Chapter 1

Wave Motion

1.1 Chapter Objectives

By the end of the chapter, students should be able to:

- Define a wave and state the principle of superposition of waves.
- State the necessary conditions for creation of standing waves.
- Derive and solve the wave equation.

1.2 Progressive Waves

Wave motion is any periodic disturbance that propagates in a given medium carrying energy with it. Suppose a periodic vibration that traverses a medium whose displacement from the reference position takes either of the following the forms,

$$u_1(t) = u_{01} \sin \omega t, \qquad u_2(t) = u_{02} \cos \omega t \tag{1.1}$$

and which is diagrammatically represented by the schematic in figure 1.1. The quantities u_{01} and

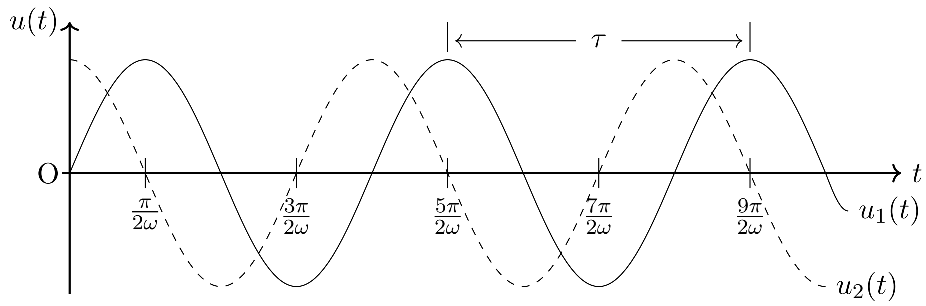

Figure 1.1: Displacement curve of a wave motion - temporal consideration

u_{02} are the amplitudes of the waves u_1 and u_2 respectively. As can be observed from the above expressions, *the amplitude of a wave is the maximum displacement from the mean position.*

1.2.1 Basic Definitions in Wave Motion

(i) Wave amplitude:
This is the maximum displacement of the wave from the mean position. In the schematic above, the mean position is the horizontal axis (i.e the time axis). It is measured in units of metres (m).

(ii) Cycle:
This is a complete alternation of the displacement $u(t)$. For a particle that starts at a given

point on the wave, it is considered to have covered a single cycle when it returns to the same point.

(iii) Period (τ):
This is the time taken by the wave to move a distance of one cycle. It can also be defined as the time it takes a particle at one of the crests (or troughs) to reach a another successive crest (or trough). The period is measured in seconds (s).
The units of the period and frequency suggest the following relationship between them,

$$f = \frac{1}{\tau} \tag{1.2}$$

(iv) Frequency of the wave (f):
This is the number of complete cycles made by the wave per second. The S.I unit of frequency is per second equivalent to a Hertz (Hz).

(v) Angular Frequency (ω):
This is closely related to the wave frequency f by the expression,

$$\omega = 2\pi f \tag{1.3}$$

For a wave propagating in a given medium, the variation of the disturbance $u(y)$ can also be considered in space. Supposing the wave travels in the $y-$ direction, the spatial variation of its displacement would take the form below (figure 1.2), The repeat length λ is called the wave length

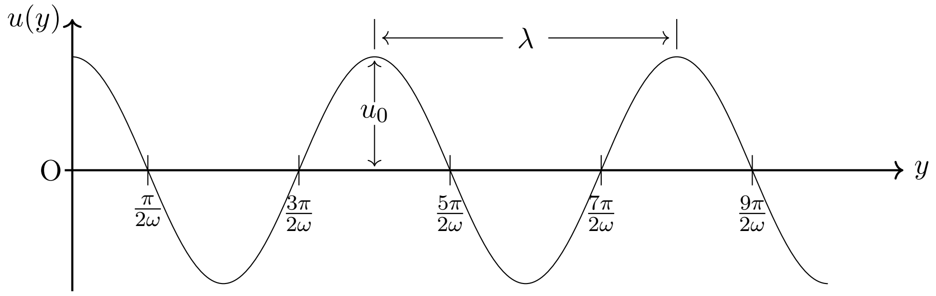

Figure 1.2: Displacement curve of a wave motion-spatial consideration. u_0 is the amplitude of the wave.

of the wave. The period τ defined earlier on can also be alternatively defined as the time taken to travel a distance equal to the wave length. With the above distance (λ) and time (τ), we can obtain the characteristic velocity,

$$v = \frac{\lambda}{\tau} = f\lambda. \tag{1.4}$$

This is referred to as the *phase velocity* of the wave, with the latter equivalence arising from the relation in equation (1.2). Any given point at a distance x, is out of phase with the origin O by a distance $\Delta\phi$(referred to as the phase difference) given by the expression,

$$\Delta\phi = \frac{2\pi}{\lambda}x, \tag{1.5}$$

where λ is the wave length.

The displacement of a wave should therefore incorporate such phase differences, and thus the general form of the displacement of a particle in a wave travelling along the positive $x-$direction takes the form,

$$u(x,t) = u_0 \sin(\omega t - \Delta\phi) \quad = \quad u_0 \sin(\omega t - kx),$$

where $k = \frac{2\pi}{\lambda}$ is called the wave number. Alternative representations of the displacement of a wave are:

$$
\begin{aligned}
u(x,t) &= u_0 \sin\left(\omega t - \frac{2\pi}{\lambda}x\right), \\
&= u_0 \sin 2\pi\left(ft - \frac{x}{\lambda}\right), \\
&= u_0 \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right). \qquad (1.6)
\end{aligned}
$$

It should however be noted that in general, a wave could be travelling in multi-dimensional space. In such cases, the wave number k becomes a vector and is best described as a propagation vector $\vec{k} = k\hat{\imath} + l\hat{\jmath} + m\hat{\mathrm{k}}$. The displacement is then given as,

$$\vec{u}(\vec{x},t) = \vec{u}_0 \sin\left(\omega t - \vec{k}\cdot\vec{x}\right), \qquad (1.7)$$

and for a wave travelling in the opposite (i.e negative) direction,

$$\vec{u}(\vec{x},t) = \vec{u}_0 \sin\left(\omega t + \vec{k}\cdot\vec{x}\right). \qquad (1.8)$$

Consider the progressive wave $u = 16.0\sin(50\pi t - 5\pi x)$ with all distances in metres and time in seconds. Find:

(i) the amplitude

(ii) the wave number

(iii) the frequency

(iv) the angular frequency

(v) The Period,

(vi) the wave length

(vii) the phase velocity,

Solution:
The first step in solving the above question is to express the displacement in the standard form of a progressive wave, which would be,

$$
\begin{aligned}
u &= 16.0\sin(50\pi t - 5\pi x) = u_0 \sin(\omega t - kx), \qquad (1.9) \\
&= 16.0 \sin 2\pi\left(25t - \frac{5}{2}x\right), \\
&= u_0 \sin 2\pi\left(ft - \frac{x}{\lambda}\right), \qquad (1.10)
\end{aligned}
$$

(i) By comparison, the amplitude, $u_0 = 16.0$ m.

(ii) By comparison, the wave number $k = 5\pi$.

(iii) By comparison, the frequency $f = 25$ Hz.

(iv) By comparison, the angular frequency $\omega = 50\pi$ Hz.

