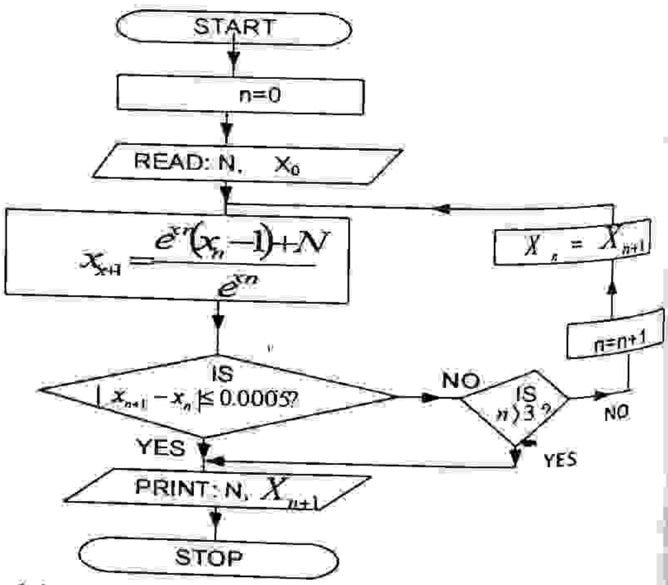
$$x_{n+1} = x_n - \left(\frac{e^{x_n} - N}{e^{x_n}}\right) = \frac{x_n e^{x_n} - e^{x_n} + N}{e^{x_n}}$$

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + N}{e^{x_n}} \qquad n = 0,1,2,3\ldots\ldots$$

Dry run

n	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	2.0	2.3533	0.3533
1	2.3533	2.3039	0.0494
2	2.3039	2.3026	0.0013
3	2.3026	2.3026	0.0000

Root =2.303

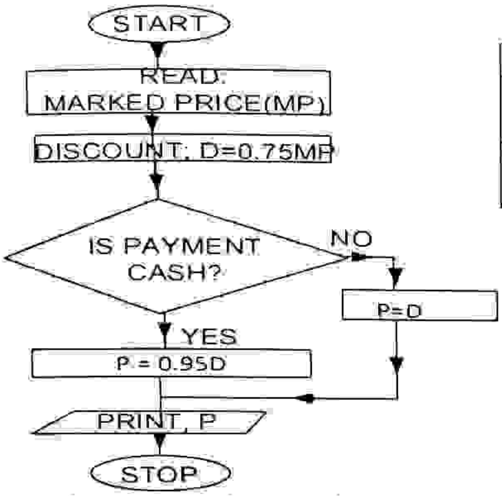


(c)

Further examples

1. A shop offers a 25% discount on all items in their store and a second discount of 5% for paying cash.
   (a) Construct a flow chart for the above information
   (b) Perform a dry run for;
   (i) A shoe 75,000/= cash (ii) A shirt 45,000/= credit

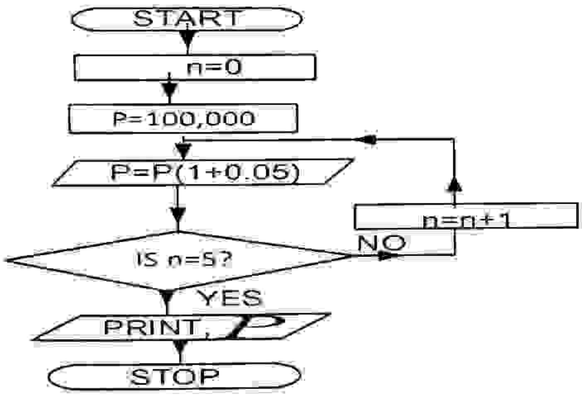
Solution



MP	D=0.75MP	Payment	cash =0.95D	Credit=D
75,000	56,250	Cash	53437.50	—
45,000	33,750	Credit	—	33750

2. Given that a man deposited 100,000/= to a bank which gives a compound interest rate of 5% . Draw a flow chart to compute the amount of money accumulated after 5 years, and perform a dry run for the flow chart

Solution



n	P	A
0	100,000	100,000
1	100,000	105,000
2	105,000	110,250
3	110,250	115,762.5
4	115,762.5	121,550.625
5	121,550.625	127,628.1563

Exercise 10f

1. (a) Show that the iterative formula base on Newton Raphson's method for approximating the cube root of a number N is given by
$x_{n+1} = {}^1/_3\left(2x_n + \frac{N}{x_n^2}\right)$ $n = 0, 1, 2 \ldots\ldots\ldots$
(b) Draw a flow chart that;
(i) Reads N and the initial approximation x_0 of the root
(ii) Computes and prints the root to three decimal places
(c) Taking $N = 54$, $x_0 = 3.7$, perform a dry run for the flow chart, give your root correct to three decimal places **An(3.780)**

2. (a) Show that the iterative formula base on Newton Raphson's method for approximating the fourth root of a number N is given by
$x_{n+1} = {}^3/_4\left(x_n + \frac{N}{3x_n^3}\right)$ $n = 0, 1, 2 \ldots\ldots\ldots$
(b) Draw a flow chart that;
(i) Reads N and the initial approximation x_0 of the root
(ii) Computes and prints the root to two decimal places
(c) Taking $N = 35$, $x_0 = 2.3$, perform a dry run for the flow chart, give your root correct to two decimal places **An(2.43)**

3. (a) Show that the iterative formula base on Newton Raphson's method for finding the root of the $x = N^{1/5}$ is given by

$$x_{n+1} = \left(\frac{4x_n^5+N}{5x_n^4}\right), \; n = 0, 1, 2 \ldots\ldots\ldots$$

(b) Draw a flow chart that;

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes the root to three decimal places

(iii) Prints the root and the number of iterations, n

(c) Taking $N = 50, x_0 = 2.2$, perform a dry run for the flow chart, give your root correct to three decimal places **An(2.187)**

4. (a) Show that the iterative formula base on Newton Raphson's method for finding the root of the $2\,lnx - x + 1 = 0$ is given by

$$x_{n+1} = x_n\left(\frac{2\,ln\,x_n - 1}{x_n - 2}\right), \; n = 0, 1, 2 \ldots\ldots\ldots$$

(b) Draw a flow chart that;

(i) Reads the initial approximation x_0 of the root

(ii) Computes and prints the root to two decimal places

(c) Taking, $x_0 = 3.4$, perform a dry run for the flow chart

5. (a) Show that the iterative formula base on Newton Raphson's method for finding the root of the $lnx + x - 2 = 0$ is given by

$$x_{n+1} = x_n\left(\frac{3 - ln\,x_n}{1 + x_n}\right), \; n = 0, 1, 2 \ldots\ldots\ldots$$

(b) Draw a flow chart that;

(i) Reads the initial approximation r of the root

(ii) Computes and prints the root of the equation, when the error is less than $1.0x10^{-4}$

(c) Taking, $r = 1.6$, perform a dry run for the flow chart.

6. (a) Show that the iterative formula base on Newton Raphson's method for finding the root of the $x = ln(x + 2)$ is given by

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + 2}{e^{x_n} - 1}, \; n = 0, 1, 2 \ldots\ldots\ldots$$

(b) Draw a flow chart that;

(i) Reads the initial approximation x_0 of the root

(ii) Computes and prints the root to three decimal places

(c) Taking, $x_0 = 1.2$, perform a dry run for the flow chart, give your root correct to three decimal places **An(1.146)**

7. (a) Show that the iterative formula base on Newton Raphson's method for finding the natural logarithm of a number N is given by

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + N}{e^{x_n}}, \; n = 0, 1, 2 \ldots\ldots\ldots$$

(b) Draw a flow chart that;

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes and prints the natural logarithm to two decimal places

(c) Taking, $N = 45, x_0 = 3.5$, perform a dry run for the flow chart, give your root correct to two decimal places **An(3.81)**

8. (a) Show that the iterative formula base on Newton Raphson's method for finding the root of the $2x^3 + 5x - 8 = 0$ is given by

$$x_{n+1} = \left(\frac{4x_n^3+8}{6x_n^2+5}\right), \; n = 0, 1, 2 \ldots\ldots\ldots$$

(b) Draw a flow chart that;

(i) Reads the initial approximation α of the root

(ii) Computes and prints the root of the equation, when the error is less than 0.001

(c) Taking, $\alpha = 1.1$, perform a dry run for the flow chart, give your root correct to three decimal places **An(1.087)**

9. A shop offers a 25% discount on all items in their store and a second discount of 5% for paying cash.

(a) Construct a flow chart for the above information

(b) Perform a dry run for;

(i) A radio 125,000/= cash

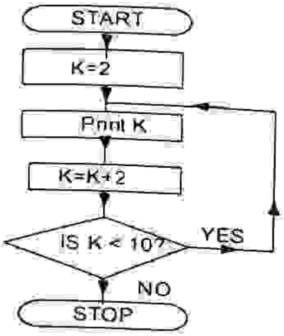
(ii) A T.V 340,000/= credit

An((i)=89.062.50, (ii)=255,000)

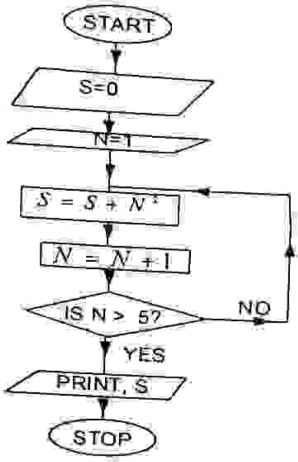
10. Given that a man deposited 120,000/= to a bank which gives a compound interest rate of 15% . Draw a flow chart to compute the amount of money accumulated after 4 years, and perform a dry run for the flow chart **An(=209880.75/=)**

11. Study the flow charts below and perform a dry run of each flow chart

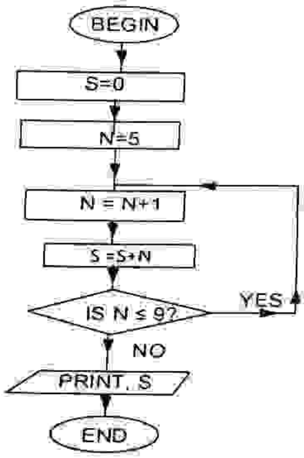
(i)



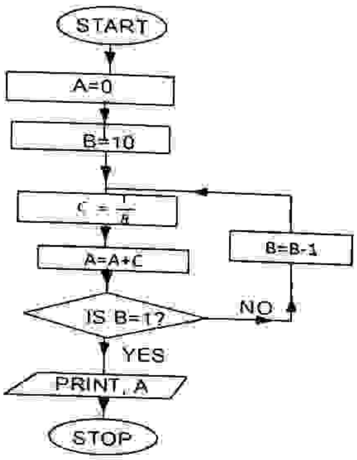
(ii)



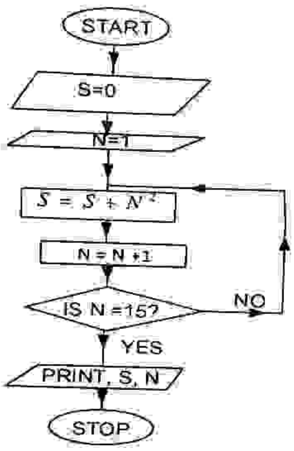
(iii)



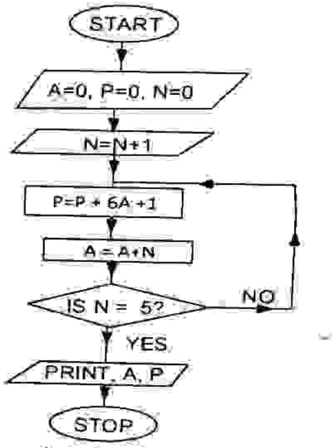
(iv)



(v)

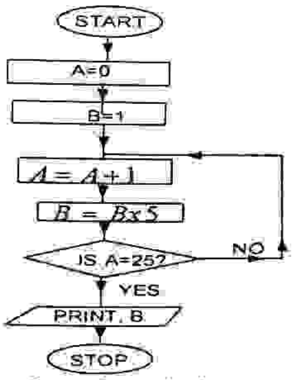


(vi)

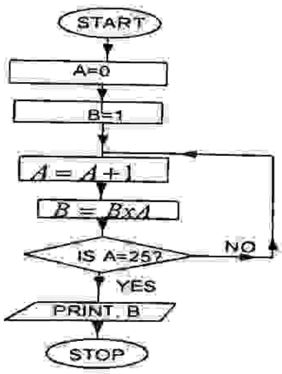


12. Study the flow charts below and perform a dry run of each flow chart and state the purpose of each flow chart

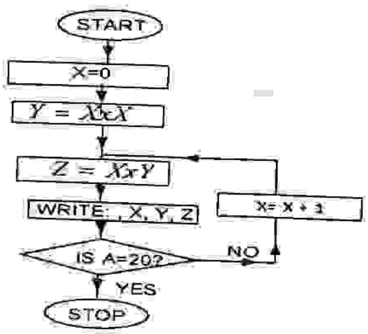
(i)



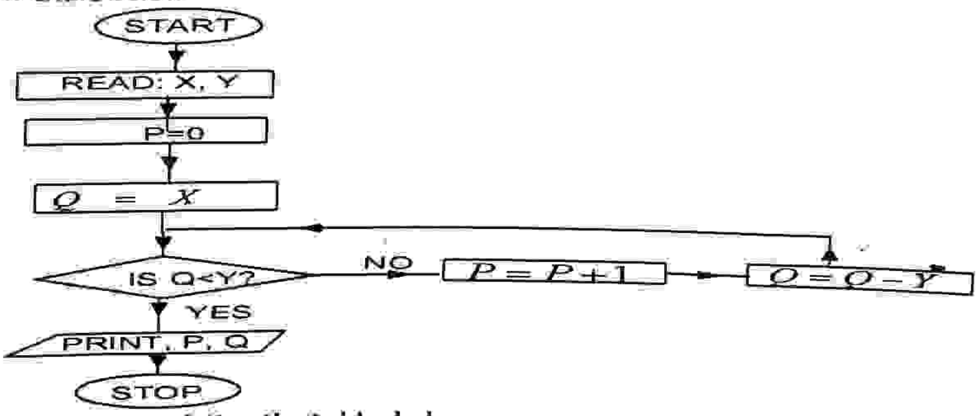
(ii)



(iii)



13. Given the flow chart below



Perform a dry run by completing the tables below

(i) X=17, Y=13

P	Q
0	17
----------	----------
----------	----------
----------	----------
----------	----------

(ii) X=50, Y=7

P	Q
----------	----------
----------	----------
----------	----------
----------	----------
----------	----------

(iii) X=9, Y=2

P	Q
----------	----------
----------	----------
----------	----------
----------	----------
----------	----------

14. Study the flow charts below and perform a dry run of each flow chart and state the purpose of each flow chart

MECHANICS

a) Kinematics

- Motion in a straight line
- Motion under gravity
- Motion on a smooth plane
- Projectile motion
- Relative motion
- Variable acceleration

(b) Dynamics

- Force and newton's laws
- Connected particles
- Work, energy and power
- Linear Momentum
- Simple harmonic motion
- Elasticity
- Circular motion

(C) Statics

- Particles in equilibrium
- Resolution and composition
- Moments
- Friction
- Coplanar forces
- Centre of gravity

CHAPTER 1: VECTORS

A vector is a quantity which has both magnitude and direction. Eg Force, displacement, velocity, acceleration and momentum

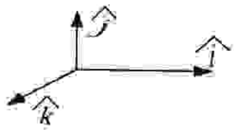
Representation of a vector

A vector is represented by a line with an arrow put to indicate the direction of the vector.

Where order of the letters shows the direction

Vectors in three dimensions

Consider the three dimensional rectangular co-ordinate system and let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be unit vectors along the x, y and z- axes



RESULTANT OF VECTORS

When several vectors $(V_1, V_2, \cdots, V_N)$ are acting on an object of mass m, the net vector, R is calculated as the vector sum

$$R = V_1 + V_2 + V_3 \dots\dots\dots\dots\dots\dots + V_N = \sum_{r=1}^{N} V_r$$

Examples

1. Find the resultant of each of the following forces
   (a) $(2\hat{i} + 3\hat{j} + 3\hat{k})N$, $(2\hat{i} + 4\hat{j} - 8\hat{k})N$ | **(b)** $(7\hat{i} - 4\hat{j} + 3\hat{k})N$, $(5\hat{i} - 2\hat{j} + 8\hat{k})N$, $(\hat{i} - \hat{k})N$,

   Solution

   (a) $R = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ -8 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -5 \end{pmatrix}$ | **(b)** $R = \begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 13 \\ -6 \\ 10 \end{pmatrix}$

2. The resultant of the forces $(5\hat{i} - 2\hat{j})N$, $(7\hat{i} + 4\hat{j})N$, $(a\hat{i} + b\hat{j})N$, and $(-3\hat{i} + 2\hat{j})N$ is a force $(5\hat{i} + 5\hat{j})N$. Find a and b. **An**$(a = -4, b = 1)$
3. The resultant of the forces $(\hat{i} - 2\hat{j} + 2a\hat{k})N$, $(2\hat{i} + \hat{j} + 4\hat{k})N$, $(b\hat{i} + 2\hat{j})N$ is $(8\hat{i} + C\hat{j} + 14\hat{k})N$. **An**$(a = 5, b = 5, c = 1)$
4. The resultant of the forces $3\hat{i} + (a - c)\hat{j}$, $(2a + 3c)\hat{i} + 5\hat{j}$, $(4\hat{i} + 6\hat{j})N$ acting on a particle is a force $(10\hat{i} + 12\hat{j})N$. Find .**Uneb 2006 No.4**
   (i) Values of a and c. | (ii) Magnitude of $(2a + 3c)\hat{i} + 5\hat{j}$,
   An$(a = 1.2, c = 0.2, 5.83N)$

MAGNITUDE OR MODULUS OF A VECTOR

This is the length of a vector

(i) Given that $\vec{R} = x\hat{i} + y\hat{j}$

$$|\vec{R}| = \sqrt{x^2 + y^2}$$

(ii) Given that $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\vec{R}| = \sqrt{x^2 + y^2 + z^2}$$

Example

Find the magnitude of the following vectors

(i) $a = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

(ii) $b = 2\hat{i} + 3\hat{j} - 6\hat{k}$

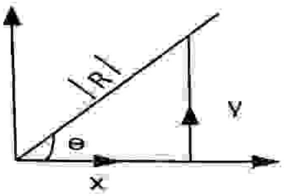
Solution

$|a| = \sqrt{4^2 + 3^2} = 5$

$|b| = \sqrt{2^2 + 3^2 + (-6)^2} = 7$

DIRECTION OF THE VECTOR

Consider $\vec{R} = x\hat{i} + y\hat{j}$



$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Examples

1. Find the magnitude and direction of the resultant of each of the following

(a) $(2\hat{i} + 3\hat{j})N$, $(5\hat{i} - 2\hat{j})N$, $(-3\hat{i} + 3\hat{j})N$

(b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}N$, $\begin{pmatrix} -6 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}N$, $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$

Solution

(a) $R = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$|R| = \sqrt{4^2 + 4^2} = 5.5669N$

Direction $\theta = tan^{-1}\left(\frac{4}{4}\right) = 45°$

The resultant force is $5.6569N$ at $45°$ above the horizontal

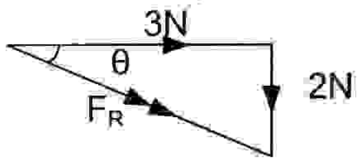
(b) $R = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -6 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$|R| = \sqrt{(-2)^2 + 0^2} = 2N$

Direction $\theta = tan^{-1}\frac{0}{2} = 180°$

(c) $R = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$|R| = \sqrt{3^2 + (-2)^2} = 3.6056N$



Direction $\theta = tan^{-1}\left(\frac{2}{3}\right) = 33.69°$

The resultant force is $3.6056N$ at $33.69°$ below the horizontal.

2. Four forces $a\hat{i} + (a-1)\hat{j}$, $3\hat{i} + 2a\hat{j}$, $(5\hat{i} - 6\hat{j})N$ and $-\hat{i} - 2\hat{j}$ act on a particle. The resultant of the forces makes an angle of $45°$ with the horizontal. Find the value of a and hence determine the magnitude of the resultant force .**Uneb 1999 No.2. An(8, 21.2N)**

UNIT VECTOR

This is a vector whose magnitude is unit (1).

Unit vector of r denoted by $\hat{r}$ is given by $\hat{r} = \frac{r}{|r|}$

Example

Find the unit vector of $a = 6\hat{i} - 2\hat{j} + 3\hat{k}$

Solution

$$\hat{a} = \frac{6\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{6^2 + (-2)^2 + 3^2}} = {}^1/_7(6\hat{i} - 2\hat{j} + 3\hat{k})$$

Parallel vectors

If a vector a and b are parallel, then one of them is a scalar multiple of the other

If a vector r of magnitude $|r|$ moves in direction $x\hat{i} + y\hat{j} + z\hat{k}$ then, $r = |r|\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$

Examples

1. Find the vector which has a magnitude of 15 units and is parallel to $16\hat{i} + 12\hat{j}$.

Solution

$v = 15x\frac{16\hat{i} + 12\hat{j}}{\sqrt{16^2 + 12^2}}$ | $= 15x\frac{16\hat{i} + 12\hat{j}}{20}$ | $= 12\hat{i} + 9\hat{j}$

2. A body of velocity v and of magnitude 20m/s moves in the direction $6\hat{i} + 8\hat{j}$. Find v

Solution

$v = 20x\frac{6\hat{i} + 8\hat{j}}{\sqrt{6^2 + 8^2}}$ | $= 20x\frac{6\hat{i} + 8\hat{j}}{10}$ | $= 12\hat{i} + 16\hat{j}$

3. A force of magnitude 12N acts on a body in the direction $2\hat{i} + \hat{j} + 2\hat{k}$. Find the force

Solution

$v = 12x\frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}$ | $= 12x\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$ | $= 8\hat{i} + 4\hat{j} + 8\hat{k}$

4. The force A of magnitude 5N acts in the direction with unit vector $\frac{1}{5}(3\hat{i} + 4\hat{j})$ and force B of magnitude 13N acts in the direction with unit vector $\frac{1}{13}(5\hat{i} - 12\hat{j})$. Find the resultant of forces A and B. **Uneb 2008 No.3**

Solution

$F_A = \frac{1}{5}(3\hat{i} + 4\hat{j})\ x\ 5 = 3\hat{i} + 4\hat{j}$

$F_B = \frac{1}{13}(5\hat{i} - 12)\ x\ 13 = 5\hat{i} - 12\hat{j}$

$R = \begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}5\\-12\end{pmatrix} = \begin{pmatrix}8\\-8\end{pmatrix}$

$|R| = \sqrt{8^2 + (-8)^2} = 11.3137N$

Exercise: 11A

1. Find the resultant of each of the following forces
   (a) $(6\hat{i} + 2\hat{j})N, (-5\hat{i} + \hat{j})N, (3\hat{i} - 3\hat{j})N$
   (b) $(2\hat{i} + 4\hat{j})N, (3\hat{i} - 5\hat{j})N, (6\hat{i} + 2\hat{j})N, (-7\hat{i} - 7\hat{j})N$
   (c) $(2\hat{i} + 3\hat{j} - 7\hat{k})N, (2\hat{i} + 5\hat{k})N, (3\hat{j} + 4\hat{k})N$
   An $(4\hat{i})N, (4\hat{i} - 6\hat{j})N, (4\hat{i} + 6\hat{j} + 2\hat{k})N$
2. The resultant of the forces $(5\hat{i} + 7\hat{j})N$, $(a\hat{i} + b\hat{j})N$, and $(-b\hat{i} - a\hat{j})N$ is a force $(11\hat{i} + 5\hat{j})N$. Find a and b. **An**$(a = 4, b = 2)$
3. Find the magnitude and direction of the resultant of each of the following
   (a) $(-2\hat{i} + 5\hat{j})N, (\hat{i} + 2\hat{j})N,$ | **(b)** $(6\hat{i} + 2\hat{j})N, (4\hat{i} - 3\hat{j})N$ | (c) $(3\hat{i} + 2\hat{j})N, (-5\hat{i} + \hat{j})N$
   An$(7.07N\ at\ 98.1°), (10.05N\ at\ 354.3°), (3.61N\ at\ 124°)$
4. A force of magnitude 50N acts on a body in the direction $24\hat{i} + 7\hat{j}$. Find the force
   An$(48\hat{i} + 14\hat{j})$.
5. Two forces F_1 and F_2 have magnitudes αN and βN and act in the direction $\hat{i} - 2\hat{j}$ and $4\hat{i} + 3\hat{j}$ respectively. Given that the resultant of F_1 and F_2 *is* $(48\hat{i} + 14\hat{j})$.Find αN and βN
   An$(\alpha = 8\sqrt{5}N$ and $50N$).
6. If $a = 3\hat{i} + 4\hat{j},\ b = 4\hat{i} + 20\hat{j}$ and $c = 5\hat{i} - 19\hat{j}$. Find the;
   (i) Resultant of a and b
   (ii) Resultant of a and c
   (iii) $|c|$
   (iv) $|a + b|$
   (v) Vector is parallel to a and has a magnitude of 15 units
   (vi) Vector is parallel to $(a + b)$ and has a magnitude of 100 units
   An (i) $(7\hat{i} + 24\hat{j})$, (ii) $(8\hat{i} - 15\hat{j})$, (iii) 19.6469 units , (iv) 25 units, **(v)** $(9\hat{i} + 12\hat{j})$, (vi) $(28\hat{i} + 96\hat{j})$,

7. If $a = 2\hat{\imath} + 5\hat{\jmath}$, $b = -7\hat{\imath} + 7\hat{\jmath}$ and $c = 14\hat{\imath}$. Find the;

   (i) Resultant of a and b
   (ii) Resultant of a, b and c
   (iii) $|b|$
   (iv) $|a + b + c|$
   (v) Vector is parallel to a and has a magnitude of $5\sqrt{29}$ units
   (vi) Vector is parallel to $(a + b + c)$ and has a magnitude of 90 units

   An (i) $(-5\hat{\imath} + 12\hat{\jmath})$, (ii) $(9\hat{\imath} + 12\hat{\jmath})$, (iii) $7\sqrt{2}$units , (iv) 15 units, **(v)** $(10\hat{\imath} + 25\hat{\jmath})$, (vi) $(54\hat{\imath} + 72\hat{\jmath})$,

8. If $a = \hat{\imath} - 3\hat{\jmath} + 2\hat{k}$, $b = 5\hat{\imath} + 4\hat{\jmath}$ and $c = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$. Find the;

   (i) Resultant of a and b
   (ii) Resultant of a, b and c
   (iii) $|a|$
   (iv) $|a + b + c|$
   (v) Vector is parallel to $(a + b + c)$ and has a magnitude of 5units

   An (i) $(6\hat{\imath} + \hat{\jmath} + 2\hat{k})$, (ii) $(9\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$, (iii) $\sqrt{14}$units , (iv) 11 units, (v) $^{5}/_{11}(9\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$,

9. If $a = 2\hat{\imath} + 7\hat{\jmath} + 7\hat{k}$, $b = 6 - 3\hat{\jmath} + 2\hat{k}$ and $c = -4\hat{\jmath} - 3\hat{k}$. Find the;

   (i) Resultant of a and b
   (ii) Resultant of a and c
   (iii) $|b|$
   (iv) $|a + b + c|$
   (v) Vector is parallel to $(a + b + c)$ and has a magnitude of 50units

   An (i) $(8\hat{\imath} + 4\hat{\jmath} + 9\hat{k})$, (ii) $(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k})$, (iii) 7units , (iv) 10 units, (v) $(40\hat{\imath} + 30\hat{k})$,

SCALAR PRODUCTS OR DOT PRODUCTS

The dot product of two vectors a and b inclined at an angle θ to each other is given by

$$a \bullet b = |a||b|\cos\theta$$

Note

If two vectors are perpendicular then the angle between them is 90° and $a \bullet b = 0$

Examples

1. If $p = \hat{\imath} - 2\hat{k}$ and $q = 3\hat{\imath} - 3\hat{\jmath} + \hat{k}$. Find;

   (i) $p.q$
   (ii) The angle between p and q

Solution

(i) $$p.q = \begin{pmatrix}1\\0\\-2\end{pmatrix}.\begin{pmatrix}3\\-3\\1\end{pmatrix} = 3 + 0 + -2 = 1$$

(ii) $$\theta = cos^{-1}\left(\frac{p.q}{|p||q|}\right) = cos^{-1}\left(\frac{1}{\sqrt{1^2+(-2)^2}\sqrt{3^2+(-3)^2+1^2}}\right) = 84.1°$$

2. If $a = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $b = \hat{\imath} + 4\hat{\jmath} + 3\hat{k}$. Find the angle between a and b

Solution

$$a.b = \begin{pmatrix}2\\-1\\3\end{pmatrix}.\begin{pmatrix}1\\4\\3\end{pmatrix} = 2 + -4 + 9 = 7$$

$$\theta = cos^{-1}\left(\frac{a.b}{|a||b|}\right) = cos^{-1}\left(\frac{7}{\sqrt{2^2 + (-1)^2 + (3)^2}\sqrt{1^2 + (4)^2 + 3^2}}\right) = 68°$$

3. If the angle between two vectors $a = x\hat{\imath} + 2\hat{\jmath}$ and $b = 3\hat{\imath} + \hat{\jmath}$ is 45°. Find the two possible values of constant x.

Solution

$$\begin{pmatrix}x\\2\end{pmatrix}.\begin{pmatrix}3\\1\end{pmatrix} = \sqrt{x^2 + 2^2}.\sqrt{3^2 + 1^2}cos45°$$

$$3x + 2 = \sqrt{x^2 + 4}.\sqrt{10}.\frac{\sqrt{2}}{2}$$

$(3x+2)^2 = (x^2+4).10.\begin{pmatrix}2\\4\end{pmatrix}$ | $x^2+3x-4=0$

$x=-4$ and $x=1$

4. If $a = 2\alpha\hat{\imath} + 7\hat{\jmath} - \hat{k}$ and $b = 3\alpha\hat{\imath} + \alpha\hat{\jmath} + 3\hat{k}$. Find the value of the scalar α if the vectors are perpendicular

Solution

$\begin{pmatrix}2\alpha\\7\\-1\end{pmatrix}.\begin{pmatrix}3\alpha\\\alpha\\3\end{pmatrix}=0$ | $6\alpha^2+7\alpha-3=0$

$\alpha = {}^1/_3$ and $\alpha = {}^3/_2$

Exercise: 11B

1. Find the scalar products for each of the following pairs of vectors.
   - (i) $a = 2\hat{\imath} + \hat{\jmath},\ b = \hat{\imath} - 3\hat{\jmath}$
   - (ii) $a = 3\hat{\imath},\ b = -2\hat{\imath} + \hat{\jmath}$
   - (iii) $a = 5\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $b = 4\hat{\imath} + 3\hat{\jmath} - 8\hat{k}$.
   - (iv) $2\hat{\imath} + 4\hat{\jmath} - 15\hat{k}$ and $-8\hat{\imath} + 2\hat{\jmath} - \hat{k}$.
   - (v) $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$
   - (vi) $\begin{pmatrix}5\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\4\end{pmatrix}$
   - (vii) $\begin{pmatrix}0\\5\\-2\end{pmatrix}$ and $\begin{pmatrix}-3\\2\\1\end{pmatrix}$
   - (viii) $\begin{pmatrix}5\\2\\7\end{pmatrix}$ and $\begin{pmatrix}1\\2\\-1\end{pmatrix}$

An(i)=-1, (ii)=-6, (iii)=39, (iv)=7, (v)=4, (vi)=6, (vii)=8, (viii)=2

2. Find the angles between each of the following pairs of vectors.
   - (i) $a = 3\hat{\imath} + 4\hat{\jmath},\ b = 5\hat{\imath} - 12\hat{\jmath}$
   - (ii) $a = 3\hat{\imath},\ b = -2\hat{\jmath}$
   - (iii) $a = 2\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$ and $b = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$.
   - (iv) $a = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $b = -\hat{\imath} + 2\hat{\jmath} - \hat{k}$.
   - (v) $\begin{pmatrix}3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-2\end{pmatrix}$
   - (vi) $\begin{pmatrix}6\\-8\end{pmatrix}$ and $\begin{pmatrix}5\\4\end{pmatrix}$
   - (vii) $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$
   - (viii) $\begin{pmatrix}2\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-1\end{pmatrix}$

An(i)=121°, (ii)=90°, (iii)=104°, (iv)=48°, (v)=82°, (vi)=92°, (vii)=120°, (viii)=73°

3. If $a = \lambda\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $b = 5\hat{\imath} - \lambda\hat{\jmath} + \hat{k}$. Find the value of the scalar λ if the vectors are perpendicular **An**$({}^1/_3)$
4. If $a = 2\hat{\imath} + \lambda\hat{\jmath} + \lambda\hat{k}$ and $b = -\lambda\hat{\imath} - \hat{k}$. Find the value of the scalar λ if the vectors are perpendicular **An**(0)
5. If $a = 4\hat{\imath} + 5\hat{\jmath}$ and $b = \alpha\hat{\imath} - 8\hat{\jmath}$. Find the value of the scalar α if the vectors are perpendicular. **An**(10)
6. If $a = 6\hat{\imath} - \hat{\jmath}$ and $b = 2\hat{\imath} + \alpha\hat{k}$. Find the value of the scalar α if the vectors are perpendicular **An**(12)
7. Given that $\begin{pmatrix}\lambda\\2+\lambda\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\4-\lambda\end{pmatrix}$ are perpendicular vectors. Find the value of the constant λ **An**(18)
8. If $a = \lambda\hat{\imath} + 8\hat{\jmath} + (3\lambda+1)\hat{k}$ and $b = (\lambda+1)\hat{\imath} + (\lambda-1)\hat{\jmath} - 2\hat{k}$. Find the value of the possible values of constant λ if the vectors are perpendicular. **An**$(2\ or - 3)$

FORCES

A force is anything which can change a body's state of rest or uniform motion in a straight line eg weight, tension, reaction, friction, resistance force

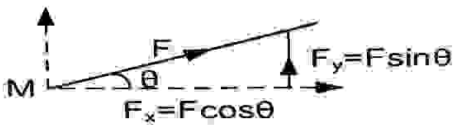
RESOLUTION OF FORCES

Components of a vector

The component of a vector is the effective value of a vector a long a particular direction. The component along any direction is the magnitude of a vector multiplied by the **cosine of the angle** between its direction and the direction of the component.

Suppose a force F pulls a body of mass m along a truck at an angle θ to the horizontal as shown below;

The effective force that makes the body move along the horizontal is the component of F along the horizontal



$$Cos\theta = \frac{F_x}{F}$$

$$\boxed{F_x = Fcos\theta}$$

$$Sin\theta = \frac{F_y}{F}$$

$$\boxed{F_y = Fsin\theta}$$

$$\boxed{\text{Resultant vector } F_R = \sqrt{F_x^2 + F_y^2}}$$

$$\boxed{\text{Direction}\alpha = tan^{-1}\left(\frac{F_y}{F_x}\right)}$$

Hints

When a vector is inclined at an angle θ to the horizontal then;

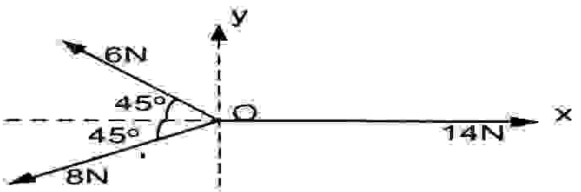
- Along the horizontal, the component of the vector is $cos\theta$
- Along the vertical, the component of the vector is $sin\ \theta$

When a vector is inclined at θ to the vertical then;

- Along the horizontal, the component of the vector is $sin\ \theta$
- Along the vertical, the component of the vector is $cos\theta$

Examples

1. Three forces are applied to a point as shown below



Calculate

a) The component in directions Ox and Oy respectively

b) Resultant force acting at O

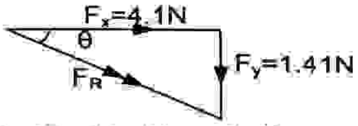
Solution

Components along Ox

$$F_x = 14 - 6cos45 - 8cos45 = 4.10N$$

Component along Oy

$$F_y = 6sin45 - 8sin45 = -1.41N$$



$$F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{4.1^2 + (-1.41)^2} = 4.34N$$

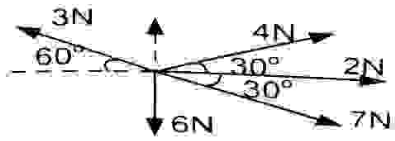
Direction $\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{1.41}{4.1}\right) = 19.0°$

Resultant force is $4.34N$ at $19.0°$ below the horizontal

2. Forces of 2N, 4N, 3N, 6N, and 7N act on a particle in the direction 0°, 30°, 120°, 270° and 330° respectively. Find the magnitude and direction of a single force represented by the above forces.

Solution

Note: the directions given involve 1,2 and 3 digits there fore they are angles and must be read anticlockwise starting from the positive x-axis

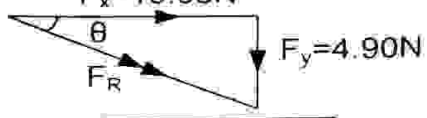


Resultant component along x-axis

$F_x = 2 + 4cos30 + 7cos30 - 3cos60 = 10.03N$

Resultant component along y-axis

$F_y = 4sin30 + 3sin60 - 7sin30 - 6 = -4.90N$



$F_R = \sqrt{10.03^2 + (-4.90)^2} = 11.16N$

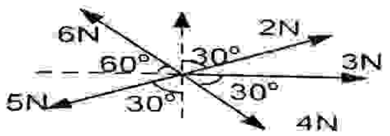
$\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{4.90}{10.03}\right) = 26.04^\circ$

The resultant force is 11.16N at 26.04° below the horizontal.

3. Forces of 2N, 3N, 4N, 5N, and 6N act on a particle in the direction 030°, 090°, 120°, 210°, and 330° respectively. Find the resultant force.

Solution

Note: the directions given involve 3 digits there fore they are bearings and must be read clockwise starting from the positive y-axis



Resultant along the x-axis

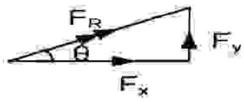
$F_x = 3 + 2sin30 + 4cos30 - 5cos30 - 6cos60$

$F_x = 1.964N$

Resultant along the y-axis

$F_y = 6sin60 + 2cos30 - 5cos30 - 4sin30$

$F_y = 0.598N$



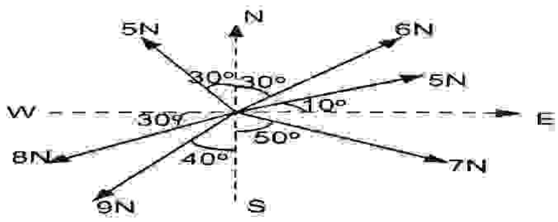
$F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{1.964^2 + 0.598^2} = 2.053N$

Direction $\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = 16.9^\circ$

The resultant force is 2.053N at 16.9° above the horizontal

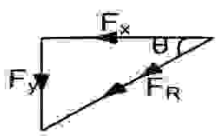
4. Forces of 6N, 5N, 7N, 8N, 5N, and 9N act pm a particle in the direction N30°E, N30°W, S50°E, N60°W, N80°E and S40°W, respectively. find the resultant force.

Solution



$F_x = 5cos10 + 6sin30 + 7sin50 - 9sin40 - 8cos50 - 5sin30 = -1.927N$

$F_y = 5cos30 + 6cos30 + 5sin10 - 8sin30 - 9cos40 - 7cos50 = -4.999N$

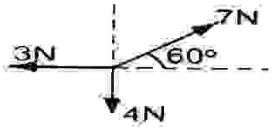


$F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{1.927^2 + 4.999^2} = 5.36N$

$\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{4.999}{1.927}\right) = 68.9^\circ$

Resultant force is 5.36N at 68.9° below horizontal

5. A particle at the origin O is acted upon by the three forces as shown below. Find the position of the particle after 2 seconds of its mass is 1kg.



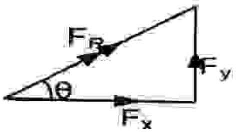
Solution

Resultant along horizontal

$$F_x = -3 + 7cos60 = 0.5N$$

Resultant along vertical

$$F_y = 7sin60 - 4 = 2.06N$$



$$F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{0.5^2 + 2.06^2} = 2.12N$$

But $F_R = ma$

$2.12 = 1a$

$a = 2.12ms^{-2}$

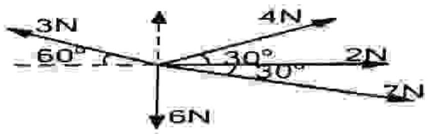
From $S = ut + ½at^2$

$u = 0 \quad t = 2s \quad a = 2.12ms^{-2}$

$S = 0x2 + ½ x2.12x2^2 = 4.24m$

6. Find the resultant of the system of forces below

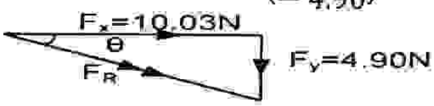
(a)



Solution

$$F_R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}4cos30\\4sin30\end{pmatrix} + \begin{pmatrix}7cos30\\-7sin30\end{pmatrix} + \begin{pmatrix}-3cos60\\3sin60\end{pmatrix} + \begin{pmatrix}0\\-6\end{pmatrix}$$

$$F_R = \begin{pmatrix}10.03\\-4.90\end{pmatrix}$$

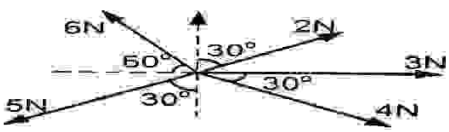


$$F_R = \sqrt{10.03^2 + 4.90^2} = 11.16N$$

$$\theta = tan^{-1}\frac{F_y}{F_x} = tan^{-1}\frac{4.90}{10.03} = 26.04°$$

The resultant force is 11.16Nat 26.04° below the horizontal.

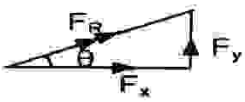
(b)



Solution

$$F_R = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}2sin30\\2cos30\end{pmatrix} + \begin{pmatrix}4cos30\\-4sin30\end{pmatrix} + \begin{pmatrix}-5cos30\\-5sin30\end{pmatrix} + \begin{pmatrix}-6cos60\\6sin60\end{pmatrix}$$

$$F_R = \begin{pmatrix}1.964\\0.598\end{pmatrix}$$



$$F_R = \sqrt{1.964^2 + 0.598^2} = 2.053N$$

Direction $\theta = tan^{-1}\frac{F_y}{F_x}$ θ = 16.9°

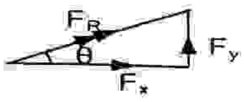
The resultant force is 2.053N at 16.9° above the horizontal

7. A body of mass 1kg is acted upon by the forces shown below. Find;
   (i) Magnitude of the resultant force
   (ii) Acceleration of the body
   (iii) Distance moved in 2s

Solution

Resultant along horizontal

$$F_R = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 7cos60 \\ 7sin60 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 2.06 \end{pmatrix}$$



$$F_R = \sqrt{0.5^2 + 2.06^2} = 2.12N$$

But $F_R = ma$

$2.12 = 1a$

$a = 2.12ms^{-2}$

From $S = ut + ½\, at^2$

$u = 0,\ t = 2s,\ a = 2.12ms^{-2}$

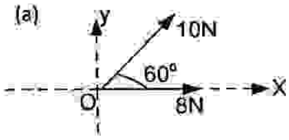
$S = 0x2 + ½\, x2.12x2^2$

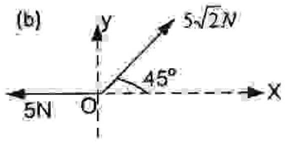
$s =$ 4.24m from the origin

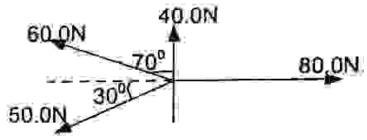
EXERCISE 11D

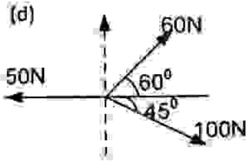
1. A force of 3N acts at 60° to a force of 5N. find the magnitude and direction of their resultant
   An(7N at 21.8° to the 5N force)
2. A force of 3N act at 90° to a force of 4N. Find the magnitude and direction of their resultant
   An(5N at 37° to the 4N force)
3. Two coplanar forces act on a point O as shown below

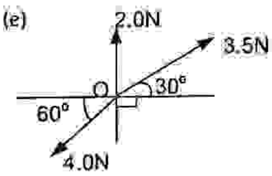
   Calculate the magnitude and direction of the resultant force
   An[12.3N at 68.0° above the horizontal]
4. Find the magnitude and direction of the forces in each of the diagrams given below

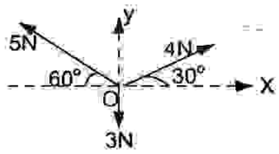
(a)


(b)


(c)


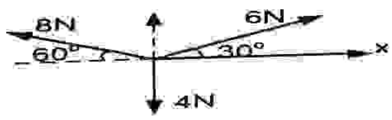
(d)


(e)


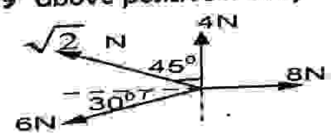
(f)


An[(a) 15.6N at 33.7° to the horizontal,
(b) 5N at 90° to the horizontal]
(c) **(40.6N at 61.0° to horizontal)**
(d) **An(54.1N at 20° below the horizontal)**
(e) An[1.07N at 15.5° above the horizontal]
(f) An[3.4N at 73.1° above the horizontal]

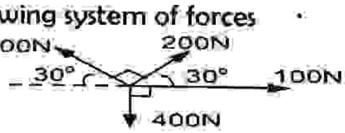
5. Find the magnitude and direction of the resultant of each of the following system of forces

(a)


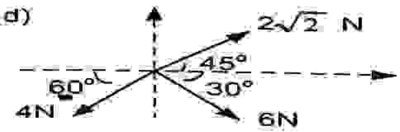
An (6N at 79° above positive x-axis)

(d)


An (2.99N at 47.9° above positive x-axis)

(c)


An (97.34N at 52.1° below positive x-axis)

(d)


An (6.851N at 319.3°)

6. The resultant of two forces PN and 3N is 7N. If the 3N force is reversed, the resultant if $\sqrt{17}N$. Find the value of P and the angle between the two forces. **An($2\sqrt{6}N$, 57.02°)**

7. Forces of 2N, 3N, 4N, 5N and 6N act on a particle in the direction 030°, 090°, 120°, 210°, and 330° respectively. Find the magnitude and direction of the resultant force. **An(1.92N at 7.8° to the horizontal)**

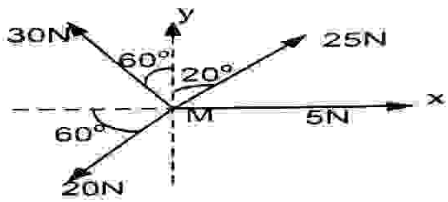
8. Forces of 7N, 2N, 2N, and 5N act on a particle in the direction 060°, 160°, 200° and 315° respectively. Find the resultant force. **An[4.14N at 52.36° below the horizontal]**

9. Find the magnitude and direction of the resultant of the forces 10N, 15N and 8N acting in the direction 030°, 150°, and 225°. **An(12.1N at 55.6° below positive x-axis)**

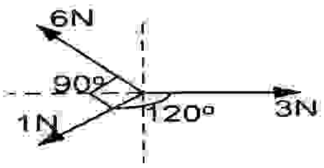
10. Forces of 2N, 1N, 3N and 4N act on a particle in the directions 0°, 90°, 270° and 330° respectively. Find the magnitude and direction of the resultant force. **An[6.77N at 36.2° below the horizontal]**

11. Four forces $(a\hat{i} - 1\hat{j})N$, $(3\hat{i} + 3a\hat{j})N$, $(5\hat{i} - 6\hat{j})N$ and $(-\hat{i} - 2\hat{j})N$ act on a particle. The resultant of the forces make an angle of 45° with horizontal. Find the value of a and hence determine the magnitude of the resultant. **An**$(a = 8, R = 15\sqrt{3})$

12. A body m of mass 6kg is acted on by forces of 5N, 20N, 25N and 30N as shown above. Find the acceleration of m. **An[5.5ms^{-2}]**

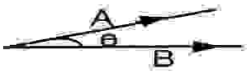


13. Three forces act on a body of mass 0.5kg as shown is the diagram. Find the position of the particle after 4 seconds. **An[3.44N, 55.2m]**

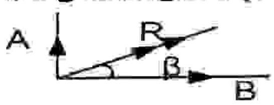


Resultant of two forces

Consider two forces A and B inclined to each other at an angle θ



(i) **θ is right angled** $(\theta = 90°)$



Resultant R is obtained from, $R^2 = A^2 + B^2$

Direction of resultant, $\beta = tan^{-1}\left(\frac{B}{A}\right)$

(ii) **θ is acute** $(0° \leq \theta \leq 90°)$

Resultant R is obtained from,

$R^2 = A^2 + B^2 - 2AB\ Cos(180 - \theta)$

Direction of resultant, $\frac{sin\,\beta}{A} = \frac{sin(180-\theta)}{R}$

$$\beta = sin^{-1}\left(\frac{A\,sin(180 - \theta)}{R}\right)$$

(iii) **θ is obtuse** $(90° \leq \theta \leq 180°)$

Resultant R is obtained from,

$R^2 = A^2 + B^2 - 2AB\ Cos(180 - \theta)$

Direction of resultant, $\frac{sin\,\beta}{A} = \frac{sin(180-\theta)}{R}$

$$\beta = sin^{-1}\left(\frac{A\,sin(180 - \theta)}{R}\right)$$

Examples

1. Two forces of magnitude 5N and 12N act on a particle with their direction inclined at 90°. Find the magnitude and direction of the resultant.

Solution

$R^2 = 5^2 + 12^2$

$R = 13N$

$\beta = tan^{-1}\left(\frac{5}{12}\right)$

$\beta = 22.6°$ to $12N$ force

2. Forces of 7N and 9N act on a particle at an angle of 60° between them. Find the magnitude and direction of the resultant

Solution

$R^2 = 7^2 + 9^2 - 2x7x9\ Cos(120°)$

$R = 13.89N$

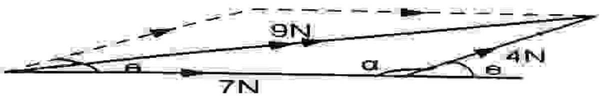
Direction of resultant, $\frac{sin\,\beta}{9} = \frac{sin(120°)}{13.89}$

$\beta = sin^{-1}\left(\frac{9sin(120°)}{13.89}\right) = 34.13°$

Resultant is 13.89N at $34.13°$ to the 7N force

3. Find the angle between a force of 7N and 4N their resultant has a magnitude of 9N

Solution



$9^2 = 7^2 + 4^2 - 2x7x4\ Cos\alpha$

$\alpha = cos^{-1}\left(-\frac{2}{7}\right) = 106.6°$

$\theta = 180 - 106.6 = 73.4°$

4. Forces of 3N and 2N act on a particle at an angle of 150° between them. Find the magnitude of the resultant force and its direction

Solution

$R^2 = 2^2 + 3^2 - 2x2x3Cos(30°)$

$R = 1.61N$

Direction of resultant, $\frac{sin\,\beta}{2} = \frac{sin(30°)}{1.62}$

$\beta = sin^{-1}\left(\frac{2\ sin(30°)}{1.61}\right) = 38.3°$

Resultant is 1.62N at $38.3°$ to the 3N force

Exercise 11C

1. Two forces of magnitude 7N and 24N act on a particle with their direction inclined at 90°. Find the magnitude and direction of the resultant **An(25N, at 16.26° with 24N force)**
2. Forces of 5N and 8N act on a particle at an angle of 50° between them. Find the magnitude and direction of the resultant **An(11.9N, at 19° with 8N force)**
3. Forces of 4N and 7N act on a particle at an angle of 25° between them. Find the magnitude and direction of the resultant **An(10.8N, at 9° with 7N force)**
4. Forces of 4N and 6N act on a particle at an angle of 60° between them. Find the magnitude and direction of the resultant **An(8.718N, at 23.4° with 6N force)**
5. Forces of 9N and 10N act on a particle at an angle of 40° between them. Find the magnitude and direction of the resultant **An(17.9N, at 18.9° with 10N force)**
6. Forces of 12N and 10N act on a particle at an angle of 105° between them. Find the magnitude and direction of the resultant **An(13.5N, at 45.7° with 12N force)**
7. Forces of 8N and 3N act on a particle at an angle of 160° between them. Find the magnitude and direction of the resultant **An(5.28N, at 11.2° with 8N force)**
8. Find the angle between a force of 10N and 4N their resultant has a magnitude of 8N **An(130.5°)**
9. The angle between a force of PN and a force of 3N is 120°. If the resultant of the two forces has a magnitude 7N, find the value of P. **An(8N)**
10. The angle between a force of QN and a force of 8N is 45°. If the resultant of the two forces has a magnitude 15N, find the value of Q **An(8.24N)**

FORCES ACTING ON A POLYGON

For any regular polygon;

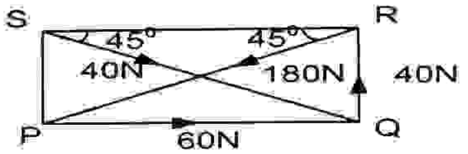
- All sides are equal
- All interior angles are equal
- All exterior angles are equal

$An\ exterior\ angle = \frac{360}{n}$ where n is number of sides

Examples

1. PQRS is a square. Forces of magnitude 60N, 40N, 180N and 40N act along the line PQ, QR, RP and SQ respectively in each case the direction of the force being given by the order of the letters. Given that SR is horizontal, determine
   (i) The magnitude of the resultant force (ii) Direction of the resultant with SR

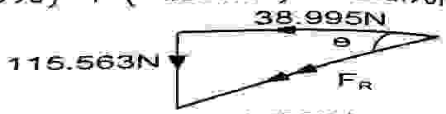
Solution



$$R = \begin{pmatrix}60\\0\end{pmatrix} + \begin{pmatrix}0\\40\end{pmatrix} + \begin{pmatrix}-180cos45\\-180sin45\end{pmatrix} + \begin{pmatrix}40cos45\\-40sin45\end{pmatrix}$$

$$R = \begin{pmatrix}-38.995\\-115.563\end{pmatrix}$$

$$R = \sqrt{(-38.995)^2 + (-115.563)^2} = 121.96N$$

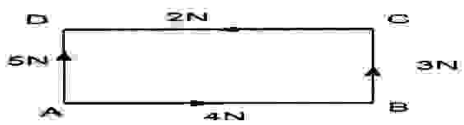


Direction $\theta = tan^{-1}\left(\frac{115.563}{38.995}\right)$ $\theta = 71.4°$

The resultant force is $121.96N$ at $71.4°$ below the negative x-axis

2. ABCD is a square. Forces of magnitude 4N, 3N, 2N and 5N act along the line AB, BC, CD and AD respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine
   (i) The magnitude of the resultant force (ii) Direction of the resultant with AB

Solution



$$R = \begin{pmatrix}4\\0\end{pmatrix} + \begin{pmatrix}0\\3\end{pmatrix} + \begin{pmatrix}-2\\0\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}2\\8\end{pmatrix}$$

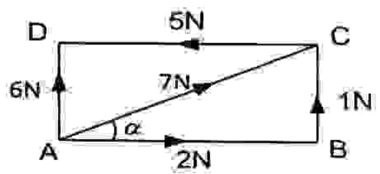
$$R = \sqrt{(2)^2 + (8)^2} = 8.25N$$

Direction $\theta = tan^{-1}\left(\frac{8}{2}\right)$ $\theta = 71.4°$

The resultant force is $8.25N$ at $71.4°$ to the horizontal

3. ABCD is a rectangle with $AB = 4cm$, and $BC = 3cm$. Forces of magnitude 2N, 1N, 5N, 6N, and 7N act along the line AB, BC, CD, AD, and AC respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine
   (i) The magnitude of the resultant force (ii) Direction of the resultant with AB

Solution

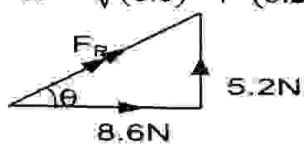


$$\tan\alpha = \frac{3}{4}$$
$$\alpha = 36.87°$$

$$R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-5\\0\end{pmatrix} + \begin{pmatrix}6\\0\end{pmatrix} + \begin{pmatrix}7cos36.87\\7sin36.87\end{pmatrix}$$

$$R = \begin{pmatrix}8.60\\5.2\end{pmatrix}$$

$$R = \sqrt{(8.6)^2 + (5.2)^2} = 10.0499N$$

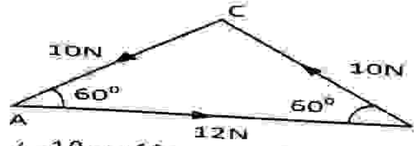


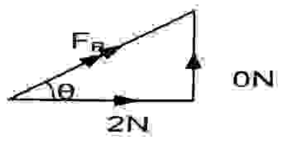
Direction $\theta = tan^{-1}\left(\frac{5.2}{8.6}\right)$ $\theta = 31.16°$

The resultant force is $10.0499N$ at $31.16°$ to the horizontal

4. ABC is an equilateral triangle. Forces of magnitude 12N, 10N and 10N act along the line AB, BC and CA, respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine
   (i) The magnitude of the resultant force (ii) Direction of the resultant with AB

Solution





$$R = \begin{pmatrix}12\\0\end{pmatrix} + \begin{pmatrix}-10\cos60\\-10\sin60\end{pmatrix} + \begin{pmatrix}-10\cos60\\10\sin60\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

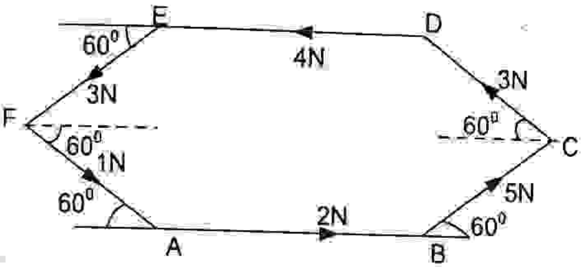
$$R = \sqrt{(2)^2 + (0)^2} = 2N$$

Direction $\theta = tan^{-1}\left(\frac{0}{0}\right)$ $\theta = 0°$

The resultant force is $2N$ parallel to AB

5. ABCDEF is a regular hexagon. Forces of magnitude 2N, 5N, 3N, 4N, 3N and 1N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude of the resultant force and direction of the resultant with AB

Solution



$$R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}5\cos60\\5\sin60\end{pmatrix} + \begin{pmatrix}-3\cos60\\3\sin60\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} + \begin{pmatrix}-3\cos60\\-3\sin60\end{pmatrix} + \begin{pmatrix}1\cos60\\-1\sin60\end{pmatrix}$$

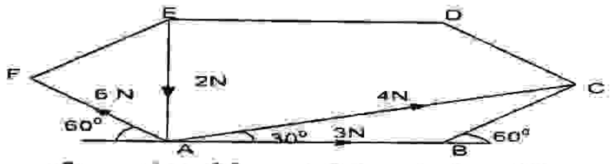
$$R = \begin{pmatrix}-2\\3.4641\end{pmatrix}$$

$$|R| = \sqrt{(-2)^2 + (3.4641)^2} = 4N$$

$$\theta = tan^{-1}\left(\frac{3.4641}{2}\right) = 60° \text{ to } AB$$

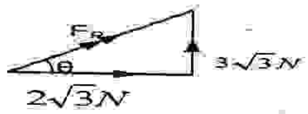
6. ABCDEF is a regular hexagon. Forces of magnitude 3N, 4N, 2N, and 6N act along the line AB, AC, EA, and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude of the resultant force and direction of the resultant with AB

Solution



$$R = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}4\cos30\\4\sin30\end{pmatrix} + \begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}-6\cos60\\6\sin60\end{pmatrix}$$

$$R = \begin{pmatrix}2\sqrt{3}\\3\sqrt{3}\end{pmatrix}$$



$$R = \sqrt{\left(2\sqrt{3}\right)^2 + \left(3\sqrt{3}\right)^2} = 6.245N$$

Direction $\theta = tan^{-1}\left(\frac{3\sqrt{3}}{2\sqrt{3}}\right)$ $\theta = 56.3°$

The resultant force is $6.245N$ at $56.3°$ above the positive x-axis

Exercise 11E

1. ABCD is a square. Forces of magnitude 6N, 4N and $2\sqrt{2}$N act along the line AD, AB and AC respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **Uneb 1998 nov/dec No.3 An(10N at 53.1° with AB)**

2. ABCD is a square of side a. Forces of magnitude 2N, 1N , $\sqrt{2}$N and 4N act along the line AB, BC, AC and DA respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **Uneb 2003 No.3 An(** 3.61N at 33.7° below AB)

3. In a square ABCD, three forces of magnitude 4N, 10N and 7N act along the line AB, AD and CA respectively in each case the direction of the force being given by the order of the letters. Determine the magnitude of the resultant force. **Uneb 2017 No.4 An(5.1388N)**

4. In an equilateral triangle PQR, three forces of magnitude 5N, 10N and 8N act along the sides PQ, QR and PR respectively. Their directions are in the order of the letters. Find the magnitude of the resultant forces. **Uneb 2018 No.7 An(16.0935N)**

5. ABCD is a rectangle. Forces of magnitude $6\sqrt{3}$N, 2N, and $4\sqrt{3}$N act along the line AB, CB, and CD respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **An**(4N at 30° to AB)

6. ABCD is a rectangle with $AB = 4m$, and $BC = 3m$. Forces of magnitude 3N, 1N and 10N act along the line AB, DC and AC respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **An(13.4N at 26.6° to AB)**

7. ABCD is a rectangle. Forces of magnitude 8N, 4N, 10N and 2N act along the line AB, CB, CD and AD respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **An(2.83N at 45° to AB)**

8. ABC is an equilateral triangle. Forces of magnitude 10N, 10N and 10N act along the line AB, BC and AC respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude of the resultant force. **An(20N at 60° to AB)**

9. ABC is an equilateral triangle. Forces of magnitude 5N, 9N and 7N act along the line AB, BC and CA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **An**($2\sqrt{3}N$ at 30° with AB)

10. ABC is an equilateral triangle. Forces of magnitude 4N, 4N and 6N act along the line AB, BC and AC respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **An(**$10N$ at 60° with AB)

11. ABCDEF is a regular hexagon. Forces of magnitude 2N, 5N, 3N, 4N, 3N and 1N act along the line AB, BC, CD, DE, EF and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force
An(6N at 60° with AB)

12. ABCDEF is a regular hexagon. Forces of magnitude 8N, 7N, 6N, 4N, 7N and 6N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force
An(12.5N at 76° with AB)

13. PQRSTU is a regular hexagon. Forces of magnitude 4N, 5N, 2N and 6N act along the line PQ, PR, PT, and PU respectively, determine the magnitude and direction of the resultant force with PQ **An(11.065N at 61.2° with PQ)**

14. ABCD is a square. Forces of magnitude 10N, 9N, 8N and 5N act along the line AB, BC, CD and AD respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **An**($2\sqrt{5}N$ at 63.43° with AB)

15. ABCD is a rectangle with $AB = 4m$, and $BC = 3m$. Forces of magnitude 3N, 10N, 4N, 6N, and 5N act along the line AB, BC, CD, DA, AC respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **An**($7.62N$ at 66.8° with AB)

EQUILIBRIUM OF FORCES

if there are several forces acting on a particle in equilibrium, then their resultant is equal to zero.

ie $F_R = \begin{pmatrix}0\\0\end{pmatrix}$

Examples

1. If each of the following sets of forces are in equilibrium find the values of **a** and **b** in each case
   - (i) $(6\hat{i} + 4\hat{j})N$, $(-2\hat{i} - 5\hat{j})$N, $(a\hat{i} + b\hat{j})$N
   - (ii) $(5\hat{i} + 4\hat{j})$N, $(3\hat{i} + \hat{j})$N and $(a\hat{i} + b\hat{j})$N
   - (iii) $(a\hat{i} + 3\hat{j})$N, $(2\hat{i} - 5\hat{j})$N and $(-7\hat{i} + b\hat{j})$N
   - (iv) $(a\hat{i} - 3b\hat{j})$N, $(b\hat{i} - 2a\hat{j})$N and $(-3\hat{i} + 8\hat{j})$N
   - (v) $(-3\hat{i} + 2\hat{j})$N, $(4\hat{i} + 7\hat{j})$N, $(-8\hat{i} + 5\hat{j})$N and $(a\hat{i} + b\hat{j})$N
   - (vi) $(4\hat{i} + 3\hat{j} - \hat{k})$N, $(\hat{i} - 5\hat{j} + 2\hat{k})$N and $(a\hat{i} + b\hat{j} + c\hat{k})$N
   - (vii) $(-2\hat{i} + 3\hat{k})$N, $(4\hat{j} - 7\hat{k})$N and $(a\hat{i} + b\hat{j} + c\hat{k})$N
   - (viii) $(5\hat{i} + a\hat{j} + c\hat{k})$N, $(b\hat{i} - 6\hat{j} - \hat{k})$N and $(-3\hat{i} + 2\hat{j} + c\hat{k})$N

Solution

(i) $\begin{pmatrix}6\\4\end{pmatrix} + \begin{pmatrix}-2\\-5\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$6 - 2 + a = 0$

$a = -4$

$4 - 5 + b = 0$

$b = 1$

(ii) $a = -8,\ b = -5$

(iii) $a = 5,\ b = 2$

(iv) $a = 1,\ b = 2$

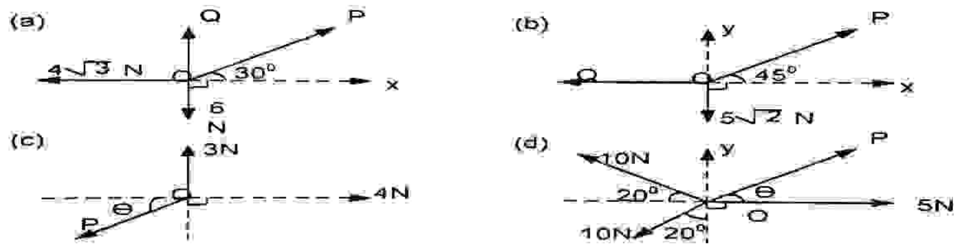
(v) $a = 7,\ b = -14$

(vi) $a = -5,\ b = 2, c = -1$

(vii) $a = 2,\ b = -4, c = 4$

(viii) $a = 4,\ b = -2, c = 0$

2. For each of the diagrams below, the particle is in equilibrium under the forces shown below. Find the unknown forces and angles



Solution

(a) $\begin{pmatrix}0\\Q\end{pmatrix} + \begin{pmatrix}Pcos30\\Psin30\end{pmatrix} + \begin{pmatrix}0\\-6\end{pmatrix} + \begin{pmatrix}-4\sqrt{3}\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$0 + Pcos30 + 0 - 4\sqrt{3} = 0$

$P = 8N$

$Q + Psin30 - 6 + 0 = 0$

$Q = 2N$

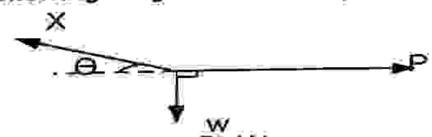
(b) $P = 10N, Q = 5\sqrt{2}N$

(c) $P = 5N, \theta = 36.9°$

(d) $P = (5\sqrt{3} + 4)$N, $Q = (5 + 4\sqrt{3})$N

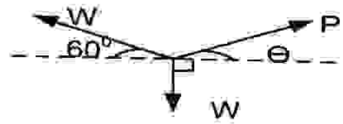
3. Each of the following diagrams shows a particle in equilibrium under the forces shows

(a)



Prove PtanΘ=W

(b)



Prove than tanΘ= $2\sqrt{3}$

Exercise 11D

1.

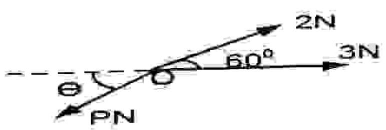
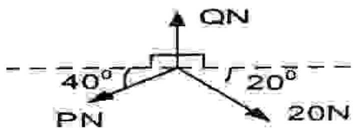


Diagram below shows three coplanar forces of magnitude 2N, 3N and PN all acting at a point O in the direction shown. Given that the forces are in equilibrium, find the value of P. **An** $(4.3589N)$

2.



(i) The diagram above shows three coplanar forces in equilibrium. Find the value of P and Q

(ii) If the direction of Q is now reversed, find the magnitude and direction of the resultant of the three forces. **An((i) 24.5N, 22.6N, (ii) 45.2N)**

3. Three forces F_1, F_2 and F_3 act on a particle and $F_1 = (-3\hat{\imath} + 7\hat{\jmath})N$, $F_2 = (\hat{\imath} - \hat{\jmath})N$ and $F_3 = (p\hat{\imath} + q\hat{\jmath})N$.

(a) If the particle is in equilibrium, find the value of p and q

(b) Find the magnitude of the resultant of F_1 and F_2 and determine the direction of that resultant

An ((a) $2, -6$**, (b)** $6.3246N, 71.57°$**)**

4. The diagram below shows three forces FN, 4N and 8N acting on a particle

If the forces are in equilibrium, find the value of;

(i) θ

(ii) F **An** $(6.928N, 60°)$

5. The diagram below shows three forces PN, 8N and 5N acting on a particle

Find the value of θ and P

(i) If the forces are in equilibrium,

(ii) If the resultant of the forces is 10N due north

An $((i)9.4354N, 32°(ii)17N, 61.92°)$

6. Force of acting on a particle are $(2\hat{\imath} + 3\hat{\jmath})N$, $(4\hat{\imath} - 7\hat{\jmath})N$, $(-5\hat{\imath} + 4\hat{\jmath})N$ and $(\hat{\imath} - \hat{\jmath})N$. find

(i) The resultant of the forces

(ii) The fifth force that should be added to the system of forces so that they are in equilibrium

Uneb 1988 No.6 **An.** $((2\hat{\imath} - \hat{\jmath})N, (-2\hat{\imath} - \hat{\jmath})N$

7. Forces of 6N, 5N, 7N, 8N, 5N and 9N act on particle in the direction N30°E, N30°W, S50°E, N60°W, N80°E and S40°W respectively. Find the additional force that will keep the system of forces in equilibrium.

An(5.358N at 68.92° above the positive axis)

8. Forces of 7N, 2N, 4N and 5N act on particle in the direction 060°, 160°, 200° and 315° respectively. Find the additional force that will keep the system of forces in equilibrium.

An(2.3125N at 37.18° below the negative axis)

9. Forces of 2N, 1N, 3N and 4N act on particle in the direction 0°, 90°, 270° and 330° respectively. Find the additional force that will keep the system of forces in equilibrium.

An(6.8N at 36° above the negative axis)

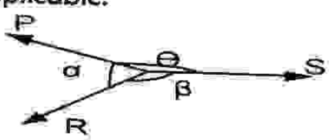
10. Forces of 6N, 5N, 7N, 4N, $3\sqrt{2}$N and $7\sqrt{2}$N act in direction AB, CB, CD, DA,CA and DB respectively on a square ABCD. Find the magnitude and direction of an additional force that will keep the system of forces in equilibrium.

An(19.2N at 81° above the negative axis)

11. Forces of 8N, 7N, 6N, 4N, 7N and 6N act along the sides of a regular hexagon ABCDEF in the direction AB, CB, CD, DE, EF and FA respectively. Find the magnitude and direction of an additional force that will keep the system of forces in equilibrium. **An(12.49N at 76° above AB)**

EQUILIRIUM OF THREE FORCES (LAMI'S THEOREM)

For any three forces acting on a particle in equilibrium where none of them is parallel to each other, lamis theorem is applicable.

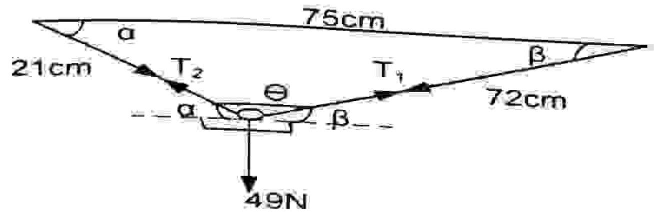


$$\frac{P}{\sin\beta} = \frac{S}{\sin\alpha} = \frac{R}{\sin\theta}$$

Examples

1. A weight of 49N is suspended by two strings of length 21cm and 72cm attached to 2 points in a the horizontal line a distance 75 cm apart. Find the tension in the strings so that the particle remains in equilibrium

Solution



By cosine rule: $75^2 = 21^2 + 72^2 - 2x21x72cos\theta$

$\theta = 90^{\circ}$

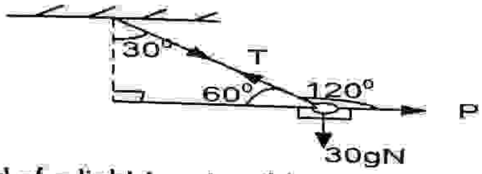
Similarly $\beta = 16.26^{\circ}$ and $\alpha = 73.74^{\circ}$

$\frac{T_1}{sin(90 + 73.74)} = \frac{49}{sin90}$ $\therefore T_1 = 13.72N$

$\frac{49}{sin90} = \frac{T_2}{sin(90 + 16.26)}$ $\therefore T_2 = 47N$

2. Mass of 30kg hangs vertically at the end of the light string. If the mass is pulled aside by the horizontal force P so that the string makes 30° with the vertical. Find the magnitude of the force P and the tension in the string so that the particle remains in equilibrium

Solution

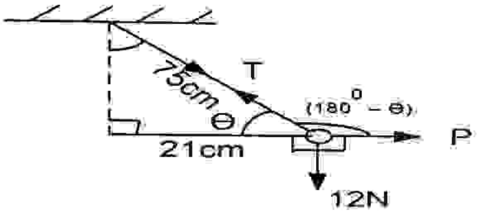


$\frac{T}{sin(90)} = \frac{30x9.8}{sin120}$ $\therefore T = 3339.47N$

$\frac{30x9.8}{sin120} = \frac{P}{sin(90 + 60)}$ $\therefore P = 169.74N$

3. One end of a light in extensible string of length 75cm is fixed to a point on a rigid pole. The particle of weight 12N is attached to the other end of the string. The particle is held 21cm away from the pole by horizontal force. Find the magnitude of this force and the tension of the sting so that that the particle is equilibrium

Solution Uneb 1996 No.1

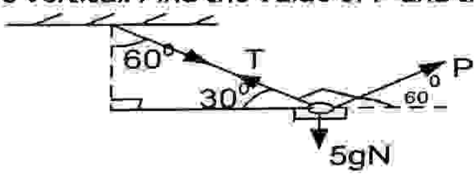


$cos\theta = \frac{21}{75}$, $\theta = 73.74^{\circ}$

$\frac{T}{sin(90)} = \frac{12}{sin(180 - \theta)}$ $\therefore T = 12.5N$

$\frac{12}{sin(180 - \theta)} = \frac{P}{sin(90 + \theta)}$ $\therefore P = 3.5N$

4. A light inextensible string AB whose end A is fixed has end B attached to a particle of mass 5kg. A force P acting perpendicular to the string is applied on the particle keeping it in equilibrium with the string inclined at 60° to the vertical. Find the value of P and the tension in the string



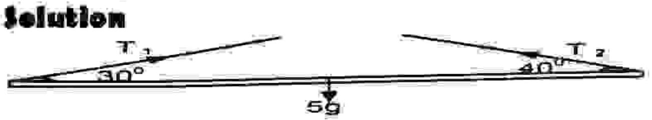
$\frac{T}{sin(90 + 60)} = \frac{5x9.8}{sin(90)}$ $\therefore T = 24.5N$

$\frac{5x9.8}{sin(90)} = \frac{P}{sin(90 + 30)}$ $\therefore P = 42.44N$

5. A non uniform beam of mass 5kg rests horizontally in equilibrium supported by two light strings attached to the ends of the beam.

The strings make angles of 30° and 40° with the beam as shown above. Find the tension in the strings

Solution



$(\rightarrow)$ $T_1cos30 = T_2cos40$

$T_1 = 0.8846T_2$

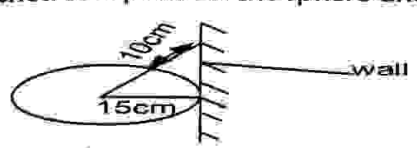
$(\uparrow)$ $T_1sin30 + T_2sin40 = 5g$

$0.8846T_2xsin30 + T_2sin40 = 5x9.8$

$T_2 = 45.159N$

$T_1 = 0.8846x45.159 = 39.947N$

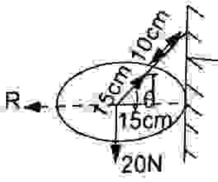
6. A sphere of weight 20N and radius 15cm rests against a smooth vertical wall. A sphere is supported in its position by a string of length 10cm attached to a point on the sphere and to a point on the wall as shown.



i) Calculate the reaction on the sphere due to the wall.

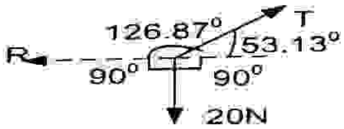
ii) Find the tension in the string

Solution



$Cos\theta = \frac{15}{28}$ $\therefore \theta = 53.13°$

Using Lami's theory



$$\frac{20}{Sin126.87} = \frac{T}{Sin90}$$

$$T = 25N$$

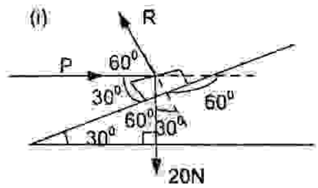
$$\frac{R}{Sin143.13} = \frac{20}{Sin126.87}$$

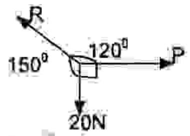
$$R = 15N$$

7. A particle of weight 20N is held at equilibrium on smooth plane incined at 30° to the horizontal by a horizontal force P.
   (i) Find the value of P and the reaction between the particle and the plane.
   (ii) If the force P is removed and the string parallel to the plane is used to hold the particle, find the tension in the string and the new value of the reaction

Solution

(i)



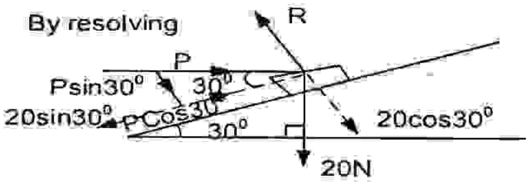


$$\frac{P}{sin(150)} = \frac{R}{sin(90)} = \frac{20}{sin(120)}$$

$R = 23.09N$ and $P = 11.55N$

Alternatively

By resolving



Parallel to the plane: $Pcos\,30 - 20sin30 = 0$

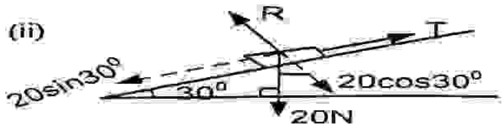
$P = 11.55N$

Perpendicular to the plane;

$R - Psin\,30 - 20COS30 = 0$

$R = 23.09N$

(ii)



Parallel to the plane: $T - 20sin30 = 0$

$T = 10N$

Perpendicular to the plane; $R - 20COS30 = 0$

$R = 17.3N$

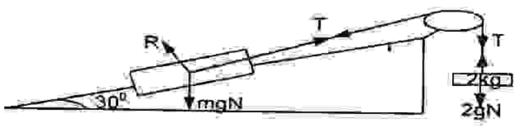
Or by lami's theorem

$$\frac{T}{sin(150)} = \frac{R}{sin(120)} = \frac{20}{sin(90)}$$

$R = 17.3N$ and $P = 10N$

8. A light inextensible string passes over a smooth fixed pulley at the top of a smooth plane inclined at 30° to th horizontal. A particle of mass 2kg is attached to one end of the string and rests vertically in equilibrium when a particle of mass m resting on the surface of the plane is attached to the other end of the. Find;
   (i) The normal reaction between m and the plane.
   (ii) The tension in the string and the value of m

Solution



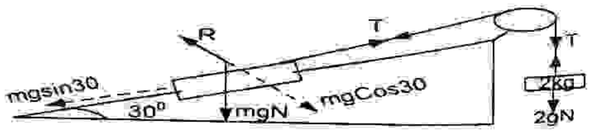
$$T - 2g = 0$$

$$T = 2x9.8 = 19.62$$

$$\frac{T}{sin(150)} = \frac{mg}{sin(90)} = \frac{R}{sin(120)}$$

$m = 4kg$ and $R = 33.98N$

Alternatively; By resolving



For 2kg mass: $T - 2g = 0$

$T = 2x9.8 = 19.62$

Parallel to the plane: $T - mgsin30 = 0$

$m = 4kg$

Perpendicular to the plane; $R - mgsin30 = 0$

$R = 33.98N$

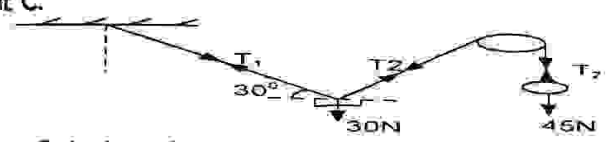
Exercise 11E

1. A particle p of mass 2kg is suspended from a fixed point O by means of a light inextensible string. The string is taut and makes an angle of 30° with the downward vertical through O and the particle is held in equilibrium by means of a horizontal force of magnitude F acting on the particle.

