SENIOR FIVE TERM 2

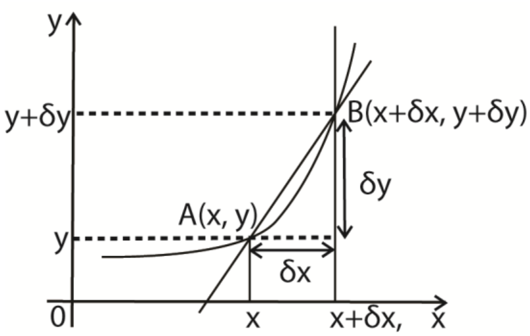
TOPIC 4/6: Differentiation 1

Competency: The learner applies differentiation techniques to solve problems in calculus and interpret their significance in real-world contexts.

Contents

Differentiation from first principles

Consider point A(x, y) lying on a curve drawn below, if another point B(x + δx, y + δy) lies in the same curve, where δx and δy are small increments in x and y respectively, the straight line AB, drawn through the curve is called a chord of the curve.



As the distance δx becomes smaller and smaller, point B moves close to A and the chord AB approaches the position of the target at A

Now, Gradient, $M_{AB} = \frac{(y+\delta y-y}{x+\delta x-x}$

$M_{AB} = \frac{\delta y}{\delta x}$

As δx tends to zero, i.e. δx →0.

$\frac{\delta y}{\delta x}$ approaches the value of the gradient of the target line at A. This value is called limiting value of $\frac{\delta y}{\delta x}$ and is written as $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$.

The limiting value of $\frac{\delta y}{\delta x}$ is called a differential coefficient or first derivative of y with respect to x which is denoted by $\frac{dy}{dx}$.

Note: the process of finding this limiting value is called differentiation.

Direct differentiation of explicit functions

Explicit functions are functions where one variable is expressed in terms of the other variable. Examples $y = x^2$, $y = x^4 + 2x$ etc.

Given the function $x = x^n$, the derivative of y with respect to x, denoted by either y' or $\frac{dy}{dx}$ is given by $y' = \frac{dy}{dx} = nx^{n-1}$.

This result applies for all rational values of n. this means that multiply the term given by the give power index and then reduce the power by one.

Note: If

(i) y = f(x) + g(x) + h(x), then

$$\frac{dy}{dx} = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big) + \frac{d}{dx}\big(h(x)\big)$$

(ii) If y = a, this is written as $y = 0a^0$,

$$\frac{dy}{dx} = 0(ax^{-1}) = 0$$

Example 1

Find the derivatives of the following with respect to x

(a) $y = x^3$

Solution

$\frac{dy}{dx} = 3x^{3-2} = 3x^2$

(b) $y = 2x^2 + 3$

Solution

$y = 2x^2 + 3x^0$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x^0)$$
$$= 2(\ 2x^{2-1}) + 0(3x^{0-1})$$
$$= 4x + 0 = 4x$$

(c) $y = \frac{1}{x}$

Solution

$y = x^{-1}$

$\frac{dy}{dx} = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

(d) y = $\sqrt{x}$

Solution

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

(e) y = $\frac{-2}{x}$

Solution

y = $-2x^{-1}$

$\frac{dy}{dx} = -2(-1x^{-1-1}) = 2x^{-2} = \frac{2}{x^2}$

(f) y = $x^4 + 3x^2 + 2$

Solution

y = $x^4 + 3x^2 + 2x^0$

$\frac{dy}{dx} = 4x^{4-1} + 2(3x^{2-1}) + 0(2x^{0-1})$

= $4x^3 + 6x + 0$

=$4x^3 + 6x$

(g) y = $\frac{3}{\sqrt{x}} - 2\sqrt{x}$

Solution

$y = 3x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}\left(3x^{-\frac{1}{2}-1} - \frac{1}{2}\left(2x^{\frac{1}{2}-1}\right)\right)$

$-\frac{3}{2}x^{\frac{-3}{2}} - x^{-\frac{1}{2}} = -\frac{3}{2x^{\frac{2}{3}}} + \frac{1}{x^{\frac{1}{2}}}$

(h) y = x4(x + 1)

solution

y= $x^5 + x^4$

$\frac{dy}{dx} = 5x^{5-1} + 4x^{4-1} = 5x^4 + 4x^3$

(i) y =$6\sqrt{x}(x^2 - 2x)$

Solution

$y = 6x^{\frac{5}{2}} - 12x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{5}{2}\left(6x^{\frac{5}{2}-1}\right) - \frac{3}{2}\left(12x^{\frac{3}{2}-1}\right)$

$15x^{\frac{3}{2}} - 18x^{\frac{1}{2}}$

Revision exercise 1

Find the derivatives of the following with respect to x

(a) y = $3x^2$ [6x]

(b) y = $2x^4 + 2$ [$8x^3$]

(c) y = b [0]

(d) y = $\frac{9}{2x^3}$ $\left[-\frac{27}{2x^4}\right]$

(e) y = $2x^{-2}$ [$-4x^{-3}$]

(f) y = $\frac{-3}{4x^4}$ $\left[\frac{3}{x^5}\right]$

(g) y = $\sqrt[1]{x}$ $\left[\frac{1}{4x^{\frac{3}{4}}}\right]$

(h) y = $\frac{4}{5\sqrt{x}}$ $\left[\frac{2}{5x^{\frac{3}{2}}}\right]$

(i) y = $\frac{-6}{\sqrt[3]{x}}$ $\left[\frac{2}{x^{\frac{4}{3}}}\right]$

(j) $6\sqrt{x}(x^3 - 2x + 1)$ $\left[21x^{\frac{5}{2}} - 18x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}}\right]$

Direct differentiation of explicit functions from first principles

There are four basic steps followed when differentiating functions from first principles.

Given the function y = f(x), the steps are

(i) Add small changes in x and y to the function y = f(x) i.e. y + δy = f(x + δx)

(ii) Subtract y = f(x) from the established function in step one above i.e. δy = f(x+δx)- f(x)

(iii) Divide the function in step (ii) by δx i.e. $\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$

(iv) Find the limit of the above quotient when δx→0. This is the derivative required

Example 2

Differentiated y = 2x respect to x from first principles

Solution

y = 2x

y + δy = 2(x + δx)

δy = 2(x + δx) – 2x

δy = 2δx

$\frac{\delta y}{\delta x}$ = 2

Differentiation of polynomial functions from first principles

These are functions in terms of y = ax^n where n is both rational and irrational numbers.

Example 3

Differentiated the following with respect to x from first principles

(a) y = x^2

Solution

y = x^2

y + δy = $(x + \delta x)^2$

δy = $(x + \delta x)^2 - x^2$ ………………….. (i)

Eqn. (i) is difference of two squares expression

δy = (x +δx + x)(x + δx – x)

$\delta y = (2x + \delta x)\delta x = 2x\delta x + (\delta x)^2$

$\frac{\delta y}{\delta x} = 2x + \delta x$

$\frac{\delta y}{\delta x} = \max_{\delta x\to 0}\frac{dy}{dx} = 2x$

$\therefore \frac{dy}{dx} = 2x$

(b) y = $\sqrt{x}$

Solution

y = $\sqrt{x}$

y + $\delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

Dividing through by δx

$\frac{dy}{dx} = \frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}$

Rationalizing the numerator on the RHS

$\frac{\delta y}{\delta x} = \frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}\left(\frac{(\sqrt{x+\delta x}+\sqrt{x})}{(\sqrt{x+\delta x}+\sqrt{x})}\right)$

$\frac{\delta y}{\delta x} = \frac{x+\delta x-x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})} = \frac{\delta x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}$

$\frac{\delta y}{\delta x} = \frac{1}{(\sqrt{x+\delta x}+\sqrt{x})}$

$\frac{\delta y}{\delta x} = \max_{\delta x\to 0}\frac{dy}{dx} = \frac{1}{(\sqrt{x}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

(c) y = $\frac{1}{x^2}$

Solution

y = $\frac{1}{x^2}$

y + δy = $\frac{1}{(x + \delta x)^2}$

$\delta y = \frac{1}{(x + \delta x)^2} - y$

$\delta y = \frac{1}{(x + \delta x)^2} - \frac{1}{x^2}$

$\delta y = \frac{x^2-(x + \delta x)^2}{x^2(x + \delta x)^2} = \frac{(x+x+dx)(x-x-\delta x}{x^2(x + \delta x)^2}$

$\delta y = \frac{(2x+\delta x)(-\delta x)}{x^2(x + \delta x)^2} = \frac{-2x\delta x-(\delta x)^2}{x^2(x + \delta x)^2}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{-2x-\delta x}{x^2(x + \delta x)^2}$

$\frac{\delta y}{\delta x} = \max_{\delta x\to 0}\frac{dy}{dx} = \frac{-2x}{x^4} = \frac{-2}{x^3}$

$\therefore \frac{\delta y}{\delta x} = \frac{-2}{x^3}$

(d) y = $2x^3$

Solution

y = $2x^3$

y + $\delta y = 2(x + \delta x)^3$

$\delta y = 2(x + \delta x)^3 - 2x^3$

$\delta y = 2x^3 + 6x^2\delta x + 6x(\delta x)^2 - 2x^3$

$\delta y = 6x^2\delta x + 6x(\delta x)^2$

$\frac{\delta y}{\delta x} = 6x^2 + 6x\delta x$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = 6x^2$

$\therefore \frac{dy}{dx} = 6x^2$

(e) y = $\frac{x}{1+x^2}$

Solution

y = = $\frac{x}{1+x^2}$

y + δy = $\frac{x+\delta d}{1+(x+\delta x)^2}$

$\delta y = \frac{x+\delta d}{1+(x+\delta x)^2} - \frac{x}{1+x^2}$

$\delta y = \frac{(x+\delta d)(1+x^2) - x(1+(x+\delta x)^2)}{(1+x^2)(1+(x+\delta x)^2)}$

$\delta y = \frac{x+x^3+\delta x+x^2\delta x - x - x^3 - 2x^2\delta x - x(\delta x)^2}{(1+x^2)(1+(x+\delta x)^2)}$

$\delta y = \frac{\delta x - x^2\delta x - x(\delta x)^2}{(1+x^2)(1+(x+\delta x)^2)}$