(v) The Period,

$$T = \frac{1}{f} = \frac{1}{15} \text{ s.}$$

(vi) The wave length: using the relation,

$$\begin{aligned} k = 5\pi &= \frac{2\pi}{\lambda}, \\ \rightarrow \lambda &= \frac{2\pi}{k}, \\ &= \frac{2\pi}{5\pi} = 0.4 \text{ m}. \end{aligned}$$

(vii) The phase velocity,

$$\upsilon = \frac{\omega}{k} = \frac{50\pi}{5\pi} = 10 \text{ ms}^{-1}.$$

1.2.2 Intensity of a wave

Suppose a particle of mass m, of the medium in which a wave is propagating. Let $u = u_0 \sin(\omega t - kx)$ be its displacement. The instantaneous velocity of the particle is given by the time derivative of its displacement as,

$$\upsilon = \frac{\partial u}{\partial t} = \omega u_0 \cos(\omega t - kx), \tag{1.11}$$

from which we obtain the instantaneous kinetic energy as,

$$\text{K.E} = \frac{1}{2} m v^2 = \frac{1}{2} m\omega^2 u_0^2 \cos^2(\omega t - kx), \tag{1.12}$$

As observed in the expression in equation (1.12), the kinetic energy is varying with time and hence the meaningful value of the kinetic energy to consider, is the time average (or mean) over a given time interval. A suitable time interval is the period of the wave motion τ,

$$\begin{aligned} \langle \text{K.E} \rangle &= \left\langle \frac{1}{2} m\omega^2 u_0^2 \cos^2(\omega t - kx) \right\rangle, \\ &= \frac{1}{2} m\omega^2 u_0^2 \left\langle \cos^2(\omega t - kx) \right\rangle_\tau = \frac{1}{4} m\omega^2 u_0^2, \end{aligned} \tag{1.13}$$

where we have used the result,

$$\left\langle \cos^2(\omega t - kx) \right\rangle_\tau = \frac{1}{2} \tag{1.14}$$

The average potential energy (P.E) of a particle undergoing simple harmonic motion, as is the case for wave motion, is equal to the average kinetic energy (K.E). Thus,

$$\langle \text{P.E} \rangle = \langle \text{K.E} \rangle = \frac{1}{4} m\omega^2 u_0^2 \tag{1.15}$$

The total average energy is thus twice the kinetic energy given by,

$$\langle \text{E} \rangle = \frac{1}{2} m\omega^2 u_0^2. \tag{1.16}$$

Suppose ρ and V are the density and volume respectively, of the substance through which the wave propagates. The average energy per unit volume is obtained as,

$$\left\langle \frac{\text{E}}{V} \right\rangle = \frac{1}{2} \rho\omega^2 u_0^2. \tag{1.17}$$

By writing the volume V as, $V = Al$ where l is the length of the stretch in the direction of propagation of the wave and A is the cross sectional area. We therefore obtain,

$$\begin{aligned} \frac{1}{A} \frac{d}{dt} \langle E \rangle &= \frac{1}{2} \rho \cdot \frac{dl}{dt} \cdot \omega^2 u_0^2, \\ &= \frac{1}{2} \rho\upsilon\omega^2 u_0^2 \end{aligned} \tag{1.18}$$

The term on the left hand side of equation (1.18) is the amount of energy passing through a cross sectional area of 1 m^2 in one second and is referred to as the *Intensity* of the wave. Considering that $\upsilon = f\lambda$, the intensity can further be written as,

$$I = \frac{1}{A}\frac{d}{dt}\langle E\rangle = \frac{1}{2}\rho f\lambda\omega^2 u_0^2 \qquad (1.19)$$

On substituting for $\omega = 2\pi f$, the wave intensity in equation (1.19) simplifies to,

$$\begin{aligned} I &= \frac{1}{2}\rho f\lambda 4\pi^2 f^2 u_0^2, \\ &= 2\pi^2\rho\lambda u_0^2 f^3, \\ \Rightarrow I &\propto \rho, \lambda, u_0^2, f^3 \qquad (1.20) \end{aligned}$$

The intensity of a wave is thus found to depend on the density of the medium, wavelength of the wave, amplitude of the wave and its frequency. It is from this result that we shall later on use the square of the wave amplitude as a proxy for the intensity of the wave. It should however be noted that while the square of the amplitude is a good approximation to the intensity, the most accurate value to use involves the wavelength, density of the material medium and frequency of the wave.

Example
A source of sound of frequency 1 kHz radiates isotropically, power of 2 W in air. Find the amplitude of the sound at a distance of 10 m from the source, given that the density of air at 20° C is 1.29 kgm^{-3}, and that the speed of sound in air at this temperature is 343 ms^{-1}.

Solution:
The intensity of sound at a point 10 m from the source is,

$$\begin{aligned} I &= \frac{1}{A}\frac{dE}{dt} = \frac{P}{4\pi r^2}, \\ &= \frac{2}{4\pi \times 10^2} = 1.59 \times 10^{-3}\mathrm{Wm}^{-2}. \end{aligned}$$

But we have also shown that,

$$\begin{aligned} I &= 2\pi^2\rho\upsilon f^2 u_0^2, \\ \rightarrow u_0 &= \left[\frac{I}{2\pi^2\rho\upsilon f^2}\right]^{\frac{1}{2}}, \end{aligned}$$

and provided in the question is that: $f = 1 \times 10^3$ kgm^{-3}, $\upsilon = 343$ ms^{-1} and $\rho = 1.29$ kgm^{-3}, such that,

$$\begin{aligned} u_0 &= \left[\frac{1.59 \times 10^{-3}}{2\pi^2 \times 1.29 \times 343 \times (1 \times 10^3)^2}\right]^{\frac{1}{2}}, \\ &= 4.27 \times 10^{-9}\mathrm{m}. \end{aligned}$$

1.3 The Principle of Superposition

For more than a single disturbance (e.g a wave) propagating through a medium at the same instant, the resultant disturbance at any given point due to the disturbances is the vector sum of the multiple disturbances.

As an example, suppose two disturbances (waves), $u_1 = u_{01}\sin(\omega_1 t - k_1 x)$ and $u_2 = u_{02}\sin(\omega_2 t - k_2 x)$ are propagating in the same medium. If the wave amplitudes u_{01} and u_{02} are equal, the resultant disturbance u due to u_1 and u_2 is given as,

$$\begin{aligned} u &= u_1 + u_2 = u_0 \sin(\omega_1 t - k_1 x) + u_0 \sin(\omega_2 t - k_2 x), \\ &= 2u_0 \sin\left(\frac{[\omega_1+\omega_2]t - [k_1+k_2]x}{2}\right)\cos\left(\frac{[\omega_1-\omega_2]t - [k_1-k_2]x}{2}\right), \end{aligned} \tag{1.21}$$

where we have used the trigonometric identity, $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

1.3.1 Formation of Beats

Consider the superposition of the two waves above with $u_{01} = u_{02} = u_0$. If we further suppose that the frequencies and wave numbers of the two waves are nearly equal, the resultant disturbance takes the form,

$$\begin{aligned} u &= 2u_0 \sin\left(\frac{[\omega_1+\omega_2]t - [k_1+k_2]x}{2}\right)\cos\left(\frac{[\omega_1-\omega_2]t - [k_1-k_2]x}{2}\right), \\ &= 2u_0 \sin 2\pi\left(\left[\frac{f_1+f_2}{2}\right]t - \left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]\frac{x}{2}\right)\cos 2\pi\left(\left[\frac{f_1-f_2}{2}\right]t - \left[\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right]\frac{x}{2}\right). \end{aligned}$$

Without any loss of generality but for simplicity reasons, we consider the case above at the origin, that is, $x = 0$ the resultant disturbance becomes,

$$\begin{aligned} u &= 2u_0 \sin 2\pi\left(\frac{[f_1+f_2]t}{2}\right)\cos 2\pi\left(\frac{[f_1-f_2]t}{2}\right), \\ &= 2u_0 \sin\left(\pi[f_1+f_2]t\right)\cos\left(\pi[f_1-f_2]t\right), \\ &= \left[2u_0 \cos\left(\pi[f_1-f_2]t\right)\right]\sin\left(\pi[f_1+f_2]t\right), \\ &= u_0' \sin 2\pi f t, \quad \text{with} \quad f = \frac{1}{2}(f_1+f_2), \end{aligned}$$

The resultant wave motion is thus given by,

$$\boxed{u = u_0' \sin 2\pi f t} \tag{1.22}$$

with a modified, time varying amplitude given by,

$$u_0' = 2u_0 \cos\left(\pi[f_1-f_2]t\right). \tag{1.23}$$

The resultant wave in equation (1.22) is a function involving a variation of large frequency $f = 1/2(f_1+f_2)$ encapsulated with a low frequency (equal to $f_1 - f_2$) variation. The resultant wave in equation (1.22) is illustrated in figure 1.3: The intensity (I) of the resultant wave is given as,