(i) The value of F (*ii*) The tension in the string
An([i] 11.3161N [ii] 22.6321N)

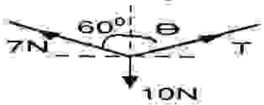
2. A particle of mass 3kg lies on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The particle is held in equilibrium by a horizontal force of magnitude PN. The line of action of this force is in the same vertical plane as a line of greatest slope of the inclined plane. find the value p **An(22.05N)**

3. The figure below shows a light inextensible string ABCD which is fixed at one end A. A small weight of 30N is attached at B and the other end a weight of 45N is freely running over a smooth peg at C.



Calculate the value of T_1 and T_2 if the system is in equilibrium. **An** $(T_1 = T_2 = 45N)$

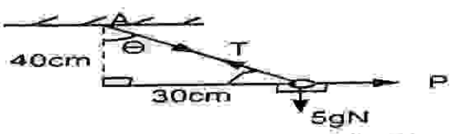
4. The diagram below shows a body of weight 10N supported in equilibrium by two light inextensible strings. The tensions in the strings are 7N and T and the angles the string makes with the upward vertical are 60° and θ respectively.



Find T and θ **An** $(T = 8.89N, \theta = 43°)$

5. A particle of weight 8N is attached to a point B of a light inextensible string AB. It hangs in equilibrium with point A fixed and AB at an angle of 30° to the downward vertical. A force F at B acting at right angles to AB, keeps the particle in equilibrium. Find the magnitude of the force F and the tension in the string .**Uneb 1998 Mar No.4.** **An**$(4N, 4\sqrt{3}N)$

6. The diagram shows a light inextensible string with one end fixed at A and a mass of 5kg suspended at the other end.



The mass is held in equilibrium at an angle θ to the downward vertical by a horizontal force P. find

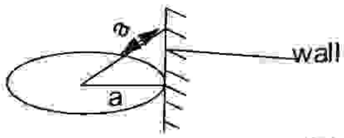
(i) The value of θ

(ii) The magnitude of the force P and the tension T.
An $(\theta = 36.9°, P = 36.75N, T = 61.25N)$

7. A sphere of mass 5kg and radius 63cm rests against a smooth vertical wall. A sphere is supported in its position by a string of length 24cm attached to a point on the sphere and to a point on the wall as shown.

Find the tension in the string. **An(71.05N)**

8. A sphere of weight W and radius a rests against a smooth vertical wall. A sphere is supported in its position by a string of length a attached to a point on the sphere and to a point on the wall as shown.



Find the tension in the string. **An**$(\frac{2w}{\sqrt{3}})$

9. A particle whose weight is 50N is Suspended by a light string which is 35° to the vertical under the action of horizontal force F. Find
(a) The tension in the string
(b) Force F **An(61.0N, 35.0N)**

10. A particle of weight W rests on a smooth plane which is inclined at 40° to horizontal. The particle is prevented from slipping by a force of 50.0N acting parallel to the plane and up a line of greatest slope. Calculate
(a) W (*b*) Reaction due to the plane
An(77.8N, 59.6N)

11. A mass of 2kg is suspended by two light inextensible strings. One making an angle of 60° with the upward vertical and the other 30° with the upward vertical. Find the tension in each string. **An** $(= 9.8N, = 17.0N)$

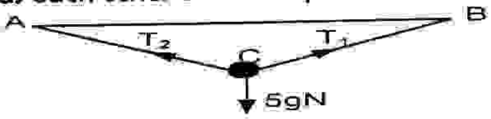
12. A heavy uniform rod of weight W is hung from a point by two equal strings, one attached to each end f the rod. A body of weight w is hung half way between A and the centre of the rod. Prove that the ratio of the tension in the string is $\frac{2W+3w}{2W+w}$

13. A non-uniform beam AB of length 8m and its weight of 10N acts from a point G between A and B such that $AG = 6m$. The beam is supported horizontally by strings attached at A and B. The string attached to A makes an angle of 30° with AB. Find the angle that the string attached to B makes with BA and find the tension in the strings. **An(60°, 5N, 8.66N)**

14. A light inextensible string of length 40cm has its upper end fixed to a point A and carries a mass of 2kg at its lower end. A horizontal force applied to

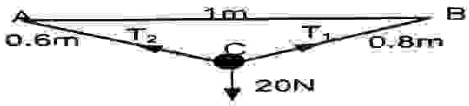
the mass keeps it in equilibrium, 20cm form the vertical through A. find the magnitude of this horizontal force and the tension in the string **An** $(= 11.3N, = 22.6N)$

15. The diagram shows a body of mass 5kg supported by two light inextensible strings, the other end of which are attached to two points A and B on the same level as each other end 7m apart



The body rests in equilibrium at, 3m vertically below AB. If $C\hat{B}A=45°$, find T_1 and T_2 the tension in the strings **An** $(= 35N, = 28\sqrt{2}N)$

16. The diagram shows a body of weight 20N suspended by two light inextensible strings of length 0.6m and 0.8m from two points 1m apart on a horizontal beam.



The body rests in equilibrium. find T_1 and T_2 the tension in the strings **An** $(= 16N, = 12N)$

17. A light inextensible string of length 50cm has its upper end fixed at point A and carries a particle of mass 8kg at its lower end. A horizontal force P applied to the particle keeps it in equilibrium 30cm from the vertical through A. find the magnitude of P and the tension in the string **An** $(= 58.8N, = 98N)$

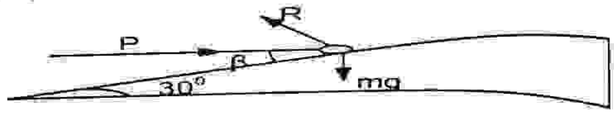
18. A particle is in equilibrium under the action of forces 4N due north, 8N due west, $5\sqrt{2}$ N south east and P. find the magnitude and direction of P. **An** $(= 3.16N, N71.6°E)$

19. A particle of mas 0.3kg lies on a smooth surface which is inclined at α to the horizontal, where $tan\alpha = \frac{3}{4}$. The particle is held in equilibrium by a horizontal forces force of magnitude PN. The line of action of this force is in the same vertical plane as a line of greatest slope of the inclined plane. Find the value of P **An** $(2.2N)$

20. A force acting parallel to and up a line of greatest slopes holds a particle of mass 10kg in equilibrium on a smooth plane which is inclined at 30° to the horizontal. Find the magnitude of the force and of the normal reaction between the particle and the plane **An** $(= 49N, = 84.9N)$

21. A horizontal force P holds a particle of mass 10kg in equilibrium on a smooth plane which is inclined at 30° to the horizontal. Find the magnitude of the force and of the normal reaction between the particle and the plane **An** $(= 56.6N, = 113N)$

22. A force P holds a particle of mass mkg in equilibrium on a smooth plane which is inclined at 30° to the horizontal.



If P makes an angle β with the plane, Find β when R the normal reaction between the particle and the plane is 1.5mg **An** $(51.7°)$

23. A particle of mass 5kg is held at equilibrium on smooth plane inclined at $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ to the horizontal by a horizontal force P. Find the value of P and the reaction between the particle and the plane. **Uneb 2001 No.5**
An $(= 28.3 = 56.6N,)$

24. (a) A particle of mass of 3kg is attached to the lower end B of a light inextensible string. The upper end A of the string is fixed to a point on the ceiling of a roof. A horizontal force of 22N and upward vertical force of 4.9N act upon the particle making it to be in equilibrium, with the string making an angle α with the vertical. Find the value of α and the tension in the string.
(b) A non- uniform rod of mass 9kg rests horizontally in equilibrium supported by two light inextensible strings tied to the ends of the rod. The strings make angles of 50° and 60° with the rod. Calculate the tension in the strings **Uneb 2002, No.12. An((a)41.9°, 33N, (b) 60.33N, 46.93N)**

25. A particle of weight W rests on a smooth inclined at 30° to the horizontal and is held in equilibrium by a sting inclined at 30° to the plane. Show that the tension in the string is $\frac{W\sqrt{3}}{3}$

26. A particle of mas 3kg lying on a smooth surface which is inclined at θ to the horizontal is attached to a light inextensible string which passes up the plane, along the ;line of greatest slope over a smooth pulley at the top carries a 1kg mass freely suspended at its other end. If the system rests in equilibrium, find
   (i) Value of θ
   (ii) The tension in the string
   (iii) The normal reaction between the particle and the plane **An** $(19.5°, 9.8N, 27.7N)$

27. A 5kg mass lies on a smooth horizontal table. A light inextensible string attached to this mass passes up and over a smooth pulley and carries a freely

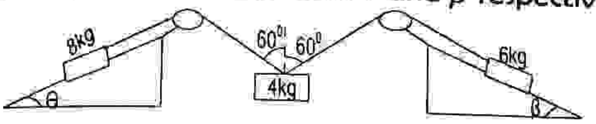
suspended mass of 5kg at its other end. The part of the string between the mass on the table and the pulley makes an angle of 25° with the horizontal. The system is kept in equilibrium by a horizontal force applied to the mass on the table. Find the;

(i) magnitude of this horizontal force,

(ii) tension in the string and

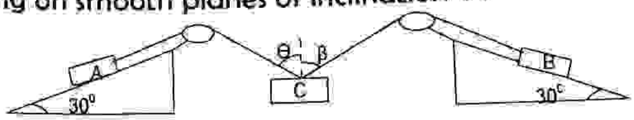
(iii) normal reaction between the table and the mass resting on it **An** $(44.4N, 49N, 28.3N)$

28. The diagram below shows masses of 8kg and 6kg lying on smooth planes of inclination θ and β respectively



Light inextensible strings attached to these masses pass along the lines of greatest slopes over smooth pulleys and are connected to a 4kg mass hanging freely. The strings both make an angle of 60° with the upward vertical as shown above. If the system rest in equilibrium find θ and β. **An** $(\theta = 30°, \beta = 41.8°)$

29. The diagram below shows masses A and B each lying on smooth planes of inclination 30°



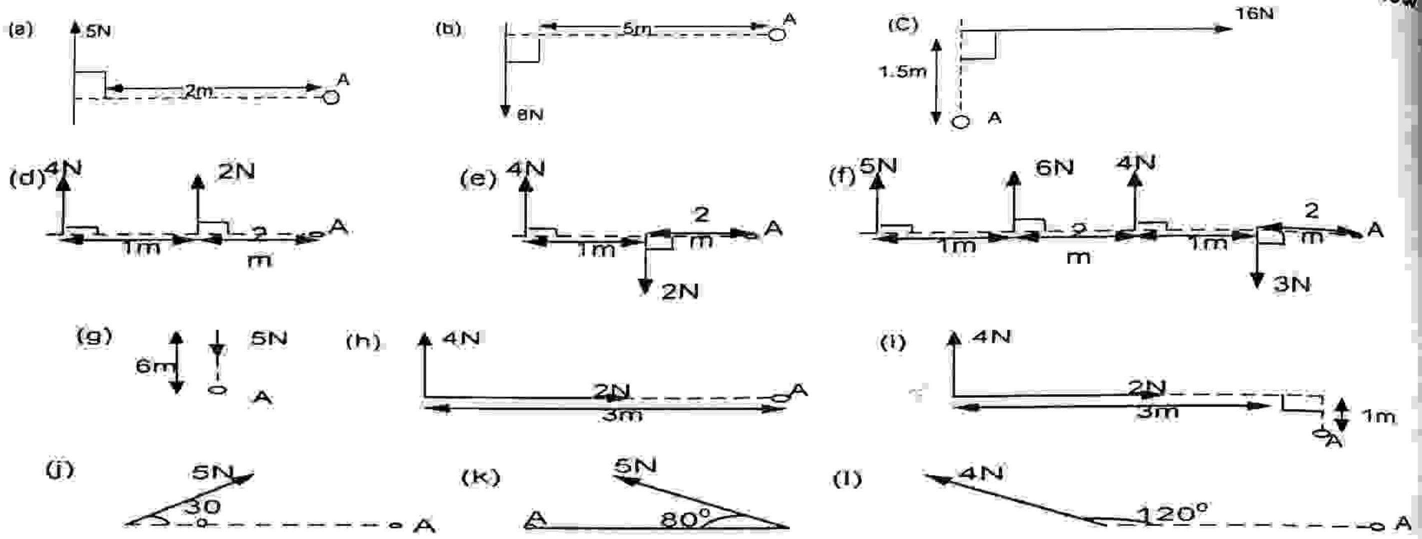
Light inextensible strings attached to A and B pass along the lines of greatest slopes, over smooth pulleys and are connected to a third mass C hanging freely. The strings make angle of θ and β with the upward vertical as shown above. If A, B and C have masses 2m, m and m respectively and the system rest in equilibrium, show that $sin\theta = 2sin\beta$ and $cos\beta + 2cos\theta = 2$. Hence find θ and β. **An** $(29.0°, 75.5°)$

This is the product of a force and perpendicular distance from the pivot to the line of action of the force. The unit of moments is Nm

The moment of F about point O is= Fxd

Examples

For each of the questions below, find the sum of the moments about points A of the forces shown below



Solution

(a)

$\overset{\frown}{A} = 5x2 = 10Nm\ clockwise$

(b)

$\overset{\frown}{A} = 8x5 = 40Nm\ anticlockwise$

(c)

$\overset{\frown}{A} = 1.5x16 = 24Nm\ clockwise$

(d)

$\overset{\frown}{A} = (2x2) + (4x3) = 16Nm\ clockwise$

(e)

$\overset{\frown}{A} = (4x3) - (2x2) = 8Nm\ clockwise$

(f)

$\overset{\frown}{A}$ =(5x5)+(6x4)+(4x3)-(3x2)=55Nm clockwise

(g)

$\overset{\frown}{A} = 5x0 = 0Nm$

(h)

$\overset{\frown}{A} = (4x3) + (2x0) = 12Nm\ clockwise$

(i)

$\overset{\frown}{A} = (2x1) + (4x3) = 14Nm\ clockwise$

(j)

$\overset{\frown}{A} = 5sin30°x10 = 25Nm\ clockwise$

(k)

$\overset{\frown}{A} = 5sin80°x2 = 9.85Nm\ clockwise$

(l)

$\overset{\frown}{A} = 4sin60°x10 = 34.6Nm\ clockwise$

MATRIX APPROACH OF FINDING SUM OF MOMENTS ABOUT THE ORIGIN

If forces $(a_1\hat{\imath} + b_1\hat{\jmath})N$, $(a_2\hat{\imath} + b_2\hat{\jmath})N, \ldots\ldots\ldots\ldots\ldots (a_n\hat{\imath} + b_n\hat{\jmath})N$ act on the body at points (x_1, y_1), $(x_2, y_2), \ldots\ldots\ldots\ldots\ldots (x_n, y_n)$. The sum of the moments about the origin is;

$$G = \begin{vmatrix} x_1 & a_1 \\ y_1 & b_1 \end{vmatrix} + \begin{vmatrix} x_2 & a_2 \\ y_2 & b_2 \end{vmatrix} + \cdots \ldots\ldots\ldots\ldots\ldots \begin{vmatrix} x_n & a_n \\ y_n & b_n \end{vmatrix}$$

$$G = (b_1x_1 - a_1y_1) + (b_2x_2 - a_2y_2) + \cdots \ldots\ldots\ldots (b_nx_n - a_ny_n)$$

Note

If G is positive, the sum of moments will be anticlockwise and if G is negative then the sum of moments will be clockwise

Examples

1. Find the moment about the origin of a force of $4\bar{j}N$ acting at a point which has position vector $5\bar{\imath}m$

   Solution

$$G = \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = 20Nm\ anticlockwise$$

2. Find the moment about the origin of a force of $4\bar{j}N$ acting at a point which has position vector $-5\bar{\imath}m$

   Solution

$$G = \begin{vmatrix} -5 & 0 \\ 0 & 4 \end{vmatrix} = -20Nm = 20Nm\ clockwise$$

3. Forces of $(2\hat{\imath} - 3\hat{j})N$, $(4\hat{\imath} + \hat{j})N$ and $(5\hat{\imath} - 3\hat{j})N$ act on a body at points with Cartesian co-ordinates $(1,1), (2,4), and\ (-1,3)$ respectively. Find the sum of moments of the forces about the origin is;

   Solution

$$G = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 5 \\ 3 & -3 \end{vmatrix}$$

$$G = (1x - 3 - 1x2) + (2x1 - 4x4) + (-3x - 1 - 3x5) = -31\ Nm = 31Nm\ clockwise$$

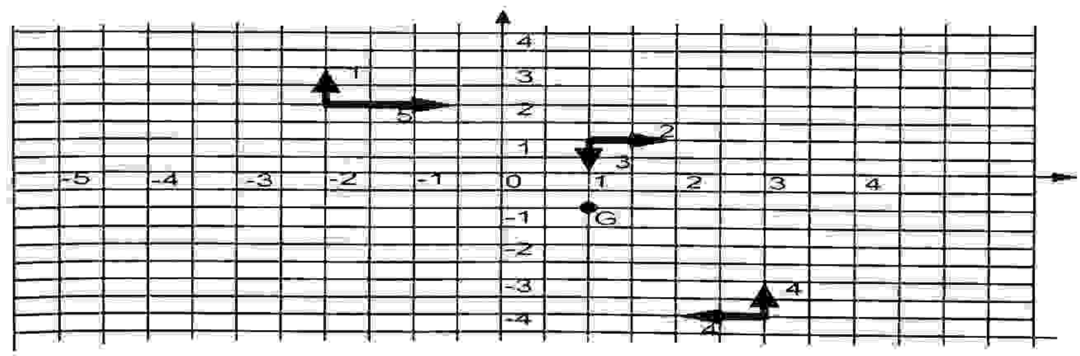
4. Forces of $(2\hat{\imath} - 3\hat{j})N$, $(5\hat{\imath} + \hat{j})N$ and $(-4\hat{\imath} + 4\hat{j})N$ act on a body at points with position vectors $(\hat{\imath} + \hat{j})$, $(-2\hat{\imath} + 2\hat{j})$ and $(3\hat{\imath} - 4\hat{j})$ respectively. Find the sum of moments of the forces about the
   (i) origin
   (ii) point with position vector$(\hat{\imath} - \hat{j})$

   Solution

   (a) $G = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} -2 & 5 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ -4 & 4 \end{vmatrix}$

   $G = (1x - 3 - 1x2) + (-2x1 - 5x2) + (3x4 - -4x - 4) = -21\ Nm = 21Nm\ clockwise$

   (b)



$$G = (5x3) + (1x3) + (2x0) + (2x2) + (4x3) - (4x2) = 26Nm\ clockwise$$

Exercise 12A

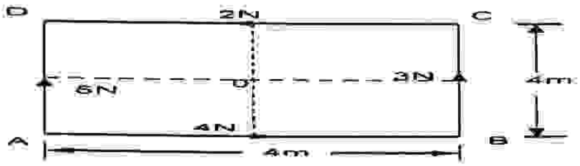
1. Find the moment about the origin of a force of $3\bar{\imath}\ N$ acting at a point which has position vector $(2\bar{\imath} + 3\bar{\imath})m$. **An**($9Nm\ clockwise$)
2. Find the moment about the origin of a force of $(4\bar{\imath} + 2\bar{j})\ N$ acting at a point which has position vector $(3\bar{\imath} + 2\bar{j})m$ **An**($2Nm\ clockwise$)
3. A force of $(3\bar{\imath} - 2\bar{j})\ N$ act at a point which has position vector $(5\bar{\imath} + \bar{j})m$. Find the moment about the point which has a position vector $(\bar{\imath} + 2\bar{j})m$ **An**($5Nm\ clockwise$)
4. A force of $(2\bar{\imath} + \bar{j})\ N$ act at a point which has position vector $(2\bar{\imath} + 2\bar{j})m$ and a force of $5\bar{\imath}\ N$ act at a point which has position vector $(-2\bar{\imath} + \bar{j})m$. Find the sum of moments of these forces about the origin. **An**($7Nm\ clockwise$)
5. A force of $(3\bar{\imath} + 2\bar{j})\ N$ act at a point which has position vector $(5\bar{\imath} + \bar{j})m$ and a force of $(\bar{\imath} + \bar{j})\ N$ act at a point which has position vector $(2\bar{\imath} + \bar{j})m$. Find the sum of moments of these forces about the point which has a position vector $(\bar{\imath} + 3\bar{j})m$ **An**($17Nm\ anti\ clockwise$)

MOMENT OF FORCES ACTING ON A POLYGON

Examples

1. ABCD is a square of side 4m. Forces of magnitude 4N, 3N, 2N and 5N act along the line AB, BC, CD and AD respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about
   (i) Centre O of the square | (ii) Point A

   Solution

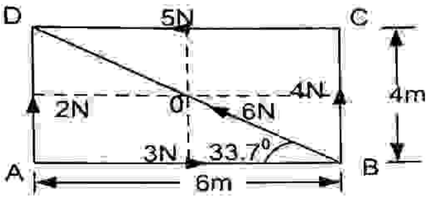


↺O $G = (4x2) + (3x2) + (2x2) - (5x2)$
$= 8\ Nm\ anticlockwise$

↺A $G = (4x0) + (3x4) + (2x4) - (5x0)$
$= 20\ Nm\ anticlockwise$

2. ABCD is a rectangle where $AB = 6m$ and $BC = 4m$. Forces of magnitude 3N, 4N, 5N, 2N and 6N act along the line AB, CD, AD and BD respectively in each case the direction of the force being given by the order of the letters. Find the sum of moments of the forces about
   (i) Centre O of the square | (ii) Point A

   Solution

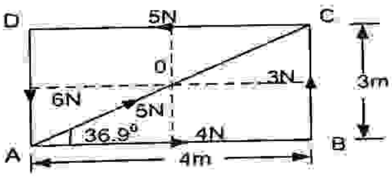


↺O $G = (3x2) + (4x3) + (5x2) - (2x3) +$
$(6\ sin33.7°x3) - (6\ cos33.7°x2)$
$= 22\ Nm\ anticlockwise$

↺A $G = (3x0) + (4x6) + (5x4) - (2x0) +$
$(6\ sin33.7°x6) - (6\ cos33.7°x0)$
$= 63.97\ Nm\ anticlockwise$

3. ABCD is a rectangle with $AB = 4m$, and $BC = 3m$. Forces of magnitude 4N, 3N, 5N, 6N, and 5N act along the line AB, BC, CD, DA, and AC respectively, in each case the direction of the force being given by the order of the letters. . Find the sum of moments of the forces about
   (i) Centre O of the square | (ii) Point A

   Solution

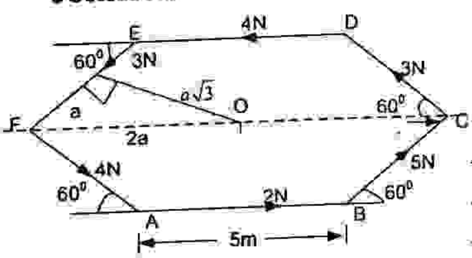


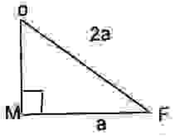
↺O $= (4x1.5) + (3x2) + (5x1.5) + (6x2) -$
$(5sin36.9°x2) + (5cos36.9°x1.5)$
$G = 31.49\ Nm\ anticlockwise$

↺A $G = (3x4) + (5x3)$
$G = 27\ Nm\ anticlockwise$

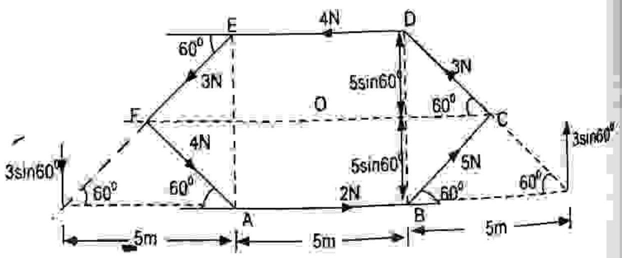
4. ABCDEF is a regular hexagon of side 5m. Forces of magnitude 2N, 5N, 3N, 4N, 3N and 4N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about
   (i) Centre O of the hexagon | (ii) Point A

Solution





$MO = \sqrt{(2a)^2 - a^2}$
$MO = \sqrt{4a^2 - a^2}$
$MO = a\sqrt{3}$

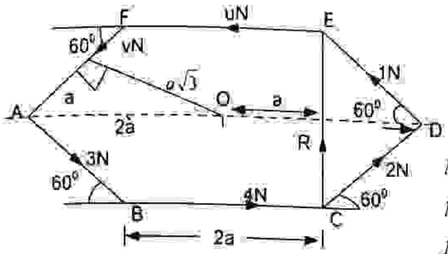


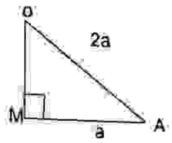
↺O $G = (2 + 5 + 3 + 4 + 3 + 4)a\sqrt{3} = 21x2.5\sqrt{3}$
$G = 52.5\sqrt{3}\ Nm = 90.933Nm\ anticlockwise$

↺A $G = (5sin60°x5) + (3sin60°x10) +$
$(4x10sin60°) + (3sin60°x5)$
$= 55\sqrt{3}\ Nm = 95.26Nm\ anticlockwise$

5. ABCDEF is a regular hexagon of side 2a. Forces of magnitude 3N, 4N, 2N, 1N, uN and vN act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Find the resultant of the value of u and v if the resultant of the six forces acts along CE

Solution





$MO = \sqrt{(2a)^2 - a^2}$

$MO = \sqrt{4a^2 - a^2}$

$MO = a\sqrt{3}$

Since the resultant is (↑), its (→) component is zero

(→) $4 - u + (2 - 1 - v + 3)cos60 = 0$

$(4 - v)x0.5 = u - 4$

$2u + v = 12 \ldots\ldots\ldots(i)$

(↑) $(2 + 1 - v - 3)sin60 = R$

$-v\dfrac{\sqrt{3}}{2} = R \ldots\ldots\ldots(ii)$

$Ra = (3 + 4 + 2 + 1 + u + v)a\sqrt{3}$

$R = (10 + u + v)\sqrt{3}\ldots\ldots\ldots(iii)$

(ii)=(iii); $-v\frac{\sqrt{3}}{2} = (10 + u + v)\sqrt{3}$

$2u + 3v = -20 \ldots\ldots\ldots(iv)$

(iv)-(i); $2v = -32$

$v = -16$

Also $2u + v = 12$

$2u - 16 = 12$

$u = 14$

Exercise 12B

1. Forces of magnitude 3N, 4N, 5N and 6N act on a rectangle along the line AB, BC, CD and DA respectively in each case the direction of the force being given by the order of the letters. Given that BC is horizontal, Find **Uneb 2005 No.7**
   (i) Magnitude and direction of the resultant force.
   (ii) couple at the centre of the rectangle of sides 2m by 4m. **An**$(2\sqrt{2}N, 135° \text{ to } BC, 26Nm$
2. Forces of 2N, 3N, 4N, and 5N act along the sides of a square ABCD of side 4m in the direction AB, BC, CD, and AD respectively. Find the sum of moments of the forces about
   (iii) The center of square.
   (iv) Point A **An(8Nm, 28Nm)**
3. Forces of 5N, 6N, 4N, 7N, 6N, and 8N act in the direction AB, BC, CD, DA, AC, DB respectively of a square ABCD of side 6m. Find the sum of moments of the forces about
   (i) The center of the square.
   (ii) Point A **An(66Nm, 26Nm)**
4. A B C D is a rectangle with $AB = 8cm$ and $BC = 6cm$. Forces of 4N, 5N, 3N, 6N, and 8N act in the direction AB, BC, CD, AD and BD respectively of the rectangle ABCD, find the sum of moments of the force about
   a) The center of the rectangle
   b) Point A **An(17Nm, 96.4Nm)**
5. ABCDEF is a regular hexagon 2m. Forces of magnitude 5N, 2N, 6N, 4N, 8N and 3N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about Point A: **An** $(29\sqrt{3}\ Nm)$
6. ABCDEF is a regular hexagon 3m. Forces of magnitude 4N, 5N, 1N, 3N, 7N and 2N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about Point A **An** $(30\sqrt{3}\ Nm)$
7. ABCDEF is a regular hexagon 4m. Forces of magnitude 8N, 4N, 7N, 4N, 6N and 5N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about Point A. **An** $(64\sqrt{3}\ Nm)$
8. ABCDEF is a regular hexagon 4m. Forces of magnitude 5N, 6N, 7N, 4N, 5N and 8N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about
   (i) Centre O of the hexagon (ii) Point A
   An $(70\sqrt{3}\ Nm, 66\sqrt{3}Nm)$
9. ABCDEF is a regular hexagon 3m. Forces of magnitude 3N, 1N, 2N, 5N, 6N and 4N act along the line AB, BC, CD, ED, EF and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about
   (i) Centre O of the hexagon (ii) Point A
   An $(4.5\sqrt{3}\ Nm\ )$

COUPLE

These are equal forces acting in opposite direction

Conditions for forces to form a couple

Force reduce to a couple if;

- Resultant force is zero
- If the sum of moments about a point is not zero

Examples

1. Forces of $(-5\bar{\imath} - \bar{\jmath})\ N$, $-3\bar{\jmath}$ N and $(5\bar{\imath} + 4\bar{\jmath})\ N$ act a body at a point with position vectors $(\bar{\imath} - \bar{\jmath})m$, $(2\bar{\imath} + \bar{\jmath})\ m$ and $(4\bar{\imath} - 5\bar{\jmath})m$ respectively. Show that these forces reduce to a couple

Solution

$$R = \begin{pmatrix}-5\\-1\end{pmatrix} + \begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

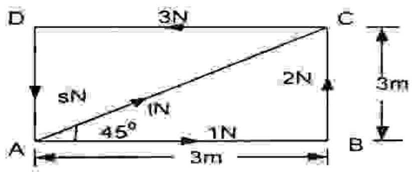
(a) $G = \begin{vmatrix}1 & -5\\-1 & -1\end{vmatrix} + \begin{vmatrix}2 & 0\\1 & -3\end{vmatrix} + \begin{vmatrix}4 & 5\\-5 & 4\end{vmatrix}$

$$G = (1x - 1 - -5x - 1) + (-3x2 - 1x0) + (4x4 - 5x - 5) = 29\ Nm$$

Since the resultant force is zero and $G \neq 0$, then the forces reduce to a couple

2. ABCD is a square of side 3m. Forces of magnitude 1N, 2N, 3N, sN, and tN act along the line AB, BC, CD, DA, and AC respectively, in each case the direction of the force being given by the order of the letters. Taking AB as horizontal and BC as vertical, find the values of s and t so that the resultant of the forces is a couple.

Solution



$$R = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}0\\-s\end{pmatrix} + \begin{pmatrix}tcos45\\tsin45\end{pmatrix}$$

$$R = \begin{pmatrix}0\\0\end{pmatrix}$$

$(\rightarrow)\quad tcos45 = 2$

$$t = \frac{2}{cos45} = 2\sqrt{2}N$$

$(\uparrow)\quad 2 - s + tsin45 = 0$

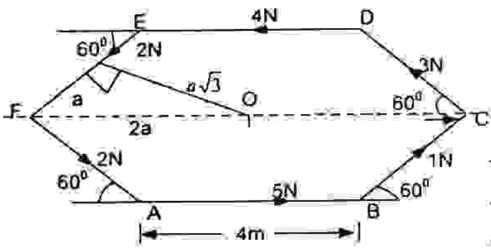
$$s = 2 + 2\sqrt{2}sin45 = 4N$$

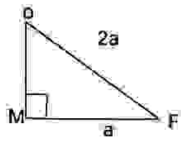
It must also be shown that $G \neq 0$

A $\quad G = 2x3 + 3x3 = 15Nm$

3. ABCDEF is a regular hexagon of side 4m. Forces of magnitude 5N, 1N, 3N, 4N, 2N and 2N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Show that these forces reduce to a couple

Solution





$MO = \sqrt{(2a)^2 - a^2}$

$MO = \sqrt{4a^2 - a^2}$

$MO = a\sqrt{3}$

$(\rightarrow)\ 5 - 4 + (1 - 3 - 2 + 2)cos60 = 0$

$(\uparrow)\ (1 + 3 - 2 - 2)sin60 = 0$

O $\quad G = (5 + 1 + 3 + 4 + 2)a\sqrt{3}$

$$= 17x2\sqrt{3}$$

$G = 34\sqrt{3}\ Nm = 58.89Nm\ anticlockwise$

Since $R = 0, and\ G \neq 0$, then it's a couple

4. Forces of 6N, 8N, 6N, and 8N act along the sides of a rectangle A B C D where $AB = 8m$ and $BC = 6m$ in the directions AB, BC, CD and DA respectively.
   a) Show that the forces reduce to a couple
   b) Find the moment of the couple about point A **An(100Nm)**
5. Forces of 5N, 3N, 5N and 3N act along the sides of a square A B C D of side 4m in the direction AB, BC, CD ad DA respectively
   a) Show that the forces form a couple
   b) Find the moment of the couple about point A **An(32Nm)**

Exercise 12C

1. ABCD is a rectangle with $AB = 6m$ and $BC = 2m$. A force of 3N acts along each of the four sides AB, BC, CD and DA in the directions indicated by the order of the letters. Show that the forces form a couple and find its moment. **An(21Nm)**
2. ABCD is a rectangle with $AB = 6m$ and $BC = 2m$. Forces of 5N, 5N, XN, and XN act along CB, AD, AB and CD respectively. The direction of the forces are given by the order of the letters. If the system is in equilibrium , Find X **An(15N)**
3. A B C D is a square of side 40cm. Forces of 20N, 5N, and 20N act along the sides AB, BC and CD respectively ad a force Y acts along A. the directions of the forces are given by the order of the letters . If the system is equivalent to a couple. Find the magnitude of Y and the moment of the couple. **An(15N, 14Nm)**
4. A B C D is a square of side 60cm. Forces of 6N, 2N, 6N and 2N act along the sides AB CB CD and AD respectively, in the directions indicated by the order of the letters .
   Show that the forces form the couple that must be applied to the system in order to produce equilibrium. **An(2.4Nm)**
5. A force of $(3\hat{\imath} - 5\hat{\jmath})$ N acts at the point which has position vector $(6\hat{\imath} + \hat{\jmath})$m and a force of $(-3\hat{\imath} + 5\hat{\jmath})$ N acts at the point which has position vector $(4\hat{\imath} + \hat{\jmath})$m. Show that the force reduce to a couple and find the moment of the couple. **An(10Nm)**
6. A force of $(4\hat{\imath} + 3\hat{\jmath})$N acts at the point which has position vector $(6\hat{\imath} + 3\hat{\jmath})$m and a force of $(-4\hat{\imath} - 3\hat{\jmath})$N acts at the point which has position vector $(3\hat{\imath} - \hat{\jmath})$m. Show that these force reduce to a couple and find the moment of the couple **An(7Nm)**
7. Force of $(\hat{\imath} + \hat{\jmath})$N , $(-4\hat{\imath} + \hat{\jmath})$N and $(3\hat{\imath} - 2\hat{\jmath})$N act at the points having position vectors $(2\hat{\imath} + 2\hat{\jmath})$m, $(-\hat{\imath} + 4\hat{\jmath})$N and $(4\hat{\imath} - 2\hat{\jmath})$m respectively. Show that these forces reduce to a couple and find the moment of the couple. **An(13Nm)**
8. Forces of $(a\hat{\imath}+b\hat{\jmath})$N and $(6\hat{\imath}-4\hat{\jmath})$N act at the points having vectors $(-2\hat{\imath}-2\hat{\jmath})$m and $(3\hat{\imath}-\hat{\jmath})$ respectively. If these forces reduce to a couple, find a and b and the moment of the couple. **An(-6, 4 and 26Nm)**

LINE OF ACTION OF THE RESULTANT OF FORCES

The equation of the line of action is given by

$$G = \begin{vmatrix} x & X \\ y & Y \end{vmatrix} = xY - yX$$

$$\boxed{G - xY + yX = 0}$$

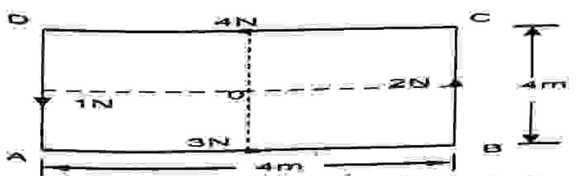
Note

The line of action cuts the horizontal when $y = 0$ and cuts the vertical axis when $x = 0$

Examples

1. Forces of 3N, 2N, 4N and 1N act along the sides of a square A B C D of side 4m in the direction AB, BC, CD and DA respectively. Find the;
   a) Magnitude and direction of the resultant
   b) Equation of the line of action
   c) Point where the line of action of the resultant of the forces cuts AB

Solution



(i) $R = \binom{3}{0} + \binom{0}{2} + \binom{-4}{0} + \binom{0}{-1} = \binom{-1}{1}$

$|R| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}N$

Direction, $\theta = tan^{-1}\left(\frac{1}{1}\right) = 45° \text{ to } AB$

$G = (4x4) + (4x2) = 24\ Nm\ anticlockwise$

Equation is $24 - x - y = 0$

(ii) Line cuts AB when $y = 0$

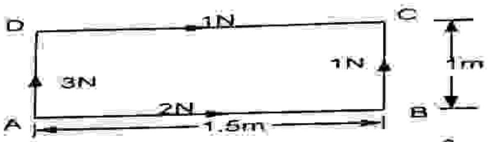
$24 - x - 0 = 0$

x = 24m From A

2. Forces of 2N, 1N, 1N and 3N act along the sides of a rectangle A B C D of side $AB = 1.5m$ and $AD = 1m$ in the direction AB, BC, DC and AD respectively. Find the;

a) Magnitude and direction of the resultant
b) Equation of the line of action
c) Point where the line of action of the resultant of the forces cuts AB

Solution



(i) $R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$

$|R| = \sqrt{(3)^2 + 4^2} = 5N$

Direction, $\theta = tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$ to AB

A: $G = (1x1.5) - (1x1) = 0.5Nm$ anticlockwise

$0.5 - 4x + 3y = 0$

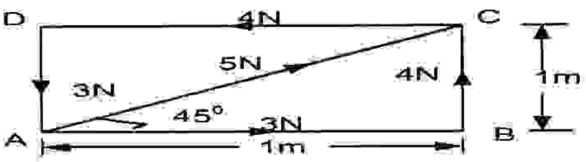
(ii) Line cuts AB when $y = 0$

$0.5 - 4x + 0 = 0$

$x = \frac{1}{8}$m From A

3. Five Forces of magnitude 3N, 4N, 4N, 3N and 5N act along AB, BC, CD, DA and AC respectively of a square of side 1m. The direction of the forces being in the order of the letters. Taking AB and AD as horizontal and vertical axis respectively. Find **Uneb 2016 No.9**
   (i) the magnitude and the direction of the resultant force
   (ii) Equation of the line of action
   (iii) Point where the line of action cuts AB.

Solution



$R = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}0\\4\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} + \begin{pmatrix}0\\-3\end{pmatrix} + \begin{pmatrix}5cos45\\5sin45\end{pmatrix}$

$R = \begin{pmatrix}2.54\\4.54\end{pmatrix}$

$|R| = \sqrt{(2.54)^2 + 4.54^2} = 5.196N$

Direction, $\theta = tan^{-1}\left(\frac{4.54}{2.54}\right) = 60.8^\circ$ to AB

A: $G = (4x1) + (4x1) = 8Nm$ anticlockwise

$8 - 4.54x + 2.54y = 0$

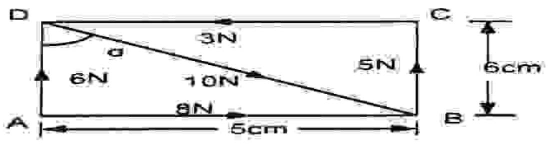
(i) Line cuts AB when $y = 0$

$8 - 4.54x + 0 = 0$

$x = 1.76$m From A

4. A B C D is a rectangle with $AB = DC = 5cm$ and $AD = BC$ =6cm. Forces of magnitude 8N, 5N, 3N, 6N and 10N act along AB, BC, CD, AD and DB of a rectangle respectively. Find
   (i) the magnitude and the direction of the single force that could replace this system of forces
   (ii) Equation of the line of action
   (iii) where its line of action cuts AB.

Solution



$\alpha = tan^{-1}\left(\frac{5}{6}\right) = 39.8^\circ$

(i) $R = \begin{pmatrix}8\\0\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}0\\6\end{pmatrix} + \begin{pmatrix}10sin39.8\\-10cos39.8\end{pmatrix}$

$= \begin{pmatrix}5.64\\3.32\end{pmatrix}$

$|R| = \sqrt{(5.64)^2 + 3.32^2} = 6.54N$

Direction, $\theta = tan^{-1}\left(\frac{3.32}{5.64}\right) = 30.5^\circ$ to AB

A: $G = (5x0.05) + (3x0.06) - 10sin39.8x0.06$

$= 0.046\ Nm$ anticlockwise

$0.046 - 3.32x + 5.64y = 0$

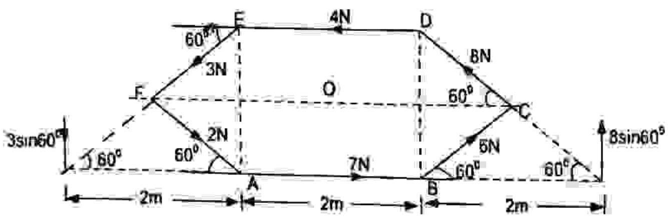
(ii) Line cuts AB when $y = 0$

$0.046 - 3.32x = 0$

$x = 0.014$m From A

5. ABCDEF is a regular hexagon 2m. Forces of magnitude 7N, 6N, 8N, 4N, 3N and 2N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the;
   (i) the magnitude and the direction of the resultant force
   (ii) where its line of action of the resultant cuts AB.

Solution



$R = \begin{pmatrix}7\\0\end{pmatrix} + \begin{pmatrix}6cos60\\6sin60\end{pmatrix} + \begin{pmatrix}-8cos60\\8sin60\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix}$
$+ \begin{pmatrix}-3cos60\\-3sin60\end{pmatrix} + \begin{pmatrix}2cos60\\-2sin60\end{pmatrix}$

$R = \begin{pmatrix}1.5\\7.794\end{pmatrix}$

$$|R| = \sqrt{(1.5)^2 + (7.794)^2} = 7.94N$$

$$\theta = tan^{-1}\left(\frac{7.794}{1.5}\right) = 79.11° \text{ to } AB$$

$$G = (6sin60°x2) + (8in60°x4) + (4x4sin60°) + (3sin60°x2)$$

$$G = 33\sqrt{3}\ Nm = 57.158Nm \text{ anticlockwise}$$

$$G = \begin{vmatrix} x & X \\ y & Y \end{vmatrix}$$

$$57.158 = \begin{vmatrix} x & 1.5 \\ y & 7.794 \end{vmatrix}$$

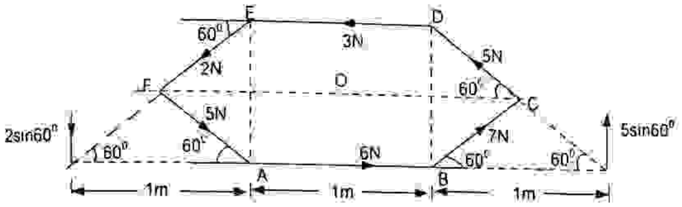
$$57.158 = 7.794x - 1.5y$$

Line cuts AB when $y = 0$

$$x = \frac{57.158}{7.794} = 7.334m$$

6. ABCDEF is a regular hexagon 1m. Forces of magnitude 6N, 7N, 5N, 3N, 2N and 5N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the ;
   (i) the magnitude and the direction of the resultant force
   (ii) where its line of action of the resultant cuts AB.

Solution



$$R = \binom{6}{0} + \binom{7cos60}{7sin60} + \binom{-5cos60}{5sin60} + \binom{-3}{0} + \binom{-2cos60}{-2sin60} + \binom{5cos60}{-5sin60}$$

$$R = \binom{5.5}{4.33}$$

$$|R| = \sqrt{(5.5)^2 + (4.33)^2} = 7.0N$$

$$\theta = tan^{-1}\left(\frac{4.33}{5.5}\right) = 38.21° \text{ to } AB$$

$$G = (7sin60°x1) + (5in60°x2) + (3x2sin60°) + (2sin60°x1)$$

$$G = 12.5\sqrt{3}\ Nm = 21.65Nm \text{ anticlockwise}$$

$$G = \begin{vmatrix} x & X \\ y & Y \end{vmatrix}$$

$$21.65 = \begin{vmatrix} x & 5.5 \\ y & 4.33 \end{vmatrix}$$

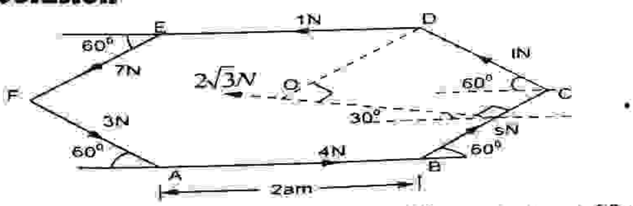
$$21.65 = 4.33x - 5.5y$$

Line cuts AB when $y = 0$

$$x = \frac{21.65}{4.33} = 5m$$

7. The centre of a regular hexagon ABCDEF of side 2a is O. Forces of magnitude 4N, sN, tN, 1N, 7N and 3N act along the sides AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters.; **Uneb 2010 No.13**
   (i) Given that the resultant of these six forces is of magnitude $2\sqrt{3}N$ acting in a direction perpendicular to BC, determine the value of s and t
   (ii) Show that the sum of moments of the forces about O is $27a\sqrt{3}Nm$
   (iii) If the mid point of BC is M, find the equation of the line of action of the resultant, refer to OM as x-axis and OD as y-axis

Solution



$$\binom{-2\sqrt{3}\cos 30}{2\sqrt{3}\sin 30} = \binom{4}{0} + \binom{s\ cos60}{s\ sin60} + \binom{-t\ cos60}{t\ sin60} + \binom{-1}{0} + \binom{-7cos60}{-7sin60} + \binom{3cos60}{-3sin60}$$

$$\begin{pmatrix} -2\sqrt{3}x\frac{\sqrt{3}}{2} \\ 2\sqrt{3}x\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 + \frac{s}{2} - \frac{t}{2} - 1 - 3.5 + 1.5 \\ \frac{s\sqrt{3}}{2} + \frac{t\sqrt{3}}{2} - \frac{10\sqrt{3}}{2} \end{pmatrix}$$

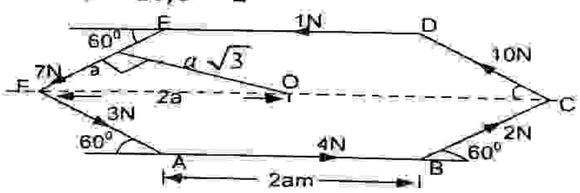
$$-2\sqrt{3}x\frac{\sqrt{3}}{2} = \frac{s}{2} - \frac{t}{2} + 1$$

$$s = t - 8 \ldots\ldots (i)$$

$$2\sqrt{3}x\frac{1}{2} = \frac{s\sqrt{3}}{2} + \frac{t\sqrt{3}}{2} - \frac{10\sqrt{3}}{2}$$

$$2 = s + t - 10 \ldots\ldots (ii)$$

$$t = 10, s = 2$$



$$G = (4 + 2 + 10 + 1 + 7 + 3)a\sqrt{3}$$

$$G = 27a\sqrt{3}\ Nm$$

Since the resultant acts along the direction of OM, then

$$X = 2\sqrt{3}, Y = 0$$

$$G = \begin{vmatrix} x & X \\ y & Y \end{vmatrix} = \begin{vmatrix} x & 2\sqrt{3} \\ y & 0 \end{vmatrix}$$

$27a\sqrt{3} = -2\sqrt{3}\,y$

Line cuts AB when $y = -13.5a$

Exercise 12D

1. A B C D is a square of side 5m. Forces of 4N, 6N, 8N and 10N act along BD, DC CA and CB respectively. When a force P acts along AD and a force Q acts along AB, the whole system is equivalent to a couple. Find the magnitude of P and Q and the moment of the couple. **An(12.8N, 2.49N, 65.9Nm)**
2. A B C D is a square of side a meters. Forces of 1N, 4N, 3N and 6N act along AB, CB, DC, and AD respectively. Calculate the magnitude and direction of the single force that could replace this system of forces and find where its line of action cuts AB. **An(4.47N, 26.6° *to AB*, 3.5a from A)**
3. A B C D is a rectangle with $AB = 3m$ and $< CAB$ =30°. Forces of 10N, 20N, and 20N act along AC, AD and DB respectively. Calculate the magnitude and the direction of the single force that could replace this system of of forces and find where its line of action cuts AB. **An(30N, 30° *to AB*, 2m from A)**
4. ABCD is a rectangle with $AB = 5m$, and $BC = 3m$. Forces of magnitude 2N, 4N, 3N and 11N act along the line AB, BC , DC and DA respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the;
   (i) Magnitude and direction of the resultant
   (ii) Equation of the line of action of the resultant
   (iii) Distance from A where the line of action of the resultant force cuts AB

   An$(8.6N, 11 + 7x + 5y = 0, 1.57m)$
5. ABCD is a rectangle with $AB = 4m$, and $BC = 3m$. Forces of magnitude 3N, 5N, 6N , 4N and 7N act along the line AB, BC, CD, DA and AC respectively, in each case the direction of the force being given by the order of the letters. Find the;
   (i) Magnitude and direction of the resultant
   (ii) Distance from A where the line of action of the resultant force cuts AB **An**$(5.81N, 7.31m)$
6. ABCD is square of side 2m. Forces of magnitude 10N, 9N, 8N, 7N and 5N act along the line AB, BC, CD, DA and AC respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the ;
   (i) Magnitude and direction of the resultant
   (ii) Sum of the moments about A
   (iii) Distance from A where the line of action of the resultant force cuts AB

   An$(20.305N, 34Nm, 1.74m)$
7. ABCDEF is a regular hexagon 4m. Forces of magnitude 6N, 4N, 7N, 8N, 4N and 2N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the resultant force being given by the order of the letters. Given that AB is horizontal, Find the;
   (i) Magnitude and direction of the resultant
   (ii) Distance from A where the line of action of the resultant force cuts AB **An** $(7.55N, 21.71m)$
8. ABCDEF is a regular hexagon 3m. Forces of magnitude 5N, 6N, 2N, 3N, 6N and 1N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the;
   (i) Magnitude and direction of the resultant
   (ii) Distance from A where the line of action of the resultant force cuts AB

   An $(4.583N, 13.2m)$
9. PQRSTUP is a regular hexagon 2m. Forces of magnitude 9N, 5N, 7N, 3N, 1N and 4N act along the line PQ, QR, RS, ST, TU and UP respectively, in each case the direction of the force being given by the order of the letters. Given that PQ is horizontal, Find the ; **Uneb 2000 No.16**
   (i) Magnitude and direction of the resultant force
   (ii) Point where the line of action of the resultant cuts PQ

   An $(8.9N\ at\ 43°\ to\ PQ,\ 7.43m)$
10. ABCDEF is a regular hexagon of side 2m. Forces of magnitude 2N, 3N, 4N, and 5N act along the line AC, AE, AF and ED respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the;
    (i) Magnitude and direction of the resultant force
    (ii) Point where the line of action of the resultant cuts AB **An(8.84N, 57.6° *to AB*, 2.32m from A)**
11. ABCDEF is a regular hexagon of side 2m. Forces of magnitude 3N, 4N, 2N, 1N, 2N and 6N act along the line AB, BC, DC, ED, EF and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the;
    (i) Magnitude of the resultant force
    (ii) Equation of the line of action of the resultant force **An(6N,** , $y = \sqrt{3}x$**)**

PARALLEL FORCES IN EQUILIBRIUM

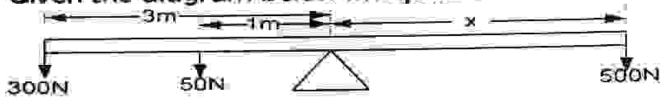
Condition for a body to be in equilibrium

When a system of parallel forces act on a body then it will be in equilibrium when;

1. Sum of the forces acting in one direction are equal to the sum of forces acting in opposite direction.
2. Sum of the clock wise moments about a point are equal to the sum of the anti clock wise moment about the same point.

Examples

1. Given the diagram below in equilibrium



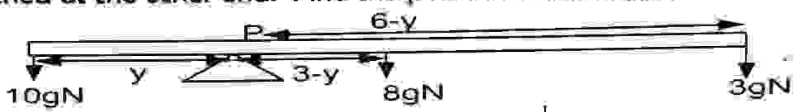
Find x

Solution

P: $(500 \times x) = (300 \times 3) + (50 \times 1)$

$x = \frac{950}{500} = 1.9\text{m}$

2. A uniform beam AB of length 6m and mass 8kg has a mass of 10kg attached at one end and a mass of 3kg attached at the other end. Find the position of the support if the beam rests horizontally



Solution:

P: $10g\,xy = 8g\,x\,(3-y) + 3gx(6-y)$

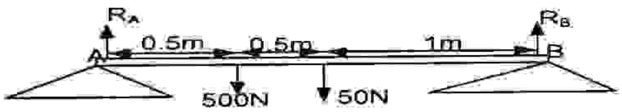
$10y = 24 - 8y + 18 - 3y$

$y = \frac{42}{21} = 2m$

2m from a 10kg mass

3. A uniform beam of weight 50N and length 2m rests horizontally on two supports pivoted at each end. A load of weight 500N is placed 0.5m from one end. Find the reaction on each support.

Solution



B: $R_A \times 2 = 50 \times 1 + 500 \times 1.5$

$2R_A = 50 + 750$

$R_A = 400N$

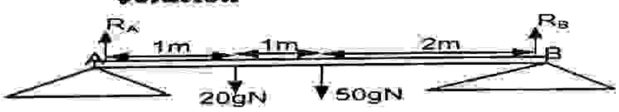
Also: $R_A + R_B = 500N + 50N$

$400 + R_B = 550$

$R_B = 150N$

4. A uniform beam of weight 50kg and length 4m rests horizontally on two supports pivoted at each end. A load of mass 20kg is placed 1m from one end. Find the reaction on each support.

Solution



A: $R_B \times 4 = 20g \times 1 + 50g \times 2$

$R_B = 30gN = 294N$

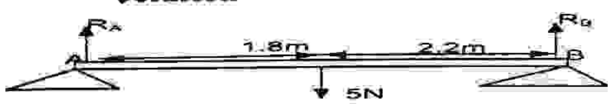
Also: $R_A + R_B = 20gN + 50gN$

$R_A + 30g = 70g$.

$R_A = 40gN = 392N$

5. A non- uniform beam AB of length 4m has its weight 5N acting at a point 1.8m from end A. The beam rests horizontally on two supports pivoted at each end. Find the reaction on each support.

Solution



A: $R_B \times 4 = 5x1.8$

$R_B = 2.25N$

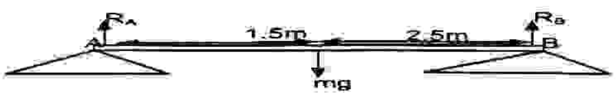
Also: $R_A + R_B = 5$

$R_A + 2.25 = 5$.

$R_A = 2.75N$

6. A non- uniform beam AB of length 4m rests in horizontal position on vertical supports at A and B. The centre of gravity is at 1.5m from end A. the reaction at B is 37.5N. Find the ;
   (i) Mass of the beam
   (ii) reaction at A.

Solution



A: $37.5 \times 4 = mgx1.5$

$m = 10.2kg$

Also: $R_A + 37.5 = mg$

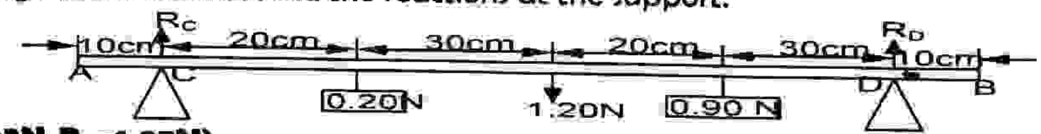
$R_A + 37.5 = 10.2x9.8$.

$R_A = 62.5N$

Exercise:12E

1. A uniform beam AB of length 10m rests horizontally on two supports at A and B. if the beam has a mass of 20kg, find the reactions at the supports. **(An 98N, 98N)**
2. A uniform beam AB of length 14m and mass 20kg rests horizontally on two supports, one at A and the other at C which is 4m from B. Find the reactions at the supports. **(An 58.8N, 137.2N)**
3. A uniform beam AB of length 10m and mass 20kg rests horizontally on two supports, one at A and the other at C which is 2m from B. If a weight of mass 20kg is attached to the beam at a point 6m from A. Find the pressures on the supports. **(An 392.4N, 196.2N)**
4. A uniform beam AB of length 4m and mass 10kg rests horizontally on two supports, one at A and the other at C which is 1m from B. where must a boy of mass 50kg stand on the beam so that the reactions at the supports are equal. **(An 1.4m from A)**
5. A uniform beam AB of length 12m and mass 12kg rests on two supports at A and B. A particle of mass 3kg is tied at a point 2m from A. At what distance must a particle of mass 4kg be tied so that the reactions at the supports are equal. **(An 9m from A)**
6. A playground sea saw consists of a uniform beam of length 4m supported at its mid-point. If a girl of mass 25kg sits at one end of the sea saw, find where her brother of mass 40kg must sit if the sea saw is to balance horizontally. **(An 75cm** from A**)**
7. Three uniform rods of mass 2kg, 4kg and 8kg and each of length 20cm are joined together in the order mentioned to form one long rigid rod of length 60cm. This rod is then suspended horizontally by a vertical string attached to the rod at a point y cm from its mid-point. Find the value of y and the tension in the string. **(An 8.57cm, 137.2N)**
8. A broom consists of a uniform broom stick of length 120cm and mass 4kg and a broom head of mass 6kg attached at the other end. Find where a support should be placed so that the broom will balance horizontally. **(An 24cm from head)**
9. A non-uniform beam AB of length 4m rests horizontally on two supports, one at A and the other at B. The reactions at these supports are 5gN and 3gN respectively. If instead the rod were to rest horizontally on one support, find how far from A this support would have to be place. **(An 1.5m from A)**
10. A uniform beam AB of mass 80g is pivoted at the 30cm from A, a force of 10N is placed on the beam at the 80cm from end A and a string is tied at the 40cm from end B so that the beam rests horizontally. Find the tension in the string. **(An 17.2N)**
11. A uniform beam AB of length 100cm is pivoted at 60cm from end B. the beam rests horizontally when a mass at A is 35g. Calculate the mass (m) of the beam. **(An0.14kg)**
12. A uniform meter rule pivoted at 10cm mark balances when a mass of 400N is suspended at the 0cm mark. If the system is in equilibrium. Find the mass of the ruler. **(An =10kg)**
13. Two boys are carrying a uniform ladder of weight 800N, if the boys hold the ladder at 2m and 3m respectively from the center of gravity, calculate the weight that each boy support.

    (An R_A = 480N, R_B =320N)
14. A painter of weight 330N trestles on a 4m plank of weight 60N on which he is to stand while painting a wall. He puts a tin of paint of weight 20N at a distance of 0.5m from the left end of the plank and stands at 1.5m from the right end of the plank. If the trestles are at a distance of 1m from both ends. Find the reactions at the trestles. **(An R_1 =137.5N, R_2 =272.5N)**
15. A uniform wooden lath AB, 120 cm long and weighing 1.20 N rests on two sharp edged supports C and D placed 10 cm from each end of the lath respectively. A 0.20 N weight hang 30 cm from A and a 0.90 N weight hangs 40 cm from B. Find the reactions at the support.



An(R_C= 1.03N,R_D=1.27N)

CHAPTER 3: MOTION IN A STRAIGHT LINE

Distance and displacement

Distance is the length between 2 fixed point **Displacement** is the distance covered in a specific direction

Speed and velocity

Speed is the rate of change of distance with time

Velocity is the rate of change of displacement with time

Average speed and average velocity

$$average\ speed = \frac{total\ distance}{total\ time} \quad \Big| \quad average\ velocity = \frac{total\ displacement}{total\ time}$$

Examples

1. Find the distance travelled in $5s$ by a body moving with a constant speed of $3.2ms^{-1}$

Solution

$$constant\ speed = \frac{distance}{time} \quad \Big| \quad 3.2 = \frac{distance}{5} \quad \Big| \quad distance = 16m$$

2. If a man runs 1500m race in 3minutes and 33s. find his average speed for the race

Solution

$$average\ speed = \frac{T.\ distance}{T.\ time} \quad \Big| \quad average\ speed = \frac{1500}{(3x60+33)} = 7.04ms^{-1}$$

3. A man walks 150m due north, in a time of 70s and then 50m due south in a time of 30s find
   (i) Average speed
   (ii) Average velocity

Solution

(i) $average\ speed = \frac{Total\ distance}{Total\ time}$

$average\ speed = \frac{150+50}{(70+30)} = 2ms^{-1}$

(ii) $average\ velocity = \frac{Total\ displacemnt}{Total\ time}$

$average\ velocity = \frac{150-50}{(70+30)} = 1ms^{-1}$ due north

4. A, B and C are three points in that order on a straight line with $AB = 5km$ and $BC = 4km$. A man runs from A to B at $20kmh^{-1}$ and then walks from B to C at $8kmh^{-1}$. Find
   (i) Total time taken to travel from A to C
   (ii) The average speed of the man for the journey from A to C

Solution

(i) $constant\ speed = \frac{distance}{time}$

$t_{AB} = \frac{5}{20} = 0.25h$ and $t_{BC} = \frac{4}{8} = 0.5h$

Total time= $0.25 + 0.5 = 0.75h$

(ii) $average\ speed = \frac{Total\ distance}{Total\ time}$

average speed $= \frac{5+4}{(0.75)} = 12\ kmh^{-1}$

5. A, B and C are three points in that order on a straight line with $AB = 60m$ and $BC = 80m$. A man walks from A to B at an average speed of $10ms^{-1}$ and then walks from B to C in a time of $4s$ and then returns to B. The average speed for the whole journey is $5ms^{-1}$. Find
   (i) Average speed of the man in the second stage of the motion $(ie\ B \to C)$. **An** $5ms^{-1}$
   (ii) The average speed of the man in moving from A to C **An** $8ms^{-1}$
   (iii) The time taken for the third stage of the motion $(ie\ C \to B)$. **An** $10s$
   (iv) Average velocity for the complete motion **An** $3ms^{-1}$

Acceleration

$$\text{Acceleration} = \frac{change\ in\ velocity}{time} \quad \Big| \quad a = \frac{v-u}{t}$$

EQUATIONS OF UNIFORM ACCELERATION

1st equation

Suppose a body moving in a straight line with uniform acceleration a, increases its velocity from u to v in a time t, then from definition of acceleration

$a = \frac{v-u}{t}$ | $at = v - u$ | $\boxed{v = u + at}$ 1

2nd equation

Suppose an object with velocity u moves with uniform acceleration for a time t and attains a velocity v, the distance s travelled by the object is given by S = average velocity x time

$S = \left(\frac{v+u}{2}\right)t$ But $v = u + at$

$S = \frac{(u + at + u)}{2}t$

$S = \frac{2ut + at^2}{2}$

$\boxed{S = ut + \frac{1}{2}at^2}$ 2

3rd equation

S = average velocity x time

$S = \left(\frac{v+u}{2}\right)t$ But $t = \frac{v-u}{a}$

$S = \left(\frac{v+u}{2}\right)\left(\frac{v-u}{a}\right)$

$S = \frac{v^2-u^2}{2a}$

$\boxed{v^2 = u^2 + 2as}$ 3

Examples:

1. A car starts from rest and accelerates uniformly at 1.5ms^{-2} until it attains a speed of 30 ms^{-1}. Find the distance the car travels during this motion and the time taken.

Solution

$v^2 = u^2 + 2as$ | $s = \frac{30^2 - 0^2}{2x1.5} = 300m$ | $t = \frac{v-u}{a} = \frac{30-0}{1.5} = 20s$

2. A car is initially at rest at a point O. The car moves away from O in a straight line, accelerating at 4 ms^{-2}. Find how far the car,
   (i) is from O after 2s
   (ii) is from O after 3s
   (iii) travels in the third second

Solution

$S = ut + \frac{1}{2}at^2$

When $t = 2s$, $S = 0x2 + \frac{1}{2}x4x2^2 = 8m$

When $t = 3s$, $S = 0x3 + \frac{1}{2}x4x3^2 = 18m$

distance traveled in the 3^{rd} $= 18 - 8 = 10m$

3. A particle is projected away from an origin O with initial velocity of 0.25 ms^{-1}. The particle travels in a straight line and accelerates in a straight line and accelerates at 1.5 ms^{-2}, Find
   (i) How far the particle is from O after 4s
   (ii) The distance travelled by the particle during the fourth second after projection

Solution

(i) $S = ut + \frac{1}{2}at^2$

When $t = 4s$, $S = 0.25x4 + \frac{1}{2}x1.5x4^2 = 7.5m$

(ii) When $t = 3s$, $S = 0.25x3 + \frac{1}{2}x1.5x3^2 = 13m$

distance traveled in the 4^{rd} $= 13 - 7.4 = 5.5m$

4. At $t = 0s$, a body is projected from an origin O with an initial velocity of 10 ms^{-1}. The body moves along a straight line with a constant deceleration of 2 ms^{-2}.
   (a) Find the displacement of the body from O after $t = 7s$
   (b) How far from O does the body come to rest and what time does it take to come to rest

Solution

(a) $S = ut + \frac{1}{2}at^2$

When $t = 7s$, $S = 10x7 + \frac{1}{2}x(-2)x7^2 = 21m$

(b) $v^2 = u^2 + 2as$

$$s = \frac{0^2 - 10^2}{2x - 2} = 25m$$

$$t = \frac{v - u}{a} = \frac{0 - 10}{-2} = 5s$$

5. A car is being driven along a road at a steady speed of 25 ms^{-1}, when suddenly the driver notices a fallen tree on the road 65m ahead. The driver immediately applies brakes giving the car a constant retardation of 5 ms^{-2}
   (a) How in front of the tree does the car come to rest
   (b) If the driver had not reacted immediately and the brakes were applied one second later, with what speed would the car have hit the tree.

Solution

(a) $v^2 = u^2 + 2as$

$$s = \frac{0^2 - 25^2}{2x - 5} = 62.5m$$

Distance in front of the tree = $65 - 62.5 = 2.5m$

(b) $1s$ before brake were applied;

$$constant\ speed = \frac{distance}{time}$$

$$25 = \frac{distance}{1}$$

$distance = 25m$

Remaining distance; $65 - 25 = 40m$

When brake were applied: $v^2 = u^2 + 2as$

$$v^2 = 25^2 + 2x - 5x40$$

$$v = 15ms^{-1}$$

6. A particle travels in a straight line with a uniform acceleration. The particle passes through three points A, B and C lying in that order on the line, at time $t = 0, t = 2s$, and $t = 5s$ respectively. If BC= $30m$ and the speed of the particle when at B is $7ms^{-1}$, find the acceleration of the particle and its speed when at A

Solution

For AB: $v = u + at$

$7 = u + 2a \ldots\ldots\ldots (i)$

For BC: $s = ut + \frac{1}{2}at^2$

$$30 = 7x3 + \frac{1}{2}xax3^2$$

a= $2ms^{-2}$

but $7 = u + 2a \ldots\ldots\ldots (i)$

$7 = u + 2x2$

u= $3ms^{-1}$

7. A taxi approaching a stage runs two successive half kilometers in $16s$ and $20s$ respectively. Assuming the retardation to be uniform. Find
   (i) Initial speed of the taxi
   (ii) The further distance, the taxi runs before stopping

Solution

(i) For AB: $s = ut + \frac{1}{2}at^2$

$$500 = 16u + \frac{1}{2}a(16)^2$$

$125 = 4u + 32a \ldots\ldots\ldots (i)$

For BC: $1000 = 36u + \frac{1}{2}a(36)^2$

$250 = 9u + 162a \ldots\ldots (ii)$

$9x\ (i) - 4x(ii)$

$-125 = 360a$

a= $-\frac{25}{72}ms^{-2}$

but $125 = 4u + 32a$

$$125 = 4u + 32\left(-\frac{25}{72}\right)$$

u= $34.028ms^{-1}$

(ii) $v^2 = u^2 + 2as$

$$s = \frac{0^2 - 34.028^2}{2x\left(-\frac{25}{72}\right)} = 1667.36m$$

Extra distance = $1667.36 - 1000 = 667.36m$

8. An over loaded taxi travelling at a constant velocity of 90kmh^{-1}overtake a stationery traffic police car. 2s later, the police car sets off in pursuit , accelerating at a uniform rate of 6ms^{-2}. How far does the traffic car travel before catching up with the taxi. **Uneb 1999 no.4**

Solution

For taxi: $S_T = ut + \frac{1}{2}at^2$

Since it moves with a constant velocity a =0, $u = \frac{90x1000}{3600} = 25m/s$

$S_T = 25t$---------------(1)

For car: $S_C = ut + \frac{1}{2}at^2$

If car is to catch up with taxi then it must travel faster i.e it will take a time of $(t-2)s$

$$S_c = 0x(t-2) + \frac{1}{2}x6(t-2)^2$$

$$S_c = 3t^2 - 12t + 12 \ldots\ldots(2)$$

For car to catch taxi then

$$S_T = S_B$$

$$25t = 3t^2 - 12t + 12$$

$$3t^2 - 37t + 12 = 0$$

$$t = \frac{37 \pm \sqrt{37^2 - 4x12x3}}{2x3}$$

$t = 12s$ or $t = \frac{1}{3}s$

Since the car leaves 2s later then time 12s is correct since it gives a positive value

$$S_T = 25t$$

$$S_T = 25x12$$

$$S_B = 300m$$

9. A lorry starts from a point A and moves along a straight horizontal road with a constant acceleration of $2ms^{-2}$. At the same time a car moving with a speed of $20ms^{-1}$ and a constant acceleration of $3ms^{-2}$ is 400m behind the point A and moving in the same direction as the lorry. Find;

(i) How far from A the car over takes the lorry

(ii) The speed of the lorry when it is being over taken

Solution

(a) For lorry: $S = ut + \frac{1}{2}at^2$

$$S_L = 0xt + \frac{1}{2}x2xt^2$$

$$S_L = t^2 \text{---------(1)}$$

For car: $S_C = ut + \frac{1}{2}at^2$ (iii)

(iv)

$$(400 + S_L) = 20t + \frac{3}{2}xt^2 \quad \text{(v)}$$

(vi)

$$(400 + t^2) = 20t + \frac{3}{2}xt^2 \quad \text{(vii)}$$

$$t^2 + 40t - 800 = 0$$

$$t = \frac{-40 \pm \sqrt{40^2 - 4x(-800)}}{2}$$

$t = 14.64s$ or $t = -54.64s$

Hence $t = 14.64s$

$$S_L = 14.64^2 = 214.33m$$

(b) $v = u + at$

$$v = 0 + 2x14.64$$

$$v = 29.28ms^{-1}$$

Exercise 13A

1. Two cyclists A and B are 36m apart on a straight road. Cyclist B starts from rest with an acceleration of 6 ms^{-2} while A is in pursuit of B with a velocity of 20 ms^{-1} and acceleration of 4 ms^{-2}. Find the times when A over takes B.

2. The speed of a boda- boda rider decreases from 90kmh^{-1} to 18 kmh^{-1} in a distane of 120m. find the speed of the rider when it had covered a distance of 50m. **Uneb 2013 no3**

3. (a) Show that the final velocity v of a body which starts with an initial velocity u and moves with a uniform acceleration a, hence covering a distance x, is given by $v = [u^2 + 2ax]^{1/2}$

(b) Find the value of x in (a) if $v = 30m/s$, $u = 10m/s$ and $5m/s^2$ **Uneb 2012 no4**

4. P, Q and R are points on a straight road such that $PQ = 20$m and $QR = 55$m. A cyclist moving with uniform acceleration passes P and then notices that it takes him 10s and 15s to travel between P and Q, Q and R respectively. Find the acceleration. **Uneb 2010 no2**

5. A car travels from kampala to Jinja and back. Its average speed on the return journey is 4 kmh^{-1} greater than that on the out ward journey and it takes 12 minutes less. Given that Kampala and Jinja are 80km apart, find the average speed on the outward journey. **Uneb 2009 no.2**

6. Car A travelling at 35m/s along a straight horizontal road, accelerates uniformly at $0.4ms^{-2}$. At the same time, another car B moving at 44m/s and accelerating uniformly at $0.5ms^{-2}$ is 200m behind A

(i) Find the time taken before car B over takes car A. **An**= 20s

(ii) Speed with which B over takes A

7. A car is being driven along a road at 72 kmh^{-1} notices a fallen tree on the road 800m ahead and suddenly reduces the speed to 36 kmh^{-1} by applying brakes. For how long were the brakes applied. **Uneb 2002 no3 An(53.33s)**

8. A train starts from station A with a uniform acceleration of $0.2ms^{-2}$ for 2 minutes and attains a maximum speed and moves uniformly for 15 minutes. It is then brought to rest at a constant retardation of $\frac{5}{3}ms^{-2}$ at station B. find the distance between A and B **Uneb 2003 no8 An(232112.8m)**

9. A motorcycle decelerated uniformly from 20kmh^{-1} to 8 kmh^{-1}in travelling 896m. find the rate of deceleration. **Uneb1998 march no3 An(0.0145m/s²)**

10. A car can accelerates from rest to 30 ms^{-1} in a distance of 25m. Find the acceleration of the car. **An(18ms⁻²)**

11. A body moves with a uniform acceleration and covers a distance of 27m in 3s, it then moves with a uniform velocity and covers a distance of 60m in 5s. Find the initial velocity and the acceleration of the body. **An**($6ms^{-1}, 2ms^{-2}$)

12. The manufacturer of a new car claims that the car can accelerate from rest to 90 kmh^{-1} in 10 s Find the acceleration of the car. **An(2.5ms^{-2})**

13. A car travelling at 20 ms^{-1} retards when brakes are applied covering a distance of 30m. what is the retardation and how for long are brakes applied **An(** $6\frac{2}{3}$**ms^{-2}, 3s)**

14. A taxi which is moving with a uniform acceleration is observed to take 20s and 30s to travel successive 400m. Find;
    (i) Initial speed of the taxi
    (ii) The further distance, the taxi runs before stopping **An(** $\frac{68}{3}$**ms^{-1}, 163.3m)**

15. A train decreases from 96kmh^{-1} to 24 kmh^{-1} in a distance of 0.8km. Find
    (a) For how long the brakes are applied.
    (b) How much longer it would take to come to rest **An(** 48s, **16s)**

16. A cyclist riding at $5ms^{-1}$ passes a motor car just as it begins to move in the same direction. The car maintains an acceleration of $0.4ms^{-2}$ for 20s, and then moves uniformly. How far will the car travel before overtaking the cyclist. **An(** 133.33m**)**

17. A train passes another train on a parallel track, the first train running at a uniform speed of 60 kmh^{-1} and the second is running at a speed of 15kmh^{-1} with an acceleration of $0.15ms^{-2}$. How long will it take before the second train catches the first again and how far will they have travelled in this interval. **An(** 166.67s, 2.78km**)**

18. A cyclist A riding at 16 kmh^{-1} is overtaken by cyclist B riding at 20 kmh^{-1}.
    (i) If A immediately increases his speed with uniform acceleration, find the speed he catches up with B
    (ii) If A increases his speed to 22 kmh^{-1} and maintains this speed and catches B after covering 200m, find his acceleration **An(** $24kmh^{-1}, 0.0694ms^{-2}$**)**

19. A particle moving in a straight line with a uniform acceleration passes $a\ ms^{-2}$ certain point with a velocity $u\ ms^{-1}$. 3 s later another particle, moving in the same straight line with a constant acceleration $^4/_3\ a\ ms^{-2}$, passes the same point with a velocity of $^1/_3\ u\ ms^{-1}$. The first particle is overtaken by the second particle when their velocities are $8.1ms^{-1}$ and $9.3ms^{-1}$ respectively. Find the;
    (iii) value of u and a
    (iv) Distance travelled from the point **An(** $u = 0.9ms^{-1}, a = 0.15ms^{-2}, 216m$**)**

20. P, Q and R are points on a straight road such that $PQ = 2$km and $QR = 2$km. A cyclist moving with uniform acceleration passes P and then notices that it takes him 100s and 150s to travel between P and Q, Q and R respectively. Find
    (i) The acceleration.
    (ii) How far beyond R the cyclist travels before coming to rest **An(**$-0.0533ms^{-2}, 820m$**)**

21. A train is accelerated and passes successive kilometer marks with velocities in 10kmh^{-1} and 20kmh^{-1} respectively. Find
    (i) Velocity when it passes the next kilometer mark
    (ii) Time take for each of these two intervals of 1km **An(** $10\sqrt{7}\ kmh^{-1}, 240s, 154.8s$**)**

VERTICAL MOTION UNDER GRAVITY

When a body is projected **vertically downwards** , its subjected to an acceleration of $9.8ms^{-2}$ ie

$$a = g = 9.8ms^{-2}$$

Equations of motion become

$$v = u + g\,t \qquad h = ut + \frac{1}{2}gt^2 \qquad v^2 = u^2 + 2gh$$

When a body is projected **vertically upwards** , its subjected to a retardation of $9.8ms^{-2}$ ie

$a = -g = -9.8ms^{-2}$

Equations of motion become

$$v = u - g\,t \qquad h = ut - \frac{1}{2}gt^2 \qquad v^2 = u^2 - 2gh$$

Maximum/ greatest height

When a particle is projected vertically upwards, the final velocity is 0m/s at its maximum height

$$v^2 = u^2 - 2gh$$
$$0 = u^2 - 2gh_{max}$$
$$h_{max} = \frac{u^2}{2g}$$

Time to reach maximum height

$$v = u - gt$$
$$0 = u - gt$$
$$t = \frac{u}{g}$$

Time of flight

$$T = \frac{2u}{g}$$

Examples

1. A stone is dropped from a point which is 40m above the ground. Find the time taken for the stone to reach the ground

Solution

$$h = ut + \frac{1}{2}gt^2$$
$$40 = 0xt + \frac{1}{2}x9.8t^2$$
$$t = \sqrt{\frac{80}{9.8}} = 2.857s$$

2. A ball is thrown vertically upwards with an initial speed of $30ms^{-1}$. Calculate.
   i) Time taken to return to the thrower
   ii) Maximum height reached

solution

(i) From $v = u - gt$

At max height $v = 0$

$0 = 30 - 9.81t$

$t = 3.06s$

Time taken to reach maximum height = $3.06s$

But the total time ta ken to return to the thrower = 2t

$= 2x3.06 = 6.12s$

(ii) $v^2 = u^2 - 2gh$

at max height v=0m/s, u=30m/s,

$0^2 = 30^2 - 2x9.8h_{max}$

$h_{max} = 45.92m$

3. A particle is projected from the ground level vertically upwards with velocity of $19.6ms^{-1}$. Find
   i) The greatest height attained
   ii) Time taken by the particle to reach maximum height
   iii) Time of flight

Solution

At greatest height $v = 0m/s$

$v^2 = u^2 - 2gh$

$h_{max} = \frac{19.6^2}{2x9.8} = 19.58m$

ii) From $v = u + gt$

$v = 0$ at max height

$0 = 19.6 - 9.8t$

$t = 1.998s$

t = 2.0s Time to maximum height = $2.0s$

iii) Time of flight = $2x$ time to max height

$= 2x2 = 4.0s$

4. A particle is projected vertically upwards with velocity of 10m/s. After 2s another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Find the height above the level of projection where the particles meet.