$\frac{\delta y}{\delta x} = \frac{1-x^2-x\delta x}{(1+x^2)(1+(x+\delta x)^2)}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)(1+x^2)} = \frac{1-x^2}{(1+x^2)^2}$

$\therefore \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$

(f) y=x^n

Solution

y=x^n

$y + \delta y = (x+\delta x)^n$

$\delta y = (x+\delta x)^n - x^n$

Since n is assumed to be positive, we expand $(x+\delta x)^n$ using binomial expansion

$\delta y = x^n + \binom{n}{1} x^{n-1}\delta x + \binom{n}{2} x^{n-2}(\delta x)^2 + \cdots + -x^n$

$\delta y = nx^{n-1}\delta x + \binom{n}{2} x^{n-2}(\delta x)^2 + \cdots + (\delta x)^n$

$\frac{\delta y}{\delta x} = nx^{n-1} + \binom{n}{2} x^{n-2}\delta x + \cdots + (\delta x)^{n-1}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = nx^{n-1}$

$\therefore \frac{dy}{dx} = nx^{n-1}$

Revision exercise 2

Differentiated the following with respect to x from first principles

(a) y = $3x^2$ [6x]

(b) y = $2x^4 + 2$ [$8x^3$]

(c) y = b [0]

(d) y = $\frac{9}{2x^3}$ $\left[-\frac{27}{2x^4}\right]$

(e) y = $2x^{-2}$ [$-4x^{-3}$]

(f) y = $\frac{-3}{4x^4}$ $\left[\frac{3}{x^5}\right]$

(g) y = $\sqrt[1]{x}$ $\left[\frac{1}{4x^{\frac{3}{4}}}\right]$

(h) y = $\frac{4}{5\sqrt{x}}$ $\left[\frac{2}{5x^{\frac{3}{2}}}\right]$

(i) y = $\frac{-6}{\sqrt[3]{x}}$ $\left[\frac{2}{x^{\frac{4}{3}}}\right]$

(j) $y = 6\sqrt{x}(x^3 - 2x + 1)$ $\left[21x^{\frac{5}{2}} - 18x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}}\right]$

(k) y = $X^3 + x^2$ [$3x^2 + 2x$]

(l) $y = \frac{2}{\sqrt{(x+2)}}$ $\left[\frac{1}{\sqrt{(x+2)}}\right]$

(m) $y = 4x + 2x^2$ $[4 + 4x]$

Differentiation of product and quotient of a function

Given the function y = uv and that u and v are functions of x, the derivatives of y with respect to x is done from first principles.

Let δx be a small increment in x and let δu, δv and δy be the resulting small increment in u, v and y

y = uv

$y + \delta y = (u + \delta u)(v + \delta v)$

$\delta y = (u + \delta u)(v + \delta v) - uv$

$= u\delta v + v\delta u + \delta u\delta v$

Dividing through by δx

$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u\delta v}{\delta x}$

As δx → 0; δu → 0; δv → 0 and δy → 0

⇨ $\frac{\delta y}{\delta x} \to \frac{dy}{dx}; \frac{\delta u}{\delta x} \to \frac{du}{dx}; \frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\frac{\delta u\delta v}{\delta x} \to 0$

$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

This can also be expressed as (uv)’ = u’v + uv’

Example 4

Differentiate the following functions with respect to x.

(a) $x^2(x+2)^3$

Here u = x^2 and v = $(x+2)^3$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3(x + 2)^2$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{\delta y}{\delta x} = 2x(x + 2)^3 + 3x^2(x + 2)^2$

$= (x + 2)^2(2x^2 + 4x + 3x^2)$

$= (x + 2)^2(5x^2 + 4x)$

$= x(x + 2)^2(5x + 4)$

$\therefore \frac{\delta}{\delta x}(x^2(x + 2)^3 = x(x + 2)^2(5x + 4)$

(b) $(x + 2)^3(1 - x^2)^4$

u = $(x + 2)^3$ and $v =$ $(1 - x^2)^4$

$\frac{du}{dx} = 3(x + 2)^2$ and

$\frac{dv}{dx} = 4(1 - x^2)^3(-2x) = -8x(1 - x^2)^3$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -8x(x+2)^3(1-x^2)^3 + 3(1-x^2)^4(x+2)^2$

$= (1-x^2)^3(x+2)^2\,[-8x(x+2) + 3(1-x^2)]$

$= (1-x^2)^3(x+2)^2\,[-8x^2 - 16x + 3 - 3x^2]$
$= (1-x^2)^3(x+2)^2\,(3 - 16x - 11x^2)$

$\therefore \frac{\delta}{\delta x}\{(x+2)^3(1-x^2)^4\} = (1-x^2)^3(x+2)^2\,(3-16x-11x^2)$

(c) $7x^2\sqrt{x^2-1}$

u = $7x^2$ and v = $(x^2-1)^{\frac{1}{2}}$

$\frac{du}{dx} = 14x \text{ and } \frac{dv}{dx} = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x) = x(x^2-1)^{-\frac{1}{2}}$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 7x^2\left[x(x^2-1)^{-\frac{1}{2}}\right] + 14x(x^2-1)^{\frac{1}{2}}$
$= 7x^2\left[\frac{x}{(x^2-1)^{\frac{1}{2}}}\right] + 14x(x^2-1)^{\frac{1}{2}}$
$= \frac{7x^3+14x(x^2-1)}{(x^2-1)^{\frac{1}{2}}} = \frac{21x^3-14x)}{(x^2-1)^{\frac{1}{2}}} = \frac{7x(3x^2-2)}{(x^2-1)^{\frac{1}{2}}}$

$\therefore \frac{\delta}{\delta x}\left(7x^2\sqrt{x^2-1}\right) = \frac{7x(3x^2-2}{\sqrt{x^2-1}}$

(d) $2x^4(3x^2-6x+2)^3$
$u = 2x^4$ and $v = (3x^2-6x+2)^1$

$\frac{du}{dx} = 8x^3 \text{ and } \frac{dv}{dx} = 3(3x^2-6x+2)^3(6x-6)$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 2x^4[3(3x^2-6x+2)^2(6x-6)] + 8x^3(3x^2-6x+2)^3$
$= 4x^{\,3}\,(3x^2-6x+2)^2\,\{9x^2 - 9x + 6x^2 - 12x + 4\}$
$= 4x^{\,3}\,(3x^2-6x+2)^2\,\{15x^2 - 21x + 4\}$
$\therefore \frac{\delta}{\delta x}(2x^4(3x^2-6x+2)^3) = 4x^{\,3}\,(3x^2-6x+2)^2\,(15x^2-21x+4)$

(e) $\sqrt{(6+x)}\sqrt{(3-2x)}$

$u = (6+x)^{\frac{1}{2}}$ and $v = (3-2x)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}(6+x)^{-\frac{1}{2}} \text{ and } \frac{dv}{dx} = \frac{1}{2}(3-2x)^{-\frac{1}{2}}(-2) = -(3-2x)^{-\frac{1}{2}}$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{-\sqrt{(6+x)}}{\sqrt{3-2x}} + \frac{\sqrt{(3-2x)}}{2\sqrt{(6+x)}} = \frac{-12-2x+3-2x}{2\sqrt{3-2x}\sqrt{(6+x)}} = \frac{-(9x+4)}{2\sqrt{3-2x}\sqrt{(6+x)}}$$

$$\therefore \frac{d}{dx}\left(\sqrt{(6+x)}\sqrt{(3-2x)}\right) = \frac{-(9x+4)}{2\sqrt{3-2x}\sqrt{(6+x)}}$$

Quotient rule in differentiation

This is an extension of the product rule

Given the function y = $\frac{u}{v}$

Then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \ or \ \left(\frac{u}{v}\right)' = \frac{vu' - uv\prime}{v^2}$$

Example 5

Differentiate the following with respect to x

(a) $\frac{x^2+6}{2x-7}$

$u = x^2 + 6$ and $v = 2x - 7$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 2$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(2x-7).x-(x^2+6).2}{(2x-7)^2}$

$= \frac{2(2x^2-7x-x^2-6)}{(2x-7)^2} = \frac{2(x^2-7x-6)}{(2x-7)^2}$

$\therefore \frac{d}{dx}\left(\frac{x^2+6}{2x-7}\right) = \frac{2(x^2-7x-6)}{(2x-7)^2}$

(b) $\frac{x}{(x^2+4)^3}$

u = u and v = $(x^2 + 4)^3$

$\frac{du}{dx} = 1$

$\frac{dv}{dx} = 2(x^2 + 4)^2 . 2x = 6x(x^2 + 4)^2$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(x^2+4)^3 - 6x^2(x^2+4)^2}{((x^2+4)^3)^2}$

$= \frac{(4-5x^2)}{(x^2+4)^6}$

Revision Exercise 3

1. Find the derivatives of each of the following

a. $\frac{2x+1}{3x-4}$ $\left[\frac{-11}{(3x-4)^2}\right]$

b. $\frac{3x-4}{2x+1}$ $\left[\frac{11}{(2x+1)^2}\right]$

c. $\frac{x^2-3}{2x+1}$ $\left[\frac{-2(x^2+1+3)}{(2x+1)^2}\right]$

d. $\frac{2x+1}{x^2-3}$ $\left[\frac{-2(x^2+1+3)}{(x^2-3)^2}\right]$

e. $\sqrt{\frac{x^3}{x^2-1}}$ $\left[\frac{\sqrt{x}(x^2-3)}{2\sqrt{x^2-1}}\right]$

f. $\sqrt{\frac{3+x}{2-3x}}$ $\left[\frac{11}{2\sqrt{(3+x)}\sqrt{(2-3x)}}\right]$

g. $\frac{\sqrt{x}+1}{\sqrt{x}-1}$ $\left[-\frac{1}{\sqrt{x}(\sqrt{x}-1)^2}\right]$

h. $\frac{2x}{\sqrt{x}+1}$ $\left[\frac{\sqrt{x}+2}{(\sqrt{x}+1)^2}\right]$

i. $\frac{x^2+1}{3x-1}$ $\left[\frac{3x \quad -2x-3}{(3x-1)^2}\right]$

j. $\frac{x(x-1)^3}{x-3}$ $\left[\frac{3(x^2-4x+1)(x-1)^2}{(x-3)^2}\right]$

2. Show that

(a) $\frac{d}{dx}\left(\frac{x(x-3)^3}{(x+3)(x+5)^2}\right)^2 = \frac{2x(x-3)^5(x^3+27x^2+69x-45)}{(x+3)^3(x+5)^5}$