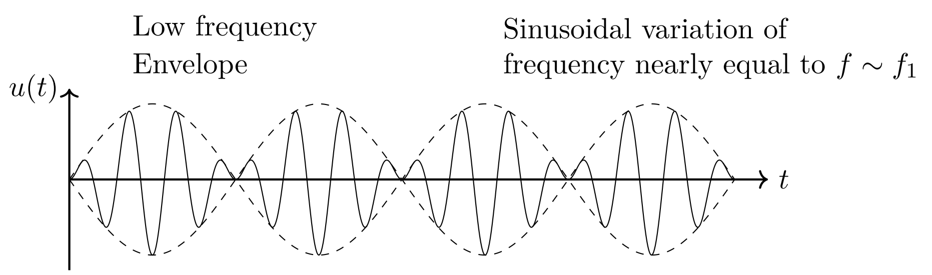

Figure 1.3: A schematic showing the formation of beats. The pattern between any two consecutive nodes is referred to as a wave packet.

$$\begin{aligned} I \propto {u_o'}^2 &= 4u_o^2 \cos^2\left(\pi[f_1-f_2]t\right), \\ &= 4u_o^2 \cdot \frac{1}{2}\left[1 + \cos 2\pi[f_1-f_2]t\right], \\ &= 2u_o^2\left[1 + \cos 2\pi[f_1-f_2]t\right], \end{aligned} \tag{1.24}$$

where we have made use of the trigonometric identity,

$$\cos^2 \pi[f_1 - f_2]t = \frac{1}{2}\left[1 + \cos 2\pi[f_1 - f_2]t\right] \cdot$$

The maximum values of amplitude (and hence intensities) of the superposed waves are obtained at times T_n given by,

$$T_n = \frac{1}{f_1 - f_2} \cdot n, \quad n = 1, 2, \ldots \tag{1.25}$$

The time T is referred to as the beat period and the corresponding frequency is the beat frequency, that is, $|f_1 - f_2|$. The intensity thus goes through periodic rises and falls, a phenomenon referred to as *beats*.

From equation (1.16), the energy carried by a wave is also found to be proportional to the square of the amplitude. This implies that for a standing wave, the energy is confined to the regions between consecutive nodes where the amplitude is non-zero. At the positions of nodes, the energy carried by the wave is zero since the wave amplitude is zero.

1.3.2 Interference

From equation (1.24), we notice the first term is the sum of two equal intensities of non-interacting waves. The second term shows the effect of interaction of the two waves. Consequently, the second term is called the *interference part* and is determined by the phase difference $(f_1 - f_2)$.

When $(f_1 - f_2) = n, \; n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$, the interference part will be positive and we refer to this situation as *constructive interference*. On the other hand, for $n = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, the interference part will be negative. This situation is referred to as *destructive interference*. Thus, we conclude this section that when waves combine their energy, the resulting intensity may increase or decrease.

Example I: The two waves: $u_1 = 2.0\sin(100\pi t - 1.2x)$ and $u_2 = 2.0\sin(104\pi t - 0.4x)$ propagate simultaneously in a medium. How many beats are heard after 4 s?

Solution: Comparing the wave equations with the standard wave equation, $u = u_0 \sin(2\pi f t - kx)$, we find that:

For u_1: $f_1 = 50$ Hz,

For u_2: $f_2 = 52$ Hz

The beat frequency is $f = |f_1 - f_2| = 2$ Hz. The number of beats heard after 4 s is $4 \times f = 8$.

Example II: Two tuning forks are sounded simultaneously producing beats of frequency 10 Hz. Given that the frequency of one of the tuning forks is 303 Hz. Find the frequency of the other tuning fork.

Solution: Using

$$\begin{aligned} f &= |f_1 - f_2|, \\ \rightarrow f_1 &= f + f_2 = 10 + 303 = 313 \text{ Hz}, \\ \rightarrow f_2 &= f_1 - f = 303 - 10 = 293 \text{ Hz} \end{aligned}$$

From the above, the other frequency could be either 313 Hz or 293 Hz.

1.4 Diffraction

In the section above, the superposition of only two waves is considered leading to formation of beats. In the general framework, the same dynamics also lead to the phenomenon of interference. However, the natural question to follow is what if it is more than two waves that propagate simultaneously is a given medium, what would the resultant disturbance be?

The answer to the above question is diffraction. The term *Diffraction* has been coined to refer to the superposition of more than two waves which simultaneously travel in a given medium. As such, there is no fundamental difference between the two phenomena. To discuss diffraction, we shall consider the wave equation in a slightly modified form so as to emphasize the phase difference between any corresponding waves. Consider the superposition of n waves with equal amplitudes u_0 but whose phases differ. The waves have the form:

$$\begin{aligned} u_1 &= u_0 \cos(\omega t), \\ u_2 &= u_0 \cos(\omega t + \vartheta), \\ u_3 &= u_0 \cos(\omega t + 2\vartheta), \\ u_4 &= u_0 \cos(\omega t + 3\vartheta), \\ \vdots &= \vdots \\ u_n &= u_0 \cos(\omega t + [n-1]\vartheta), \end{aligned} \tag{1.26}$$

where the phase difference between successive waves u_n and u_{n+1} is constant and is equal to ϑ. We seek to find the amplitude of the resultant disturbance when the waves $u_i, \quad i = 1, 2, 3, \ldots, n$ simultaneously propagate through a medium. From the principle of superposition, the resultant wave is the sum of the individual disturbances given by,

$$\begin{aligned} u_r &= u_1 + u_2 + u_3 + \cdots + u_n, \\ &= \sum_{i=1}^{n} u_i \end{aligned} \tag{1.27}$$

To evaluate the summation in equation (1.27), we shall consider the sum involving the first eight elements and then generalize the result to apply for n waves. We do a trick here by adding odd and even terms separately as follows:

$$u_1 + u_3 = u_0 [\cos(\omega t) + \cos(\omega t + 2\vartheta)] = 2u_0 \cos(\omega t + \vartheta) \cos\vartheta, \tag{1.28}$$

$$u_2 + u_4 = u_0 [\cos(\omega t + \vartheta) + \cos(\omega t + 3\vartheta)] = 2u_0 \cos(\omega t + 2\vartheta) \cos\vartheta, \tag{1.29}$$

$$u_5 + u_7 = u_0 [\cos(\omega t + 4\vartheta) + \cos(\omega t + 6\vartheta)] = 2u_0 \cos(\omega t + 5\vartheta) \cos\vartheta, \tag{1.30}$$

$$u_6 + u_8 = u_0 [\cos(\omega t + 5\vartheta) + \cos(\omega t + 7\vartheta)] = 2u_0 \cos(\omega t + 6\vartheta) \cos\vartheta, . \tag{1.31}$$

To this end, the superposition of the first eight waves is,

$$\begin{aligned} u_8 &= \sum_{i=1}^{8} u_i, \\ &= 2u_0 \cos\vartheta \left[\cos(\omega t + \vartheta) + \cos(\omega t + 5\vartheta) + \cos(\omega t + 2\vartheta) + \cos(\omega t + 6\vartheta)\right], \\ &= 4u_0 \cos\vartheta \cos 2\vartheta \left[\cos(\omega t + 3\vartheta) + \cos(\omega t + 4\vartheta)\right], \end{aligned} \tag{1.32}$$

where we have used the trigonometric identities:

$$\begin{aligned} \cos(\omega t + \vartheta) + \cos(\omega t + 5\vartheta) &= 2\cos(\omega t + 3\vartheta)\cos 2\vartheta, \\ \cos(\omega t + 2\vartheta) + \cos(\omega t + 6\vartheta) &= 2\cos(\omega t + 4\vartheta)\cos 2\vartheta. \end{aligned}$$