Solution

Let $T = time\ taken\ by\ the\ 1^{st}$

$T - 2 = time\ taken\ by\ the\ 2^{st}$

$$h = 10T - \frac{1}{2}gT^2 \ldots\ldots(i)$$

$$h = 10(T-2) - \frac{1}{2}g(T-2)^2 \ldots\ldots(ii)$$

$$10T - \frac{1}{2}gT^2 = 10(T-2) - \frac{1}{2}g(T-2)^2$$

$$T = \frac{20 + 2g}{2g} = 2.02s$$

$$h = 10T - \frac{1}{2}gT^2$$

$$h = 10x2.02 - \frac{1}{2}x9.8x(2.02)^2$$

$$h = 0.206m$$

5. A particle is projected vertically upwards with velocity of um/s. After $t\ seconds$ another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Prove that the particles collides after $\left(\frac{t}{2}+\frac{u}{g}\right)s$. Hence show that they will meet at a height of $\frac{4u^2-2(gt)^2}{8g}$

Exercise 13B

1. A stone is thrown vertically upwards with a velocity of $21ms^{-1}$. Calculate the: **UNEB 2018** No.1
   (a) maximum height attained by the stone
   (b) time the stone takes to reach the maximum height. **An (i)=22.5m, (ii)=2.143s**
2. A particle is projected vertically upwards with a velocity of $21ms^{-1}$. How long it take to reach a point 280m below the point of projection. **An(10s)**
3. A particle is projected vertically upwards with a velocity of $17.5ms^{-1}$. Find ;
   (a) How high the particle will go
   (b) What time elapses before its at a height of 10m. **An(15.6m, $\frac{5}{7}$s, $\frac{22}{7}s$)**
4. A particle is projected vertically upwards with a velocity of $24.5ms^{-1}$. Find ;
   (a) When its velocity is $4.9ms^{-1}$
   (b) How long it takes to return to the point of projection
   (c) At what time it will be 19.6m above the point of projection. **An(2s, 5s, $1s$ and $4s$)**
5. A particle is projected vertically upwards with a velocity of $35ms^{-1}$. Find ;
   (a) How long it takes to reach the greatest height
   (b) Distance it ascends during the 3rd second of the motion. **An($3\frac{4}{7}$s, $10.5m$)**
6. Two objects are dropped from a cliff of height H. The second is dropped when the first has travelled a distance, D. Prove that the instant when the first object reaches the bottom, the second is a distance $2\sqrt{DH} - D$ from the top of the cliff
7. A particle is projected vertically upwards with velocity of um/s. After $t\ seconds$ another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Prove that the particles collide with each having a velocity of $\frac{1}{2}gt$.
8. A particle is projected vertically upwards with velocity of 28m/s. After 2s another particle is projected vertically upwards from the same point of projection and with an initial velocity of 21m/s. Find when the two bodies are at the same height and the velocity of each body at that instant. **An(4.9s after the first body started, 20s^{-1}, 7.4ms^{-1})**
9. A small iron ball is dropped from the top of a vertical cliff. Find
   (a) how far it falls in 10show long it takes to fall 100m
   (b) Its velocity after falling 100m
   An(490m, 4.5s, 44. $3ms^{-1}$)
10. A stone is thrown vertically upwards with an initial velocity of 14m/s from the top of a tower 35m high. Find.
   (a) The highest point above the ground that the stone reached
   (b) Speed of the stone on hitting the ground
   (c) The time the stone was in air. **An(45m, 29.7ms^{-1}, 4.5s)**

CHAPTER 4: FORCE AND NEWTON'S LAWS OF MOTION

LAW I : Everybody continues in its state of rest or uniform motion in **a straight line** unless acted upon by an external force.

LAW II: The rate of change of momentum of a body is directly proportional to the applied force and takes place in the direction of the force.

$F = m\frac{(v-u)}{t}$ But $a = \frac{v-u}{t}$

$F = ma$

Note: F must be the resultant force

Examples:

1. Find the acceleration produced when a body of mass 5kg experiences a resultant force of 10N.

Solution

$F = ma$

$10 = 5a$

$a = 2ms^{-2}$

2. A car of mass 600kg travels a distance of 24m while uniformly accelerating from rest to 12m/s.
   (i) Find the acceleration of the car
   (ii) Determine the accelerating force.

Solution

$v^2 = u^2 + 2as$

$12^2 = 0^2 + 2xax24$

$a = 3ms^{-2}$

$F = ma$

$F = 600x3a$

$a = 1800N$

3. A body of mass 500g experiences a resultant force of 3N. find
   (a) Acceleration produced
   (b) Distance travelled by the body while increasing it speed from $1ms^{-1}$ to $7ms^{-1}$ **An($6ms^{-2}$, 4m)**

Solution

$F = ma$

$3 = 0.5xa$

$a = 6ms^{-2}$

$v^2 = u^2 + 2as$

$7^2 = 1^2 + 2x6xs$

$s = 4m$

4. Two forces of magnitude 12N and 9N act on a particle producing an acceleration of 3.65ms^{-2}. The forces act an angle of 60° to each other. Find the mass of the particle. **Uneb 2004 No.6 An(5kg)**

Solution

$R^2 = 12^2 + 9^2 - 2x12x9\,Cos(120°)$

$R = 18.25N$

$F = ma$

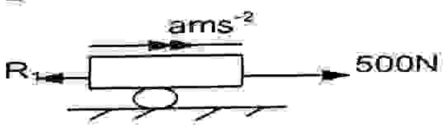
$18.25 = 3.65xm$

$m = 5kg$

WHEN RESISTANCE/ FRICTION IS INVOLVED

1. A car moves along a level road at a constant velocity of 22m/s. if its engine is exerting a forward force of 500N, what resistance is the car experiencing

Solution



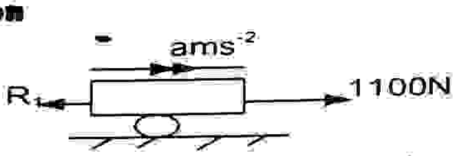
$F = ma$

$500 - R_1 = mx0$

$R_1 = 500N$

2. A car of mass 500kg moves along a level road with an acceleration of 2ms^{-2}. If its engine is exerting a forward force of 1100N, what resistance is the car experiencing

Solution



$F = ma$

$1100 - R_1 = 500x20$

$R_1 = 100N$

3. A van of mass 2 tones moves along a level road against resistance of 700N. if its engine is exerting a forward force of 2200N, find the acceleration of the van

Solution

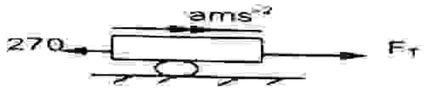
$F = ma$

$2200 - 700 = 2000xa$

$a = 0.75ms^{-2}$

4. Find the constant force necessary to accelerate a car of mass 1000kg from 15m/s to 20m/s in 10s against a resistance of 270N

Solution



$v = u + at$

$20 = 15 + 10a$

$a = 0.5ms^{-2}$

$F = ma$

$F_T - 270 = 1000x0.5$

$F_T = 770N$

5. Find the constant force necessary to accelerate a car of mass 600kg from rest to 25m/s in 12s, if the resistance to motion is
   (a) Zero
   (b) 350N **An(1250N, 1600N).**
6. A train of mass 60 tones is travelling at 40m/s when the brakes are applied. If the resultant braking force is 40kN, find the distance the train travels before coming to rest. **An(1200m).**
7. A train of mass 100 tones starts form rest at station A and accelerates uniformly at $1ms^{-2}$ until it attains a speed of 30m/s. it maintains this speed for a further 90s and then the brakes are applied, producing a resultant braking force of 52kN. If the train comes to rest at station B, find the distance between the two stations. **An(4050m).**

CALCULATIONS INVOLVING VECTOR FORM

1. Find the resultant force required to make a body of mass 2kg accelerates at $(5\bar{\imath} + 2\bar{\jmath})ms^{-2}$

Solution

$F = ma$

$F = 2\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}10\\4\end{pmatrix}N$

2. Find the acceleration produced in a body of mass 500g is subjected to forces of $(4\bar{\imath} + 2\bar{\jmath})N$ and $(-\bar{\imath} + \bar{\jmath})N$

Solution

$F = ma$

$\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} = 0.5a$

$a = \begin{pmatrix}6\\6\end{pmatrix}ms^{-2}$

3. Find the magnitude of the acceleration produced in a body of mass 2kg subjected to forces of $(2\bar{\imath} - 3\bar{\jmath} + 4\bar{k})N$ and $(\bar{\imath} + 5\bar{\jmath} + 2\bar{k})N$

Solution

$F = ma$

$\begin{pmatrix}2\\-3\\4\end{pmatrix} + \begin{pmatrix}1\\5\\2\end{pmatrix} = 2a$

$a = \begin{pmatrix}1.5\\1\\3\end{pmatrix}ms^{-2}$

$|a| = \sqrt{1.5^2 + 1^2 + 3^2} = 3.5\ ms^{-2}$

4. Forces of $(10\bar{\imath} + 2\bar{\jmath})N$ and $(a\bar{\imath} + b\bar{\jmath})N$ acting on a body of mass 500g causing it to accelerate at $(24\bar{\imath} + 3\bar{\jmath})ms^{-2}$. Find the constants a and b. **An**$(a = 2, b = -0.5)$
5. Forces of $(a\bar{\imath} + b\bar{\jmath} + c\bar{k})N$ and $(2\bar{\imath} - 3\bar{\jmath} + \bar{k})N$ acting on a body of mass 2kg causing it to accelerate at $(4\bar{\imath} + \bar{k})ms^{-2}$. Find the constants a, b and c. **An**$(a = 6, b = 3, c = 1)$
6. A particle pf mass 2.5kg is acted upon by a resultant force of magnitude 15N acting in the direction $(2\bar{\imath} - \bar{\jmath} - 2\bar{k})$. Find the magnitude of the acceleration

Solution

$$F = 15x\frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$= 15x\frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}$$

$$F = 10\hat{\imath} - 5\hat{\jmath} - 10\hat{k}$$

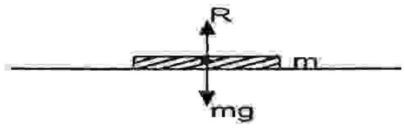
$F = ma$

$\begin{pmatrix}10\\-5\\-10\end{pmatrix} = 2.5a$

$a = \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix} ms^{-2}$ | $|a| = \sqrt{4^2 + (-2)^2 + (-4)^2} = 6 \text{ ms}^{-2}$

LAW III: To every action there is an equal but opposite reactions.

1. Consider a body of mass **m** placed on a smooth horizontal surface

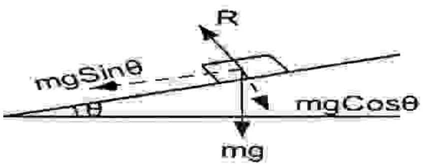


$\boxed{R = mg}$

R = normal reaction

Mg = gravitational pull [weight]

2. Mass m placed on a smooth inclined plane of angle of inclination θ



$\boxed{R = mgCos\theta}$

NB:

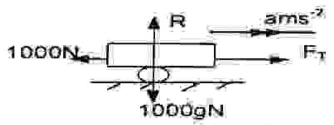
- ❖ All objects placed on, or moving on an inclined plane experience a force $mgsin\theta$ **down** the plane. [It doesn't matter what direction the body is moving]
- ❖ If the plane is **rough** the body experiences a frictional force whose direction is opposite to the direction of motion.

MOTION ON A HORIZONTAL PLANE

Examples:

1. A car of mass 1000kg is accelerating at 2ms⁻². If the resistance to the motion is 1000N.
   (i) Find the normal reaction of the car on the road surface
   (ii) What accelerating force acts on the car?

Solution

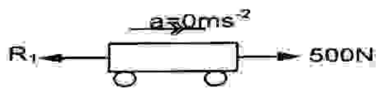


(c) $R = 10000gN$
$R = 1000x9.8N$
$R = 9800N$

(vi) $F = ma$
$F_T - 1000 = 1000x2$
$F_T = 3000N$

2. A car moves along a level road at a constant velocity of 22m/s. If its engine is exerting a forward force of 500N, what resistance is the car experiencing

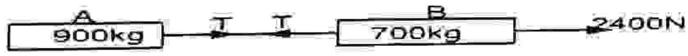
Solution



$500 - R_1 = ma$

But $a = 0$ since it moves with constant velocity
$500 - R_1 = 0$
$R_1 = 500N$

3. A car of mass 900kg tows a caravan of mass 700kg along a level road. The engine of the car exerts a forward force of 2400N and there is no resistance to the motion, find the acceleration produced and the tension in the tow bar



Solution

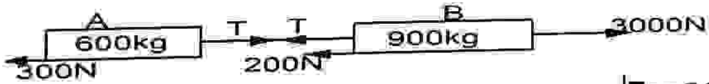
For 700kg: $2400 - T = 700a \ldots\ldots (1)$

For 900kg: $T = 900\ a \ldots\ldots (2)$

$eqn(1) + eqn(2)$

$2400 = 1600a$
$a = 1.5ms^{-2}$
$T = 900\ a$
$T = 900x1.5 = 1350N$

4. A car of mass 900kg tows a trailer of mass 600kg along a level road by means of a rigid bar. The car exerts experiences a resistance of 200N and the trailer a resistance of 300N, if the car engine exerts a forward force of 3kN, find the acceleration produced and the tension in the tow bar



Solution

For 900kg: $3000 - (T + 200) = 900a \ldots (1)$

For 600kg: $T - 300 = 600a \ldots\ldots(2)$

$(1) + (2):\quad 2500 = 1500a$

$a = 1.67ms^{-2}$

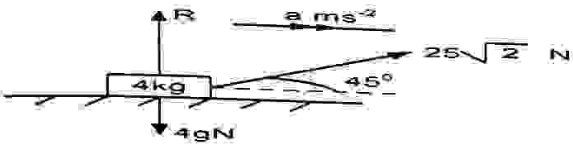
$T - 300 = 600a$

$T = 600x1.67 + 300 = 1300N$

FORCE INCLINED AT ANGLE TO THE HORIZONTAL

1. A body of mass 4kg is acted upon by force of $25\sqrt{2}N$ which is inclined at 45° to a smooth horizontal surface. Find the acceleration of the body and the normal reaction between the body and the surface

Solution



$(\rightarrow)\quad 24\sqrt{2}cos45 = 4a$

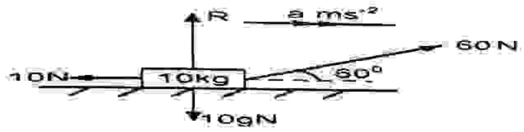
$a = 6.25ms^{-2}$

$(\uparrow)\quad R + 24\sqrt{2}Sin45 - 4g = 0$

$R = 14.2N$

2. A body of mass 10kg is initially at rest on a rough horizontal surface. It is pulled along the surface by a constant force of 60N inclined at 60° above the horizontal. If the resistance to motion totals 10N, find the acceleration of the body and the distance travelled in the first 3s

Solution



$(\rightarrow)\quad 60cos60 - 10 = 10a$

$a = 2ms^{-2}$

$s = ut + \frac{1}{2}at^2$

$s = 0x3 + \frac{1}{2}x2x(3)^2 = 9m$

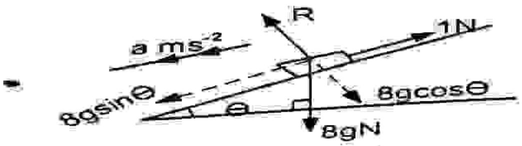
Exercise 14A

1. A railway engine of mass 100 tones is attached to a line of truck of total mass 80 tones. Assuming there is no resistance to motion, find the tension in the coupling between the engine and the leading truck when the train
   (a) has an acceleration of 0.020ms^{-2}
   (b) Is moving at constant velocity
   An(25.6kN).
2. A body of mass 5kg, initially at rest on a smooth horizontal surface is pulled along the surface by a constant force P inclined at 45° above the horizontal. In the first 5 seconds of motion, the body moves a distance of 10m along the surface. Find the;
   (i) Acceleration of the body
   (ii) Magnitude of P
   (iii) Normal reaction between the body and the surface **An**$(0.8ms^{-2}, 4\sqrt{2}N, 45N)$
3. A body of mass mkg, initially at rest on a smooth horizontal surface is pulled along the surface by a constant force P inclined at θ above the horizontal. Show that the body moves a distance in time t along the surface given by $\frac{Pt^2cos\theta}{2m}$
4. A body of mass mkg, initially at rest on a rough horizontal surface is pulled along the surface by a constant force P inclined at θ above the horizontal. If the mass acquire velocity v in a distance d. Show that the resistance to motion is given by $Pcos\theta - \frac{mv^2}{2d}$

MOTION ON AN INCLINED PLANE

1. A body of mass 8kg is released from rest on the surface of a plane inclined at $1\ in\ 40$. If the resistance to motion is 1N, find the acceleration of the body and the speed it acquired after 6 seconds.

Solution



$sin\theta = \frac{1}{40}$

$F = ma$

$8gsin\theta - 1 = 8a$

$a = 0.12ms^{-2}$

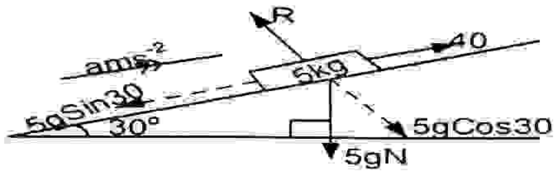
$v = u + at$

$v = 0 + 0.12x6 = 0.72ms^{-1}$

2. A body of mass 5kg is pulled up a smooth plane inclined at 30° to the horizontal by a force of 40N acting parallel to the plane. Find

a) Acceleration of the body b) Force exerted on the body by the plane

Solution



(a) Resolving parallel to the plane: $F = ma$

$$40 - 5gsin30 = ma$$

$$40 - 5x9.81sin30 = 5xa$$

$$1.5.475 = 5a$$

$$a = 3.095ms^{-2}$$

(b) Force exerted on the body by the plane is the normal reaction

$$R = 5gcos30$$

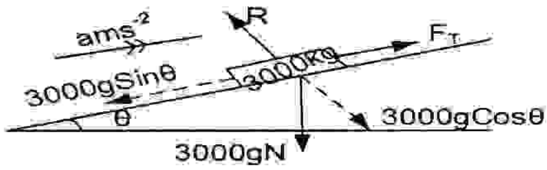
$$R = 5x9.81cos30 = 42.4N$$

3. A lorry of mass 3 tones travelling at 90km/hr starts to climb an incline of 1 in 5. Assuming the tractive pull between its tyres and the road remains constant and that its velocity reduces to 54km/h in a distance of 500m. Find the tractive pull

Solution

$$u = 90km/h = \frac{90x1000}{36000} = 25ms^{-1}$$

$$v = \frac{54km}{h} = \frac{54x1000}{36000} = 15ms^{-1}$$



Resolving along the plane

$$F - 3000gsin\theta = 3000a$$

$$F_T - 3000x9.81x\frac{1}{5} = 3000a$$

$F_T - 5886 = 3000a$....................(i)

But $v^2 = u^2 + 2as$

$$15^2 = 25^2 + 2ax500$$

$$a = -0.4ms^{-2}$$

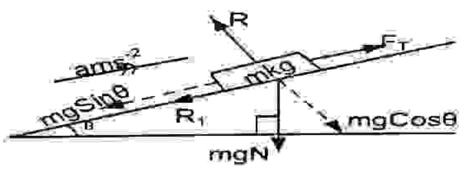
put into (i): $F_T - 5886 = 3000a$

$$F_T = -3000x0.4 + 5886 = 4686N$$

The tractive force is 4686N

4. A train travelling uniformly at 72km/h begins an ascent on 1 in 75. The tractive force which the engine exerts during the ascent is constant at 24.5kN, the resistance due to friction and air is also constant at 14.7kN, given the mass of the whole train is 225 tones. Find the distance a train moves up the plane before coming to rest.

Solution



1 in 75 means $sin\theta = \frac{1}{75}$ ∴ $\theta = 0.76°$

$$F_T - (mgsin\theta + R_1 = ma$$

$$24500 - (225000x9.81x\frac{1}{75} + 14700) = 22500a$$

$$a = -0.087ms^{-2}$$

$v^2 = u^2 + 2as$ [$v = om/s$ comes to rest]

$$u = 72km/h = \frac{72x1000}{36000} = 20ms^{-1}$$

$$0^2 = 20^2 + 2(-0.087)s$$

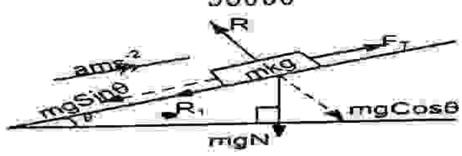
$$-400 = -0.174s$$

$$S = 2298.85m$$

5. The resistance to motion of the car is $\frac{1}{60}$ of its weight. If the car starts to climb a hill of 1 in 150 at 72km/h with the engine turned off. Find how far the car will go up the hill before it comes to rest

Solution

$$u = 72km/h = \frac{72x1000}{36000} = 20ms^{-1}$$



Resolving along the plane

$$F_T - (R_1 + mgsin\theta) = ma$$

$$0 - \left(\frac{1}{60}mg + mgx\frac{1}{150}\right) = ma$$

$$a = -\frac{210x9.8}{9000} = -0.229ms^{-2}$$

But $v^2 = u^2 + 2as$

$$0^2 = 20^2 - 2x0.229s$$

$$s = \frac{400}{2x0.229} = 873.36m$$

Exercise 14B

1. A mass 5kg is initially at the bottom of a smooth slope which is inclined at $sin^{-1}\left(\frac{3}{5}\right)$ to the horizontal. The mass is pushed up the slope by a horizontal force 50N, find
   (i) the normal reaction between the mass and the plane
   (ii) calculate the acceleration up the slope
   (iii) how far up the slope with the mass travel in the first 4s. **An(69.2N, 2.12ms^{-2}, 16.96N)**
2. A body of mass 100kg is released from rest at the top of a smooth slope which is inclined at 30° to the horizontal. Find;
   (i) Velocity of the body when it has travelled 20m down the slope
   (ii) Velocity, if the mass of the body was 50kg. **An(14ms^{-1}, 14ms^{-1})**
3. A body of mass 20kg is released from rest at the top of a smooth slope which is inclined at 30° to the horizontal. If the body accelerates down the slope at 3ms^{-2}, find the constant resistance to motion experienced by the body **An(38N)**
4. A body of mass 20kg is released from rest at the top of a rough slope which is inclined at 30° to the horizontal. 6s later the body has a velocity of 21ms^{-1} down the slope, find the constant resistance to motion experienced by the body **An(28N)**
5. A car of 1 tonne accelerates from 36kmh to $72\ kmh^{-1}$ while moving $0.5kmh^{-1}$ up a road inclined at an angle of α to the horizontal, where $sin\alpha = \frac{1}{20}$. If the total resistive force to its motion is 0.3kN, find the driving force of the car engine **An(1009N).**
6. A railway truck of mass 6.0 tonnes moves with an acceleration of 0.050ms^{-2} down a track which is inclined to the horizontal at an angle α where $sin\alpha = \frac{1}{120}$. Find the resistance to motion **An(2.0x10^{2}N).**
7. A body of mass 5.0kg is pulled along a smooth horizontal ground by means of force of 40N acting at 60° above the horizontal. Find
   (a) Acceleration of the body
   (b) Force the body exerts on the ground **An(4.0ms^{-2}, 15.4N).**
8. A body of mass 3.0kg slides down a plane which is inclined at 30° to the horizontal. Find the acceleration of the body , if:
   (a) The plane is smooth
   (b) There is a frictional resistance of 9.0N **An(5.0ms^{-2}, 2.0ms^{-2}).**
9. A car of mass 1000kg tows a caravan of mass 600kg up a road which rises 1m vertically for every 20m of its length. There are constant frictional resistance of 200N and 100N to the motion of the car and to the motion of the caravan respectively. The combination has an acceleration of 1.2ms^{-2} with the engine exerting a constant driving force. ($Take\ g = 10ms^{2}$)Find
   (a) Driving force
   (b) Tension in the tow- bar **An(3.02kN, 1.12kN).**
10. Find the time interval between a particle reaching the bottom of a smooth slope of length 5m and inclination 1 in 98, and another particle reaching the bottom of a smooth slope of length 6m and inclination 1 in 70. Both particle are released from rests at the top of their respective slopes at the same time **An(0.74s)**
11. A body of mass 5kg is pulled along a rough slope which is inclined at 30° to the horizontal by a force of 50N. The mass starts from rest and after 4s the pulling force ceases. If the resistance to motion is 20N throughout, find the total distance travelled before the mass comes to rest again **An(10m)**
12. A body of mass mkg is released from rest at the top of a smooth slope which is inclined at θ to the horizontal. Show that its velocity, when it has travelled a distance s down the slope is given by $\sqrt{2gsin\theta}$
13. A body of mass mkg is released from the top of a rough slope which is inclined at θ to the horizontal. After time t, the mass has travelled a distance d down the slope. Show that the resistance to motion experienced by the body is
$$\frac{m}{t^2}(gt^2 sin\theta - 2d).$$
14. A body travels along a line of greatest slope of a smooth plane inclined at angle θ to the horizontal, where $sin\theta = \left(\frac{3}{5}\right)$. The particle starts from a point P where it is given an initial speed of 15ms^{-1} up the plane to point Q. given that PQ is 12m,
   (i) Calculate the speed of the particle when it first passes Q
   (ii) Calculate also the time for which the particle s above the level of Q **An(9ms^{-1}, 3s)**
15. The pull exerted by an engine is 1/80 of the weight of the whole train and the maximum brake force which can be exerted is 1/30 of the weight of the

train. Find the time in which the train travels from the rest up a slope of 1 in 240 and 4800m along, if the brakes are applied when the engine is switched off . **An(379s).**

16. The resistance to the motion of the train due to friction is equal to 1/160 of the weight of the train, if the train is travelling on a level road at $72kmh^{-1}$ and comes to the foot of an incline of 1 in 150 and steam is then turned off, how far will the train go up the incline before it comes to rest. **An(1579.99m)**

17. A carton of mass 0.4kg is thrown across a table with a velocity of $25ms^{-1}$. The resistance of the table to its motion is 50N. **Uneb march 1998 No2**
    (i) How far will it travel before coming to rest
    (ii) What be the resistance if it only travels 2m. **An(2.5m, 62.5N)**

18. A particle of mass 5kg resting on a smooth plane inclined at $tan^{-1}(\frac{1}{\sqrt{3}})$ to the horizontal. Find the magnitude of the horizontal force required to keep the particle in equilibrium and the normal reaction to the plane. **Uneb** 2001 No.5 **An (56.58N)**

19. The engine of a train exerts a force of 3,500N on a train of mass 240 tonnes and draws up a slope of 1 in 120 against resistances totaling to 160N per tonne. Find the acceleration of the train. **Uneb** 2008, No6 **An**($0.00417ms^{-2}$)

20. A car of mass 2.5 metric tonnes is drawn up a slope of 1 in 10 from rest with an acceleration of $1.2ms^{-2}$ against a constant frictional resistance of $\frac{1}{100}$ of the weight of the vehicle using a cable. Find the tension in the cable. **Uneb** 2006, No6 **An**(5695N)

CHAPTER 5: FRICTION

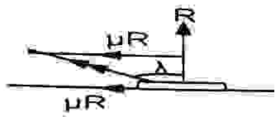
This is a force that opposes relative motion or attempted motion between two bodies in contact

Frictional force is given by $\boxed{F = \mu R}$

At limiting equilibrium, the body is on the point of moving (slip or slide) and frictional force is maximum

Angle of friction

This is the angle between the resultant force and the normal reaction force

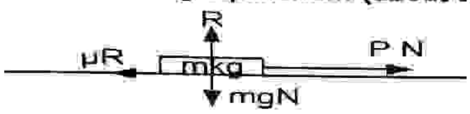


$$\tan\lambda = \frac{\mu R}{R}$$

$$\boxed{\tan\lambda = \mu}$$

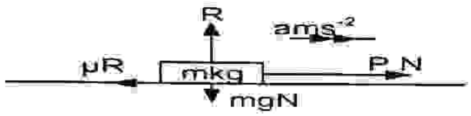
A HORIZONTAL PLANE

(i) At limiting equilibrium (about to slip or slid)



$$P = \mu R$$
$$P = \mu m x 9.8$$

(ii) In motion



$$F = ma$$
$$P - \mu R = ma$$
$$P - \mu x(mx9.8) = ma$$

Examples

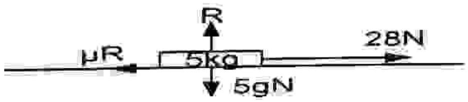
1. Calculate the maximum frictional force which can act when a block of mas 3kg rests on a rough horizontal surface, the coefficient of friction between the surface being 0.2

Solution

$$F = \mu R$$
$$F = 0.2(3x9.8) = 5.88N$$

2. When a horizontal force of 28N is applied to a body of mass 5kg which is resting on a rough horizontal plane, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane

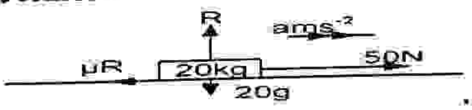
Solution



$$28 = \mu R$$
$$28 = \mu(5x9.8)$$
$$\mu = 0.57$$

3. A block of mass 20kg rests on a rough horizontal plane. The coefficient of friction between the block and the plane is 0.25. If a horizontal force of 50N acts on a body, find the acceleration of the body

Solution

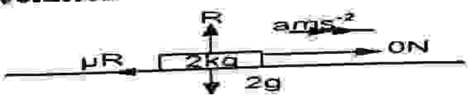


$$F = ma$$

$$50 - \mu R = 20a$$
$$50 - 0.25x(20x9.8) = 20a$$
$$a = 0.05ms^{-2}$$

4. A block of mass 2kg sliding along a smooth horizontal surface at a constant speed of 2m/s. When the mass encounters a rough horizontal surface of coefficient of friction 0.2, it comes to rest. Find the distance the body will move across the rough surface before it comes to rest.

Solution



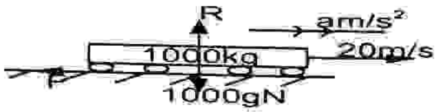
$$F = ma$$
$$0 - \mu R = 20a$$

$$-0.2x(2x9.8) = 2a$$
$$a = -1.96ms^{-2}$$
$$s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 20^2}{2x - 1.96} = 1.02m$$

5. A car of mass 1000kg moving along a straight road with a speed of $72kmh^{-1}$ is brought to rest by a speedy application of brakes in a distance of 50m. Find the coefficient of kinetic friction between the tyres and the road.

Solution

$$u = \frac{72x1000}{3600} = 20m/s$$



$$v^2 = u^2 + 2as$$
$$0 = 20^2 + 2ax50$$
$$a = -4ms^{-2}$$
$$F = ma$$
$$ma = \mu R$$
$$4x1000 = 1000x9.8\mu$$
$$\mu = 0.41$$

Alt: Work done against friction=loss in k.e

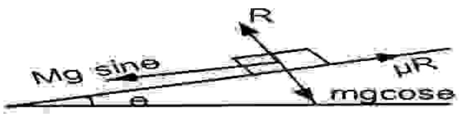
$$\mu(m)gxs = \frac{1}{2}(m)v^2$$
$$\mu x9.81x50 = \frac{1}{2}x(20)^2$$
$$\mu = 0.408$$

Exercise 15A

1. When a horizontal force of 0.245N is applied to a body of mass 250g which is resting on a rough horizontal plane, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane. **An**(0.1)
2. A body of mass 40kg is resting on a rough horizontal plane and can just be moved by a force of 98N acting horizontally. Find the coefficient of friction **An(0.25)**
3. A block of mass 0.5kg rests on a rough horizontal plane. The coefficient of friction between the block and the table is 0.1. When a horizontal force of 1N acts on the block, find;
   (i) Frictional force experienced by the block
   (ii) Acceleration with which the block will move.
   An$(0.49N, 1.02ms^{-2})$
4. When a horizontal force of 37N is applied to the body of mass 10kg which is resting on a rough horizontal surface, the body moves along the surface with an acceleration$1.25ms^{-2}$. Find the coefficient of friction between the body and the surface **An**(0.25)

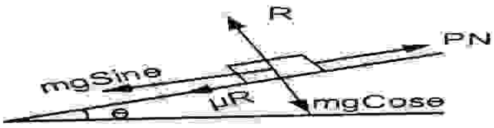
AN INCLINED PLANE

(i) At limiting equilibrium (about to slip down or slid down)



$$mgsin\theta = \mu R$$
$$mgsin\theta = \mu mcos\theta$$
$$\mu = tan\theta$$

(ii) A force P applied parallel to and up the plane to just move the particle upwards (prevent it from moving upwards)

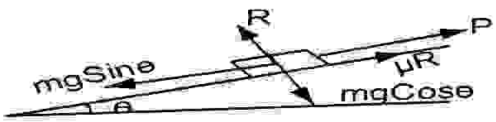


Normal to the plane: $mgcos\theta = R$

Parallel to the plane: $mgsin\theta + \mu R = P$

$$P = mgsin\theta + \mu mgcos\theta$$

(iii) A force P applied parallel to and up the plane so that the particle is on the point of moving downwards (prevent it from moving downwards)

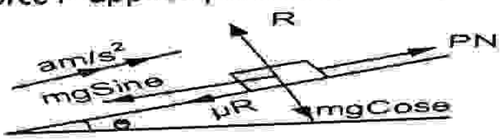


Normal to the plane: $mgcos\theta = R$

Parallel to the plane: $mgsin\theta = P + \mu R$

$$P = mgsin\theta - \mu mgcos\theta$$

(iv) A force P applied parallel to and up the plane move the particle upwards



Normal to the plane: $mgcos\theta = R$

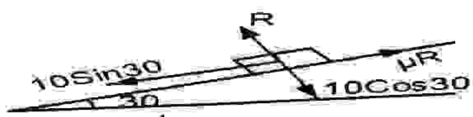
Parallel to the plane: $P - (mgsin\theta + \mu R) = ma$

$$P - mgsin\theta - \mu mgcos\theta = ma$$

Examples

1. A particle of weight 10N rests on a rough plane inclined at 30° to the horizontal and is just about to slip. Find the value of coefficient of friction between the plane and particle

Solution

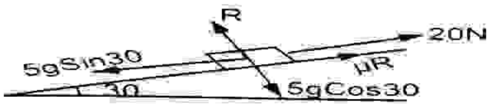


At limiting equilibrium

$$R = 10cos30$$
$$\mu R = 10\ sin30$$
$$\mu(10cos30) = 10\ sin30$$
$$\mu = 0.5774$$

2. A body of mass 5kg lies on a rough plane which is inclined at 35° to the horizontal. When a force of 20N is applied to the body parallel to and up the plane, the body is on the point of moving down the plane. Find the coefficient of friction between the body and the plane

Solution



At limiting equilibrium

$R = 5gcos30$
$20 + \mu R = 5g\ sin30$
$20 + \mu(5gcos30) = 5g\ sin30$
$\mu = 0.106$

3. A block of wood of mass 150g rests on an inclined plane. If the coefficient of friction between the surface of contact is 0.3. Find the force parallel to the plane necessary to prevent slipping when the angle of the plane to the horizontal is 30°.

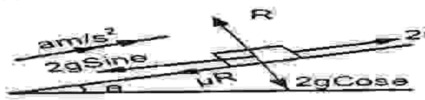
Solution

At limiting equilibrium

but $R = mgcos30$
$P + \mu R = mg\ sin30$
$P + 0.3x0.15x9.8cos30 = 0.15x9.8sin30$
$P = 0.353N$

4. A body of mass 2kg lies on a rough plane which is incline at $sin^{-1}\left(\frac{5}{13}\right)$ to the horizontal. A force of 20N is applied to the body, parallel to and up the plane. If the body accelerates up the plane at $1.5ms^{-1}$, find the coefficient of friction between the body and the plane

Solution

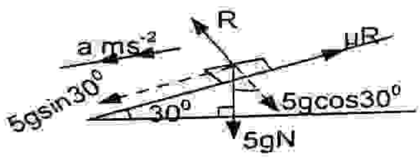


$Sin\ \theta = \frac{5}{13},\ \ cos\theta = \frac{12}{13}$
$R = 2gcos\theta$

$F = ma$
$20 - (2gsin\theta + \mu R) = 2a$
$20 - \left(2x9.8x\frac{5}{13} + \mu x2x9.8x\frac{12}{13}\right) = 2x1.5$
$\mu = 0.523$

5. A body of mass 5kg is released from rest on a rough surface of a plane inclined at 30° to the horizontal. If the body takes $2\frac{1}{2}s$ to acquire a speed of $4ms^{-1}$ from rest, find the frictional force and the coefficient of friction

Solution



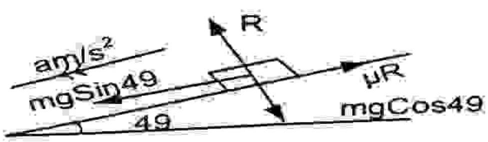
$v = u + at$

$4 = 0 + ax2\frac{1}{2}$
$a = 1.6ms^{-2}$
$F = ma$
$5gsin30 - \mu R = 5x1.6$
$\mu R = 16.5N$

Frictional force= $16.5N$
$\mu R = 16.5N$
$\mu = \frac{16.5}{5gcos30} = 0.243$

6. A car of mass 500kg moves from rest with the engine switched off down a road which is inclined at an angle 49° to the horizontal
   a)Calculate the normal reaction
   b)If the coefficient of friction between the tyres and surface of the road is 0.32. Find the acceleration of the car

Solution

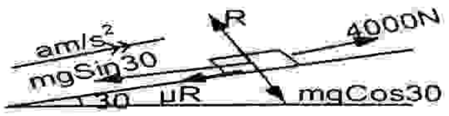


$R = mgcos49$

R= $500x9.8cos49 = 3217.97N$
a) $F = ma$
$mgsin49 - \mu R = 500a$
$500x9.8sin49 - 0.32x3217.97 = 500a$
$a = 5.34ms^{-2}$.

7. A car of mass 1000kg climbs a plane which is inclined at 30° to the horizontal. The speed of the car at the bottom of the incline is $36kmh^{-1}$. If the coefficient of friction between the plane and the car tyres is 0.3 and engine exerts a force of 4000N how far up the incline does the car move in 5s?

Solution



$u = 36kmh^{-1} = \frac{36x\,1000}{3600} = 10ms^{-2}$

$F = ma$

$4000 - (mgsin30 + \mu R) = ma$

$4000 - (1000x9.8sin30 + 0.3mgcos30) = 1000a$

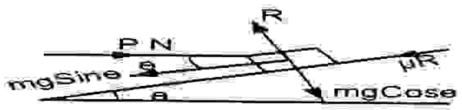
$a = -3.45ms^{-2}$

$S = ut + ½at^2$

$S = 10x5 + \frac{1}{2}x - 3.45x5^2 = 6.9m$

HORIZONTAL FORCE ON INCLINED PLANES

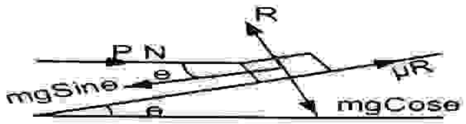
(i) A horizontal force P required to just move the particle upwards (prevent it from moving upwards)



Normal to the plane: $mgcos\theta + Psin\theta = R$

Parallel to the plane: $mgsin\theta + \mu R = Pcos\theta$

(ii) A horizontal force P required to prevent the particle from moving downwards



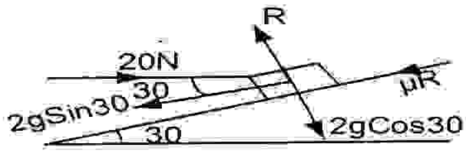
Normal to the plane: $mgcos\theta + Psin\theta = R$

Parallel to the plane: $mgsin\theta - \mu R = Pcos\theta$

Examples

1. A body of mass 2kg lies on a rough plane which is inclined at 30° to the horizontal. When a horizontal force of 20N is applied to the body in an attempt to push it up the plane, the body is found to be on the point of moving up the plane. Find the coefficient of friction between the body and the plane

Solution



At limiting equilibrium

$R = 2gcos30 + 20Sin30$

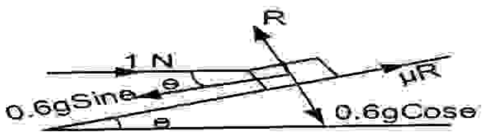
$20\cos 30 = \mu R + 2g\,sin30$

$20\cos 30 = \mu(2gcos30 + 20Sin30) + 2g\,sin30$

$\mu = 0.279$

2. A horizontal force of 1N is just sufficient to prevent a brick of mass 600g sliding down a rough plane which is inclined at $Sin^{-1}\left(\frac{5}{13}\right)$ to the horizontal. Find the coefficient of friction between the brick and the plane

Solution



$Sin\theta = \frac{5}{13}, \; cos\theta = \frac{12}{13}$

At limiting equilibrium

$R = 0.6gcos\theta + 1xSin\theta$

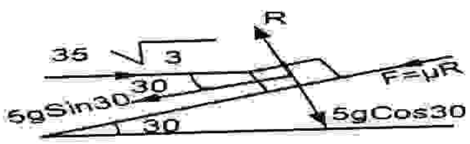
$1xcos\,\theta + \mu R = 0.6g\,sin\theta$

$1x\cos\theta + \mu(0.6gcos\theta + Sin\theta) = 0.6g\,sin\theta$

$\mu = 0.23$

3. A body of mass 5kg is initially at rest at the bottom of a rough inclined plane of length 6.3m. The plane is inclined at 30° to the horizontal and coefficient of friction between the body and the plane is $^{1}/_{2\sqrt{3}}$. A constant horizontal force of $35\sqrt{3}$ N is applied to the body causing it to accelerate up the plane. Find the time taken for the body to reach the top and its speed on arrival.

Solution



$R = 5gcos30 + 35\sqrt{3}Sin30$

$35\sqrt{3}\cos 30 - (\mu R + 5g\,sin30) = 5a$

$a = 1.39ms^{-2}$

$s = ut + {}^{1}/_{2}at^2$

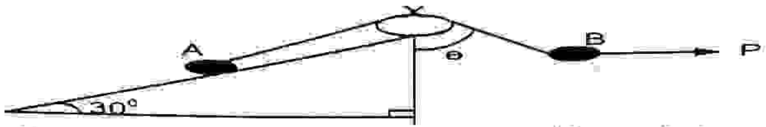
$6.3 = 0xt + {}^{1}/_{2}x1.39t^2$

$t = 3s$

$v = u + at$

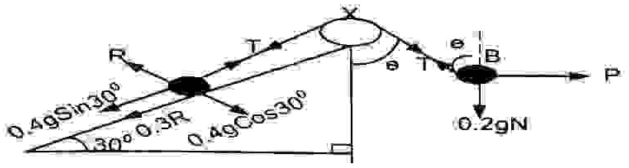
$v = 0 + 1.39x3 = 4.17m/s$

5. Particle A of mass 0.4kg and B of mass 0.2kg are attached to the end of a light inextensible sting. Particle A rests in equilibrium on a rough plane which is inclined at 30° to the horizontal. The string passes over a smooth pulley X at the top if the plane, and the part AX of the string is parallel to a line of greatest slope of the plane. Particle B is held in equilibrium by means of a horizontal force, P in such a way that, part XB of the sting makes angle θ with the vertical



The points A, X, B lie in the same vertical plane. The coefficient of friction between A and the sloping plane is 0.3. Given that A is about to slip up the plane, find the value of θ and the magnitude of the horizontal force P

Solution



At limiting equilibrium

0.4kg mass: $R = 0.4gcos30$

$T = 0.2gsin30 + 0.3R$

$T = 0.2gsin30 + 0.3x0.4gcos30$

$T = 2.98N$

$0.4kg\ mass: Tcos\theta = 0.2g$

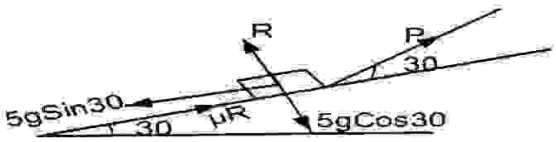
$(\uparrow)\quad \theta = cos^{-1}\left(\frac{0.2x9.8}{2.98}\right) = 48.9°$

$(\rightarrow)\quad P = Tsin\theta$

P = 2.98sin48.9 = 2.245N

6. A body of mass 5kg is placed on a rough plane inclined at an angle of 30° to the horizontal. The coefficient of friction between the plane and the body is 0.4. Find the maximum force that can be applied to keep the body in equilibrium, if the force acts upwards at an angle of 30° to the line of greatest slope

Solution



At limiting equilibrium

$R = 5gcos30 - PSin30$

$P\ cos\ 30 + \mu R = 5g\ sin30$

$P\ cos\ 30 + \mu(5gcos30 - PSin30) = 5g\ sin30$

$P(cos\ 30 - 0.4sin30) = 5g\ sin30 - 0.4x5g\ cos30$

$P = 11.3N$

Exercise: 15B

1. A body of weight W is held in limiting equilibrium on a rough slope inclined at 60° to the horizontal by a force P at an angle of 30° to the slope. The coefficient of friction between the body and the surface is 0.2. show that $P = W$
2. A body of mass 5kg is placed on a rough plane inclined at an angle of 30° to the horizontal. The angle of friction between the plane and the body is 20°. Find the maximum force that can be applied to make the body to just be on the point of moving up the plane, if the force acts upwards at an angle of 30° to the line of greatest slope. **An(38.91N)**
3. A body of mass 4kg lies on a rough plane which is inclined at 16° to the horizontal. A force of 1N applied parallel to the plane is just sufficient to prevent the body sliding down the plane.
   (i) Find the coefficient of friction between the body and the plane.
   (ii) With the body at the top of the plane, the applied force is removed. Find the time taken for the body to reach the bottom of the plane, if the length of the plane is 2m **An (μ = 0.26, 4s)**
4. A force F acting parallel to and up a rough plane of inclination θ, is just sufficient to prevent a body of mass m from sliding down the plane. A force of 4F acting parallel to and up the same rough plane causes the mass m to be on the point of moving up the plane. If μ is the coeffienect of friction between the mass and the plane, show that $5\mu = 3\ tan\theta$
5. A body of mass 3kg is released from a rough surface which is inclined at $Sin^{-1}\left(\frac{3}{5}\right)$ to the horizontal. If after 2.5s the body has acquired a velocity of 4.9m/s down the surface. Find the coefficient of friction between the body and the surface **An (μ = 0.5)**
6. A particle of weight 4.9N resting on a rough inclined plane of angle equal to $tan^{-1}(\frac{5}{12})$ is acted upon by a horizontal force of 8N. If the particle is on the point of moving up the plane, find

coefficient of friction between the particle and the plane. **An** ($\mu = 0.72$)

7. A particle of weight 10N rests in with a rough plane inclined at 30° to the horizontal an is about to slip. Find the value of coefficient of friction between the plane and the particle **An** ($\mu = \frac{1}{\sqrt{3}}$)
8. A particle of mass 12kg slides from rest down a plane inclined at 50° to the horizontal. If the coefficient of friction between the particle and the plane is 0.4, calculate the acceleration of the particle **An** ($4.99ms^{-2}$)
9. A box of mass 2kg rests on a rough inclined plane of angle 25°. The coefficient of friction between the box and the plane is 0.4.. Find the least force applied parallel to the plane which would move the box up the plane. **Uneb 1997 No.6**
   An[15.39N]
10. A particle of mass 0.5kg is released from rest and slides down a rough plane inclined at 30° to the horizontal. It takes 6 seconds to go 3 meter. **Uneb 1997 No.14**
    i. Find the coefficient of friction between the particle and the plane
    ii. What minimum horizontal force is needed to prevent the particle from moving?
    An[0.557, 0.064N]
11. A particle of weight 10N is placed on a rough plane inclined at 30° to the horizontal, the coefficient of friction between the particle and the plane is 0.5. Find the horizontal force required;
    i. To prevent the particle from sliding down
    ii. To make the particle just about move up.
    An[0.6N, 15.2N]
12. A parcel of mass 2kg is placed on a rough plane inclined at 45° to the horizontal, the coefficient of friction between the parcel and the plane is 0.25. Find the force that must be applied parallel to the plane so that the parcel is just.
    i. Prevented from sliding down the plane
    ii. On the point of moving up the plane.
    An[10.39N, 17.32N]
13. A box of mass 6kg is placed on a rough plane inclined at 45° to the horizontal, the coefficient of friction between the parcel and the plane is 0.5. Find the horizontal force that must be applied to the plane so that the box is;
    i. Just Prevented from sliding down the plane
    ii. Just On the point of moving up the plane.
    iii. Moves up the plane with an acceleration of $2\sqrt{2}\ ms^{-2}$ **An[19.6N, 176.4N, 224.4N]**
14. The length of an incline plane is 5m and the height is 3m. A force of 49N acting parallel to the plane will prevent a mass of 10kg from sliding down. Find the coefficient of friction **An(0.125)**
15. A 2kg body lies on a plane of inclination 60°. The coefficient of friction between the body and the plane is 0.25. find the least horizontal force which prevents the body form sliding down the plane **An(20.27N)**
16. A force F acting parallel to and up a rough plane of inclination θ, is just sufficient to prevent a body of mass m from sliding down the plane. A horizontal force of 4F acting on the same rough plane causes the mass m to be on the point of moving up the plane. If μ is the coeffienect of friction between the mass and the plane, show that
    $$5\mu\, tan^2\theta - 3(\mu^2 + 1)\, tan\theta + 5\mu = 0$$
17. A particle of mass 2m rests on a rough plane inclined to the horizontal at an angle of rests on a rough plane inclined to the horizontal at an angle if $tan^{-1}(3\mu)$ where μ is the coefficient of friction between the particle and the plane. The particle is acted upon by a force of P Newton's.
    (a) Given that the force acts along the line of greatest slope and that the particle is on the point of slipping up, show that the maximum force possible to maintain the particle in equilibrium is $P_{max} = \frac{8\mu mg}{\sqrt{1+9\mu^2}}$
    (b) Given that the force acts horizontally in a vertical plane through a line of greatest slope and that the particle is on the point of sliding down the plane, show thqt the minimum force required to maintain the particle in equilibrium is
    $$P_{min} = \frac{4\mu mg}{1 + 3\mu^2}$$
18. A vehicle of mass 2.5 metric tones is drawn up a slope of 1 in 10 from rest with an acceleration of $1.2ms^{-2}$ against constant frictional resistance of $\frac{1}{100}$ of the weight of the vehicle using a cable. Find the tension in the cable
    An(5695N) **(Uneb 2006 Qn 6)**
19. A particle of mass mkg is projected with a velocity of 10m/s up a rough plane of inclination 30° to the horizontal. The coefficient of friction between the particle and the plane is 0.25. Find how far up the plane the particle travels
    An(7.12m) **(Uneb 2007 Qn 16)**

20. A body of mass 8kg rests on a rough plane inclined at θ to the horizontal. If the coefficient of friction is μ, find the least horizontal force in terms of μ, θ and g which will hold the body in equilibrium. **(Uneb 2009 Qn 5)**

$\frac{8g\,(tan\theta-\mu)}{1+\mu tan\theta}$

21. A carton of mass 3kg rests on a rough plane inclined at an angle of 30° to the horizontal. The coefficient of friction between the carton and the plane is $\frac{1}{3}$. Find a horizontal force that should be applied to make the cartoon just about to move up the plane. **An(33.16N)** **(Uneb 2010 Qn 5)**

22. A particle of mass 2kg rests in limiting equilibrium on a rough plane inclined at 30° to the horizontal. Find the value of coefficient of friction **An(0.577)** **(Uneb 2016 Qn 7)**

CHAPTER 6: CONNECTED PARTICLES

SIMPLE CONNECTIONS

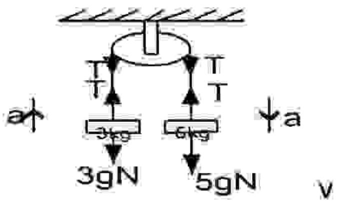
When two particles are connected by a light inextensible string passing over a smooth pulley and allowed to move freely, then as long as the string is taut, the following must be observed.

- Acceleration of particles is the same
- The tension in the uninterrupted string is constant
- The tensions in interrupted strings are different.

Examples

1. Two particles of masses 5kg and 3kg are connected by a light inelastic string passing over a smooth fixed pulley. Find;
   (i) Acceleration of the particles
   (ii) The tension in the string
   (iii) The force on the pulley

Solution



$F = ma$

For 5kg mass: $5g - T = 5a$..........(i)

For 3kg mass: $T - 3g = 3a$ ....(ii)

Adding (i) and (ii)

$2g = 8a$

$a = \frac{2x9.8}{8} = 2.45ms^{-2}$

ii) $T - 3g = 3a$

$T = 3x2.45 + 3x9.8 = 36.78N$

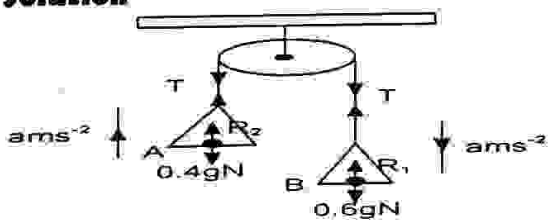
iii) Force on the pulley

$R = 2T = 2x36.78 = 73.56N$

Force on the pulley is $73.56N$

1. An inextensible string attached to two scale pans A and B each of weight 20g passes over a smooth fixed pulley. Particles of weight 3.8N and 5.8N are placed in pans A and B respectively, if the system is released from rest (take $g = 10ms^{-2}$). Find the:
   (i) Tension in the string
   (ii) Reaction of the scale pan holding the 3.8N weight

Solution



Mass of scale pan $= 20g = 0.02kg$

Total mass of A$= \frac{3.8}{10} + 0.02 = 0.4kg$

Total mass of B$= \frac{5.8}{10} + 0.02 = 0.6kg$

$F = ma$

For 0.6kg mass: $0.6g - T = 0.6a$..........(i)

For 0.4kg mass: $T - 0.4g = 0.4a$ ....(ii)

Adding (i) and (ii): $0.2g = a$

$a = 0.2x10 = 2ms^{-2}$

$T - 0.4g = 0.4a$

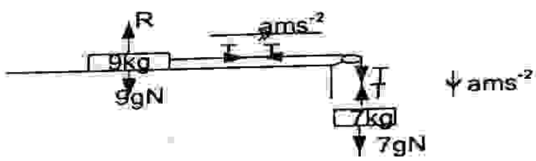
$T = 0.4x2 + 0.4x10 = 4.8N$

For the scale pan A; $R_2 - 3.8 = 0.28a$

$R_2 = 3.8 + 2x3.8 = 4.56N$

2. A mass of 9kg resting on a smooth horizontal table is connected by a light string passing over a smooth pulley at the edge of the table, to the pulley is a 7kg mass hanging freely 1.5m above the ground; find
   (i) Common acceleration
   (ii) The tension in the string
   (iii) The force on the pulley in the system if its allowed to move freely.
   (iv) Time taken for the 7kg mass to hit the ground

Solution



$F = ma$

For 7kg mass: $7g - T = 7a$..........(i)

For 9kg mass: $T = 9a$.........................(ii)

(ii) + (i): $7g = 16a$

$$a = \frac{7g}{16} = \frac{7x9.8}{16} = 4.29ms^{-2}$$

(ii) Tension : $T = 9a = 9 \times 4.29 = 38.61N$

(iii) The force on the pulley

$F^2 = \sqrt{T^2 + T^2} = T\sqrt{2} = 38.61\sqrt{2}$

Force on the pulley $= 54.603N$

(iv) $s = ut + \frac{1}{2}at^2$

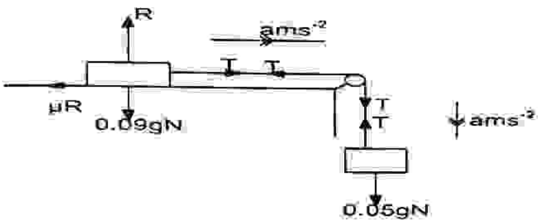
$$1.5 = 0xt + \frac{1}{2}x4.29xt^2$$

$$t = 0.84s$$

3. A mass of 90g resting on a rough horizontal table is connected by a light inextensible string passing over a smooth pulley at the edge of the table attached to a 50g mass hanging freely. The coefficient of friction between the 90g mass and the table is $\frac{1}{3}$ and the system is released from rest, find

(i) Common acceleration (ii) The tension in the string

Solution



$$F = ma$$

For 50g mass: $0.05g - T = 0.05a$...........(i)

For 90g mass: $T - \mu R = 0.09a$

$T - \frac{1}{3}(0.09x9.8) = 0.09a$................(ii)

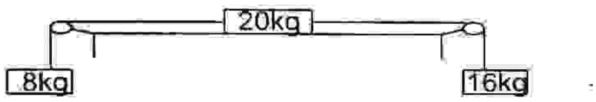
(ii) + (i): $0.05g - \frac{1}{3}(0.09g) = 0.14a$

$$a = \frac{0.02g}{0.14} = 1.4ms^{-2}$$

(ii) Tension : $0.05g - T = 0.05a$

$$T = 0.05x9.8 - 0.05 \times 1.4 = 0.42N$$

4.



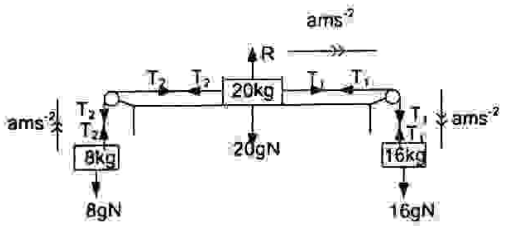
The figure shows a block of mass 20 kg resting on a smooth horizontal table. Its connected by inelastic strings which pass over fixed pulleys at the edges of the table to two loads of masses 8kg and 16kg which hang vertically. When the system is released freely, Calculate;

(i) Acceleration of 16kg mass

(ii) Tension in each string

(iii) Reaction on each pulley

Solution



For 16kg mass $16g - T_1 = 16a$..........[1]

For 20kg mass $T_1 - T_2 = 20a$............[2]

For 8kg mass $T_2 - 8g = 8a$...............[3]

Adding 1 and 2: $16g - T_2 = 36a$.............[x]

And 3 and x: $8g = 44a$

$a = \frac{8x\,9.8}{44}$ ∴ $a = 1.782ms^{-2}$

ii) Tension in each string

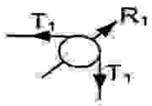
$$16g - T_1 = 16a$$

$$T_1 = 16x9.8 - 16x1.782 = 128.291N$$

$$T_2 - 8g = 8a$$

$$T_2 = 8x1.782 + 8x9.8 = 92.656N$$

iii) Reaction on each pulley



$$R_1^2 = T_1^2 + T_1^2 = T_1\sqrt{2}$$

$$R_1 = 128.291x\sqrt{2} = 181.431N$$

$$R_2 = T_2\sqrt{2} = 92.656\sqrt{2} = 131.035N$$

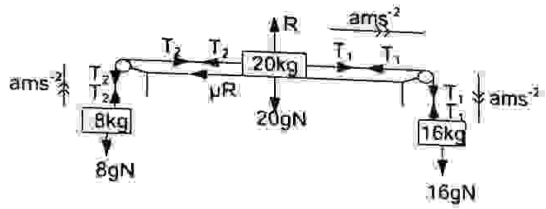
6. 5

The figure shows a block of mass 20 kg resting on a rough horizontal table of coefficient of friction 0.21. Its connected by inelastic strings which pass over fixed pulleys at the edges of the table to two loads of masses 8kg and 16kg which hang vertically. When the system is released freely, Calculate;

(i) Acceleration of 16kg mass

(ii) Tension in each string

Solution



For 16kg mass $16g - T_1 = 16a$..........[1]

For 20kg mass $T_1 - T_2 - 20g\mu = 20a$...........[2]

For 8kg mass $T_2 - 8g = 8a$...............[3]

Adding 1 and 2: $16g - T_2 - 20g\mu = 36a$.............[x]

And 3 and x: $8g - 20g\mu = 44a$

$a = \frac{8x\,9.81 - 20x9.8x0.21}{44} \therefore a = 0.846ms^{-2}$

ii) Tension in each string

$16g - T_1 = 16a$

$T_1 = 16x9.8 - 16x0.846 = 128.416N$

$T_2 - 8g = 8a$

$T_2 = 8x1.784 + 8x9.8 = 92.752N$

Exercise 16A

1. Two particles of masses 7kg and 3kg are connected by a light inelastic string passing over a smooth fixed pulley. Find;
   a. Acceleration of the particles
   b. The tension in the string
   c. The force on the pulley

   An(3.92ms^{-2} , 41.16N, 82.32N)

2. Two particles of masses 6kg and 2kg are connected by a light inextensible string passing over a smooth fixed pulley. With the masses hanging vertically, the system is released from rest. Find;
   (i) Acceleration of the particles
   (ii) The tension in the string
   (iii) Distance moved by the 6kg mass in the first 2 seconds of motion

   An(4.9ms^{-2} , 3N, 9.8m)

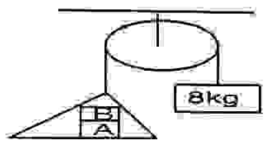
3. A man of mass 70kg and a bucket of bricks of mass 100kg are tied to the opposite ends of a rope which passes over a frictionless pulley so that they hang vertically downwards
   (a) what is the tension in the section of the section of rope supporting the man
   (b) What is the acceleration of the bucket

   An(807.06N, 1.73ms^{-2})

4. Two particles of masses 200g and 300g are connected to a light inelastic string passing over a smooth pulley, when released freely find;
   (i) Common acceleration
   (ii) The tension in the string
   (iii) The force on the pulley

   An[1.96ms^{-2}] [2.35N] [4.71N]

5. The diagram shows a particle of mass 8kg connected to a light scale pan by a light inextensible string which passes over a smooth fixed pulley. The scale pan holds two blocks A and B of mass 3kg and 4kg respectively, with B resting on top of A



if the system is released from rest, find;
   (i) Acceleration of the system
   (ii) The reaction between A and B

   An(0.653ms^{-2} , 41.813N)

6. A mass of 5kg is placed on a smooth horizontal table and connected by a light string to a 3kg mass passing over a smooth pulley at the edge of the table and hanging freely. If the system is allowed to move, calculate;
   a) The common acceleration of the masses
   b) The tension in the string
   c) The force acting on the pulley

   An[3.68m/s^2,18.4N, 26N]

7. A mass of 3kg resting on a smooth horizontal table. It is attached by a light inextensible string passing over a smooth pulley at the edge of the table, to another mass of 2kg hanging freely 2.1m above the ground; find
   (i) Common acceleration
   (ii) The tension in the string
   (iii) The force on the pulley in the system if its allowed to move freely.
   (iv) Velocity with which the 2kg mass will hit the ground

   An[3.92ms^{-2}, 11.76N, 16.63N, 4.06m/s]

8. A mass of 5kg rests on a rough horizontal table and is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a 3kg mass hanging freely. The coefficient of friction between the 5kg mass and the table is 0.25 and the system is released from rest, find
   (i) Common acceleration
   (ii) The tension in the string

   An[2.144ms^{-2}, 22.97N]

9. A mass of 1kg rests on a rough horizontal table and is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a 500g mass hanging freely. The coefficient of friction between the 1kg mass and the table is 0.1 and the system is released from rest, find
(ii) Common acceleration
(iii) The tension in the string
An[2.61ms^{-2}, 3.593N]

10. Two objects of mass 3kg and 5kg are attached to the ends of a cord which passes over a fixed frictionless pulley placed at 4.5m above the floor. The objects are held at rest with 3 kg mass touching the floor and the 5kg mass at 4m above the ground and then released, what is;
(i) The acceleration of the system
(ii) The tension of the cord.
(iii) Time will elapse before the 5kg object hits the floor
An(2.45ms^{-2}, 36.75N, 1.81s).

11.


The diagram shows a particle A of mass $2kg$ resting on a rough horizontal table of coefficient of friction 0.5. It is attached to particles B of mass $5kg$ and C of mass $3kg$ by light inextensible strings hanging over light smooth pulleys. If the system is allowed to move from rest, find the common acceleration of the particle and the tension in each string.
An[0.98ms^{-2}, 32.37N, 44.15N]

12.


The diagram shows a particle A of mass $3kg$ resting on a rough horizontal table of coefficient of friction 0.5. It is attached to particles B of mass $4kg$ and C of mass $6kg$ by light inextensible strings hanging over light smooth pulleys. If the system is allowed to move from rest, find the common acceleration of the particle and the tension in each string.
An [0.754ms^{-2}, 54.277N, 31.662N]

13.


The diagram shows a particle A of mass $5kg$ resting on a rough horizontal table. It is attached to particles B of mass $3kg$ and C of mass $2kg$ by light inextensible strings hanging over light smooth pulleys. If the system is released from rest, body B descends with an acceleration of 0.28ms^{-2}, find the c coefficient of friction between the body A and the surface of the table **An[$\frac{1}{7}$]**

14.

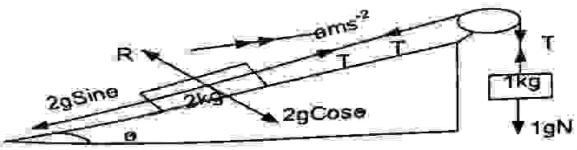

The diagram shows a particle A of mass $10kg$ resting on a smooth horizontal table. It is attached to particles B of mass $4kg$ and C of mass $7kg$ by light inextensible strings hanging over light smooth pulleys. If the system is allowed to move from rest, find the common acceleration of the particle and the tension in each string. **An [1.4ms^{-2}, 44.8N, 58.8N**

CONNECTED PARTICLES ON INCLINED PLANES

Examples

1. A mass of 2kg lies on a smooth plane of inclination 1 in 3. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope, over a smooth pulley fixed at the top of the plane is a freely suspended mass of 1kg at it's other end.
If the system is released from rest, find the;
(i) acceleration of the masses
(ii) Tension in the string
(iii) distance each particle travels in the first 2 seconds

Solution



$$sin\theta = \frac{1}{3} \qquad F = ma$$

For 2kg mass: $T - 2gsin\theta = 2a$..............(i)

For 1kg mass: $1g - T = 1a$....................(ii)

(ii) + (i): $1g - 2gsin\theta = 3a$

$$a = \frac{(9.8\text{-}2x9.8x^1/_3)}{3} = 1.089ms^{-2}$$

(ii) Tension : $1g - T = 1a$

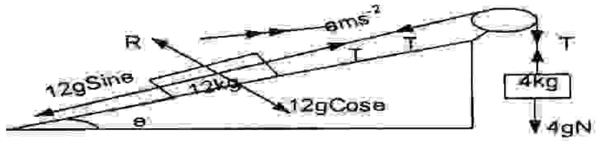
$T = 1x9.8 - 1x\ 1.089 = 8.711N$

(iii) $s = ut + \frac{1}{2}at^2$

$$S = 0x2 + \frac{1}{2}x1.089x2^2 = 2.178m$$

2. A mass of 12kg lies on a smooth incline plane 6m long and 1m high. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope, over a smooth pulley fixed at the top of the plane is a freely suspended mass of 4kg at it's other end.
If the system is released from rest, find the;
   (i) acceleration of the system
   (ii) Tension in the string
   (iii) Velocity with which the 4kg mass will hit the ground
   (iv) Time the 4kg mass takes to hit the ground **Uneb 2013, No 10**

Solution



$$sin\theta = \frac{1}{6} \qquad F = ma$$

For 12kg mass: $T - 12gsin\theta = 12a$..............(i)

For 4kg mass: $4g - T = 4a$...................(ii)

(ii) + (i): $4g - 12gsin\theta = 16a$

$$a = \frac{(4x9.8\text{-}12x9.8x^1/_6)}{16} = 1.225ms^{-2}$$

(ii) Tension : $4g - T = 4a$

$T = 4x9.8 - 4x\ 1.225 = 34.3N$

(iii) $v^2 = u^2 + 2as$

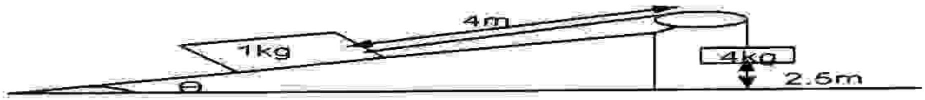
$v = \sqrt{0^2 + 2x1.225x1} = 1.57m/s$

$v = u + at$

$1.57 = 0 + 1.225xt$

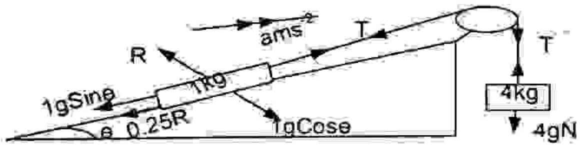
$t = 1.28s$

3. Amass of 1kg lies on a rough inclined plane coefficient of friction 0.25. One end of a light inextensible string is fastened to and 1kg mass and passes up the line of greatest slope over a smooth fixed pulley at the top of the plane and the other end of a sting is tied to a mass of 4kg hanging freely



The plane makes an angle θ with the horizontal where $\sin\theta = \frac{3}{5}$. When the system released from rest, find;
   (i) The acceleration of the system
   (ii) Tension in the string
   (iii) Velocity with which the 4kg mass hits the floor.

Solution



$$sin\theta = \frac{3}{5} \qquad F = ma$$

For 1kg mass: $T - 1gsin\theta - 0.25R = 1a$.....(i)

For 4kg mass: $4g - T = 4a$...................(ii)

(ii) + (i): $4g - 1gsin\theta - 0.25x1gcos\theta = 5a$

$$a = \frac{(4x9.8\text{-}1x9.8x^3/_5\text{-}0.25x9.8x^4/_5)}{5} = 6.272ms^{-2}$$

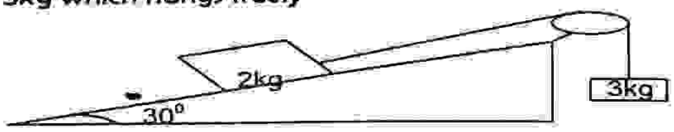
(ii) Tension : $4g - T = 4a$

$T = 4x9.8 - 4x\ 6.272 = 14.112N$

(iii) $v^2 = u^2 + 2as$

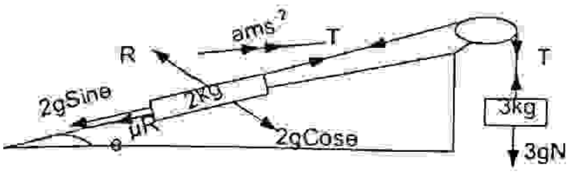
$v = \sqrt{0^2 + 2x6.272x2.5} = 5.6m/s$

4. A particle of mass 2kg on a rough plane inclined at 30° to the horizontal is attached by means of a light inextensible string passing over a smooth pulley at the top edge of the plane to a particle of mass 3kg which hangs freely



If the system is released from rest with above parts of the string taut, the 3kg mass travels a distance of 0.75m before it attains a speed of 2.25m/s. Find the;
   (i) Acceleration of the system
   (ii) Coefficient of friction between the plane and the 2kg mass
   (iii) Tension in the string. **Uneb 2003, No11**

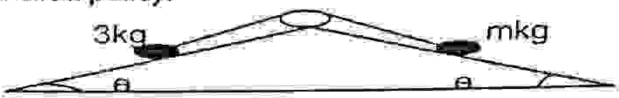
Solution



(i) $v^2 = u^2 + 2as$ $2.25^2 = 0^2 + 2x0.75s$ $a = 3.375ms^{-2}$	(ii) $F = ma$ For 2kg mass: $T - 2gsin\theta - \mu R = 2a$.....(i) For 3kg mass: $3g - T = 3a$...................(ii) (ii) + (i): $3g - 2gsin\theta - \mu x2gcos\theta = 5a$ $\mu = \frac{(3x9.8-2x9.8xsin30-5x3.375)}{2x9.8\ cos30} = 0.161$ (iii) Tension : $3g - T = 3a$ $T = 3x9.8 - 3x3.375 = 19.275N$

DOUBLE INCLINED PLANES

1. The diagram below shows two smooth fixed slopes each inclined at an angle θ to the horizontal where $sin\theta = 0.6$. two particles of mass 3kg and mkg, where m< 3 are connected by a light inextensible string passing over a smooth fixed pulley.

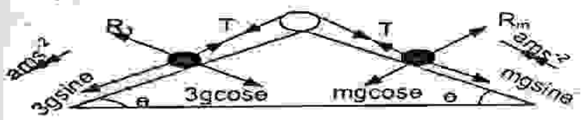


The particles are released from rest with a string taut. After traveling a distance of 1.08m, the speed of the particles is 1.8m/s. calculate;

(i) Acceleration of the particles (ii) Tension in the string

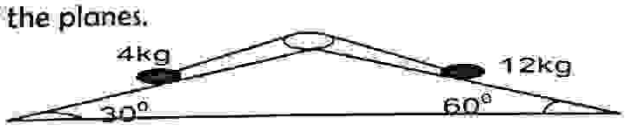
(iii) Value of m

Solution



(i) $v^2 = u^2 + 2as$ $1.8^2 = 0^2 + 2ax1.08$ $a = 1.5ms^{-2}$	**(ii)** $F = ma$ For 3kg mass: $3gsin\theta - T = 3a$ $T = 3x9.8x0.6 - 3x1.5 = 13.14N$ (iii) For mkg mass: $T - mgsin\theta = ma$ $13.14 = m(9.8x0.6 + 1.5)$ $m = 1.78kg$

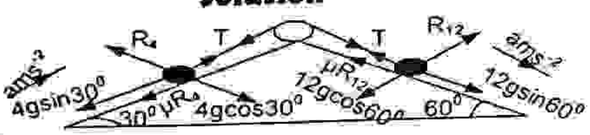
2. Two rough planes inclined at 30° and 60° to the horizontal and of thee same height are placed back to back. Masses of 4kg and 12kg are placed on the faces and connected by a light sting passing over a smooth pulley at the top of the planes.



If the coefficient of friction is 0.5 on both faces , find the;

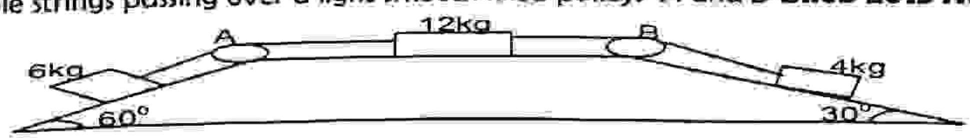
(i) Acceleration of the system (ii) Tension in the strings

Solution



(i) $F = ma$ For 4kg mass: $T - 4gsin30 - \mu R_4 = 4a$.....(i) 3kg mass: $12gsin60 - T - \mu R_{12} = 12a$.......(ii)	(ii) + (i) $12gsin60 - 4gsin30 - \mu x(4gcos30 + 12cos60) = 16a$ $a = 2.25ms^{-2}$ (i) Tension : $T - 4gsin30 - \mu R_4 = 4a$ $T = 4x2.25 + 4x9.8sin30 + 0.5x4cos30$ $T = 45.54N$

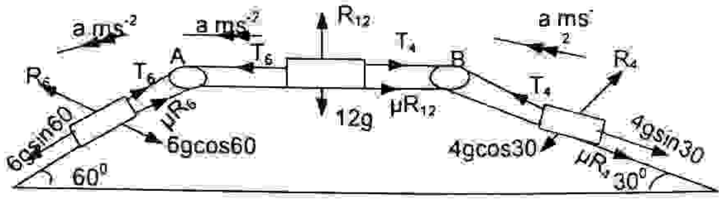
3. The diagram below shows a 12kg mass on a horizontal rough plane. The 6kg and 4kg masses are on rough planes inclined at angles of 60° and 30° respectively. The masses are connected to each other by a light inextensible strings passing over a light smooth fixed pulleys A and B **Uneb 2015 No.11**



The planes are equally rough with coefficient of friction of $\frac{1}{12}$. If the system is released from rest, find the;

(i) Acceleration of the system (ii) Tension in the strings

Solution



$F = ma$

For 4kg mass: $T_4 - 4gsin30 - \mu R_4 = 4a$.....(i)

12kg mass: $T_6 - T_4 - \mu R_{12} = 12a$.......(ii)

6kg mass: $6gsin60 - T_6 - \mu R_6 = 6a$.........(iii)

(ii) + (i)

$T_6 - 4gsin30 - \mu x(4gcos30 + 12g) = 16a$

$T_6 - 2g - \mu x(2g\sqrt{3} + 12g) = 16a$.........(iv)

(iii) + (iv)

$6gsin60 - 2g - \mu x(2g\sqrt{3} + 15g) = 16a$

$3g\sqrt{3} - 2g - \frac{1}{12}x(2g\sqrt{3} + 15g) = 16a$

$a = 0.738ms^{-2}$

(ii) Tension :

$T_4 - 4gsin30 - \frac{1}{12}x4gcos30 = 4x0.738$

$T_4 = 25.38N$

$T_6 - 2g - \mu x(2g\sqrt{3} + 12g) = 16a$

$T_6 - 2g - \frac{1}{12}x(2g\sqrt{3} + 12g) = 16x0.738$

$T_6 = 44.04N$

Exercise 16B

1. A mass of 2kg lies on a smooth incline plane 9m long and 3m high. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope, over a smooth pulley fixed at the top of the plane is a freely suspended mass of 1kg at it's other end. If the system is released from rest, find the;
   a. acceleration of the system
   b. Tension in the string
   c. Velocity with which the 1kg mass will hit the ground
   d. Time the 1kg mass takes to hit the ground

   An (1.089**ms**$^{-2}$ **8.711N, 2.556m/s, 2.347s)**

2. A mass of 15kg lies on a smooth plane of inclination 1 in 49. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope, pver a smooth pulley fixed at the top of the plane is a freely suspended mass of 10kg at it's other end.
   If the system is released from rest, find the acceleration of the masses and the distance each travels in the first 2 seconds **An (3.8ms**$^{-2}$ **, 7.6m)**

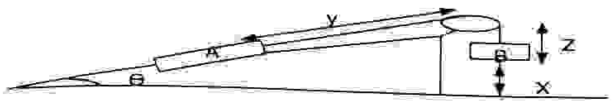
3. Amass of $2kg$ lies on a rough plane which is inclined at 30° to the horizontal. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope and over a sooth pulley fixed at the top of the slope, a freely suspended mass of 5kg is attached to its other end.
   The system is released from rest as the $2kg$ mass accelerates up the slope, it experiences a constant resistance to motion of 14N down the slope due to friction. Find the tension of the string **An(31N)**

4. A mass of $10kg$ lies on a smooth plane which is inclined at θ to the horizontal. The mass is 5m from the top, measured along the plane. One end of a light in extensible string is attached to this mass, the string passes up the line of greatest slope and over a smooth pulley fixed at the top of the slope. The other end is attached to a freely suspended mass of 15kg. This 15kg mass is 4m above the floor. The system is released from rest and the string first goes slack $1\frac{3}{7}$s later, Find the value of θ
   An(**30°**)

5. One of two identical masses lies on a smooth plane, which is inclined at $sin^{-1}(\frac{1}{4})$ to the horizontal and is $2m$ from the top. A light inextensible string attached to this mass passes along the line of greatest slope over a smooth pulley fixed at the top of the incline, the other end carries the other mass hanging freely $1m$ above the floor. If the system is released from Test, find the taken for the hanging mass to reach the floor An(**0.663s)**

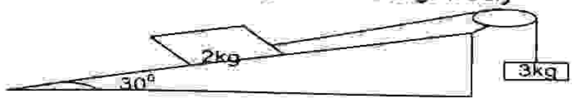
6. The body A of mass 4kg lies on a smooth slope and body B of mass 3kg is freely suspended.

The pulley is smooth and the string is light end inextensible



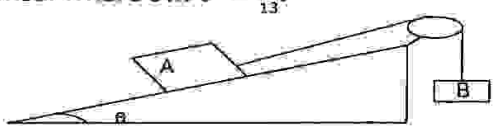
With θ=30°, a body A will accelerate up the slope. If $y = 3m$ and $x = 2.8m$ find the velocity with which A hits the pulley

7. A particle of mass 2kg on a smooth plane inclined at 30° to the horizontal is attached by means of a light inextensible string passing over a smooth pulley at the top edge of the plane to a particle of mass 3kg which hangs freely



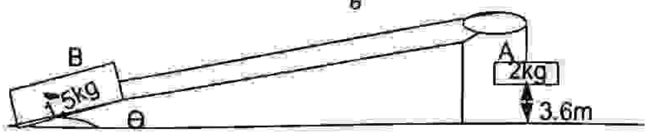
If the system is released from rest with above parts of the string taut, find the speed acquired by the particles when both have moved a distance of 1m An(**2.8 ms^{-1}**)

8. A body A of mass 13kg lying on a rough inclined plane, co-efficient of friction, μ. From A, a light inextensible string passes up the line of greatest slope and over a smooth fixed pulley to a body B of mass m kg hanging freely, the plane makes an angle, θ with the horizontal where $sin\ \theta = \frac{5}{13}$.



When $m = 1kg$ and the system is released from rest, B has an upward acceleration of ams^{-2}. When $m = 11kg$ and the system released from rest, B has a down ward acceleration of ams^{-2}. Find a and μ An(**1.96ms^{-2}, 0.1)**

9. A particle a of mass 2kg and B of mass 1.5kg are connected by a light in extensible string passing over a smooth pulley The system is released from rest with A at height of 3.6m above the horizontal ground and B at the foot of a smooth slope inclined at an angle θ to the horizontal where sin $\theta = \frac{1}{6}$

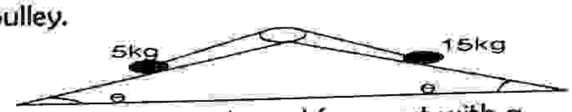


Calculate

(i) the magnitude of the particle **An(4.9m/s)**

(ii) the speed with which A reaches the ground. **An(5.94ms^{-2})**

10. The diagram below shows two smooth fixed slopes each inclined at an angle θ to the horizontal where $sin\theta = \frac{3}{5}$. Two particles of mass 5kg and $15kg$ are connected by a light inextensible string passing over a smooth fixed pulley.