Differentiation of functions by use of chain rule

Chain rule is a rule used to differentiate a function of a function i.e. if y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Example 6

Find $\frac{dy}{dx}$ of each of the following using chain rule

(a) $(x + 5)^3$

Let u = (x + 5); thus y = u^3

$\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 1$

Using chain rule; $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 3u^2.1 = 3u^2$

Substituting for u

$\frac{dy}{dx} = 3(x+5)^2$

$\therefore \frac{d}{dx}(x+5)^3 = 3(x+5)^2$

(b) $(2x-5)^{10}$

Let $u = 2x - 5$ so that y = u^{10}

$\frac{du}{dx} = 2$ and $\frac{dy}{du} = 10u^9$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 10u^9.2 = 20u^9$

Substituting for u

$\frac{dy}{dx} = 20(2x-5)^9$

$\therefore \frac{d}{dx}(2x-5)^{10} = 20(2x-5)^9$

(c) $(x^2+x-1)^4$

Let u = $x^2 + x - 1$ so that y = u^4

$\frac{du}{dx} = 2x + 1$ and $\frac{dy}{du} = 4u^3$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 4(2x+1)(x^2+x-1)^3$

$\therefore \frac{d}{dx}(x^2+x-1)^4 = 4(2x+1)(x^2+x-1)^3$

Revision exercise 4

1. Differentiate each of the following with respect to x using chain rule

(a)	$2(1-x)^5$	$[-10(1-x)^4]$
(b)	$(x^2+3)^4$	$[8x(x^2+3)^3]$
(c)	$\frac{1}{3-7x}$	$\left[\frac{7}{(3-7x)^2}\right]$
(d)	$\sqrt{6x+1}$	$\left[\frac{3}{\sqrt{6x+1}}\right]$
(e)	$(6x^2-5)^4$	$[48x(6x^2-5)^3]$
(f)	$(2x-5)^{-3}$	$[-6(2x-5)^{-4}]$
(g)	$(3x+2)^{-1}$	$[-3(3x+2)^{-2}]$
(h)	$(x^2+3)^{-2}$	$[-4x(x^2+3)^{-3}]$
(i)	$(5-2x^3)^{-1}$	$[6x^2(5-2x^3)^{-2}]$
(j)	$\frac{1}{3+4x}$	$\left[\frac{-4}{(3+4x)^2}\right]$
(k)	$(2x-1)^{\frac{1}{2}}$	$\left[\frac{1}{\sqrt{2x-1}}\right]$
(l)	$(6-x)^{\frac{1}{3}}$	$\left[\frac{-1}{3(6-x)^{\frac{2}{3}}}\right]$
(m)	$(x^3-2)^{\frac{2}{3}}$	$\left[\frac{2x^2}{(6-x)^{\frac{1}{3}}}\right]$
(n)	$(4-x^5)^{-\frac{1}{5}}$	$\left[\frac{x^4}{(4-x^5)^{\frac{6}{5}}}\right]$
(o)	$\sqrt{x^3-6x}$	$\left[\frac{3(x^2-2)}{2\sqrt{x^3-6x}}\right]$
(p)	$\frac{1}{x^2-3x+5}$	$\left[\frac{3-2x^2}{(x^2-3x+5)^2}\right]$
(q)	$\frac{3x-1}{\sqrt{x^2+1}}$	$\left[\frac{x+3}{(x^2+1)^{\frac{3}{2}}}\right]$

Differentiation of parametric equations

Parametric equations are expressed in terms of a third variable say t such as $y = t^2$ and $x = 2t + 1$, here the parametric variable is t. Chain rule is often used to find the derivatives of these equations.

Example 7

Find the derivatives of the following in terms of parameter t.

(a) $y = 3t^2 + 2t$, $x = 1-2t$

$\frac{dy}{dt} = 6t + 2$ and $\frac{dx}{dt} = -2$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= (6t + 2).\frac{1}{-2}$

$= -(3t + 1)$

(b) $y = (1 + 2t)^3$, $x = t^3$

$\frac{dy}{dt} = 6(1 + 2t)^2$ and $\frac{dx}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{6(1+2t)^2}{3t^2} = \frac{2(1+2t)^2}{t^2}$

(c) $x = t^2$, $y = 4t-1$

$\frac{dy}{dt} = 4$ and $\frac{dx}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{4}{2t} = \frac{2}{t}$

(d) $x = \frac{2}{3+\sqrt{t}}$, $y = \sqrt{t}$

$x = 2\left(3 + t^{\frac{1}{2}}\right)^{-1}$

$\frac{dx}{dt} = -2\left(3 + t^{\frac{1}{2}}\right)^{-2}.\frac{1}{2}t^{-\frac{1}{2}} = \frac{-1}{\left(3+t^{\frac{1}{2}}\right)^2 t^{\frac{1}{2}}}$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{1}{2\sqrt{t}}.-\left(3 + t^{\frac{1}{2}}\right)^2 t^{\frac{1}{2}} = .\frac{-\left(3+t^{\frac{1}{2}}\right)^2}{2}$

Revision exercise 5

Find $\frac{dy}{dx}$ for each of the following

(a) $x=2\sqrt{t}, y = 5t - 4$ $\left[5\sqrt{t}\right]$

(b) $x=4\sqrt{t} - t, y = t^2 - 2\sqrt{t}$ $\left[\frac{2\sqrt{t^3}-1}{2-\sqrt{t}}\right]$

(c) $x = \frac{2}{\sqrt[3]{3t-4}}, y = \sqrt[3]{6t+1}$ $\left[\sqrt[3]{\frac{(3t-4)^4}{(6t+1)^2}}\right]$

(d) $x = t + 5$, $y = t^2 - 2t$ $[2(t-1)]$

(e) $x = t^6$, $y = 6t^3 - 5$ $[3t^{-3}]$

(f) $x = \sqrt{t-1}, y = \frac{1}{t}$ $\left[\frac{-2\sqrt{t-1}}{t^2}\right]$

(g) $x = t^2(3t-1), y = \sqrt{3t+4}$

$\left[\frac{3}{2\sqrt{3t+4}(9t^2-2t)}\right]$

Differentiation of implicit function

The functions given in the form y = f(x) such as y = 2x, y = x^5 + 3x etc. are known as explicit functions whereas functions that cannot be expressed in the form y = f(x) such as y^2+2xy = 5, x^2 + 5xy + y^2 =4 etc. are known as implicit functions because y cannot be expressed easily in terms of x.

When differentiating such functions with respect to x or y, we consider each of the individual terms in the equation given

Example 8

Find $\frac{dy}{dx}$ for each of the following functions.

(a) $x^2 - 6y^3 + y = 0$

$\frac{d}{dx}(x^2) - \frac{d}{dx}(6y^2) + \frac{d}{dx}(y) = 0$

$2x - 18y^2\frac{dy}{dx} + \frac{dy}{dx} = 0$

$\frac{dy}{dx}(18y^2 - 1) = 2x$

$\frac{dy}{dx} = \frac{2x}{18y^2-1}$

(b) $x^2y = 5x + 2$

$\frac{d}{dx}(x^2y) = \frac{d}{dx}(5x) + \frac{d}{dx}(2)$

$\frac{d}{dx}(x^2y)$ is done by use of product rule

$x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) = \frac{d}{dx}(5x) + \frac{d}{dx}(2)$

$x^2\frac{dy}{dx} + 2xy = 5$

$\frac{dy}{dx} = \frac{5-2xy}{x^2}$

(c) $(x + y)^5 - 7x^2 = 0$

$\frac{d}{dx}(x + y)^5 - \frac{d}{dx}7x^2 = 0$

$5(x + y)^4\frac{d}{dx}(x + y) - 14x = 0$

$5(x + y)^4\left(1 + \frac{dy}{dx}\right) = 14x$

$\frac{dy}{dx} = \frac{14x}{5(x+y)^4} - 1$

$= \frac{14x-5(x+y)^4}{5(x+y)^4}$

(d) $y^2 + x^3 - y^3 + 6 = 3y$

$\frac{d}{dx}y^2 + \frac{d}{dx}x^3 - \frac{d}{dx}y^3 + \frac{d}{dx}6 = \frac{d}{dx}3y$

$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} + 0 = 3\frac{dy}{dx}$

$3x^2 = \frac{dy}{dx}(3y^2 - 2y + 3)$

$\frac{dy}{dx} = \frac{3x^2}{(3y^2-2y+3)}$

(e) $y^2 + x^3 - xy + cosy = 0$

$\frac{d}{dx}y^2 + \frac{d}{dx}x^3 - x\frac{d}{dx}y - y\frac{d}{dx}x + \frac{d}{dx}cosy = 0$

$2y\frac{dy}{dx} + 2x^2 - x\frac{dy}{dx} - y - siny\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y - x - siny) = y - 2x^2$

$\frac{dy}{dx} = \frac{y-2x^2}{(2y-x-siny)}$

Revision exercise 6

1. Find $\frac{dy}{dx}$ for each of the following functions

(a) $\frac{x^3}{x+y} = 2$ $\left[\frac{3x^2-2}{2}\right]$

(b) $2x - y^3 = 3xy$ $\left[\frac{2-3y}{3x+3y^2}\right]$

(c) $x^6 - 5xy^3 = 9xy$ $\left[\frac{6x^5-y^2-9y}{3x(3+5y^3)}\right]$

(d) $\frac{x^2}{x+y} = 2x$ $\left[\frac{x+y}{x}\right]$

(e) $\frac{y}{x^2-7y^3} = x^5$ $\left[\frac{7x^4(x^2-5y^3)}{1+21x^5y^2}\right]$

(f) $\sqrt{x} + \sqrt{y}$ $\left[\sqrt{\frac{y}{x}}\right]$

(g) $\frac{y}{x} + \frac{x}{y} = 1$ $\left[\frac{y}{x}\right]$

(h) $x^2 + xy + y^2 - 3x - y = 3$ $\left[\frac{3-2x-y}{x+2y-1}\right]$

(i) $y^2 - 5xy + 8x^2 = 2$ $\left[\frac{5y-16x}{2y-5x}\right]$

2. For each of the following find the gradient of the stated curve at the point specified,

(a) $xy^2 - 6y = 8 \; at \; (2,1)$ $\left[\frac{1}{10}\right]$

(b) $3y^4 - 7xy^2 - 12y = 5 \; at(-2,1)$ $\left[\frac{1}{4}\right]$

(c) $\frac{x^2}{x-y} = 8 \; at \; (4,2)$ $[0]$

(d) $\frac{2}{x} + \frac{5}{y} = 2xy \; at \left(\frac{1}{2}, 5\right)$ $[-15]$

(e) $(x + 2y)^4 = 1 \; at \; (5, -2)$ $\left[-\frac{1}{2}\right]$

(f) $x^2 + 6y^2 = 10 \; at \; (2, -1)$ $\left[\frac{1}{3}\right]$

(g) $x^3 + 4xy = 15 + y^2 \; at \; (2, 1)$ $\left[-2\frac{2}{3}\right]$

Second derivative

Suppose y is a function of x, the first derivative of y with respect to x is denoted as $\frac{dy}{dx}$ or f'(x)

The result of differentiating $\frac{dy}{dx}$ with respect to x is the second derivative denoted by $\frac{d^2y}{dx^2}$ or f''(x)

Note that If $\frac{d^2y}{dx^2}$ is used to determine the natures of stationary points

A stationary point on a curve occurs when $\frac{dy}{dx} = 0$ Once you have established where there is a stationary point, the type of stationary point (maximum, minimum or point of inflexion) can be determined using the second derivative.