Also,

$$\cos(\omega t + 3\vartheta) + \cos(\omega t + 4\vartheta) = 2\cos\left(\omega t + \frac{7}{2}\vartheta\right)\cos\frac{\vartheta}{2},$$

such that,

$$u_8 = 8u_0 \cos 2\vartheta \cos\vartheta \cos\frac{\vartheta}{2}\left[\cos\left(\omega t + \frac{7}{2}\vartheta\right)\right]. \tag{1.33}$$

In what follows, we concentrate on simplifying the amplitude factor of the resultant wave in equation (1.33). We start by multiplying it by $1 = \frac{\sin\frac{\vartheta}{2}}{\sin\frac{\vartheta}{2}}$,

$$\begin{aligned} u_{08} &= 8u_0 \cos 2\vartheta \cos\vartheta \cos\frac{\vartheta}{2} \times \frac{\sin\frac{\vartheta}{2}}{\sin\frac{\vartheta}{2}} = \frac{4u_0}{\sin\frac{\vartheta}{2}} \cos 2\vartheta \cos\vartheta \cdot 2\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}, \\ &= \frac{4u_0}{\sin\frac{\vartheta}{2}} \cos 2\vartheta \cos\vartheta \sin\vartheta = \frac{2u_0}{\sin\frac{\vartheta}{2}} \cos 2\vartheta \cdot 2\cos\vartheta\sin\vartheta, \\ &= \frac{2u_0}{\sin\frac{\vartheta}{2}} \cos 2\vartheta \sin 2\vartheta = \frac{u_0}{\sin\frac{\vartheta}{2}} \cdot 2\cos 2\vartheta \sin 2\vartheta = u_0 \frac{\sin 4\vartheta}{\sin\frac{\vartheta}{2}}, \end{aligned} \tag{1.34}$$

where we have used the identities,

$$\begin{aligned} 2\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2} &= \sin\vartheta, \quad \text{and,} \\ 2\cos\vartheta\sin\vartheta &= \sin 2\vartheta, \\ 2\cos 2\vartheta \sin 2\vartheta &= \sin 4\vartheta. \end{aligned}$$

The resultant disturbance involving the first eight waves is thus,

$$u_8 = u_0 \frac{\sin 4\vartheta}{\sin\frac{\vartheta}{2}} \left[\cos\left(\omega t + \frac{7}{2}\vartheta\right)\right]. \tag{1.35}$$

We can therefore generalize the result in equation (1.35) to obtain the resultant disturbance of the first n waves as:

$$u_n = u_0 \frac{\sin\frac{n}{2}\vartheta}{\sin\frac{\vartheta}{2}} \left[\cos\left(\omega t + \frac{[n-1]}{2}\vartheta\right)\right]. \tag{1.36}$$

Complex Analysis Approach

Alternatively, the resultant disturbance could be obtained by considering a change to complex numbers as follows:

$$\begin{aligned} u_r &= \sum_{i=1}^{n} u_i, \\ &= u_0 \sum_{i=1}^{n} \cos(\omega t + [n-1]\vartheta). \end{aligned} \tag{1.37}$$

We now consider the trigonometric identity:

$$\cos\phi = \frac{1}{2}\left[e^{i\phi} + e^{-i\phi}\right], \tag{1.38}$$

and re write the expression of the resultant disturbance as:

$$\begin{aligned} u_r &= u_0\left[\cos\omega t + \cos(\omega t+\vartheta) + \cos(\omega t + 2\vartheta) + \cdots + \cos(\omega t + [n-1]\vartheta)\right], \\ &= \frac{u_0}{2}\Big[\left(e^{i\omega t} + e^{-i\omega t}\right) + \left(e^{i(\omega t+\vartheta)} + e^{-i(\omega t+\vartheta)}\right) + \left(e^{i(\omega t+2\vartheta)} + e^{-i(\omega t+2\vartheta)}\right) + \cdots \\ &\quad \cdots + \left(e^{i(\omega t+[n-1]\vartheta)} + e^{-i(\omega t+[n-1]\vartheta)}\right)\Big], \\ &= \frac{u_0}{2}\left[e^{i\omega t} + e^{i(\omega t+\vartheta)} + e^{i(\omega t+2\vartheta)} + \cdots + e^{i(\omega t+[n-1]\vartheta)}\right] + \\ &+ \frac{u_0}{2}\left[e^{-i\omega t} + e^{-i(\omega t+\vartheta)} + e^{-i(\omega t+2\vartheta)} + \cdots + e^{-i(\omega t+[n-1]\vartheta)}\right]. \end{aligned} \tag{1.39}$$

One quickly notices that the terms in the square parenthesis in equation (1.39) are geometric progressions (G.P) where each term differs from the next one by a constant ratio r. In general, geometric progressions have the form:

$$a + ar + ar^2 + \cdots ar^{n-1},$$

and the sum of the first n terms is obtained as:

$$S_n = a\left[\frac{1-r^n}{1-r}\right], \tag{1.40}$$

where a is the first term of the progression and r is the common ration. With this, equation (1.39) can be written as:

$$u_r = T_1 + T_2, \quad \text{where,} \tag{1.41}$$

$$\begin{aligned} T_1 &= \frac{u_0}{2}\left[e^{i\omega t} + e^{i(\omega t+\vartheta)} + e^{i(\omega t+2\vartheta)} + \cdots + e^{i(\omega t+[n-1]\vartheta)}\right], \\ &= \frac{1}{2}u_0 e^{i\omega t}\left[1 + e^{i\vartheta} + e^{2i\vartheta} + \cdots + e^{i[n-1]\vartheta}\right], \end{aligned}$$

$$\therefore T_1 = \frac{1}{2}u_0 e^{i\omega t}\cdot\left[\frac{1-e^{in\vartheta}}{1-e^{i\vartheta}}\right], \tag{1.42}$$

where the term in the square parentheses has been treated as a geometric progression with the first term $a = 1$ and common ratio $r = e^{i\vartheta}$. Also,

$$\begin{aligned} T_2 &= \frac{u_0}{2}\left[e^{-i\omega t} + e^{-i(\omega t+\vartheta)} + e^{-i(\omega t+2\vartheta)} + \cdots + e^{-i(\omega t+[n-1]\vartheta)}\right], \\ &= \frac{1}{2}u_0 e^{-i\omega t}\left[1 + e^{-i\vartheta} + e^{-2i\vartheta} + \cdots + e^{-i[n-1]\vartheta}\right], \end{aligned}$$

$$\therefore T_2 = \frac{1}{2}u_0 e^{-i\omega t}\left[\frac{1-e^{-in\vartheta}}{1-e^{-i\vartheta}}\right], \tag{1.43}$$

the term in the last parentheses arises as a sum of the geometric progression with the first term $a = 1$ and common ratio $r = e^{-i\vartheta}$. From equations (1.42) and (1.43), we obtain:

$$u_r = T_1 + T_2 = \frac{1}{2}u_0 e^{i\omega t}\cdot\left[\frac{1-e^{in\vartheta}}{1-e^{i\vartheta}}\right] + \frac{1}{2}u_0 e^{-i\omega t}\left[\frac{1-e^{-in\vartheta}}{1-e^{-i\vartheta}}\right]. \tag{1.44}$$

In what follows, we solve out the different components simplifying them and we shall eventually substitute them back to obtain the total disturbance u_r.