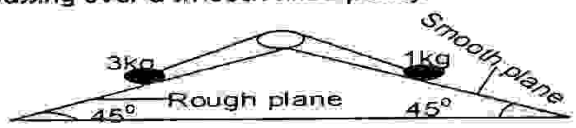
The particles are released from rest with a string taut. calculate;

(i) Acceleration of the particles

(ii) Tension in the string

An(2.94ms^{-2}, 44.1N)

11. The diagram below shows two smooth plane and a rough plane both inclined at 45° to the horizontal. Two particles of mass 1kg and $3kg$ are connected by a light inextensible string passing over a smooth fixed pulley.



The particles are released from rest with a string taut. Calculate;

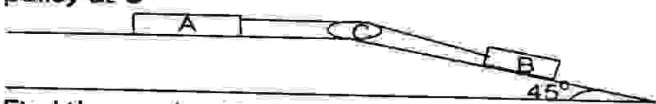
(i) Acceleration of the particles

(ii) Tension in the string

(iii) Coefficient of friction

An(1.4ms^{-2}, 8.48N, 0.4)

12. In the diagram particles A and B are of masses 10kg and 8kg respectively and rest on planes as shown below. They are connected by a light in extensible string passing over a smooth fixed pulley at C



Find the acceleration in the system and the tension in the string if,

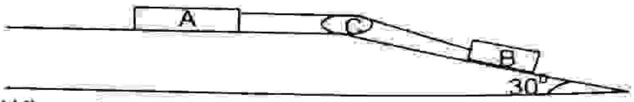
(i) The particles are contact with smooth planes

(ii) The particles are in contact with rough planes with coefficient of friction 0.25

An(3.08ms^{-2}, 30.8N, 0.95ms^{-2}, 33.98N)

13. In the diagram particles A and B are of masses 2.4kg and 3.6kg respectively. A rest on a rough horizontal plane (coefficient of friction 0.5), it is connected by a light in extensible string passing over a smooth fixed pulley at C to a particle B

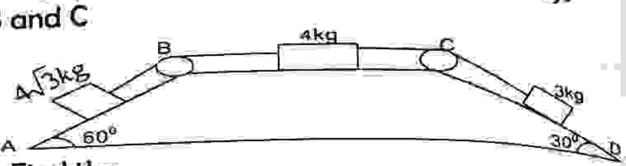
resting on a smooth plane inclined at 30° to the horizontal



When the system is released from rest, find the
(i) acceleration in the system and the tension in the string
(ii) The force on the pulley at C
(iii) The velocity of the A mass after 2 seconds

An(0.98ms^{-2}, 14.112N, 7.3049N, 1.96ms^{-1})

14. The diagram below shows a 4kg mass on a horizontal rough plane with coefficient of friction 0.25. The $4\sqrt{3}$ kg mass rests on a smooth plane inclined at angles of 60° to the horizontal while the 3kg rests on a rough plane inclined at an angle 30° to the horizontal and coefficient of friction $\frac{1}{\sqrt{3}}$. The masses are connected to each other by a light inextensible strings passing over a light smooth fixed pulleys B and C



(a) Find the
  (i) Acceleration of the system
  (ii) Tension in the strings
(b) Work done against frictional forces when the particles each moved 0.5m

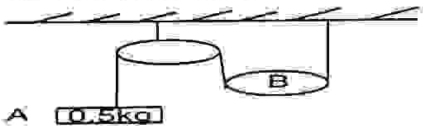
An(1.407ms^{-2}, 49.051N, 33.622N, 12.25J)

MULTIPLE CONNECTIONS

- Acceleration of a particles moving between two portions of the string is equal to half the net acceleration of the particle(s) attached to the end of the string
- The tension in the uninterrupted string is constant
- The tensions in interrupted strings are different.

Examples

1. The diagram below shows particle A of mass 0.5kg attached to one end of a light inextensible string passing over a fixed light pulley and under a movable light pulley B. the other end of the string is fixed as shown below. **Uneb 1997**



(i) What mass should be attached at B for the system to be equilibrium

(ii) If B is 0.8kg, what are the acceleration of particle A and pulley B

(iii) Find the tension in the string

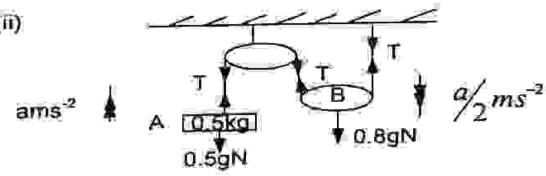
Solution

a. At equilibrium: $T = 0.5g$

$$mg = 2T$$

$$mg = 2x0.5g = 1kg$$

(ii)



F = ma

For 0.8kg mass: $0.8g - 2T = 0.8x\frac{a}{2}$..............(i)

For 0.5kg mass: $T - 0.5g = 0.5a$............(ii)

$$T = 0.5g + 0.5a$$

$$0.8g - 2T = 0.4a$$

$$0.8g - 2(0.5g + 0.5a) = 0.4a$$

$$a = \frac{0.2g}{1.4} = \frac{0.2x9.8}{1.4} = 1.4ms^{-2}$$

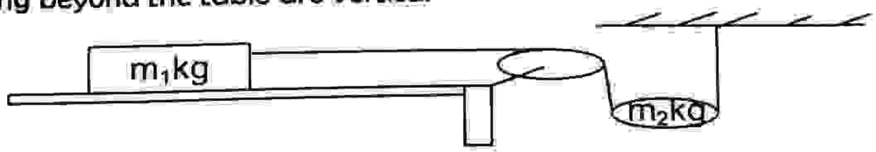
acceleration of particle A is $1.4ms^{-2}$

acceleration of pulley B is $0.7ms^{-2}$ (ii)

Tension : $T = 0.5g + 0.5a$

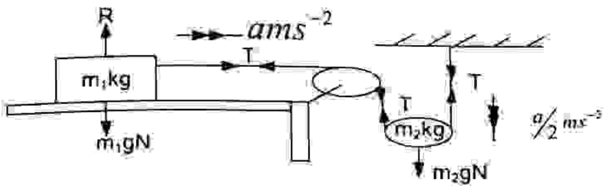
$$T = 0.5x9.8 + 0.5x1.4 = 42N$$

2. A particle of mass m_1 on a smooth horizontal table is tied to one end of the string which passes over a fixed pulley at the edge and then under a movable pulley of mass m_2, its other end being fixed so that the parts of the string beyond the table are vertical



Show that m_2 descends with an acceleration of $\frac{m_2 g}{4m_1+m_2}$

Solution



$$F = ma$$

For $m_1$kg mass: $T = m_1 a$..............(i)

For $m_2$kg mass: $m_2 g - 2T = m_2 \frac{a}{2}$.............(ii)

$$m_2 g - 2T = m_2 \frac{a}{2}$$

$$m_2 g - 2m_1 a = m_2 \frac{a}{2}$$

$$m_2 g = \frac{m_2 a + 4m_1 a}{2}$$

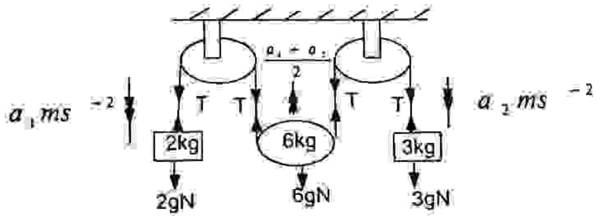
$$a = \frac{2m_2 g}{4m_1 + m_2}$$

Acceleration of $m_2 = \frac{a}{2} = \frac{m_2 g}{4m_1 + m_2}$

3. A string has a load of mass 2kg attached at one end. The string passes over a smooth fixed pulley under a movable pulley of mass 6kg over another fixed pulley and has a load of mass 3kg attached to its end

Find the acceleration of the movable pulley and the tension in the string.

Solution



$$F = ma$$

For 2kg mass: $2g - T = 2a_1$..............(1)

For 3kg mass: $3g - T = 3a_2$.............(2)

For 6kg mass: $2T - 6g = 6x\,^1/_2(a_1 + a_2)$.......(3)

$(2) - (1)$: $g = 3a_2 - 2a_1$ .... (i)

$2x(2) + (3)$: $0 = 9a_2 + 3a_1$ .... (ii)

$3x(i) - (ii)$: $3g = -9a_1$

$$a_1 = \frac{-g}{3} = -3.267ms^{-2}$$

$$0 = 9a_2 + 3(-3.267)$$

$$a_2 = 1.089\ ms^{-2}$$

acceleration of pulley $= {}^1/_2(a_1 + a_2)$

$= {}^1/_2(-3.267 + 1.089) = -1.089ms^{-2}$

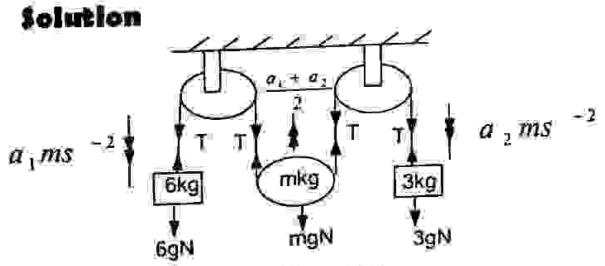
(ii) Tension : $T = 2g - 2a_1$

$T = 2x9.8 - 2x - 3.267 = 26.134N$

4. In the pulley system below, A is a heavy pulley which is free to move.

Find the mass of pulley A if it does not move upwards or downwards when the system is released from rest

Solution



$$F = ma$$

For 6kg mass: $6g - T = 6a_1$..............(1)

For 3kg mass: $3g - T = 3a_2$.............(2)

For mkg mass: $2T - mg = 0$.......(3)

$${}^1/_2(a_1 + a_2) = 0$$

$$a_1 = -a_2$$

$6g - T = -6a_2$ .... (i)

$3g - T = 3a_2$ .... (ii)

$(i) - (ii)$

$$3g = -9a_2$$

$$a_2 = \frac{-g}{3} = -3.267ms^{-2}$$

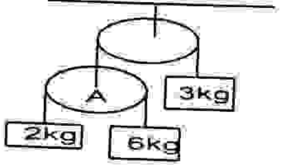
$$3g - T = 3a_2$$

$$T = 3x9.8 - 3(-3.267) = 39.201$$

$$2T - mg = 0$$

$$m = \frac{2x39.201}{9.8} = 8kg$$

5. The diagram shows a fixed pulley carrying a string which has a mass of 3kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 6kg at one and a mass of 2kg at the other end



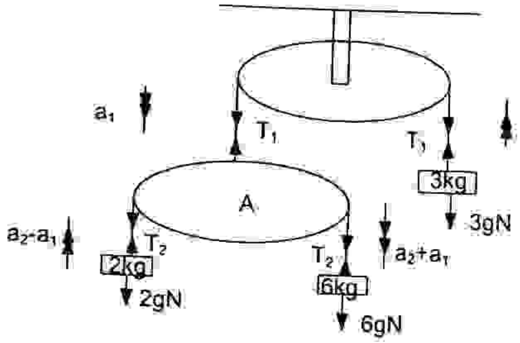
Find

(i) Acceleration of pulley A

(ii) Acceleration of 2kg, 6kg and 3kg masses

(iii) The tension in the string

Solution



For 3kg mass: $T_1 - 3g = 3a_1$..............(1)

For 6kg mass: $6g - T_2 = 6(a_2 + a_1)$.............(2)

For 2kg mass: $T_2 - 2g = 2(a_2 - a_1)$.............(3)

For Pulley A: $2T_2 - T_1 = 0xa_1$.............(4)

$eq\ (2) + eq(3): 4g = 8a_2 + 4a_1 \dots\dots\dots (i)$

$eq\ (1) + eq(4): 2T_2 - 3g = 3a_1 \dots\dots\dots (ii)$

$2xeq\ (3) - eq(ii): -g = 4a_2 - 7a_1 \dots\dots\dots (iii)$

$eq\ (iii) - eq(i): -18a_1 = -6g$

$$a_1 = \frac{9.8}{3} = 3.27ms^{-2}$$

$$4g = 8a_2 + 4a_1$$

$$a_2 = \frac{9.8 - 3.27}{2} = 3.27ms^{-2}$$

Acceleration of pulley A = $3.27ms^{-2}$

Acceleration of 2kg = $3.27 - 3.27 = 0ms^{-2}$

Acceleration of 6kg = $3.27 + 3.27 = 6.54ms^{-2}$

Acceleration of 3kg = $3.27ms^{-2}$

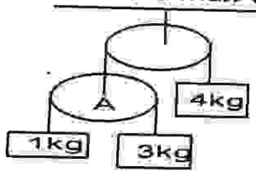
$$T_1 - 3g = 3a_1$$

$$T_1 = 3x3.27 + 3x9.8 = 39.21N$$

$$2T_2 - T_1 = 0xa_1$$

$$T_2 = \frac{39.21}{2} = 19.61N$$

6. The diagram shows a fixed pulley carrying a string which has a mass of 4kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 3kg at one and a mass of 1kg at the other end



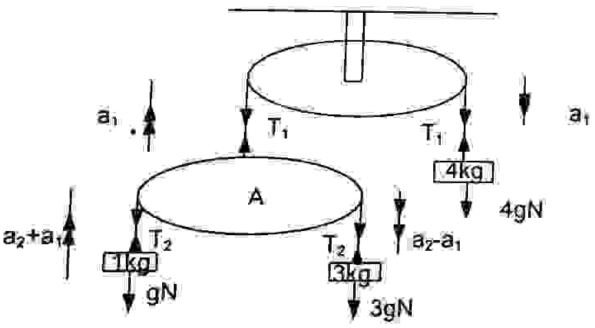
Find

(i) Acceleration of pulley A

(ii) Acceleration of 1kg, 3kg and 4kg masses

(iii) The tension in the string

Solution



For 4kg mass: $4g - T_1 = 4a_1$..............(1)

For 3kg mass: $3g - T_2 = 3(a_2 - a_1)$.............(2)

For 1kg mass: $T_2 - g = a_2 + a_1$.............(3)

For Pulley A: $T_1 - 2T_2 = 0xa_1$.............(4)

$eq\ (2) + eq(3): g = 2a_2 - a_1 \dots\dots\dots (i)$

$eq\ (1) + eq(4): 4g - 2T_2 = 4a_1 \dots\dots\dots (ii)$

$2xeq\ (3) + eq(ii): 2g = 2a_2 + 6a_1 \dots\dots\dots (iii)$

$eq\ (iii) - eq(i): 7a_1 = g$

$$a_1 = \frac{9.8}{7} = 1.4ms^{-2}$$

$$g = 2a_2 - a_1$$

$$a_2 = \frac{1.4 + 9.8}{2} = 5.6ms^{-2}$$

Acceleration of pulley A = $1.4ms^{-2}$

Acceleration of 1kg = $5.6 + 1.4 = 7ms^{-2}$

Acceleration of 3kg = $5.6 - 1.4 = 4.2ms^{-2}$

Acceleration of 4kg = $1.4ms^{-2}$

$$4g - T_1 = 4a_1$$

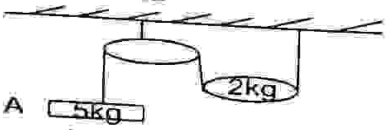
$$T_1 = 4x9.8 - 4x1.4 = 33.6N$$

$$T_1 - 2T_2 = 0xa_1$$

$$T_2 = \frac{33.6}{2} = 16.8N$$

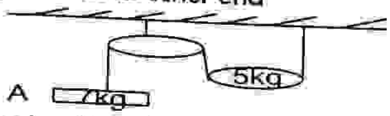
Exercise 16C

1. A string with one end fixed passes under a movable pulley of mass 2kg, over a fixed pulley and carries a 5kg mass at its other end



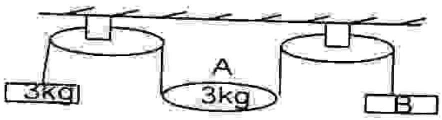
Find the acceleration of the movable pulley and the tension on the string.
An(3.56ms⁻², 13.36N)

2. A string with one end fixed passes under a movable pulley of mass 5kg, over a fixed smooth pulley and carries a 7kg mass at its other end



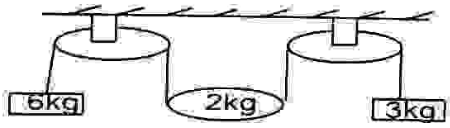
Find the acceleration of the movable pulley and the particle. **An(2.673ms⁻², 5.145 ms⁻²)**

3. In the pulley system below, A is a heavy pulley which is free to move.



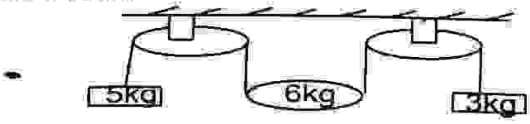
Find the mass of load B, if it does not move upwards or downwards when the system is released from rest.
An(1kg)

4. Two particles of mass 3kg and 6kg are connected by a light inextensible string passing over two fixed smooth pulleys and under a heavy smooth movable pulley of mass 2kg, the position of the string not in contact are vertical



if the system is released from rest, find
(i) Acceleration of the movable pulley
(ii) Tension in the string
An(5.88 ms⁻², 15.68N)

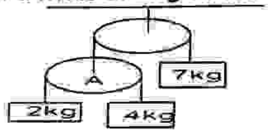
5. Two particles of mass 3kg and 5kg are connected by a light inextensible string passing over two fixed smooth pulleys and under a heavy smooth movable pulley of mass 6kg, the position of the string not in contact are vertical



if the system is released from rest, find
(i) Acceleration of the movable pulley
(ii) Tension in the string

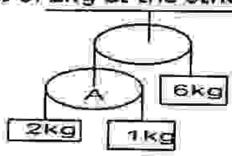
An(1.089 ms⁻², 32.667N)

6. The diagram shows a fixed pulley carrying a string which has a mass of 7kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 4kg at one and a mass of 2kg at the other end



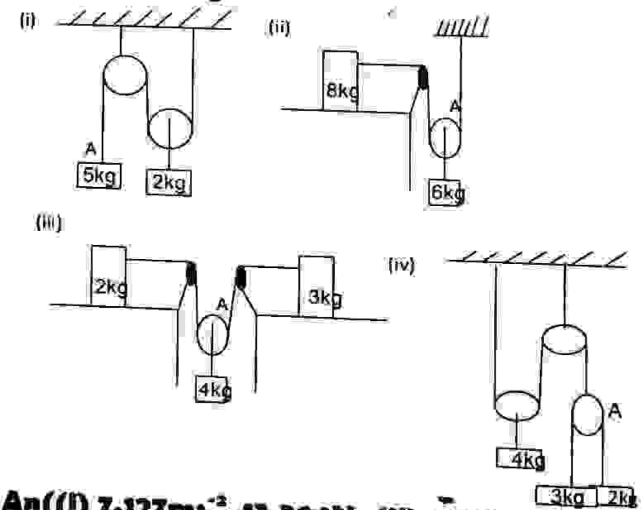
Find
(i) Acceleration of 4kg masses
(ii) The tension in the string **An(2.38ms⁻², 59.33N, 29.66N)**

7. The diagram shows a fixed pulley carrying a string which has a mass of 7kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 4kg at one and a mass of 2kg at the other end



Find
(i) Acceleration of 1kg masses
(ii) The tension in the string
An(8.2923ms⁻², 18.0923N, 36.1846N)

8. For each of the systems below; all the strings are light and inextensible, all pulleys are light and smooth and all surfaces are smooth.
In each case find the acceleration of A and the tension in the string



An((i) 7.127ms⁻², 13.364N, (ii) 1.547ms⁻², 24.758N, (iii) 3.364ms⁻², 10.691N, (iv) 4.731ms⁻², 12.166N, 24.331N)

WORK DONE BY A CONSTANT FORCE

Work is said to be done when energy is transferred from one system to another
When a block of mass m rests on a smooth horizontal

When a constant force F acts on the block and displaces it by s, then the work done by F is given by

$$W = Fs$$

Examples:

1. Find the work done against gravity when a body of mass 5kg is moved through a vertical distance of 2m
Solution
$W = Fs = mgs$ | $W = 5x9.8x2 = 98J$ |

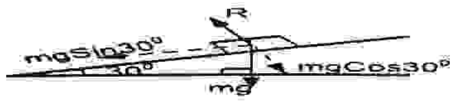
2. A man building a wall lifts 50 bricks through a vertical distance of 3m. If each brick has a mass of 4kg, how much work does the man do against gravity
Solution
$W = Fs = mgs$ | $W = 50x4x9.8x3 = 5880J$ |

3. A body of mass 2kg is moved vertically upwards at a constant speed of 5m/s. Find how much work is done against gravity in each second
Solution
$W = Fs = mgs$ | $W = 2x9.8x5x1 = 98J$ |

4. A box is pulled a distance of 8m across a horizontal surface against resistance totaling 7N. If the box moves with uniform velocity, find the work done against the resistance.
Solution
$W = Fs$ | $W = 8x7$ | w = 56J

5. A horizontal force pulls a body of mass 5kg a distance of 8m across a rough horizontal surface, coefficient of friction 0.25. The body moves with uniform velocity, find the work done against friction
Solution
$W = \mu Rs$ | $W = 0.25x5x9.8x8$ | w = 98J

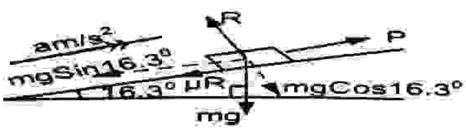
6. A block of mass 5kg is released from rest on a smooth plane inclined at an angle of 30° to the horizontal and slides through 10m. Find the work done by the gravitation force.
Solution



$W = mgsin30xd$
$W = 10x5x9.81sin30 = 245.25J$

7. A rough surface is inclined at $tan^{-1}\left(\frac{7}{24}\right)$ to the horizontal. A body of mass 5kg lies on the surface and is pulled at a uniform speed a distance of 75cm up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is $\frac{5}{12}$. Find:
a) Work done against gravity
b) Work done against friction

Solution



$\theta = tan^{-1}\left(\frac{7}{24}\right) = 16.3°$

a) Work done against friction
$W = \mu Rd$ But $R = mgcos\theta$

$$W = \mu mgcos\theta d$$
$$W = \frac{5}{12}x5x\frac{75}{100}x9.81cos16.3 = 14.71J$$

b) Work done against gravity

$$W = mgsin\ \theta xd$$
$$W = 5x9.81sin16.3x\frac{75}{100} = 10.35J$$

Exercise 17A

1. Find the work done against gravity when a body of mass 1kg is raised through a vertical distance of 3m. **An(29.4J)**

2. Find the work done against gravity when a person of mass 80kg climbs a vertical distance of 25m. **An(19600J)**

A body of mass 200g is moved vertically upwards at a constant speed of 2m/s. Find how much work is done against gravity in each second **An(3.92J)**

A body of mass 10kg is pulled a distance of 20m across a horizontal surface against resistance totaling 40N. If the body moves with a uniform velocity, find the work done against the resistance **An(800J)**

A horizontal force pulls a body of mass 3kg a distance of 20m across a rough horizontal surface, coefficient of friction $\frac{2}{7}$. The body moves with a uniform velocity and the only resitance is that due to friction. Find the work done. **An(168J)**

A horizontal force drags a body of mass 4kg a distance of 10m across a rough horizontal floor at a constant speed. The work done against friction is 49J. Find the coefficient of friction between the body and the surface. **An(0.125)**

A block of mass 15kg rests on a smooth plane inclined at an angle of 30° to the horizontal. The block is pulled at a uniform speed a distance of 10m up the line of greatest slope. Find the work done by the gravitation force. **An(735J)**

A surface is inclined at $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body of mass 50kg lies on the surface and is pulled at a uniform speed a distance of 5m up the line of greatest slope against a resistance totaling to 50N. Find;

a) Work done against gravity
b) Work done against friction

An(1470J, 250J)

9. A rough surface is inclined at $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. A body of mass 130kg lies on the surface and is pulled at a uniform speed a distance of 50m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is $\frac{2}{7}$.
Find;
   a) The frictional force acting
   b) Work done against friction
   c) Work done against gravity

   An(336J, 16800J, 24500J)

10. A rough surface is inclined at 30° to the horizontal. A body of mass 100kg lies on the surface and is pulled at a uniform speed a distance of 20m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is 0.1. Find:
   a) Work done against friction
   b) Work done against gravity

   An(1700J, 9800J)

11. A rough surface is inclined at $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body of mass 50kg lies on the surface and is pulled at a uniform speed a distance of 15m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is $\frac{1}{3}$. Find the total work done on the body. **An(6370J)**

12. A rough surface is inclined at an angle θ to the horizontal. A body of mass mkg lies on the surface and is pulled at a uniform speed a distance of x meters up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is μ. Show that the total work done on the body is $mgx(sin\theta + \mu cos\theta)$

13. A particle of mass 15 kg is pulled up a smooth slope by a light inextensible string parallel to the slope. The slope is 10.5 m long and inclined at $sin^{-1}\left(\frac{4}{7}\right)$ to the horizontal. The acceleration of the particle is $0.98ms^{-2}$. Determine the:
   (a) tension in the string.
   (b) work done against gravity when the particle reaches the end of the slope.

   UNEB 2018 No.4 An(98.7N, 882J)

WORK-ENERGY THEOREM

It states that the work done by the net force acting on a body is equal to the change in its kinetic energy.
Consider a body of mass m accelerated from velocity, u by a constant force, F so that in a distance, s it gains velocity v

$a = \frac{v^2-u^2}{2s}$ ---------------------- [1]

resultant $force\ F = ma = \frac{m(v^2-u^2)}{2s}$

But work done $= Fxs$

$W = \frac{m(v^2-u^2)}{2s}s = \frac{m(v^2-u^2)}{2}$

$\boxed{W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2}$ work-energy theorem.

Examples

1. A and B are two points 4m apart on a smooth horizontal surface. A body of mass 2kg is initially at rest at A and is pushed by a force of constant magnitude acting in the direction from A to B. The body reaches B with speed of 4m/s. Find the magnitude of the force.

Solution

$$a = \frac{v^2 - u^2}{2xs} = \frac{4^2 - 0^2}{2x4} = 2ms^{-2}$$
$$F = ma = 2x2 = 4N$$

Alternatively : $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$$Fx4 = \frac{1}{2}x2x(4^2 - 0^2)$$
$$F = 4N$$

2. A car mass 1000kg moving at 50ms⁻¹ skid to rest in 4s under a constant retardation. Calculate the magnitude of the work done by the force of friction

Solution

a) Using $v = u + at$

$0 = 50 + 4a$

$a = -12.5m/s^2$

Frictional force $= ma$

$= 1000x - 12. = 12500N$

$S = ut + ½ at^2$

$S = 50x4 + ½ x - 12.5x4^2$

$S = 100m$

W= $FxS = 12500x100$

Work done $= 1.25x10^6 J$

Alternatively

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

W= $\frac{1}{2}x1000x50^2 - \frac{1}{2}x1000x0^2$

Work done $= 1.25x10^6 J$

3. A constant force pushes a mass of 4kg in a straight line across a smooth horizontal surface. The body passes through a point A with a speed of 5m/s and then through a point B with a speed of 8m/s. B is 6m form A. Find the magnitude of the force acting on the mass.

Solution

$$a = \frac{v^2 - u^2}{2xs} = \frac{8^2 - 5^2}{2x6} = 3.25ms^{-2}$$

$F = ma = 4x3.25 = 13N$

OR $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$$Fx6 = \frac{1}{2}x4x(8^2 - 5^2)$$
$$F = 13N$$

4. A body of mass 4kg is moving with an initial velocity of 5m/s on a plane. The kinetic energy of the body is reduced by 16J in a distance of 40m. Find the deceleration of the body. **Uneb 2016**

Solution

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$
$$16 = \frac{1}{2}x4x(5^2 - v^2)$$

$$v^2 = 17$$
$$a = \frac{v^2 - u^2}{2xs} = \frac{17 - 5^2}{2x40}$$

$a = -0.1ms^{-2}$

5. A body of mass 5kg moves in a straight line across a horizontal surface against a constant resistance of magnitude 10N. The body passes through point A and then comes to rest at point B, 9m from A. Find the speed of the body when it is at A

Solution

$$F = ma$$
$$-10 = 5a$$
$$a = -2ms^{-2}$$
$$v^2 = u^2 + 2as$$

$$0^2 = u^2 + 2x - 2x9$$
$$u = 6ms^{-1}$$

OR : $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$$10x9 = \frac{1}{2}x5x(u^2 - 0^2)$$
$$u = 6ms^{-1}$$

6. A body of mass 5kg slides over a rough horizontal surface. In sliding 5m, the speed of the body decrease from 8m/s to 6m/s, find

(i) Frictional force

(j) Coefficient of friction

Solution

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$
$$Fx5 = \frac{1}{2}x5x(8^2 - 6^2)$$
$$F = 14N$$

$$F = \mu R$$
$$\mu = \frac{14}{5x9.8} = 0.286$$

Alternatively $v^2 = u^2 + 2as$

$$a = \frac{6^2 - 8^2}{2x5} = -2.8ms^{-2}$$
$$F = ma = 5x2.8 = 14N$$

7. A bullet of mass 15g is fired towards a fixed wooden block and enters the block when travelling horizontally at 400m/s. It comes to rest after penetrating a distance of 25cm. find the

(i) work done against resistance of the wood

(ii) Magnitude of the resistance

Solution

(i) $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$W = \frac{1}{2}x0.015x(400^2 - 0^2) = 1200J$

(ii) $W = FxS$

$1200 = Fx0.25$

$F = 4800N$

8. A particle of mass 2kg is released from rest and falls freely under gravity. Find its speed when it has fallen a distance of 10m

Solution

$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$2x9.8x10 = \frac{1}{2}x2x(v^2 - 0^2)$

$v = 14m/s$

9. A particle of mass 5kg falls vertically against a constant resistance. The particle passes through two points A and B 2.5m apart with A above B. Its speed is 2m/s when passing through A and 6m/s when passing through B. Find the magnitude of the resistance

Solution

$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

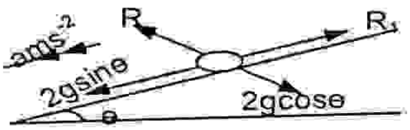
$(5g - R)x2.5 = \frac{1}{2}x5x(6^2 - 2^2)$

$R = 17N$

Incline planes

1. A rough slope of length 5m is inclined at angle of 30° to the horizontal. A body of mas 2kg is released from rest at the top of the slope and travels down the slope against a constant resistance. The body reaches the bottom of the slope with speed of 2m/s. find the magnitude of the resistance

Solution



$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$(2gsin\theta - R)x5 = \frac{1}{2}x2x(2^2 - 0^2)$

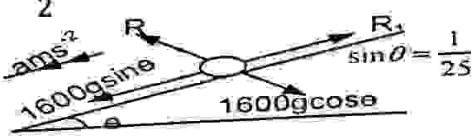
$R = 9N$

2. A car of mass 1600kg slides down a hill of slope 1 in 25. When the car descends 200m along at the hill, its speed increases from $3ms^{-1}$ to $10ms^{-1}$. Calculate
   (i) The change in the total kinetic energy
   (ii) Average value of resistance to motion **UNEB 1992, No.6**

Solution

(i) $\Delta k.e = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$= \frac{1}{2}x1600(10^2 - 3^2) = 72{,}800J$



$v^2 = u^2 + 2as$

$a = \frac{10^2 - 3^2}{2x200} = 0.228ms^{-2}$

using $F = ma$

$1600g\ sin\theta - R_1 = 1600a$

$R_1 = 1600x9.8x\frac{1}{25} - 1600x0.228 = 262.4N$

OR $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$(1600gsin\theta - R)x200 = \frac{1}{2}x1600(10^2 - 3^2)$

$R = 263.2N$

Exercise 17B

1. A carton of mass 0.4kg is thrown across a table with a velocity of 25m/s. The resistance of the table to its motion is 50N. How far will it travel before coming to rest. What must be the resistance if it travels only 2m. **An**$(2.5m, 62.5N)$
2. A and B are two points 3m apart on a smooth horizontal surface. A body of mass 6kg is initially at rest at A and is pushed towards B with a constant force of 9N. find the speed of the body when it reaches B. **An**$(3ms^{-1})$
3. A constant force of magnitude 8N pushes a body of mass 4kg in a straight line across a smooth horizontal surface. The body passes through a point A with a speed of 4m/s and then through a point B 5m form A. find the speed of the body at B. **An**$(3ms^{-1})$
4. A particle of mass 100g moves in a straight line across a horizontal surface against a resistance of constant magnitude. The particle passes through a point A with a speed of 7m/s and then through B with a speed of 3m/s. B being 2m form A. find the magnitude off the résistance. **An**$(1N)$

5. A and B are two points 15m apart in the same vertical line, with A above B. A body of mass 5kg is released form rest at A and falls vertically against a constant resistance of 25N. Find the speed of the body when it passes B. **An**$(12ms^{-1})$

6. A particle of mass 6kg is released from rest and falls freely under gravity. Find its speed when it has fallen a distance of 90m **An**$(42m/s)$

7. A particle of mass 6kg is released from rest and falls freely under gravity. Find the distance it has fallen when its speed is 7m/s **An**$(2.5m)$

8. A body of mass 3kg is projected vertically upwards from a point A with speed 4m/s. The body passes through a point B 5m below A. Find the speed of the body at B. **An**$(10.7m/s)$

9. A body of mass 10kg is projected vertically upwards from a point A with speed 4m/s. The body reaches its highest point and then falls, passing through a point B 5m below A. find the speed of the body at B. **An**$(10.7m/s)$

10. A particle of mass 2kg falls vertically against a constant resistance of 14N. The particle passes through two points A and B with a speed of 3m/s and 10m/s respectively. Find the distance AB **An**$(16.25m)$

11. A bullet of mass 8g is fired towards a fixed wooden block and enters the block when travelling horizontally at 300m/s. How far does the bullet penetrate if the wood provides a constant resistance of 1800N **An**$(20cm)$

12. A bullet of mass 50g travelling horizontally at 100ms^{-1} strikes as stationary block of wood and coming to rests through a distance of 5m. Calculate the average resistance of the block to the motion of the bullet. **An[50N]**

13. A bullet of mass 50g travelling horizontally at 500ms^{-1} strikes as stationary block of wood and after travelling 10cm, it emerges from the block travelling at 100ms^{-1}. Calculate the average resistance of the block to the motion of the bullet. **An[60000N]**

14. A bullet of mass 20g travelling horizontally at 210ms^{-1} strikes as stationary block of wood of thickness 0.1m and emerges from the block travelling at 50ms^{-1}. Calculate the average resistance of the block to the motion of the bullet. **An[4160N]**

15. A smooth slope is inclined at $tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A block of mass 4kg is released from rest at the top of the slope and travels down the slope, reaching the bottom of the slope with speed of 78m/s, find the length if the slope **An**$(4.17m)$

16. Point A is situated at the bottom of a smooth slope inclined at angle of $tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. A body is projected form A with a speed of 14m/s along and up a line of greatest and the body first comes to rest a t a point B. Find the distance AB **An**$(26cm)$

17. A rough slope of length 10m is inclined at angle of $tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A block of mass 50kg is released from rest at the top of the slope and travels down the slope, reaching the bottom of the slope with speed of 8m/s, find the
    (i) magnitude of the frictional force
    (ii) Work done by the frictional force
    (iii) Coefficient of friction
    An$(134N, 1340J, 0.342)$

18. Point A is situated at the bottom of a rough slope of length 10m is inclined at angle of $tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body is projected form A with a speed of 14m/s along and up a line of greatest slope. The coefficient of friction between the body and the plane is 0.25. The body first comes to rest a t a point B. Find the distance AB **An**$(12.5m)$

19. Point A is situated at the bottom of a rough slope of length 10m is inclined at angle of 45° to the horizontal. A body of mass 0.5kg is projected form A with a speed of 14m/s along and up a line of greatest slope. The coefficient of friction between the body and the plane is $\frac{3}{7}$. The body first comes to rest at point B a distance $4\sqrt{2}$ m from A, before returning to A find;
    (i) Work done against friction when the body moves from A to B
    (ii) The initial speed of the body
    (iii) Work done against friction when the body moves from A to B and back to A
    (iv) Speed of the body on return to A
    An$(8.4J, 10.58ms^{-1}, 16.8J, 6.69ms^{-1})$

20. Point A is situated at the bottom of a rough slope inclined at angle of θ to the horizontal. A body of mass. mkg is projected form A along and up a line of greatest slope. The coefficient of friction between the body and the plane is μ. The body first comes to rest at point B a distance x from A, before returning to A show;
    (i) Work done against friction when the body moves from A to B and back to A is given by $2\mu mgx\ cos\theta$
    (ii) The initial speed of the body $\sqrt{2gx\ (sin\theta + \mu cos\theta)}$
    (iii) Speed of the body on return to A Is $\sqrt{2gx\ (sin\theta - \mu cos\theta)}$

POWER

It's the rate of doing work.

Power $= \frac{\text{Work done}}{\text{time}}$

P= $\frac{F x d}{t}$

P= $Fx\ \frac{d}{t}$

$P = Fxv$

$F = \frac{P}{v}$

Examples

1. What is the average rate at which work must be done in lifting a mass of 100kg a vertical distance of 5m in 7s

Solution

P= $\frac{F x d}{t}$

P= $\frac{100x9.8x5}{7}$

$P = 700W$

2. What is the rate at which work must be done in lifting a mass of 500kg vertically at a constant speed of 3m/s

Solution

P= $\frac{F x d}{t}$

P= $500x9.8x3$

$P = 14700W$

Motion of cars

Consider a car being driven along a road, the forward or tractive force F_T moves the car is supplied by the engine working a constant rate of P watts

Power $= \frac{\text{Work done}}{\text{time}}$

P= $\frac{F_T x d}{t}$

P= $F_T x\ \frac{d}{t}$

$P = F_T x v$

$F_T = \frac{P}{v}$

Numerical examples

1. A car is driven along a level road against a constant resistance to motion of 400N. Find the maximum speed at which the car can move when its engine works at a steady rate of 4kW.

Solution

$F = ma$ at constant speed $a = 0ms^{-2}$

$F_T - R = mx0$

$\frac{4000}{v} - 400 = mx0$

$v = 10m/s$

2. A car is working at 5kW and is travelling at a constant speed of 72km/h. Find the resistance to motion

Solution Uneb 2007 No16

$V = 72km/h = 20m/s$

$F = ma$ at constant speed $a = 0ms^{-2}$

$F_T - R = mx0$,

$\frac{5000}{20} - R = 0$

$R = 250N$

3. A cyclist travels along a level road at a constant speed of 8m/s. If the resistance to motion is 50N, find the rate at which the cyclist is working

Solution

$F = ma$ at constant speed $a = 0ms^{-2}$

$F_T - R = mx0$,

$\frac{P}{8} - 50 = 0$

$P = 400W$

4. A car of mass 900kg is driven along a level road against a constant resistance to motion of 300N. with the engine f car working at steady rate of 12kW, find
   - (i) Acceleration of the car when its speed is 10m/s
   - (ii) The maximum speed of the car

   Solution

   (i) $F = ma$

   $F_T - R = ma$

   $\frac{12000}{10} - 300 = 900a$

   $a = 1ms^{-2}$

   (ii) at maximum speed $a = 0ms^{-2}$

   $F_T - R = mx0$

   $\frac{12000}{v} - 300 = mx0$

   $v = 40m/s$

5. A car of mass 800kg is driven along a level road against a constant resistance to motion of 200N. With the engine working at a steady rate of 14kW Find the
   - (i) Acceleration of the car when its speed is 10m/s

(ii) maximum speed at which the car can move

Solution

$$F = ma$$
$$F_T - R = mxa$$
$$\frac{14000}{10} - 200 = 800xa$$
$$a = 1.5ms^{-2}$$

$F = ma$ at constant speed $a = 0ms^{-2}$

$$F_T - R = mx0$$
$$\frac{14000}{v} - 200 = mx0$$
$$v = 70m/s$$

6. A force on a particle of mass 15kg moves it along a straight line with a velocity of 10m/s. The rate at which work is done by the force is 50W. If the particle starts from rest, determine the time it takes to move a distance of 100m

Solution Uneb 2000 No7

$$F_T = ma$$
$$\frac{50}{10} = 15a$$
$$a = 0.33ms^{-2}$$
$$s = ut + \frac{1}{2}at^2$$
$$100 = 0t + \frac{1}{2}x0.33xt^2$$
$$t = 24.51s$$

7. A car of mass 500kg has an engine of maximum power 2.5kW.
   (a) Calculate the force resisting the motion of the car when it is travelling at its maximum speed of 72km/h on a level road
   (b) If the resistance remains unaltered, find the acceleration of the car when traveling at 36km/h on a level road with the engine working at the same rate

Solution

(i) $V = 72km/h = 20m/s$

$$F = ma$$
$$F_T - R = ma$$
$$\frac{2500}{20} - R = 500x0$$
$$R = 125N$$

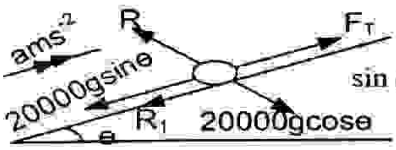
(ii) $F_T - R = mx0$

$$\frac{2500}{10} - 125 = 500xa$$
$$a = 0.25ms^{-2}$$

Inclined planes

1. A train of mass 20000kg moves at a constant speed of 72kmh^{-1} up a straight incline against a frictional force of 128. The incline is such that the train rises vertically one meter for every 100m travelled along the incline. Calculates the necessary power developed by the train.

Solution



$\sin\theta = \frac{1}{100}$

$$F_T - (mgSin\theta + R_1) = ma$$

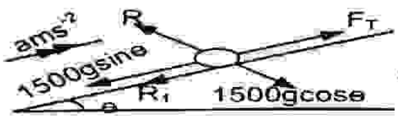
$a = 0ms^{-2}$ constant speed

$$\frac{P}{v} - (20000x9.8x\frac{1}{100} + 128) = 0$$
$$\frac{P}{20} = 2088N$$

Power= 41760W

2. A car of mass 1.5 metric tones moves with a constant speed of 6m/s up a slope of inclination $sin^{-1}\left(\frac{1}{7}\right)$. Given that the engine of the car is working at a constant rate of 18kW. Find the resistance to the motion

Solution



$\sin\theta = \frac{1}{7}$

$$F_T - (mgSin\theta + R_1) = ma$$

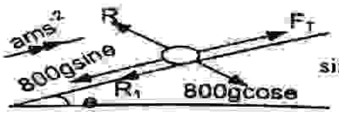
$a = 0ms^{-2}$constant speed

$$\frac{18000}{6} - (1500x9.8x\frac{1}{7} + R_1) = 0$$
$$R_1 = 900N$$

3. A car of mass 800kg with the engine working at a constant rate of 15kW climbs a hill of inclination 1 in 98 against a constant resistance to motion of 420N. Find the
   (iii) Acceleration of a car up a hill when travelling with a speed of 10m/s
   (iv) Maximum speed of the car up the hill

Solution

(i)



$\sin\theta = \frac{1}{98}$

$$F_T - (mgSin\theta + R_1) = ma$$
$$\frac{15000}{10} - (800x9.8x\frac{1}{98} + 420) = 800a$$

$$a = 1.25ms^{-2}$$

(i) $F_T - (mgSin\theta + R_1) = ma$

$a = 0ms^{-2} maximum\ speed$

$$\frac{15000}{v} - (800x9.8x\frac{1}{98} + 420) = 0$$

$$v = 30m/s$$

4. A car of mass 1000kg has a maximum speed of 40m/s on a level road and the engine is working at 32kW against a constant resistance.
   (i) Find the resistance to the motion of the car
   (ii) Given that the resistance in both cases varies as the speed, find the rate at which the engine must work for the car to ascend a slope of 1 in 98 at a constant speed of 20m/s

Solution

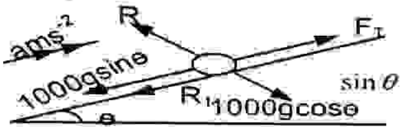
(i) $F = ma$

$$F_T - R_1 = ma$$

$$\frac{32000}{40} - R_1 = mx0$$

$$R_1 = 800N$$

(ii)



$\sin\theta = \frac{1}{98}$

(ii) $R_1 = kv$

$$k = \frac{800}{40} = 20$$

$$\frac{P}{20} - 1000g\ sin\theta - R_1 = 1000a$$

$$\frac{P}{20} - 1000x9.8x\frac{1}{98} - 20x20 = 1000x0$$

$$P = 10kW$$

5. A car of mass 1000kg has a maximum speed of 150km/h on a level rough road and the engine is working at 60kW.
   (i) Find the coefficient of friction between the car and the road if all resistance is due to friction
   (ii) Given that the tractive force remains unaltered and the non-gravitational resistance in both cases varies as square of the speed, find the greatest slope on which a speed of 120km/h could be maintained

Solution Uneb 2005 No13b

(i) $V = 150km/h = 41.67m/s$

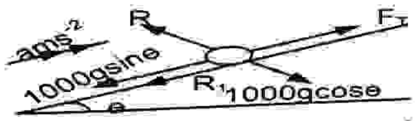
$$F = ma$$

$$F_T - \mu R = ma$$

$$\frac{60000}{41.67} - \mu x1000x9.8 = mx0$$

$$\mu = 0.147$$

(ii)



$$\mu R = kv^2$$

$$k = \frac{0.147x1000x9.8}{(41.67)^2} = 0.8297$$

$$u = 120km/h = 33.33m/s$$

$$\frac{60000}{41.67} - 1000g\ sin\theta - R_1 = 1000a$$

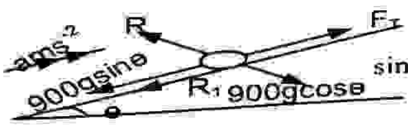
$$\frac{60000}{41.67} - 1000g\ sin\theta - 0.829x(33.33)^2 = 1000x0$$

$$\theta = 3.04°$$

6. A car of mass 900kg with the engine working at a constant rate of 7.35kW climbs a hill of inclination 1 in 63 against a constant resistance to motion. Find the
   (i) Resistance to motion when the car is travelling with a constant speed of 15m/s
   (ii) Maximum speed of the car when travelling down the same slope with the engine working at the same rate as before and the resistance to motion unchanged.

Solution

(i)



$\sin\theta = \frac{1}{63}$

$$F_T - (mgSin\theta + R_1) = ma$$

$$\frac{7350}{15} - (900x9.8x\frac{1}{63} + R_1) = 900x0$$

$$R_1 = 350N$$

(i) $F_T + (mgSin\theta - R_1) = ma$

$a = 0ms^{-2} maximum\ speed$

$$\frac{7350}{v} + (900x9.8x\frac{1}{63} - 350) = 0$$

$$v = 35m/s$$

7. A car of mass mkg has an engine which works at a constant rate of 2H kW. The car has a constant speed of Vm/s along a horizontal road.
   (a) Find in terms of m, H, V and θ the acceleration of the car when travelling
      (i) Up a road of inclination θ with a speed of $^3/_4 V\ ms^{-1}$

(ii) Down the same road with a speed of $^{3}/_{5}\,V\,ms^{-1}$, the resistance to the motion of the car apart from the gravitational force, being contant

(b) If the acceleration in a/9ii/0 above is 3 times that of a(i) above, find the angle oof inclination θ of the raod

(c) If the car continues directly up the road, incase a(i) above, who that its maximum speed is $^{12}/_{13}\,V\,ms^{-1}$ **Uneb 2004 No13**

An $(i)\ a = \frac{2000H - 3mvg\sin\theta}{3mv}$ $(ii)\ a = \frac{4000H + 3mvg\sin\theta}{3mv}$

Exercise 17C

1. A man of mass 75kg climbs 300m in 30 minutes. At what rate is he working **An[125W]**
2. A crane lifts an iron girder of mass 400kg at a steady speed of $2.0ms^{-1}$. At what rate is the carne working **An[8000W]**
3. What is the power output of a cyclist moving at a steady speed of $5.0ms^{-1}$ along a level road against a resistance of 20N **An[100W]**
4. A car is driven along a level road against a constant resistance to motion of 400N. Find the maximum speed at which the car can move when its engine works at a steady rate of 8.8kW **An[$22ms^{-1}$]**
5. What is the maximum speed which a car can travel along road when its engine is developing 24kW and there is a resistance to motion of 800N **An[$30ms^{-1}$]**
6. A car is working at 14kW and is travelling at a constant speed of 75m/s along a level road. Find the resistance to motion **An[400N]**
7. A car of mass 1000kg is driven along a level road against a constant resistance to motion of 200N. with the engine f car working at steady rate of 8kW, find
   (i) Acceleration of the car when its speed is 5m/s
   (ii) The maximum speed of the car **An($1.4m/s^2$, 40m/s)**
8. A train of mass 100 tonnes has an engine of maximum power 60kW.
   (a) Calculate the force resisting the motion of the car when it is travelling at its maximum speed of 108km/h on a level road
   (b) If the resistance remains unaltered, find the acceleration of the car when traveling at 54km/h on a level road with the engine working at the same rate **An(2000N, $0.02m/s^2$)**
9. A cyclist of mass 75kg moves on a level road and working at a rate of 210W against a constant resistance of 21 N.
   (i) Find the maximum speed that the cyclist can attain
   (ii) With the resistance and the rate of working unchanged, find the maximum speed the cyclist can ascend a slope of inclination 1 in 15 **An(10m/s, 3m/s)**
10. A car of mass 900kg moves on a level road at a maximum speed of 48m/s against a constant resistance of 350N.
    (i) Find the rate at which the engine is working
    (ii) With the resistance and the rate of working unchanged, find the maximum speed the car can ascend a slope of inclination 1 in 18 **An(16.8kW, 20m/s)**
11. A car of mass 100 metric tones moves with a constant speed of 654km/h up a slope of inclination $\sin^{-1}\left(\frac{1}{50}\right)$. Given that the engine of the car is working at a constant rate of 369kW. Find the resistance to the motion **An[5000N]**
12. A man of mass 70kg rides a bicycle of mass 15kg at a steady speed of $4.0ms^{-1}$ up a road which rises 1.0m fro every 20m of its length. What power is the cyclist developing if there is a constant resistance to motion of 20N. **An[250W]**
13. A lorry of mass 2000kg moving at 10m/s on a horizontal surface is brought to rest in a distance of 12.5m by the brakes being applied.
    (a) Calculate the average retarding force
    (b) What power must the engine produce if the lorry is to travel up a hill of 1 in 10 at a constant speed of 10m/s, frictional resistance being 200N. **An[8000N, 22000W]**
14. A car of mass 2 tones moves from rest down. A road of inclination $\sin^{-1}\left(\frac{1}{20}\right)$ to the horizontal. Given that the engine develops a power of 64.8kW when it is travelling at a speed of $54kmh^{-1}$ and the resistance to motion is 500N, find the acceleration. **An[$2.4m/s^2$]**
15. A car is driven at a uniform speed of $48kmh^{-1}$ up a smooth incline of 1 in 8. If the total mass of the car is 800kg and the resistance are neglected calculate the power at which the car is working. **An[1.31×10^4W]**
16. With its engine working at a constant rate of 9.8kW, a car of mass 800kg can descend a

slope of 1 in 56 at twice the steady speed that it can ascend the same slope, the resistances to motion remaining the same through it. Find the magnitude of the resistance and the speed if ascent **An[420N, 17.5m/s]**

17. The engine of a lorry of mass 5,000kg is working at a steady rate of 350kW against a constant resistance force of 1,000N. The lorry ascends a slope of inclination θ° to the horizontal. If the maximum speed of the lorry is $20ms^{-1}$. Find the value of θ **Uneb 2017 No8.** **An** 19.68°

PUMP RAISING AND EJECTING WATER.

Consider a pump which is used to raise water from a source and then eject it at a given speed. The work done per second gives the rate (power) at which the pump is working.

$$\boxed{work\ done\ per\ second = P.E\ given\ to\ water\ per\ second + K.E\ given\ to\ water\ per\ second}$$

Example

1. A pump raises water through a height of 3.0m at a rate of 300kg per minute and delivers it with a velocity of $8.0ms^{-1}$. Calculate the power output of the pump

Solution

$$Power = P.E\ given\ to\ water\ per\ second + K.E\ given\ to\ water\ per\ second$$

$$work\ done\ per\ second = (mass\ per\ second x g x h) + \left(\frac{1}{2} x mass\ per\ second x v^2\right)$$

$$work\ done\ per\ second = \left(\frac{300}{60} x9.81x3\right) + \left(\frac{1}{2} x \frac{300}{60} x(8)^2\right)$$

$$work\ done\ per\ second = 310J$$

2. A pump draws $6m^3$ of water of density $1000kgm^{-3}$ from a well 9m below the ground in every minute, and issues it at a speed of $12\ ms^{-1}$. Find the rate at which the pump is working

Solution

$$power = P.E\ given\ to\ water\ per\ second + K.E\ given\ to\ water\ per\ second$$

$$work\ done\ per\ second = (mass\ per\ second x g x h) + \left(\frac{1}{2} x mass\ per\ second x v^2\right)$$

$$work\ done\ per\ second = \left(\frac{6}{60} x1000x9.8x9\right) + \left(\frac{1}{2} x \frac{6}{60} x1000x12^2\right)$$

$$Power = 16020W$$

3. A pump raises $2m^3$ of water through a vertical distance of 10m in one and half minutes, and discharges it at a speed of 2.5m/s. Show that the power developed is approximately 2.25kW. **Uneb 2005 No13a**

Solution

$$Power = P.E\ given\ to\ water\ per\ second + K.E\ given\ to\ water\ per\ second$$

$$Power = (mass\ per\ second x g x h) + \left(\frac{1}{2} x mass\ per\ second x v^2\right)$$

$$Powerwork\ done\ per\ second = \left(\frac{2x1000}{90} x9.8x10\right) + \left(\frac{1}{2} x \frac{2x1000}{90} x2.5^2\right)$$

$$Power = 2247.22W = 2.25kW$$

4. A pump draws $3.6m^3$ of water of density $1000kgm^{-3}$ from a well 5m below the ground in every minute, and issues it at ground level r a pipe of cross-sectional area $40cm^2$. Find
   (i) The speed with which water leaves the pipe
   (ii) The rate at which the pump is working
   (iii) If the pump is only 80% efficient, find the rate at which it must work
   (iv) Find the power wasted

Solution

i) $volume\ per\ second = area\ x\ velocity$

$$\frac{3.6}{60} = 40x10^{-4}\ v$$

$$v = 15\ ms^{-1}$$

ii) Mass per second $= volume\ per\ second\ x\rho = \frac{3.6}{60} x1000 = 60kgs^{-1}$

$$work\ done\ per\ second = P.E\ given\ to\ water\ per\ second + K.E\ given\ to\ water\ per\ second$$

$$work\ done\ per\ second = (mass\ per\ second x g x h) + \left(\frac{1}{2} x mass\ per\ second x v^2\right)$$

$$work\ done\ per\ second = (60x9.81x5) + \left(\frac{1}{2}x60x15^2\right)$$

$$Power = 9693W$$

iii) Efficiency $= \frac{power\ output}{power\ input}x100\%$

$80\% = \frac{9693}{power\ input}x100\%$

$power\ input = 12116.25W$

iv) Power wasted= power output - power input

Power wasted= $12116.25 - 9693 = 2423.25W$

EXERCISE: 17D

1. A pump with a power output of 600W raises water from a lake a height of 3.0m and delivers it with a velocity of 6.0ms^{-1}. What mass of water is removed from the lake in one minute **An[7500kg]**
2. In every minute a machine pumps 300kg of water along a horizontal hose from rest at one end to eject at a speed of 4m/s at the other. Find the average rate at which the machine is working **An[40W]**
3. In every minute a pump draws $6m^3$ of water from a well and issues it at a speed of 5m/s from a nozzle situated 4m above the level from which the water was drawn. Find the average rate at which the pump is working **An[5.17kW]**
4. In every minute a pump draws $5m^3$ of water from a well and issues it at a speed of 6m/s from a nozzle situated 6m above the level from which the water was drawn. Find the average rate at which the pump is working **An[6.4kW]**
5. A pump draws water from a tank and issues it at a speed of 8m/s from the end of a pipe of cross-sectional are $0.01m^2$, situated a 10m above the level from which the water is drawn. Find
   (i) The mass of water issued from the pipe in each second
   (ii) Rate at which the pump is working **An[80kg, 10.4kW]**
6. A pump draws water from a tank and issues it at a speed of 10m/s from the end of a pipe of cross-sectional are $5cm^2$, situated a 4m above the level from which the water is drawn. Find the rate at which the pump is working **An[446W]**
7. In every minute a pump draws $2.4m^3$ of water from a well 5m below ground, and issues it at ground level through a pipe of cross-sectional are $50cm^2$. Find the
   (i) Speed with which the water leaves the pipe
   (ii) Rate at which the pump is working, if its only 75% efficient **An[8m/s, 3.24kW]**
8. In each minute a pump working at 3.48kW raises $1.5m^3$ of water from an underground tank and issues it from the end of a pipe situated at ground level. The water leaves the pipe with a speed 10m/s and the pump is 50%. Find ,
   (i) The area of cross-section of the pipe
   (ii) The depth below ground level from which the water is drawn **An[25cm^2, 2m]**
9. In each minute a pump working at 825W draws $0.3m^3$ of water from a well and issues from a nozzle situated 5m above the level from which the water was drawn. If the pump is 60%. Find ,
   (i) The velocity with which water was ejected
   (ii) The area of cross-section of the nozzle **An[10m/s, 5cm^2]**
10. A pump raises 75kg of water a vertical distance of 20m in 14 seconds find the average rate at which the pump is working. **An[1.05kW]**

CHAPTER 8: LINEAR MOMENTUM

This is the product of mass of a body and its velocity

$$Mmonetum = mass\ x\ velocity$$

Collisions

Case 1

Consider two bodies A and B with body A having a mass of M_A, initial velocity U_A , and body B having a mass of M_B, initial velocity U_B, after collision body A has a final velocity V_A and body B has a final velocity V_B

$$\boxed{M_AU_A + M_BU_B = M_AV_A + M_BV_B}$$

$$\boxed{loss\ in\ k.e = \left(\frac{1}{2}M_AU_A{}^2 + \frac{1}{2}M_BU_B{}^2\right) - \left(\frac{1}{2}M_AV_A{}^2 + \frac{1}{2}M_BV_B{}^2\right)}$$

Case 2

Consider two bodies A and B with body A having a mass of M_A, initial velocity U_A , and body B having a mass of M_B, initial velocity U_B, after collision body A and body B stick together and move with a common velocity V.

$$\boxed{M_AU_A + M_BU_B = (M_A + M_B)V}$$

$$\boxed{loss\ in\ k.e = \frac{1}{2}(M_AU_A + M_BU_B) - \frac{1}{2}(M_A + M_B)V^2}$$

Examples:

1. A trolley of mass 3kg travelling at a velocity of 4m/s collide with another trolley of mass 1kg which is at rest . At what velocity do the two bodies move together after collision

Solution

$M_AU_A + M_BU_B = (M_A + M_B)V$

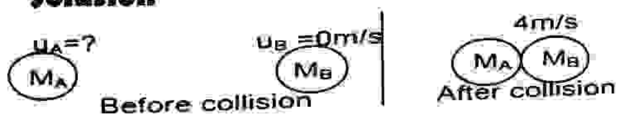
$(3x4) + (1x0) = (3 + 1)V$

$12 = 4V$

$V = \frac{12}{4} = 3ms^{-1}$

2. An object of mass 10kg collides with a stationery object of mass 5kg. If the objects stick together and move forward with a velocity of 4m/s. What was original velocity of the moving object?

Solution



$M_AU_A + M_BU_B = (M_A + M_B)V$

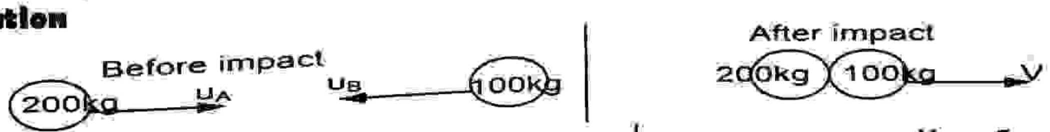
$(10xU_A) + (5x0) = (10 + 5)4$

$10U_A$ = 150x4

$u_A = \frac{600}{10} = 60m/s$

3. Two bodies of masses 200kg and 100kg travel towards each other with velocities of 20m/s and 25m/s respectively and join to form one body on collision. Find the common velocity.

Solution



$M_AU_A + M_BU_B = (M_A + M_B)V$

$(200x20) + (100x - 25) = (200 + 100)v$

$V = 5ms^{-1}$

Common velocity is $5ms^{-1}$

4. A bullet of mass 50g travelling horizontaly at 80m/s hits ablock of wood of mass 10kg resting on a smooth horizontal plane. If the bullet emerges with a speed of 50m/s, ifnd the speed with which the block moves

Solution Uneb 2016 No 8

$M_AU_A + M_BU_B = M_AV_A + M_BV_B$

$0.05x80 + 10x0 = 0.05x50 + 10xv$

$v = 0.15m/s$

5. A particle of mass 2kg moving with a speed 10m/s collides with a stationary particle of mass 7kg. Immediatel after impact, the particle move with the same speed but in opposite directions. Find the loss in kinectic energy

Uneb 2007 No 4

Solution

$$M_AU_A + M_BU_B = M_AV_A + M_BV_B$$
$$2x10 + 7x0 = 2x - v + 7xv$$
$$v = 4m/s$$
$$k.e\ before = \frac{1}{2}x2x10^2 + \frac{1}{2}x7x0^2 = 100J$$

$$k.e\ after = \frac{1}{2}x2x4^2 + \frac{1}{2}x7x4^2 = 72$$
$$loss\ in\ k.e = 100 - 72 = 28J$$

6. Two particles are moving towards each other along a straight line. The first particle hasss a mass of 0.2kg and moving with a velocity 4m/s and then the second hass a mass of 0.4kg moving with a velocity of 3m/s. on collision, the first particle reverses its direction and movees with a velocity of 2.5m/s. find the percentage loss in kinecti energy **Uneb 1997 NO13**

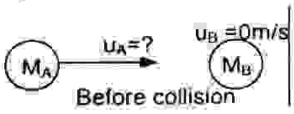
Solution

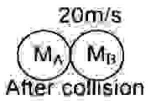
$$M_AU_A + M_BU_B = M_AV_A + M_BV_B$$
$$0.2x4 + 0.4x - 3 = 0.2x - 2.5 + 0.4xv$$
$$v = 0.25m/s$$
$$k.e\ before = \frac{1}{2}x0.2x4^2 + \frac{1}{2}x0.4x(-3)^2 = 3.4J$$

$$k.e\ after = \frac{1}{2}x0.2x(-2.5)^2 + \frac{1}{2}x0.4x0.25^2$$
$$= 0.6375J$$
$$\%loss\ in\ k.e = \frac{3.4 - 0.6375}{3.4}x100\% = \frac{2.762}{3.4}$$
$$= 81.25\%$$

8. A bullet of mass 20g is fired into a block of wood of mass 400g lying on a smooth horizontal surface. If the bullet and the wood move together with the speed of 20m/s. Calculate
   (i) The speed with which the bullet hits the wood
   (ii) The kinetic energy lost

Solution





By the law of conservation of momentum

$$M_AU_A + M_BU_B = (M_A + M_B)V$$
$$(0.02xU_A) + (0.4x0) = (0.02 + 0.4)x20$$
$$0.02U_A = 0.42x20$$
$$u_A = \frac{8.4}{0.02} = 420m/s$$

∴ The original velocity of the bullet was 420m/s

$$Initial\ k.e_i = \tfrac{1}{2}M_AU_A^2 + \tfrac{1}{2}M_BU_B^2$$
$$k.e_i = \frac{1}{2}\ x\ 0.02x\ 420^2 + \frac{1}{2}x0.4x0^2 = 1764J$$
$$Final\ k.e_f = \tfrac{1}{2}(M_A + M_B)V^2$$
$$k.e_f = \tfrac{1}{2}x(0.02 + 0.04)x(20)^2 = 84J$$

Loss in kinetic energy = $k.e_i - k.e_f$
$$= 1764 - 84 = 1680J$$

Gun and a bullet

When a bullet of mass M_b is fired with a muzzle velocity of V_b from a gun of mass M_g, the gun jerks back ward with a recoil velocity of V_g

$$\boxed{M_gV_g = M_bV_b}$$

1. A bullet of mass $15g$ is fired from a rifle of mass $3kg$ with a muzzle velocity of 100m/s. Find the recoil velocity of the rifle.

Solution

$$M_gV_g = M_bV_b$$
$$3V_g = 15\ x\ 10^{-3}x100$$
$$V_g = \frac{1.5}{3} = 0.5m/s$$

2. A gun of mass 3000kg fires horizontally a shell at an initial velocity of 300m/s. If the recoil of the gun is brought to rest by a constant opposing force of 9000N in 2 second, find the;

(a) (i) initial velocity of the recoil gun
(ii) mass of the shell
(iii) Gain in kinetic energy of the shell just after firing
(b) (i) displacement of the gun
(ii) work done against the opposing force

Solution

(a) (i) $F = ma$

$$-9000 = 3000a$$
$$a = -3ms^{-2}$$
$$v = u + at$$
$$0 = u - 3x2$$
$$u = 6m/s$$

(ii) $M_g V_g = M_b V_b$

$$3000x6 = M_b x300$$

$$M_b = 60kg$$

(iii) Gain in k.e= $\frac{1}{2}M_b V_b^2 = \frac{1}{2}x60x300^2$
$= 2.7x10^6 J$

(b) (i) $v^2 = u^2 + 2as$

$$0^2 = 6^2 - 2x3s$$
$$s = 6m$$

$(ii)\ w = Fs = 9000x6 = 54{,}000J$

Exercises 18

1. A bullet of mass 0.1kg travelling horizontally at 420m/s hits a block of wood of mass 2kg resting on a smooth horizontal plane. If the bullet becomes embedded on the block, find the speed with which the block moves after impact. **An(20m/s)**
2. A 2kg object moving with a velocity of 8m/s collides with a 3kg object moving with a velocity 6ms^{-1} along the same direction. If the collision is completely inelastic, calculate the decrease in kinetic energy collision. **An [2.4J]**
3. A particle A of mass 150g lies at rest in a smooth horizontal surface. A second particle B of mass 100g is projected along the surface with speed um/s and collides directly with A. on collision the masses coalesce and move on with speed 4m/s. Find the value of u an the loss in kinetic energy of the system during impact **An [10m/s, 3J]**
4. Two bodies A and B of mass 2kg and 4kg moving with velocities of 8m/s and 5m/s respectively collide and move on in the same direction. Object A's new velocity is 6m/s.
   i. Find the velocity of B after collision
   ii. Calculate the percentage loss in kinetic energy. **An(6m/s, 5.26%)**
5. Two smooth sphere A and B of equal radii and masses 3kg and 1.5kg respectively are travelling along the same horizontal line in opposite direction. The speeds of A and B are 6m/s and 2m/s respectively. The sphere collide and after collision B reverses its direction and moves with a speed of 4m/s. Find the velocity of A after collision. **An(3m/s)**
6. Two smooth spheres A and B of equal radii and masses 180g and 100g respectively are travelling along the same horizontal line. The initial speeds of A and B are 2m/s and 6m/s respectively. The spheres collide and after collision both spheres reverse their directions and B moves with a speed of 3m/s. Find the speed of A after collision and loss in kinetic energy of the system. **An(3m/s, 0.9J)**
7. A particle of mass 2kg moving with speed 10ms^{-1} collides with a stationary particle of mass 7kg. Immediately after impact the particles move with the same speeds but in opposite directions. Find the loss in kinetic energy during collision. **An(28J)**
8. Two identical rail way trucks are travelling in the same direction along the same straight piece of track with constant speeds of 6m/s and 2m/s. The faster truck catches up with the other one and on collision, the two trucks couple together. Find the common speed of the trucks after collision **An(8m/s)**
9. A 2kg object moving with a velocity of 6ms^{-1} collides with a stationary object of mass 1kg. If the collision is perfectly elastic, calculate the velocity of each object after collision. **An[2ms^{-1}, 8ms^{-1}]**
10. A van of mass of mass 1200kg and a lorry of mass 3200 kg collide. Just before the crash they are moving directly towards each other and each has a speed of 12m/s. Immediately after the crash they move with the same velocity. Find the loss in kinetic energy. **An(251,000J)**
11. A bullet of mass $20g$ is fired from a rifle of mass $2.5kg$. The bullet leaves the gun with a velocity of 500m/s. Find the recoil velocity of the rifle. **An(4m/s)**
12. A bullet of mass 5kg is fired from a rifle of mass 2000kg. The bullet leaves the gun with a velocity of 400m/s. Find the recoil velocity of the rifle. **An(1m/s)**

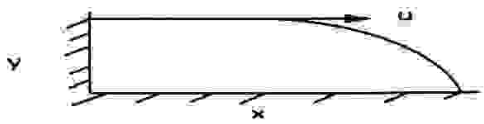
CHAPTER 9: PROJECTILE MOTION

This is the motion of a body which after being given an initial velocity moves freely under the influence of gravity.

TERMS USED IN PROJECTILES

1. **Angle of projection θ.** Angle the initial velocity makes with the horizontal.
2. **MAXIMUM HEIGHT [GREATEST HEIGHT].** The greatest height reached by projectile.
3. **TIME OF FLIGHT [T].** Time taken for a projectile to complete motion.
   Note: The time of flight is twice the time to maximum height
4. **RANGE [R].** Horizontal distance covered by projectile.
5. **MAXIMUM RANGE [R_{max}].** The greatest horizontal distance covered.
6. **A TRAJECTORY.** A trajectory is a path described by a projectile.

a. An object projected horizontally from a height above the ground



Horizontal motion: $u_x = u, a = 0$

$s = ut + ½at^2$

$x = ut \dots\dots (1)$

$v = u + at$

$v_x = u \dots\dots\dots (2)$

vertical motion: $u_y = 0, a = g = -9.8ms^{-2}$

$s = ut + ½at^2$

$y = {}^1/_2\, gt^2 \dots\dots (3)$

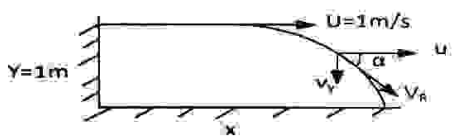
$v = u + at$

$v_y = gt \dots\dots\dots (4)$

Examples

1. A ball rolls off the edge of a table top 1m high above the floor with a horizontal velocity 1ms⁻¹. Find;
   i) The time it takes to hit the floor
   ii) The horizontal distance it covered
   iii) The velocity when it hits the floor

Solution



u=1ms⁻¹ θ=0° y=-1m below the point of projection

(i) vertical motion: $y = {}^1/_2\, gt^2$

$-1 = {}^1/_2 x - 9.8t^2$

$-1 = -4.9t^2$

$t = 0.4518s$

(ii) $x = ut = 1x0.4518m = 0.4518m$

velocity when it hits the ground

$v_x = u = 1m/s$

$v_y = gt = 9.8x0.4518 = -4.428m/s$

$v = \sqrt{V_x^2 + V_y^2} = \sqrt{(1)^2 + (-4.428)^2}$

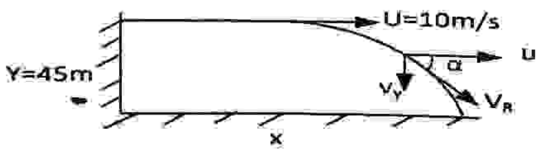
$V = 4.54ms^{-1}$

$\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{4.428}{1}\right) = 77.3°$

The velocity is $4.5ms^{-1}$ at 77.3°to the horizontal

2. A ball is thrown forward horizontally from the top of a cliff with a velocity of 10m/s. the height of a cliff above the ground is 45m. calculate
   i) Time to reach the ground
   ii) Distance from the cliff where the ball hits the ground
   iii) Direction of the ball just before it hits the ground

Solution



vertical motion: $y = {}^1/_2\, gt^2$

$-45 = {}^1/_2 x - 9.8t^2$

$t = 3.03s$

ii) $x = ut = 10x3.03m = 3.03m$

iv) velocity when it hits the ground

$v_x = u = 10m/s$

$v_y = gt = -9.8x3.03 = -29.694m/s$

$v = \sqrt{V_x^2 + V_y^2}$

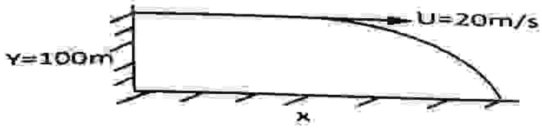
$V = \sqrt{(10)^2 + (-29.694)^2} = 31.333ms^{-1}$

$\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{29.694}{10}\right) = 71.4°$

The velocity is $31.333ms^{-1}$ at 71.4°to the horizontal

3. An object is projected horizontally at a speed 20 ms^{-1} from a height 100 m. Find:
   (a) the time of flight
   (b) the horizontal range
   (c) its velocity on reaching the ground

Solution:



(a) Vertical motion, $y = {}^1/_2 gt^2$

$-100 = {}^1/_2 x - 9.8t^2$

$-100 = -4.9t^2$

$t = 4.52s$

Horizontal motion, $x = ut = 20x4.52 = 90.4m$

Find resultant velocity v_R

$v_x = u = 20m/s$

$v_y = gt = -9.8x4.52 = -32.7m/s$

$v_R = \sqrt{V_x^2 + V_y^2} = \sqrt{(20)^2 + (-32.7)^2} = 38.3ms^{-1}$

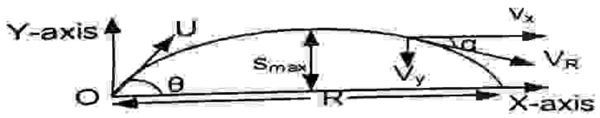
$\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{32.7}{20}\right) = 58.5°$

Exercise: 19A

1. A pencil is accidentally knocked off the edge of a horizontal desktop. The height of the desk is 64.8cm and the pencil hits the floor a horizontal distance of 32.4cm from the edge of the desk, What was the speed of the pencil as it left the desk. **An[0.9ms^{-1}]**
2. An aero plane moving horizontally at 150ms^{-1} releases a bomb at a height of 500m. the hits the intended target. What was the horizontal distance of aero plane from the target when the bomb was released? **An(1500m)**
3. A bomb is dropped from an aero plane when it is directly above a target at a height of 1402.5m. the aero plane is moving horizontally with a speed of 500kmh^{-1}. Determine whether the bomb will hit the target. **An (misses target by 2347.2m)**
4. A projectile is fired horizontally from the top of a cliff 250m high. The projectile lands 1.414x10^{3}m from the bottom of the cliff. Find the
   i) Initial speed of the projectile
   ii) Velocity of the projectile just before it hits the ground
   An [198m/s, 210m/s at 19.5°]

b. Objects projected upwards from the ground at an angle to the horizontal

Suppose an object is projected with velocity u at an angle θ from a horizontal ground.