If $\frac{d^2y}{dx^2}$ is positive, then it is a minimum point

If $\frac{d^2y}{dx^2}$ is negative, then it is a maximum point

If $\frac{d^2y}{dx^2}$ =0 then it could be maximum, maximum or point of inflection

Example 9

Determine the second derivative of each of the following

(a) x^4

$\frac{dy}{dx} = 4x^3$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3) = 12x^2$

(b) $x^2(1 - \mathrm{x})^2$

$x^2(1 - \mathrm{x})^2 = x^2(1 - 2\mathrm{x} + x^2)$

$= x^2 - 2x^3 + x^4$

$\frac{dy}{dx} = 2x - 6x^2 + 4x^3$

$\frac{d^2y}{dx^2} = 2 - 12x + 12x^2$

(c) If $x^2 + 3xy - y^2 = 3$, find $\frac{dy}{dx}$and $\frac{d^2y}{dx^2}$ at the point (1,1)

$2x + 3y + 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x+3y}{2y-3x}$

At (1,1)

$\frac{dy}{dx} = \frac{2(1)+3(1)}{2(1)-3(1)} = -5$

$\frac{d^2y}{dx^2} = \frac{(2y-3x)\left(2+3\frac{dy}{dx}\right)-(2x+3y)\left(2\frac{dy}{dx}-3\right)}{(2y-3x)^2}$

Substituting for x =1, y= 1and $\frac{dy}{dx} = -5$

$\frac{d^2y}{dx^2} = \frac{(2-3)(2+3(-5))-(2+3)(2(-5)-3)}{(2-3)^2}$

$= \frac{(-1)(-13)-(5)(-13)}{(-1)^2}$

$= \frac{13+65}{1} = 78$

Example 10 (parametric equation)

Find $\frac{d^2y}{dx^2}$ in terms of t if

(a) $x = a(t^2 - 1) and\ y = 2a(t + 1),$

$\frac{dx}{dt} = 2at$

$\frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{2a}{2at} = \frac{1}{t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$

$= \frac{d}{dx}(t^{-1}).\frac{dt}{dx}$

$= \frac{-1}{t^2}.\frac{1}{2at}$

$= \frac{-1}{2at^3}$

Revision exercise 7

1. Find $\frac{d^2y}{dx^2}$ of each of the following

(a) $\frac{x^2}{1+x}$ $\left[\frac{2}{(1+x)^3}\right]$

2. Find $\frac{d^2y}{dx^2}$ in terms of t or θ if

(a) $x = \frac{1+t^2}{1-t},\ y = \frac{2t}{1-t}$ $\left[-4\left(\frac{1-t}{1+2t-t^2}\right)^3\right]$

(b) $x = t + 3, y = t^2 + 4$ [2]

(c) $x = 3 - 2t^2, y = \frac{1}{t}$ $\left[\frac{3}{16t^5}\right]$

(d) $x = t^2 + 2t, y = t^2 - 3t$ $\left[\frac{3}{4(t+1)}\right]$

3. Given that y = cot5x, show that $\frac{d^2y}{dx^2} + 10y\frac{dy}{dx} = 0$

4. Given that $x = 1 - sint$ and $y = 1 - cost$ show that $y^2\frac{d^2y}{dx^2} + 1 = 0$

Differentiation of exponential functions

An exponential function is the function given in the form $y = e^x$, where y is said to be an exponential function of x.

These are differentiated using product and quotient rules.

Example 11

Differentiate each of the following with respect to x

(a) e^x

$\frac{d}{dx}(e^x) = e^x$

(b) e^{3x^2}

$Let\ u = 3x^2 and\ y = e^u$

$\frac{du}{dx} = 6x\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 6x.e^u$

$= 6xe^{3x^2}$

(c) e^{3x}

Let $u = 3x$ => $y = e^u$

$\frac{du}{dx} = 3\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = e^u.3$

$= 3e^{3x}$

(d) $y = 2e^{x^2+1}$

Let $u = x^2 + 1$ => $y = 2e^u$

$\frac{du}{dx} = 2x\ and\ \frac{dy}{du} = 2e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2e^u.2x$

$= 4xe^{x^2+1}$

(e) $\frac{e^{-\frac{1}{2}\sqrt{x}}}{x^2}$

$\frac{dy}{dx} = \frac{x^2\frac{d}{dx}e^{-\frac{1}{2}\sqrt{x}} - e^{-\frac{1}{2}\sqrt{x}}\frac{d}{x}(x^2)}{x^4}$

$= \frac{e^{-\frac{1}{2}\sqrt{x}}\left(\frac{x^2}{4\sqrt{x}} + \frac{2x}{1}\right)}{x^4}$

$= \frac{e^{-\frac{1}{2}\sqrt{x}}(x + 8\sqrt{x})}{4x^{\frac{7}{3}}}$

Differentiation of logarithmic functions

Logarithms of numbers to base e is called natural logarithm or napeilian logarithm.

The natural logarithm of a number say x is denoted by $\log_e x$ or Inx

Let y $= \log_e x$

$e^y = x$

$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)}{e^y} = \frac{\frac{d}{dx}(e^y)}{e^y}$

Example 10

Differentiate with respect to x

(a) Inx

Let y = Inx

$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)}{x} = \frac{1}{x}$

(b) In(1+2x)

Let y = In(1 + 2x)

$\frac{dy}{dx} = \frac{\frac{d}{dx}(1+2x)}{1+2x} = \frac{2}{1+2x}$

(c) In(1- x)

Let y = In(1 – x)

$\frac{dy}{dx} = \frac{\frac{d}{dx}(1-x)}{1-x} = \frac{-1}{1-x}$

(d) $In(4x^3)$

Let $y = In(4x^3)$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(4x^3)}{4x^3} = \frac{12x^2}{4x^3} = \frac{3}{x}$

(e) $2y^2$

Let $q = 2y^2$

$Inq = 2y^2$=$2In(2y)$

$\frac{1}{q}\frac{dt}{dy} = 2\frac{\frac{d}{dy}(2y)}{2y} = \frac{2}{y}$

$\frac{dq}{dy} = \frac{2q}{y} = \frac{4y^2}{y} = 4y$

But $\frac{dq}{dy} = \frac{dq}{dy}.\frac{dy}{dx}$

$\frac{dq}{dx} = 4y\frac{dy}{dx}$

(f) Iny

Let q = Iny

$\frac{dq}{dy} = \frac{1}{y}$

But $\frac{dq}{dy} = \frac{dq}{dy}.\frac{dy}{dx}$

$\frac{dq}{dy} = \frac{1}{y}.\frac{dy}{dx}$

(g) 2^x

Let y = 2^x

Iny =In2^x=xIn2

$\frac{1}{y}\frac{dy}{dx} = In2$

$\frac{dy}{dx} = yIn2 = 2^x In2$

(h) 2^{x^2}

$Iny = In2^{x^2} = x^2 In2$

$\frac{1}{y}\frac{dy}{dx} = 2xIn2$

$\frac{dy}{dx} = y2xIn2 = 2^{x^2}2xIn2$

(i) $3x^2.3^x$

Let y =$3x^2.3^x$

$Iny = In3x^2.3^x$

$= In3 + Inx^2 + In3^x$

$= In3 + 2Inx + xIn3$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + Inx = \frac{2+xInx}{x}$

$\frac{dy}{dx} = y\frac{2+xInx}{x} = 3x^2.3^x\left(\frac{2+xInx}{x}\right)$

$= 3x.3^x(2 + xInx)$

(j) $\sqrt[3]{\frac{x+1}{x-1}}$

$Let\ y = \sqrt[3]{\frac{x+1}{x-1}}$

$y^3 = \frac{x+1}{x-1}$

$Iny^3 = In(x+1) - In(x-1)$

$\frac{3y^2}{y^3}\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

$\frac{dy}{dx} = \frac{y}{3}.\frac{-2}{(x+1)(x-1)}$

$= \frac{(x+1)^{\frac{1}{3}}}{3(x-1)^{\frac{1}{3}}}.\frac{-2}{(x+1)(x-1)}$

$= \frac{-2}{3(x+1)^{\frac{2}{3}}(x-1)^{\frac{4}{3}}}$

Revision exercise 8

1. Differentiate with respect x

(a) e^{2y} $\left[2e^{2y}\frac{dy}{dx}\right]$

(b) $4x^2+\frac{2}{e^{x^2}}$ $\left[8x-\frac{4x}{e^{x^2}}\right]$

(c) xe^{-x} $[e^{-x}-xe^{-x}]$

(d) $\frac{\sqrt{x^2+1}}{(2x-1)^2}$ $\left[\frac{2x^2+x+4}{(x^2+1)^{\frac{1}{2}}(2x-1)^3}\right]$