$$1 - e^{in\vartheta} = e^{\frac{in\vartheta}{2}}\left[e^{\frac{-in\vartheta}{2}} - e^{\frac{in\vartheta}{2}}\right] = -e^{\frac{in\vartheta}{2}}\left[e^{\frac{in\vartheta}{2}} - e^{\frac{-in\vartheta}{2}}\right], \tag{1.45}$$

$$1 - e^{-in\vartheta} = e^{\frac{-in\vartheta}{2}}\left[e^{\frac{in\vartheta}{2}} - e^{\frac{-in\vartheta}{2}}\right], \tag{1.46}$$

$$1 - e^{i\vartheta} = e^{\frac{i\vartheta}{2}}\left[e^{\frac{-i\vartheta}{2}} - e^{\frac{i\vartheta}{2}}\right] = -e^{\frac{i\vartheta}{2}}\left[e^{\frac{i\vartheta}{2}} - e^{\frac{-i\vartheta}{2}}\right], \tag{1.47}$$

$$1 - e^{-i\vartheta} = e^{\frac{-i\vartheta}{2}}\left[e^{\frac{i\vartheta}{2}} - e^{\frac{-i\vartheta}{2}}\right]. \tag{1.48}$$

substituting equations (1.45 - 1.48) into (1.44) yields:

$$\begin{aligned} u_r &= \frac{u_0/2}{\left[e^{\frac{i\vartheta}{2}} - e^{\frac{-i\vartheta}{2}}\right]}\cdot\left[e^{\frac{in\vartheta}{2}} - e^{\frac{-in\vartheta}{2}}\right]\cdot\left[e^{i(\omega t+\frac{[n-1]\vartheta}{2})} - e^{-i(\omega t+\frac{[n-1]\vartheta}{2})}\right], \\ &= \frac{u_0/2}{2i\sin\frac{\vartheta}{2}}\cdot 2i\sin\frac{n\vartheta}{2}\cdot 2\cos\left(\omega t + \frac{[n-1]\vartheta}{2}\right), \end{aligned}$$

$$\therefore u_r = u_0\frac{\sin\frac{n}{2}\vartheta}{\sin\frac{\vartheta}{2}}\left[\cos\left(\omega t + \frac{[n-1]}{2}\vartheta\right)\right], \tag{1.49}$$

which is the same result as that obtained earlier in equation (1.36) using a purely trigonometric approach. The one outstanding feature of this solution is that the amplitude of the resultant disturbance is a composite of trigonometric functions. This is unlike the constant amplitude u_0 of the individual waves that superposed to give rise to the eventual disturbance. The amplitude of the superposed waves given by:

$$u_0' = u_0 \frac{\sin \frac{n}{2}\vartheta}{\sin \frac{\vartheta}{2}}, \tag{1.50}$$

and its graphical representation is shown n figure 1.4.

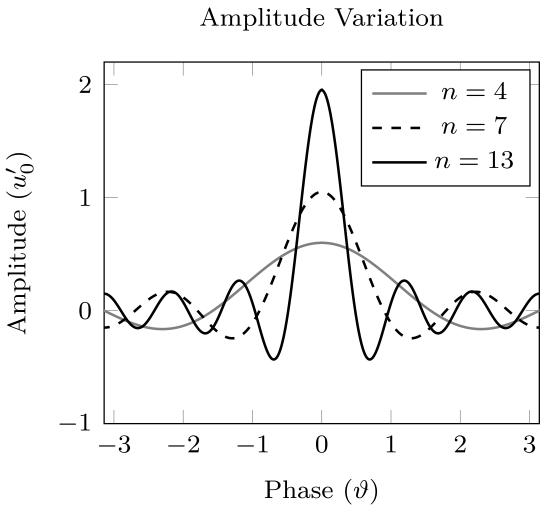

Figure 1.4: An illustration of the variation of the amplitude with the phase angle ϑ. Here, $n = 4$ (continuous gray line), $n = 7$ (black dashed line) and $n = 7$ (continuous black line) with ϑ set to vary from $-\pi$ to π, that is, $-\pi \leq \vartheta \leq \pi$.

The resultant intensity which is known to be proportional to the square of the wave amplitude is thus obtained to be:

$$I_n \sim u_{0n}^2 = \left[u_0 \frac{\sin \frac{n}{2}\vartheta}{\sin \frac{\vartheta}{2}} \right]^2 = u_0^2 \frac{\sin^2 \frac{n}{2}\vartheta}{\sin^2 \frac{\vartheta}{2}}. \tag{1.51}$$

The variation of the intensity in equation (1.51) is illustrated in figure 1.5 for three different values of n. It is observed that the intensity of the net disturbance is highest for large values of n and reduces for low values of n. This is as expected because larger values of n imply more waves propagating through the medium and since each wave carries its own energy, more energy is also brought into the system hence the high intensity.

Further, to verify our assertion that diffraction and interference are equivalent except for the number of individual superposing disturbances, we consider the intensity in equation (1.51) for $n = 2$:

$$I_2 = u_0^2 \frac{\sin^2 \frac{2}{2}\vartheta}{\sin^2 \frac{\vartheta}{2}} = I_0 \frac{\sin^2 \vartheta}{\sin^2 \frac{\vartheta}{2}}, \tag{1.52}$$

where $I_0 = u_0^2$ is the intensity of a single wave whose amplitude is u_0. Thus,

$$\begin{aligned} I_2 &= I_0 \frac{[\sin \vartheta]^2}{\sin^2 \frac{\vartheta}{2}}, \\ &= I_0 \frac{[2 \sin \frac{\vartheta}{2} \cos \frac{\vartheta}{2}]^2}{\sin^2 \frac{\vartheta}{2}} = 4 I_0 \cos^2 \frac{\vartheta}{2}, \\ &= 2 I_0 \left[1 + \cos \vartheta\right], \end{aligned} \tag{1.53}$$

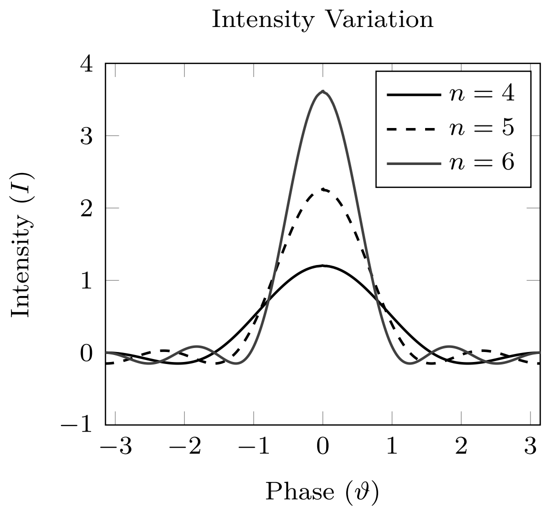

Figure 1.5: An illustration of the variation of the intensity I with the phase angle ϑ. Here, $n = 4$ (continuous black line), $n = 5$ (black dashed line) and $n = 6$ (dark gray line) with ϑ set to vary from $-\pi$ to π, that is, $-\pi \leq \vartheta \leq \pi$.

where we have used the trigonometric identity,

$$\cos^2 \frac{\vartheta}{2} = \frac{1}{2} [1 + \cos \vartheta] \,. \tag{1.54}$$

We note that the result in equation (1.53) is similar to that which was obtained when we considered the interference that results when two waves are superposed. We therefore conclude this section with two confirmations:

(i) That diffraction and interference are equivalent.

(ii) That the net disturbance of n waves propagating through the same medium is given by the expression in equation (1.36) and the resulting intensity in equation (1.51).

1.5 Doppler Effect

This is the apparent change in frequency detected by an observer when there is relative motion between the source and the observer. In each of the following cases, we consider the velocity of the wave to be c. The velocity of the wave relative to the source is $u = c - u_s$. From $u = f\lambda_a$, where

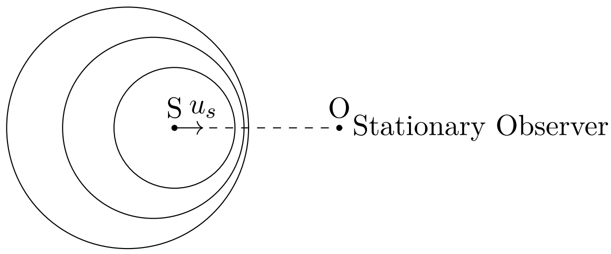

Figure 1.6: The source S moving towards a stationary observer O with velocity u_s.