Horizontally: $u = u_x = ucos\,\theta$, $a = 0$

$v = u + at$

$\boxed{v_x = u\cos\theta}$

$s = ut + {}^1/_2 at^2$

$\boxed{x = u\cos\theta\, t}$

Vertically: $u = u_y = usin\,\theta$, $a = -g = -9.8ms^{-2}$

$v = u + at$

$\boxed{v_y = u\sin\theta - gt}$

$s = ut + {}^1/_2 at^2$

$\boxed{y = u\sin\theta t - \frac{1}{2}gt^2}$

Examples

1. A Particle is projected with a velocity of 30ms^{-1} at an angle of elevation of 30°. Find
   i) The greatest height reached
   ii) The time of flight
   iii) Horizontal range
   iv) The velocity and direction of motion at a height of 4m on its way downwards

Solution

Vertically: $u = u_y = usin\,\theta$ $a = -g = -9.8ms^{-2}$

$at\ max\ height\ v = 0$

$$v^2 = u^2 + 2as$$

$$0^2 = (30sin30)^2 - 2x9.8H$$

$$H = 11.47m$$

at time of flight $s = 0$

$$s = ut - \frac{1}{2}gt^2$$

$$0 = 30\sin 30t - \frac{1}{2}x9.8t^2$$

$$0 = \left(30\sin 30 - \frac{1}{2}x9.8T\right)T$$

Either $T = 0$ or

$$30\sin 30 - \frac{1}{2}x9.8T = 0$$

$$T = 3.0612s$$

horizontally : $u = u_x = ucos\ \theta$ and $a = 0$

$$s = ut + {}^1/_2\,at^2$$

$$R = ucos\ \theta xT$$

$$R = (30cos\ 30)x30.0612 = 79.5329m$$

For vertical motion : $y = usin\theta t - ½\,gt^2$

$$4 = 30sin30t - ½\,9.81x\,t^2$$

$$4.905t^2 - 15t + 4 = 0$$

$$t = 2.76s \text{ or } t = 0.30s$$

The value of t=0.30s is the correct time since it's the smaller value for which the body moves upwards.

$$v_x = ucos\theta = 30cos\ 30 = 25.98m/s$$

$$v_y = usin\theta - gt$$

$$v_y = 305x30 - 9.81x0.30 = 12.06m/s$$

$$v = \sqrt{V_x^2 + V_y^2} = \sqrt{25.98^2 + 12.06^2} = 28.64m/s$$

Direction : $\alpha = tan^{-1}\frac{V_y}{V_x}$

$\alpha = tan^{-1}\frac{12.06}{25.98}$ α =24.9° to the horizontal

Velocity is 28.64m/s at 24.9° to horizontal

2. A projectile is fired with a velocity of 320m/s at an angle of 30° to the horizontal. Find
   (i) time to reach the greatest height (ii) horizontal range

Solution

i) At max height v=0,

$$v = usin\theta - gt$$

$$0 = 320sin30 - 9.81t$$

$$t = \frac{320sin30}{9.81} = 16.31s$$

ii) range $R = ucos\theta xT$

T=twice time to max height

R= $320cos30x\ [2x16.31] = 9039.92m$

3. A projectile fired at an angle of 60° above the horizontal strikes a building 30m away at a point 15m above the point of projection. Find
   (i) Speed of projection (ii) Velocity when it strikes a building

Solution

i) Horizontal distance at time t : $x = ucos\theta t$

$$30 = utcos60$$

$$t = \frac{60}{u}$$

Also vertical distance at any time t

$$y = usin\theta - \frac{1}{2}gt^2$$

$$15 = usin60x\frac{60}{u} - \frac{1}{2}x9.81\left(\frac{60}{u}\right)^2$$

$$15 = 51.9615 - \frac{4.905\ x\ 3600}{u^2}$$

$$u = \sqrt{477.74} = 21.86m/s$$

ii)but since t $= \frac{60}{u} = \frac{60}{21.86} = 2.75s$

$$v_x = ucos\theta = 21.86cos60 = 10.93m/s$$

$$v_y = usin\theta - gt$$

$$v_y = 21.81sin60 - 9.81x2.75$$

$$v_y = -8.09m/s$$

$$v = \sqrt{V_x^2 + V_y^2}$$

$$v = \sqrt{10.93^2 + (-8.09)^2} = 13.60m/s$$

$$\alpha = tan^{-1}\frac{V_y}{V_x} = tan^{-1}\frac{8.09}{10.9} = 36.58°$$

The velocity is 13.60m/s at 36.58° to the horizontal

4. A body is projected at an angle of 60° above horizontal and passes through a net after 10s. Find the horizontal and vertical distance moved by the body after it, was projected at a speed of 20m/s

Solution

4. A stone is projected at an angle of 30° to the horizontal with a velocity of 60m/s. Taking $g = 10ms^{-2}$, calculate;
   (a) The time taken for the particle to reach its maximum height
   (b) Maximum height (C) Time taken for flight
   (d) Horizontal range. **An[3s, 45m, 6s, 312m]**
5. A particle is projected a velocity of 25m/s at an angle of 30° above the horizontal. Find the horizontal and vertical component of the velocity 2.5s after projection. Hence find the speed and direction of the motion of the particle at that time **An[21.7m/s, 12m/s, 24.8m/s at 24° below the horiozontal]**
6. A particle is projected from an origin O and has an initial velocity of $30\sqrt{2}m/s$ at angle 45° above the horizontal. Find the horizontal and vertical component of the displacement 2s after projection. Hence find the distance of the motion of the particle at that time **An[60m, 40.4m, 72.3m]**
7. Calculate the range of a projectile which is fired at an angle of 45° to the horizontal with a speed of 20m/s . **An [40.77m]**
8. A particle is projected at an angle α to the horizontal with a speed u m/s, just clears a vertical wall 4m high and 10m away from the point of projection when travelling horizontally. Find the angle of projection. **An [38.66°]** **Uneb** 1998 No.15
9. A stone is thrown upwards from level ground towards a vertical pole, 4m high and 30m away from the point of projection. It just passes the pole in one second. Find **Uneb 2002 No**.16
   i) Its initial speed and angle of projection **An [31.29m/s, 16.5°]**
   ii) The distance beyond the pole where the particle will fall **An [24.47m]**
10. A particle projected from a poi t on the level ground has a horizontal range of 240m and time of flight of 6s. Find the magnitude and direction of the velocity of projection. (Taking $g = 10ms^{-2}$) **An [50m/s, 36.9°]**
11. A particle is projected with a velocity of 30m/s at an angle of 40° above the horizontal plane. find ;
   a) The time for which the particle is in the air.
   b) The horizontal distance it travels **An [3.9s, 22.9m/s]**
12. A particle is projected from ground level with an initial speed of 28m/s and during the course of motion it must not go higher than 10m above the ground level. Find the angle of projection that would allow the particle to go as high as possible. **An [30°]**
13. A body is projected with a velocity of 200ms⁻¹ at an angle of 30° above the horizontal. Calculate
   (a) Time taken to reach the maximum height
   (b) Its velocity after 16s **An [10.2s, 183m/s at 19.1°]**
14. A foot ball is kicked from a point O on a level ground. 2s later the football just clears a vertical wall of height 2.4m. If O is 22m from the wall, Find the velocity with which the ball is kicked. **An [15.6m/s at 45° above the horizontal]**
15. A particle is projected from a level ground in such a way that its horizontal and vertical components of velocity are $20ms^{-1}$ and $10ms^{-1}$ respectively. Find
   (a) Maximum height of the particle
   (b) Its horizontal distance from the point of projection when it returns to the ground
   (c) The magnitude and direction of the velocity on landing **An [5.0m, 40m, 22.4m/s at 26.6° below horizontal]**
16. A particle is projected form a horizontal plane and has an initial velocity of 49m/s at an angle of 30° above the horizontal. For how long is the particle at least 19.6m above the level of the plane. **An(3s)**
17. A particle is projected with a speed of $25ms^{-1}$ at 30° above the horizontal. Find;
   (a) Time taken to reach the height point of trajectory
   (b) The magnitude and direction of the velocity after 2.0s **An [1.25s, 22.9m/s at 19.1° below horizontal]**
18. A particle is projected at 84m/s to hit a point 360m away and on the same horizontal level as the point of projection. Find the two possible angles of projection. **An**(15°, 75°)
19. A golfer hits a golf ball at 30m/s and wishes it to land at appoint 45m away, on the same horizontal level as the starting point. Find the two possible angles of projection. **An**(14.7°, 75.3°)
20. A hammer thrown in athletics consists of a metal sphere of mass 7.26kg with a wire handle attached, the mass of which can be neglected. In a certain attempt it is thrown with an initial velocity which makes an angle of 45° with the

horizontal and its flight takes 4.00s. stating any assumptions find;

(i) The horizontal distance travelled
(ii) Kinetic energy of the sphere just before it strikes the ground **An [80.0m, $2.90x10^3$J]**

21. A stone thrown upwards at an angle θ to the horizontal with speed ums^{-1} just clears a vertical wall 4m high and 10m from the point of projection when travelling horizontally. Find the angle of projection **An[38.66°]**

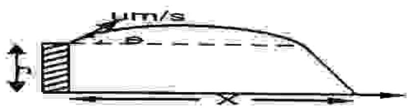
22. A particle is projected from a horizontal ground and has an initial speed of 35m/s. When the ball is travelling horizontally, it strikes a vertical wall. If the wall is 25m form the point of projection, find the two possible angles f projection **An**$(11.8°, 78.2°)$

23. A particle is projected from a point O and passes through a point A when the particle is travelling horizontally. If A is 10m horizontally and 8m vertically from O, find the magnitude and direction of the velocity of projection. **An**$(14.8ms^{-1}, 58°$ ***above the horizontal***$)$

24. A stone is projected at an angle of 20° to the horizontal and just clears a wall which is 10m high and 30m form the point of projection. Find the;
i) Speed of projection
ii) Angle which the stone makes with the horizontal as it clears the wall
An[73.78m/s, 16.9°]

b. Objects projected upwards from a point above the ground at an angle to the horizontal

Suppose an object is projected with velocity u at an angle θ from a height h



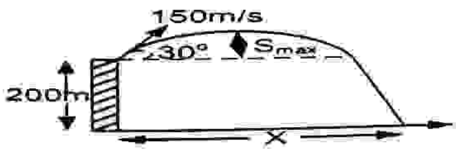
Horizontal: $u_x = ucos\theta,\ a = 0$

Vertical: $u_y = usin\theta,\ a = -g = -9.8$

Examples

1. A bullet is fired from a gun placed at a height of 200m with a velocity of 150ms^{-1} at an angle of 30° to the horizontal find
i) Maximum height attained
ii) Time taken for the bullet to hit the ground

Solution



(i) At max $height\ v = 0$

$v^2 = u^2 + 2as$

$0^2 = (150sin30)^2 - 2x9.8H$

$H = 286.70m$

The max height attained is 286.70m from the point of projection

(ii) Time taken for the bullet to hit the ground

Vertical motion : $y = usin\theta t - ½gt^2$

$y = -200m$ since its below the point of projection

$-200 = 150sin30T - ½x9.8T^2$

$4.9T^2 - 75T - 200 = 0$

$$T = \frac{75 \pm \sqrt{(-75)^2 - 4x4.9x - 200}}{2x4.9}$$

$t = 17.61s\ or\ t = -2.32s$

Time taken is $17.61s$

2. A particle is projected at an angle of elevation of 40° with a speed of 36m/s. If the point of projection is 0.5m above the horizontal ground, it just clears a wall which is 70m on the horizontal plane from the point of projection. find; **Uneb 2013 No.14**
(i) Time taken for the particle to reach the wall
(ii) Height of the wall
(iii) Find the maximum height reached by the particle from the point of projection

Solution

(i) $x = ucos\theta xt$

$70 = 36(cos40)t$

$t = 2.54s$

(ii) $y = usin\theta t - ½gt^2$

$h = 36sin(40)x2.54 - ½x9.8x(2.54)^2 = 27.16m$

height of the wall$= 27.16 + 0.5 = 27.66m$

(iii) At max $height\ v = 0$

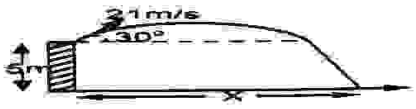
$v^2 = u^2 + 2as$

$0^2 = (36sin40)^2 - 2x9.8H$

$H = 27.32m$

3. A particle is projected at an angle of elevation of 30° with a speed of 21m/s. If the point of projection is 5m above the horizontal ground, find the horizontal distance that the particle travels before striking the ground. **Uneb 2006 No.16a**

Solution



$y = -5m$ since it's below the point of projection

For vertical motion: $y = usin\theta t - ½ gt^2$

$-5 = 21sin30T - \frac{9.8T^2}{2}$

$4.9T^2 - 10.5T - 5 = 0$

$T = \frac{10.5 \pm \sqrt{(-10.5)^2 - 4x4.9x - 5}}{2x4.9}$

T= $2.54s$ or T = $-0.40s$

Time of flight $T = 2.54s$

For horizontal motion

$x = ucos\theta xT = 21(cos30)\,x2.54 = 46.19m$

The horizontal distance = $46.19m$

Alternatively $y = xtan\theta - \frac{g\,x^2(1+tan^2\theta)}{2\,u^2}$

$-5 = xtan30 - \frac{9.8\,x^2(1 + tan^2 30)}{2\,x21^2}$

$0.0148x^2 - 0.5774x - 5 = 0$

$x = 45.15m$

EXERCISE 19C

1. A stone is thrown from the edge of a vertical cliff with a velocity of 50m/s at an angle of $tan^{-1}\frac{7}{24}$ above the horizontal. The stone strikes the sea at a point 240m from the foot of the cliff. Find the time for which the stone is in air and the height of the cliff **An[5s, 52.5m]**
2. A particle is projected with a velocity of 10ms⁻¹ at an angle of 45° to the horizontal, it hits the ground at a point which is 3m below its point of projection. Find the time from which it is in the air and the horizontal distance covered by the particle in this time **An[1.76s, 12.42m]**
3. A batsman hits a ball with a velocity of 17.5m/s at an angle of $tan^{-1}\frac{3}{4}$ above the horizontal, the ball initially being 60cm above the level ground. The ball is caught by a fielder standing 28m from the batsman. Find the time taken for the ball to reach the fielder and the height above the ground at which he takes the catch **An[2s, 2m]**
4. A vertical tower stands on a level ground. A stone is thrown from the top of the tower and has an initial velocity of 24.5m at an angle of $tan^{-1}\left(\frac{4}{3}\right)$ above the horizontal. The stone strikes the ground at a point 73.5m from the foot of the tower. Find the time taken for the stone to reach the ground and the height of the tower **An[5s, 24.5m]**
5. A stone is thrown from the top of a vertical cliff, 100m above sea level. The initial velocity of the stone is 13m/s at an angle of elevation of $tan^{-1}\frac{5}{12}$. Find the tiem taken for the stone to reach the sea and its horizontal distance from the cliff at that time. (Taking $g = 10ms^{-2}$) **An[5s, 60m]**
6. A golfer hits a golf ball with a velocity of 30m/s at an angle of $tan^{-1}\frac{4}{3}$ above the horizontal. The ball lands on green 5m below the level form which it was struck. Find the horizontal distance travelled by the ball. (Taking $g = 10ms^{-2}$) **An[90m]**
7. A pebble is thrown from the top of a cliff at a speed of 10m/s and at 30° above the horizontal. it hits the sea below the cliff 6.0s later , find;
   a) The height of the cliff . **An[150m, 52m]**
   b) The distance from the base of the cliff at which the pebble falls into the sea.
8. An arrow is fired from a point at a height 1.5m above horizontal. It has an initial velocity of 12m/s at an angle of 30° above the horizontal. The arrow hits a target at a height of 1m above the horizontal ground. find;
   (i) Time taken for the arrow to hit the target
   (ii) Horizontal distance between where the arrow is fired and the target
   (iii) Speed of the arrow when it hit the target
   An[1.3s, 13.51m, 12.39m/s]

Standard equations of projectiles

Suppose an object is projected with velocity u at an angle θ from a horizontal ground.

1. MAXIMUM HEIGHT [GREATEST HEIGHT] [H]

For vertical motion : at max height v=0,

$u_y = u\sin\theta,\ a = -g,\ s = H$

$v_y^2 = u_y^2 + 2gs$

$0 = (u\sin\theta)^2 - 2gH$

$2gH = u^2\sin^2\theta$

$$H = \frac{u^2\sin^2\theta}{2g}$$

Note : $\sin^2\theta = (\sin\theta)^2$ but $\sin^2\theta \neq \sin\theta^2$

2. TIME TO REACH MAX HEIGHT [t]

Vertically $v = u_y + at$ at max height v=0

$u_y = u\sin\theta,\ a = g$

$0 = u\sin\theta - gt$

$$t = \frac{u\sin\theta}{g}$$

3. TIME OF FLIGHT [T]

Vertically: $S_y = u_y t + \frac{1}{2}at^2$

at point A when the projectile return to the plane S_y=0,

t=T(time of flight), $a = -g$ $u_y = u\sin\theta$

$0 = u\sin\theta T - \frac{gT^2}{2}$

$T\left(u\sin\theta - \frac{gT}{2}\right) = 0$

Either $T = 0$ or $\left(u\sin\theta - \frac{gT}{2}\right) = 0$

$\left(u\sin\theta - \frac{gT}{2}\right) = 0$

$u\sin\theta = \frac{gT}{2}$

$$T = \frac{2u\sin\theta}{g}$$

Note: The time of flight is twice the time to maximum height

4. RANGE [R]

Horizontally : $S_x = u_x t + \frac{1}{2}at^2$

$u_x = u\cos\theta$, a=0 (constant velocity), t=T

$R = u\cos\theta\, T + \frac{1}{2} x 0 x T^2$

$R = u\cos\theta T$

But $T = \frac{2u\sin\theta}{g}$

$R = \frac{u^2\, 2\sin\theta\cos\theta}{g}$

But from trigonometry $2\sin\theta\cos\theta = \sin 2\theta$

$$R = \frac{u^2\sin 2\theta}{g}$$

5. MAXIMUM RANGE [R_{max}]

For maximum range sin2θ =1, R=R_{max}

2 θ= $\sin^{-1}(1)$

$2\theta = 90°$

$R_{max} = \frac{u^2\sin 90}{g}$

$$R_{max} = \frac{u^2}{g}$$

6. EQUATION OF A TRAJECTORY

A trajectory is expressed in terms of horizontal distance x and vertical distance y.

For horizontal motion at any time t

$x = u\cos\theta t$

$t = \frac{x}{u\cos\theta}$ ----------------[1]

For vertical motion at any time t

$y = u\sin\theta t - \frac{1}{2}gt^2$ -------------[2]

Putting t into equation [2]

$y = u\sin\theta \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Either $y = x\tan\theta - \frac{g x^2 \sec^2\theta}{2u^2}$

Or $y = x\tan\theta - \frac{g x^2(1+\tan^2\theta)}{2u^2}$

Examples:

1. A golfer hits a ball with a velocity of $44.1 m/s$ at an angle of $\sin^{-1}\left(\frac{3}{5}\right)$ above the horizontal. The ball lands on the green at a point which is level with point of projection. Find the time for which the golf ball was in air.

Solution

$$T = \frac{2u\sin\theta}{g}$$

$$T = \frac{2 \times 44.1 \times \frac{3}{5}}{9.8} = 5.4s$$

2. A ball is projected from a horizontal ground and has an initial velocity of 20m/s at an angle of elevation $tan^{-1}\left(\frac{7}{24}\right)$. When the ball is traveling horizontally it strikes a vertical wall. How high above the ground level does the impact occur.

Solution

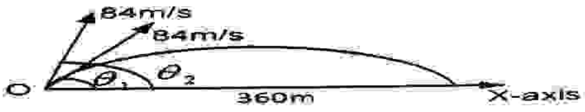
Projectile travels horizontally at maximum height

$$H = \frac{u^2 \sin^2\theta}{2g}$$

$$H = \frac{(20)^2 \sin^2\left(tan^{-1}\left(\frac{7}{24}\right)\right)}{2x9.8} = 1.6m$$

3. A particle is projected from a point on a horizontal ground at a speed of $84ms^{-1}$. If the particle hits a point 360 m away and on the same horizontal plane as the point of projection, find their:
(a) angles of projections (b) maximum heights (c) times of flight

Solution:



(a) $R = \frac{u^2 \sin 2\theta}{g}$

$$360 = \frac{(84)^2 \sin 2\theta}{9.8}$$

$\Rightarrow \theta = 15°, 75°$

(b) $H = \frac{u^2 \sin^2\theta}{2g}$

$$H_1 = \frac{84^2 \sin^2 15}{2x9.8} = 24.1\,m$$

$$H_2 = \frac{84^2 \sin^2 15}{2x9.8} = 335.9\text{ m}$$

(c) $T = \frac{2u\sin\theta}{g}$

$$T_1 = \frac{2x84sin15}{9.8} = 4.44\,s$$

$$T_2 = \frac{2x84sin75}{9.8} = 16.56\text{ s}$$

4. A gun has its barrel set at an angle of elevation of 15°. The gun fires a shell with an initial speed of 210m/s. Find the;
a. horizontal range of the shell. b. Maximum range

Solution

(i) $R = \frac{u^2 \sin 2\theta}{g} = \frac{(210)^2 \sin 2x15}{9.8} = 2250m$

(ii) $R_{max} = \frac{u^2}{g} = \frac{(210)^2}{9.8} = 4500m$

5. A stone thrown upwards at an angle θ to the horiozontal with speeds um/s just clears a vertical wall 4m high and 10m from the point of projection when travelling horizontally. Find the angle of projection

Solution

Projectile travels horizontally at maximum height

$$H = \frac{u^2 \sin^2\theta}{2g}$$

$$4 = \frac{u^2 \sin^2\theta}{2g}$$

$$8g = u^2 \sin^2\theta \ldots\ldots(i)$$

Also $x = ucos\theta xt$

$$t = \frac{u\sin\theta}{g}$$

$$x = ucos\theta\left(\frac{u\sin\theta}{g}\right)$$

$$10g = u^2 cos\theta\, sin\theta \ldots\ldots(ii)$$

$(i) \div (ii)$

$$tan\theta = \frac{8}{10}$$

$$\theta = 38.7°$$

6. If the horizontal range of a particle with a velocity u is R, show that the greatest height H is satisfied by the equation $16gH^2 - 8Hu^2 + gR^2 = 0$

Solution

$$H = \frac{u^2 \sin^2\theta}{2g}$$

$$\sin^2\theta = \frac{2gH}{u^2} \ldots\ldots(i)$$

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$gR = u^2 2\cos\theta \sin\theta$$

$$cos\theta = \frac{gR}{2u^2 \sin\theta}$$

$$cos^2\theta = \frac{(gR)^2}{4u^4 \sin^2\theta}$$

$$cos^2\theta = \frac{(gR)^2}{4u^4\left(\frac{2gH}{u^2}\right)}$$

$$cos^2\theta = \frac{gR^2}{8u^2H} \ldots\ldots(ii)$$

$$cos^2\theta + \sin^2\theta = 1$$

$$\frac{gR^2}{8u^2H}+\frac{2gH}{u^2}=1$$

$$gR^2+16gH^2=8u^2H$$

$$16gH^2-8Hu^2+gR^2=0$$

7. A ball is projected from point A and falls at a point which is in level with A and at a distance of 160m from A. The greatest height of the ball attained is 40m. Find the : **Uneb 2015 No.13**
   a. Angle and velocity at which the ball is projected
   b. Time taken for the ball to attain its greatest height.

Solution

$$H=\frac{u^2\,Sin^2\theta}{2g}$$

$$40=\frac{u^2\,Sin^2\theta}{2g}$$

$$Sin^2\theta=\frac{784}{u^2}\ldots\ldots(i)$$

$$R=\frac{u^2\,Sin\,2\theta}{g}$$

$$9.8x160=u^2 2\,cos\theta Sin\,\theta$$

$$cos\theta=\frac{784}{u^2 Sin\,\theta}$$

$$cos^2\theta=\frac{614656}{u^4 Sin^2\,\theta}$$

$$cos^2\theta=\frac{614656}{u^4\left(\frac{784}{u^2}\right)}$$

$$cos^2\theta=\frac{784}{u^2}\ldots\ldots(ii)$$

$$cos^2\theta+Sin^2\theta=1$$

$$\frac{784}{u^2}+\frac{784}{u^2}=1$$

$$u^2=1568$$

$$u=39.6m/s$$

$$Sin^2\theta=\frac{784}{u^2}$$

$$Sin^2\theta=\frac{784}{(39.6)^2}$$

$$Sin\theta=\sqrt{\frac{784}{(39.6)^2}}$$

$$\theta=45°$$

$$t=\frac{u\,sin\theta}{g}$$

$$t=\frac{39.6x\,sin45}{9.8}=2.85s$$

8. A boy throws a ball at an initial speed of 40m/s at an angle of elevation, θ. Show taking g to be $10ms^{-2}$, that the times of flight corresponding to a horizontal range of 80m are positive roots of equation $T^4-64T^2+256=0$ **Uneb 2006 No.16 b**

Solution

$$R=\frac{u^2\,2sin\theta\,cos\theta}{g}$$

$$80=\frac{40^2x2sin\theta\,cos\theta}{10}$$

$$sin\theta\,cos\theta=0.25$$

$$sin\theta=\frac{1}{4\,cos\theta}\ldots(i)$$

$$x=ucos\theta t$$

$$80=40xcos\theta xT$$

$$cos\theta=\frac{2}{T}\ldots\ldots(ii)$$

but $sin^2\theta+cos^2\theta=1$

$$\left(\frac{1}{4\,cos\theta}\right)^2+\left(\frac{2}{T}\right)^2=1$$

$$\left(\frac{1}{4\,x\,{}^2/_T}\right)^2+\left(\frac{2}{T}\right)^2=1$$

$$\left(\frac{T}{8}\right)^2+\left(\frac{2}{T}\right)^2=1$$

$$\frac{T^2}{64}+\frac{4}{T^2}=1$$

$$\frac{T^4+256}{64T^2}=1$$

$$T^4-64T^2+256=0$$

9. An object is projected from the ground making angle $arc\;tan3/4$ with the horizontal. The particle passes through a point 20 m horizontally and 10 m vertically from the point of projection. Find the speed of projection.

Solution:

$\tan\theta=3/4$, $x=20$ and $y=10$

$$y=x\tan\theta-\frac{gx^2}{2u^2}(1+tan^2\theta)$$

$$10=20\times\frac{3}{4}-\frac{g\times20^2}{2u^2}\left[1+\left(\frac{3}{4}\right)^2\right]$$

$\Rightarrow$u $=24.8\ ms^{-1}$

10. A particle projected from a point O on a horizontal ground moves freely under gravity and hits the ground again at A. Taking O as the origin, the equation of the path of the particle is $60y=20\sqrt{3}x-x^2$ where x and y are measured in meters. Determine the; **Uneb 2008 No.11**
   a. Initial speed and angle of projection
   b. Distance OA

Solution

(i) $60y=20\sqrt{3}x-x^2$

$$y=\frac{\sqrt{3}}{3}x-\frac{x^2}{60}$$

Comparing with

$$y=xtan\theta-\frac{g\,x^2(1+tan^2\theta)}{2\,u^2}$$

$$tan\theta=\frac{\sqrt{3}}{3}$$

$$\boldsymbol{\theta=30°}$$

$\frac{g(1+\tan^2\theta)}{2u^2} = \frac{1}{60}$

$\frac{9.8x(1+{}^3/_9)}{2u^2} = \frac{1}{60}$

$u = 19.8m/s$

(ii) At A $y = 0$

$0 = 20\sqrt{3}x - x^2$

$0 = (20\sqrt{3} - x)x$

$x = 20\sqrt{3}m$

11. A particle is projected from a point O on the level ground, with initial speed 30m/s to pass through a point which is a horizontal distance **40m** from O and a distance **10m** vertically above the level O

(i) show that there are two possible angles of projection

(ii) If these angles are α and β, prove that $tan(\alpha+\beta) = -4$. take $g = 10ms^{-2}$

Solution

$$y = xtan\theta - \frac{g\,x^2(1+tan^2\theta)}{2\,u^2}$$

$$10 = 40tan\theta - \frac{10x40^2(1+tan^2\theta)}{2x(30)^2}$$

$$10 = 40tan\theta - \frac{80}{9}(1+tan^2\theta)$$

$$8tan^2\theta - 36tan\theta + 17 = 0$$

since its a quadratic equation in $tan\theta$, its has two roots and two values of $\theta < 90$

$$tan\alpha + tan\beta = \frac{36}{8} \ldots\ldots\ldots (i)$$

$$tan\alpha x tan\beta = \frac{17}{8} \ldots\ldots\ldots (ii)$$

$$tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha x tan\beta} \ldots\ldots (iii)$$

$$tan(\alpha+\beta) = \frac{\frac{36}{8}}{1-\left(\frac{17}{8}\right)} = \frac{\frac{36}{8}}{\left(\frac{8-17}{8}\right)}$$

$$= \left(\frac{36}{8}\right) x \frac{8}{-9}$$

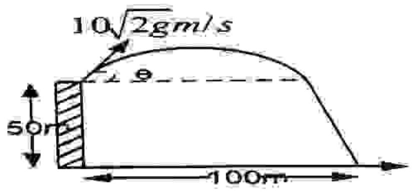
$$tan(\alpha+\beta) = -4$$

12. A particle is projected with a speed of $10\sqrt{2g}\ m/s$ from the top of a cliff 50m high. The particle hits the see at a distance 100m from the vertical through the point of projection. **Uneb 1998 No.10**

(i) Show that there are two possible directions of projection which are perpendicular

(ii) Determine the time taken from the point of projection in each case.

Solution



OR $y = xtan\theta - \frac{g\,x^2(1+tan^2\theta)}{2\,u^2}$

$$-50 = 100tan\theta - \frac{gx100^2(1+tan^2\theta)}{2x100x2g}$$

$$-50 = 100tan\theta - 25(1+tan^2\theta)$$

$$tan^2\theta - 4tan\theta - 1 = 0$$

$$tan\theta = \frac{4 \pm \sqrt{16-4x1x-1}}{2}$$

$$tan\theta_1 = 2+\sqrt{5}$$

$$tan\theta_2 = 2-\sqrt{5}$$

$$tan\theta_1 . tan\theta_1 = (2+\sqrt{5})(2-\sqrt{5}) = -1$$

Hence hey perpendicular

For horizontal motion

$x = ucos\theta xt$

$tan\theta_1 = 2+\sqrt{5}$

$\theta_1 = 76.72°$

$$t = \frac{100}{10\sqrt{2g}\,(cos76.72)} = 9.83s$$

$tan\theta_2 = 2-\sqrt{5}$

$\theta_1 = -13.28°$

$$t = \frac{100}{10\sqrt{2g}\,(cos-13.28)} = 2.32s$$

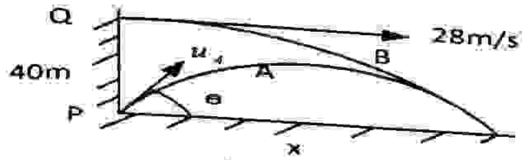
Different points of projection

1. Two objects A and B are projected simultaneously from different points. A is projected upwards at an angle from point P from the ground, and B at speed 28 ms-1 horizontally from point Q 40 m vertically above P. If the objects hit the same point on the ground, find:

(a) time taken and distance from P to where they hit

(b) speed and angle of projection of A.

Solution:



(a) For B : $y = \frac{1}{2} x g t^2$

$$-40 = \frac{1}{2} x - 9.8t^2$$

$$t = \frac{20}{7} s$$

$$x = ut = 28x\frac{20}{7} = 80m$$

(b) For A : $x = u\cos\theta x t$

$$80 = u_A \cos\theta x \frac{20}{7}$$

$$u_A \cos\theta = 28 \ldots\ldots\ldots (i)$$

$$s = ut + \frac{1}{2} at^2$$

$$0 = u_A \sin\theta x \frac{20}{7} - \frac{1}{2} x 9.8 \left(\frac{20}{7}\right)^2$$

$$u_A \sin\theta = 14 \ldots\ldots (ii)$$

$(ii) \div (i)$: $\tan\theta = 0.5$

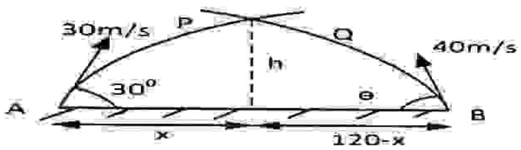
$$\theta = 26.6°$$

$$u_A \sin 26.6 = 14$$

$$u_A = 31.3 m/s$$

2. A particle P is projected from a point A with initial velocity 30m/s at an angle of elevation 30° to the horizontal. At the same instant a particle Q is projected in opposite direction with initial speed of 40m/s from a point at the same level with A and 120m from A. Given that the particles collide. Find:
   (i) Angle of projection of Q
   (ii) Time when collision occurs

Solution



$$s = ut + \frac{1}{2} at^2$$

At A: $h = 30\sin 30 x t - \frac{1}{2} x 9.8t^2 \ldots\ldots (i)$

At B: $h = 40\sin\theta x t - \frac{1}{2} x 9.8t^2 \ldots\ldots (ii)$

$(ii) \div (i)$: $40\sin\theta x t = 30\sin 30 x t$

$$\theta = 24.5°$$

For A : $x = 30\cos 30 x t \ldots\ldots\ldots (iii)$

For B : $120 - x = 40\cos 24.5 x t \ldots\ldots\ldots (iv)$

$$120 - 30\cos 30 x t = 40\cos 24.5 x t$$

$$t = 1.9s$$

3. Two objects A and B are projected simultaneously from the same point on a horizontal ground. A is projected with speed 28 ms^{-1} at elevation 45° and B at speed 35 ms^{-1}. The objects land at points 15 m apart with A beyond B. Find their ranges and the two possible angles of projection of B.

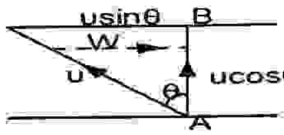
CHAPTER 10: VECTOR MECHANICS

(i) CROSSING THE RIVER

There are three cases to consider when crossing a river

a. Case I (shortest route)

If the water is not still and the boat man wishes to cross **directly opposite** to the starting point. In order to cross point A to another point B directly opposite A (perpendicularly), then the course set by the boat must be upstream of the river.



u is the speed of the boat in still water,
w is the speed of the running water
At point B: $u\sin\theta = w$

$\sin\theta = \frac{w}{u}$

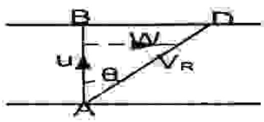
$\theta = \sin^{-1}\frac{w}{u}$

θ is the direction to the vertical but the direction to the bank is $(90-\theta)^\circ$

$\text{Time taken} = \frac{AB}{u\cos\theta}$

b. Case II. The shortest time/as quickly as possible

If the boat man wishes to cross the river as quickly as possible, then he should steer his boat directly from A to B as shown. The river pushes him down stream.



Time to cross the river $t = \frac{AB}{u}$

Distance covered downstream is = $w \times t$

Or distance downstream = $w\frac{AB}{u}$

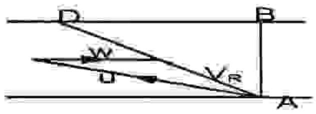
$\tan\theta = \frac{w}{u}$ $\theta = \tan^{-1}\frac{w}{u}$

The resultant velocity downstream V_R

$V_R^2 = W^2 + u^2$

$V_R = \sqrt{w^2 + u^2}$

C. Case III



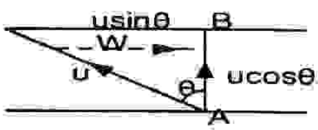
Resultant velocity $\vec{V_R} = \vec{W} + \vec{U}$

Examples

1. A man who can swim at 6km/h in still water would like to swim between two directly opposite points on the banks of the river 300m wide flowing at 3km/hr. Find the time he would take to do this.

Solution

U=6km/hr W= 3km/hr
AB=300m AB = 0.3km



$\sin\theta = \frac{w}{u}$ $\theta = \sin^{-1}\frac{3}{6}$ $\theta = 30^\circ$

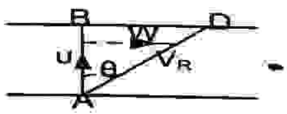
Time taken $= \frac{AB}{u\cos\theta} = \frac{0.3}{6\cos 30}$

Time $= 0.058hrs = 3.46minute$

He must swim at 30° to AB in order to cross directly and it will take 3.46minutes

2. A man who can swim at 2m/s at in still water wishes to swim across a river 120m wide as quickly as possible. If the river flows at 0.5m/s, find the time the man takes to cross and how far down streams he travels.

Solution



U=2m/s w=0.5m/s AB=120m

$t = \frac{AB}{u} = \frac{120}{2} = 60s$

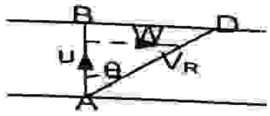
Distance downstream $= wt = 0.5 \times 60 = 30m$

3. A boat can travel at 3.5m/s in still water. A river is 80m wide and the current flows at 2m/s, calculate
   a) The shortest time to cross the river and the distance downstream that the boat is carried.

b) The course that must be set to a point exactly opposite the starting point and the time taken for crossing

Solution

a) U=3.5m/s, w=2m/s AB=80m



Shortest time t $= \frac{AB}{u} = \frac{80}{3.5} = 22.95$

Distance downstream BD= wt

$= 2x22.9$

Distance downstream BD=45.8m

b. U=3.5m/s, w=2m/s, AB=80

$Sin\theta = \frac{w}{u}$

$\theta = Sin^{-1}\frac{2}{3.5}$ 34.8°

The course must be 34.8° to AB.

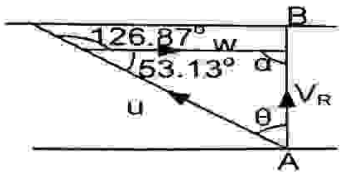
Time for crossing **Time taken** $= \frac{AB}{uCos\theta}$

Time taken $= \frac{80}{3.5Cos34.8} = 27.8s$

4. A man who can swim at 8m/s in still water crosses a river by steering at an angle of 126.87° to the water current. If the river is 75m wide and flows at 5m/s, find;
   (i) The velocity with which the person crosses the river
   (ii) The time he takes to do this

Solution

u=8m/s w=5m/s AB=75m



α is not 90°

Using cosine rule

$V_R^2 = 8^2 + 5^2 - 2\,u.wcos53.13$

$V_R = \sqrt{8^2 + 5^2 - 2\,x8x5cos53.13}$

$V_R = 6.4m/s$

The person crosses with 6.4m/s.

ii) **Time taken** $= \frac{AB}{uCos\theta}$

But $V_R = Ucos\theta$

Time $= \frac{75}{6.4} = 11.72\ seconds$

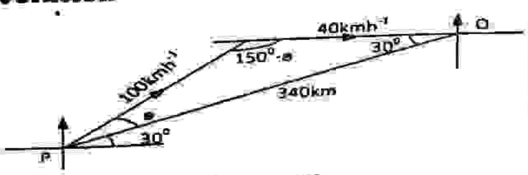
WIND BLOWING

- Locate the start and final destinations
- Join the direction of the blowing wind to the final destination
- Join the speed in still from start destination to the starting point of the direction of the wind

Examples

1. An aircraft which can travel at $100km/h$ in still air is to be flown from town P to town Q, 340km apart with Q on a bearing of 060° from P. If there is a steady wind of $40km/h$ blowing from the west, find the course set and the time taken for the flight.

Solution



$\frac{100}{sin30°} = \frac{40}{sin\theta}$

$\theta = 11.54°$

Course $N78.46°E$

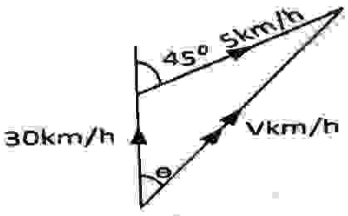
$\frac{V_{PQ}}{sin150° - 11.54°} = \frac{40}{sin11.54°}$

$V_{PQ} = 132.5941km/h$

Time $= \frac{PQ}{V_{PQ}} = \frac{340}{132.5941} = 2.5642hr$

2. A ship is heading due north at a speed of $30kmh^{-1}$. Water in the lake is moving in the north-east direction at an average speed of $5kmh^{-1}$. Calculate the;
   (i) Velocity of the ship
   (ii) Distance off course the ship will be after 40 minutes

Solution



$$V = \sqrt{30^2 + 5^2 - 2x30x5cos135°} = 33.72kmh^{-1}$$

$$\frac{33.72}{(sin135°)} = \frac{5}{sin\theta}$$

$$\theta = 60°$$

Velocity of the ship is $33.72kmh^{-1}$due $N60°E$

Distance, $d = vsin\theta xt$

$$d = 33.72sin60°x\frac{40}{60} = 2.35km$$

Exercise 20A

1. A man who an row at 0.9m/s in still water wishes to cross the river of width 1000m as quickly as possible. If the current flows at a rate of 0.3m/s. Find the time taken for the journey. Determine the direction in which he should point the boat and position of the boat where he lands **An [1111.11s, 71.57° to the bank, 333.33 downstream]**
2. A man swims at 5kmh^{-1} in still water. Find the time it takes the man to swim across the river 250m wide, flowing at 3kmh^{-1}, if he swims so as to cross the river;
   (i) By the shortest route **An [225s]**
   (ii) In the quickest time **An[180s]**
3. A boy can swim in still water at 1m/s, he swims across the river flowing at 0.6m/s which is 300m wide, find the time he takes;
   (i) If he travels the shortest possible distance
   (ii) If he travels as quickly as possible and the distance travelled downstream. **[375s,180m]**
4. A boy wishes to swim across a river 100m wide as quickly as possible. The river flows at 3km/h and the boy can swim at 4km/h in still water. Find the time that the boy takes to cross the river and how far downstream he travels. **An [90s,75m].**
5. An aircraft which can travel at $200km/h$ in still air is to be flown from town A to town B, 500km apart with B on a bearing of 060° from A. If there is a steady wind of $40km/h$ blowing from the west, find the course set and the time to the nearest minute taken for the flight. **An [$N54.3°E$,2h 8mins].**
6. An air craft flying at $250km/h$ in still air is to be flown from P to Q situated 300km form it on a bearing of 320°. If there is a wind of $50km/h$ blowing from 030°, find the course of the pilot that must be set so as to reach B and the time taken to the nearest minute. **An [$N29.2°W$,1h 19mins].**
7. A boat traveling at $5m/s$ in the direction 030° in still water is blown by wind moving at 8m/s from a bearing of 150°. Calculate the speed and the course the boat will be steered **UNEB 2000 No 4** **An [111.8°,7m/s].**

RELATIVE MOTION

It comprises of:
1-Relative velocity
2-Relative path

(a) Relative velocity

This is the velocity a body would have as seen by an observer on another body. Suppose A and B are two moving bodies, the velocity of A relative to B is the velocity of A as it appears to an observer on B.

It's denoted by $_AV_B = V_A - V_B$

Note that $_AV_B \neq {_BV_A}$ since $[_BV_A = V_B - V_A]$

Numerical calculations

There are two methods used in calculations

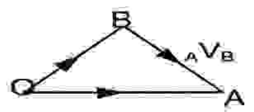
- Geometric method
- Vector method

1. Geometrical method

In this method make sure that the velocities of moving objects originates from a common point and their relative velocity closes to form a triangle of velocities.

Note

The direction of the relative velocity must be from the observer because it's the one observing where the other object is moving.



Apply either cosine formula or sine formula to obtain the unknown quantities

i.e $a^2 = b^2 + c^2 - 2bc\,CosA$ and $\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC}$

2. Vector method

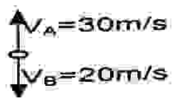
Find component of velocity for each object separately

Therefore $_AV_B = V_A - V_B$

Examples

1. Particle A is moving due to north at $30ms^{-1}$ and particle B is moving due south at 20m/s. find the velocity of A relative to B.

Solution



$_AV_B = V_A - V_B$

$_AV_B = \begin{pmatrix}0\\30\end{pmatrix} - \begin{pmatrix}0\\-20\end{pmatrix} = \begin{pmatrix}0\\50\end{pmatrix}$

$|_AV_B| = \sqrt{0^2 + 50^2} = 50$m/s due north

2. A cruiser is moving at 30km/hr due north and a battleship is moving at 20km/hr due north, find the velocity of the cruiser relative to the battleship.

Solution

$V_C = \begin{pmatrix}0\\30\end{pmatrix}$ $V_B = \begin{pmatrix}0\\20\end{pmatrix}$

$_CV_B = V_C - V_B$

$_CV_B = \begin{pmatrix}0\\30\end{pmatrix} - \begin{pmatrix}0\\20\end{pmatrix} = \begin{pmatrix}0\\10\end{pmatrix}$

$|_CV_B| = \sqrt{0^2 + 10^2}$

$_CV_B$ =10km/h due north

3. A particle A has a velocity of $(4i + 6j - 5k)$ m/s while particle B has a velocity of $(-10i - 2j + 6k)$ m/s. find the velocity of A relative to B

Solution

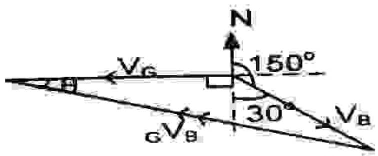
$_AV_B = V_A - V_B = \begin{pmatrix}4\\6\\-5\end{pmatrix} - \begin{pmatrix}-10\\-2\\6\end{pmatrix}$

$= \begin{pmatrix}14\\8\\-11\end{pmatrix}$

$_AV_B = (14i + 8j - 11k)$m/s

4. A boy runs at 5km/h due west and a girl runs 12km/hr at a bearing of 150°. Find the velocity of the girl relative to the boy.

Method I [geometrical]



The arrow of relative velocity is from the boy since he is the observer

$(_GV_B) = V_B^2 + V_G^2 - 2\,V_B V_G \cos 20^\circ$

$(_GV_B)^2 = 5^2 + 12^2 - 2x12x5cos120$

$(_GV_B)^2 = 229$

$_GV_B = \sqrt{229} = 15.13kmhr^{-1}$

Direction: using sine rule

$\frac{V_B}{Sin\theta} = \frac{GVB}{Sin120}$

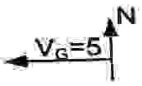
$\frac{12}{Sin\theta} = \frac{15.31}{Sin120}$

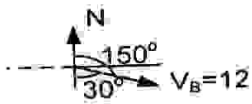
$Sin\theta = \frac{12Sin120}{15.13}$

$\theta = sin^{-1}\left(\frac{12sin120}{15.31}\right) = 43.4^\circ$

The relative velocity is 15.13km/hr at 43.4° below the western direction or S46.6°W

Method II (vector)



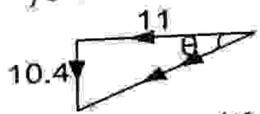


$_GV_B = V_G - V_B$

$_GV_B = \begin{pmatrix}-5\\0\end{pmatrix} - \begin{pmatrix}12Sin30\\-12Cos30\end{pmatrix} = \begin{pmatrix}-11\\-10.4\end{pmatrix}$

$/_GV_B/ = \sqrt{(-11)^2 + (-10.4)^2}$

$/_GV_B/ = 15.14km/hr$



$\theta = tan^{-1}\left(\frac{10.4}{11}\right) = 43.4^\circ$

Relative velocity is 15.14km/hr at 43.4°below the horizontal.

5. Plane A is flying due north at 40km/hr while plane B is flying in the direction N30°E at 30km/hr. Find the velocity of A relative to B.

Solution

$_AV_B = V_A - V_B$

$_AV_B = \begin{pmatrix}0\\40\end{pmatrix} - \begin{pmatrix}30Sin30\\-30Cos30\end{pmatrix} = \begin{pmatrix}-15\\14.02\end{pmatrix}$

$/_AV_B/ = \sqrt{(-15)^2 + (14.02)^2}$

$/_AV_B/ = 20.53km/hr$

$\theta = tan^{-1}\left(\frac{14.02}{15}\right) = 43.07^\circ$

The relative velocity is 20.53 at N46.93°W

6. Ship P is steaming at 60km/hr due east while ship Q is steaming in the direction N60°W at 50km/hr. Find the velocity of P relative to Q.

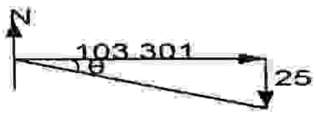
Solution

$V_p = \begin{pmatrix}60\\0\end{pmatrix}$ $V_q = \begin{pmatrix}-50Sin60\\50Cos60\end{pmatrix}$

$pVq = Vp - Vq$

$pVq = \begin{pmatrix}60\\0\end{pmatrix} - \begin{pmatrix}-50Sin60\\50Cos60\end{pmatrix} = \begin{pmatrix}103.301\\-25\end{pmatrix}$

$/pVq/ = \sqrt{(103.301)^2 + (-25)^2} = 106.3km/hr$



$\theta = tan^{-1}\left(\frac{25}{103.301}\right) = 13.60^\circ$

Direction $S(90 - 13.6)^\circ E$

Relative velocity is 106.3km/hr at S76.4°E

Finding true velocity

1. To a cyclist riding due north at 40km/hr, a steady wind appears to blow from west at 30km/hr. find the true velocity of the wind.

Solution

$Vc = \begin{pmatrix}0\\40\end{pmatrix}$ $wVc = \begin{pmatrix}30\\0\end{pmatrix}$ Vw=?

$wVc = Vw - Vc$

$\begin{pmatrix}30\\0\end{pmatrix} = V_w - \begin{pmatrix}0\\40\end{pmatrix}$

$V_w = \begin{pmatrix}30\\40\end{pmatrix}$

$V_w = \sqrt{30^2 + 40^2} = 50km/hr$

$\theta = tan^{-1}\left(\frac{40}{30}\right) = 53.13^\circ$

Direction $N(90 - 53.13)^\circ E = N36.87^\circ E$

To a motorist travelling due north at 40km/h, a steady wind appears to blow from N60°E at 50km/h
(i) Find the true velocity of the wind
(ii) If the wind velocity and direction remains constant but the speed of the motorist is increased, find his speed when the wind appears to be blowing from the direction $N45°E$

Solution

(i) $Vm = \begin{pmatrix} 0 \\ 40 \end{pmatrix}$ $wVm = \begin{pmatrix} 50sin60 \\ -50cos60 \end{pmatrix}$ $V_w = ?$

wVm = Vw − Vm

$$\begin{pmatrix} 50sin60 \\ -50cos60 \end{pmatrix} = V_w - \begin{pmatrix} 0 \\ 40 \end{pmatrix}$$

$$V_w = \begin{pmatrix} -43.5 \\ 15 \end{pmatrix}$$

$$V_w = \sqrt{(-43.5)^2 + 15^2}$$

$$V_w = 46km/hr$$

$$\theta = tan^{-1}\left(\frac{15}{43.5}\right) = 19.02°$$

Direction $N(90 - 19.02)°W$

N71°W

(ii) $Vm = \begin{pmatrix} 0 \\ a \end{pmatrix}$ $wVm = \begin{pmatrix} -bsin45 \\ -bcos45 \end{pmatrix}$

$$V_w = \begin{pmatrix} 46sin71 \\ -46cos71 \end{pmatrix}$$

wVm = Vw − Vm

$$\begin{pmatrix} -bsin45 \\ -bcos45 \end{pmatrix} = \begin{pmatrix} 46sin71 \\ -46cos71 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix}$$

i $components$: $-bsin45 = 46sin71$

$b = 61.5096$

j $components$: $-bcos45 = -46cos71 - a$

$a = 58.47$

$Vm = 58.47km/h$

3. To a man travelling due north at 10km/h, a steady wind appears to blow from East. When he travels in direction N60°W at 8km/h, it appears to come from south. Find the true velocity of the wind

Solution

$Vm = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$ $wVm = \begin{pmatrix} -a \\ 0 \end{pmatrix}$ $V_w = ?$

wVm = Vw − Vm

$$\begin{pmatrix} -a \\ 0 \end{pmatrix} = V_w - \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

$$V_w = \begin{pmatrix} -a \\ 10 \end{pmatrix} \ldots\ldots (i)$$

Also $Vm = \begin{pmatrix} -8sin60 \\ 8cos60 \end{pmatrix}$, $wVm = \begin{pmatrix} 0 \\ b \end{pmatrix}$, $V_w = ?$

wVm = Vw − Vm

$$\begin{pmatrix} 0 \\ b \end{pmatrix} = V_w - \begin{pmatrix} -8sin60 \\ 8cos60 \end{pmatrix}$$

$$V_w = \begin{pmatrix} -8sin60 \\ b + 8cos60 \end{pmatrix} \ldots\ldots (ii)$$

$$\begin{pmatrix} -a \\ 10 \end{pmatrix} = \begin{pmatrix} -8sin60 \\ b + 8cos60 \end{pmatrix}$$

$$a = 8sin60 = 4\sqrt{3}$$

$$10 = b + 8cos60$$

$$b = 6$$

$$V_w = \begin{pmatrix} -a \\ 10 \end{pmatrix} = \begin{pmatrix} -4\sqrt{3} \\ 10 \end{pmatrix}$$

$$V_w = \sqrt{\left(-4\sqrt{3}\right)^2 + 10^2} = 12.17km/hr$$

$$\theta = tan^{-1}\left(\frac{10}{4\sqrt{3}}\right) = 55.3°$$

Direction $N(90 - 55.3)°W$ =N34.7°W

4. To a cyclist riding due north at 40km/h, a steady wind appears to blow East wards. On reducing his speed to 30km/h but moving in the same direction, the wind appears to come from south west. Find the true velocity of the wind.

Solution

$Vc = \begin{pmatrix} 0 \\ 40 \end{pmatrix}$ $wVc = \begin{pmatrix} a \\ 0 \end{pmatrix}$ $V_w = ?$

wVc = Vw − Vc

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = V_w - \begin{pmatrix} 0 \\ 40 \end{pmatrix}$$

$$V_w = \begin{pmatrix} a \\ 40 \end{pmatrix} \ldots\ldots (i)$$

Also : $Vc = \begin{pmatrix} 0 \\ 30 \end{pmatrix}$ $wVc = \begin{pmatrix} bsin45 \\ bcos45 \end{pmatrix}$ $V_w = ?$

wVc = Vw − Vc

$$\begin{pmatrix} bsin45 \\ bcos45 \end{pmatrix} = V_w - \begin{pmatrix} 0 \\ 30 \end{pmatrix}$$

$$V_w = \begin{pmatrix} bsin45 \\ 30 + bcos45 \end{pmatrix} \ldots\ldots (ii)$$

$$\begin{pmatrix} a \\ 40 \end{pmatrix} = \begin{pmatrix} bsin45 \\ 30 + bcos45 \end{pmatrix}$$

$$40 = 30 + bcos45$$

$$b = 10\sqrt{2}$$

$$a = bsin45 = 10$$

$$V_w = \begin{pmatrix} a \\ 40 \end{pmatrix} = \begin{pmatrix} 10 \\ 40 \end{pmatrix}$$

$$V_w = \sqrt{(10)^2 + 40^2} = 41.23km/hr$$

$$\theta = tan^{-1}\left(\frac{40}{10}\right) = 75.96°$$

Direction $N(90 - 76)°E$ =N14°E

Exercise 20B

1. Car A is moving East wards at 20m/s and car B is moving Northwards at 10m/s. find the
   i) Velocity of A relative to B
   ii) Velocity of B relative to A

   An [$10\sqrt{5}$ m/s, $10\sqrt{5}$ m/s]

2. A yatch and a trawler leave a harbor at 8am. The yatch travels due west aat 10km/h and the trawler due east at 20km/h
   (i) What is the velocity of the trawler relative to the yatch
   (ii) How far apart are the boats at 9:30am

   An [30km/hr east, 45km]

3. At 10:30am a car travelling at 25m/s due east overtakes a motor bike travelling at 10m/s due east. What is the velocity of the car relative to the motor bike and how far apart are the vehicles at 10:31am **An [15m/s east, 900m]**

4. Bird A has a velocity of $(7i + 3j + 10k)$m/s while bird B has a velocity of $(6i - 17k)$ m/s. Find the velocity of B relative to A. **An** $(-i + 3j - 27k)$m/s

5. Joe rides his horse with a velocity $\begin{pmatrix} 5 \\ 24 \end{pmatrix}$km/h while Jill is riding her horse with velocity $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$km/h
   (ii) Find Joe's velocity as seen by Jill
   (iii) What is Jill's velocity as seen by Joe.

   An (i) $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$km/h, (ii) $\begin{pmatrix} 0 \\ -12 \end{pmatrix}$km/h

6. In EPL football match, a ball is moving at 5m/s in the direction of N45°E and the player is running due north at 8m/s. Find the velocity of the ball relative to the player.
   An[5.69m/s at S38.38°E].

7. An aircraft is flying at 250km/h in the direction $N60°E$ and a second aircraft is flying at 200km/h the direction $N20°W$. Find the velocity of the first air craft as seen by the pilot of the second aircraft. **An [292km/h at S77.9°E]**

8. A ship is sailing south East at 20km/h and a second ship is sailing due west at 25km/hr. Find the magnitude and direction of the velocity of the first ship relative to the second. **An [41.62km/h at S70.13°E]**

9. What is the velocity of a cruiser moving at 20km/h due north as seen by an observer on a liner moving at 15km/h in a direction $N30°W$
   An [10.3km/h at N46.9°E]

10. A car is being driven at 20m/s on a bearing of 040°. Wind is blowing from 330° with a speed pf 10m/s. find the velocity of the wind as experienced by the driver of the car. **An [48.13m/s at S18.13°W]**

11. An air craft is moving at 250km/h in direction N 60°E. the second aircraft is moving at 200km/h in a direction N 20°W. Find the velocity of the first aircraft as seen by the pilot of the second aircraft **An [292km/h at S77.9°E]**

12. To a pilot of a bomber aircraft travelling with a velocity $\begin{pmatrix} 150 \\ -200 \end{pmatrix} km/h$, a fighter aircraft appears to have a velocity of $\begin{pmatrix} 150 \\ 440 \end{pmatrix} km/h.$ Find the true velocity of the fighter.

13. To a pigeon flying with a velocity of $(-2i + 3j + k)$m/s, a hawk appears to have a velocity of $(i - 5j - 10k)$m/s. Find the true velocity of the hawk. **An** $(-i - 2j - 9k)$m/s

14. To a cyclist riding at 3m/s due east, the wind appears to come from the south with speed $3\sqrt{3}$m/s. Find the true speed and the direction of the wind. **An**(6m/s from $S30°W$)

15. To the pilot of an aircraft A, travelling at 300km/h due south, it appears that an aircraft B is travelling at 600km/h in a direction $N60°W$. Find the true speed and direction of the aircraft B **An**(520kmh west)

16. Jane is riding her horse at 5km/h due north and sees Suzan riding her horse apparently with velocity 4km/h, N60°E. Find Suzan's true velocity **An**(7.81km/h $N26.3°E$)

17. An eagle flying at 8m/s on a bearing of 240° sees a chick apparently running at 5m/s on bearing 300°. Find the true velocity of the chick. **An**(11.4m/s at $262.4°$)

18. A train is travelling at 80km/h in direction $N15°E$. A passenger on the train observes a palne apparently moving at 125km/h in direction $N50°E$. Find the true velocity of the plane. **An**(196km/h $N36.5°E$)

19. To a passenger on a boat which is travelling at 20km/h on a bearing of 230°, the wind seems to be blowing from 250° at 12km/h. Find the true velocity of the wind **An**(9.64km/h $N24.8°E$)

20. To an athlete jogging at 12km/h in a direction $N10°E$, wind seems to come from a direction $N20°W$ at 15km/h. Find the true velocity of the wind. **An**(7.57km/h $N72.5°W$)

21. On a particular day wind is blowing N30°E at a velocity of 4m/s and a motorist is driving at 40m/s in the direction of S60°E

a) Find the velocity of the wind relative to motorist **An [40.2m/s at N54.28°W]**

b) If the motorist changes the direction maintaining his speed and the wind appears to blow due East. What is the new direction of the motorist? **An[N85.03°W]**

22. A, B, and C are three aircrafts. A has velocity $(200i + 170j)$m/s. To the pilot of A it appears that B has a velocity $(50i - 270j)$m/s. To the pilot of B it appears that C has a velocity $(50i + 170j)$m/s. Find the velocities of B and C. **An** $(250i - 100j)$m/s, $(300i + 70j)$m/s.

23. To a bird fling due east at 10m/s, the wind seems to come from the south. When the bird alters its direction of flight to $N30°E$ with out altering its speed, the wind seems to come from the north-west. Find the true velocity of the wind. **An**(10.6m/s, from $S69.9°W$)

24. To an observer on a trawler moving at 12km/h in a direction $S30°W$, the wind appears to come from $N60°W$. To an obersver on a ferry moving at 15km/h in a direction $S80°E$, the wind appears to come from the north. Find the true velocity of the wind **An**(26.8km/h $N33.4°W$)

(b) RELATIVE PATH

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB respectively.

Position of A after time t is

$R_{At} = OA + t \times V_A$

Position of B after time t is

$R_{Bt} = OB + t \times V_B$

Relative path

$({}_AR_B)_t = R_{At} - R_{Bt}$

$({}_AR_B)_t = (OA + tV_A) - (OB + tV_B)$

$({}_AR_B)_t = (OA - OB) + t(V_A - V_B)$

$({}_AR_B)_t = (OA - OB) + t({}_AV_B)$

Examples

1. A car A and B are moving with their respective velocities $2i - j$ and $i + 3j$, if their position vectors are $4i + j$ and $2i - 3j$ respectively. Find the path of A relative to B

i) At any time t ii) At t=2s

Solution

i) $V_A = \begin{pmatrix}2\\-1\end{pmatrix}$ $V_B = \begin{pmatrix}1\\3\end{pmatrix}$

$OA = \begin{pmatrix}4\\1\end{pmatrix}$ $OB = \begin{pmatrix}2\\-3\end{pmatrix}$

$({}_AR_B)_t = (OA - OB) + t({}_AV_B)$

$({}_AR_B)_t = \left[\begin{pmatrix}4\\1\end{pmatrix} - \begin{pmatrix}2\\-3\end{pmatrix}\right] + t\left[\begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix}\right]$

$({}_AR_B)_t = \begin{pmatrix}2\\4\end{pmatrix} + t\begin{pmatrix}1\\-4\end{pmatrix}$

ii) When t=2 ${}_AR_B = \begin{pmatrix}2\\4\end{pmatrix} + 2\begin{pmatrix}1\\-4\end{pmatrix}$

$= \begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}2\\-8\end{pmatrix} = \begin{pmatrix}4\\-4\end{pmatrix}$

2. The velocities of two ships P and Q are $(i + 6j)km/h$ and $(-i + 3j)km/h$. At a certain instant, the displacement between the two ships is F $(7i + 4j)km$. Find the: **Uneb 2002 No.15a**

(iv) Relative velocity of ship P to Q

(v) Magnitude of displacement between ships P and Q after 2 hours

Solution

$V_p = \begin{pmatrix}1\\6\end{pmatrix}$ $V_q = \begin{pmatrix}-1\\3\end{pmatrix}$

$pVq = Vp - Vq$

$pVq = \begin{pmatrix}1\\6\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$

${}_PR_Q = (OP - OQ) + pVqxt$

${}_PR_Q = \begin{pmatrix}7\\4\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}$

${}_PR_Q = \begin{pmatrix}7\\4\end{pmatrix} + 2x\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}11\\10\end{pmatrix}$

$|{}_PR_Q(t)| = \sqrt{(11)^2 + (10)^2} = 14.87km$

3. Two ships A and B move simultaneously with velocities 20km/hr and 40km/hr respectively. Ship A moves in the northern directions while ship B moves in N60°E. Initially ship B is 10km due west of A. determine

a) The relative velocity of A to B b) The relative path of A to B

Solution

a)

$V_A = \begin{pmatrix} 0 \\ 20 \end{pmatrix}$ $V_B = \begin{pmatrix} 40Sin60 \\ 40Sin60 \end{pmatrix}$

${}_AV_B = V_A - V_B$

${}_AV_B = \begin{pmatrix} 0 \\ 20 \end{pmatrix} - \begin{pmatrix} 40Sin60 \\ 40Sin60 \end{pmatrix} = \begin{pmatrix} -34.64 \\ 0 \end{pmatrix}$

${}_AV_B = 34.64km/hr$

b)

${}_AR_B = (OA - OB) + t({}_AV_B)$

$OB = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $OA = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

${}_AR_B = \left[\begin{pmatrix} 10 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] + t\begin{pmatrix} -34.64 \\ 0 \end{pmatrix}$

${}_AR_B = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + t\begin{pmatrix} -34.64 \\ 0 \end{pmatrix}$

DISTANCE AND TIME OF CLOSEST APPROACH
(SHORTEST DISTANCE AND TIME TO SHORTEST DISTANCE)

When two particles are moving simultaneously with specific velocities, time will come when they are closest to each other **without** colliding

Numerical calculations

There three methods used

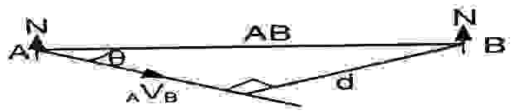
- Geometrical
- Vector
- Differential

1. Geometrical

It works when the information given is either a bearing (direction) or in two dimensions i.e. only **i** and **j** components

Procedure

- Draw a diagram showing the initial positions of the particles
- Consider the motion of one body relative to another i.e. A relative to B
- Represent the velocities of the bodies on the diagram with their directions specified.
- Superimpose the relative velocity to the body being observed
- Shortest distance is perpendicular to the relative velocity as shown below;



Shortest distance

$d = ABsin\theta$

${}_AV_B$ is a velocity, therefore to express it as a distance, it becomes $/{}_AV_B/\ t$

But $cos\theta = \frac{/AVB/\ t}{AB}$ *Hence* $/{}_AV_B/\ t = ABcos\theta$

Time to shortest distance

$t = \frac{A\,B\,Cos\theta}{/AVB/}$

2. Vector

Consider particles A and B moving with velocities V_A and V_B from point with positions vectors OA and OB respectively.

Then **shortest distance** **d= $/{}_AR_B/$**

For minimum distance to be attained then $\mathbf{{}_AV_B \cdot {}_AR_B = 0}$ This gives the time

Or time $= \frac{/AB\ .\ AVB/}{/AVB/^2}$ Where $\mathbf{AB \cdot {}_AV_B}$ is a dot product

3. Differential

The minimum distance is reached when $\frac{d}{dt}/{}_AR_B/^2 = 0$ This gives the time

Minimum distance $\mathbf{d} = /{}_AR_B/$

Examples

1. A particle P moves with a constant velocity $(i + j)$ passes a point with position vector $3i + 2j$. At the same instant particle Q passes through a point whose position vector is $i + j$ moving with a constant velocity of $(4i - 2j)$. Find: **Uneb 1988 No.5**
   i) Displacement of P relative to Q after t seconds
   ii) Time when P and Q are closest together and closest distance at that time

Solution:

a) $OP = \begin{pmatrix}3\\-2\end{pmatrix}$ $V_P = \begin{pmatrix}1\\1\end{pmatrix}$

$OQ = \begin{pmatrix}1\\4\end{pmatrix}$ $V_Q = \begin{pmatrix}4\\-2\end{pmatrix}$

${}_PR_Q = OP - OQ$

${}_PR_Q = \begin{pmatrix}3\\-2\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}2\\-6\end{pmatrix}$

${}_PV_Q = V_P - V_Q$

${}_PV_Q = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}-3\\3\end{pmatrix}$

${}_PR_Q(t) = (OP - OQ) + ({}_PV_Q)t$

${}_PR_Q(t) = \begin{pmatrix}2\\-6\end{pmatrix} + \begin{pmatrix}-3\\3\end{pmatrix}t = \begin{pmatrix}2-3t\\-6+3t\end{pmatrix}$

For minimum distance

${}_PV_Q \cdot {}_PR_Q(t) = 0$

$\begin{pmatrix}-3\\3\end{pmatrix} \cdot \begin{pmatrix}2-3t\\-6+3t\end{pmatrix} = 0$

$-24 + 18t = 0$

$t = 1.33\ units$

b) Shortest distance d=/${}_QR_P$/

${}_PR_Q(t) = \begin{pmatrix}2-3t\\-6+3t\end{pmatrix}$

$t = 1.33$

${}_PR_Q(t) = \begin{pmatrix}2-3x1.33\\-6+3x1.33\end{pmatrix} = \begin{pmatrix}-1.99\\-2.01\end{pmatrix}$

$/{}_PR_Q(t)/ = \sqrt{(-1.99)^2 + (-2.01)^2} = 2.83units$

2. A particle P starts from rest from a point with position vector $2j + 2k$ with a velocity $(j + k)$m/s. A second particle Q starts at the same time from a point whose position vector is $-11i - 2j - 7k$ with a velocity of $(2i + j + 2k)$m/s . Find;
   i) The shortest distance between the particles
   ii) The time when the particles are closest together
   iii) How far each has travelled by this time

Solution:

Method I vector

i) $OP = \begin{pmatrix}0\\2\\2\end{pmatrix}$ $V_P = \begin{pmatrix}0\\1\\1\end{pmatrix}m/s$

$OQ = \begin{pmatrix}-11\\-2\\-7\end{pmatrix}$ $V_Q = \begin{pmatrix}2\\1\\2\end{pmatrix}$m/s

${}_PV_Q = V_P - V_Q$

${}_PV_Q = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\-1\end{pmatrix}$

${}_PR_Q(t) = (OP - OQ) + ({}_PV_Q)t$

${}_PR_Q(t) = \left[\begin{pmatrix}0\\2\\2\end{pmatrix} - \begin{pmatrix}-11\\-2\\-7\end{pmatrix}\right] + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$

${}_PR_Q(t) = \begin{pmatrix}11\\4\\9\end{pmatrix} + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$

For minimum distance

${}_PV_Q \cdot {}_PR_Q(t) = 0$

$\begin{pmatrix}-2\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix} = 0$

$-22 + 4t + 0 - 9 + t = 0$

$t = \frac{31}{5}$ $\therefore\ t = 6.2s$

ii) Shortest distance d=/${}_PR_Q(t)$/

${}_PR_Q(t) = \begin{pmatrix}11\\4\\9\end{pmatrix} + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$

t=6.2

${}_PR_Q(6.2) = \begin{pmatrix}11\\4\\9\end{pmatrix} + \begin{pmatrix}-2\\0\\-1\end{pmatrix}6.2 = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$

$/{}_PR_Q(t)/ = \sqrt{(-1.4)^2 + 4^2 + 2.8^2}$ =5.08m

iii) How far each has travelled

$R_P = OP + V_Pt$

$R_P = \begin{pmatrix}0\\2\\2\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}6.2 = \begin{pmatrix}0\\8.2\\8.2\end{pmatrix}$

$/R_P/ = \sqrt{0^2 + 8.2^2 + 8.2^2}$

$/R_P/$= 11.6m

$R_Q = OQ + V_Qt$

$R_Q = \begin{pmatrix}-11\\-2\\-7\end{pmatrix} + \begin{pmatrix}2\\0\\1\end{pmatrix}6.2 = \begin{pmatrix}1.4\\4.2\\5.4\end{pmatrix}$

$/R_Q/ = \sqrt{1.4^2 + 4.2^2 + 5.2^2}$= 6.8m

Method II (differential)

$\frac{d}{dt}|_PR_Q|^2 = 0$

$_PR_Q = \begin{pmatrix}11\\4\\9\end{pmatrix} + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$

$|_PR_Q|^2 = (11-2t)^2 + (4)^2 + (9-t)^2$

$|_PR_Q|^2 = 121 - 44t + 4t^2 + 16 + 81 - 18t + t^2$

$|_PR_Q|^2 = 218 - 62t + 5t^2$

$\frac{d}{dt}|_PR_Q|^2 = -62 + 10t$

$\frac{d}{dt}|_PR_Q|^2 = 0$

$-62 + 10t = 0$

$t = 6.2s$

Minimum Distance d = $|_PR_Q|$

d= $\begin{pmatrix}11-2x6.2\\4\\9-6.2\end{pmatrix} = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$

d= $\sqrt{(-1.4)^2 + (4)^2 + (2.8)^2}$ = 5.08m

3. A particle P moves with a constant velocity $(2i + 3j + 8k)$ passes a point with position vector $6i - 11j + 4k$. At the same instant particle Q passes through a point whose position vector is $i - 2j + 5k$ moving with a constant velocity of $(3i + 4j - 7k)$. Find; **Uneb 2005 No.14**
   i) Position and velocity of Q relative to P at that instant
   ii) The shortest distance between the particles
   iii) Time that elapses before the particles are nearest each other

Solution:

c) $OP = \begin{pmatrix}6\\-11\\4\end{pmatrix}$ Vp = $\begin{pmatrix}2\\3\\8\end{pmatrix}$

$OQ = \begin{pmatrix}1\\-2\\5\end{pmatrix}$ $V_Q = \begin{pmatrix}3\\4\\-7\end{pmatrix}$

$_QR_P = OQ - OP$

$_QR_P = \begin{pmatrix}1\\-2\\5\end{pmatrix} - \begin{pmatrix}6\\-11\\4\end{pmatrix} = \begin{pmatrix}-5\\9\\1\end{pmatrix}$

$_QV_P = V_Q - V_P$

$_QV_P = \begin{pmatrix}3\\4\\-7\end{pmatrix} - \begin{pmatrix}2\\3\\8\end{pmatrix} = \begin{pmatrix}1\\1\\-15\end{pmatrix}$

$_QR_P = (OP - OQ) + (_QV_P)t$

$_QR_P = \begin{pmatrix}-5\\9\\1\end{pmatrix} + \begin{pmatrix}1\\1\\-15\end{pmatrix}t = \begin{pmatrix}-5+t\\9+t\\1-15t\end{pmatrix}$

For minimum distance : $_QV_P . _QR_P = 0$

$\begin{pmatrix}1\\1\\-15\end{pmatrix} . \begin{pmatrix}-5+t\\9+t\\1-15t\end{pmatrix} = 0$

$22t - 11 = 0$

$t = 0.0485\ units$

d) Shortest distance d= $|_QR_P|$, $t = 0.0485$

$_QR_P = \begin{pmatrix}-5+t\\9+t\\1-15t\end{pmatrix} = \begin{pmatrix}-5+0.0485\\9+0.0485\\1-15x0.0485\end{pmatrix} = \begin{pmatrix}-4.9515\\9.0485\\-14.9515\end{pmatrix}$

$|_QR_P| = \sqrt{(-4.9515)^2 + 9.0485^2 + (-14.9515)^2}$

$|_QR_P| = 10.32 units$

4. Two particles P and Q move with a constant velocities $(4i + j - 2k)m/s$ and $(6i + 3k)m/s$ respectively. Initially P is at a point whose position vector $(is\ i - 20j + 21k\ )m$ and Q is at a point whose position vector is $(i + 3k)m$. Find; **Uneb 2009 No.16**
   i) Time for which the distance between P and Q is least
   ii) Distance of P from the origin at the time when the distance between P and Q is least
   iii) Least distance between P and Q **An(2.2s, 28.8m, 24.14m)**

5. At 12 noon the position vectors r and velocity vectors v of two ships A and B are as follows

$r_A = (-9\bar{i} + 6\bar{j})km,$ $V_A = (3\bar{i} + 12\bar{j})kmh^{-1}$

$r_B = (16\bar{i} + 6\bar{j})km,$ $V_B = (-9\bar{i} + 3\bar{j})kmh^{-1}$

   i. Find how far apart the ships are at 12 noon
   ii. Assuming velocities do not change, find the least distance between the ships in the subsequent motion
   iii. Find when their distance of closest approach occurs and the position vectors of A and B at that time

Solution:

a) $OA = \begin{pmatrix}-9\\6\end{pmatrix}$ $V_A = \begin{pmatrix}3\\12\end{pmatrix} kmh^{-1}$ | $OB = \begin{pmatrix}16\\6\end{pmatrix}$ $V_B = \begin{pmatrix}-9\\3\end{pmatrix} kmh^{-1}$

$_AR_B = OA - OB$

$_AR_B = \binom{-9}{6} - \binom{16}{6} = \binom{-25}{0}$

$/_AR_B/ = \sqrt{(-25)^2 + 0^2} = 25\ km$ *apart*

(b) $_AV_B = V_A - V_B$

$_AV_B = \binom{3}{12} - \binom{-9}{3} = \binom{12}{9}$

$_AR_B = (OA - OB) + (_AV_B)t$

$_AR_B = \binom{-25}{0} + \binom{12}{9} t$

For minimum distance $_AV_B . _AR_B = 0$

$\binom{12}{9} \cdot \binom{-25 + 12t}{9t} = 0$

$-300 + 144t + 81t = 0$

$t = \frac{4}{3}$ hours

Shortest distance d=/$_PR_Q$/, $t = \frac{4}{3}$

$_AR_B = \binom{-25 + 12t}{9t} = \binom{-25 + 12x\frac{4}{3}}{9x\frac{4}{3}} = \binom{-9}{12}$

$/_AR_B/ = \sqrt{(-9)^2 + 12^2} = 15km$

(c) It occurs at $\frac{4}{3}x60\ mins + 12{:}00$

$= 80\ mins + 12{:}00 = 1{:}20\ pm$

How far each has travelled

$R_A = OA + V_A xt$

$R_A = \binom{-9}{6} + \binom{3}{12} x \frac{4}{3} = \binom{-5}{22}$

$R_B = OB + V_B xt$

$R_B = \binom{16}{6} + \binom{-9}{3} x \frac{4}{3} = \binom{4}{10}$

6. At a certain time, the position vectors r and velocity vectors v of two ships A and B are as follows

$r_A = (20\bar{j})km,\quad V_A = (9\bar{i} - 2\bar{j})kmh^{-1}$ at 14:00 hours

$r_B = (\bar{i} + 4\bar{j})km,\quad V_B = (4\bar{i} + 8\bar{j})kmh^{-1}$ at 15:00 hours

Assuming velocities do not change. Find

(a) the position vector of A at 15:00 hour

(b) the least distance between A and B in the subsequent motion

(c) time at which this least separation occurs

Solution:

a) $OA = \binom{0}{20}\quad V_A = \binom{9}{-2} kmh^{-1}$ at 14:00 hours

$R_A = OA + V_A xt$

at 15:00 hours $\quad R_A = \binom{0}{20} + \binom{9}{-2} x = \binom{9}{18} km$

ii. $_AV_B = V_A - V_B$

$_AV_B = \binom{9}{-2} - \binom{4}{8} = \binom{5}{-10}$

$_AR_B = (OA - OB) + (_AV_B)t$

$_AR_B = \left[\binom{9}{18} - \binom{1}{4}\right] + \binom{5}{-10} t = \binom{8 + 5t}{14 - 10t}$

For minimum distance

$_AV_B . _AR_B = 0$

$\binom{5}{-10} \cdot \binom{8 + 5t}{14 - 10t} = 0$

$40 + 25t - 140 + 100t = 0$

$t =$ 0.8 hours

Shortest distance d=/$_PR_Q$/, $t = 0.8$

$_AR_B = \binom{8 + 5t}{14 - 10t} = \binom{8 + 5x0.8}{14 - 10x0.8} = \binom{12}{6}$

$/_AR_B/ = \sqrt{(12)^2 + 6^2} = 13.42km$

b. It occurs at $0.8x60\ mins + 15{:}00$

$= 48\ mins + 15{:}00 = 15{:}48\ hours$

7. Initially two ships A and B are 65km apart with B due East of A. A is moving due East at 10km/hr and B due south at 24km/hr. the two ships continue moving with these velocities. Find the least distance between the ships in the subsequent motion and the time taken to the nearest minute for such a situation to occur. **Uneb 1996 No.10**

Solution

But $_AV_B = \binom{10}{24}$

A |——65km——| B

$A = \binom{0}{0}\qquad B = \binom{65}{0}$

$_AR_B = (OA - OB) + {_AV_B} t$

$= \left[\binom{0}{0} - \binom{65}{0}\right] + t\binom{10}{24}$

$_AR_B = \binom{-65 + 10t}{24t}$

For least distance $(_AV_B . _AR_B) = 0$

$\binom{10}{24} \cdot \binom{-65 + 10t}{24t} = 0$

$-650 + 100t + 576t = 0$

$\therefore t = 0.96hrs$

least distance d = /$_AR_B$/

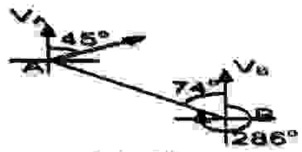
$_AR_B = [(OA)-(OB)] + {_AV_B}\ t$

$_AR_B = \binom{-65 + [10x0.96]}{24x0.96} = \binom{-55.4}{23.04}$

$/_AR_B/ = \sqrt{(-55.4)^2 + 23.04^2} = 60km$

8. At noon a boat A is 30km from boat B and its direction from B is 286°. Boat A is moving in the North east direction at 16km/hr and boat B is moving in the northern direction at 10km/hr. Determine when they are closest to each other. What is the distance between them. **Uneb 1997 No.16**

Solution For relative velocity



$V_A = \begin{pmatrix}16sin45\\16cos45\end{pmatrix}$ $V_B = \begin{pmatrix}0\\10\end{pmatrix}$

${}_AV_B = V_A - V_B$

${}_AV_B = \begin{pmatrix}16sin45\\16cos45\end{pmatrix} - \begin{pmatrix}0\\10\end{pmatrix} = \begin{pmatrix}11.314\\1.314\end{pmatrix}$

$/{}_AV_B/ = \sqrt{11.314^2 + 1.314^2} = 11.39km/hr$

$\alpha = tan^{-1}\left(\frac{1.314}{11.314}\right) = 6.62^0$

$N(90 - 6.62)°E$

Relative velocity is 11.39km/hr at N 83.38°E

SKETCH

$d = ABsin\theta$

$d = 30sin\,(6.62 + 16) = 30sin\,(22.63)$

$d = 11.54km$

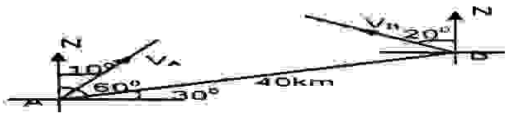
Time $t = \frac{A\,B\,Cos\theta}{/AVB/} = \frac{30\,Cos22.62}{11.39} = 2.43hrs$

Time is 2.43hours from noon or 2 hours and 25.8 minutes

It occurs 2:26pm at a distance of 11.54km

9. Two planes A and B are both flying above the pacific ocean. Plane A is flying on a course of 010° at a speed of 300km/h and plane B is flying on a course of 340° at 200km/h. At a certain instant, plane B is 40 km from plane A. plane A is then on a bearing of 060°. After what time will they come closest together and what will their minimum distance apart **Uneb 2004 No.16**

Solution



$V_A = \begin{pmatrix}300sin10\\300cos10\end{pmatrix}$ $V_B = \begin{pmatrix}-200sin20\\200cos20\end{pmatrix}$

${}_AV_B = V_A - V_B$

${}_AV_B = \begin{pmatrix}300sin10\\300cos10\end{pmatrix} - \begin{pmatrix}-200sin20\\200cos20\end{pmatrix} = \begin{pmatrix}120.4985\\107.5038\end{pmatrix}$

$A = \begin{pmatrix}0\\0\end{pmatrix}$ $B = \begin{pmatrix}40cos30\\40sin30\end{pmatrix}$

${}_AR_B = (OA - OB) + {}_AV_Bt$

$= \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}40cos30\\40sin30\end{pmatrix}\right] + t\begin{pmatrix}120.4985\\107.5038\end{pmatrix}$

${}_AR_B = \begin{pmatrix}-34.641 + 120.4985t\\-20 + 107.5038t\end{pmatrix}$

${}_AV_B \,.\, {}_AR_B = 0$

$\begin{pmatrix}120.4985\\107.5038\end{pmatrix}.\begin{pmatrix}-34.641 + 120.4985t\\-20 + 107.5038t\end{pmatrix} 0$

$-6324.2645 + 26076.9555t = 0$

$t = 0.2425hrs$

least distance d = $/{}_AR_B/$

${}_AR_B = \begin{pmatrix}-34.641 + 120.4985x0.2425\\-20 + 107.5038x0.2425\end{pmatrix}$

$= \begin{pmatrix}5.4202\\6.0692\end{pmatrix}$

$/{}_AR_B/ = \sqrt{(5.4202)^2 + 6.0692^2} = 8.14km$

Exercise 20C

1. A ship A is 8km due North of Ship B, ship A is moving at 150kmh⁻¹ due west while B is moving at 200km/h due N30°W. After what time will they be nearest together and how far apart will they be. **An(2.22km, 0.043hrs)**
2. The point P is 50km west of Q. Two air crafts A and B fly simultaneously from p and q velocities are 400km/h N50°E and 500km/h N20°W respectively. Find;
   (i) The closest distance between the air crafts
   (ii) The time of flight up to this point
   An(20.35km, 5.24 minutes)
3. Ship A steams North-west at 60km/h whereas B steams southwards at 50km/h, initially ship B was 80km due north of A. find:
   (i) The velocity of A relative to B
   (ii) The time taken for the shortest distance to be reached
   (iii) The shortest distance between A and B.
   An(101.675km/h at N24.7°W, 42.9minutes, 33.282km)
4. At 8am, two ships A and B are 11km apart with B due west of A. A and B move with a constant velocities $(-4i + 3j)km/h$ and $(2i + 4j)km/h$ respectively. Find the;

i) Least distance between the two ships in the subsequent motion
ii) Time to the nearest minute, at which this situation occurs **An(1.81km, 9:47am)**