(e) $\frac{x^2e^x}{(x-1)^3}$ $\left[\frac{xe^x(x^2-2x-2)}{(x-1)^4}\right]$

(f) $\frac{(x-1)(2-3x)}{(1+x)(x+2)}$ $\left[\frac{2(8-4x-7x^2)}{(1+x)^2(x+2)^2}\right]$

(g) $In(1+x^2)$ $\left[\frac{2x}{1+x^2}\right]$

(h) $In(x^3-2)$ $\left[\frac{3x^2}{x^3-2}\right]$

(i) $In(e^x+4)$ $\left[\frac{e^x}{e^x+4}\right]$

(j) $In(\sqrt{x})$ $\left[\frac{1}{2x}\right]$

(k) $(3-2Inx)^3$ $\left[\frac{-6(3-2Inx)^2}{x}\right]$

(l) x^2Inx $[x(1+2Inx)]$

(m) $xIn(1+x)$ $\left[\frac{x}{1+x}+In(1+x)\right]$

(n) $x^2In(3+2x)$ $\left[\frac{2x^2}{3+2x}+2xIn(3+2x)\right]$

(o) $\frac{x}{Inx}$ $\left[\frac{Inx-1}{(Inx)^2}\right]$

(p) 7^x $[7^xInx]$

(q) 2^{x^2} $[x2^{x^2}In4]$

(r) 3^{2x-1} $\left[\frac{2}{3}(3^{2x})In3\right]$

(s) e^{Inx} $[1]$

2. Given that $y=xe^{2x}$, show that $x\frac{dy}{dx}=(2x+1)y$

3. Given that $y=\frac{e^x}{e^x+1}$, show that $(1+e^x)\frac{dy}{dx}-y=0$

4. Given that $y=\frac{e^{x^2}}{x}$, show that $\frac{dy}{dx}=\frac{2e^{x^2}-y}{x}$

5. Given that e^x-e^{-x}, show that $\left(\frac{dy}{dx}\right)^2-y^2=4$

6. Given that $Ae^{4x}+Be^{-4x}$, where A and B are constants show that $\frac{d^2y}{dx^2}-16y=0$

7. Given that $y=In(Inx)$, show that $(Inx)\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{x^2}=0$

8. Given that $y=In\left(\frac{1+x}{1-x}\right)$, show that $(1-x^2)\frac{dy}{dx}-2=0$

9. Given that $y=\frac{In(1+x)}{x^2}$, show that $x^2\frac{dy}{dx}+2xy=\frac{1}{1+x}$

10. Given that $y=In(I1+e^x)$, show that $\frac{d^2y}{dx^2}=e^x\left(1-\frac{dy}{dx}\right)^2$

11. Given that $y=e^{3x}sin2x$, show that $\frac{d^2y}{dx^2}+13y=6\frac{dy}{dx}$

Revision exercise 9 (mixed exercise on diffeerentiation)

1. Differentiate the following with respect to x

(a) $\frac{x^3}{\sqrt{(1-2x^2)}}$ $\left[\frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}\right]$

(b) (x – 0.5) e^{2x} $[2x\,e^{2x}]$

(c) $\frac{(x+1)^2}{(x+4)^3}$ $\left[\frac{(5-x)(x+1)}{(x+4)^4}\right]$

(d) $\frac{3x+4}{\sqrt{2x^2+3x-2}}$ $\left[\frac{-(7x+4)}{(2x^2+3x-2)^{\frac{3}{2}}}\right]$

(e) $\log_e\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$ $\left[\frac{1}{1-x^2}\right]$

(f) e^{ax^2} $[2e^{ax^2}]$

(g) $(1-2x)^{-\frac{1}{2}}$ $\left[\frac{2x}{1-2x^2}\right]$

(h) $(x+1)^{\frac{1}{2}}(x+2)^2$ $\left[\frac{(5x+6)(x+2)}{2(x+1)^{\frac{1}{2}}}\right]$

(i) $\frac{2x^2+3x}{(x-4)^2}$ $\left[\frac{(x-4)(4x+3)-2(2x^2+3x)}{(x-4)^3}\right]$

(j) $\frac{3x-1}{\sqrt{x^2+1}}$ $\left[\frac{x+3}{(x^2+1)^{\frac{3}{2}}}\right]$

(k) $\left(\frac{1+2x}{1+x}\right)^2$ $\left[\frac{2(1+2x)}{(1+x)^3}\right]$

2. If $y=\sqrt{x}$ show that $\frac{dy}{dx}=\frac{1}{\sqrt{(x+\delta)}+\sqrt{x}}$

3. If $y = \sqrt{(5x^2+)}$, show that
   $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$
4. Given $y = In\left(1 - \frac{1}{u}\right)^{\frac{1}{2}}, 2u = \left(x - \frac{1}{x}\right)$, show that $\frac{dy}{dx} = \frac{(x+1)}{(x^2+1)(x-1)}$
5. If $y = e^{-t}\cos(t + \beta)$, show that
   $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
6. Given that x = $\frac{t^2}{1+t^3}$ and y =$\frac{t^3}{1+t^3}$, find$\frac{d^2y}{dx^2}$.
   $\left[\frac{6}{t}\left(\frac{1+t^3}{2-t^3}\right)^3\right]$
7. Differentiate y = 2x^2 + 3 from first principles
   $[4x]$

Application of Differentiation

Differentiation is helpful in various application including

- Displacement, velocity and acceleration given as a function of time
- Rates of changes
- Small angles
- Tangents and normal
- Turning points and stationary points
- Maclaurin's theory
- Curve sketching

Application of differentiation on Displacement, velocity and acceleration

Displacement

Displacement is the distance covered by a particle/body in a specified direction.

The displacement ® of a particle is said to be maximum or minimum when $\frac{d}{dt}(r) = 0$ this enables us to obtain the time when r is maximum or minimum. Hence

$r_{\max}$ $_{or\ r_{min}}$ is the value $|r|$

Velocity

This is the rate of change of displacement or $v = \frac{d}{dt}(r)$ where r is displacement.

The velocity of a partice is maximum or minimum when $\frac{d}{dt}(v) = 0$, this enables us to obtain the time when v is maximum or minimum. Hence

$v_{\max}$ $_{or\ v_{min}}$ is the value $|v|$

Acceleration, a

This is the rate of change of velocity or $a = \frac{dv}{dt}$. The acceleration of a particle is minimum or maximum when $\frac{d}{dt}(a) = 0$

Example 11

(a) The distance, s meters of a particle from a fixed point is given by $s\ =\ t^2(t^2 + 6) - 4t(t-1)(t+1)$, where t is the time in seconds.

Find the velocity and acceleration of the particle when t 1s.

Solution

$$s\ =\ t^2(t^2 + 6) - 4t(t-1)(t+1)$$
$$= t^4 + 6t^2 - 4t(t^2 - 1)$$
$$= t^4 + 6t^2 - 4t^3 + 4t$$

Velocity = $\frac{ds}{tt} = 4t^3 + 12t - 12t^2 + 4$

Whet t = 1

$v = 4 + 12 - 12 + 4 = 8ms^{-1}$

Acceleration $= \frac{dv}{dt} = 12t^2 + 12 - 24t$

When t= 1

$a = 12 + 12 - 24 = 0ms^{-2}$

(b) A particle moves along a straight line OX so that its displacement x meters from the origin, O at time t second is given by

$$x = 4t^3 - 18t^2 + 24t$$

Find

(i) when and where the velocity of the particle is zero

$x = 4t^3 - 18t^2 + 24t$

$v = \frac{dx}{dt} = 12t^2 - 36t + 24$

For v = 0

$12t^2 - 36t + 24 = 0$

$t^2 - 3t + 2 = 0$

$(t - 1)(t - 2) = 0$

Either t = 1 or t= 2

$\therefore velocity = 0\ when$

$t = 1s\ or\ t = 2s$

When t = 1s

$x = 4(1)^3 - 18(1)^2 + 24(1)$

$x = 4 - 18 + 24 = 10m$

When t= 2

$x = 4(2)^3 - 18(12)^2 + 24(2)$

$x = 32 - 72 + 48 = 8m$

(ii) its acceleration at these instants

$a = \frac{dv}{dt} = \frac{d}{dt}(12t^2 - 36t + 24)$

$= 24t - 36$

When $t = 1s$,

$a = 24 - 36 = -12ms^2$

When $t = 2s$,

$a = 48 - 36 = 12ms^2$

(iii) its velocity when its acceleration is zero.

Acceleration is zero when $\frac{dv}{dt} = 0$

$24t - 36 = 0$

$t = \frac{36}{24} = \frac{3}{2}s$

Velocity v

$= 12\left(\frac{3}{2}\right)^2 - 36\left(\frac{3}{2}\right) = 24$

$= -3ms^{-1}$

i.e. the particle is moving in opposite direction.

(c) A particle of mass 5kg moves such that

$s = \begin{pmatrix} 2 - cos3t \\ 6sin2t \end{pmatrix}$

(i) Show that the particle never crosses the y-axis

For any point on the y-axis , x= 0

$2 - cos3t = 0$

cos3t = 2

$3t = cos^{-1}(2)$

$t = \frac{1}{3}cos^{-1}(2)$

Since $cos^{-1}(2)$ has no value, the particle does not cross y-axis

(ii) Find the velocity of the particle when $t = \frac{\pi}{6}$

$v = \frac{dx}{dt} = \frac{d}{dx}\begin{pmatrix} 2 - cos3t \\ 6sin2t \end{pmatrix}$

$= \begin{pmatrix} 3sin3t \\ 12cos2t \end{pmatrix}$

$At\ t = \frac{\pi}{6}$

$v = \begin{pmatrix} 3sin\frac{3\pi}{6} \\ 12cos\frac{2\pi}{6} \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} ms^{-1}$

(d) The acceleration of a car t s after starting from rest is $\frac{75 - 10t - t^2}{20} ms^{-2}$ until the instant when this expression vanishes. After this instant, the speed of this car remains constant. Find the maximum acceleration.