λ_a is the apparent wave length. The apparent wave length of waves emitted by the source is,

$$\lambda_a = \frac{u}{f} = \frac{c - u_s}{f} \cdot \tag{1.55}$$

Since the observer is stationary (figure 1.6), the velocity of waves relative to the observer O is c and thus the apparent frequency to the observer is,

$$\begin{aligned} f_{app} &= \frac{v}{\lambda_a} = \frac{c}{\lambda_a} = \frac{c}{\frac{c-u_s}{f}}, \\ \rightarrow f_{app} &= \left[\frac{c}{c-u_s}\right] f. \end{aligned} \quad (1.56)$$

In equation (1.56), $\frac{c}{c-u_s} > 1$ hence $f_{app} > f$. The apparent frequency of waves with respect to the observer is higher as the source approaches the observer. This explains the high pitch (frequency) of the siren made by an ambulance or police car as it approaches an a stationary observer.

When the ambulance reaches the observer with a very pitch, the frequency tends to decrease as the siren recedes away from the observer (figure 1.7). In this case, the relative velocity of the waves with respect to the source is $u = c + u_s$ and thus using $f = u/\lambda_a$, we obtain the apparent wave length as,

$$\lambda_a = \frac{c+u_s}{f}. \quad (1.57)$$

The relative velocity of the waves with respect to the observer is $v = c$ and thus the apparent

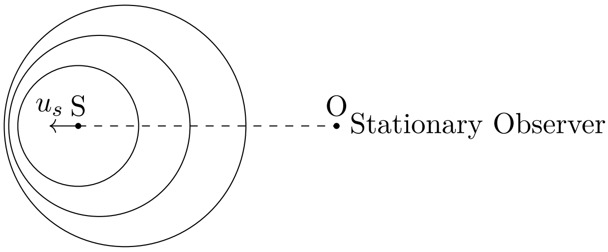

Figure 1.7: The source S moving away from the observer O with velocity u_s.

frequency of waves with respect to the observer is obtained from $v = f_{app}\lambda_a$ as,

$$f_{app} = \frac{v}{\lambda_a} = \frac{c}{\frac{c+u_s}{f}} = \left[\frac{c}{c+u_s}\right] f. \quad (1.58)$$

Since $c + u_s > c$, we see that $f_{app} < f$ and thus as the siren recedes away from the observer, the frequency appears to go down.

The next scenario involves an observer in motion moving toward a source of waves which is also approaching him/her (1.8). Here, the relative velocity of the waves with respect to the source is $u = c - u_s$. The apparent wavelength of the waves reaching the observer is,

$$\lambda_a = \frac{c-u_s}{f} \quad (1.59)$$

Instead, the relative velocity of the waves with respect to the observer is $v = c + u_o$, and thus using $v = f_{app}\lambda_a$, we obtain the apparent frequency as,

$$\begin{aligned} f_{app} &= \frac{v}{\lambda_a} = \frac{c+u_o}{\frac{c-u_s}{f}}, \\ f_{app} &= \left[\frac{c+u_o}{c-u_s}\right] f. \end{aligned} \quad (1.60)$$

Following similar steps as above, we obtain for the source and observer moving away from each

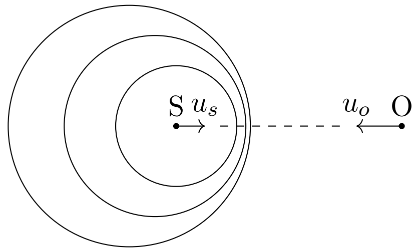

Figure 1.8: The source S moving with velocity u_s towards an observer O moving with velocity u_o.

other with velocities u_s and u_o respectively (figure 1.9), that the apparent wavelength is obtained as,

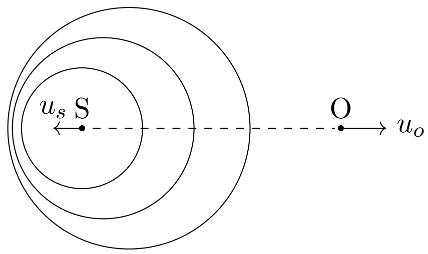

Figure 1.9: The source S moving away from the observer O with velocity u_s.

$$f_{app} = \left[\frac{c+u_o}{c+u_s}\right] f. \tag{1.61}$$

The above four scenarios can be combined into a compact formula accounting for each case as follows:

$$f_{app} = \left[\frac{c \pm u_o}{c \pm u_s}\right] f \tag{1.62}$$

1.6 Standing Waves

A special but important consideration and application of the principle of superposition, involves two waves of equal amplitudes, frequencies and wave numbers but travelling in opposite directions. Let the two waves be $u_1 = u_0 \sin(kx - \omega t)$ and $u_2 = u_0 \sin(kx + \omega t)$. Using the principle of superposition, the resultant disturbance due to the two waves simultaneously propagating through a given medium is given by,

$$\begin{aligned} u &= u_1 + u_2 = u_0 \left[\sin(kx - \omega t) + \sin(kx + \omega t)\right], \\ &= 2u_0 \sin(kx) \cos\omega t, \\ &= [u_0'] \cos\omega t, \end{aligned} \tag{1.63}$$

where the amplitude,

$$u_0' = 2u_0 \sin kx. \tag{1.64}$$

It is quickly realized that this is nolonger a progressive wave but instead a wave whose amplitude u_0' depends on the position. This resultant disturbance is referred to as a standing wave.

In equation (1.64), the special positions at which the amplitude u_0' vanishes, that is, $u_0' = 0$ are called *Nodes* characterized by the condition,

$$\begin{aligned} \sin kx &= 0, \quad kx_n = n\pi, \quad n = 1, 2, \ldots \\ \rightarrow x_n &= \frac{n\pi}{k} = n\frac{\lambda}{2}, \quad n = 1, 2, \ldots \end{aligned} \tag{1.65}$$

where we have used the definition of a wave number $k = \frac{2\pi}{\lambda}$ with λ being the wave length. The distance d between successive nodes is,

$$\begin{aligned} d &= x_{n+1} - x_n = [n+1]\frac{\lambda}{2} - n\frac{\lambda}{2}, \\ &= n\frac{\lambda}{2} + \frac{\lambda}{2} - n\frac{\lambda}{2}, \\ d &= \frac{\lambda}{2}, \end{aligned} \tag{1.66}$$

Thus the distance between successive nodes is half the wave length.

The positions where the resultant wave amplitude takes on maximum values are called *Antinodes*, characterized by,

$$\begin{aligned} \sin kx &= 1, \\ kx_n &= \left[n + \frac{1}{2}\right]\pi, \quad n = 0, 1, 2, 3, \ldots \\ \to x_n &= \left[n + \frac{1}{2}\right]\frac{\pi}{k} = \left[n + \frac{1}{2}\right]\frac{\pi}{\frac{2\pi}{\lambda}}, \\ x_n &= \left[n + \frac{1}{2}\right]\frac{\lambda}{2}, \quad n = 0, 1, 2, 3, \ldots \end{aligned} \tag{1.67}$$

The distance d' between successive antinodes is given as,

$$\begin{aligned} d' &= x_{n+1} - x_n = \left[(n+1) + \frac{1}{2}\right]\frac{\lambda}{2} - \left[n + \frac{1}{2}\right]\frac{\lambda}{2}, \\ &= \left[n + \frac{1}{2}\right]\frac{\lambda}{2} + \frac{\lambda}{2} - \left[n + \frac{1}{2}\right]\frac{\lambda}{2}, \\ \to d' &= \frac{\lambda}{2}, \end{aligned} \tag{1.68}$$

Thus the distance between successive antinodes is a half the wave length. Using equations (1.65) and (1.67), we obtain the distance D between successive nodes and antinodes as,

$$\begin{aligned} D &= \left[n + \frac{1}{2}\right]\frac{\lambda}{2} - n\frac{\lambda}{2}, \\ &= n\frac{\lambda}{2} + \frac{\lambda}{4} - n\frac{\lambda}{2}, \\ \to D &= \frac{\lambda}{4}, \end{aligned} \tag{1.69}$$

hence the distance between neighbouring nodes and antinodes is a quarter of the wave length.