5. At 7:30am, two ships A and B are 8km apart with B due north of A. A and B move with a constant velocities $(12j)km/h$ and $(-5i)km/h$ respectively. Find the;
   i) Least distance between the two ships in the subsequent motion
   ii) Time to the nearest minute, at which this situation occurs **An(3.08km, 8:04pm)**
6. A and B are two tankers at 13;00hours, tanker B has a position vector of $(4i + 8j)km$ relative to A. A and B move with a constant velocities $(6i + 9j)km/h$ and $(-3i + 6j)km/h$ respectively. Find the;
   i) Least distance between the two ships in the subsequent motion
   ii) Time to the nearest minute, at which this situation occurs **An(6.32km, 13:40hours)**
7. At 12 noon the position vectors r and velocity vectors v of two ships A and B are as follows
   $r_A = (5\bar{\imath} + \bar{\jmath})km, \quad V_A = (7\bar{\imath} + 3\bar{\jmath})kmh^{-1}$
   $r_B = (8\bar{\imath} + 7\bar{\jmath})km, \quad V_B = (2\bar{\imath} - \bar{\jmath})kmh^{-1}$
   i. Assuming velocities do not change, find the least distance between the ships in the subsequent motion
   ii. Find when their distance of closest approach occurs **An(2.81km, 12:57pm)**
8. At a certain time, the position vectors r and velocity vectors v of two ships A and B are as follows
   $r_A = (3i + \bar{\jmath})km, V_A = (2\bar{\imath} + 3\bar{\jmath})kmh^{-1}$ at 11:00 am
   $r_B = (2\bar{\imath} - \bar{\jmath})km, V_B = (3\bar{\imath} + 7\bar{\jmath})kmh^{-1}$ at 12:00 noon
   Assuming velocities do not change. Find
   (a) the position vector of A at 12:00 noon
   (b) distance between the ships at 12:00noon
   (c) the least distance between A and B in the subsequent motion
   (d) time at which this least separation occurs
   An($(5i + 4\bar{\jmath})km$, **5.83km, 1.70km, 1:21pm)**
9. At 12 noon the position vectors r and velocity vectors v of battleships B and cruiser C are as follows
   $r_B = (13\bar{\imath} + 5\bar{\jmath})km, \quad V_B = (3\bar{\imath} - 10\bar{\jmath})kmh^{-1}$
   $r_C = (3\bar{\imath} - 5\bar{\jmath})km, \quad V_C = (15\bar{\imath} + 14\bar{\jmath})kmh^{-1}$
   i. Assuming velocities do not change, find the least distance between the ships in the subsequent motion
   ii. The battle ship has guns with a range of up to 5km, find the length of time during which the cruiser is within range of the battleship's guns **An(4.47km, 10minutes)**
10. At a certain time, the position vectors r and velocity vectors v of two ships A and B are as follows
   $r_A = (-2i + 3\bar{\jmath})km, V_A = (12\bar{\imath} - 4\bar{\jmath})kmh^{-1}$ at 11:45 am
   $r_B = (8\bar{\imath} + 7\bar{\jmath})km, V_B = (2\bar{\imath} - 14\bar{\jmath})kmh^{-1}$ at 12:00 noon
   Assuming velocities do not change. Find
   (a) the least distance between A and B in the subsequent motion
   (b) length of time for which A is within range, if ship B has guns within a range of up to 2km
   An(1.41km, 12minutes)
11. At time $t = 0$, the position vectors r and velocity vectors v of two ships A and B are as follows

$$r_A = \begin{pmatrix}1\\2\\3\end{pmatrix}m, \qquad V_A = \begin{pmatrix}-6\\0\\1\end{pmatrix}m/s$$

$$r_B = \begin{pmatrix}4\\-14\\1\end{pmatrix}m, \qquad V_B = \begin{pmatrix}-5\\1\\7\end{pmatrix}m/s$$

   Assuming velocities do not change, find
   (i) The positon vector of B relative to A at time t seconds
   (ii) the least distance between the ships in the subsequent motion
   (iii) Find when their distance of closest approach occurs **An((ii)**$\frac{25}{33}s$**, 15.9m)**
12. At time $t = 0$, the position vectors r and velocity vectors v of two ships A and B are as follows

$$r_A = (\beta)m, \qquad V_A = \begin{pmatrix}2\\1\\-5\end{pmatrix}m/s$$

$$r_B = (2\beta)m, \qquad V_B = \begin{pmatrix}1\\-5\\1\end{pmatrix}m/s$$

Where β is a constant and assuming velocities do not change.
Show that the least distance between the ships in the subsequent motion is $\frac{\beta}{73}$ and their distance of closest approach is $\frac{6\beta\sqrt{2}}{\sqrt{73}}$ $meters$ **An((ii)**$\frac{25}{33}s$**, 15.9m)**

13. A lizard lies in wait at point A, position vector $r_A = \begin{pmatrix} 65 \\ 40 \\ 0 \end{pmatrix} cm$. At tiem $t = 0$ seconds a fly has a position vector r_F and velocity vector V_F given by $r_F = \begin{pmatrix} 37 \\ 16 \\ 22 \end{pmatrix} cm$ and $V_F = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} cm/s$.
If the fly were to continue with this velocity, find the closest distance it would come to the lizard and the value of t when this occurs
An($\sqrt{374}cm$, 7)

14. Two particles A and B move with constant velocities $(\beta i + 3j + 30k)m/s$ and $(4i - 2j - 15k)m/s$ respectively, where β is constant . At time $t = 0$ the position vectors of A and B are $(2i + j - 15k)m$ and $(-i + 4j + 12k)m$ respectively.
   (a) Find the value of β such that A and B will collide and the value of t when this collision occurs
   (b) In the particular case when $\beta = 2$, find the least distance between A and B
   An($\beta = -1, t = 0.6, 1.80m$)

15. With respect to a stationary bird watcher, the position vectors , r and velocity vectors V of the a pigeon and a bird of prey, at time $t = 0$ seconds, were as follows

$$r_{pigeon} = (11\hat{i} + 38\hat{j} + 11\hat{k})m$$
$$V_{pigeon} = (3\hat{i} + 6\hat{j} - \hat{k})m/s$$
$$r_{bird\ of\ prey} = (4\hat{i} - 60\hat{j} + 88\hat{k})m$$
$$V_{bird\ of\ prey} = (4\hat{i} + 20\hat{j} - 12\hat{k})m/s$$

   (a) Assuming the above velocities are maintained, determine the least distance between each bird and the bird watcher, for $t \geq 0$ and value of t for which this least distance occurs
   (b) Again assuming the above velocities are maintained, show that the bird of prey will intercept the pigeon and find the value of t when this would occur
   (c) When $t = 6$ the pigeon suddenly changes its velcotiy to $(6\hat{j} - \hat{k})m/s$ If the bird of prey does not alter its velocity what is the least distance between the birds in the subsequent motion
   An((a) pigeon, $\sqrt{1686}m$, when $t = 0$, bird of prey $20\sqrt{6}m$ when $t = 4$, (b) 7, (c) 2.9m)

16. Two air crafts A and B are flying at the same altitude with velocities 180m/s due east and 240m/s due north respectively. Initially B is 5km due south of A. Given that the aircrafts do not change their velocities, find the shortest distance between the air craft in the subsequent motion and the time taken for such a situation to occur.
An(3km, 13.333s)

17. A road running north-south crosses a road running east-west at a junction O. Initially Paul is on the east west road, 1.7km west of O and is cycling towards O at 15km/h. At the same time John is at O cycling due north at 8km/h. If Paul and John do not alter their velocities, find the least distance they are apart in the subsequent motion and the time taken for that situation to occur **An(800m, 318s)**

18. At 7:30am two ships A and B have velocities $15km/h$, $N30°E$ and 20km/h due east respectively, with B 5km due west of A. In the subsequent motion A and B do not alter their velcoities. Find the distance between A and B when they are closest together and the time at which this situation occurs to the nearest minute
An(3.6km, 7:42am)

19. A road running north south crosses a road running east west at a junction O. Peter cycles towards O from the west at 3m/s as Tom cycles towards O from the south at 4m/s. Initially Peter is 600m from O and Tom is 250m from O.
   (a) If Tom and Peter do not alter their velocities, find the least distance they are apart during the motion and the time take to reach that situation.
   (b) How far and in what direction are Tom and Peter then from O **An(330m, 112s, 198m north 264m west)**

20. Two air crafts A and B are flying, at the same altitude, with velocity 200m/s, $N30°E$ and $300m/s, N50°W$ respectively. Initially A and B are 2km apart with B on a bearing $S70°E$ from A. Given that A and B do not alter their velcoities, find the least distance of separation between the two air craft in the subsequent motion and the time taken to reach such a situation **An(571m, 5.8s)**

21. At 15:00 hours a trawler is 10km due east of a launch. The trawler maintains a steady 10km/h on a bearing 180° and the launch maintains a steady 20km/h on a bearing 071°
   (a) Find the minimum distance the boats are apart in the subsequent motion, and the time at which this occurs

(b) Find to the nearest minute, the length of time for which the two boats ae within 8km of each other.
An(6.58km, 13:18 hours, 22 minutes)

22. At 12 noon a ship A moving with a constant velocity of $20.4kmh^{-1}$ in the direction $N\theta°E$, where $tan\theta = \frac{1}{5}$. A second ship B is $15km$ due north of A. Ship B is moving with constant velocity of 5 kmh^{-1} in the direction $S\alpha°W$, where $tan\alpha = \frac{3}{4}$. If the shortest distance between the ship is $4.2km$, find the time to the nearest minute when the distance between the ship is shortest. **UNEB 2017 No. 16 An(0.376h, 12:35 pm,)**

23. At 10:00 am, ship A and ship B are 16 km apart. Ship A is on a bearing $N35^0E$ from ship B. Ship A is travelling at 14 kmh^{-1} on a bearing $S29^0E$. Ship B is travelling at 17 kmh^{-1} on a bearing $N50^0E$. Determine the; **UNEB 2015 No. 16**
(a) velocity of ship B relative to ship A.
(b) closest distance between the two ships and the time when it occurs.
An(23.9964km/h E74.9°N, 10.38Am, 5.4562km)

COURSE OF CLOSEST APPROACH

If A is to pass as close as possible to B, then velocity of A must be perpendicular to the relative velocity ${}_AV_B.V_A = 0$

Examples

1. Two particles P and Q initially at positions $(3i + 2j)m$ and $(13i + 2j\ )m$ respectively begin moving. Particle P moves with a constant velocity $(2i + 6j)$m/s. A second particle Q moves with a constant velocity of $(5j)$m/s . **UNEB 2003 No.16**
   (a) Find;
      i) The time when the particles are closest together
      ii) Bearing of particle P from Q when they are closest to each other
   (b) Given that half the time, the two particles are moving closest to each other, particle P reduces its speed to half its original speed, in the direction to approach particle Q and the velocity of Q remains unchanged, find the direction of particle P

Solution:

a) $OP = \begin{pmatrix}3\\2\end{pmatrix}$ $\quad V_P = \begin{pmatrix}2\\6\end{pmatrix}ms^{-1}$

$OQ = \begin{pmatrix}13\\2\end{pmatrix}$ $\quad V_Q = \begin{pmatrix}0\\5\end{pmatrix}ms^{-1}$

${}_PV_Q = V_P - V_Q$

${}_PV_Q = \begin{pmatrix}2\\6\end{pmatrix} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

${}_PR_Q = (OP - OQ) + ({}_PV_Q)t$

${}_PR_Q = \left[\begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}13\\2\end{pmatrix}\right] + \begin{pmatrix}2\\1\end{pmatrix}t = \begin{pmatrix}-10+2t\\t\end{pmatrix}$

For minimum distance

${}_PV_Q.\,{}_PR_Q = 0$

$\begin{pmatrix}2\\1\end{pmatrix}.\begin{pmatrix}-10+2t\\t\end{pmatrix} = 0$

$-20 + 4t + t = 0$

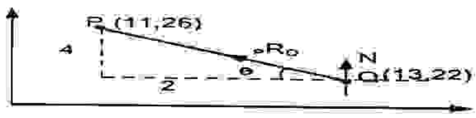
$t = 4\,s$

$R_P = OP + V_Pxt$

$R_P = \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}2\\6\end{pmatrix}x\,4 = \begin{pmatrix}11\\26\end{pmatrix}$

$R_Q = OQ + V_Qxt$

$R_Q = \begin{pmatrix}13\\2\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix}x\,4 = \begin{pmatrix}13\\22\end{pmatrix}$



$tan\theta = \frac{4}{2}$ $\quad \theta = 63.4°$

$N26.6°W$

(c) At $t = 2s$ $V_P = \begin{pmatrix}1\\3\end{pmatrix}ms^{-1}$

$/V_P/ = \sqrt{1^2+3^2} = \sqrt{10}m/s$

Let P move at angle θ to x-axis

${}_PV_Q = \sqrt{10}\begin{pmatrix}cos\theta\\sin\theta\end{pmatrix} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}\sqrt{10}cos\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}$

If P is to approach Q, then

${}_PV_Q.V_P = 0$

$\begin{pmatrix}\sqrt{10}cos\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}.\begin{pmatrix}\sqrt{10}cos\theta\\\sqrt{10}sin\theta\end{pmatrix} = 0$

$10cos^2\theta + 10sin^2\theta - 5\sqrt{10}sin\theta = 0$

$sin\theta = \frac{10}{5\sqrt{10}}$ $\quad \theta = 39.2°$

$N50.8°E$

2. A motor boat B is travelling at a constant velocity of 10m/s due east and motor boat A is traveling at a constant speed of 8m/s. Initially A and B are 600m apart with A due south of B. find ;

(i) Course that A should set in order to get close as possible to B

(ii) Closest distance and time taken for the situation to occur

Solution

(i) Let A move at angle θ to x-axis

$$_AV_B = \begin{pmatrix}8cos\theta\\8sin\theta\end{pmatrix} - \begin{pmatrix}10\\0\end{pmatrix} = \begin{pmatrix}8cos\theta - 10\\8sin\theta\end{pmatrix}$$

If A is to approach B, then

$$_AV_B.V_A = 0$$

$$\begin{pmatrix}8cos\theta - 10\\8sin\theta\end{pmatrix}.\begin{pmatrix}8cos\theta\\8sin\theta\end{pmatrix} = 0$$

$$64cos^2\theta + 64sin^2\theta - 80cos\theta = 0$$

$$cos\theta = \frac{64}{80} \qquad \theta = 36.9°$$

$$N53.1°E$$

(ii) $OA = \begin{pmatrix}0\\0\end{pmatrix} \qquad OB = \begin{pmatrix}0\\600\end{pmatrix}$

$$_AR_B = (OA - OB) + {_AV_B}t$$

$$= \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}0\\600\end{pmatrix}\right] + t\begin{pmatrix}8cos36.9 - 10\\8sin36.9\end{pmatrix}$$

$$_AR_B = \begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix}$$

$$_AV_B . {_AR_B} = 0$$

$$\begin{pmatrix}-3.603\\4.803\end{pmatrix}.\begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix} = 0$$

$$12.9816t - 2881.8 + 23.0688t = 0$$

$$t = \frac{2881.8}{36.0504} \quad \therefore \ t = 80s$$

least distance d = $/{_AR_B}/$

$$_AR_B = \begin{pmatrix}-3.603x80\\-600 + 4.803x80\end{pmatrix} = \begin{pmatrix}-288.24\\-215.76\end{pmatrix}$$

$$/{_AR_B}/ = \sqrt{(-288.24)^2 + (-215.76)^2} = 360m$$

3. A motor boat B is travelling at a constant velocity of 13km/h due north and motor boat A is traveling at a constant speed of 12km/h. Initially A and B are 5.2km apart with A due west of B. find ;

(i) Course that A should set in order to get as close as possible to B

(ii) Closest distance and time taken for the situation to occur

Solution

(i) Let A move at angle θ to x-axis

$$_AV_B = \begin{pmatrix}12cos\theta\\12\ sin\theta\end{pmatrix} - \begin{pmatrix}0\\13\end{pmatrix} = \begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}$$

If A is to approach B, then

$$_AV_B.V_A = 0$$

$$\begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}.\begin{pmatrix}12cos\theta\\12sin\theta\end{pmatrix} = 0$$

$$144cos^2\theta + 144sin^2\theta - 156sin\theta = 0$$

$$sin\theta = \frac{12}{13} \qquad \theta = 67.4°$$

$$N22.6°E$$

(ii) $OA = \begin{pmatrix}0\\0\end{pmatrix} \qquad OB = \begin{pmatrix}52\\0\end{pmatrix}$

$$_AR_B = (OA - OB) + {_AV_B}t$$

$$= \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}52\\0\end{pmatrix}\right] + t\begin{pmatrix}12cos67.4\\12sin67.4 - 13\end{pmatrix}$$

$$_AR_B = \begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix}$$

$$_AV_B . {_AR_B} = 0$$

$$\begin{pmatrix}4.612\\-1.921\end{pmatrix}.\begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix} = 0$$

$$3.6902t - 23.9824 + 21.271t = 0$$

$$t = \frac{23.9824}{24.9612} \quad \therefore \ t = 0.961h$$

least distance d = $/{_AR_B}/$

$$_AR_B = \begin{pmatrix}-5.2 + 4.612x0.96\\-1.921x0.96\end{pmatrix} = \begin{pmatrix}-0.7725\\-1.8442\end{pmatrix}$$

$$/{_AR_B}/ = \sqrt{(-0.7725)^2 + (-1.8442)^2} = 2km$$

4. Two air craft A and B are flying at the same altitude with A initially 10km due north of B and flying at a constant speed of 300m/s on a bearing of 060°. If B flies at a constant speed of 200m/s. find ;

(i) Course that B should set in order to get as close as possible to A

(ii) Closest distance and time taken for the situation to occur

Solution

(i) Let B move at angle θ to x-axis

$$_BV_A = \begin{pmatrix}200cos\theta\\200\ sin\theta\end{pmatrix} - \begin{pmatrix}300sin60\\300cos60\end{pmatrix} = \begin{pmatrix}200cos\theta - 150\sqrt{3}\\200sin\theta - 150\end{pmatrix}$$

If A is to approach B, then

$$_BV_A.V_B = 0$$

$$\begin{pmatrix}200cos\theta - 150\sqrt{3}\\200sin\theta - 150\end{pmatrix}.\begin{pmatrix}200cos\theta\\200sin\theta\end{pmatrix} = 0$$

$$40000cos^2\theta + 40000sin^2\theta - 30000\sqrt{3}\,cos\theta - 30000\,sin\theta = 0$$

$$3sin\theta + 3\sqrt{3}\,cos\theta = 4$$

Using $Rsin(\theta + \alpha) = 4$

$Rsin\alpha = 3\sqrt{3} \ldots (i)$
$Rcos\alpha = 3 \ldots (ii)$

$$\alpha = tan^{-1}\left(\frac{3\sqrt{3}}{3}\right) = 60^\circ$$

$Rsin(\theta + \alpha) = 4$
$6sin(\theta + 60^\circ) = 4$
$\theta + 60^\circ = 41.8^\circ$
$\theta = -11.8^\circ$
$N78.2^\circ E$

(ii) $OA = \begin{pmatrix} 0 \\ 10000 \end{pmatrix}$ $OB = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

${}_BR_A = (OB - OA) + {}_BV_A t$

$$= \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 10000 \end{pmatrix}\right] + t\begin{pmatrix} 200cos(-11.8) - 150\sqrt{3} \\ 200sin(-11.8) - 150 \end{pmatrix}$$

$${}_BR_A = \begin{pmatrix} -64.034t \\ 10000 - 190.8992t \end{pmatrix}$$

${}_BV_A \cdot {}_BR_A = 0$

$$\begin{pmatrix} -64.034 \\ -190.8992 \end{pmatrix} \cdot \begin{pmatrix} -64.034t \\ 10000 - 190.8992t \end{pmatrix} = 0$$

$4100.353t - 1908992 + 36438.6867t = 0$

$$t = \frac{1908992}{40539.03968} \quad \therefore \ t = 47.1s$$

least distance d = $/{}_AR_B/$

$${}_AR_B = \begin{pmatrix} -64.034x47.1 \\ 10000 - 190.8992x47.1 \end{pmatrix} = \begin{pmatrix} -3,016.0014 \\ 1008.6477 \end{pmatrix}$$

$$/{}_AR_B/ = \sqrt{(-3,016.0014)^2 + (1008.64772)^2} = 3.18km$$

INTERCEPTION AND COLLISIONS

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB respectively.

For collision to occur $R_{At} = R_{Bt}$

Position of A after time t is

$R_{At} = OA + t \ x \ V_A$

Position of B after time t is

$R_{Bt} = OB + t \ x \ V_B$

$OA + t \ x \ V_A = OB + t \ x \ V_B$

$(OA - OB) + t({}_AV_B) = 0$

${}_AR_{Bt} = 0$

SHOWING THAT PARTICLES COLLIDE

Equate the corresponding unit vector form both side to show that y is constant in both directions

For vectors in three dimensions at least any two unit vectors be constant

Examples

1. The position vectors $r_A = (5\bar{i} - 3\bar{j} + 4\bar{k})m$ and $r_B = (7\bar{i} + 5\bar{j} - 2\bar{k})m$ are for two particles with velocities $v_A = (2\bar{i} + 5\bar{j} + 3\bar{k})m/s$ and $v_B = (-3\bar{i} - 15\bar{j} + 18\bar{k})m/s$ respectively. Show that if the velocities remain constant, a collision will occur

Solution:

(i) $OA + t \ x \ V_A = OB + t \ x \ V_B$

$$\begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} t = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -15 \\ 18 \end{pmatrix} t$$

$$\begin{pmatrix} -2 \\ -8 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ -20 \\ 15 \end{pmatrix} t$$

Along the i direction; $-2 = -5t$

$t = 0.4s$

Along the j direction; $-8 = -20t$

$t = 0.4h$

Along the k direction; $6 = 15t$

$t = 0.4h$

Since t is constant in all directions collision occurred

2. At 12 noon the position vectors r and velocity vectors v of two ships A and B are as follows

$r_A = (\bar{i} + 7\bar{j})km,$ $V_A = (6\bar{i} + 2\bar{j})kmh^{-1}$

$r_B = (6\bar{i} + 4\bar{j})km,$ $V_B = (-4\bar{i} + 8\bar{j})kmh^{-1}$

Assuming velocities do not change,

(i) Show that collision will occur

(ii) Find the time at which collision occur

(iii) Find the position vector of the location during collision

Solution:

(i) $OA + t \ x \ V_A = OB + t \ x \ V_B$

$$\begin{pmatrix} 1 \\ 7 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} t = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ 8 \end{pmatrix} t$$

$$\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \end{pmatrix} t$$

Along the i direction; $-5 = -10t$

$t = 0.5h$

Along the j direction; $3 = 6t$

$t = 0.5h$

Since t is constant in all directions collision occurred

(ii) It occurs at $0.5x60\ mins + 12:00$
$= 30\ mins + 12:00$
$12:30\ pm$

How far each has travelled

(iii) $R_A = OA + V_A xt$

$R_A = \binom{1}{7} + \binom{6}{2}x0.5 = \binom{4}{8}$

3. At 11:30am a battle ship is at a place with position vectors $(-6\bar{i} + 12\bar{j})km$ and is moving with velocity vector$(16\bar{i} - 4\bar{j})km/h$. At 12:00 noon a cruiser is at a place with position vectors $(12\bar{i} - 15\bar{j})km$ and is moving with velocity vector $(8\bar{i} + 16\bar{j})km/h$ Assuming velocities do not change.
   (i) Show that collision will occur
   (ii) Find the time at which collision occur
   (iii) Find the position vector of the location during collision

Solution:

(i) At 12:00 R_A : $= OA + 0.5\ x\ V_A + tx\ V_A$

$= \binom{-6}{12} + \binom{16}{-4}x0.5 + \binom{16}{-4}t = \binom{2}{10} + \binom{16}{-4}t$

$OA + t\ x\ V_A = OB + t\ x\ V_B$

$\binom{2}{10} + \binom{16}{-4}t = \binom{12}{-15} + \binom{8}{16}t$

$\binom{-10}{25} = \binom{-8}{20}t$

Along the i direction; $-10 = -8t$

$t = 1.25h$

Along the j direction; $25 = 20t$

$t = 1.25h$

Since t is constant in all directions collision occurred

(ii) It occurs at $1.25x60\ mins + 12:00$
$= 75\ mins + 12:00$
$1:15\ pm$

How far each has travelled

(iii) $R_{AB} = OA + V_B xt$

$R_B = \binom{12}{-15} + \binom{8}{16}x1.25 = \binom{22}{5}$

4. The position vectors of two particles are $r_1 = (4i - 2j)t + (3i + j)t^2$ and $r_2 = (10i + 4j) + (5i - 2j)t$ respectively. Show that the two particles will collide. Find their speed at the time of collision.

Uneb 2002 No.15b

Solution

$r_1 = r_2$

$\binom{4}{-2}t + \binom{3}{1}t^2 = \binom{10}{4} + \binom{5}{-2}t$

Along the i direction ; $4t + 3t^2 = 10 + 5t$

$3t^2 - t - 10 = 0$

$t = 2$ or $t = -\frac{5}{3}$

Along the j direction ; $-2t + t^2 = 4 - 2t$

$t^2 = 4$

$t = \pm 2$

Particles collide when $t = 2\ units$

$v = \frac{dr}{dt}$

$v_1 = \frac{d}{dt}[(4i - 2j)t + (3i + j)t^2]$

$v_1 = (4\bar{i} - 2\bar{j}) + (6\bar{i} + 2\bar{j})t$

At $t = 2$: $v_1 = (4\bar{i} - 2\bar{j}) + (6\bar{i} + 2\bar{j})x2$

$v_1 = (16\bar{i} + 2\bar{j})$

$speed, v_1 = \sqrt{(16)^2 + (2)^2} = \sqrt{260}$

$v_2 = \frac{d}{dt}[(10i + 4j) + (5i - 2j)t]$

$v_2 = (5\bar{i} - 2\bar{j})$

$speed, v_2 = \sqrt{(5)^2 + (-2)^2} = \sqrt{29}$

5. At 12:00 noon two ships A and B are 10km apart with B due east of A. A is travelling $N60°E$ at a speed of 20km/h and ship B is travelling due north at 10 km/h. Show that, if the two ships do not change their velocities, they collide and find to the nearest minute when the collision occurs

Solution

$OA + t\ x\ V_A = OB + t\ x\ V_B$

$\binom{0}{0} + \binom{20sin60}{20cos60}t = \binom{10}{0} + \binom{0}{10}t$

$\binom{10\sqrt{3}}{10}t = \binom{10}{0} + \binom{0}{10}t$

Along the i direction ; $10\sqrt{3}t = 10$

$t = 0.5774h$

Along the j direction; $10t = 10t$

$\therefore t = 34.6minutes$

6. At 11pm two ships A and B are 10km apart with B due north of A. A is travelling north east at a speed of 18km/h and ship B is travelling due east at $9\sqrt{2}\ km/h$. Show that, if the two ships do not change their velocities, they collide and find to the nearest minute when the collision occurs

Solution

(i) Show that ship A and B will collide and find the when and where the collision occurs
(ii) Find the position vector of C when A and B collide and find how far C is from the collision
(iii) When the collision occurs, C immediately changes its course but not its speed and streams direct to the scene. When does C arrive

5. At 12noon the position vectors r and velocity vectors v of three ships A, B and C are as follows

$$r_A = (10.5\hat{i} + 6\hat{j})km, \qquad V_A = (9\hat{i} + 18\hat{j})kmh^{-1}$$
$$r_B = (7\hat{i} + 20\hat{j})km, \qquad V_B = (12\hat{i} + 6\hat{j})kmh^{-1}$$
$$r_C = (10\hat{i} + 15\hat{j})km, \qquad V_C = (6\hat{i} + 12\hat{j})kmh^{-1}$$

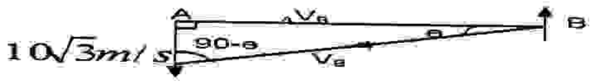
Assuming velocities do not change,
(i) Show that ship A and B will collide and find the when and where the collision occurs
(ii) When the collision occurs, C immediately changes its course but not its speed and streams direct to the scene. When does C arrive

COURSE OF INTERCEPTION

Suppose particle A moving with a speed V_A is to intercept particle B moving with a speed V_B, then;
- Draw a sketch a diagram showing the initial position and velocities of the two particles
- For interception to occur, the relative velocity must be in the direction if the initial displacements of the particles

$$t = \frac{AB}{AV_B}$$

1. At any instant a body A travelling south at $10\sqrt{3}m/s$ is 150m west of B. Show that B will intercept A if B is travelling S30°W at 20m/s and find the time that elapses before the collision occurs
Solution



$$\frac{\sin 90}{20} = \frac{\sin\theta}{10\sqrt{3}}$$
$$\theta = 60°$$

Bearing $S30°W$

Also $\frac{\sin 90}{20} = \frac{\sin(90-\theta)}{AV_B}$

$$AV_B = 20x\sin(90-60) = 10m/s$$
$$t = \frac{AB}{AV_B} = \frac{150}{10} = 15s$$

2. At 12:00 noon two ships A and B are 12km apart with B on a bearing of 140° from A. ship A moves at 30km/h to intercept B which is travelling at 20km/h on a bearing of 340°. Find the;
(i) Direction A should set order to intercept B (ii) Time taken for the interception to occur
Solution

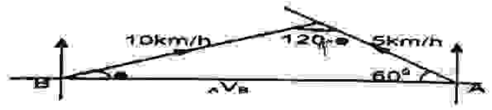
$$\frac{\sin 20}{30} = \frac{\sin(\theta)}{20}$$

$$\theta = 13.2°$$
Bearing $E\ (50-13.2)S$
$E36.8°S$ or $S53.2°E$

Also; $\frac{\sin 20}{30} = \frac{\sin(180-[20+13.2])}{AV_B}$

$$AV_B = 48.03km/h$$
$$t = \frac{AB}{AV_B} = \frac{12}{48.03} = 0.250h = 15minutes$$

3. At 9:00 am two ships A and B are 15km apart with B on a bearing of 270° from A. ship A moves at 50km/h on a bearing of 330°. If the maximum speed of B is 10km/h, Find the;
(i) Direction B should set order to intercept A as soon as possible
(ii) Time taken for the interception to occur
Solution



$$\frac{\sin 60}{10} = \frac{\sin(\theta)}{5}$$

$$\theta = 25.7°$$
Bearing $N\ (90-25.7)E = N64.3°E$

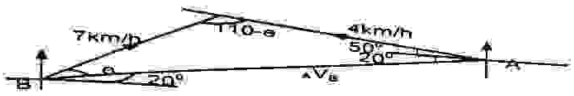
Also; $\frac{\sin 60}{10} = \frac{\sin(120-\theta)}{AV_B}$

$$AV_B = 11.515km/h$$

$$t = \frac{AB}{AV_B} = \frac{15}{11.515} = 1.303h = 78minutes$$

4. At 12:00 noon two ships A and B are 12km apart with B on a bearing of 250° from A. ship A moves at 4km/h on a bearing of 320°. If the maximum speed of B is 7km/h. Find the;
   (i) Direction B should set order to intercept A as soon as possible
   (ii) Time taken for the interception to occur

Solution



$$\frac{sin70}{7} = \frac{sin(\theta)}{4}$$
$$\theta = 32.5°$$

Bearing $N\ (90 - 52.5)E = N37.5°E$

Also $\frac{sin70}{7} = \frac{sin(110-\theta)}{AV_B}$

$$AV_B = 7.273km/h$$
$$t = \frac{AB}{AV_B} = \frac{12}{7.273} = 1.65h = 99minutes$$

Exercise 20E

1. In gulf water, a battleship steaming at 16km/h is 5km southwest of a submarine. Find the course which the submarine should set in order to intercept the battleship, if its speed is 12km/h. **An(** $N15°W$**)**
2. A boy hits a ball at 15m/s in a direction S80°W. A girl 45m and S65OW from the boy runs at 6m/s to intercept the ball. Assuming the velocities remain constant, find in what direction the girl must run to intercept the ball as quickly as possible and how long does it takes her. **An**$(N24.7°E, 2.36s)$
3. A helicopter sets off from its base and flies at 50m/s to intercept a ship which, when the helicopter sets off, is at a distance of 5km on a bearing 335° from the base. The ship is travelling at 10m/s on a bearing 095°. Find the course that the helicopter pilot should set if he is to intercept the ship as quickly as possible and the time interval between the helicopter taking off and it reaching the ship **An**$(N15°W, 92.2s)$
4. A life boast sets out from a harbor at 9:10pm to go to the assistance of a yatch which is, at the time, 5km due south of the harbor and drifting due west at 8km/h. If the life boat travels at 20km/h, find;
   (i) Course the life boat should set so as to reach the yatch as quickly as possible
   (ii) Time when the life boat arrives
   An$(S23.6°W, 9{:}17pm)$
5. A coast guard vessel wishes to intercept a yatch suspected of smuggling. At 1am the yatch is 10km due east of the coast guard vessel and is travelling due north at 15km/h. If the coast guard vessel travels at 20km/h,
   (i) in what direction should it steer in order to intercept the yatch
   (ii) when would this interception occur
   An$(N41.4°E, 1{:}45am)$
6. The driver of a speed boat travelling at 75km/h wishes to intercept a yatche travelling at 20km/h in direction $N40°E$. Initially the speed boat is positioned 10km from the yatch in a bearing $S30°E$. Find;
   (i) Course the speed boat should set so as to reach the yatch as quickly as possible
   (ii) Time when the interception occurs
   An$(N15.5°W, 9minutes\ and\ 7seconds)$
7. A batsman hits a ball at 15m/s in a direction $S80°W$ A fielder , 45m and $S65°W$ form the batmans, runs at 6m/s to intercet the ball. Assuming the velocities remain unchanged,
   (i) find in what direction the fielder must run to intercept the ball as quickly as possible
   (ii) How long did it take him,
   An$(N24.7°E, 2.4s)$

CHAPTER 11: CENTRE OF GRAVITY

This is the point where the resultant force due to attraction acts

General formula for COG

Consider a system of particle of weight $W_1, W_2, \ldots\ldots\ldots\ldots W_n$ located at the points with coordinates $x_1 y_1,\ x_2 y_2, \ldots\ldots\ldots\ldots x_n y_n$ in the $x - y$ plane.

The resultant of weight $W_1 + W_2 + \ldots\ldots\ldots + W_n$ have a C.O.G at a point G$(\bar{x}, \bar{y})$

Taking moments along the y-axis

$$(W_1 + W_2 + \ldots\ldots\ldots + W_n)\bar{x} = W_1 x_1 + W_2 x_2 + \ldots\ldots\ldots + W_n x_n$$

$$\bar{x} = \frac{\sum W_i x_i}{\sum W_i}$$

Similarly: Taking moments along the X-axis : $\bar{y} = \frac{\sum W_i y_i}{\sum W_i}$

Alternatively : $$\binom{\bar{x}}{\bar{y}} = \frac{\left(W_1\binom{x_1}{y_1} + W_2\binom{x_2}{y_2} + \ldots\ldots\ldots + W_n\binom{x_n}{y_n}\right)}{(W_1 + W_2 + \ldots\ldots\ldots + W_n)}$$

If the masses are given then use

$$\binom{\bar{x}}{\bar{y}} = \frac{\left(m_1 g\binom{x_1}{y_1} + m_2 g\binom{x_2}{y_2} + \cdots\ldots\ldots\ldots\ m_n g\binom{x_n}{y_n}\right)}{(m_1 g + m_2 g + \cdots\ldots\ldots\ldots\ m_n g)}$$

Note :

(i) If all particles in question lie along the x-axis, the y coordinates of the center of gravity of the system of force is zero ie $\bar{y} = 0$

(ii) If all particles in question lie along the y-axis, the x coordinates of the center of gravity of the system of force is zero ie $\bar{x} = 0$

Examples

1. Find the position of the center of gravity of three particles of masses 1kg, 5kg, 2kg which lie on the y-axis at the points (0,2), (0,4) and (0,5) respectively . **UNEB 1996 NO.9(a)**

Solution

$$\binom{\bar{x}}{\bar{y}} = \frac{1g\binom{0}{2} + 5g\binom{0}{4} + 2g\binom{0}{5}}{(1g + 5g + 2g)}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\binom{0}{2} + \binom{0}{20} + \binom{0}{10}}{8}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\binom{0}{32}}{8} = \binom{0}{4}$$

2. Find the co-ordinates of the center of gravity of four particles of masses 5kg, 2kg, 2kg and 3kg which are situated at (3,1), (4,3), (5,2) and (-3,1) respectively

Solution

$$\binom{\bar{x}}{\bar{y}} = \frac{5g\binom{3}{1} + 2g\binom{4}{3} + 2g\binom{5}{2} + 3g\binom{-3}{1}}{(5g + 2g + 2g + 3g)}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\binom{24}{18}}{12}$$

$$(\bar{x}, \bar{y}) = (2, 1.5)$$

3. Three particles of masses 2kg, 1kg and 3kg are situated at (4,3), (1,0) and (a,b) respectively. If the centre of gravity of the system lies at (0,2), find the values of a and b

Solution

$$\binom{0}{2} = \frac{2g\binom{4}{3} + 1g\binom{1}{0} + 3g\binom{a}{b}}{(2g + 1g + 3g)}$$

$$\binom{0}{2} = \frac{\binom{9 + 3a}{6 + 3b}}{6}$$

$$0 = \frac{9 + 3a}{6}$$

$$a = -3$$

$$2 = \frac{6 + 3b}{6}$$

$$a = 2$$

4. Find the co-ordinates of the center of gravity of four particles of masses 50kg, 60kg, 20kg and 20kg which are situated at $(6\bar{\imath} + 6\bar{\jmath})$, $(3\bar{\imath} + 5\bar{\jmath})$, $(7\bar{\imath} + 3\bar{\jmath})$, and $(2\bar{\imath} - \bar{\jmath})$ respectively

Solution

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{50g\begin{pmatrix}6\\6\end{pmatrix} + 60g\begin{pmatrix}3\\5\end{pmatrix} + 20g\begin{pmatrix}7\\3\end{pmatrix} + 20g\begin{pmatrix}2\\-1\end{pmatrix}}{(50g + 60g + 20g + 20g)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\begin{pmatrix}660\\760\end{pmatrix}}{150}$$

$(\bar{x}, \bar{y}) = (4.4, 5.07)$

5. Find the co-ordinates of the center of gravity of four particles of weights 6N, 5N, 4N and 7N which are situated at $(\bar{\imath} + 2\bar{\jmath})$, $(2\bar{\imath})$, $(3\bar{\jmath})$, and $(4\bar{\imath} + 2\bar{\jmath})$ respectively

Solution

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{6\begin{pmatrix}1\\2\end{pmatrix} + 5\begin{pmatrix}2\\0\end{pmatrix} + 4\begin{pmatrix}0\\3\end{pmatrix} + 7\begin{pmatrix}4\\2\end{pmatrix}}{(6+5+4+7)} = \frac{\begin{pmatrix}44\\38\end{pmatrix}}{22}$$

$(\bar{x}, \bar{y}) = (2, 1.73)$

6. The rectangle $EFGH$ has $EF = 3m$ and $EH = 2m$, particles of masses 20g, 30g, 60g and 10g are placed at the mid-point of the sides EF, FG, GH, and EH respectively. Find the distance of the centre of gravity of the system from each of the line EF and EH.

Solution

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{0.02\begin{pmatrix}1.5\\0\end{pmatrix} + 0.03\begin{pmatrix}3\\1\end{pmatrix} + 0.06\begin{pmatrix}1.5\\2\end{pmatrix} + 0.01\begin{pmatrix}0\\1\end{pmatrix}}{(0.02 + 0.03 + 0.06 + 0.01)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\begin{pmatrix}0.21\\0.16\end{pmatrix}}{0.12}$$

$(\bar{x}, \bar{y}) = (1.75, 1.33)$

Exercise 21A

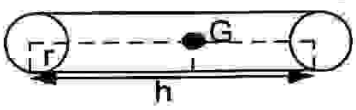
1. Find the co-ordinates of the center of gravity of four particles of masses 60g, 30g, 70g and 40g which are situated at $(4\bar{\imath} + 3\bar{\jmath})$, $(6\bar{\imath} + 5\bar{\jmath})$, $(-6\bar{\imath} + 5\bar{\jmath})$, and $(-5\bar{\imath} - 2\bar{\jmath})$ respectively. **An**$(-1,3)$
2. The rectangle $ABCD$ has $AB = 4cm$ and $AD = 2cm$, particles of masses 3kg, 5kg, 1kg and 7kg are placed at the points A, B, C, and D respectively. Find the distance of the centre of gravity of the system from each of the line AB and AD. **An**$(1cm, 1.5cm)$
3. Find the position of the center of gravity of four particles of masses 5kg, 6kg, 2kg and 2kg which are situated at $(5\bar{\imath} - 7\bar{\jmath})$, $(-3\bar{\imath} + 2\bar{\jmath})$, $(3\bar{\imath} - 5\bar{\jmath})$, and $(\bar{\imath} - 6\bar{\jmath})$ respectively. **An**$(\bar{\imath} - 3\bar{\jmath})$
4. Particles of weight 1N, 2N, 3N and 4N which are situated at $(6\bar{\imath})$, $(\bar{\imath} - 5\bar{\jmath})$, $(3\bar{\imath} + 2\bar{\jmath})$, and $(a\bar{\imath} + b\bar{\jmath})$ respectively. If the centre of gravity of this system lies at the points with position vector $(2.5\bar{\imath} - 2\bar{\jmath})$. Find the value of a and b. **An**$(2, -4)$
5. Find the position of the center of gravity of four particles of weight 2N, 1N, 5N and 2N which are situated at $(4, -5)$, $(1,2)$, $(3, -6)$, and $(0,3)$ respectively. **An**$(2.4, -3.2)$
6. Particles of mass 1kg, 2kg, and mkg which are situated at $(5,2)$, $(1,5)$, $(1, -2)$, respectively. If the centre of gravity of this system lies at $(2, \bar{y})$. Find the value of m and $\bar{y}$ **An**$(m = 1kg, \bar{y} = 2.5)$
7. Particles of mass 2kg, 1kg, and 3kg which are situated on y-axis at $(0,7)$, $(0,4)$, $(0, -2)$, respectively. Where must a 6kg mass be placed to ensure that the centre of gravity of this system lies at the origin. **An**$(0, -2)$
8. Particles of weight 5N, 4N, and 3N which are situated at $(-5,0)$, $(4,0.5)$, $(-4, -3)$, respectively. Where must a 7N particle be placed to ensure that the centre of gravity of this system lies at the origin. **An**$(3, 1)$

Centre of gravity of some laminas and solids

1 C.O.G of a rectangle

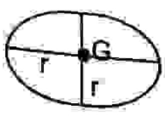
$G\left(\frac{AB}{2}, \frac{AD}{2}\right)$

2 C.O.G of a Solid Cylinder



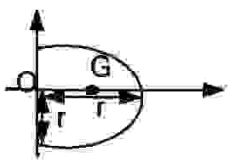
$G\left(r, \frac{h}{2}\right)$

3 C.O.G of a circle



$G(r, r)$

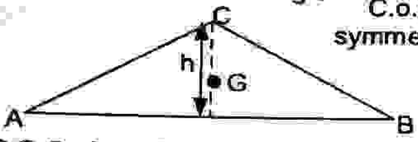
4 C.O.G of semi-circle



C.o.g lies along the lien of symmetry at a distance $\frac{4r}{3\pi}$ from straight edge O

$G\left(\frac{4r}{3\pi}, r\right)$

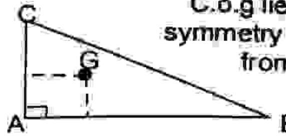
5 C.O.G of isoscele triangle



C.o.g lies along the line of symmetry at a distance $\frac{h}{3}$ from straight edge AB

$G\left(\frac{AB}{2}, \frac{h}{3}\right)$

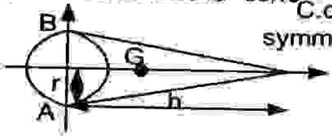
6 C.O.G of right angled triangle



C.o.g lies along the line of symmetry at a distance $\frac{AB}{3}$ and $\frac{AC}{3}$ from right angles

$G\left(\frac{AB}{3}, \frac{AC}{3}\right)$

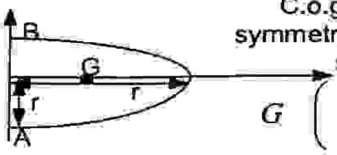
7 C.O.G of a solid cone



C.o.g lies along the line of symmetry at a distance $\frac{h}{4}$ from straight edge AB

$G\left(\frac{h}{4}, r\right)$

8C.O.G of a solid hemisphere



C.o.g lies along the line of symmetry at a distance $\frac{3r}{8}$ from straight edge AB

$G\left(\frac{3r}{8}, r\right)$

Examples

1. Find the position of the center of gravity of a uniform lamina in form of a triangle whose co-ordinates are

(i)	(0,0), (2,6), and (4,0).	(iii)	(0,0), (0,6), and (6,0).	(v)	(1,0), (5,0), and (0,6).
(ii)	(0,3), (3,0), and (6,3).	(iv)	(0,0), (0,6), and (3,0).	(vi)	(3,0), (6,0), and (0,6).

Solution

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{3}(0+2+4, 0+6+0)$$

$$= (2,2)$$

$$(ii) = (3,2)\ (iii) = (2,2)\ \ (iv) = (1,2)\ \ (v) = (2,2)\ \ (vi) = (3,2)$$

CENTRE OF GRAVITY OF COMPOSITE LAMINAE AND SOLIDS

1. The figure below shows a lamina formed by joining together a rectangular solid and triangular solid. Find the C.O.G of the composite lamina from side AE and AB

Solution

lamina	Area	Weight	Distance of C.O.G from AE	AB
ABDE	32cm²	32 W	4	2
BDC	12cm²	12 W	10	2
composite	44cm²	44 W	$\bar{x}$	$\bar{y}$

c.o.g from AE; $44W\bar{x} = 12Wx10 + 32Wx4$

$\bar{x} = 5.64cm$

c.o.g from AB; $44W\bar{y} = 12Wx2 + 32Wx2$

$\bar{y} = 2cm$

2. The figure below shows a lamina formed by joining together a rectangular solid to a semicircular solid. Find the C.O.G of the composite lamina from side AD and AB

Solution

W is the weight per unit area

lamina	Area	Weight	Distance of C.O.G from AD	AB
ABCD	70cm²	70 W	5	3.5
Semi-cirlce	19.24cm²	19.24 W	$10+\frac{4x3.5}{3\pi}=11.49$	3.5
composite	89.24cm²	89.24 W	$\bar{x}$	$\bar{y}$

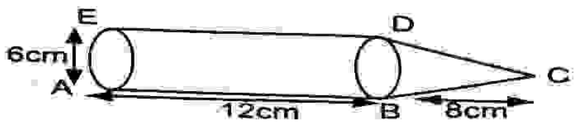
c.o.g from AD; $89.24W\bar{x} = 19.24Wx\ 11.49 + 70Wx5$

$\bar{x} = 6.4cm$

c.o.g from AB; $89.24W\bar{y} = 19.24Wx3.5 + 70Wx3.5$

$\bar{y} = 3.5cm$

3. The figure below shows a lamina formed by joining together a cylindrical solid to a conical solid. Find the C.O.G of the composite lamina from side AE and AB



Solution

W is the weight per unit volume

lamina	volume	Weight	Distance of C.O.G from AE	AB
cylinder	339.29cm³	339.29 W	6	3
cone	75.40cm³	75.4 W	$12 + \frac{8}{4} = 14$	3
composite	414.69cm²	414.69 W	$\bar{x}$	$\bar{y}$

co.g from AE; $414.69W\bar{x} = 75.4W \times 14 + 339.29W \times 6$

$\bar{x} = 7.45cm$

co.g from AB; $414.69W\bar{y} = 75.4W \times 3 + 339.29W \times 3$

$\bar{y} = 3cm$

4. Find the centre of gravity of the following from line AF and AB

(i)

An(4.36cm, 3.78cm)

(ii)

An(4cm, 3.3cm)

5. A body of consists of a solid hemisphere of radius r joined to a solid right circular cone of base radius r and perpendicular h. The pane surface of the cone and hemisphere coincide and both solids are made of the same uniform material. Show that the C.O.G of the body lies on the axis of symmetry at a distance $\frac{3r^2-h^2}{4(h+2r)}$ from the base of the cone

Solution

W is weight per unit volume

lamina	Weight	C.O.G from y axis
Cone	$\frac{1}{3}\pi r^2 hW$	$\frac{3}{4}h$
hemisphere	$\frac{2}{3}\pi r^3 W$	$h + \frac{3r}{8}$
Composite	$\frac{1}{3}\pi r^2(h+2r)W$	$\bar{x}$

$$\frac{1}{3}\pi r^2(h+2r)W\bar{x} = \frac{2}{3}\pi r^3\left(h + \frac{3r}{8}\right)W + \frac{1}{3}\pi r^2 h\left(\frac{3h}{4}\right)W$$

$$(h+2r)\bar{x} = 2r\left(h + \frac{3r}{8}\right) + \frac{3h^2}{4}$$

$$\bar{x} = \frac{3h^2 + 8hr + 3r^2}{4(h+2r)}$$

Centre of gravity from the base= $\frac{3h^2+8hr+3r^2}{4(h+2r)} - h$

$$= \frac{3r^2 - h^2}{4(h+2r)}$$

Exercise 21B

8. Find the position of the center of gravity of a uniform lamina in form of a triangle whose (2,2), (4,6), and (0,3) respectively. **UNEB 2007 NO.2**

An$(2, \frac{11}{3})$

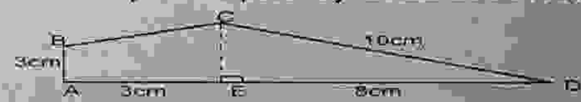
9. Find the position of the center of gravity of particles of weight 12N, 8N and 4N which are situated at $(1,-3)$, $(0,2)$, and $(1,3)$ respectively

10. Particles of masses 5kg, 2kg , 3kg and 2kg are situated at $(3i - j)$, $(2i + 3j)$, $(-2i + 5j)$, and $(-i - 2j)$ respectively. Find the position vector of their centre of gravity

11. The figure below is a lamina formed by welding together a rectangular metal sheet and a semicircular metal sheet.

Find the position of the centre of gravity of the lamina from side OA

12. Find the co-ordinate if the centre of mass of the lamina shown below. Take A as the origin and AD, AB as x and y-axis respectively. **UNEB 2008 NO.8**



An(4.227cm, 2.12cm)

13. ABCD is a trapezium in which AB and CD are parallel and length a and b respectively

Prove that the distance of the centre of mass from AB is $\frac{1}{3}h\left(\frac{a+2b}{a+b}\right)$ where h is the distance between AB and CD

14. The figure ABCD below shows a metal sheet of uniform material cut in the shape of a trapezium $\overline{AB} = x$, $\overline{CD} = y$, $\overline{AF} = a$, $\overline{EB} = b$ and h is the vertical distance between AB and CD

W is the weight per unit volume

lamina	volume	Weight	Distance of C.O.G from AE	AB
cylinder	339.29cm³	339.29 W	6	3
cone	75.40cm³	75.4 W	$12+\frac{8}{4}=14$	3
composite	414.69cm³	414.69 W	$\bar{x}$	$\bar{y}$

c.o.g from AE: $414.69W\bar{x} = 75.4Wx\,14 + 339.29Wx6$

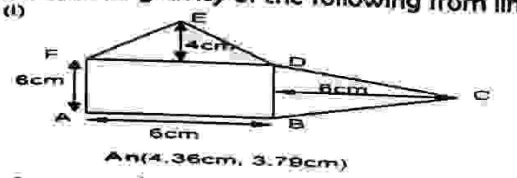
$\bar{x} = 7.45cm$

c.o.g from AB: $414.69W\bar{y} = 75.4Wx\,3 + 339.29Wx3$

$\bar{y} = 3cm$

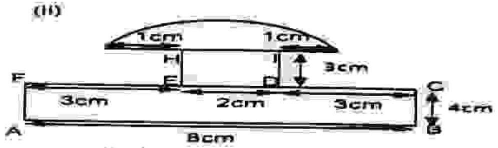
4. Find the centre of gravity of the following from line AF and AB

(i)



An(4.36cm, 3.79cm)

(ii)



An(4cm, 3.3cm)

5. A body of consists of a solid hemisphere of radius r joined to a solid right circular cone of base radius r and perpendicular h. The pane surface of the cone and hemisphere coincide and both solids are made of the same uniform material. Show that the C.O.G of the body lies on the axis of symmetry at a distance $\frac{3r^2-h^2}{4(h+2r)}$ from the base of the cone

Solution

W is weight per unit volume

lamina	Weight	C.O.G from y axis
Cone	$\frac{1}{3}\pi r^2hW$	$\frac{3}{4}h$
hemisphere	$\frac{2}{3}\pi r^3W$	$h+\frac{3r}{8}$
Composite	$\frac{1}{3}\pi r^2(h+2r)W$	$\bar{x}$

$$\frac{1}{3}\pi r^2(h+2r)W\bar{x} = \frac{2}{3}\pi r^3\left(h+\frac{3r}{8}\right)W + \frac{1}{3}\pi r^2h\left(\frac{3h}{4}\right)W$$

$$(h+2r)\bar{x} = 2r\left(h+\frac{3r}{8}\right) + \frac{3h^2}{4}$$

$$\bar{x} = \frac{3h^2+8hr+3r^2}{4(h+2r)}$$

Centre of gravity from the base $= \frac{3h^2+8hr+3r^2}{4(h+2r)} - h$

$$= \frac{3r^2-h^2}{4(h+2r)}$$

Exercise 21B

8. Find the position of the center of gravity of a uniform lamina in form of a triangle whose (2,2), (4,6), and (0,3) respectively. **UNEB 2007 NO.2**
An$(2,\frac{11}{3})$

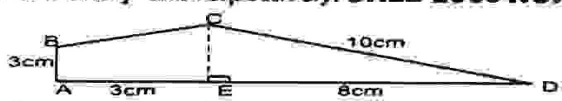
9. Find the position of the center of gravity of particles of weight 12N, 8N and 4N which are situated at $(1,-3)$, $(0,2)$, and $(1,3)$ respectively

10. Particles of masses 5kg, 2kg , 3kg and 2kg are situated at $(3\bar{\imath}-\bar{\jmath})$, $(2\bar{\imath}+3\bar{\jmath})$, $(-2\bar{\imath}+5\bar{\jmath})$, and $(-\bar{\imath}-2\bar{\jmath})$ respectively. Find the position vector of their centre of gravity

11. The figure below is a lamina formed by welding together a rectangular metal sheet and a semicircular metal sheet.

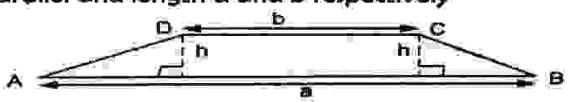
Find the position of the centre of gravity of the lamina from side OA

12. Find the co-ordinate if the centre of mass of the lamina shown below. Take A as the origin and AD, AB as x and y-axis respectively. **UNEB 2008 NO.8**



An(4.227cm, 2.12cm)

13. ABCD is a trapezium in which AB and CD are parallel and length a and b respectively

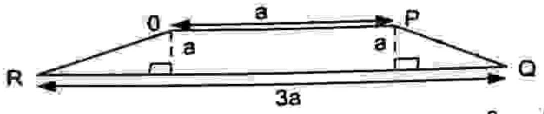


Prove that the distance of the centre of mass from AB is $\frac{1}{3}h\left(\frac{a+2b}{a+b}\right)$ where h is the distance between AB and CD

14. The figure ABCD below shows a metal sheet of uniform material cut in the shape of a trapezium $\overline{AB} = x$, $\overline{CD} = y$, $\overline{AF} = a$, $\overline{EB} = b$ and h is the vertical distance between AB and CD

Prove that the distance of the centre of gravity of the sheet is at a distance $\frac{1}{3}h\left(\frac{3y+a+b}{x+y}\right)$ from side AB

15. The figure OPQR below shows a metal sheet of uniform material cut in the shape of a trapezium $\overline{OP} = a, \overline{RQ} = 3a$, and the vertical height of P from $RQ = a$ **UNEB 2006 NO.5**



Calculate the centre of mass of OPQR **An**$(\frac{3}{2}a, \frac{5}{12}a)$.

16. The diagram shows two uniform squares ABFG and BCDE joined together. The mass per unit area of BCDE is twice that of ABFG

Find the distance of the centre of gravity of composite body from AB and AG. **An**$(1\frac{2}{3}m, 3m)$.

17. ABCD is a uniform rectangular lamina in which $AB = p$ and $BC = 3p$. The points E is on AD such that $ED = 3q$
    i. Show that G, the centre of gravity of the trapezium ABCE is distant $\frac{3p^2-3pq+q^2}{2p-q}$ from AB and find its distance from BC
    ii. When the trapezium is suspended from E, the edge BC is horizontal, prove that $q = {}^1/_2 p(3-\sqrt{3})$

18. Two uniform square lamina, each of side 3m are joined together to a rectangular lamina 6m by 3m. The squares are not made of the same material and the mass per unit area of one of them is twice that of the other. Find the distance of the c.o.g of the composite body from the edge of the square. **An**(0.5m).

19. Two solid cubes, one of side 4cm and the other of side 2cm are made of the same uniform material. The smaller cube is glued centrally to one of the faces of the larger cube as shown below

Find how far the centre of gravity of the composite body is from the common surface of the cubes **An**$(1\frac{2}{3}cm)$.

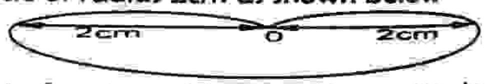
20. A uniform semicircular lamina of radius 6cm is joined to another uniform semicircular lamina of radius 3cm. The centre of the straight edges of each lamina coincide, but the lamina do not overlap. If the two laminae are made of the same material, find the position of the C.OG of the composite lamina formed. **An**(On axis of symmetry, $\frac{28}{5\pi}$cm into the larger semicircle from common diameter).

21. A uniform semicircular lamina of radius 6cm is joined to another uniform semicircular lamina of radius 3cm. The centre of the straight edges of each lamina coincide, but the lamina do not overlap. If the smaller semicircular is made of the material having mass per unit area equal to twice that of the larger semicircular lamina, find the position of the C.OG of the composite lamina formed.
    An(On axis of symmetry, $\frac{4}{\pi}$ cm into the larger semicircle from common diameter).

22. A solid right circular cylinder has a base of radius of 3cm and a height of 6cm. the cylinder's circular top form the base of a solid right circular cone of base radius 3cm and perpendicular height 4cm. the cylinder and the cone are made from the same uniform material. Find the position of the C.O.G of the composite body . **An**(On axis of symmetry, $3\frac{8}{11}$ cm above the base of cylinder).

23. A body consists of a solid hemisphere of radius 4cm joined to a solid right circular cone of radius 4cm and perpendicular height 12cm. The plane surface of the cone and hemisphere coincides and both solids are made of the same uniform material. Find the position of the C.O.G of the body. **An**(On axis of symmetry, $10.8cm$ from tip of the cone)

24. A badge is cut from a uniform thin sheet of metal. The badge is formed by joining the diameter of two semicircle, each of radius 1cm to the diameter of semicircle of radius 2cm as shown below



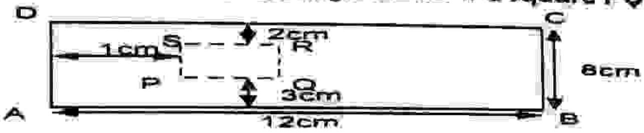
The point of contact of two smaller semicircle is O. Determine in terms of π, the distance from O of the centre of mass of the badge. **An**$(\frac{4}{3\pi}cm)$.

25. A child's toy consists of a solid uniform hemisphere of radius r and a solid right circular cone of base radius r and height h. The base radius of the solids are glued together. If the density of the hemisphere is k times that of the cone.
    (i) show that the distance from the vertex of the cone to the centre of gravity of the toy is $\frac{kr(3r+8h)+3h^2}{4(2kr+h)}$
    (ii) If the toy is suspended from a point on the rim of the common base and rests in equilibrium with the axis of the cone inclined at an angle of θ to the downward vertical. Show that $tan\theta = \frac{4r(2kr+h)}{h^2-3kr^2}$

CENTRE OF GRAVITY OF A LAMINA WHOSE PART HAS BEEN REMOVED (REMAINDER)

Examples

1. Find the centre of gravity of the remainder of the rectangle ABCD if a square PQRS is removed as shown below



Solution

W is the weight per unit area

lamina	area	Weight	Distance of C.O.G from AD	AB
ABCD	$96cm^2$	96 W	6	4
PQRS	$9cm^2$	9 W	2.5	4.5
Remainder	$87cm^2$	87 W	$\bar{x}$	$\bar{y}$

c.o.g from AD; $87W\bar{x} = 96Wx\,6 - 9Wx2.5$

$\bar{x} = 6.36cm$

c.o.g from AB; $87W\bar{y} = 96Wx\,4 - 9Wx4.5$

$\bar{y} = 3.95cm$

2. Find the centre of gravity of the remainder of the square ABCD if a circle of radius r is removed as shown below

Solution

W is the weight per unit area

lamina	area	Weight	Distance of C.O.G from AD	AB
ABCD	$100cm^2$	100 W	5	5
circle	$28.27cm^2$	28.27 W	4	6
Remainder	$71.73cm^2$	71.73 W	$\bar{x}$	$\bar{y}$

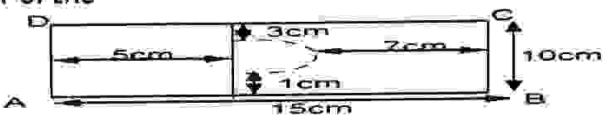
c.o.g from AD: $71.73W\bar{x} = 100Wx\,5 - 28.27Wx4$

$\bar{x} = 5.39cm$

c.o.g from AB; $71.73W\bar{y} = 100Wx\,5 - 28.27Wx6$

$\bar{y} = 4.61cm$

3. A rectangle ABCD is of $AB = 15cm$ and $AD = 10cm$. If the semi-circle is removed as shown below. Find the centre of gravity of the remainder of the



Solution

W is the weight per unit area

lamina	Area	Weight	Distance of C.O.G from AD	AB
ABCD	$150cm^2$	150 W	7.5	5
Semi-circle	$14.14cm^2$	14.14 W	$5+\frac{4x3}{3\pi}=6.27$	4
remainder	$135.86cm^2$	135.86 W	$\bar{x}$	$\bar{y}$

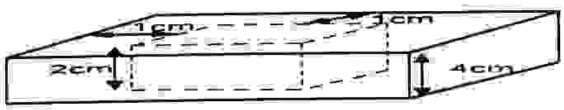
c.o.g from AD; $135.86W\bar{x} = 150Wx\,7.5 - 14.14Wx6.27$

$\bar{x} = 7.63cm$

c.o.g from AB: $135.86W\bar{y} = 150Wx\,5 - 14.14Wx4$

$\bar{y} = 5.1cm$

4. A solid cube of side 4cm is made from a uniform material. From this a smaller cube of side is removed as shown below



Find the position of the centre of gravity of the remaining body

Solution

W is the weight per unit volume

lamina	volume	Weight	Distance of C.O.G from AD	AB
Big cube	$64cm^3$	64 W	2	2
Small cube	$8cm^3$	8 W	2	3
remainder	$56\ cm^3$	56 W	$\bar{x}$	$\bar{y}$

c.o.g from AD; $56W\bar{x} = 64Wx\,2 - 8Wx2$

$\bar{x} = 2cm$

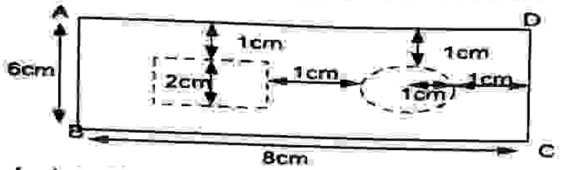
c.o.g from AB; $56W\bar{y} = 64Wx\,2 - 8Wx3$

$\bar{y} = 1.86cm$

Exercise 21C

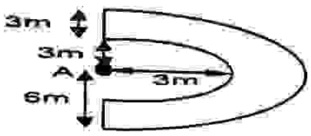
1. A circular lamina, made of uniform material has its centre at the origin and radius of 6cm. Two smaller circles are cut from this circle, one of radius 1cm and centre (-1,-3) and the other of radius 3cm and centre (1,2). Find the co-ordinates of the centre of gravity of the remaining shape. An $\left(\frac{-4}{3}, \frac{-15}{26}\right)$
2. ABCD is a uniform rectangular sheet of card board of length 8cm and width 6cm. A square

and a circular hole are cut off from the card board as shown above. **UNEB 1999 NO.16**



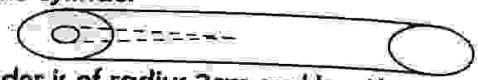
Calculate the position of the C.O.G of the remaining sheet **An** $(3.944cm, 2.825cm)$

3. The diagram shows a uniform semi-circular lamina of radius 6m with a semi-circular portion of radius 3m missing



Find the distance of the C.O.G of the remaining shape from the point A shown above $An\left(\frac{28}{\pi}\right)$

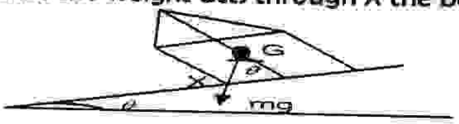
4. A conical hole is made in one end of a right circular cylinder. The axis of symmetry of the cone is the same as that of the cylinder



The cylinder is of radius 2cm and length 6cm. the conical hole penetrates 4cm into the cylinder and the circular hole at the end of the cylinder is of radius 1.5cm. find the position of the C.O.G of the remaining body **An** (On axis of symmetry, $2\frac{5}{17}$ *cm from undrilled end*).

TOPPLING

Consider a body which is resting on a slop which is rough enough to prevent slipping.
When the angle of the slope is such that the weight acts through X the body will be on the point of toppling



Example

1. The figure below shows a uniform hemispherical solid of radius 30cm with a cylindrical hole of radius 5cm and height 10cm centrally drilled in it

(a) Find the distance of centre of gravity of the figure from a flat surface
(b) If the figure is placed on an inclined plane with the flat surface in contact with the plane, calculate the angle that should be inclined, before toppling occurs assuming that sliding does not occur

Solution

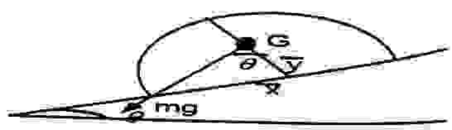
lamina	Volume	Weight	C.O.G from Flat face	C.O.G from Y-axis
hemisphere	$\frac{2}{3}\pi(30)^3 = 56548\ 668$	56548 668 W	$\frac{3x30}{8} = 11.25$	30cm
Cylinder	$\pi(5)^2 x10 = 785\ 398$	785.398W	5cm	30cm
Remainder	55763.270	55763.270W	$\bar{y}$	$\bar{x}$

c.o.g from flat face

$$55763.27\bar{y} = 56548.668x11.25 - 785.398x5$$
$$\bar{y} = 11.34$$

c.o.g from y-axis

$$55763.27\bar{x} = 56548.668x30 - 785.398x30$$
$$\bar{x} = 30$$



$$tan\theta = {}^{\bar{x}}/_{\bar{y}} = \frac{30}{11.34} = 69.3°$$

2. A body consists of a uniform solid cylinder of mass 6m, base radius r and height 2r, attached to a plane face of a uniform solid cone of mass 4m base radius r and height r

(i) Find the position of the centre of gravity of the body
(ii) The body is now placed with its plane face AE in contact with a horizontal table. The surface of the table is rough enough to prevent the body slipping as the table is slowly tilted. Find the angle through which the table has been tilted when the body is on the point of toppling

Solution

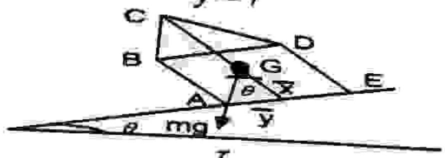
lamina	Weight	Distance of C.O.G from AF	AB
cylinder	6mg	r	r
cone	4mg	$2r + \frac{r}{4} = \frac{9r}{4}$	r
composite	10mg	$\bar{x}$	$\bar{y}$

c.o.g from AE; $10mg\bar{x} = 4mgx\,\frac{9r}{4} + 6mgxr$

$$\bar{x} = 1.5r$$

c.o.g from AB; $10mg\bar{y} = 4mgx\,r + 6mgxr$

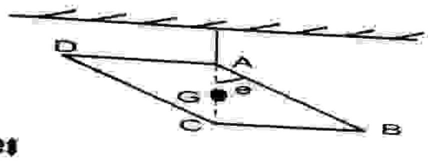
$$\bar{y} = r$$



$$tan\theta = \frac{r}{1.5r} = 33.7°$$

EQUILIBRIUM OF SUSPENDED LAMINA

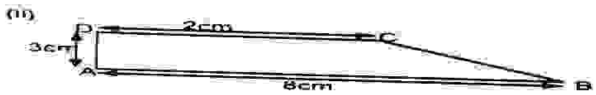
Consider a lamina freely suspended from point A. The centre of gravity passes through the point of suspension



$$tan\,\theta = \frac{BC}{AB}$$

Examples

1. The uniform laminae below are freely suspended from point A. In each case find the angle that the line AB makes with the vertical



Solution

W is the weight per unit area

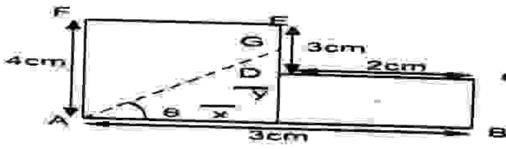
lamina	area	Weight	Distance of C.O.G from AF	AB
AFEP	$4cm^2$	4 W	0.5	2
PBCD	$2cm^2$	2 W	1+1=2	0.5
Composite	$6cm^2$	6 W	$\bar{x}$	$\bar{y}$

c.o.g from AF; $6W\bar{x} = 4Wx\;0.5 + 2Wx2$

$$\bar{x} = 1cm$$

c.o.g from AB; $6W\bar{y} = 4Wx\;2 + 2Wx0.5$

$$\bar{y} = 1.5cm$$



$$tan\,\theta = \frac{1.5}{1} = 56.3°$$

Solution

W is the weight per unit area

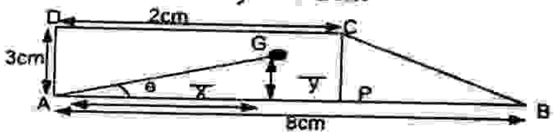
lamina	area	Weight	Distance of C.O.G from AD	AB
ADCP	$6cm^2$	6 W	1	1.5
PCB	$9cm^2$	9 W	2+1/3x6=4	1
Composite	$15cm^2$	15 W	$\bar{x}$	$\bar{y}$

c.o.g from AF; $15W\bar{x} = 9Wx\;4 + 6Wx1$

$$\bar{x} = 2.8cm$$

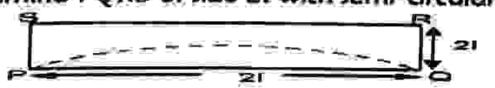
c.o.g from AB; $15W\bar{y} = 9Wx\;1 + 6Wx1.5$

$$\bar{y} = 1.2cm$$



$$tan\,\theta = \frac{1.2}{2.8} = 23.2°$$

2. The figure below shows a uniform lamina $PQRS$ of side $2l$ with semi-circular lamina cut off as show below



(a) Show that the distance of the centre of gravity of the figure from PQ is $\frac{20l}{3(8-\pi)}$

(b) The figure is freely suspended from the point R. find the angle that RS makes with the vertical

Solution

W is the weight per unit area

lamina	Area	Weight	Distance of C O G from PQ
PQRS	$4l^2$	$4l^2$ W	l
Semi-circle	$\frac{1}{2}\pi l^2$	$\frac{1}{2}\pi l^2 W$	$\frac{4l}{3\pi}$
remainder	$\left(4-\frac{1}{2}\pi\right)l^2$	$\left(4-\frac{1}{2}\pi\right)l^2 W$	$\bar{y}$

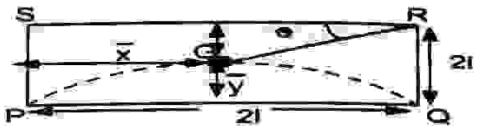
c.o.g from PQ: $\left(4-\frac{\pi}{2}\right)l^2\bar{y} = 4l^2 x\, l - \frac{\pi}{2}l^2 x \frac{4\,l}{3\pi}$

$$\left(\frac{8-\pi}{2}\right)\bar{y} = 4l - \frac{2\,l}{3}$$

$$\left(\frac{8-\pi}{2}\right)\bar{y} = \frac{10\,l}{3}$$

$$\bar{y} = \frac{20\,l}{3(8-\pi)}$$

c.o.g from AB



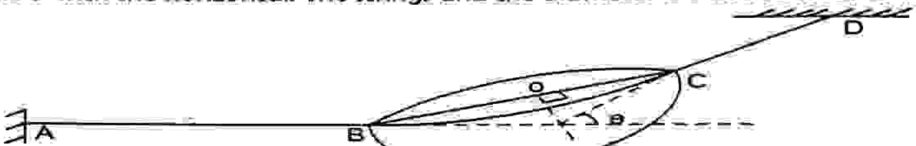
C.O.G from RS= $2l - \frac{20\,l}{3(8-\pi)} = \frac{48\,l-6l\pi-20l}{3(8-\pi)}$

$$= \frac{28\,l - 6l\pi}{3(8-\pi)}$$

$$\tan\theta = \frac{\frac{28\,l-6l\pi}{3(8-\pi)}}{l} = \frac{28 - 6\pi}{3(8-\pi)}$$

$$= 32.12^{\circ}$$

3. A uniform sloid hemisphere is kept in equilibrium in space by two in elastic strings, AB which is horizontal and CD which makes an angle θ with the horizontal. The strings and the diameter BC lie in the same vertical plane.

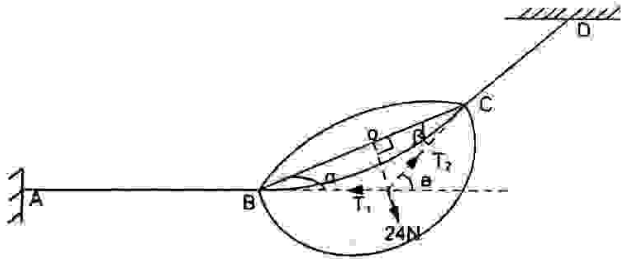


Given that the weight of the hemisphere is 24N, show that

(i) $\tan\theta = \frac{48}{55}$

(ii) Tension in strings CD and AB are 36.5N and 27.5N respectively

Solution



$\tan\alpha = \frac{3r}{8}\big/ r = \frac{3}{8}$ and $\tan\beta = \frac{3r}{8}\big/ r = \frac{3}{8}$

$$\theta = \alpha + \beta$$

$$\tan\theta = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta}$$

$$\tan\theta = \frac{\frac{3}{8}+\frac{3}{8}}{1-\frac{3}{8}x\frac{3}{8}} = \frac{3}{4}\Big/\frac{55}{64} = \frac{48}{55}$$

$(\rightarrow)\ T_2\cos\theta = T_1 \ldots\ldots (i)$

$(\uparrow)\ T_2\sin\theta = 24 \ldots\ldots\ldots (ii)$

$\frac{T_2\sin\theta}{T_2\cos\theta} = \frac{24}{T_1}$ $\quad\therefore \tan\theta = \frac{24}{T_1}$

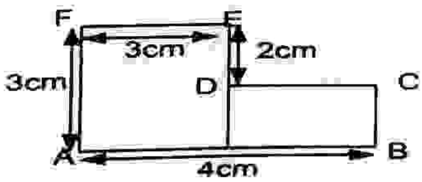
$$T_1 = \frac{24}{\tan\theta} = 24\Big/\frac{48}{55} = 27.5N$$

$$T_2 = \frac{24}{\sin\theta} = \frac{24}{\sin\left(\tan^{-1}\frac{48}{55}\right)} = 36.5N$$

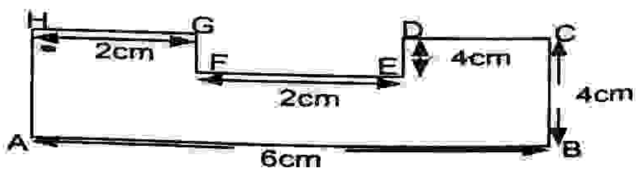
Exercise 21D

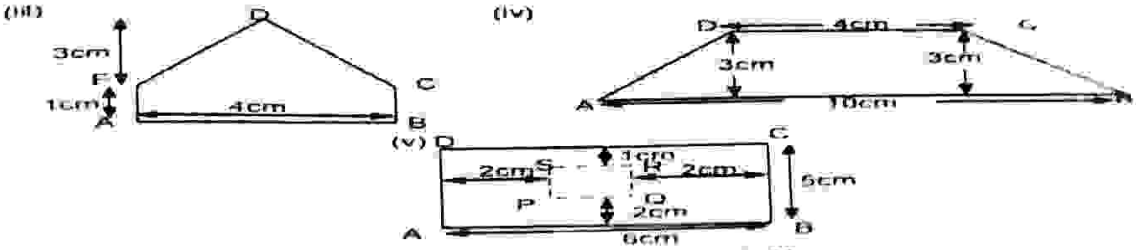
1. The uniform laminae below are freely suspended from point A. In each case find the angle that the line AB makes with the vertical

(i)



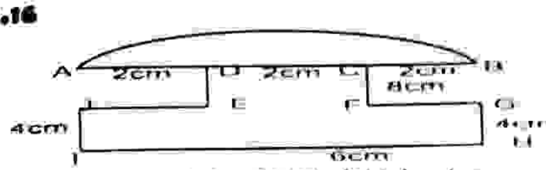
(ii)





An $(i) = 39.5°, (ii) = 31°\ \ (iii) = 35\ \ (iv) = 14.4°\ \ (v) = 38.9°$

2. The figure ABCDEFGHIJ shows a symmetrical composite lamina made up of a semicircle radius 3cm, a rectangle CDEF 2cm by 8cm and a nother rectangle GHIJ 6cm by 4cm **UNEB Mar 1998 NO.16**



Find the distance of the C.O.G of this lamina from IH. If the lamina is suspended from H by means of a peg through a hole, calculate the angle of inclination of HG to the vertical **An** $6.72cm, 24.1°$.)

3. (a) A, B, C and D are the points (0,0), (10,0), (7,4) and (3,4) respectively. If AB, BC, CD and DA are made of a thin wire of uniform mass, find the co-ordinates of centre of gravity. **UNEB 1994 NO.6**

(b)(i) If instead AB is uniform lamina, find its centre of gravity

(ii) If lamina is hung from B, find the angle AB makes with vertical **An** (a)(5, 1.5), (b)(i)(5, 1.7), 18.8°.)

4. (a) Prove that the centre of mass of a solid cone is $\frac{1}{4}$ of the vertical height from the base. **UNEB 2001 NO.15**

(b) The figure ABCDE below shows a solid cone of radius r, height h, joined to a solid cylinder of same material with the same radius and height H.

(i) If the centre of mass of the whole solid lies in the plane of the cone where the two solids are joined, find H.

(ii) If instead $H = h$ and $r = \frac{1}{2}h$, find the angle AB makes with the horizontal, if the body is hanged from A. **An** $H = \frac{\sqrt{6}}{6}h, 32.01°$.)