Solution

A is maximum when $\frac{d(a)}{dt} = 0$

$\frac{d}{dt}\left(\frac{75+10t-t^2}{20}\right) = \frac{10-2t}{20}$

A is maximum when $\frac{10-2t}{20} = 0$

t = 5s

$a_{max} = \frac{75+10(5)-(5)^2}{20} = \frac{100}{20} = 5ms^{-2}$

(e) The distance s m of a particle from a fixed point is given by $s = t^2(t^2 + 6))$ where t is the time. Find the velocity and acceleration of the particles when t = 1s

Solution

$s = t^2(t^2 + 6)$

$= t^4 + 6t^2$

$v = \frac{d(s)}{dt} = \frac{d}{dt}(t^4 + 6t^2)$

$= 4t^3 + 12t$

$at\ t = 1s$

$v = 4(1)^3 + 12(1) = 16ms^{-1}$

$a = \frac{d(v)}{dt} = \frac{d}{dt}(4t^3 + 12t\ )$

$= 12t^2 + 12$

$at\ t = 1s$

$a = 12(1)^2 + 12 = 24ms^{-2}$

Revision exercise displacement, velocity and acceleration

1. A ball is thrown vertically upwards and its height after t seconds is h m where $h = 25.2t - 4.9t^2$
   Find
   (a) its height and velocity after 3s
   (b) when it is momentarily at rest
   (c) the greatest height reached
   (d) the distance moved in the 3rd second
   (e) the acceleration when $t = 2\frac{4}{7}$
   $\begin{bmatrix} (a)\quad 31.5m, -4.2ms^{-1}; \\ (b)\ t = 2\frac{4}{7}; (c)32.4m; (d)2.5m; \\ (e) - 9.8ms^2(constant) \end{bmatrix}$
2. A particle moves along a straight line in such a way that its distance s m from the origin after t s is given by $s = 7t + 12t^2$.
   (a) What does it travel in the 9th second?
   (b) What are its velocity and acceleration at the end of 9th second?
   $[(a)211s; (b)223cms^{-1}\ (c)24ms^{-2}]$
3. A point moves along a straight line OX so that its distance x from the point O at t s is given by $s = t^3 - 6t^2 + 9t$.Find
   (a) at what times and in what position the point will have zero velocity.
   (b) its acceleration at those instants
   (c) its velocity when its acceleration is zero.
   $\begin{bmatrix} (a)1s, 3s, 4cm, 0; (b) - 6, 6cms^{-2}; \\ (c) - 3cms^{-1} \end{bmatrix}$
4. A particle moves in a straight line so that after t s it is 5m from a fixed point O on the line where$s\ = t^4 + 3t^2$. Find
   (a) The acceleration when t = 1, t = 2 and t = 3s.
   (b) The average acceleration between t = 1 and t =3s
   $[(a)18, 54, 114ms^{-1}; (b)58ms^{-2}]$
5. A particle moves a long a straight line so that after t s, its distance from a fixed point O on the line is 5m where $s = t^3 - 3t^2 + 2t$
   (a) When is the particle is at O?
   (b) What is the velocity and acceleration at these times?
   (c) What is the average acceleration between t = 0 and t = 2s.
   $\begin{bmatrix} (a)after\ 0, 1, 2s; (b)2, -1, 2ms^{-1}; \\ -6, 0, 6ms^{-2}; (c)0ms^{-1}; (d)0ms^{-1} \end{bmatrix}$

Application of differentiation on rates of change in measurement

This deals with aspects that vary with others.

Example 12

(a) The side of a cube is increasing at the rate of $0.3ms^{-1}$.
Find the rate of volume when the length is 5m.

Solution

Let L = length of each side of the cube.

$v = L^3$

$\frac{dv}{dL} = 3L^2$

$\frac{dL}{dt} = 0.3$

$\frac{dv}{dt} = \frac{dv}{dL}.\frac{dL}{dt} = 3L^2.0.3 = 0.9L^2$

When L= 5m

$$\frac{dv}{dt} = 0.9(5)^2 = 22.5m^3s^{-1}$$

(b) The volume of a cube is increasing at the rate of 2m^{3}s^{-1}. Find the rate of change of the side when its side is 10m.

$v = L^3$

$\frac{dv}{dL} = 3L^2$

$\frac{dv}{dt} = \frac{dv}{dL}.\frac{dL}{dt}$

But $\frac{dv}{dt} = 2$

$2 = 3L^2.\frac{dL}{dt}$

$\frac{dL}{dt} = \frac{2}{3L^2}$

When L= 10m

$\frac{dL}{dt} = \frac{2}{3(10)^2} = 0.007ms^{-1}$

(c) The volume of a cube increases uniformly at $a^3m^3s^{-1}$. Find an expression for the rate of increase of the surface area when the area of a face is b^2m^2.

Solution

Let L = side of the cube

A = surface area

V = volume of the cube

$v = L^3$

$\frac{dv}{dL} = 3L^2$

$\frac{dv}{dt} = \frac{dv}{dL}.\frac{dL}{dt}$

But $\frac{dv}{dt} = a^2$

$2 = 3L^2.\frac{dL}{dt}$

$\frac{dL}{dt} = \frac{a^3}{3L^2}$

For face area = $b^2 = L^2$ since L=b

Surface area of a cube = $6b^2$

$A = 6L^2$

$\frac{dA}{dL} = 12L$

$But\ \frac{dA}{dt} = \frac{dA}{dL}.\frac{dL}{dt}$

$= 12L.\frac{a^3}{3L^2} = \frac{4a^3}{L} = \frac{4a^3}{b}$

(d) A spherical balloon is inflated at a rate of 5m^{3}s^{-1}. Find the rate of increase of radius when the radius is 3m.

Solution

If v and r are the volume and radius of the sphere at time t, then

$v = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

$\frac{dv}{dr} = 5$

But $\frac{dv}{dt} = \frac{dv}{dr}.\frac{dr}{dt}$

$5 = 4\pi r^2.\frac{dr}{dt}$

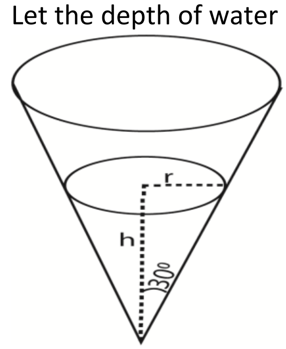
$\frac{dr}{dt} = \frac{5}{4\pi r^2}$

When r=3

$\frac{dr}{dt} = \frac{5}{4\pi(3)^2} = \frac{5}{36\pi}ms^{-1}$

(e) A hollow can of semi-vertical angle 30^0 is held with its vertex downwards. Water is poured into the cone at the rate of 3m^{3}s^{-1}. Find the rate at which the depth of water in the cone is increasing when the depth is 5m.

Let the depth of water in the cone be h m



From the diagram above

$\tan 30 = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$

$\frac{dv}{dh} = \frac{\pi h^2}{3}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$3 = \frac{\pi h^2}{3}.\frac{dh}{dt}$

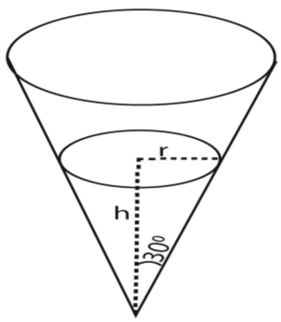
$\frac{dh}{dt} = \frac{9}{\pi h^2}$

When h = 5

$\frac{dh}{dt} = \frac{9}{\pi(5)^2} = \frac{9}{25\pi} ms^{-1}$

∴the rate of change of height is $\frac{9}{25\pi} ms^{-1}$

(f) An inverted cone with a vertical angle of 60^0 is collecting water leaking from a tap at a rate of $2m^3s^{-1}$. If the height of water collected in the cone is 10m, find the rate at which the surface are of eater is increasing.



$\tan 30 = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$

$\frac{dv}{dh} = \frac{\pi h^2}{3}$

Let A = surface area

$A = \pi r^2$

Substituting for r

$A = \pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{\pi h^2}{3}$

$\frac{dA}{dh} = \frac{2}{3}\pi h$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$2 = \frac{\pi h^2}{3}.\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{6}{\pi h^2}$

Also, $\frac{dA}{dt} = \frac{dA}{dh}.\frac{dh}{dt} = \frac{2}{3}\pi h . \frac{6}{\pi h^2} = \frac{4}{h}$

Substituting for h =10

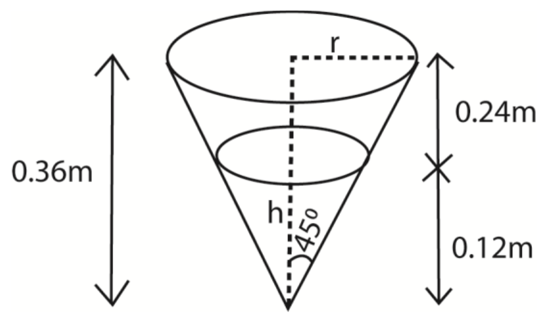
$\frac{dA}{dt} = \frac{4}{10} = 0.4m^2s^{-1}$

∴the rate at which the surface area is changing is $0.4m^2s^{-1}$.

(g) A hollow circular cone with vertical angle 90^0 and height 0.36m is inverted and filled with water. This water begins to leak away through a small hole in the vertex. If the level of the water begins to sink at a rate of 0.01m in 120s, and the water continues to leak a way at the same rate, at what rate is the level sinking when the water is 0.24m from the top?