1.7 Applications of Standing Waves

1.7.1 Standing Waves on Strings

Here, we consider a string of length ℓ fixed at both ends and of uniform cross sectional area. The string is then plucked at each of the following lengths:

1. String plucked half way between the fixed ends (1.10).
   The wave generated in the string is reflected to travel in the opposite direction. The reflected wave is shown as the dashed line. We quickly note the relationship between the wave length (λ) and the length (ℓ) of the string, namely,

$$\ell = \frac{1}{2}\lambda. \tag{1.70}$$

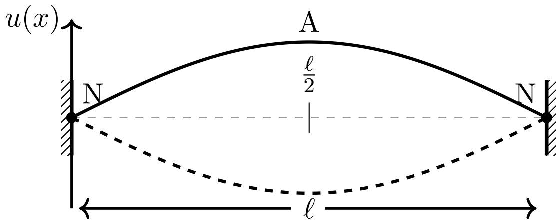

Figure 1.10: A string plucked at a position halfway its total length ℓ. The fixed ends of the string become node (N) and the plucked position becomes an antinode (A).

The frequency of the resulting note is obtained as,

$$f_1 = \frac{\upsilon}{\lambda} = \frac{\upsilon}{2\ell}, \tag{1.71}$$

where υ is the wave velocity. The frequency f_1 is known as the fundamental mode.

2. String plucked a quarter way from one of the fixed ends (Figure 1.11). Here, the wave length

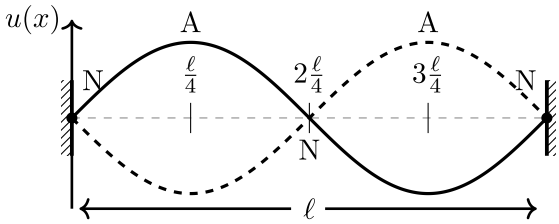

Figure 1.11: A string plucked at a position a quarter way its total length ℓ from one of the two fixed ends. The reflected wave travelling in the opposite direction is shown with a dashed line.

is equal to the length of the string,

$$\ell = \lambda \tag{1.72}$$

and the corresponding frequency is,

$$\begin{aligned} f_2 &= \frac{\upsilon}{\lambda} = \frac{\upsilon}{l} = 2 \cdot \frac{\upsilon}{2l}, \\ \rightarrow f_2 &= 2f_1. \end{aligned} \tag{1.73}$$

The resulting frequency is thus, twice the fundamental frequency.

3. String plucked at a position a sixth way from one of the fixed ends (1.12). From the illustra-

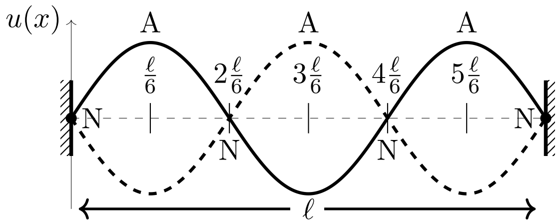

Figure 1.12: A string plucked at a position an sixth times of its total length ℓ from one of the two fixed ends. The generated wave is shown by the continuous line while the reflected one is shown by the dashed line.

tion, we see that,

$$\ell = \frac{3}{2}\lambda, \tag{1.74}$$

and the corresponding frequency is,

$$f_3 = \frac{v}{\lambda} = \frac{v}{\frac{2l}{3}} = 3\frac{v}{2l},$$
$$\rightarrow f_3 = 3 \cdot \frac{v}{2l} = 3f_1 \cdot \quad (1.75)$$

4. String plucked at a position an eighth way from one of the fixed ends (1.13). From the

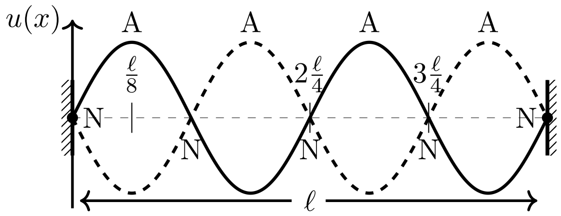

Figure 1.13: A string plucked at a position an eighth times of its total length ℓ from one of the two fixed ends. The generated wave is shown by the continuous line while the reflected one is shown by the dashed line.

illustration, we see that,

$$\ell = 2\lambda \quad (1.76)$$

and the corresponding frequency is,

$$f_4 = \frac{v}{\lambda} = \frac{v}{\frac{l}{2}} = 2\frac{v}{l},$$
$$\rightarrow f_4 = 4 \cdot \frac{v}{2l} = 4f_1 \quad (1.77)$$

5. The above cases can be joined together for easy comparison (Figure 1.14). From this figure, we learn that when a player plucks the string of an instrument like a guitar, the frequency of the note depends on the location where the string is plucked. High frequency notes are obtained by plucking the strings at locations near the fixed end while low frequency notes are reached at, by plucking the strings at locations far away from the fixed end.

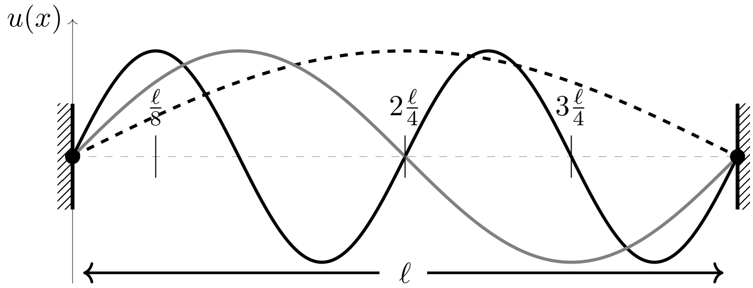

Figure 1.14: A schematic of a string plucked at different lengths from one of the fixed ends: The string is plucked in the middle i.e $\frac{\ell}{2}$ (dashed black line), a quarter way from one of the fixed ends i.e $\frac{\ell}{4}$ (continuous dark gray line), plucked at a location an eighth of the total length i.e $\frac{\ell}{8}$ (continuous black line).

In general the n^{th} frequency can be obtained in terms of the fundamental mode f_1 by generalizing the above four frequency relations as,

$$f_n = nf_1 \quad (1.78)$$

The higher frequencies can thus be estimated once the fundamental frequency f_1 is known.

1.7.2 Standing Waves in pipes

When a pipe is either closed at one of or both ends and waves are made to propagate through it, standing waves are formed. This is because the closed end(s) reflect the original incident waves leading to superposition of the incident and the reflected waves hence forming standing waves. The frequency of the resulting waves depend on the length of the pipe as described in the following scenarios.

Pipes closed at one end and free on the other

1. A pipe closed at one end showing the first allowed mode. The boundary condition at the free

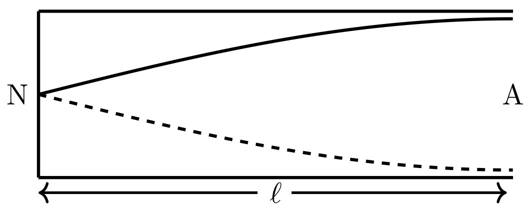

Figure 1.15: A pipe closed at one end showing the first allowed mode, also called the fundamental mode. A represents the position of antinodes while N represents positions of nodes.

end of the pipe restricts that an antinode be made their while nodes are always formed at the closed end. It is observed that the length of the pipe is equal to a quarter of the wave length, that is,

$$\ell = \frac{1}{4}\lambda. \tag{1.79}$$

The frequency of the waves, also called the fundamental mode is thus given as,

$$f_1 = \frac{v}{\lambda} = \frac{v}{4\ell}. \tag{1.80}$$

This is the lowest acceptable frequency of waves in the pipe subject to the constraints that one end is closed while the other is left open.