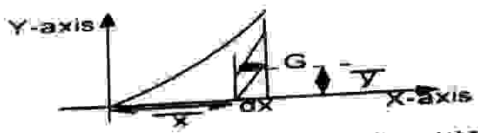
3. A body consists of a uniform solid cylinder of base radius r and height hcm, attached to a plane face of a uniform solid cone of base radius r and height 4cm

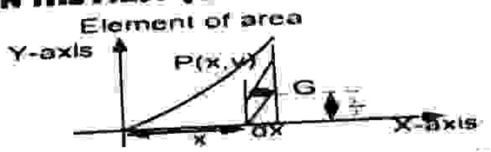
(i) Show that the distance of the centre of gravity of the solid from AE is $\frac{3h^2+6h+8}{6h+8}$

(ii) The body is now placed with its plane face AE in contact with a rough plane inclined at 45 to the horizontal, it will be at the point of toppling. Find the angle through which the table has been tilted when the body is on the point of toppling. If the radius of the cylinder is $2\,^3/_4\ cm$. Find the value of h

CENTRE OF GRAVITY OF THE LAMINA WHOSE AREA IS BOUNDED

(I) C.O.G OF THE AREA BOUNDED IN THE FIRST QUADRANT





- Taking moments about the y-axis: $W\bar{x}\int ydx = W\int xydx$

Also Taking moments about the x-axis: $W\bar{y}\int ydx = W\int \frac{y}{2}ydx$

Where W – weight per unit area

Examples

1. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$, the x-axis and the line $x = 4$

Solution

$$W\bar{x}\int_0^4 ydx = W\int_0^4 xydx$$

$$\bar{x}\int_0^4 x^2dx = \int_0^4 x(x^2)dx$$

$$\bar{x}\left[\frac{x^3}{3}\right]_0^4 = \left[\frac{x^4}{4}\right]_0^4$$

$$\bar{x} = 3$$

$$W\bar{y}\int_0^4 ydx = W\int_0^4 \frac{y^2}{2}dx$$

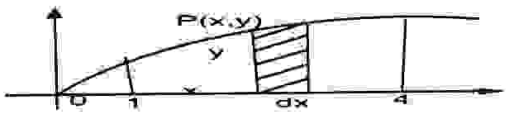
$$\bar{y}\int_0^4 x^2dx = \frac{1}{2}\int_0^4 x^4dx$$

$$\bar{y}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^4$$

$$\bar{y} = 4.8$$

2. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y^2 = 9x$, the x-axis and the line $x = 1$ and $x = 4$ and lying in the first quadrant

Solution



$$W\bar{x}\int_1^4 ydx = W\int_1^4 xydx$$

$$\bar{x}\int_1^4 3x^{1/2}\,dx = \int_1^4 x(3x^{1/2})dx$$

$$3\,\bar{x}\left[\frac{x^{3/2}}{2}\right]_1^4 = 5\left[\frac{x^{5/2}}{2}\right]_1^4$$

$$\bar{x} = 2.66$$

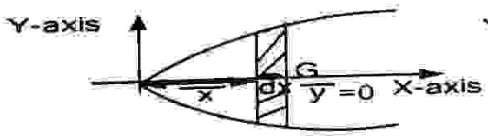
$$W\bar{y}\int_1^4 ydx = W\int_1^4 \frac{y^2}{2}dx$$

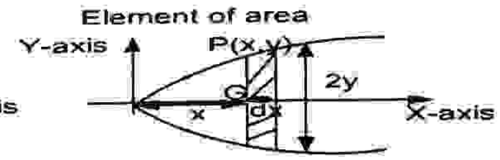
$$\bar{y}\int_1^4 3x^{1/2}dx = \frac{1}{2}\int_1^4 9x\,dx$$

$$\bar{x}\left[\frac{x^{3/2}}{2}\right]_1^4 = \frac{3}{2}\left[\frac{x^2}{2}\right]_1^4$$

$$\bar{y} = 2.41$$

(*ii*) C.O.G OF THE AREA BOUNDED IN THE FIRST AND FOURTH QUADRANT





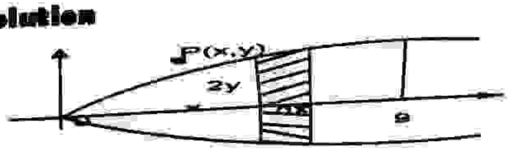
Taking moments about the y-axis: $W\bar{x}\int 2ydx = W\int x(2y)dx$

Where W – weight per unit area

Examples

1. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y^2 = 4x$, and the line $x = 9$

Solution



$$W\bar{x}\int_0^9 2ydx = W\int_0^9 x(2y)dx$$

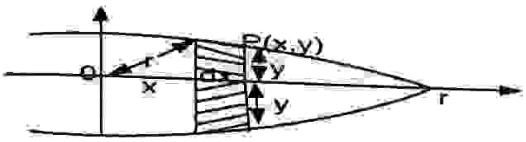
$$\bar{x}\int_0^9 x^{1/2}\,dx = \int_0^9 x(x^{1/2})dx$$

$$3\bar{x}\left[\frac{x^{3/2}}{2}\right]_0^9 = 5\left[\frac{x^{5/2}}{2}\right]_0^9$$

$$\bar{x} = 5.4$$
$$(\bar{x}, \bar{y}) = (5.4, 0)$$

2. Show that the position of C.O.G of a uniform semi-circular lamina of radius r is $\frac{4r}{3\pi}$ from the straight edge

Solution



Area of semi circle= area of element of semicircle

$$W\frac{1}{2}\pi r^2\bar{x} = W\int_0^r x(2y)dx$$

But a semi cycle is part of a circle of radius r whose equation is $x^2 + y^2 = r^2$

$$y = (r^2 - x^2)^{1/2}$$

$$\frac{1}{2}\pi r^2\bar{x} = 2\int_0^r x\left((r^2 - x^2)^{1/2}\right)dx$$

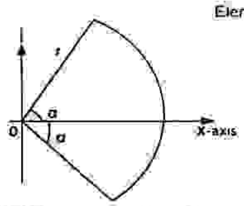
$$\frac{1}{2}\pi r^2\bar{x} = 2\left[-\frac{(r^2 - x^2)^{3/2}}{3}\right]_0^r$$

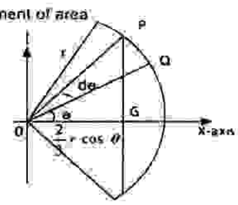
$$\frac{1}{2}\pi r^2\bar{x} = \frac{2r^3}{3}$$

$$\bar{x} = \frac{4r}{3\pi}$$

3. Show that the centre of gravity of a uniform lamina in the shape of <u>a sector of a circle</u> of radius r and subtending an angle 2α at the centre O is given by $\frac{2}{3}\frac{r\sin\alpha}{\alpha}$ from O.

Solution





The strip OPQ approximates a triangle and its C.O.G is at a distance $\frac{2}{3}r$ from O. The distance of C.O.G from O is therefore, $\frac{2}{3}r\cos\theta$

Area of a sector= element of the area of sector

$$W\frac{1}{2}r^2(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha} xydx$$

$$\frac{1}{2}r^2(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha}\frac{2}{3}r\cos\theta\left(\frac{1}{2}r^2\right)d\theta$$

$$\alpha\bar{x} = \frac{1}{3}\int_{-\alpha}^{\alpha} r\cos\theta d\theta$$

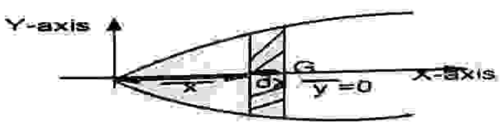
$$\alpha\bar{x} = \frac{r}{3}[\sin\theta]_{-\alpha}^{+\alpha}$$

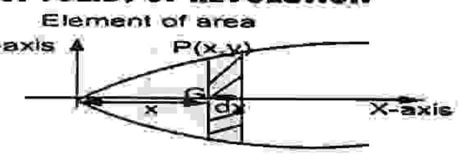
$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}$$

For a complete semi –circle, $\alpha = \frac{\pi}{2}$ there fore

$$\bar{x} = \frac{4r}{3\pi}$$

CENTRE OF GRAVITY OF SOLIDS OF REVOLUTION





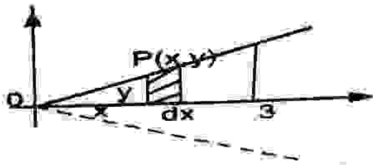
Taking moments about the y-axis: $\bar{x}W\pi\int y^2dx = W\pi\int xy^2dx$

Where W – weight per unit volume

Examples

1. Find the centre of gravity of the solid generated by rotating about the x-axis, the area under $y = x$ from $x = 0$ and $x = 3$

Solution



$$\bar{x}W\pi\int y^2dx = W\pi\int xy^2dx$$

$$\bar{x}\int_0^3 x^2\,dx = \int_0^3 x(x^2)dx$$

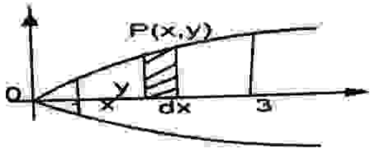
$$\bar{x}\left[\frac{x^3}{3}\right]_0^3 = \left[\frac{x^4}{4}\right]_0^3$$

$$\bar{x} = 2.25$$

$$(\bar{x},\bar{y}) = (2.25,0)$$

2. Find the centre of gravity of the solid generated by rotating about the x-axis, the area bounded by $y^2 = 5x$, the x-axis, the lines $x = 1$ and $x = 3$ and lies in the first quadrant

Solution



$$\bar{x}W\pi\int y^2dx = W\pi\int xy^2dx$$

$$\bar{x}\int_1^3 5x\,dx = \int_1^3 x(5x)dx$$

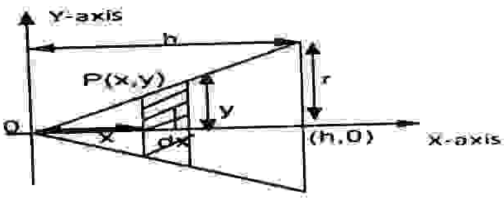
$$\bar{x}\left[\frac{x^2}{2}\right]_1^3 = \left[\frac{x^3}{3}\right]_1^3$$

$$\bar{x} = 2.17$$

$$(\bar{x},\bar{y}) = (2.17,0)$$

4. Show that the position of C.O.G of a uniform solid right circular cone of base radius r and height h is given by $\frac{h}{4}$ from the straight edge

Solution



Volume of a cone= volume of element of a cone

$$\bar{x}W\frac{1}{3}\pi r^2h = W\pi\int xy^2dx$$

$$\bar{x}\frac{1}{3}r^2h = \int_0^h xy^2dx$$

From similarity $\frac{y}{x} = \frac{r}{h}$

$$y = \frac{r}{h}x$$

$$\bar{x}\frac{1}{3}r^2h = \int_0^h x\left(\frac{r}{h}x\right)^2dx$$

$$\bar{x}\frac{1}{3}r^2h = \left(\frac{r^2}{h^2}\right)\left[\frac{x^4}{4}\right]_0^h$$

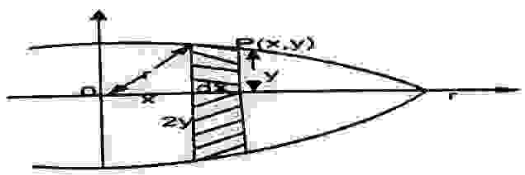
$$\bar{x} = \frac{3h}{4}$$

From the straight edge

$$\bar{x} = h - \frac{3h}{4} = \frac{h}{4}$$

5. Show that the position of C.O.G of a uniform solid hemisphere of radius r is $\frac{3r}{8}$ from the straight edge

Solution



$$W\frac{2}{3}\pi r^3\bar{x} = W\pi\int_0^r xy^2dx$$

But $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$$\frac{2}{3}\pi r^3\bar{x} = \pi\int_0^r x(r^2 - x^2)dx$$

$$\frac{2}{3}r^3\bar{x} = \left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r$$

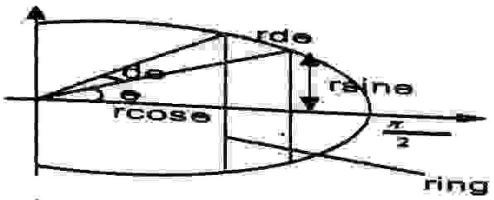
$$\frac{2}{3}r^3\bar{x} = \frac{r^4}{2} - \frac{r^4}{4}$$

$$\bar{x} = \frac{3r}{8}$$

SURFACE OF REVOLUTION

1. Show that the centre of gravity of a uniform thin hemispherical cup of radius r is at a distance $\frac{r}{2}$ from the base. **Uneb 2012 No.13**

Solution



Surface area of a hemisphere= element of the surface

$$W2\pi r^2\bar{x} = W2\pi\int_0^{\frac{\pi}{2}} xydx$$

$$W2\pi r^2\bar{x} = W2\pi\int_0^{\frac{\pi}{2}} (rsin\theta\ xrcos\theta)rd\theta$$

$$W2\pi r^2\bar{x} = W\pi r^3\int_0^{\frac{\pi}{2}} 2sin\theta\ cos\theta d\theta$$

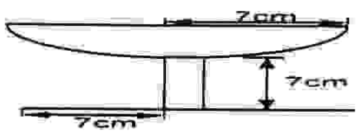
$$\bar{x} = \frac{1}{2}r\int_0^{\frac{\pi}{2}} sin2\theta d\theta$$

$$\bar{x} = \frac{1}{2}r\left[-\frac{1}{2}cos2\theta\right]_0^{\frac{\pi}{2}}$$

$$\bar{x} = -\frac{1}{4}r\left[cos2\left(\frac{\pi}{2}\right) - cos2(0)\right]_0^{\frac{\pi}{2}}$$

$$\bar{x} = -\frac{1}{4}r(-1-1) = \frac{r}{2}$$

2. The figure below is made up of a thin hemispherical cup of radius 7cm. it is welded to a stem of length 7cm and then to a circular base of the same material and of radius 7cm. The base of the stem is one-quarter that of the cup



Find the distance from the base of the centre of gravity of the figure

Solution

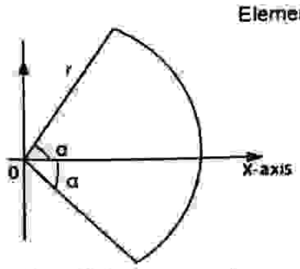
lamina	Area	Weight	C.o.g from base
Circular base	$\pi r^2 = \pi[7]^2 = 49\pi$	$49\pi w$	0
stem	$98\ \pi/4$	$24.5\pi w$	3.5
Hemispherical cup	$2\pi r^2 = 2\pi[7]^2 = 98\pi$	$98\pi w$	10.5
composite	171.5π	$171.5\pi w$	$\bar{x}$

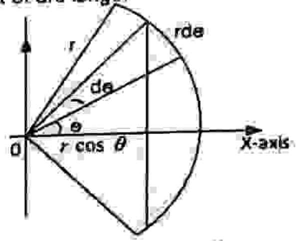
$$171.5\pi W\bar{x} = 98\pi Wx\ 10.5 + 24.5\pi Wx3.5$$

$$\bar{x} = 6.5cm$$

3. Show that the centre of gravity of a uniform lamina in the shape of an arc of a circle of radius r and subtending an angle 2α at the centre O is given by $\frac{rsin\alpha}{\alpha}$ from O.

Solution





Length of the arc= element of the arc length

$$Wr(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha} xdx$$

$$Wr(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha}(rcos\theta)rd\theta$$

$$2\alpha\bar{x} = \int_{-\alpha}^{\alpha} rcos\theta d\theta$$

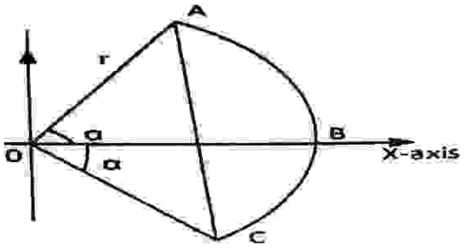
$$2\alpha\bar{x} = r[\sin\theta]_{-\alpha}^{+\alpha}$$

$$\bar{x} = \frac{rsin\alpha}{\alpha}$$

For a semi –circle arc, $\alpha = \frac{\pi}{2}$ there fore

$$\bar{x} = \frac{2r}{\pi}$$

4.



In the figure above ABC is a segment of a circle of radius r and centre O, subtending an angle 2α at the O. Show that the centre of gravity of a segment of a circle is given by $\frac{4}{3}\frac{r\sin^3\alpha}{2\alpha - \sin 2\alpha}$ from O.

Solution

Body	Weight	C.O.G from O
Sector	$wr^2\alpha$	$\frac{2r}{3}\frac{\sin\alpha}{\alpha}$
Triangle	$w\frac{1}{2}r^2\sin 2\alpha$	$\frac{2r}{3}\cos\alpha$
Segment	$wr^2\left(\alpha - \frac{1}{2}\sin 2\alpha\right)$	$\bar{x}$

$$r^2\left(\alpha - \frac{1}{2}\sin 2\alpha\right)\bar{x} = r^2\alpha.\frac{2}{3}\frac{r\sin\alpha}{\alpha} - \frac{1}{2}r^2\sin 2\alpha.\frac{2}{3}r\cos\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r(\sin\alpha - \cos^2\alpha\sin\alpha)}{\alpha - {}^1/_2\sin 2\alpha}$$

$$\bar{x} = \frac{4}{3}\frac{r\sin^3\alpha}{2\alpha - \sin 2\alpha}$$

Exercise 21E

1. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$, the x-axis and the line $x = 2$ **An**$(1.5, 1.2)$ Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = 2x - x^2$, and the x-axis **An**$(1.0, 0.4)$
2. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2 + 2$, the x-axis and the line $x = 1$ and $x = 2$ **An**$(1.56, 2.25)$
3. Find the co-ordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y^2 = 8x$, the x-axis and the line $x = 2$ and $x = 8$ **An**$(5.31, 3.21)$
4. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$ and the line $y = 3x$ **An**$(1.5, 3.6)$
5. Find the co-ordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y = 4 - x^2, y = 3x^2$ and the y-axis **An**$(0.375, 2.2)$
6. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^3$, the x-axis and the line $x = 3$ **An**$(2.4, 7.14)$
7. The area enclosed by the curve $y^2 = x$, the x-axis, the line $x = 4$ and lying in the first quadrant is rotated about the x-axis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(2.67, 0)$
8. The area enclosed by the curve $y^2 = x$, the x-axis, the line $x = 2, x = 4$ and lying in the first quadrant is rotated about the x-axis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(3.39, 0)$ **UNEB 1996 NO.9(b)**
9. The area enclosed by the curve $y^2 = x$, the x-axis, the line $= 0, x = 2$, and $x = 4$ is rotated about the x-asis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(3.39, 0)$
10. The area enclosed by the curve $y = x^2 + 3$, the x-axis, the y-axis and the line $x = 2$ is rotated about the x-asis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(1.30, 0)$
11. The area enclosed by the curve $y = x^3$, the x-axis and the line $x = 3$ is rotated about the x-asis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(2.625, 0)$

CHAPTER 12: VARIABLE ACCELERATION

Let r = displacement, v = velocity and a = acceleration.

Differential calculus

If r, v and a are functions of t:

$velocity, \underset{\smile}{v} = \frac{dr}{dt}$ where r is displacement | $acceleration, \underset{\smile}{a} = \frac{dv}{dt} = \frac{d^2r}{dt^2}$ where v is velocity

$differentiation: displacement \Rightarrow velocity \Rightarrow acceleration$

Examples:

1. A particle moves a long a straight line such that after t seconds its displacement from a fixed point is r meters where $r = 8t^2\hat{\imath} - t^4\hat{\jmath}$. Find
   (a) Velocity after t second | (b) Velocity after 1s | (c) Speed after 1s

   Solution

   (a) $v = \frac{dr}{dt} = \frac{d}{dt}(8t^2\hat{\imath} - t^4\hat{\jmath})$

   $v = 16t\hat{\imath} - 4t^3\hat{\jmath}$

   (b) when $t = 1s$

   $v = 16x1\hat{\imath} - 4(1)^3\hat{\jmath} = 16\hat{\imath} - 4\hat{\jmath}$

   Speed $|v| = \sqrt{16^2 + (-4)^2} = 16.49m/s$

2. A particle moves a long a straight line such that after t seconds its displacement from a fixed point is s meters where $s = 2sint\ \hat{\imath} + 3cost\hat{\jmath}$. Find
   (a) acceleration after t second
   (b) acceleration after $\frac{\pi}{2}s$
   (c) magnitude of the acceleration after $\frac{\pi}{2}s$

   Solution

   (a) $v = \frac{ds}{dt} = \frac{d}{dt}(2sint\ \hat{\imath} + 3cost\hat{\jmath})$

   $v = 2cost\ \hat{\imath} - 3sint\hat{\jmath}$

   $a = \frac{dv}{dt} = \frac{d}{dt}(2cost\ \hat{\imath} - 3sint\hat{\jmath})$

   $a = -2sint\ \hat{\imath} - 3cost\hat{\jmath}$

   (b) **When** $t = \frac{\pi}{2}s$

   $a = -2sin\left(\frac{\pi}{2}\right)\ \hat{\imath} - 3cos\left(\frac{\pi}{2}\right)\hat{\jmath} = -2\ \hat{\imath}\ ms^{-2}$

   (c) $|a| = \sqrt{(-2)^2 + (0)^2} = 2\ ms^{-2}$

3. A particle moves a long a straight line such that after t seconds its displacement from a fixed point is s meters where $s = \begin{pmatrix} sin2t \\ t+1 \\ cost + sint \end{pmatrix}$. Find
   (a) Velocity when $t = \frac{\pi}{2}s$ | (b) speed when $t = \frac{\pi}{2}s$ | (c) acceleration after $\frac{\pi}{2}s$

   Solution

   (a) $v = \frac{ds}{dt} = \frac{d}{dt}(sin2t\ \hat{\imath} + [t+1]\hat{\jmath} + [cost + sint]\hat{k})$

   $v = 2cos2t\ \hat{\imath} + \hat{\jmath} + [sint - cost]\hat{k}$

   When $t = \frac{\pi}{2}s$

   $v = 2cos2\left(\frac{\pi}{2}\right)\ \hat{\imath} + \hat{\jmath} + \left[sin\frac{\pi}{2} - cos\frac{\pi}{2}\right]\hat{k}$

   $v = -2\ \hat{\imath} + \hat{\jmath} - \hat{k}$

   (b) $|v| = \sqrt{(-2)^2 + (1)^2 + (-1)^2} = 2.45\ ms^{-1}$

   (c) $a = \frac{dv}{dt} = \frac{d}{dt}(2cos2t\ \hat{\imath} + \hat{\jmath} + [sint - cost]\hat{k})$

   $a = -4sin2t\ \hat{\imath} + [cost + sint]\hat{k}$

   When $t = \frac{\pi}{2}s$

   $a = -4sin2\left(\frac{\pi}{2}\right)\ \hat{\imath} + \left[cos\frac{\pi}{2} + sin\frac{\pi}{2}\right]\hat{k}$

   $a = \hat{k}ms^{-2}$

4. A particle moves in the x-y plane such that its position vector at any time t is given by $r = (3t^2 - 1)\hat{\imath} + (4t^3 + t - 1)\hat{\jmath}$. Find **Uneb 2002 No.8**
   (a) Speed after $t = 2s$ | (b) Magnitude of the acceleration after $t = 2s$

   Solution

   (a) $v = \frac{dr}{dt} = \frac{d}{dt}\{(3t^2 - 1)\hat{\imath} + (4t^3 + t - 1)\hat{\jmath}\}$

   $v = 6t\hat{\imath} + (12t^2 + 1)\hat{\jmath}$

   when $t = 2s$

   $v = 6x2\hat{\imath} + (12x2^2 + 1)\hat{\jmath} = 12\hat{\imath} + 49\hat{\jmath}$

   Speed $|v| = \sqrt{12^2 + (49)^2} = 50.45m/s$

   (b) $a = \frac{dv}{dt} = \frac{d}{dt}\{6t\hat{\imath} + (12t^2 + 1)\hat{\jmath}\}$

   $a = 6\ \hat{\imath} + 24t\hat{\jmath}$

   When $t = 2s$; $a = 6\ \hat{\imath} + 24x2\hat{\jmath} = 6\ \hat{\imath} + 48\hat{\jmath}$

   $|a| = \sqrt{(6)^2 + (48)^2} = 48.37ms^{-2}$

Exercise 22A

1. The displacement of a particle after t seconds is given by $r = t^3\hat{\imath} + 9t\hat{\jmath}$. Find the speed when $t = 2s$ **An** $15\ ms^{-1}$
2. The displacement of a particle after t seconds is given by $s = 2\sqrt{3}sint\hat{\imath} + 8cost\hat{\jmath}$. Find the speed when $t = \frac{\pi}{6}s$ **An** $5\ ms^{-1}$
3. The displacement of a particle after t seconds is given by $r = 8t^3\hat{\imath} + 2t^2\hat{\jmath}$. Find the
   (i) acceleration when $t = 1s$
   (ii) magnitude of the acceleration when $t = 1s$
   An. $(6\hat{\imath} + 4\hat{\jmath})\ ms^{-2}$, $7.21\ ms^{-2}$
4. The velocity of a particle after t seconds is given by $v = 2t^2\hat{\imath} + 6\hat{\jmath}$. Find the magnitude of acceleration when $t = 3s$ **An.** $12\ ms^{-2}$
5. The velocity of a particle after t seconds is given by $v = sin2t\hat{\imath} - cost\hat{\jmath}$. Find the
   (i) acceleration when $t = \frac{\pi}{6}s$ **An.** $(-2\hat{\imath} + \hat{\jmath})\ ms^{-2}$,
   (ii) magnitude of acceleration when $t = \frac{\pi}{6}s$ **An.** $2.24\ ms^{-2}$
6. A particle of mass 6kg moves such that its displacement $S = \begin{pmatrix} t^2 - 5 \\ t^2 - 3t + 2 \end{pmatrix} m$. Find the
   (a) velocity after time t **An.** $(2t\hat{\imath} + [2t - 3]\hat{\jmath})\ ms^{-1}$,
   (b) speed of the particle at $t = 2s$ **An.** $15\ ms^{-1}$
   (c) acceleration and hence determine the force acting on the particle **An.** $(2\hat{\imath} + 2\hat{\jmath})\ ms^{-2}$, $(12\hat{\imath} + 12\hat{\jmath})\ N$
7. A particle of mass 2kg moves such that its displacement $S = \begin{pmatrix} t^2 - 4t - 5 \\ t^2 - 4t + 3 \end{pmatrix} m$. Find the
   (a) speed of the particle at $t = 2s$ **An.** $0\ ms^{-1}$
   (b) force acting on the particle **An.** , $(4\hat{\imath} + 4\hat{\jmath})\ N$
8. A particle of mass 4kg moves such that its displacement $S = (t^3 - t^2 - 4t + 3)\hat{\imath} + (t^3 - 2t^2 + 3t - 7)\hat{\jmath}$. Find the
   (a) speed of the particle at $t = 4s$ **An.** $50.21\ ms^{-1}$
   (b) magnitude of the force acting on the particle $t = \frac{2}{3}s$ **An.** , $8\ N$
9. A particle of mass 0.5kg moves such that its displacement $r = \begin{pmatrix} 4sin2t \\ 2cost - 1 \end{pmatrix} m$. Find the
   (a) velocity of the particle at $t = \frac{\pi}{6}s$ **An.** $(4\hat{\imath} - \hat{\jmath})\ ms^{-1}$
   (b) force acting on the particle at any time t **An.** $-8sin2t\hat{\imath} - cost\hat{\jmath}$,
10. A particle of mass 2kg moves such that its displacement $r = (2 - cos3t)\hat{\imath} + (6sin2t)\hat{\jmath}$. Find the
   (a) velocity of the particle at $t = \frac{\pi}{6}s$ **An.** $(3\hat{\imath} + 6\hat{\jmath})\ ms^{-1}$
   (b) force acting on the particle at $t = \pi s$ **An.** , $-18\hat{\imath}\ N$
11. A particle moves such that its displacement $s = (2sint + sin2t)\hat{\imath} + (4cost + cos2t)\hat{\jmath}$. . Find the
   (a) velocity of the particle at $t = \frac{\pi}{3}s$ **An.** $-3\sqrt{3}\hat{\jmath}\ ms^{-1}$
   (b) acceleration of the particle at $t = \frac{\pi}{2}s$ **An.** , $(-2\hat{\imath} + 4\hat{\jmath})\ ms^{-2}$

Integral calculus

If r, v or a are functions of time t:

$\boxed{velocity, \underset{\sim}{v} = \int adt + \underset{\sim}{c}}$ and $\boxed{displacement, \underset{\sim}{r} = \int vdt + \underset{\sim}{c}}$

integration: acceleration ⇒ velocity ⇒ displacement

Examples

1. The velocity of the particle $v = 3t^2\hat{\imath} + 10t\hat{\jmath}$. Given that the displacement is $4\hat{\imath} - 4\hat{\jmath}$ at $t = 0$. Find the distance of the body from the origin when $t = 2s$

Solution

$$r = \int vdt + c$$
$$r = \int \{3t^2\hat{\imath} + 10t\hat{\jmath}\}dt + c$$
$$r = t^3\hat{\imath} + 5t^2\hat{\jmath} + c$$

At $t = 0$, $r = 4\hat{\imath} - 4\hat{\jmath}$
$$4\hat{\imath} - 4\hat{\jmath} = 0^3\hat{\imath} + 5x0^2\hat{\jmath} + c$$
$$c = 4\hat{\imath} - 4\hat{\jmath}$$
$$r = (t^3 + 4)\hat{\imath} + (5t^2 - 4)\hat{\jmath}$$

When $t = 2s$
$$r = (2^3 + 4)\hat{\imath} + (5x2^2 - 4)\hat{\jmath}$$
$$r = (12\hat{\imath} + 16\hat{\jmath})m$$
$$|r| = \sqrt{(12)^2 + (16)^2} = 20m$$

2. A particle is accelerated from rest at the origin with an acceleration of $(2t + 4)\ ms^{-2}$. Find
   (i) Velocity attained after $t = 2s$
   (ii) Distance travelled at $t = 1s$

Solution

$$v = \int a dt + c$$
$$v = \int (2t + 4)dt + c$$
$$v = t^2 + 4t + c$$
At $t = 0, v = 0$
$$0 = 0^2 + 4x0 + c$$
$$c = 0$$
$$v = t^2 + 4t$$

When $t = 2s$
$$v = 2^2 + 4x2 = 12m/s$$
$$r = \int v dt + c = \int (t^2 + 4t)dt + c$$
$$r = \frac{t^3}{2} + 2t^2 + c$$
At $t = 0, r = 0$

$$0 = \frac{0^3}{2} + 2x0^2 + c$$
$$c = 0$$
$$r = \frac{t^3}{2} + 2t^2$$
When $t = 1s$
$$r = \frac{1^3}{2} + 2x1^2 = 2.33m$$

3. A particle is accelerated from rest at the origin with an acceleration of $(4t + 2)\hat{\imath} - 3\hat{\jmath}$. Find

(i) Velocity attained after $t = 3s$

(ii) Speed at $t = 3s$

Solution

$$v = \int a dt + c$$
$$v = \int \{(4t + 2)\hat{\imath} - 3\hat{\jmath}\}dt + c$$
$$v = (2t^2 + 2t)\hat{\imath} - 3t\hat{\jmath} + c$$
At $t = 0, v = 0$: $v = (2x0^2 + 2x0)\hat{\imath} - 3x0\hat{\jmath} + c$
$$c = 0$$
$$v = (2t^2 + 2t)\hat{\imath} - 3t\hat{\jmath}$$
When $t = 3s$: $v = (2x3^2 + 2x3)\hat{\imath} - 3x3\hat{\jmath}$
$$v = (24\hat{\imath} - 9\hat{\jmath})m/s$$
$$|v| = \sqrt{(24)^2 + (-9)^2} = 25.63ms^{-1}$$

4. A particle starts from rest origin (0,0). Its acceleration in ms^{-2} at time t second is given by $a = 6t\hat{\imath} - 4\hat{\jmath}$. Find its speed when $t = 2$ second **Uneb 2014 No.2**

Solution

$$v = \int a dt + c$$
$$v = \int \{6t\hat{\imath} - 4\hat{\jmath}\}dt + c = 3t^2\hat{\imath} - 4t\hat{\jmath} + c$$
At $t = 0, v = 0$: $v = 3x0^2\hat{\imath} - 4x0\hat{\jmath} + c$
$$c = 0$$
$$v = 3t^2\hat{\imath} - 4t\hat{\jmath}$$
When $t = 2s$
$$v = 3x2^2\hat{\imath} - 4x2\hat{\jmath} = (12\hat{\imath} - 8\hat{\jmath})m/s$$
$$|v| = \sqrt{(12)^2 + (-8)^2} = 14.42ms^{-1}$$

5. An object of mass 5kg is initially at rest at a point whose position vector is $-2\hat{\imath} + \hat{\jmath}$. If it is acted upon by a force $F = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$. Find

(i) The acceleration

(ii) Its speed after 3 second

(iii) Its distance from the origin after 3 seconds

Solution

(i) $F = ma$
$$a = \frac{1}{5}(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k})ms^{-2}$$
(ii) $v = \int a dt + c = \frac{1}{5}\int(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k})dt + c$
$$v = \frac{1}{5}(2t\hat{\imath} + 3t\hat{\jmath} - 4t\hat{k}) + c$$
At $t = 0, v = 0$: $v = \frac{1}{5}(2x0\hat{\imath} + 3x0\hat{\jmath} - 4x0\hat{k}) + c$
$$c = 0$$
$$v = \frac{1}{5}(2t\hat{\imath} + 3t\hat{\jmath} - 4t\hat{k})$$
When $t = 3s$: $v = \frac{1}{5}(2x3\hat{\imath} + 3x3\hat{\jmath} - 4x3\hat{k})$
$$|v| = \frac{1}{5}\sqrt{(6)^2 + (9)^2 + (-12)^2} = 3.23ms^{-1}$$
(iii) $r = \int v dt + c$
$$r = \int \frac{1}{5}(2t\hat{\imath} + 3t\hat{\jmath} - 4t\hat{k})dt + c$$
$$r = \frac{1}{5}(t^2\hat{\imath} + 1.5t^2\hat{\jmath} - 2t^2\hat{k}) + c$$
At $t = 0$, $r = -2\hat{\imath} + \hat{\jmath}$
$$-2\hat{\imath} + \hat{\jmath} = \frac{1}{5}(0^2\hat{\imath} + 1.5x0^2\hat{\jmath} - 2x0^2\hat{k}) + c$$
$$c = -2\hat{\imath} + \hat{\jmath}$$
$$r = \frac{1}{5}([t^2 - 10]\hat{\imath} + [1.5t^2 + 5]\hat{\jmath} - 2t^2\hat{k})$$
When $t = 3s$
$$r = \frac{1}{5}([3^2 - 10]\hat{\imath} + [1.5x3^2 + 5]\hat{\jmath} - 2x3^2\hat{k})$$
$$r = \frac{1}{5}(-\hat{\imath} + 18.5\hat{\jmath} - 18\hat{k})m$$
$$|r| = \frac{1}{5}\sqrt{(-1)^2 + (18.5)^2 + (-18)^2} = 5.166m$$

6. A particle starts from rest at a point (2,0,0) and moves such that its acceleration in ms^{-2} at time t second is given by $a = [16cos4t\hat{\imath} + 8sin2t\hat{\jmath} + (sint - 2sin2t)\hat{k}]\ ms^{-2}$. Find the: **Uneb 2016 No.13**

(a) speed when $t = \frac{\pi}{4}$

(b) Distance from origin when $t = \frac{\pi}{4}$

Solution

$$v = \int a\,dt + C = \int \begin{pmatrix} 16cos4t \\ 8sin2t \\ sint - 2sin2t \end{pmatrix} dt + c$$

$$v = \begin{pmatrix} 4sin4t \\ -4cos2t \\ -cost + cos2t \end{pmatrix} + c$$

At $t = 0, v = 0;$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4sin4x0 \\ -4cos2x0 \\ -cos0 + cos2x0 \end{pmatrix} + c$

$$c = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$v = \begin{pmatrix} 4sin4t \\ 4 - 4cos2t \\ -cost + cos2t \end{pmatrix}$$

When $t = \frac{\pi}{4}$: $v = \begin{pmatrix} 4sin4x\frac{\pi}{4} \\ 4 - 4cos2x\frac{\pi}{4} \\ -cosx\frac{\pi}{4} + cos2x\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -\sqrt{2}/_2 \end{pmatrix}$

$$|v| = \sqrt{(0)^2 + (4)^2 + \left(-\sqrt{2}/_2\right)^2} = 4.062ms^{-1}$$

$$r = \int v\,dt + c = \int \begin{pmatrix} 4sin4t \\ 4 - 4cos2t \\ -cost + cos2t \end{pmatrix} dt + c$$

$$r = \begin{pmatrix} -cos4t \\ 4t - 2sin2t \\ -sint + 0.5sin2t \end{pmatrix} + c$$

At $t = 0, r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -cos4x0 \\ 4x0 - 2sin2x0 \\ -sin0 + 0.5sin2x0 \end{pmatrix} + c$$

$$c = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$r = \begin{pmatrix} 3 - cos4t \\ 4t - 2sin2t \\ -sint + 0.5sin2t \end{pmatrix}$$

When $t = \frac{\pi}{4}$

$$r = \begin{pmatrix} 3 - cos4x\frac{\pi}{4} \\ 4x\frac{\pi}{4} - 2sin2x\frac{\pi}{4} \\ -sin\frac{\pi}{4} + 0.5sin2x\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 4 \\ 1.1416 \\ -0.2071 \end{pmatrix}$$

$$|r| = \sqrt{(4)^2 + (1.1416)^2 + (-0.2071)^2}$$
$$= 4.1649m$$

8. A particle of mass 4 kg starts from rest at the origin. It acted upon by a force $F = (2t\bar{\imath} + 3t^2\bar{\jmath} + 5\bar{k})N$. Find the work done by the force F after 3 seconds.

Solution

$$F = ma$$

$$a = \frac{1}{4}(2t\bar{\imath} + 3t^2\bar{\jmath} + 5\bar{k})ms^{-2}$$

$$v = \int a\,dt + c = \frac{1}{4}\int(2t\bar{\imath} + 3t^2\bar{\jmath} + 5\bar{k})dt + c$$

$$v = \frac{1}{4}(t^2\bar{\imath} + 3t^3\bar{\jmath} + 5t\bar{k}) + c$$

At $t = 0, v = 0$: $v = \frac{1}{4}(0^2\bar{\imath} + 3x0^3\bar{\jmath} + 5x0\bar{k}) + c$

$$c = 0$$

$$v = \frac{1}{4}(t^2\bar{\imath} + 3t^3\bar{\jmath} + 5t\bar{k})$$

$$r = \int \frac{1}{4}(t^2\bar{\imath} + 3t^3\bar{\jmath} + 5t\bar{k})dt + c$$

$$r = \frac{1}{4}\left(\frac{1}{3}t^3\bar{\imath} + \frac{3}{4}t^4\bar{\jmath} + \frac{5}{2}t^2\bar{k}\right) + c$$

At $t = 0$, $r = 0$

$$0 = \frac{1}{4}\left(\frac{1}{3}x0^3\bar{\imath} + \frac{3}{4}x0^4\bar{\jmath} + \frac{5}{2}x0^2\bar{k}\right) + c$$

$$c = 0$$

$$r = \frac{1}{4}\left(\frac{1}{3}t^3\bar{\imath} + \frac{3}{4}t^4\bar{\jmath} + \frac{5}{2}t^2\bar{k}\right)$$

$W = F.d$

At t= 3s: $F = \begin{pmatrix} 2x3 \\ 3x3^2 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 27 \\ 5 \end{pmatrix}N$

$$d = \frac{1}{4}\begin{pmatrix} \frac{1}{3}x3^3 \\ \frac{3}{4}x3^4 \\ \frac{5}{2}x3^2 \end{pmatrix} = \begin{pmatrix} 2.25 \\ 15.1875 \\ 5.625 \end{pmatrix}m$$

$$W = \begin{pmatrix} 6 \\ 27 \\ 5 \end{pmatrix}.\begin{pmatrix} 2.25 \\ 15.1875 \\ 5.625 \end{pmatrix} = 451.6875J$$

Exercise 22B

1. A particle of mass 4 kg starts from rest at a point $(2\hat{\imath} - 3\hat{\jmath} + \hat{k})m$. It moves with acceleration $a = (4 + 2\hat{\jmath} - 3\bar{k})ms^{-2}$ when a constant force F acts on it. Find the: **Uneb 2018 No.10**

   (a) force F.

(b) velocity at any time t.
(c) work done by the force F after 6 seconds.

An $\begin{pmatrix}16\\8\\-12\end{pmatrix}N$, $\begin{pmatrix}4t\\2t\\-3t\end{pmatrix}m/s$, $2088J$

2. A particle of mass 3kg is acted upon by a force $F = (6\hat{\imath} - 36t^2\hat{\jmath} + 54t\hat{k})N$ at time t. At time $t = 0$, the particle is at the point with a potion vector $\hat{\imath} - 5\hat{\jmath} - \hat{k}$ and its velocity is $(3\hat{\imath} + 3\hat{\jmath})m/s$. Determine the, **Uneb 2017 No.10**
   i. Position vector of the particle at a time $t = 1$ second
   ii. Distance of the particle from the origin at time $t = 1$ second. **An** $\begin{pmatrix}5\\-3\\2\end{pmatrix}m$, $6.1644m$
3. The acceleration of a particle is $6t\hat{\imath} + 2\hat{\jmath}$. Given that the velocity is $(4\hat{\imath} - \hat{\jmath})m/s$ and displacement is $(2\hat{\imath} + 3\hat{\jmath})m$ when $t = 1s$. Find the displacement when $t = 3s$ **An** $(30\hat{\imath} + 5\hat{\jmath})m$
4. If the velocity of a particle is $4cos2t\hat{\imath} + 2sin2t\hat{\jmath}$, given that the displacement is $6\hat{\imath} - 2\hat{\jmath}$ when $t = \frac{\pi}{4}s$. Find the distance of the body from origin when $t = \pi s$ **An** $5m$
5. If the acceleration of a particle is $9sin3t\hat{\imath} + 2cost\hat{\jmath}$, and the body is initially at rest. Find its velocity when $t = \frac{\pi}{6}s$. **An** $(3\hat{\imath} + \hat{\jmath})m/s$
6. If the acceleration of a particle is $6sin6t\hat{\imath} + 9cos3t\hat{\jmath}$, given that the velocity is $(\hat{\imath} + 3\hat{\jmath})m/s$ and displacement is $(5\hat{\imath} + 2\hat{\jmath})m$ when $t = \frac{\pi}{6}s$. Find the displacement when $t = \frac{\pi}{3}s$ **An** $(5\hat{\imath} + 3\hat{\jmath})m$
7. If the acceleration of a particle is $6t\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$, given that the velocity is $(3\hat{\imath} + 6\hat{\jmath} - 3\hat{k})m/s$ and displacement is $(2\hat{\imath} + 5\hat{\jmath} - 2\hat{k})m$ when $t = 1s$. Find the;
   (i) Velocity when $t = 2s$
   (ii) displacement when $t = 3s$

   An $(12\hat{\imath} + 12\hat{\jmath} - 5\hat{k})m/s$, $(28\hat{\imath} + 29\hat{\jmath} - 12\hat{k})m$
8. If the acceleration of a particle is $2\hat{\imath} + 6\hat{\jmath} + 12t^2\hat{k}$, given that the velocity is $(3\hat{\imath} + \hat{k})m/s$ and displacement is $(-\hat{\imath} + \hat{k})m$ when $t = 1s$. Find the;
   (i) Velocity when $t = 1s$
   (ii) displacement when $t = 2s$

   An $(-\hat{\imath} + 3\hat{\jmath} + 5\hat{k})m/s$, $(-3\hat{\imath} + 8\hat{\jmath} + 19\hat{k})m$
9. If the acceleration of a particle is $6t\hat{\imath} - 2\hat{k}$, given that the velocity is $(\hat{\imath} + 12\hat{\jmath} - 4\hat{k})m/s$ and displacement is $(3\hat{\imath} + 6\hat{\jmath})m$ when $t = 2s$. Find the;
   (i) Velocity when $t = 4s$
   (ii) displacement when $t = 3s$

   An $(37\hat{\imath} + 12\hat{\jmath} - 8\hat{k})m/s$, $(11\hat{\imath} + 18\hat{\jmath} - 5\hat{k})m$
10. The velocity of the particle $v = 4t^3\hat{\imath} + 6t\hat{\jmath} - 3t^2\hat{k}$. Given that the displacement is $14\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$ at $t = 1$. Find the;
    (i) Acceleration when $t = 3s$
    (ii) displacement when $t = 0s$

    An $(108\hat{\imath} + 6\hat{\jmath} - 18\hat{k})ms^{-2}$, $(13\hat{\imath} + 3\hat{\jmath} - 2\hat{k})m$
11. The velocity of the particle $v = (3t^2 - 10t)\hat{\imath} + 2\hat{\jmath} - 6t\hat{k}$. Given that the displacement is $-9\hat{\imath} + 3\hat{\jmath} - 13\hat{k}$ at $t = 2$. Find the;
    (i) Acceleration when $t = 5s$
    (ii) displacement when $t = 3s$

    An $(20\hat{\imath} - 6\hat{k})ms^{-2}$, $(15\hat{\imath} + 5\hat{\jmath} - 28\hat{k})m$
12. The particle starts from rest at the origin moving with a velocity of $v = \begin{pmatrix}2cos2t + 11\\3sin3t\\4\end{pmatrix}$ Find the;
    (i) Speed when $t = \frac{\pi}{6}s$
    (ii) displacement when $t = \frac{\pi}{2}s$
    (iii) Acceleration when $t = \pi s$

    An $13m/s$, $(1.5\pi\hat{\imath} + \hat{\jmath} - 2\pi\hat{k})m$, $(9\hat{\jmath})m$
13. A particle starts from rest at a point (2,0,0) and moves such that its acceleration in ms^{-2} at time t second is given by $a = [16cos4t\hat{\imath} + 8sin2t\hat{\jmath} + (sint - 2sin2t)\hat{k}]\,ms^{-2}$. Find the;
    (a) Acceleration when $t = \pi$
    (b) Velocity when $t = \frac{\pi}{2}$
    (c) Displacement from origin when $t = \frac{\pi}{4}$

    An $(16\hat{\imath})ms^{-2}$, $(8\hat{\imath} - \hat{k})m/s$

CHAPTER 13: COPLANAR FORCES (RIGID BODIES)

A rigid body is one in which distances between its various parts remain fixed

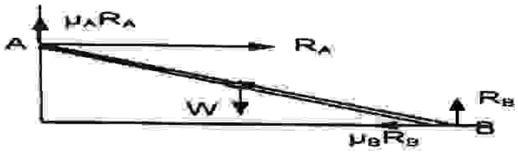
Equilibrium of a rigid body

(i) Sum of Force acting in one direction is equal to sum of forces acting in opposite direction
(ii) Sum of Clockwise moments about a point is equal to sum of anticlockwise moments about the same point

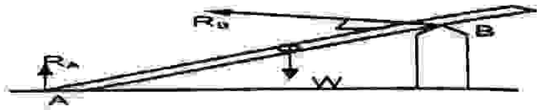
Points to note

- When a rigid body rests in contact with a smooth wall and a string tied at the base, the reaction on the body is perpendicular to the surface

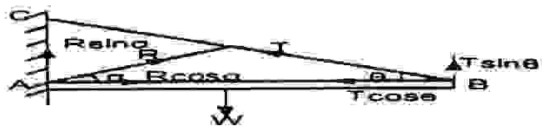
- When a rigid body rest in contact with a rough wall and rough ground



- A rigid body resting against a smooth peg or bar, the reaction on the rigid body is perpendicular to the bar



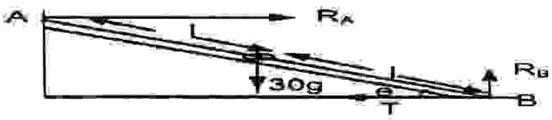
- When a rod is hinged, the reaction at the hinge acts at an angle to either the horizontal or vertical



SMOOTH CONTACTS AT THE LADDER

1. A uniform ladder AB of mass 30kg rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal inextensible string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 60° to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. Find
   (i) Tension in the string
   (ii) Normal reactions at points A and B

Solution



(i) (↑) $R_B = 30g = 30x9.8 = 294N$

(→) $R_A = T$ ....(ii)

B↶ $R_A\ x\ 2l\sin\theta = 30gxl\cos\theta$ ......(iii)

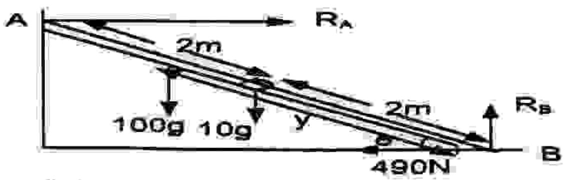
$R_A\ x\ 2l\sin60 = 30gxl\cos60$

$R_A = 84.87N$

$T = 84.87N$

2. A uniform ladder AB of mass 10kg and length 4m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium at an angle $tan^{-1}(2)$ to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. A man of mass 100kg ascends the ladder.
   (i) If the string will break when the tension exceeds 490N, find how far up the ladder the man can go before this occurs
   (ii) What tension must the string be capable of withstanding if the man is to reach the top of the ladder

Solution



(i) (→) $R_A = 490N$ ....(i)

B↶ $R_A\ x\ 4\sin\theta = 10gx2\cos\theta + 100gy\cos\theta$

$R_A\ x\ 4\tan\theta = 20g + 100gy$

$490\ x\ 4\tan\theta = 20g + 100gy$

$y = \frac{490x4x2-20g}{100g} = 3.8m$

(ii) (→) $R_A = T$ ....(i)

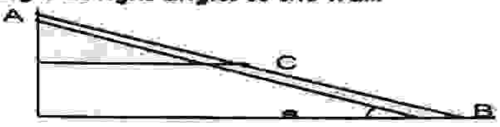
B↶ $R_A\ x\ 4\sin\theta = 10gx2\cos\theta + 100gx4\cos\theta$

$R_A\ x\ 4\tan\theta = 20g + 400g$

$R_A = 514.5N$

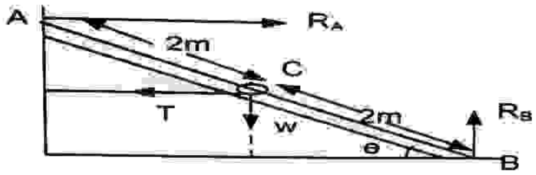
T = 514.5N

3. A uniform ladder AB of weight W and length 4m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The ladder is kept in equilibrium inclined at θ to the ground by a light horizontal string attached to the wall and to a point C on the ladder. The vertical plane containing the ladder and the string is at right angles to the wall.



If $\tan\theta = 2$, find the tension in the string when BC is of length 3m

Solution



(↑) $R_B = W \ldots\ldots(i)$

(→) $R_A = T \ldots\ldots(ii)$

B↶ $R_A \; x \; 4\sin\theta = Wx2\cos\theta + Tx3\sin\theta \ldots\ldots(iii)$

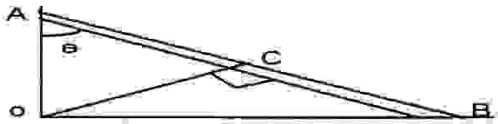
$R_A \; x \; 4\tan\theta = 2W + Tx3\tan\theta$

$R_A = \frac{2W+Tx3\tan\theta}{4\tan\theta} = \frac{2W+Tx3x2}{4x2} = \frac{W+3T}{4}$

$\frac{W+3T}{4} = T$

$T = W$

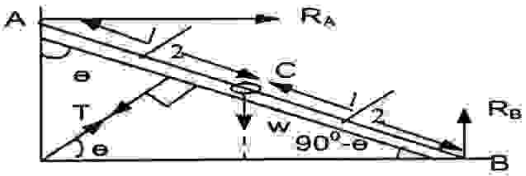
4. A uniform rod AB of weight W and length l rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal table. The ladder is kept in equilibrium inclined at θ to the wall by a light inextensible string OC. C being a point on AB such that OC is perpendicular to AB and O on the point of intersection of the wall and the table. Angle AOB is 90° **UNEB 2000 No.13**



Find;

(i) Tension in the string

Solution



(↑) $R_B = W + T\sin\theta \ldots\ldots(i)$

(→) $R_A = T\cos\theta \ldots\ldots(ii)$

Taking moments about O

$R_B \; x \; l\sin\theta = Wx\, {}^{l}/_{2}\sin\theta + R_A \; x \; l\cos\theta$

$(W + T\sin\theta)l\sin\theta = W\, {}^{l}/_{2}\sin\theta + T\cos\theta \; l\cos\theta$

$T(\cos^2\theta - \sin^2\theta) = \frac{W(2\sin\theta - \sin\theta)}{2}$

$T = \frac{W\sin\theta}{2(\cos^2\theta - \sin^2\theta)}$

(j) Reactions at A and B in terms of θ and W

$R_A = T\cos\theta = \left[\frac{W\sin\theta}{2(\cos^2\theta - \sin^2\theta)}\right]\cos\theta$

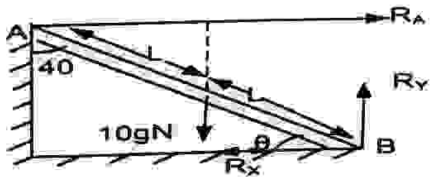
$R_A = \frac{W\sin\theta\cos\theta}{2(\cos^2\theta - \sin^2\theta)}$

$R_B = W + T\sin\theta = W + \left[\frac{W\sin\theta}{2(\cos^2\theta - \sin^2\theta)}\right]\sin\theta$

$R_B = \left[\frac{2\cos^2\theta - 2\sin^2\theta + \sin^2\theta}{2(\cos^2\theta - \sin^2\theta)}\right]W = \frac{W(2\cos^2\theta - \sin^2\theta)}{2(\cos^2\theta - \sin^2\theta)}$

5. A uniform rod AB of mass 10kg is smoothly hinged at B and rests in a vertical plane with the end A against a smooth vertical wall. If the rod makes an angle of 40° with the wall, find the reaction on the wall and the magnitude of the reaction at B

Solution



let length of the ladder be $2L$

$\theta = 90° - 40° = 50°$

Taking moments about B

$R_A \; x \; 2L\sin\theta = 10g \; x \; L\cos\theta$

$R_A \; x \; 2L\sin 50 = 10x9.8l\cos 50$

$R_A = 41.12N$

(↑): $R_Y = 10gN = 10x9.8 = 98N$

(→): $R_X = R_A$

$\therefore R_X = 41.12$

$R = \sqrt{(R_X)^2 + (R_Y)^2}$

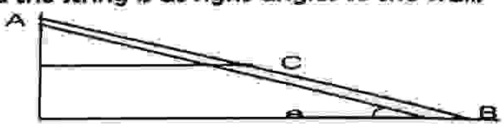
$R = \sqrt{(41.12)^2 + (98)^2} = 106.28N$

$\alpha = \tan^{-1}\left(\frac{R_Y}{R_X}\right) = \tan^{-1}\left(\frac{98}{41.12}\right) = 67.24°$

Reaction at B is $106.28N$ at 67.24° to the beam.

Exercise 23A

1. A uniform ladder AB of mass 10kg and length 4m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 40° to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. Find
   (i) Tension in the string
   (ii) Normal reactions at points A and B
   An($R_A = 58.4N, T = 58.4N$
2. A uniform ladder AB of mass 30kg and length 6m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal inextensible string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 70° to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. Find
   (i) Tension in the string
   (ii) Normal reactions at points A and B.
   An(53.3N, 294N, 53.5N)
3. A uniform ladder AB of mass 30kg rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal inextensible string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 80° to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. Find
   (i) Tension in the string
   (ii) Normal reactions at points A and B.
   An(25.92N, 294N, 25.92N)
4. A uniform ladder AB of mass 8kg rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The base of the ladder is 1m from the floor and the top of the ladder is 2m from the floor. A light horizontal inextensible string, which has one end attached to B and the other end attached to the a point on the wall vertically below the top of the ladder and 1m above the floor, keeps the ladder in equilibrium. Find the tension in the string. **An(27.7N)**
5. A uniform ladder AB of weight W and length 2l rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The ladder is held in position by a light horizontal inextensible string, which has one end attached to B and the other end attached to a point on the wall vertically below the top of the ladder, keeps the ladder in equilibrium. Find the tension in the string. **An**($\frac{W}{2\sqrt{3}}$)
6. A uniform ladder AB of weight W and length 4m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The ladder is kept in equilibrium inclined at θ to the ground by a light horizontal string attached to the wall and to a point C on the ladder. The vertical plane containing the ladder and the string is at right angles to the wall.



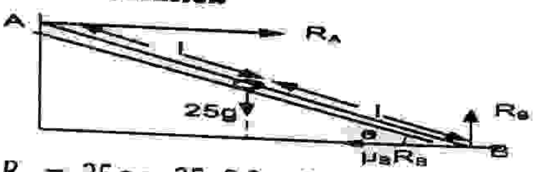
If $\tan\theta = 2$, find the tension in the string when BC is of length
   (i) 1m
   (ii) 2m **An**($\frac{W}{3}N, \frac{W}{2}N$)
7. A uniform ladder AB of mass mkg and length 2l rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The ladder held in position by a light horizontal inextensible string, which has one end attached to B and the other end attached to a point on the wall vertically below the top of the ladder, keeps it in equilibrium inclined at angle θ to the vertical.

(a) A man of mass 4m kg stands on the ladder at a distance of $^l/_2$ from the bottom of the ladder. Find
   (i) The tension in the rope
   (ii) Normal reaction at the bottom of the ladder
   (iii) Normal reaction at the top of the ladder.
(b) If the maximum tension which the rope can bear without breaking is 4mg tanθ, find how far up the ladder the man can safely climb
   An($\frac{3mg\tan\theta}{2}, 5mg, \frac{3mg\tan\theta}{2}, {}^{7l}/_{4}$)

ROUGH CONTACT AT THE FOOT AND SMOOTH CONTACT AT THE TOP OF THE LADDER

1. A uniform ladder AB of mass 25kg rests in limiting equilibrium with the top end against a smooth vertical wall and its base on a rough horizontal floor. If the ladder makes an angle of 75° with the horizontal. Find:
   (i) Magnitude of the normal reaction at the floor
   (ii) Coefficient of friction between the floor and the ladder

Solution



(i) (↑) $R_B = 25g = 25x9.8 = 245N$

(→) $R_A = \mu_B R_B$ ……(ii)

$245\mu_B = R_A$

Taking moments about B

$R_A\ x\ 2lsin\theta = 25gxlcos\theta$ …..(iii)

$R_A\ x\ 2lsin75 = 25gxlcos75$

$R_A = 32.824N$

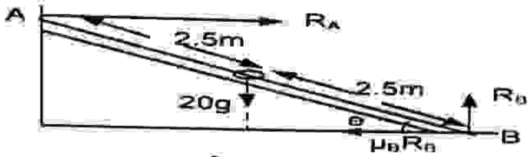
$245\mu_B = R_A$

$245\mu_B = 32.824$

$\mu_B = 0.134$

2. A uniform ladder which is 5m long and has a mass of 20kg leans with its upper end against a smooth vertical wall and its lower end on a rough ground. The bottom of the ladder is 3m from the wall. Calculate
   (i) the frictional force between the ladder and the ground
   (ii) the coefficient of friction

Solution



$Cos\theta = \frac{3}{5}$ ∴ $\theta = 53.13°$

(i) (↑) $R_B = 20g = 20x9.8 = 196N$

(→) $R_A = \mu_B R_B$ ……(ii)

$196\mu_B = R_A$

Taking moments about B

$R_A\ x\ 5sin\theta = 20gx2.5cos\theta$ …..(iii)

$R_A\ x5sin53.13° = 20gx2.5cos53.13°$

$R_A = 73.5N$

Frictional force $= 73.5N$

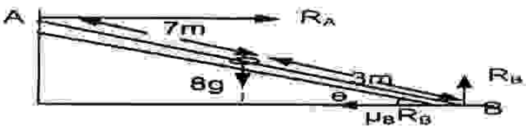
(ii) $196\mu_B = R_A$

$196\mu_B = 73.5$

$\mu_B = 0.375$

3. A non uniform ladder AB 10m long and mass 8kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground and the upper end resting against a smooth vertical wall. If the centre of gravity of the ladder is 3m from the foot of the ladder and the ladder makes an angle of 30° with horizontal, find the
   (i) Coefficient of friction between the ladder and the ground
   (ii) Reaction at the wall **UNEB 2004 No.15**

Solution



(i) (↑) $R_B = 8g = 8x9.8 = 78.4N$

(→) $R_A = \mu_B R_B$ ……(ii)

$78.4\mu_B = R_A$

Taking moments about B

$R_A\ x\ 10sin\theta = 8gx3cos\theta$ …..(iii)

$R_A\ 10sin30 = 8gcos30$

$R_A = 40.738N$

$78.4\mu_B = R_A$

$78.4\mu_B = 40.738$

$\mu_B = 0.5196$

4. A uniform ladder AB 10m long and mass 30kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground and the upper end resting against a smooth vertical wall. If the ladder makes an angle of 60° with horizontal, with a man of mass 90kg standing on the ladder at a point 7.5m from its base. find the;
   (i) Magnitude of the normal reaction and of the frictional force at the ground
   (ii) The minimum value for the coefficient of friction between the ladder and the ground that would enable the man to climb to the top of the ladder

Solution

(ii) (↑) $R_B = 30g + 90g = 120x9.8 = 1176N$

(→) $R_A = \mu_B R_B$ ……(ii)

$1176\mu_B = R_A$

Taking moments about B

$R_A\ x\ 10sin60 = 30gx5cos60 + 90gx7.5cos60$

$R_A\ 10sin60 = 825gcos60$

$R_A = 466.788N$

$1176\mu_B = R_A$

$1176\mu_B = 466.788$

$\mu_B = 0.397$

(iii) $R_A\ x\ 10sin60 = 30gx5cos60 + 90gx10cos60$

$R_A\ 10sin60 = 1050gcos60$

$R_A = 594.093N$

$1176\mu_B = R_A$

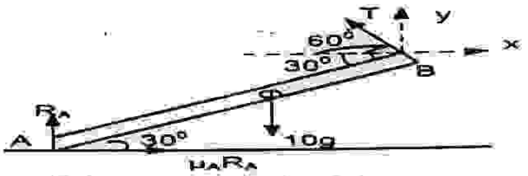
$1176\mu_B = 594.093$

$\mu_B = 0.5052$

ROUGH CONTACT AT THE FOOT OF THE LADDER

1. A uniform pole AB of mass 10kg has its lower end A on a rough horizontal ground and being raised to vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is at right angles to the pole and the pole is 30° to the horizontal, find;
   (i) Tension in the rope | (ii) Coefficient of friction on the ground

Solution



(i) Taking moments about A

$T\ x\ 2l\ = 10gxlcos30$

$T\ = 42.44N$

(ii) $(\uparrow)\ R_A + Tsin60 = 10g$

$R_A + 42.44xsin60 = 10x9.8$

$R_A = 61.25N$

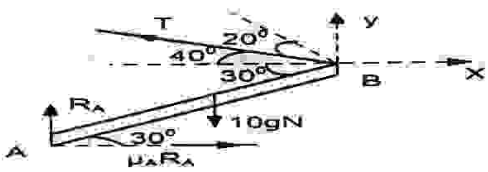
$(\rightarrow)\ Tcos60 = \mu_A R_A \ldots\ldots(iii)$

$42.44cos60 = \mu_A x61.25$

$\mu_A = 0.346$

2. A uniform pole AB of mass 10kg has its lower end A on a rough horizontal ground and being raised to vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is makes an angle of 70° to the pole and the pole is 30° to the horizontal, find;
   (i) Tension in the rope | (ii) Coefficient of friction on the ground

Solution



(i) Taking moments about A

$2l\ xT\ cos20 = 10gxlcos30$

$T\ = 45.157N$

(ii) $(\uparrow)\ R_A + Tsin40 = 10g$

$R_A + 45.157xsin40 = 10x9.8$

$R_A = 68.974N$

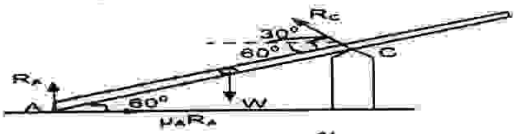
$(\rightarrow)\ Tcos40 = \mu_A R_A \ldots\ldots(iii)$

$45.157cos40 = 68.974\mu_A$

$\mu_A = 0.5015$

3. A uniform ladder AB of length 2l rests in limiting equilibrium with its lower end A resting on a rough horizontal ground. A point C on the beam rests against a smooth support. AC is of length $\frac{3l}{2}$ with C higher than A and AC makes an angle of 60° with the horizontal. Find the coefficient of friction between the ladder and the ground

Solution



Taking moments about A: $R_C\ x\ \frac{3l}{2} = Wlcos60$

$R_C = \frac{2Wcos60}{3} \ldots\ldots(i)$

$(\uparrow)\ R_A + R_C sin30 = W \ldots\ldots(ii)$

$R_A + \frac{2Wcos60}{3}sin30 = W$

$R_A = \frac{(3-2sin30\ cos60)W}{3}$

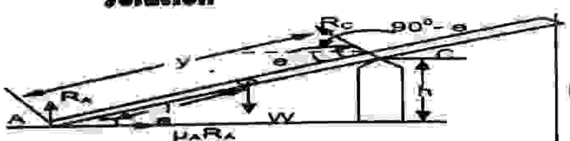
$(\rightarrow)\ R_C cos30 = \mu_A R_A \ldots\ldots(iii)$

$\frac{2Wcos60}{3}cos30 = \mu_A \frac{(3-2sin30\ cos60)W}{3}$

$\mu_A = \frac{2\cos 30\cos 60}{(3-2sin30\ cos60)} = 0.346$

4. A uniform rod of length 2l inclined at an angle θ to the horizontal rests in a vertical plane against a smooth horizontal bar at a height h above the ground. Given that the lower end of the rod is on a rough ground and the rod is about to slip. Show that the coefficient of friction between the rod and the ground is $\frac{l\ sin^2\theta\ cos\theta}{h-l\ cos^2\theta\ sin\theta}$

Solution



Taking moments about A: $R_C\ x\ y\ = Wlcos\theta$

$R_C = \frac{Wlcos\theta}{y} \ldots\ldots(i)$

$(\uparrow)\ R_A + R_C sin(90-\theta) = W \ldots\ldots(ii)$

$R_A + \frac{Wlcos\theta}{y}cos\theta = W$

$R_A = \frac{(y-l\ cos^2\theta)W}{y}$

$(\rightarrow)\ R_C cos(90-\theta) = \mu_A R_A \ldots\ldots(iii)$

$\frac{Wlcos\theta}{y}sin\theta = \mu_A \frac{(y-l\ cos^2\theta)W}{y}$

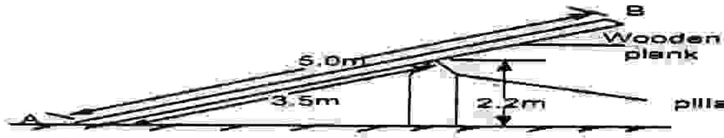
$\mu_A = \frac{l\cos\theta\ sin\theta}{(y-l\ cos^2\theta)}$

but $sin\theta = \frac{h}{y}$ $\Rightarrow y = \frac{h}{sin\theta}$

$$\mu_A = \frac{l\cos\theta\sin\theta}{\left(\frac{h}{\sin\theta} - l\cos^2\theta\right)}$$

$$\mu_A = \frac{l\sin^2\theta\cos\theta}{h - l\cos^2\theta\sin\theta}$$

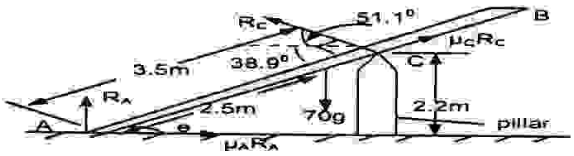
5. The diagram below shows a uniform wooden plank AB of mass 70kg and length 5m. The end A rests on a rough horizontal ground. The plank is in contact with the top of a rough pillar at C. The height of the pillar is $2.2m$ and $\overline{AC} = 3.5m$



Given that the coefficient of friction at the ground is 0.6 and the plank is just about to slip, find the

(i) Angle the plank makes with the ground at A
(ii) Normal reaction at A and normal reaction at C
(iii) Coefficient of friction at C

Solution



(i)

$$\sin\theta = \frac{2.2}{3.5} \quad \theta = 38.9°$$

(ii) Taking moments about A

$R_C\ x\ 3.5\ = 70gx2.5cos38.9$

$R_C\ = 381.34N$

(↑) $R_A + \mu_C R_C sin38.9 + R_C sin51.1 = 70g\ ...(i)$

$R_A = 70g - \mu_C 381.34sin38.9 - 381.34sin51.1$

$R_A = 389.2248 - 239.4674\mu_C$

(→) $R_C cos51.1 = 0.6R_A + \mu_C R_C cos38.9..(ii)$

$381.34xcos51.1 = 0.6R_A + \mu_C x381.34cos38.9$

$239.4674 = 0.6(389.2248 - 239.4674\mu_C) + 296.7752\mu_C$

$$\mu_c = \frac{5.93252}{153.0948} = 0.0388$$

$R_A = 389.2248 - 239.4674x0.0388$

$R_A = 379.933N$

Exercise 23B

1. A uniform ladder AB rests in limiting equilibrium with the top end against a smooth vertical wall and its base on a rough horizontal floor of coefficient of friction. If the ladder makes an angle of $\theta$with the floor. Show that $2\mu tan\theta = 1$

2. A non-uniform ladder AB 10m long and mass 8kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground of angle of friction 17° and the upper end resting against a smooth vertical wall. If the centre of gravity is at point C and the ladder makes an angle of 63° with horizontal, find the length AC **An(6m)**

3. A uniform ladder of mass 30kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground of coefficient of friction 0.4 and the upper end resting against a smooth vertical wall. If the ladder makes an angle of 60° with horizontal. find the Magnitude of the of the frictional force at the ground. **An(158.4N)**

4. A uniform pole AB of mass 100kg has its lower end A on a rough horizontal ground and being raised to vertical position by a rope attached to B. the rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is at right angles to the pole and the pole is 20° to the horizontal, find;
   (i) Normal reaction on the ground
   (ii) Frictional force on the ground **An(157N, 547N)**

5. A non-uniform pole AB of mass 50kg has its centre of gravity at the point of trisection of its length near to B. The pole has its lower end A on a rough horizontal ground and being raised into a vertical position by a rope attached to B. the rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is at right angles to the pole and the pole is 30° to the horizontal, find;
   (i) Normal reaction on the ground
   (ii) Frictional force on the ground **An(141N, 245N)**

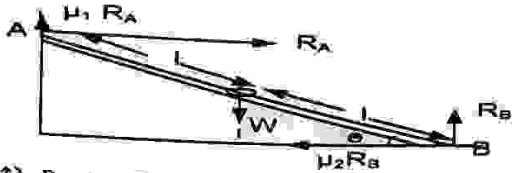
6. A ladder 12m long and weighing 200N is placed 60° to the horizontal with one end B leaning against the smooth vertical wall and the other end A on the rough horizontal ground. Find;
   a) reaction at the wall
   b) reaction at the ground **An(57.7N, 208.2N at 73.9° to the horizontal).**

7. A uniform ladder which is 5m long and has a mass of 20kg leans with its upper end against a smooth vertical wall and its lower end on rough ground. The bottom of the ladder is 3m from the wall. Calculate the frictional forces between the ladder and ground **An(73N)**

8. A uniform pole AB has its lower end A on a rough horizontal ground of angle of friction λ and being

2. A uniform ladder rests in limiting equilibrium with the top end against a rough vertical wall with coefficient of friction μ_1 and its base on a rough horizontal floor with coefficient of friction μ_2. If the ladder makes an angle of θ with the floor, prove that $tan\theta = \frac{1-\mu_1\mu_2}{2\mu_2}$

Solution



(↑) $R_B + \mu_1 R_A = W$ ……. (i)

(→) $R_A = \mu_2 R_B$ ……. (i)

$R_B = \frac{1}{\mu_2} R_A$ put into (i)

$R_B + \mu_1 R_A = W$

$\frac{1}{\mu_2} R_A + \mu_1 R_A = W$

$\frac{(1+\mu_1\mu_2)}{\mu_2} R_A = W$

$R_A = \frac{\mu_2 W}{1+\mu_1\mu_2}$

Taking moments about B

$R_A\ x\ 2lsin\theta + \mu_1 R_A x 2lcos\theta = Wxlcos\theta$ ….. (iii)

$$\frac{\mu_2 W}{1+\mu_1\mu_2} 2lsin\theta + \mu_1 x \frac{\mu_2 W}{1+\mu_1\mu_2} 2lcos\theta = Wxlcos\theta$$

$$\frac{2\mu_2}{1+\mu_1\mu_2} sin\theta = cos\theta - \frac{2\mu_1\mu_2}{1+\mu_1\mu_2} cos\theta$$

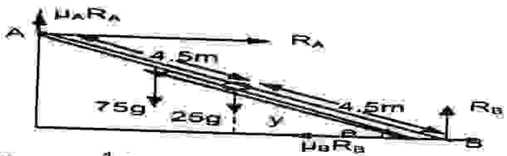
$$\frac{2\mu_2}{1+\mu_1\mu_2} sin\theta = \frac{1+\mu_1\mu_2 - 2\mu_1\mu_2}{1+\mu_1\mu_2} cos\theta$$

$$2\mu_2 sin\theta = (1-\mu_1\mu_2)cos\theta$$

$$tan\theta = \frac{1-\mu_1\mu_2}{2\mu_2}$$

3. The foot of a ladder length of 9m and mass 25kg rests on a rough horizontal surface while the upper end rests in contact with a rough vertical wall. The ladder being in vertical plane perpendicular to the wall. If the first rung is 30cm from the foot and the rest at the interval of 30cm. find the highest rung to which a man of mass 75kg can climb without causing the ladder to slip. When the ladder is inclined at 60° to the horizontal and the coefficient of friction at each end is 0.25

Solution



(↑) $R_B + \frac{1}{4} R_A = 25g + 75g$ ….. (i)

(→) $R_A = \frac{1}{4} R_B$ ……. (ii)

$R_B = 4R_A$ put into (i)

$4R_A + \frac{1}{3} R_A = 980$

$R_A = \frac{3}{13}(980) = 226.154N$

Taking moments about B

$R_A\ x\ 9sin\theta + \frac{1}{4} R_A x 9cos\theta = 25gx4.5cos\theta + 75gycos\theta$

$226.154\ x\ 9sin60 + \frac{1}{4} x 226.154x9cos60 = 25gx4.5cos60 + 75gycos60$

$y = 4m$

Number of rungs $= \frac{4}{0.3} = 13$

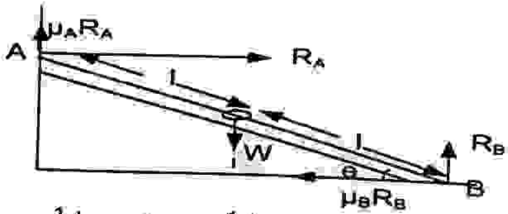
4. A uniform ladder of length 2l and weight W rests in a vertical plane with one end on a rough horizontal ground and the other against a rough vertical wall, the angle of friction being respectively $tan^{-1}(\frac{1}{3})$ and $tan^{-1}(\frac{1}{2})$.

UNEB 2005 No.16

(a) Find the inclination of the ladder to the horizontal when it is in limiting equilibrium at either end

(b) A man of weight 10 times that of the ladder begins to ascent it . How far will he climb before the ladder slips

Solution



$\mu_A = \frac{1}{3}$ and $\mu_B = \frac{1}{2}$

(↑) $R_B + \frac{1}{3} R_A = W$ ……. (i)

(→) $R_A = \frac{1}{2} R_B$ ……. (ii)

$R_B = 2R_A$ put into (i)

$2R_A + \frac{1}{3} R_A = W$

$\frac{7}{3} R_A = W$

$R_A = \frac{3W}{7}$

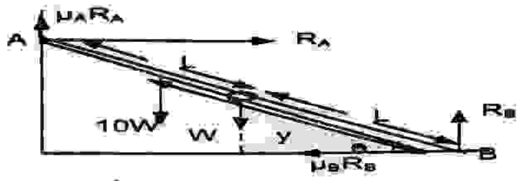
Taking moments about B

$R_A\ x\ 2lsin\theta + \frac{1}{3} R_A x 2lcos\theta = Wxlcos\theta$ ….. (iii)

$$\frac{3W}{7} x\ 2lsin\theta + \frac{1}{3} x \frac{3W}{7} x 2lcos\theta = Wxlcos\theta$$

$$\frac{6}{7} sin\theta = \frac{5}{7} cos\theta$$

$tan\theta = \frac{5}{6}$ $\quad \theta = 39.8°$



(↑) $R_B + {}^1/_3 R_A = 10W + W$ ....(i)

(→) $R_A = {}^1/_2 R_B$ ....(ii)

$R_B = 2R_A$ put into (i)

$2R_A + {}^1/_3 R_A = 11W$

$R_A = {}^{33}/_7 W$

Taking moments about B

$R_A \ 2l\sin\theta + {}^1/_3 R_A 2l\cos\theta = W l\cos\theta + 10Wy\cos\theta$

$({}^{33}/_7 W)2l\sin\theta + {}^1/_3({}^{33}/_7 W)2l\cos\theta = W l\cos\theta + 10Wy\cos\theta$

${}^{66}/_7 l\sin\theta + {}^{15}/_7 l\cos\theta = 10y\cos\theta$

Divide all through by $\cos\theta$

${}^{66}/_7 l\tan\theta + {}^{15}/_7 l = 10y$

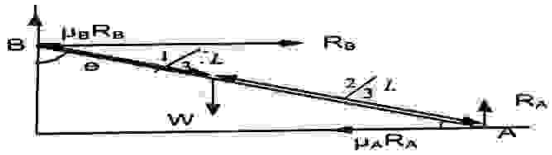
${}^{66}/_7 l({}^5/_6) + {}^{15}/_7 l = 10y$

$10l = 10y$

$y = lm$

5. A non uniform ladder AB is in limiting equilibrium with its lower end A resting on a rough horizontal ground and the upper end B resting against a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. If the centre of gravity of the ladder is at G where $AG = {}^2/_3 AB$. The coefficient of friction between the ladder and the wall is twice that between the ladder and the floor. If the ladder makes an angle θ with the wall and the angle of friction between the ladder and the floor is λ **UNEB 1990 No.7**
   (i) Prove that $4\tan\theta = 3\tan 2\lambda$
   (ii) How far can a man of equal mass as the ladder ascend without the ladder slipping given that $\theta = 45°$ and coefficient of friction between the ladder and the floor is ${}^1/_2$. **An**$({}^2/_3 AB)$

Solution



$\mu_A = \tan\lambda$ and $\mu_B = 2\tan\lambda$

(↑) $R_A + 2\tan\lambda R_B = W$ ....(i)

(→) $R_B = \tan\lambda R_A$ ....(ii) put into (i)

$R_A + 2\tan\lambda \tan\lambda R_A = W$

$R_A = \frac{W}{1+2\tan^2\lambda}$

Taking moments about B

$R_A \ x \ l\sin\theta = W\frac{l}{3}\sin\theta + \tan\lambda R_A l\cos\theta$ .....(iii)

$\frac{W}{1+2\tan^2\lambda} l\sin\theta - W\frac{l}{3}\sin\theta = \tan\lambda \frac{W}{1+2\tan^2\lambda} l\cos\theta$

$\frac{2-2\tan^2\lambda}{3(1+2\tan^2\lambda)}\sin\theta = \frac{\tan\lambda}{1+2\tan^2\lambda}\cos\theta$

$\tan\theta = \frac{3\tan\lambda}{2-2\tan^2\lambda} = \frac{3\tan\lambda}{2(1-\tan^2\lambda)}$

$\tan\theta = \frac{3}{2}x\frac{2}{2}\left[\frac{\tan\lambda}{(1-\tan^2\lambda)}\right]$

$\tan\theta = \frac{3}{4}\left[\frac{2\tan\lambda}{(1-\tan^2\lambda)}\right] = \frac{3}{4}\tan 2\lambda$

$4\tan\theta = 3\tan 2\lambda$

Exercise 23C

1. A uniform ladder rests in limiting equilibrium with the top end against a rough vertical wall with coefficient of friction 0.25 and its base on a rough horizontal floor with coefficient of friction μ. If the ladder makes on angle of 30° with the vertical. Find the value of μ **An(0.269)**
2. A non uniform ladder AB of length 6m is in limiting equilibrium with its lower end A resting on a rough horizontal ground with coefficient of friction ${}^1/_3$ and the upper end B resting against a rough vertical wall with coefficient of friction ${}^1/_4$. The ladder is in a vertical plane perpendicular to the wall. If the centre of gravity of the ladder is at C where $AC = 4m$. If the ladder makes an cute angle θ with the ground. Show that $\tan\theta = {}^{23}/_{12}$
3. A uniform ladder AB is of weight 2W and length 10m rests in limiting equilibrium with the top end against a rough vertical wall with coefficient of friction ${}^1/_3$ and its base on a rough horizontal floor with coefficient of friction ${}^1/_3$. If the ladder makes an angle of θ with the horizontal, such that $\tan\theta = {}^{16}/_{17}$. A man of weight 5W starts to climb the ladder
   (a) How far up the ladder can the man climb before slipping can occur
   (b) When a boy of weight Y stand s on the bottom rung of the ladder at A, the man is just able to climb to the top safely. Find X in terms of W **An(9m, $\frac{7W}{11}$)**
4. A non uniform ladder AB of length 12m and mass 30kg is in limiting equilibrium with its

lower end A resting on a rough horizontal ground with coefficient of friction $^1/_4$ and the upper end B resting against a rough vertical wall with coefficient of friction $^1/_5$. The ladder is in a vertical plane perpendicular to the wall. If the centre of gravity of the ladder is at its trisection of its length nearer to A.. If the ladder makes an angle θ with the horizontal such that $tan\theta = {}^9/_4$. a straight horizontal string connects A to a point at the base of the wall, vertically below B. a man of mass 90kg begins to climb the ladder.

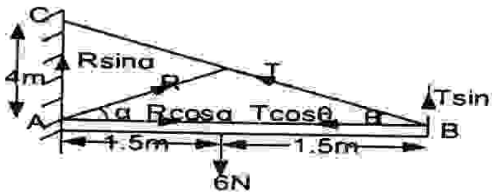
(i) How far up the ladder can the man climb without causing tension in the string

(ii) What tension must the sting be capable of withstanding if the man is to reach the top if the ladder safely **An(8m, 126N)**

BEAMS HINGED AND MAINTAINED IN A HORIZONTAL POSITION

1. A Uniform beam AB, 3.0m long and of weight 6N is hinged at a wall at A and is held stationary in a horizontal position by a rope attached to B and joined to a point C on the wall, 4.0m vertically above A. Find
   (i) the tension T in the rope
   (ii) the magnitude and direction of the Reaction R at the hinge.

Solution



$\tan\theta = \frac{4}{3}$ $\quad\theta = 53.13°$

Taking moments about A at equilibrium

$Tsin\theta \, x \, 3 = 6 \, x \, 1.5$

$(Tsin53.13) \, x \, 3 = 9$

$T = 3.75N$

$(\uparrow)$ $Rsin\alpha + Tsin\theta = 6$

$Rsin\alpha = 6 - 3.75sin53.13$

$Rsin\alpha = 3$------------------------i

$(\rightarrow)$ $Rcos\alpha = Tcos\theta$

$Rcos\alpha = 3.75cos53.13$

$Rcos\alpha = 2.238$----------------------ii

i/ii $\tan\alpha = \frac{3}{2.238}$ $\quad \alpha = 53.3°$

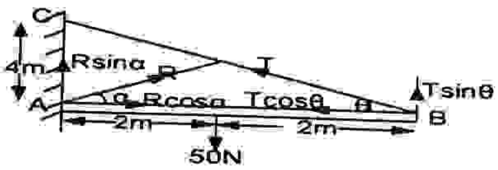
Put into 1; $Rsin53.3 = 3$

$R = 3.74N$

The reaction at A is 3.74 at 53.28° to the beam

2. A uniform beam AB of length 4m and weight 50N is freely hinged at A to a vertical wall and is held horizontal in equilibrium by a string which has one end attached at B and the other end attached to a point C on the wall, 4m above A. find
   (i) the tension T in the rope
   (ii) the magnitude and direction of the Reaction R at the hinge.