Solution

When water is full



$\tan 45^0 = \frac{r}{h}$

$1 = \frac{r}{h}$

$r = h$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi(1)^2 h = \frac{\pi h^3}{3}$

$\frac{dv}{dh} = \pi h^2$

$\frac{dh}{dt} = \frac{0.01}{120}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$= \pi h^2.\frac{0.01}{120}$

Substituting for h

$\frac{dv}{dt} = \pi(0.36)^2.\frac{0.01}{120}$……………….(i)

When water level is 0.36 - 0.24 = 0.12m

$\frac{dv}{dt} = \pi(0.12)^2.\frac{dh}{dt}$……………….(ii)

Equating (i)and (ii)

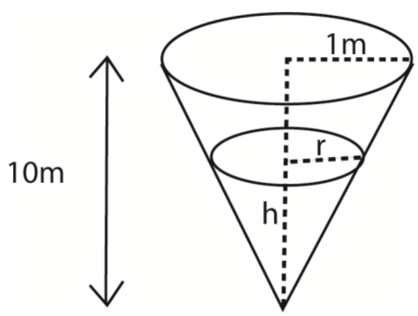
$\pi(0.36)^2.\frac{0.01}{120} = \pi(0.12)^2.\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{(0.36)^2}{(0.12)^2}.\frac{0.01}{120} = 7.5x10^{-4}ms^{-1}$

(h) A hollow right circular cone of height 10m and base radius 1m is catching the drips from a tap leaking at a rate 0.002m^{3}s^{-1}. Find the rate at which the surface area of water is increasing when water is half way up the cone

Solution

Let h and r be the height and radius of water level at time t



Expressing r in term of h, from similarity of figures,

$\frac{h}{10} = \frac{r}{1}$

$r = \frac{h}{10}$

Volume, v of a cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{10}\right)^2 h$

$= \frac{\pi h^3}{300}$

$\frac{dv}{dh} = \frac{\pi h^2}{100}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$0.002 = \frac{\pi h^2}{100}.\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{0.2}{\pi h^2}$

Surface area, A = πr^2

Substituting for r

$A = \pi\left(\frac{h}{10}\right)^2 = \frac{\pi h^2}{100}$

$\frac{dA}{dh} = \frac{\pi h}{50}$

Now $\frac{dA}{dt} = \frac{dA}{dh}.\frac{dh}{dt} = \frac{\pi h}{50}.\frac{0.2}{\pi h^2} = \frac{0.004}{h}$

When water is half way up, h=5m

$\frac{dA}{dt} = \frac{0.004}{(5)} = 0.0008m^2s^{-1}$

Revision exercise rates of change in measurements

1. The side of a square is increasing at the rate of 5cms^{-1}. Find the rate of increase of the area when the length of the side is 10cm. [100cm^{2}s^{-1}]
2. The volume of a cube is increasing at the rate of 18cm^{3}s^{-1}. Find the rate of increase of the length of a side when the volume is 125cm^3. $\left[\frac{6}{25}cms^{-1}\right]$
3. The radius of a circle is increasing at the rate of $\frac{1}{3}cms^{-1}$. Find the rate of increase of the area when the radius is 5cm. $\left[\frac{10\pi}{3}cm^2s^{-1}\right]$
4. The volume of a sphere is increasing at a rate of (12π)cm^{3}s^{-1}. Find the rate of increase of the radius when the radius is 6cm. $\left[\frac{1}{12}cms^{-1}\right]$
5. The area of a square is increasing at the rate of 7cm^{2}s^{-1}. Find the rate of increase of the length of a side when this area is 100cm^2. $\left[\frac{7}{10}cms^{-1}\right]$
6. The area of a circle is increasing at the rate of (4π)cm^{2}s^{-1}. Find the rate of increase of the radius when this radius is $\frac{1}{2}$cm. [4cms^{-1}]

7. The surface area of a sphere is increasing at a rate of $2cm^2s^{-1}$. Find the rate of increase of the radius when the surface area is (100π) cm^2? $\left[\frac{1}{20\pi}cms^{-1}\right]$
8. A boy is inflating a spherical balloon at the rat of $10cm^3s^{-1}$. Find the rate of increase of the surface area of the balloon when the radius is 5m.$[4cm^2s^{-1}]$
9. A hollow cone of semi-vertical angle 45^0 is held with its vertex pointing downwards. It receives water at a rate of $3cm^3$ per minute. Find the rate at which the depth of water in the cone is increasing when the depth is 2cm. $\left[\frac{3}{4\pi}cmmin^{-1}\right]$

Application of differentiation on Small changes

Suppose a function y = f(x) and δy and δx are increments in y and x respectively

Then as δx → 0, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

⇨ $\delta y = \frac{dy}{dx}.\delta x$

The above expression is used to find small changes in the variable x.

Example 3

(a) If $y = x^5$, find the approximate percentage increase in y due to increase of 0.1 percent in x.

$y = x^5$

$\frac{dy}{dx} = 5x^4$

But $\delta y = \frac{dy}{dx}.\delta x = 5x^4.\delta x$

$\frac{\delta y}{y} = \frac{5x^4.\delta x}{x^5} = 5\frac{\delta x}{x}$

But $\frac{\delta x}{x} = 0.1\%$

$\frac{\delta y}{y} = 5x0.1\%$ =0.5%

(b) An error of 2 ½ % is made in the measurement of the area of a circle. What percentage error results in

(i) The radius

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\delta A = \frac{dA}{dr}.\delta r$

$= 2\pi r\,.\delta r$

$\frac{\delta A}{A} = \frac{2\pi r}{\pi r^2}.\delta r = 2\frac{\delta r}{r}$

$\frac{1}{2}.\frac{\delta A}{A} = \frac{\delta r}{r}$

$\frac{\delta r}{r} = \frac{1}{2}.\frac{5}{2} = \frac{5}{4} = 1\frac{1}{4}\%$

(ii) The circumference

$c = 2\pi r$

$\frac{dc}{dr} = 2\pi$

$\delta c = \frac{dc}{dr}.\delta r = 2\pi\delta r$

$\frac{\delta c}{c} = \frac{2\pi\delta r}{2\pi r} = \frac{\delta r}{r}$

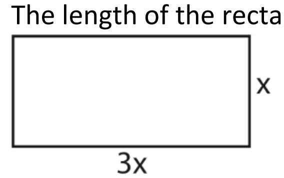
$\frac{\delta c}{c} = 1\frac{1}{4}\%$

(c) One side of a rectangle is three times the other. If the perimeter increases by 2%. What is the percentage increase in area?

Solution

Let the width of the rectangle = x

The length of the rectangle = 3x



Application of differentiation on Curve sketching (A-level)

The procedures of curve sketching depend on the nature of the curve to be sketched

Graphs of y = f(x) (Non rational functions)

For any graph of the form y = f(x) where f(x) is not linear, some or all the following steps are followed.

(a) Determine if the curve is symmetrical about either or both axes of coordinates.
- Symmetry about the x-axis occurs if the equation contains only even powers of y. here equation will be uncharged when (-y) is substituted for y. this applies to graphs of the type y^2=f(x)
- Symmetry about the y-axis occurs if the equation contains only even powers of x. Here the equation will be uncharged when (-x) is substituted for x. Here the graph is said to even i.e. f(x) =f(x). For example the graph of y = x^2. **Note** if there are odd powers of x and y then there will be no symmetry.

(b) Determine if there is symmetry about the origin. Here symmetry occurs when a change in the sign of x causes a change in the sign of y without altering its numerical value.

(c) Find the intercepts i.e. the curve cuts the x-axis at a point when y = 0 and cuts the y-axis at the point when x = 0.

(d) The curve passes through the origin if (x, y)= (0, 0)

The behaviour of the neighbour of the curve through the origin is studied by considering the ratio of $\frac{y}{x}$.

- If this ratio is small, the curve keeps close to x-axis near the origin.
- If the ratio is unity, the direction of the curve bisects the angle between the axes.
- If the ratio is large, the curve keeps near the y-axis.

Or:

We consider the behaviour of $\frac{dy}{dx}$ near the origin.

- If $\frac{dy}{dx}$ is very small, then the curve lies near the x-axis.
- If $\frac{dy}{dx}$ is large, then the curve lies near the y-axis.
- If $\frac{dy}{dx}$ is near unity, then the direction of the curve (tangent at origin) bisects the angle between the axes.

(e) Examine the behaviour of the function as x→±∞ and y →±∞ (if any)

(f) Find the turning points and their nature as well as points if inflexion (if any)
Use the second derivative
- For min point, $\frac{d^2y}{dx^2} = +ve$
- For max point, $\frac{d^2y}{dx^2} = -ve$
- Point of inflexion, $\frac{d^2y}{dx^2} = 0$

Example 1

(a) Sketch the graph of y= 5 + 4x – x^2.
Steps taken
- Finding intercepts
  x – intercept; y = 0
  0 = 5 + 4x – x^2.
  5 + 5x –x – x^2 = 0
  5(1 + x) – x(1 + x) =0
  (5 – x)(1 + x)
  Either 5 – x = 0; x = 5
  Or 1 + x = 0; x = -1
  Hence the curve cuts the x-axis at point (-1, 0) and (0, 5)
  y – intercept, when x = 0, y = 5
  Hence the curve cuts the y-axis at point (0, 5)
- As x→+∞, y →-∞ and x→-∞, y →+∞
- Finding turning point
  $\frac{dy}{dx} = 4 - 2x$

At turning point $\frac{dy}{dx} = 0$

2x – 4 = 0; x = 2

When x = 2; y= 5 + 4(2) – $(2)^2$= 9

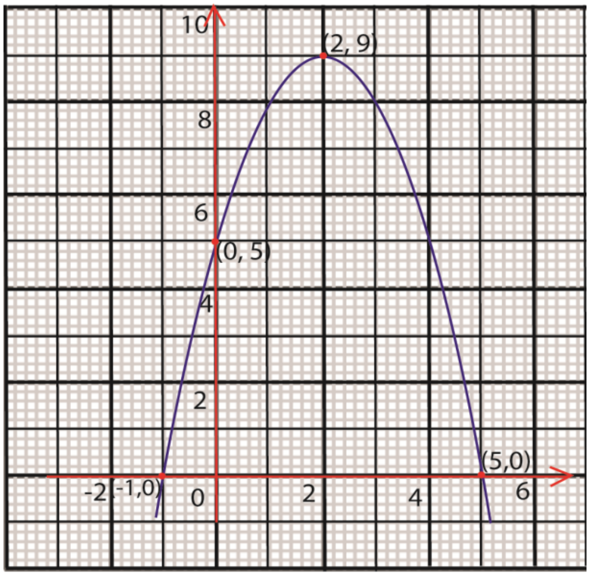
Hence turning point = (2, 9)

Finding the nature of turning point

$\frac{dy}{dx} = 4 - 2x$

$\frac{d^2y}{dx^2} = -2$

Since $\frac{d^2y}{dx^2} < 0$, hence the turning point is maximum.



(b) Sketch the curve y = $x^3 – x^2 – 5x + 6$

Steps taken

For y – intercept; x = 0, y = 0

Hence the y – intercept is (0, 6)

For x – intercept, y = 0

$x^3 – x^2 – 5x + 6 = 0$

error approach is used to find the first factor i.e. (x-2), then other factor is found by long division

$$
\begin{array}{r|l}
 & x^2 + x - 3 \\
(x-2) & x^3 - x^2 - 5x + 6 \\
 & -\ x^3 - 2x^2 \\
\hline
 & x^2 - 5x + 6 \\
 & -\ x^2 - 2x \\
\hline
 & 3x + 6 \\
 & -\ 3x + 6 \\
\hline
 & 0 + 0
\end{array}
$$

⇨ $x^3 – x^2 – 5x + 6 = (x – 2)(x^2 + x - 3) = 0$

Solving $x^2 + x - 3 = 0$

$x = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm 3.6}{2}$

x = 1.3 or -2.6

Hence the x- intercepts are (2, 0), (1.3, 0) and (-2.3, 0)

Finding turning points

$y = x^3 – x^2 – 5x + 6$

$\frac{dy}{dx} = 3x^2$-2x-5

At turning point $\frac{dy}{dx} = 0$

$3x^2$-2x-5 = (3x – 5)(x+1) = 0

Either 3x -5 = 0 x = $\frac{5}{3}$

Or x + 1 = 0; x =-1

When x = $\frac{5}{3}$;

$y = \left(\frac{5}{3}\right)^3 – \left(\frac{5}{3}\right)^2 – 5\left(\frac{5}{3}\right) + 6 = \frac{-13}{27}$

When x = -1

$y = (-1)^3 – (-1)^2 – 5(-1) + 6 = 9$

Hence turning points are $\left(\frac{5}{3}; \frac{-13}{27}\right)$ and (-1, 9)

Finding the nature of turning points

$\frac{dy}{dx} = 3x^2$-2x-5

$\frac{d^2y}{dx^2} = 6x - 2$

For$\left(\frac{5}{3};\frac{-13}{27}\right)$

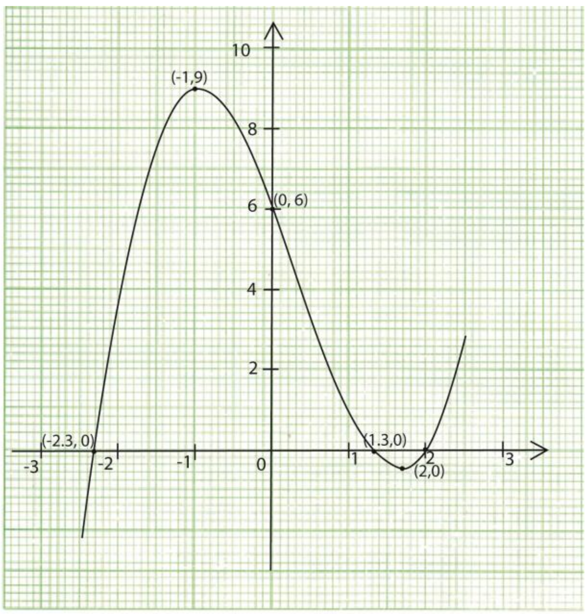
$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8\ (> 0)$

∴ $\left(\frac{5}{3};\frac{-13}{27}\right)$is minimum

For (-1, 9)

$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8\ (< 0)$

∴ $(-1; 9\ )$is maximum



(c) sketch the curve y = $x^3 - 8$

y = $x^3 - 8$

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

$3x^2 = 0$

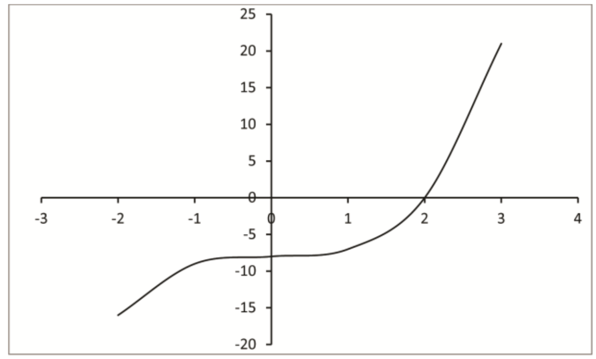
x = 0

$\frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0, x = 0$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(d) Sketch the curve $y = x^2(x - 4)$

Steps taken

- Finding the intercepts

y – intercept, (0,0)

hence y – intercept is (0, 0)

For x – intercept, y= 0

=>$x^2(x - 4) = 0$

Either x = 0 or x = 4

Hence x-intercept are (0, 0) and (4, 0)

- As x→+∞, y →+∞ and x→-∞, y →-∞
- Finding turning point(s)

$y = x^2(x - 4) = x^3 - 4x^2$

$\frac{dy}{dx} = 3x^2 - 8x$

At turning point, $\frac{dy}{dx} = 0$

⇨ $3x^2 - 8x = x(3x - 8) = 0$

Either x = 0

Or x = $\frac{8}{3}$

When x = 0; y=0

When x =$\frac{8}{3}$; =3$\left(\frac{8}{3}\right)^2 - 8\left(\frac{8}{3}\right) = \frac{-256}{27}$

Hence turning points are (0,0) and$\left(\frac{8}{3}, \frac{-256}{27}\right)$

- Finding the nature of turning points

$\frac{dy}{dx} = 3x^2 - 8x$

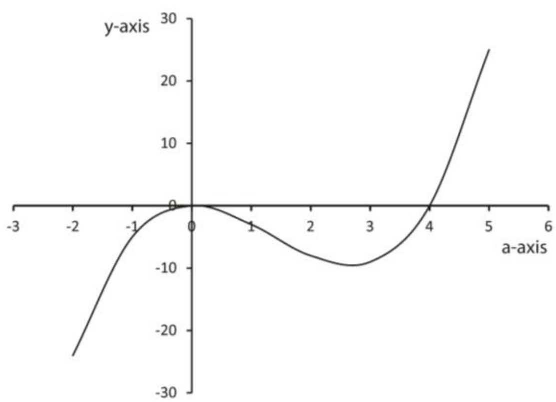
$\frac{d^2y}{dx^2} = 6x - 8$

For (0, 0); $\frac{d^2y}{dx^2} = 6(0) - 8 = -8\ (< 0)$

Hence (0, 0) is maximum

For $\left(\frac{8}{3}, \frac{-256}{27}\right)$; $\frac{d^2y}{dx^2} = 6\left(\frac{8}{3}\right) - 8 = 8\ (> 0)$

Hence $\frac{8}{3}$ is minimum



(e) Determine the nature of the turning points of the curve $y = \frac{x^2-6x+5}{(2x-1)}$. Sketch the curve for x = -2 to x = 7. State any asymptotes

Solution

A turning point $\frac{dy}{dx} = 0$

⇨ $\frac{dy}{dx} = \frac{(2x-1)(2x-6)-(x^2-6x+5)(2)}{(2x-1)^2} = 0$

$(2x – 1)(2x – 6) – (x^2 – 6x+ 5)(2) =0$

By opening bracts and simplifying

$x^2 – x – 2 = 0$

(x+1)(x – 2) = 0

Either x = -1 or x = 2

Substituting for x = -1

$y = \frac{(-1)^2-6(-1)+5}{(2(-1)-1)}$ = -4

Substituting for x = 2

$y = \frac{(2)^2-6(2)+5}{(2(2)-1)}$ = -1

Investigating the nature of turning point

For (-1, -4)

x	-2	-1	0
$\frac{dy}{dx}$	+	0	-

Maximum

Investigating the nature of turning point

For (2, -1)

x	1	2	3
$\frac{dy}{dx}$	-	0	+

Minimum

Finding the intercepts

For x- intercept, y= 0

$x^2 – 6x+ 5 =0$

(x – 5)(x – 1) =0

Either x = 5 or x= 1

Hence x-intercepts are (5, 0) and (1,0)

For y- intercept, x= 0

⇨ $y = \frac{5}{-1} = -5$

Finding asymptotes

Vertical asymptote, 2x – 1 = 0

$x = \frac{1}{2}$

Horizontal asymptote

$(2x – 1)y = x^2 – 6x+ 5$

$x^2 – 6x+ 5 -2xy + y =0$

$x^2 – (6 +2y)x+ (5 + y) =0$

For real values

$(6 + 2y)^2 \geq 4(5+y)$

$4y^2 + 20y + 16 = 0$

$y^2 + 5y + 4 = 0$

$(y + 1)(y + 4) = 0$

Either y = -1 or y = -4

For slanting asymptote

$$\frac{x}{2} - \frac{11}{4}$$

$$(2x - 1)\overline{)x^2 - 6x + 5}$$

$$x - \frac{x}{2}$$

$$\frac{11}{2}x + 5$$

$$\frac{11}{2}x + \frac{11}{2}$$

Hence the slating asymptote is $y = \frac{x}{2} - \frac{11}{4}$

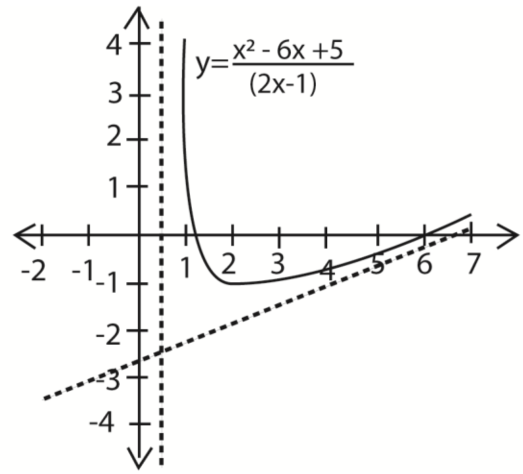
Investigating the range of x where the curve lies

x	x< ½	½ <x<1	1<x<5	x>5
x-1	-	-	+	+
x-5	-	-	-	+
(x-1)(x-5)	+	+	-	+
2x-1	-	+	+	+
y	-	+	-	+

Investigating the range of y

	Y<-4	-4<y<-1	y> -1
(y+1)(y+4)	+	-	+

Graph of $y = \frac{x^2-6x+5}{(2x-1)}$.



Revision exercise for curve sketching

1. Sketch $y = x - \frac{8}{x^2}$, show any asymptote.
2. (i) Find the Cartesian equation of the curve $x = \frac{1+t}{1-t}$ and $y = \frac{2t^2}{1-t}$= $\left[y = \frac{(x-1)^2}{1+x}\right]$
3. Given the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$
   (i) Show that the curve does not have turning points.
   (ii) Find the equation of asymptotes. Hence sketch the curve [vertical asymptote, x=4, horizontal asymptote, y=1]
   (iii) Sketch the curve
4. Sketch a graph $y = \frac{4(x-3)}{x(x+2)}$

Thank you

Dr. Bbosa Science