Solution



$\tan\theta = \frac{4}{4}$ $\quad \therefore \theta = 45°$

Taking moments about A

$Tsin\,\theta x4 = 50x2$

$Tsin45x4 = 50x2$

$T = 35.36N$

$(\uparrow)$: $Rsin\,\alpha + Tsin\,\theta = 50$

$Rsin\,\alpha + 35.36sin45 = 50$

$Rsin\,\alpha = 24.997$..............................(i)

$(\rightarrow)$: $Rcos\,\alpha = Tcos\,\theta$

$Rcos\,\alpha = 35.36cos45$

$Rcos\,\alpha = 25$..............................(ii)

(i)/(ii) $\tan\alpha = \frac{24.997}{25}$ $\therefore \alpha = 45°$

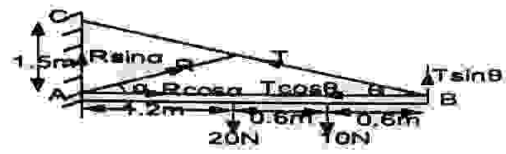
Put into (ii); $Rcos\,\alpha = 25$

$Rcos45 = 25$

$R = 35.36N$ at 45° to the beam

3. A uniform beam AB of mass 20kg and length 2.4m is hinged at a point A in a vertical wall and is maintained in a horizontal position by means of a chain attached to B and to point C in a wall 1.5m above. If the bar carries a load of 10kg at a point 1.8m from A. calculate.
   i) The tension in the chain
   ii) The magnitude and direction of the reaction between the bar and the wall

Solution



$\tan\theta = \frac{1.5}{2.4}$ $\therefore \theta = 32.01°$

Taking moments about A

$T\sin\theta \times 2.4 = 20gN \times 1.2 + 10gN \times 1.8$

$T \times 2.4\sin 32.01 = 20 \times 9.8 \times 1.2 + 10 \times 9.8 \times 1.8$

$T = 323.87N$

Tension in the chain = $323.87N$

(ii) Reaction at the wall

(↑) $R\sin\alpha + T\sin\theta = 20gN + 10gN$

$R\sin\alpha + 323.87\sin 32.01 = 30gN$

$R\sin\alpha = 122.63$……………………(i)

(→): $R\cos\alpha = T\cos\theta$

$R\cos\alpha = 32.87\cos 32.01$

$R\cos\alpha = 274.63$ ………………(ii)

(i)/(ii) $\tan\alpha = 0.446528055$

$\alpha = 24.1°$ Put α in eqn (ii)

$R\cos 24.1 = 274.63$

$R = 300.85N$

Reaction at A is 300.85 at 24.1° to the horizontal

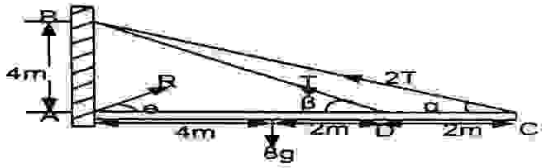
4. A uniform beam AC of mass 8kg and length 8m is hinged at A and maintained in equilibrium by two strings attached to it at points C and D as shown below. The tension in BC is twice that in BD, $\overline{AB} = 4m$, $\overline{AD} = {}^3/_4\,AC$. **UNEB 2008 No.16**

Find;

(i) Tension in the string BC

(ii) Magnitude and direction of the resultant force at the hinge

Solution



$\tan\beta = \frac{4}{6}$, $\beta = 33.7°$

$\tan\alpha = \frac{4}{8}$, $\alpha = 26.6°$

Taking moments about A

$6xT\sin\beta + 8xT\sin\alpha = 8gx4$

$6xT\sin 33.7 + 8x2T\sin 26.6 = 8gx4$

$T = 29.886N$

Tension in BC = $2x29.886 = 59.772N$

(↑): $R\sin\theta + T\sin\beta + 2T\sin\alpha = 8g$

$R\sin\theta = 35.0545 \ldots\ldots\ldots (i)$

(→): $R\cos\theta = T\cos\beta + 2T\cos\alpha$

$R\cos\theta = 78.3092 \ldots\ldots (ii)$

(i)/(ii) $\tan\theta = \frac{35.0545}{78.3092}$ $\therefore \theta = 24.12°$

Put into (ii); $R\cos\theta = 78.3092$

$R\cos 24.12 = 78.3092$

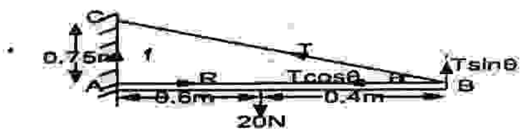
$R = 85.8N$ at 24.12°to the beam

5. A rod AB 1m long has a weight of 20N and its centre of gravity 60cm from A. it rests horizontally with A against a rough vertical wall. A string BC is fastened to the wall at C 75cm vertically above A. find the

(i) Normal and frictional forces at A. if friction is limiting the coefficient of friction

(ii) Tension in the string **UNEB 2007 No.13 a**

Solution



$\tan\theta = \frac{0.75}{1}$ $\therefore \theta = 36.9°$

Taking moments about A

$T\sin\theta x4 = 20x0.6$

$1xT\sin 36.9 = 20x0.6$

$T = 19.99N$

(↑): $f + T\sin\theta = 20$

$f + 19.99\sin 36.9 = 20$

$f = 8N$…………………………(i)

(→): $R = T\cos\theta$

$R = 19.99\cos 36.9 = 15.99N$

$f = \mu R$

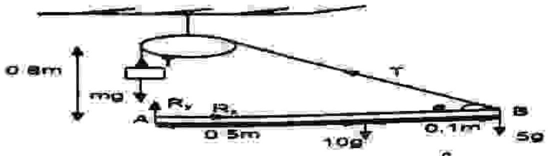
$\mu = \frac{8}{15.99} = 0.5$

5. A rod of length 0.6m long and mass 10kg is hinged at A. its center of mass is 0.5m from A. alight inextensible string attached at B passes over a fixed pulley 0.8m above A and supports a mass M hanging freely. If a mass of 5kg is attached at B so as to keep the rod in a horizontal position, find the

(i) Value of m

(ii) Reaction at the hinge **UNEB 1999 No.14**

Solution



$tan\theta = \frac{0.8}{0.6} = \frac{4}{3}, \quad sin\theta = \frac{4}{5}, \quad cos\theta = \frac{3}{5}$

For 2kg mass: $T = mg \ldots\ldots(i)$

For the beam: Taking moments about A

$$0.6xTsin\theta = 5gx0.6 + 10gx0.5 \ldots\ldots(ii)$$

$$0.6xTx\frac{4}{5} = 5gx0.6 + 10gx0.5$$

$$T = \frac{50}{3}g = 163.33N$$

$$T = mg$$

$$163.33 = 9.8m$$

$$m = 16.67kg$$

$(\uparrow)$ $Tsin\theta + R_Y = 10g + 5g$

$$163.33x\frac{4}{5} + R_Y = 15x9.8$$

$$R_Y = 16.336N$$

$(\rightarrow)$ $Tcos\theta = R_X$

$$163.33x\frac{3}{5} = R_X$$

$$R_X = 97.998N$$

$$R = \sqrt{(R_X)^2 + (R_Y)^2} = \sqrt{(97.998)^2 + (16.336)^2}$$

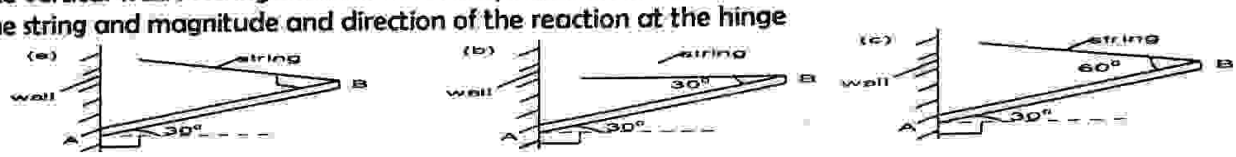
$$R = 99.35N$$

$$\alpha = tan^{-1}\left(\frac{R_Y}{R_X}\right) = tan^{-1}\left(\frac{16.336}{97.998}\right) = 9.34°$$

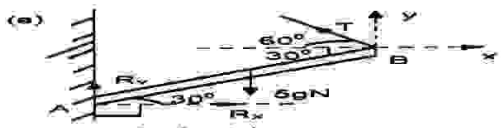
Reaction at B is $99.35N$ at $9.34°$ to the beam.

BEAMS HINGED AND MAINTAINED AT AN ANGLE

1. Each of the following diagrams shows a uniform rod of mass 5kg and length 6m freely hinged at A to the vertical wall. A string attached to B keeps the rod in equilibrium. For each case, find the tension in the string and magnitude and direction of the reaction at the hinge



Solution



Taking moments about A

$$Tx2l = 5gxlcos30$$

$$T = 21.2176N$$

$(\uparrow)$ $Tsin60 + R_Y = 5g$

$$21.2176sin60 + R_Y = 5g$$

$$R_Y = 30.625N$$

$(\rightarrow)$ $Tcos60 = R_X$

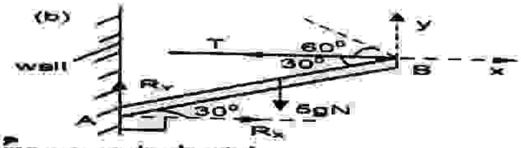
$$R_X = 21.2176cos60 = 10.6088N$$

$$R = \sqrt{(R_X)^2 + (R_Y)^2} = \sqrt{(10.6088)^2 + (30.625)^2}$$

$$R = 32.41N$$

$$\theta = tan^{-1}\left(\frac{30.625}{10.6088}\right) = 70.9°$$

Reaction is $32.41N$ at $70.9°$ to the horizontal



Taking moments about A

$$2lxTcos60 = 5gxlcos30$$

$$T = 42.4352N$$

$(\uparrow)$ $R_Y = 5g$

$$R_Y = 5g = 5x9.8 = 49N$$

$(\rightarrow)$ $T = R_X$

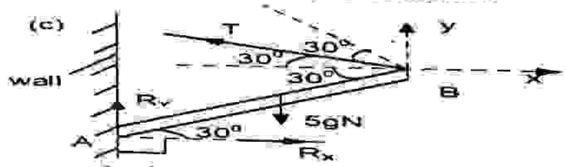
$$R_X = 42.4352N$$

$$R = \sqrt{(R_X)^2 + (R_Y)^2} = \sqrt{(42.4352)^2 + (49)^2}$$

$$R = 64.8209N$$

$$\theta = tan^{-1}\left(\frac{49}{42.4352}\right) = 49.1°$$

Reaction is $64.8209N$ at 49.1 to the horizontal



Taking moments about A

$$2lxTcos30 = 5gxlcos30$$

$$T = 24.5N$$

$(\uparrow)$ $Tsin30 + R_Y = 5g$

$$24.5sin30 + R_Y = 5g$$

$$R_Y = 36.75N$$

$(\rightarrow)$ $Tcos30 = R_X$

$$R_X = 24.5cos30 = 21.2176N$$

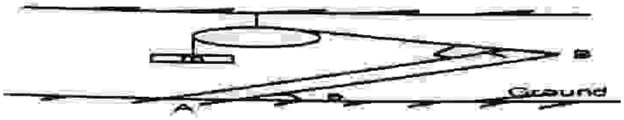
$$R = \sqrt{(R_X)^2 + (R_Y)^2} = \sqrt{(21.2176)^2 + (36.75)^2}$$

$$R = 42.4352N$$

$$\theta = tan^{-1}\left(\frac{36.75}{21.2176}\right) = 60°$$

Reaction is $42.4352N$ at $60°$ to the horizontal

2. A uniform rod AB of mass 5kg is smoothly hinged on the ground at point A. The rod making an angle θ with the horizontal ground is kept in equilibrium by a light inelastic string attached to point B. The string which makes 90° with the rod passes over a smooth fixed pulley and carries a stationary mass m of 2kg at its other end.

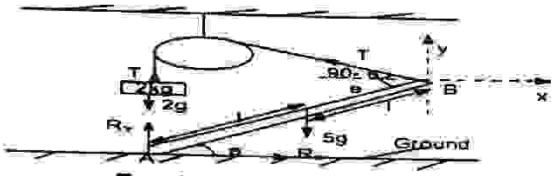


Show that

a. $cos\theta = \frac{4}{5}$

b. the magnitude of the reaction at the hinge is $\frac{49}{5}\sqrt{13}N$

Solution



For 2kg mass: $T = 2g \ldots\ldots(i)$

For the beam: Taking moments about A

$$Tx2l = 5gxlcos\theta \ldots\ldots(ii)$$

$$2gx2l = 5gxlcos\theta$$

$$cos\theta = \frac{4}{5}$$

$$(\uparrow)\ \ Tsin(90-\theta) + R_Y = 5g$$

$$Tcos\theta + R_Y = 5g$$

$$2gx\frac{4}{5} + R_Y = 5g$$

$$R_Y = \frac{17}{5}g = \frac{17}{5}x9.8 = \frac{833}{25}$$

$$(\rightarrow)\ \ Tcos(90-\theta) = R_X$$

$$Tsin\theta = R_X$$

$$2gx\frac{3}{5} = R_X$$

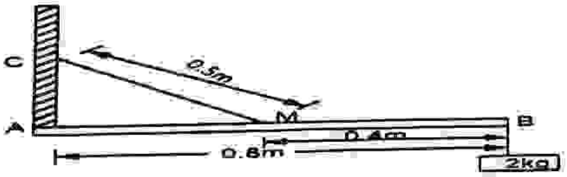
$$R_X = \frac{6}{5}g = \frac{6}{5}x9.8 = \frac{294}{25}$$

$$R = \sqrt{\left(\frac{294}{25}\right)^2 + \left(\frac{833}{25}\right)^2} = \sqrt{\frac{60025x13}{625}} = \frac{49}{5}\sqrt{13}$$

Exercise: 23D

1. A uniform beam AB of mass 5kg is freely hinged at A to a vertical wall and is held horizontal in equilibrium by a string which has one end attached at B and the other end attached to a point C on the wall above A. The string makes an angle of 30° with BA. find
   (i) the tension T in the rope
   (ii) the magnitude and direction of the Reaction R at the hinge. **An(49N, 49N at 30° with AB)**

2. The figure below shows a uniform beam of length 0.8 metres and 1 kg. The beam is hinged at A and has load of mass 2 kg attached at B.



The beam is held in a horizontal position by a light inextensible string of length 0.5 metres. The string joins the mid-point M of the beam to a point C vertically above A. Find the:
   (a) tension in the string.
   (b) magnitude and direction of the force exerted by the hinge. **UNEB 2018 No.13**
   An(81.6667N, 68.21N at 16.70° with AB)

3. A non uniform rod AB of mass 10kg has its centre of gravity at a distance $^1/_4\,AB$ from B. The rod is smoothly hinged at A. it is maintained in equilibrium at 60° above the horizontal by a light inextensible string tied at B and at right angle to AB. Calculate the magnitude and direction of the reaction at A. **UNEB 2017 No. 13 An(85.75N, at 68.2° to horizontal)**

4. A non-uniform beam AB of weight 20N and of length 4m is freely hinged at A to a vertical wall and is held horizontal in equilibrium by a string which has one end attached at B and the other end attached to a point C on the wall above A. The string makes an angle of 60° with BA. If the tension in the string is 12N, find
   (i) the magnitude and direction of the Reaction R at the hinge
   (ii) distance from A to the centre of gravity of the beam. **An(11.3N at 58° with AB, 2.[illegible]m)**

5. One end of a uniform plank of length 4m and weight 100N is hinged to the vertical wall. An inelastic rope, tied to the other end of the plank is fixed at a point 4m above the hinge. Find
   i. The tension in the rope'
   ii. The reaction of the wall on the plank

An(288.9N, 302.1N at 24.4° to horizontal)

6. A uniform diving board AB, of length 4m and mass 40kg is fixed at A to a vertical wall and is maintained in a horizontal position by means of a light strut DC. D is a point on the wall 1m below A and C is appoint on the board where $AC = 1m$. An object of mass 60kg is placed at end B.
   (a) Find the position of the centre of mass of the 60kg mass and the mass of the board AB combined
   (b) Determine
   (i) Thrust in the strut
   (ii) Magnitude of the reaction at A

An(80cm from B, 320$\sqrt{2}g$, 20$\sqrt{377}g$)

7. Two light strings are perpendicular to each other and support a particle of weight 100N. The tension in one of the strings is 40.0N. Calculate the angle this string makes with the vertical and the tension in the other string **An(66.4°, 91.7N)**
8. A uniform pole AB of weight 5W and length 8a is suspended horizontally by two vertical strings attached to it at C and D where AC=DB=a. A body of weight 9W hangs from the pole at E where ED=2a. calculate the tension in each string **An(5.5W, 8.5W)**
9. AB is a uniform rod of length 1.4m. It is pivoted at C, where AC=0.5m, and rests in horizontal equilibrium when weighs of 16N and 8N are applied at A and B respectively. Calculate
   (a) the weight of the rod **An(4N, 28N)**
   (b) the magnitude of the reaction at the pivot
10. A uniform rod AB of length 4a and weight W is smoothly hinged at its upper end, A. the rod is held at 30° to the horizontal by a string which is at 90° to the rod and attached to it at C where AC=3a, find
    (a) the tension in the string
    (b) reaction at A **An(0.58W, 0.578W)**
11. A sphere of weight 40N and radius 30cm rests against a smooth vertical wall. The sphere is supported in this position by a string of length 20cm attached to a point on the sphere and to the a point on the wall. Find
    (a) tension in the string
    (b) reaction due to the wall **An(30N, 30N at 90° to the wall)**
12. A smooth uniform rod AB of length $3a$ and weight $2w$ is pivoted at A so that it can rotate in a vertical plane. A light ring is free to slide over the rod. A light inextensible string is attached to the ring and passes over a fixed smooth peg at a point C, a height $4a$ above A and carries a particle of weight w hanging freely.

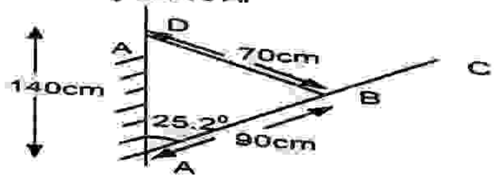
(a) Show that the angle θ that the rod makes to the vertical in equilibrium is given by $tan\theta = \frac{4}{3}$
(b) Find the magnitude of the force of the pivot on the rod at A in terms of w

An($\frac{3w\sqrt{5}}{5}$)

13. A smooth uniform rod AB of length $3a$ and weight W is pivoted at A so that it can rotate in a vertical plane. A light ring is free to slide over the rod. A light inextensible string is attached to the ring and passes over a fixed smooth peg at a point C, a height $4a$ above A and carries a particle of weight w hanging freely.

(a) Show that the angle θ that the rod makes to the vertical in equilibrium is given by $tan\theta = \frac{8w}{3W}$
(b) Find the smallest value of the ratio $\frac{w}{W}$ for which equilibrium is possible **An($\frac{\sqrt{7}}{8}$)**

14. A uniform rod AC of mass 2kg and length 120cm hangs at rest in a vertical plane with end A in contact with a vertical wall. An in elastic string of length 70cm is attached to a point B on AC such that AB is 90cm. The other end of the string is attached to the wall at a point D, 140cm vertically above A.

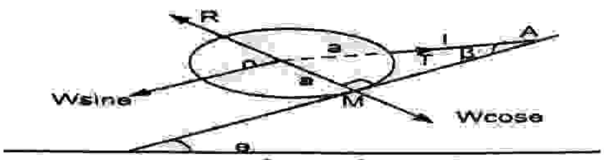


If the string is taut and angle $D\hat{A}C$ is 25.2°, find
a. Angle $D\hat{B}A$
b. The tension in the strings **An**$(121.6°, 6.53N$

BEAMS ON INCLINED PLANES

1. a sphere of radius a and weight W rests on a smooth inclined plane supported by a string of length l with one end attached to a point on the surface of the sphere and the other end fastened to a point on the plane. If the angle of inclination of the plane to the horizontal be θ. Prove that the tension of the string is $\frac{W(a+l)\sin\theta}{\sqrt{l^2+2al}}$

Solution



$$AM^2 = OA^2 - OM^2$$

$$AM = \sqrt{(a+l)^2 - a^2}$$

$$AM = \sqrt{l^2 + 2al}$$

$$\cos\beta = \frac{\sqrt{l^2+2al}}{l+a}$$

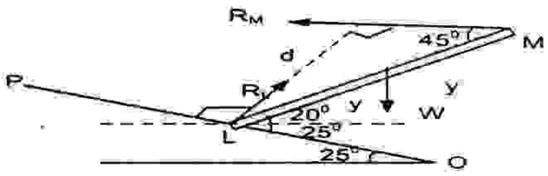
Also $W\sin\theta = T\cos\beta$

$$T = \frac{W\sin\theta}{\cos\beta} = \frac{W(a+l)\sin\theta}{\sqrt{l^2+2al}}$$

2. A uniform rod LM of weight W rests with L on a smooth plane PO of inclination 25° as shown in the diagram below

The angle between LM and the plane is 45°. What force parallel to PO applied at M will keep the rod in equilibrium? **UNEB 2007 No.13 b**

Solution



Taking moments about L

$$R_M x d = W x y\cos 20$$

But $d = 2y\sin 45$

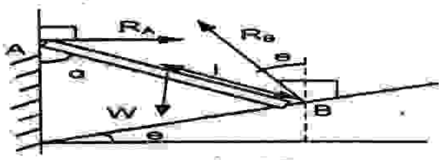
$$R_M x 2y\sin 45 = W x y\cos 20$$

$$R_M = 0.6645W$$

3. The diagram shows a uniform ladder resting in equilibrium with its top end against a smooth vertical wall and its base on a smooth inclined plane. The plane makes an angle of θ with the horizontal and the ladder makes an angle of α with the wall.

Prove that, $\tan\alpha = 2\tan\theta$

Solution



Taking moments about B

$$R_A x 2l\cos\alpha = W x l\sin\alpha$$

$$R_A = \frac{W\tan\alpha}{2} \dots\dots (i)$$

$(\uparrow)$ $R_B\cos\theta = W \dots\dots (ii)$

$(\rightarrow)$ $R_B\sin\theta = R_A \dots\dots (iii)$

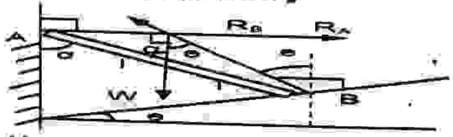
$(iii) \div (ii)$ $\frac{R_B\sin\theta}{R_B\cos\theta} = \frac{R_A}{W}$

$$W\tan\theta = R_A$$

$$W\tan\theta = \frac{W\tan\alpha}{2}$$

$$\tan\alpha = 2\tan\theta$$

Alternatively



Using cotangent rule for triangle

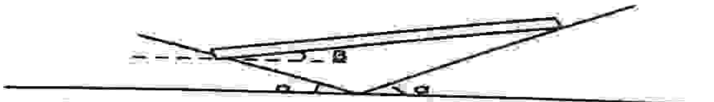
$$l\cot\theta - l\cot 90° = 2l\cot\alpha$$

$$\cot\theta = 2\cot\alpha$$

$$\frac{1}{\tan\theta} = \frac{2}{\tan\alpha}$$

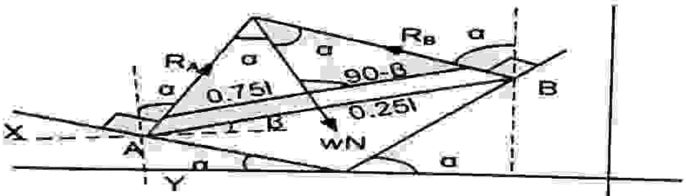
$$\tan\alpha = 2\tan\theta$$

4. A rod AB of length l has its centre of gravity at a point G where $AG = \frac{1}{4}l$. The rod rests in equilibrium in a vertical plane at an angle β to the horizontal with its ends in contact with two inclined planes whose line of intersection is perpendicular to the rod. If the planes are smooth and equally inclined at angle α to the horizontal



Show that $2tan\alpha \ tan\beta = 1$ and reaction on each plane is $\frac{w}{1+cos\alpha}$

Solution



Using cotangent rule for triangle

$$\frac{3l}{4}cot\alpha - \frac{l}{4}\ cot\alpha = \left(\frac{3l}{4} + \frac{l}{4}\right)cot(90-\beta)$$

$$0.5\ cot\alpha = tan\beta$$

$$2tan\alpha\ tan\beta = 1$$

(↑) $R_Acos\alpha + R_Bcos\alpha = w$

$$R_B = w - R_Acos\alpha \ldots\ldots(i)$$

(→) $R_Bsin\alpha = R_Asin\alpha$

$$R_B = R_A \ldots.(ii)$$

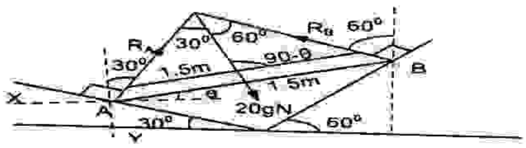
$$R_B = w - R_Bcos\alpha$$

$$R_B = \frac{w}{1+cos\alpha}$$

5. A uniform rod 3m long and of mass 20kg is placed on two smooth planes inclined at 30° and 60° to the horizontal.

Find the reaction on each plane and the inclination of the rod to the horizontal when it is in equilibrium

Solution



(↑) $R_Acos30 + R_Bcos60 = 20g$

$$R_B = 40g - R_A\sqrt{3} \ldots\ldots(i)$$

(→) $R_Bsin60 = R_Asin30$

$$R_B\sqrt{3} = R_A \ldots.(ii)$$

$$R_B = 40g - R_B\sqrt{3}x\sqrt{3}$$

$$R_B = 98N$$

$$R_A = 98\sqrt{3} = 169.74N$$

Using cotangent rule for triangle

$$1.5cot30 - 1.5cot60 = 3cot(90-\theta)$$

$$1.5\sqrt{3} - 1.5\frac{1}{\sqrt{3}} = 3tan\theta$$

$$\theta = 30°$$

Exercise 23E

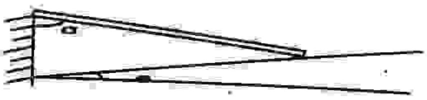
3. A uniform rod of length 2a and of mass m is placed on two smooth planes inclined at 30° and 60° to the horizontal. The normal reactions at the ends of the rods have magnitude R and S

(a) Show that $R = S\sqrt{2}$

(b) Prove that the inclination of the rod to the horizontal is $cot^{-1}(1+\sqrt{3})$

4. A uniform rod rests with its ends on two smooth planes inclined at 30° and 60° to the horizontal. A weight equal to twice that of the beam can slide along its length. Find the position of the slidding weight when a beam rests in a horizontal position **An($\frac{1}{8}$ of the length from** 30° **plane)**

5. The diagram shows a uniform ladder resting in equilibrium with its top end against a smooth vertical wall and its base on a smooth inclined plane. The plane makes an angle of θ with the horizontal and the ladder makes an angle of α with the wall.



Find α when θ is

(i) 10°

(ii) 30°

An(19.4°, 49.1°)

CHAPTER 14: SIMPLE HARMONIC MOTION

This is a periodic motion of a body whose acceleration is directly proportional to the displacement from a fixed point and is directed towards the fixed point.

$$a \propto -x$$

$$a = -\omega^2 x$$

The negative signs means the acceleration and the displacement are always in opposite direction.

MAXIMUM ACCELERATION

$a_{max} = -\omega^2 r$ where r is the acceleration

Force, F

$$F = ma = m\omega^2 x$$

Maximum Force, F_{max}

$$F = ma_{max} = m\omega^2 r$$

VELOCITY IN TERMS OF DISPLACEMENT

Velocity of a body executing S.H.M can be expressed as a function of displacement x. this is obtained from the acceleration

$a = -\omega^2 x$

$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

but $\frac{dx}{dt} = v$

$a = \frac{dv}{dx}$

$v \cdot \frac{dv}{dx} = -\omega^2 x$

$vdv = -\omega^2 xdx$

integrating both sides

$\int v\,dv = -\omega^2 \int x\,dx$

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + C$ ....[1]

Where C is a constant of integration

At momentary rest $v = 0$,

$x = r$(amplitude)

$\frac{0^2}{2} = -\frac{\omega^2 r^2}{2} + C$

$C = \frac{\omega^2 r^2}{2}$

Put into [1]

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 r^2}{2}$

$v^2 = \omega^2 r^2 - \omega^2 x^2$

$\boxed{v^2 = \omega^2(r^2 - x^2)}$

Velocity is maximum when x =0

$v^2 = \omega^2 r^2$

$\boxed{v_{max} = \omega r}$

DISPLACEMENT AT ANY TIME ,t

$\frac{dx}{dt} = v$

$\frac{dx}{dt} = \omega\sqrt{(r^2 - x^2)}$

$\int \frac{dx}{\sqrt{(r^2 - x^2)}} = \int \omega\,dt$

$sin^{-1}\frac{x}{r} = \omega t + \varepsilon$

$x = rsin(\omega t + \varepsilon)$

When timing begins at the centre, $t = 0, x = 0$

$\boxed{x = rsin\omega t}$ particle moves away from centre

When timing begins at the amplitude, $t = 0, x = r$

$\boxed{x = rcos\omega t}$ particle moves towards the centre

.

Period T

This is the time taken for one complete oscillation.

$T = \frac{distance}{speed}$

$T = \frac{2\pi}{v}$ but $v = r\omega$

$T = \frac{2\pi r}{\omega r}$

$T = \frac{2\pi}{\omega}$

Examples

1. A particles moves in a straight line with S.H.M about mean position O with a periodic time of $\frac{\pi}{2}$ s. Find the magnitude of the acceleration of the particle when 1m from O

Solution

From $a = -\omega^2 x$

Negative is ignored

$a = \left(2\pi/\frac{\pi}{2}\right)^2 x1 = 16ms^{-2}$

2. A particle moves with S.H.M about a mean position O. When the particle is 25cm from O, it is accelerating at 1m/s towards O. Find the;
   (i) Periodic time of the motion
   (ii) Magnitude of acceleration of the particle when 20cm from O

Solution

(i) $a = -\omega^2 x$

$1 = \omega^2(0.25)$

$\omega^2 = 4$

$\omega = 2\ rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi\ s$

(ii) $a = -\omega^2 x$

$a = 2^2(0.2) = 0.8ms^{-2}$

3. A particle moves with S.H.M of time period $\frac{\pi}{2}$ s and has a maximum speed of 3m/s. Find the maximum acceleration experienced by the particle

Solution

$v_{max} = \omega r$

$3 = {2\pi}/{\frac{\pi}{2}}\, r$

$r = 0.75m$

$a = -\omega^2 r$

$a = \left(2\pi/\frac{\pi}{2}\right)^2 x0.75 = 12ms^{-2}$

4. A particle moves with S.H.M about a mean position O. The amplitude of the motion is 5m and the period is 8π s. Find the
   (i) maximum speed of the particle
   (ii) speed of the particle when 3m from O

Solution

Solution

(i) $v_{max} = \omega r = \frac{2\pi}{8\pi}x5 = 1.25\ m/s$

(ii) $v^2 = \omega^2(r^2 - x^2)$

$v = \sqrt{\left(2\pi/\frac{\pi}{2}\right)^2 (5^2 - 3^2)} = 1ms^{-1}$

5. A body of mass 200g s executing S.H.M with amplitude of 20mm. The maximum force which acts upon it is 0.064N. calculate
   a) its maximum velocity
   b) its period of oscillation

Solution

$F = 0.064N$

m=200g = 0.2kg

$r = 20mm = 0.02m$

a) $v_{max} = \omega r$

But $F = ma_{max}$

0.064=0.2a_{max}

$a_{max} = 0.32m/s^2$

$a_{max} = -\omega^2 r$

0.32= $\omega^2 x0.02$

$\omega^2 = 16$

ω =4rads^{-1}

$v_{max} = \omega r = 4x0.02 = 0.08ms^{-1}$

b) T= $\frac{2\pi}{\omega}$

T= $\frac{2\pi}{4} = \frac{2x22}{4x7} = 1.57\ seconds$

6. A particle of moves with S.H.M about position O with a period of 2π seconds. It passes a point A with a velocity of 4m/s away from O. Given that $OA = 4m$, find;
   (i) the amplitude of motion
   (ii) the speed at B where $OB = 3m$ **Uneb 1989 No.8 An(5.66 m, 4.8m/s)**

Solution

i) $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\ rad/s$

Using $v^2 = \omega^2(r^2 - x^2)$

$4^2 = 1^2(r^2 - 4^2)$

$16 = (r^2 - 16)$

$r^2 = 32$

$r = 5.66m$; Amplitude = 5.66m

(iii) $v^2 = \omega^2(r^2 - x^2) = \sqrt{1^2(32 - 3^2)}$

$v = \sqrt{23} = 4.8ms^{-1}$

7. A Particle moving with S.H.M has velocities of 4ms^{-1} and 3ms^{-1} at distances of 3m and 4m respectively from equilibrium position. Find
   i) amplitude,
   ii) period,

Solution

(i) $v = 4ms^{-1}, x = 3m$ and

Using $v^2 = \omega^2(r^2 - x^2)$

$4^2 = \omega^2(r^2 - 3^2)$

$16 = \omega^2(r^2 - 9)$——— (1)

Also $v = 3ms^{-1}, x = 4m$

$3^2 = \omega^2(r^2 - 4^2)$

$9 = \omega^2(r^2 - 16)$------- (2)

Equation 1 divide by 2

$$\frac{16}{9} = \frac{\omega^2(r^2 - 9)}{\omega^2(r^2 - 16)}$$

$16(r^2 - 16) = 9(r^2 - 9)$

$r^2 = 25$

$r = 5m$; Amplitude $= 5m$

(ii) Using eqn(1)

$16 = \omega^2(5^2 - 9)$

$\omega^2 = 1$

$\omega = 1rads^{-1}$

But $T = \frac{2\pi}{\omega} = \frac{2x\frac{22}{7}}{1} = 6.28second$

8. A particle moves with s.h.m about a mean position O. The particle is initially projected from O with speed $\frac{\pi}{6}$ m/s and just reaches a point A, 2m form O.

(a) Find how far the particle is from O, 3 seconds after projection

(b) How many seconds after projection is the particle a distance of 1m from O

(i) For the first time (ii) Second time (iii) Third time

Solution

(a) At equilibrium position, $v_{max} = \omega r$

$\frac{\pi}{6} = \omega x2$

$\omega = \frac{\pi}{12} rad/s$

$x = rsin\omega t$ since particle move away from O

$x = 2sin\left(\frac{\pi}{12}x3\right) = 1.414m$

$x = rsin\omega t$

$1 = 2sin\frac{\pi}{12}t$

$\frac{\pi}{12}t = sin^{-1}(0.5)$

$\frac{\pi}{12}t = 30°, 150°, 210°$

$t = 2s, 10s, 14s$

9. A particle is released from rest at point A, 1m from a second point O. The particle accelerates towards O and moves with S.H.M of time period 12s and O as the centre of oscillation

(a) Find how far the particle is from O, 1 seconds after release

(b) How many seconds after release is the particle at the midpoint of OA

(i) For the first time (ii) Second time

Solution

(a) $x = rcos\omega t$ since particle moves towards centre

$\omega = \frac{2\pi}{12} = \frac{\pi}{6} rad/s$

$x = 1cos\left(\frac{\pi}{6}x1\right) = \frac{\sqrt{3}}{2}m$

(b) $x = rcos\omega t$

$0.5 = 1cos\frac{\pi}{6}t$

$\frac{\pi}{6}t = cos^{-1}(0.5)$

$\frac{\pi}{6}t = 60°, 300°$

$t = 2s, 10s$

10. A particle of mass 2kg moving with simple harmonic motion along the x-axis, is attracted towards the origin O by a force of 32x newton's. Initially the particle is at rest at x=20. Find the

(a) Amplitude and period of the oscillation

(b) Velocity of the particle at any time, t> 0

(c) Speed when $t = \frac{\pi}{4}s$ **Uneb 2016 No16**

Solution

(a) $F = m\omega^2 x$

$32x = 2\omega^2 x$

$\omega = 4 rad/s$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.571s$

$v^2 = \omega^2(r^2 - x^2)$

$0 = 4^2(r^2 - 20^2)$

$r = 20m$

(b) $x = rcos\omega t$

$v = \frac{d}{dt}(rcos\omega t)$

$v = -r\omega sin\omega t = -20x4sin4t$

$v = -80sin4t$

(c) Speed= $-80sin4x\frac{\pi}{4} = om/s$

11. A particle is initially released from rest at point A and performs S.H.M about mean position B. the particle just returns to A during each oscillation and $AB = 2\sqrt{2}$m. If the particle passes through B with speed $\pi\sqrt{2}\ m/s$, find

(i) the time when the particle is first travelling with a speed of $\pi m/s$

(ii) How far from B is the particle during this time

Solution

(i) At equilibrium position (B),

$$v_{max} = \omega r$$

$$\pi\sqrt{2} = \omega x 2\sqrt{2}$$

$$\omega = \frac{\pi}{2}\ rad/s$$

$$v^2 = \omega^2(r^2 - x^2)$$

$$\pi^2 = \left(\frac{\pi}{2}\right)^2\left(\left(2\sqrt{2}\right)^2 - x^2\right)$$

$$x^2 = 4$$

$$x = 2m$$

(ii) $x = r\cos\omega t$

$$2 = 2\sqrt{2}\cos\frac{\pi}{2}t$$

$$\frac{\pi}{2}t = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\frac{\pi}{2}t = 45°,$$

$$t = 0.5s$$

12. The points A, O, B, C lie in that order on a straight line with $AO = OC = 6cm$ and $OB = 5cm$. A particle performs S.H.M of period 3s and amplitude 6cm between A and C. Find the time taken for the particle to travel from A to B.

Solution

Time for AO is half the period= 1.5s

B is 5cm from O

$$x = r\cos\omega t$$

$$5 = 6\cos\frac{2\pi}{3}t$$

$$\frac{2\pi}{3}t = \cos^{-1}\left(\frac{5}{6}\right)$$

$$\frac{2\pi}{3}t = 33.6°,$$

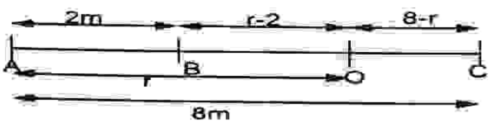
$$t = 0.28s$$

Time for AB = 1.5 − 0.28 = 1.22s

UNEB

1. A particle passes through 3 points A, B and C in that order with velocity $0m/s$, $2m/s$ and $-1m/s$ respectively. The particle is moving with S.H.M in a straight line. What is the amplitude and period of the motion If $\overline{AB} = 2m$ and $\overline{AC} = 8m$. **Uneb 1988 No.7.**

Solution



At B: $v = 2ms^{-1}$, $x = r - 2$

Using $v^2 = \omega^2(r^2 - x^2)$

$$2^2 = \omega^2(r^2 - (r-2)^2)$$

$$1 = \omega^2(r - 1) \text{------- (1)}$$

At C: $v = -1ms^{-1}$, $x = 8 - r$

$$(-1)^2 = \omega^2(r^2 - (8 - r)^2)$$

$$1 = \omega^2(16r - 64) \text{------- (2)}$$

Equation 1 divide by 2

$$\frac{1}{1} = \frac{\omega^2(r - 1)}{\omega^2(16r - 64)}$$

$$(16r - 64) = (r - 1)$$

$$r = 4.2$$

; Amplitude $= 4.2m$

Using 1 $= \omega^2(r - 1)$

$$1 = \omega^2(4.2 - 1)$$

$$\omega^2 = \frac{1}{3.2}$$

$$\omega = 0.56 rads^{-1}$$

But $T = \frac{2\pi}{\omega} = \frac{2x\frac{22}{7}}{0.56} = 11.24 second$

2. A Particle moving with S.H.M about a mean position O has velocities of 5ms^{-1} and 8ms^{-1} at distances of 16m and 12m respectively from O. Find

 i) amplitude,
 ii) period **Uneb 1993 No.5 An($4\pi\ s$)**

3. A particle is describing S.H.M in a straight line directed towards a fixed point O. When it's distance from O is 3m, its velocity is 25m/s and acceleration is 75ms^{-2}. Determine the

 a. period and amplitude of oscillation
 b. Time taken by particle to reach O.
 c. Velocity of the particle as it passes through O **Uneb 1999 NO.11 Nov/Dec**

 An($\frac{2\pi}{5}s$, $5.83m$, $0.108s$, $29.15m/s$)

4. A Particle moving with S.H.M about a mean position O has velocities of $3\sqrt{3}$ms^{-1} and 3ms^{-1} at distances of 1m and 0.268m respectively, from the end point. Find the amplitude of the motion.

 Uneb 2000 No.8 An($2m$)

5. A mass oscillates with S.H.M of period 1 second and amplitude of the oscillation is 5cm. Given that the particle begins from the centre of the motion, state the relationship between the displacement x of the mass at any time t, Hence find the first times when the mass is 3cm from its end position. **Uneb 2001 NO.14 An,** $x = rsin\omega t$ **,** $(0.066s)$

6. A particle is performing S.H.M motion with centre O, amplitude 6m and period 2π. Points B and C lie between O and A with $\overline{OB} = 1m$, $\overline{OC} = 3m$ and $\overline{OA} = 6m$. Find the least time taken while travelling from **Uneb 2004 No.1**
   (i) A to B
   (ii) A to C **An(** $1.403\ s,\ 1.047\ s)$

7. A particle moves in a straight line with S.H.M of period 5 seconds. The greatest speed is 4m/s, find the
   (i) Amplitude
   (ii) Speed when it is $\frac{6}{\pi}m$ from the centre **Uneb 2005 No.1 An(** $\frac{10}{\pi}\ m, 3.2m/s)$

8. The velocity of a particle at any time t is given by an equation $v(t) = -a\omega\ sin\omega t + b\omega cos\omega t$
   (i) Find the expression for the displacement x at any time given that $x = 0$ when time $t = 0$
   (ii) Show that the motion of the particle is simple harmonic
   Uneb 2009 No.8 An($x = acos\omega t + bsin\omega t, \ddot{x} = -\omega^2 x)$

Exercise 24A

1. A body of mass 0.30kg executes S.H.M with a period of 2.5s and amplitude of 0.04m. Determine
   (i) Maximum velocity of the body
   (ii) The maximum acceleration of the body
   An(0.101m/s, $0.25ms^{-2})$

2. A Particle moving with S.H.M about a mean positon O has velocities of $1.6ms^{-1}$ and $1.2ms^{-1}$ at distances of 60cm and 80cm respectively from O. Find
   i) amplitude ,
   ii) Period , **An(1m,** $\pi\ s)$

3. A particle moves with S.H.M in a straight line with amplitude 0.05m and period 12s. Find
   a) speed as it passes equilibrium position
   b) maximum acceleration of the particle
   An(0.026m/s, $0.014ms^{-2})$

4. A particles moves in a straight line with S.H.M about mean position O with a periodic time of $\frac{\pi}{2}\ s$ and amplitude 2m. Find the maximum speed of the particle. **An(**$8ms^{-1}$)

5. A body of mass 500g moves horizontally with S.H.M of time period of $\frac{\pi}{2}\ s$ and amplitude of 1m. Determine the magnitude of te greatest horizontal force experienced by the body during the motion **An(8N)**

6. A body of mass 100g moves horizontally with S.H.M about a mean position O. When the body is 50cm from O, the horizontal force on the body is of magnitude 5N find the time period of the motion. **An(**$\frac{\pi}{2}\ s$)

7. A particles moves in a straight line with S.H.M about mean position O with a periodic time of $\frac{\pi}{4}\ s$ and amplitude 65cm. Find how far the particle is from O when its speed is 2m/s. **An(**$60cm$)

8. A particle moves in a straight line with S.H.M about mean position O. The particle has zero velocity at appoint which is 50cm from O and a speed of 3m/s at O. Find;
   (i) The maximum speed of the particle
   (ii) The amplitude of the motion
   (iii) The time period of the motion
   An($3ms^{-1}, 50cm,\ \frac{\pi}{2}\ s$)

9. A Particle moving with S.H.M about a mean position O has velocities of $3ms^{-1}$ and $1.4ms^{-1}$ at distances of 2m and 2.4m respectively, from point O. Find the
   a. amplitude of the motion,
   b. Greatest speed attained by the particle
   An($2.5m, 5m/s)$

10. A particles is initially projected from a point A performs S.H.M about mean position A with a periodic time of $3\ s$ and amplitude 50cm. Find the
   a. maximum speed of projection
   b. Speed of the particle 2 s after projection
   c. Distance of the particle form A 2s after projection. **An(**$\frac{\pi}{3}ms^{-1}, \frac{\pi}{6}m,\ \frac{\sqrt{3}}{4}\ m$)

11. The head of a piston moves with S.H.M of amplitude $\frac{\sqrt{3}}{10}\ m$ about a mean position O. How far from O is the head of the piston when it is travelling with a speed equal to half of its maximum speed **An(** $15cm)$

12. A particle is fastened to the midpoint of a stretched spring lying on a smooth horizontal surface. The particle is set in motion so that it moves with S.H.M about a mean position O. If one metre is the

greatest distance the particle is from O during the motion. Find how far from O the particle is when it is travelling with a speed equal to four-fifth of its greatest speed **An(** $60cm$)

13. A particles performs S.H.M about mean position O with a periodic time of $3\ s$ and amplitude 6cm. Find the time it takes the particle to travel from O to a point P, a distance of 3cm from O **An(** $0.25s$)
14. A particles performs S.H.M about mean position O with a periodic time of $4\ s$ and amplitude 2cm. Find the time it takes the particle to travel from O to a point P, a distance of $\sqrt{2}$cm from O **An(** $0.5s$)
15. A particles performs S.H.M about mean position O with a periodic time of $10\ s$ and amplitude 8cm. After passing through O, the particle moves through a point A which is 2cm from O to a point B which is 6cm from O. Find the time it takes the particle to travel from A to B **An(** $0.948s$)
16. A particles performs S.H.M about mean position O with a periodic time of $3\ s$ and amplitude 3cm. After passing through O, the particle moves through a point A which is 1cm from O to a point B which is 2cm from O. Find the time it takes the particle to travel from A to B **An(** $0.186s$)
17. A particles performs S.H.M about mean position O with a periodic time of $4.5\ s$ and amplitude 6cm. After passing through O, the particle moves through a point P which is 3 from O. Find the time that ellapses before the particle next passes through P. **An(** $1.5s$)
18. The points A, O, B, C lie in that order on a straight line with $AO = OC = 4cm$ and $OB = 2cm$. A particle performs S.H.M of period 6s and amplitude 4cm between A and C. Find the time taken for the particle to travel from A to B. **An(2s)**

HOOKE'S LAW AND ELASTIC STRINGS

Hooke's law states that the tension in a stretched string is proportional to the extension from its natural(un-stretched) length

$$T = \lambda \frac{e}{l}$$

λ is modulus of elasticity of the string

Examples

1. An elastic string of natural length 4m and modulus 25N. find
   (i) The tension in the string when the extension is 20cm
   (ii) The extension of the string when the tension is 6N

   Solution

   (i) $T = \lambda \frac{e}{l}$

   $T = 25x\frac{0.2}{4} = 1.25N$

   (ii) $T = \lambda \frac{e}{l}$

   $e = 6x\frac{4}{15} = 1.6m$

2. A spring of natural length 1.6m and modulus 20N. find the thrust in the spring when it is compressed to a length of 1m

   Solution

   $T = \lambda \frac{e}{l}$

   $T = 20x\frac{0.6}{1.6} = 7.5N$

3. A body of mass mkg lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 60cm and modulus 90N. When the body moves in a horizontal circular path about O with constant speed of 3m/s, the extension in the string is 30cm. find the mass of the body

   Solution

   $T = \lambda \frac{e}{l} = 90x\frac{0.3}{0.6} = 45N$

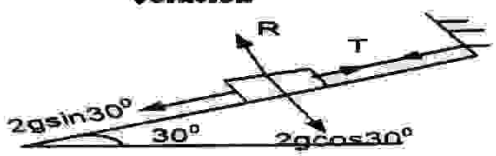
   $T = m\frac{v^2}{r}$

   $45 = mx\frac{3^2}{(0.6+0.3)}$

   m = 4.5kg

4. A smooth surface inclined at 30° to the horizontal has a body A of mass 2kg that is held at rest on the surface by a light elasti c string which has one end attached to A and the other to a fixed point on the surface 1.5m away from A up to the line of greatest slope. If the modulus of elasticity of the string is 2gN, find its natural length.

Solution



$$T = \lambda\frac{e}{l}$$

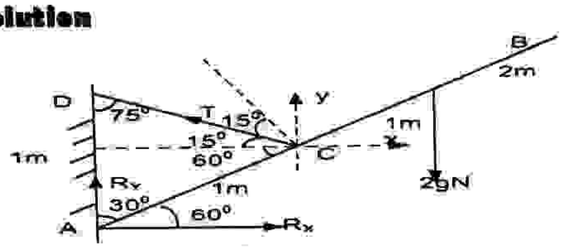
$$2gsin30 = 2gx\frac{1.5 - l}{l}$$

$$l = 1m$$

5. A uniform rod AB of length 4m and mass 2kg rests at 30° to the smooth vertical wall and is supported with end A in contact with the wall by an elastic string connecting to a point C on the rod to a point D on the wall vertically above A. if the natural length of the string is 0.4m and the distance AC and AD are 1m. find the;

(i) Tension in the string and reaction at A **(ii)** Modulus of elasticity

Solution



Taking moments about A

$$1xTcos15 = 2gx2cos60$$

$$T = 20.2914N$$

$$(\uparrow)\ Tsin15 + R_Y = 2g$$

$$20.2914sin15 + R_Y = 2g$$

$$R_Y = 14.3482\text{N}$$

$$(\rightarrow)\ Tcos15 = R_X$$

$$R_X = 20.2914os15 = 19.6N$$

$$R = \sqrt{(R_X)^2 + (R_Y)^2} = \sqrt{(14.3482)^2 + (19.6)^2}$$

$$R = 24.291N$$

Using sine rule: $\frac{l^1}{sin30} = \frac{1}{sin75}$

$$l^1 = 0.518m$$

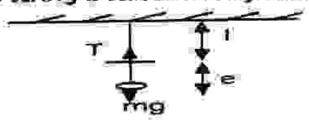
$$T = \lambda\frac{e}{l}$$

$$75.729 = \lambda\frac{0.518 - 0.4}{0.4}$$

$$\lambda = 257.58N$$

EQUILIBRIUM OF A SUSPENDED BODY

When an elastic string has one end fixed and a mass attached to its other end so that the mass is suspended in equilibrium, the string is stretched by the force due to the mass.



$$T = mg$$

$$\boxed{mg = \lambda\frac{e}{l}}$$ At equilibrium

Examples

1. A light elastic string of natural length 65cm has one end fixed and a mass of 500g freely suspended from the other. Find the modulus of elasticity of the string if the total length of the string in the equilibrium position is 85cm

Solution

$$mg = \lambda\frac{e}{l}$$

$$0.5x9.8 = \lambda x\frac{0.85 - 0.65}{0.65}$$

$$\lambda = 15.93N$$

2. A light elastic spring of natural length 1.5m has one end fixed and a mass of 400g freely suspended from the other. The modulus of the spring is 44.1N.
   (i) Find the extension of the spring when the body hangs in the equilibrium
   (ii) The mass is pulled vertically downwards a distance of 10cm and released, find the acceleration of the body when released.

Solution

(i) $mg = \lambda\frac{e}{l}$

$$0.4x9.8 = 44.1x\frac{e}{1.5}$$

$$e = 0.13m$$

(ii) $T_1 = \lambda\frac{e+x}{l}$

$$T_1 = 44.1x\frac{0.13 + 0.1}{1.5} = 6.762N$$

$$F = ma$$

$$T_1 - T = ma$$

$$6.762 - 3.92 = 0.4a$$

$$a = 7.11ms^{-2}$$

POTENTIAL ENERGY STORED IN AN ELASTIC STRING

Work done = $average\ force\ x\ extension\ = \lambda\frac{e^2}{2l}$

Examples

1. An elastic string of natural length 6.4m and modulus 55N. Find the work done in stretching it from 6.4m to a length of 6.8m.

Solution

Work done $= \lambda\frac{e^2}{2l}$

Work done $= 55x\frac{0.4^2}{2x6.4} = 0.688J$

2. An elastic string of natural length 4m is fixed at one end and is stretched to 5.6m in length by a force of 8N. Find the modulus of elasticity and find the work done.

Solution

$T = \lambda\frac{e}{l}$

$8 = \lambda\frac{(5.6-4)}{4}$

$\lambda = 20N$

Work done $= \lambda\frac{e^2}{2l} = 20x\frac{1.6^2}{2x4} = 6.4J$

3. An elastic string of natural length 1.2m and modulus of elasticity 8N is stretched until the extending force is 6N. Find the extension and work done. **Uneb 1998 No.6**

Solution

$T = \lambda\frac{e}{l}$

$6 = 8x\frac{e}{1.2}$

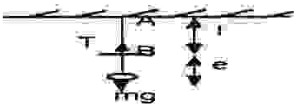
$e = 0.9m$

Work done = $average\ force\ x\ extension$

Work done $= \frac{6}{2}x0.9 = 2.7J$

4. A light elastic string of natural length 1.2m has one end fixed and a mass of 5kg freely suspended from the other. The modulus of the string is such that a 5kg mass hanging vertically would stretch the string by 15cm. The mass is held at A and allowed to fall vertically, how far from A will it come to rest.

Solution



$mg = \lambda\frac{e}{l}$

$5x9.8 = \lambda\frac{0.15}{1.2}$

$\lambda = 392N$

At $u = 0m/s, s = 1.2m, g = 9.8ms^{-2}$

$v^2 = u^2 + 2gs$

$v^2 = 0^2 + 2x9.8x12$

$v = 23.52m/s$

$w = k.e + p.e$

$\lambda\frac{e^2}{2l} = \frac{mv^2}{2} + mge$

$392x\frac{e^2}{2x1.2} = \frac{5x23.52}{2} + 5x9.8e$

$163.333e^2 - 49e - 58.5 = 0$

$e = 0.769m\ \ or\ e = -0.468m$

Depth $= 1.2 + 0.769 = 1.969$m

Exercise 25

1. An elastic string of natural length 1m and modulus 20N. Find the tension in the string when the extension is 20cm. **An(4N)**
2. A spring of natural length 50cm and modulus 10N. Find the thrust in the spring when it is compressed to a length of 40cm **An(2N)**
3. A spring of natural length 75cm and modulus 10N. Find the modulus of the spring when thrust in the spring is 5N and its extension is 25cm **An(15N)**
4. When the length of a spring is 60% of its original length, the thrust in the spring is 10N. Find the modulus of the spring. **An(25N)**
5. An elastic string of natural length 60cm and modulus 18N. Find the extension of the string when the tension in the string is 6N **An(20cm)**
6. The tension in an elastic string is 8N when extension in the string is 25cm. If the modulus of the string is 8N. Find the un-stretched length **An(25cm)**
7. A light elastic string of natural length 20cm and modulus 2gN has one end fixed and a mass of 500g freely suspended from the other. Find the total length of the string. **An(25cm)**
8. When a mass of 5kg is freely suspended from one end of a light elastic string, the other end of which is fixed, the string extends to twice its natural length. Find the modulus of the string **An(49N)**

9. A body of mass 4kg lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 64cm and modulus 25N. When the body moves in a horizontal circular path about O with constant speed of vm/s, the extension in the string is 36cm. Find v. **An(1.875m/s)**

10. A body of mass 5kg lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 2m and modulus 30N. When the body moves in a horizontal circular path about O with constant speed of 3m/s, find the extension in the string. **An(1m)**

11. An elastic string of natural length 2m and modulus 10N. Find the energy store when it is extended to a length of 3m **An(2.5J)**

12. An elastic string of natural length 1m and modulus 20N. Find the energy store when it is extended by a length of 30cm **An(0.9J)**

SIMPLE HARMONIC MOTION IN STRINGS

Examples

1. One end of a light elastic string of natural length 2m and modulus 10N is fixed to a point A on a smooth horizontal surface. A body of mass 200g is attached to the other end of the string and is held at rest at point B on the surface causing the spring to extend by 30cm. show that when released, the body will move with S.H.M. State its amplitude and find the max speed.

Solution

$$T = \lambda\frac{x}{l} = 10\frac{x}{2} = 5x$$

$F = ma$

$5x = -0.2a$

$a = -25x$

Since $a = -\omega^2 x$

$\omega^2 = 25$

$\omega = 5rad\ s^{-1}$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}s \quad and \quad r = 0.3m$$

$v_{max} = \omega r = 5x0.3 = 1.5m/s$

2. One end of a light elastic string of natural length 1m and modulus 5N is fixed to a point O on a smooth horizontal surface. A body of mass 1kg is attached to the other end. A of the string and is held at rest at point B where $OB = 1.25m$. Show that when released, the body will move with S.H.M. Find the maximum speed and total time taken from B to O.

Solution

$$T = \lambda\frac{x}{l} = 5\frac{x}{1} = 5x$$

$F = ma$

$5x = -1a$

$a = -5x$

Since $a = -\omega^2 x$

$\omega^2 = 5$

$\omega = 2.236rad\ s^{-1}$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.236} = 2.81s$$

$r = 0.25m$

$v_{max} = \omega r = 2.236x0.25 = 0.559m/s$

$$t_{OA} = \frac{distance}{speed} = \frac{1}{0.559} = 1.788s$$

Distance	1	:	0.25
time	T	:	t_{AB}

$$t_{AB} = \frac{0.25x2.81}{1} = 0.7025s$$

$t_{OB} = t_{OA} + t_{AB} = 1.788 + 0.7025 = 2.491s$

3. A light elastic string of natural length 2.4m and modulus 15N is stretched between two points A and B, 3m apart on a smooth horizontal surface. A body of mass 4kg attached to the mid-point of the string is pulled 10cm towards B and released

   (i) show that the subsequent motion is simple harmonic

   (ii) find the speed of the body when it is 158cm from A

Solution

At equilibrium $\lambda\frac{e_1}{l} = \lambda\frac{e_2}{l}$

$e_1 = e_2 \ldots\ldots(i)$

$e_1 + e_2 + 2.4 = 3$

$e_1 + e_1 + 2.4 = 3$

$2e_1 + 2.4 = 3$

$e_1 = 0.3m$

$F = ma$

$T_2 - T_1 = ma$

$$\lambda\left(\frac{0.3-x}{1.2}\right)-\lambda\left(\frac{0.3+x}{1.2}\right)=4a$$

$4.8a = -2\lambda x$

$a = -2x15x = -6.25x$

It is in the form of $a = -\omega^2 x$ hence S.H.M

$\omega^2 = 6.25$

$\omega = 2.5rad/s$

$v^2 = \omega^2(r^2 - x^2)$

when 158cm from A, $x = 8cm$

$v = \sqrt{2.5^2(0.1^2 - 0.08^2)} = 0.15m/s$

4. A particle of mass m is attached by means of light strings AP and BP of the same natural length a metres and modulus of elasticity mgN and 2mgN respectively, to points A and B on a smooth horizontal surface. The particle is released form the midpoint of $\overline{AB}$ where $\overline{AB} = 3a$. Show that the subsequent motion is simple harmonic with period $T = \left(\frac{4\pi^2 a}{3g}\right)^{1/2}$. **Uneb 2001, N.o14b**

Solution

At equilibrium $mg\frac{e_1}{a} = 2mg\frac{e_2}{a}$

$e_1 = 2e_2 \dots\dots(i)$

$e_1 + e_2 + 2a = 3a$

$2e_2 + e_2 + 2a = 3a$

$e_2 = {}^{a}/_{3}$

$e_1 = {}^{2a}/_{3}$

$F = ma$

$T_2 - T_1 = ma$

$$2mg\left(\frac{{}^{a}/_{3}-x}{a}\right)-mg\left(\frac{{}^{2a}/_{3}+x}{a}\right)=m\ddot{x}$$

$$\ddot{x} = -\frac{3g}{a}x$$

It is in the form of $a = -\omega^2 x$ hence S.H.M

$\omega^2 = \frac{3g}{a}$ $\quad\therefore \omega = \left(\frac{3g}{a}\right)^{1/2}$.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3g}{a}\right)^{1/2}} = \left(\frac{4\pi^2 a}{3g}\right)^{1/2}$$

5. A particle of mass 1.5kg lies on a smooth horizontal table and is attached to two light elastic strings fixed at points P and Q 12m apart. The strings are of natural length 4m and 5m and their modulus are λ and 2.5λ respectively.
   (a) Show that the particle stays in equilibrium at a point R mid-way between P and Q
   (b) If the particle is held at some point S in the line PQ with $PS = 4.8m$ and then released. Show that the particle performs S.H.M and find the
      (i) Period of oscillation
      (ii) Velocity when the particle is 5.5m from P **Uneb 2008, N.13**

Solution

(a) At equilibrium $\lambda\frac{e_1}{4} = 2.5\lambda\frac{e_2}{5}$

$e_1 = 2e_2 \dots\dots(i)$

$e_1 + e_2 + 4 + 5 = 12$

$2e_2 + e_2 + 9 = 12$

$e_2 = 1$

$e_1 = 2$

At mid-point R; $e_1 + 4 = e_1 + 5 = 6$

(b)

$F = ma$

$T_2 - T_1 = ma$

$$2.5\lambda\left(\frac{1-x}{5}\right)-\lambda\left(\frac{2+x}{4}\right)=15\ddot{x}$$

$$\ddot{x} = -\frac{\lambda}{2}x$$

It is in the form of $a = -\omega^2 x$ hence S.H.M

$\omega^2 = \frac{\lambda}{2}$ $\quad\therefore \omega = \left(\frac{\lambda}{2}\right)^{1/2}$.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{2}\right)^{1/2}} = \left(\frac{8\pi^2}{\lambda}\right)^{1/2}$$

$v^2 = \omega^2(r^2 - x^2)$

when 5.5m from P, $x = 0.5m$

$v = \sqrt{\frac{\lambda}{2}(1.2^2 - 0.5^2)} = \sqrt{0.595\lambda}m/s$

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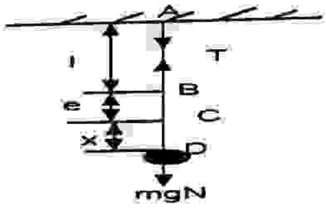
ELASTIC SPRINGS OR SPRINGS HANGING VERTICALLY

1. A particle of mass m is suspended by a string from a fixed point A and has natural length l. If the string is extended from B and C where $BC = e$ and this extension is due to weight of the body (mg). $CD = x$ is the length a particle is pulled vertically downward

Solution



At equilibrium; $T = mg$

$$T = \lambda\frac{e}{l}$$

When pulled a distance, x: $T - T_1 = ma$

$$\lambda\frac{e}{l} - \lambda\left(\frac{e+x}{l}\right) = m\ddot{x}$$

$$-\lambda\frac{x}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{\lambda}{ml}x$$

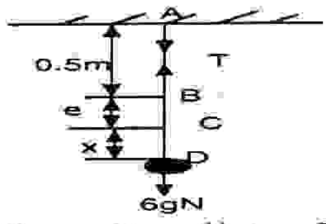
It is in the form of $a = -\omega^2 x$ hence S.H.M

$$\omega^2 = \frac{\lambda}{ml} \quad \therefore \omega = \sqrt{\frac{\lambda}{ml}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\lambda}{ml}}} = 2\pi\sqrt{\frac{ml}{\lambda}}$$

2. A light elastic spring of natural length 50cm and modulus 20gN, hangs vertically with its upper end fixed and a body of mass 6kg attached to its lower end. The body initially rests in equilibrium and then pulled down a distance of 25cm and released.
   (i) Show that the ensuing motion will simple harmonic and
   (ii) find the period of motion and the maximum speed of the body.

Solution



(i) At equilibrium; $T = mg$

$$6g = 20g x\frac{e}{0.5}$$

$$e = 0.15m$$

When pulled a distance, x:

$$T - T_1 = ma$$

$$6g - \lambda\left(\frac{e+x}{0.5}\right) = 6\ddot{x}$$

$$6g - 20g\left(\frac{0.15+x}{0.5}\right) = 6\ddot{x}$$

$$\ddot{x} = -\frac{196}{3}x$$

It is in the form of $a = -\omega^2 x$ hence S.H.M

(ii) $\omega^2 = \frac{196}{3}$

$$\omega = \left(\frac{196}{3}\right)^{1/2} = 8.083 rad/s$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{8.083} = 0.773s$$

$$v_{max} = \omega r = 8.083x0.25 = 2.021 m/s$$

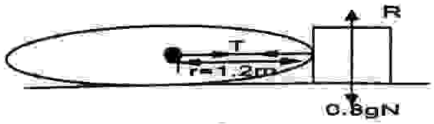
CHAPTER 15: CIRCULAR MOTION

CIRCULAR MOTION ON A SMOOTH HORIZONTAL SURFACE

Examples

1. A particle of mass 0.8kg is attached to one end of a light in extensible string of length 1.2m, the other end is fixed to a point P on a smooth horizontal table. The particles is set moving in a circular path. If the speed of the particle is 16m/s.
   (i) Determine the tension in the string and reaction on the table.
   (ii) If the string snaps, when the tension in the string exceeds 100N, find the greatest angular velocity at which the particle can travel

Solution



(i) $T = F = \frac{mv^2}{r} = \frac{0.8x16^2}{1.2} = 170.667N$

$R = 0.8gN = 0.8x9.8 = 7.84N$

(ii) $T = m\omega^2 r$

$100 = 0.8x\omega^2 x1.2$

$\omega = 10.21rads^{-1}$

2. A ball is tied on an elastic string of natural length 30m to a fixed point on a smooth horizontal table upon which a ball is describing a circle around a point at a constant speed. If the modulus of elasticity of the string is 100 times the weight of the ball and the number of revolution per minute is 20. Show that the extension in the string is approximately 4.7m

Solution

$T = F = m\omega^2 r$

$\omega = 2\pi f = 2\pi\left(\frac{20}{60}\right) = \frac{2}{3}\pi$ and $r = e + 30$

$T = m\left(\frac{2}{3}\pi\right)^2 (e + 30) \ldots\ldots(i)$

$T = \frac{\lambda}{l}e = \frac{100mg}{30}e \ldots\ldots(ii)$

$\frac{100mg}{30}e = m\left(\frac{2}{3}\pi\right)^2 (e + 30)$

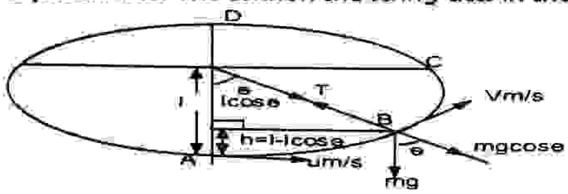
$98x3e = 4\pi^2(e + 30)$

$e = \frac{4\pi^2 x30}{294 - 4\pi^2} = 4.6532m \approx 4.7m$

MOTION IN A VERTICAL CYCLE

PARTICLE IN FOURTH QUADRANT

Consider a body of mass m attached to a string (light rod) of length l and whirled in a vertical circle with a constant speed V. If there is no air resistance to the motion, then the net force towards the centre is the centripetal force. The tension the string acts in the same way as the reaction R



At equilibrium at B: $T - mgcos\theta = \frac{mv^2}{l}$

$T = \frac{mv^2}{l} + mgcos\theta \ldots\ldots\ldots(1)$

But $v^2 = u^2 + 2as$

$a = -g, s = h = l - lcos\theta$

$\boxed{v^2 = u^2 - 2g(l - lcos\theta)} \ldots\ldots(2)$

When the particle comes momentarily to rest at some point A, then $v = 0$

$0 = u^2 - 2g(l - lcos\theta)$

$\boxed{cos\theta = 1 - \frac{u^2}{2gl}} \ldots(3)$

if the particle is attached to a rod, it can complete circles when $v > 0$ and $\theta = 180°$

$v^2 = u^2 - 2g(l - lcos\theta)$

$u^2 - 2g(l - lcos180) > 0$

$u^2 > 2g(l + l)$

$\boxed{u^2 > 4gl} \ldots\ldots(4)$

Put (2) into (1) : $T = \frac{m\,[u^2 - 2g(l - lcos\theta)]}{l} + mgcos\theta$

$T = \frac{mu^2 - 2mgl + 2mglcos\theta + mglcos\theta}{l}$

$\boxed{T = \frac{m}{l}(u^2 - 2gl + 3glcos\theta)} \ldots(5)$

if the particle is attached to a string, it can complete circles when $T > 0$ and $\theta = 180°$

$T = \frac{m}{l}(u^2 - 2gl + 3glcos\theta)$

$\frac{m}{l}(u^2 - 2gl + 3glcos180) > 0$

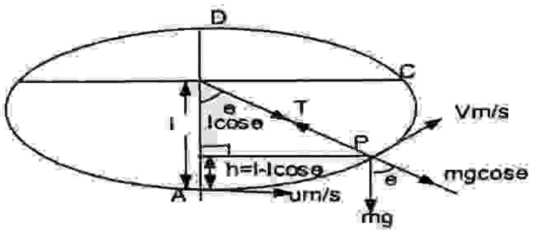
$$(u^2 - 2gl - 3gl) > 0$$

$\boxed{u^2 > 5gl}$ ...... (6)

Examples

1. A particle P of mass 5kg is suspended from a fixed point O by a light inextensible string of length 1m. The particle is projected form its lowest position at the point A, with a horizontal speed of 4m/s. when angle $AOP = 60°$. Find ;
(a) Speed at P
(b) The tension in the string at P

Solution



$$T.M.E_A = T.M.E_P$$

$$5\left(\frac{1}{2}x4^2 + 9.8x0\right) = 5\left[\frac{1}{2}xv^2 + 9.8x(1 - cos60)\right]$$

$$80 = 5xv^2 + 49$$

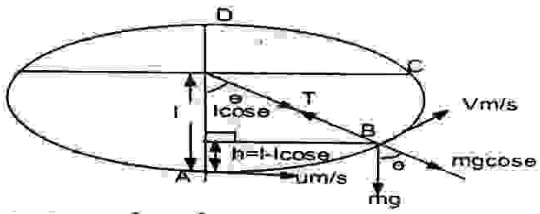
$$v = 2.49m/s$$

$$T = \frac{m\,v^2}{l} + mgcos\theta$$

$$T = \frac{5x\,(2.49)^2}{1} + 5x9.8cos60 = 55.5N$$

2. A light rod of length 2m is pivoted at one end, O and has a particle of mass 8kg attached at the other end. The rod is held vertically with the particle at point A, directly below O and the particle is given an initial horizontal speed um/s. Find ;
(a) An expression in terms of u and θ for the speed of the particle when at point B where $< AOB = \theta$
(b) Restrictions on u^2 if the particle to perform complete oscillations

Solution



At B: $v^2 = u^2 + 2as$

$a = -g, s = h = l - lcos\theta$

$v^2 = u^2 - 2g(l - lcos\theta)$

$$v = \sqrt{u^2 - 2g(l - lcos\theta)}$$

if the particle to perform complete circles $v > 0$ and $\theta = 180°$

$$v^2 = u^2 - 2g(l - lcos\theta)$$

$$u^2 - 2g(l - lcos180) > 0$$

$$u^2 > 2g(l + l)$$

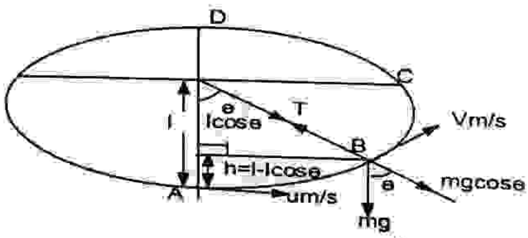
$$u^2 > 4gl$$

$$u^2 > 4x9.8x2$$

$$u^2 > 78.4$$

3. A particle P of mass 8kg is suspended form a fixed point O by an inextensible string if length 2m. The particle is projected from its lowest position A with an initial horizontal speed um/s. Find;
(a) An expression in terms of u and θ for the tension in the string when the particles is at point B where $< AOB = \theta$
(b) Restrictions on u^2 if the particle to perform complete oscillations

Solution



At equilibrium at B: $T - mgcos\theta = \frac{m\,v^2}{l}$

$T = \frac{m\,v^2}{l} + mgcos\theta$..........(1)

$v^2 = u^2 + 2as$

$a = -g, s = h = l - lcos\theta$

$v^2 = u^2 - 2g(l - lcos\theta)$

$v = \sqrt{u^2 - 2g(l - lcos\theta)}$ .... (2)

Put (2) into (1) ; $T = \frac{m\,[u^2 - 2g(l - lcos\theta)]}{l} + mgcos\theta$

$$T = \frac{mu^2 - 2mgl + 2mglcos\theta + mglcos\theta}{l}$$

$$T = \frac{m}{l}(u^2 - 2gl + 3glcos\theta)$$

for the particle to complete circles $T > 0$ and $\theta = 180°$

$$T = \frac{m}{l}(u^2 - 2gl + 3glcos\theta)$$

$$\frac{m}{l}(u^2 - 2gl + 3glcos180) > 0$$

$$(u^2 - 2gl - 3gl) > 0$$

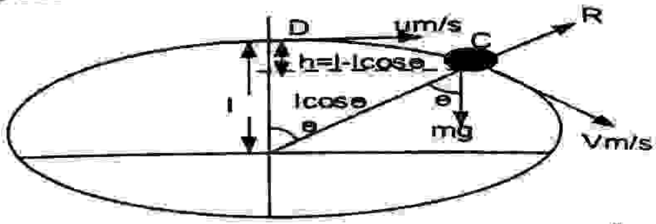
$$u^2 > 5gl$$

$$u^2 > 5x9.8x2$$

$$u^2 > 98$$

PARTICLE IN FIRST QUADRANT

Consider a body of mass rolled from the top of a sphere. The normal reaction R Acts outwards



At equilibrium at C: $mgcos\theta - R = \frac{mv^2}{l}$

$R = mgcos\theta - \frac{mv^2}{l}$..........(1)

But $v^2 = u^2 + 2as$

$a = g, s = h = l - lcos\theta$

$\boxed{v^2 = u^2 + 2g(l - lcos\theta)}$ ....(2)

Putting (2) into (1)

$R = mgcos\theta - \frac{m\left[u^2 + 2g(l - lcos\theta)\right]}{l}$

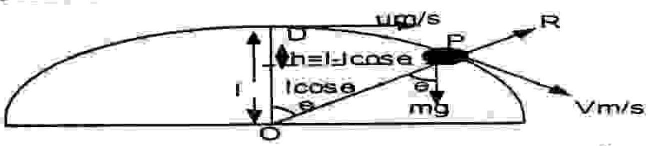
$R = \frac{mglcos\theta - mu^2 - 2mgl + 2mglcos\theta}{l}$

$\boxed{R = \frac{m}{l}(3glcos\theta - 2gl - u^2)}$ .....(3)

Example

1. A particle P of mass 5kg is slightly disturbed from rest on the top of a smooth hemisphere, radius 4m and centre O, resting with its plane face on the horizontal ground.
   (a) Show that the particle leaves the surface of the hemisphere at the point P, where the angle between the radius PO and the upward vertical is $cos^{-1}\left(\frac{2}{3}\right)$
   (b) Find the speed at P

Solution



At equilibrium at P: $mgcos\theta - R = \frac{mv^2}{l}$

$R = mgcos\theta - \frac{mv^2}{l}$..........(1)

But $v^2 = u^2 + 2as$

$u = 0,\ a = g, s = h = l - lcos\theta$

$\boxed{v^2 = 2g(l - lcos\theta)}$ ....(2)

Putting (2) into (1) ; $R = mgcos\theta - \frac{mv^2}{l}$

$R = mgcos\theta - \frac{m\left[2g(l - lcos\theta)\right]}{l}$

$R = \frac{3mglcos\theta - 2mgl}{l}$

When the particle leaves the surface of the hemisphere, $R = 0$

$0 = \frac{3mglcos\theta - 2mgl}{l}$

$3cos\theta = 2$

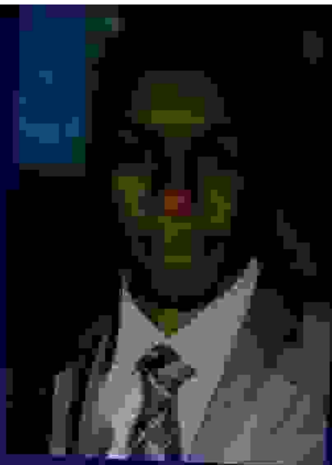
$\theta = cos^{-1}\left(\frac{2}{3}\right)$

$v^2 = 2g(l - lcos\theta)$

$v^2 = 2g\left(l - l\times\frac{2}{3}\right)$

$v^2 = \frac{2}{3}gl = \frac{2}{3}\times9.8\times4 = 26.1333$

$v = 5.1121 m/s$



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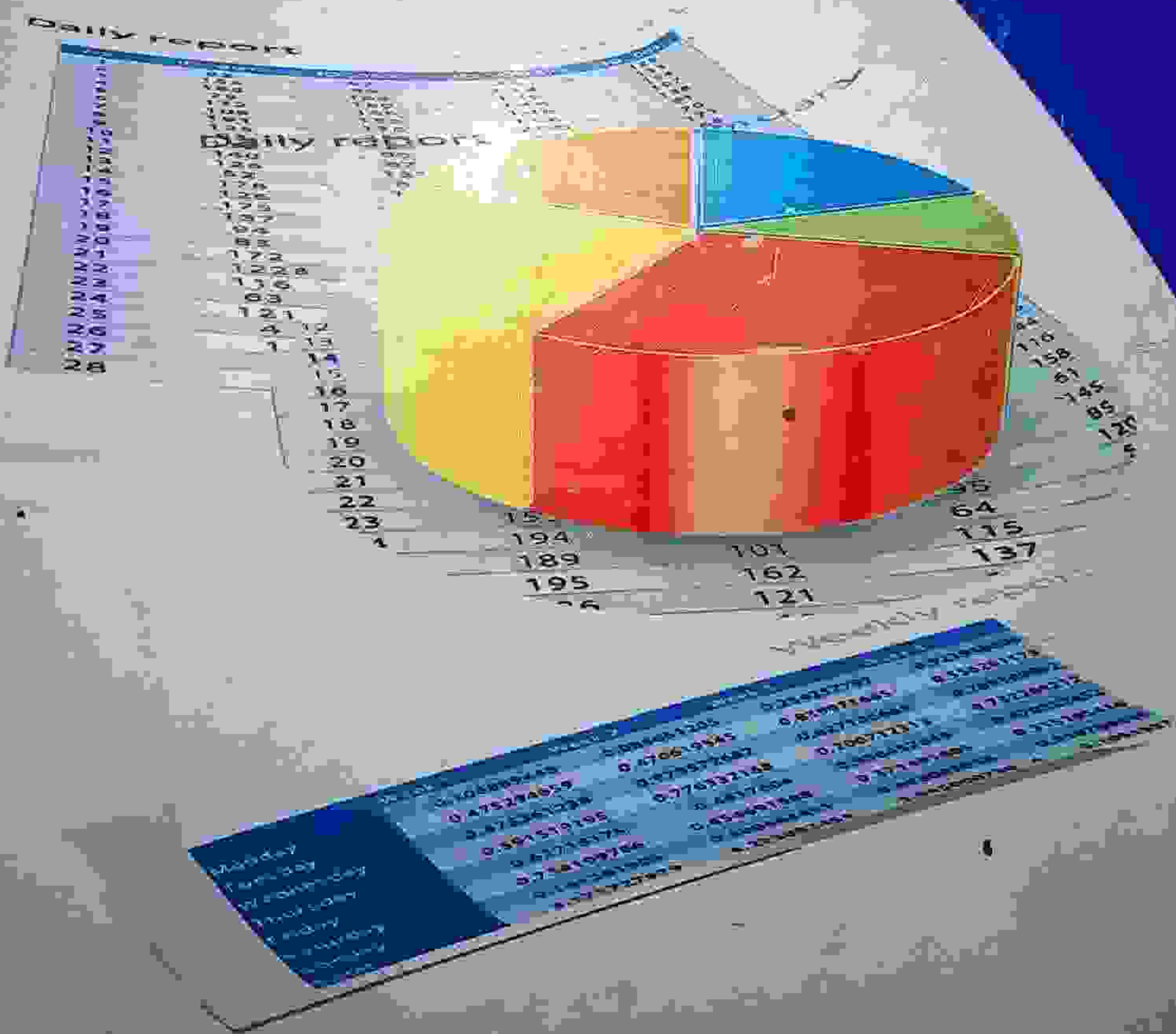

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