2. The next acceptable mode with a pipe closed at one end and open at the other is shown in Figure 1.16. Here, it is observed that the length of the tube is equal to three quarters of the

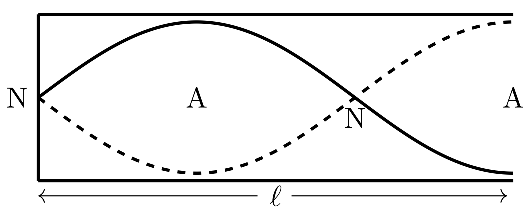

Figure 1.16: A pipe closed at one end showing the first allowed mode, also called the fundamental mode. A represents the position of antinodes while N represents positions of nodes.

wave length, that is,

$$\ell = \frac{3}{4}\lambda. \tag{1.81}$$

The resulting frequency is thus given as,

$$\begin{aligned} f_2 &= \frac{v}{\lambda} = \frac{v}{\frac{4\ell}{3}}, \\ &= 3 \cdot \frac{v}{4\ell}, \\ \rightarrow f_2 &= 3 f_1 \end{aligned} \tag{1.82}$$

3. The third acceptable mode with a pipe closed at one end and open at the other is shown in figure 1.17. The relationship between the length of the pipe ℓ and the wavelength is,

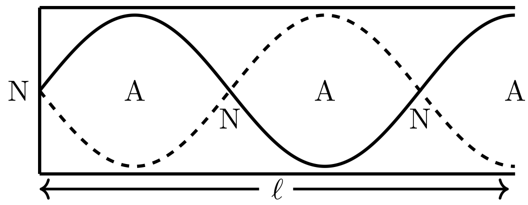

Figure 1.17: A pipe closed at one end showing the third allowed mode, also called the fundamental mode. A represents the position of antinodes while N represents positions of nodes.

$$\ell = \frac{5}{4}\lambda. \tag{1.83}$$

The corresponding frequency is,

$$\begin{aligned} f_3 &= \frac{v}{\lambda} = \frac{v}{\frac{4\ell}{5}}, \\ &= 5 \cdot \frac{v}{4\ell}, \\ f_3 &= 5 f_1. \end{aligned} \tag{1.84}$$

4. The fourth acceptable mode with a pipe closed at one end and open at the other is illustrated in figure 1.18. In this case the relationship between the pipe length ℓ and the wave length λ

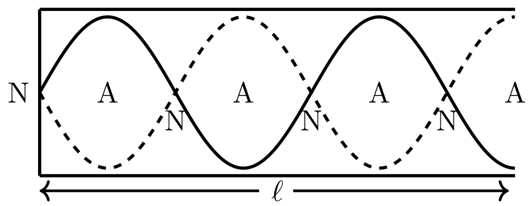

Figure 1.18: A pipe closed at one end showing the first allowed mode, also called the fundamental mode. A represents the position of antinodes while N represents positions of nodes.

is,

$$\ell = \frac{7}{4}\lambda, \tag{1.85}$$

and the resulting frequency is,

$$\begin{aligned} f_4 &= \frac{v}{\lambda} = \frac{v}{\frac{4\ell}{7}}, \\ &= 7 \cdot \frac{v}{4\ell}, \\ \rightarrow f_4 &= 7 f_1. \end{aligned} \tag{1.86}$$

5. For easy comparison and appreciation of all the acceptable modes, the illustration in figure 1.19 shows all the modes discussed above presented simultaneously. We can also state the

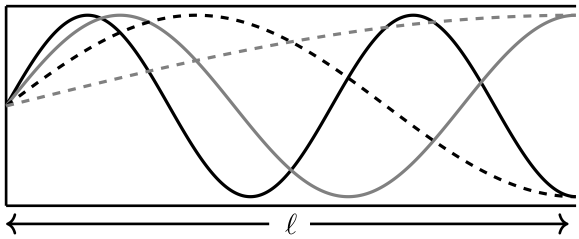

Figure 1.19: An illustration of the different modes that are acceptable in a pipe closed at one end and open on the other end. A represents the position of antinodes while N represents positions of nodes.

general relationship between higher frequencies f_n and the fundamental frequency as,

$$f_n = [2n - 1]\, f_1, \quad n = 2, 3, 4, \dots \tag{1.87}$$

Pipes open at both ends

With the pipe open at both ends, the boundary conditions of the system suggest that antinodes be formed at the free ends of the pipe, while any nodes that emerge must be formed inside the pipe.

1. The lowest acceptable mode occurs when there are antinodes at both open ends and only node inside the pipe (figure 1.20). The relationship between the length of the pipe ℓ and the

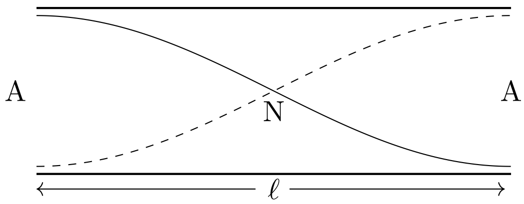

Figure 1.20: A pipe open at both ends showing the first allowed mode, also called the fundamental mode. A represents the position of antinodes while N represents positions of nodes.

wave length λ is,

$$\ell = \frac{1}{2}\lambda, \tag{1.88}$$

and the corresponding fundamental frequency f_1 is given as,

$$f_1 = \frac{v}{\lambda} = \frac{v}{2\ell}. \tag{1.89}$$

2. The second acceptable mode involves the emergency of a third and non boundary antinode and two nodes inside the pipe (figure 1.21). From the illustrations below, the length ℓ and

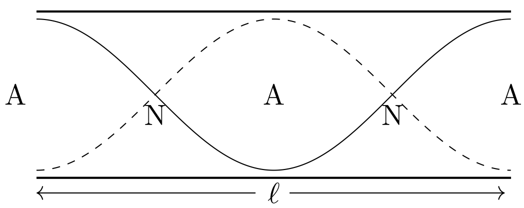

Figure 1.21: A pipe closed at one end showing the first allowed mode, also called the fundamental mode.

the wave length λ are related by,

$$\ell = \lambda, \tag{1.90}$$

and the corresponding frequency is,

$$\begin{aligned} f_2 &= \frac{\upsilon}{\lambda} = \frac{\upsilon}{\ell}, \\ &= 2 \cdot \frac{\upsilon}{2\ell}, \\ \rightarrow f_2 &= 2f_1, \end{aligned} \tag{1.91}$$

where υ is the wave velocity. Hence the first harmonic is twice the fundamental frequency.

3. The third acceptable frequency is given in figure 1.22. The length of the pipe and the

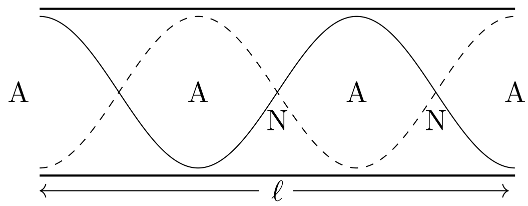

Figure 1.22: A pipe open at both ends showing the third allowed mode. A represents the position of antinodes while N represents positions of nodes.

wavelength are related by the expression,

$$\ell = \frac{3}{2}\lambda, \tag{1.92}$$

such that the mode frequency is given by,

$$\begin{aligned} f_3 &= \frac{\upsilon}{\lambda} = \frac{\upsilon}{\frac{2\ell}{3}}, \\ &= 3 \cdot \frac{\upsilon}{2\ell}, \\ f_3 &= 3f_1. \end{aligned} \tag{1.93}$$

The third harmonic is three times the fundamental mode.

4. The fourth acceptable mode involves three more antinodes plus the two at the open ends of the pipe, and four nodes inside the pipe (see Figure 1.23). Following the same steps as above,

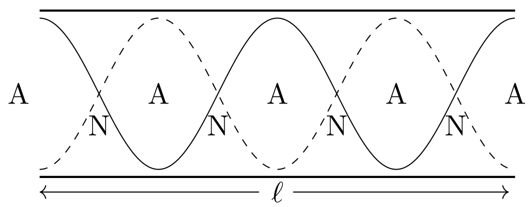

Figure 1.23: A pipe open at both ends showing the fourth acceptable mode. A represents the position of antinodes while N represents positions of nodes.

the frequency f_4 is obtained to be,

$$f_4 = 4f_1. \tag{1.94}$$

5. The modes above can be put together for easy comparison. The higher and fundamental frequencies are related by the expression;

$$f_n = nf_1 \tag{1.95}$$

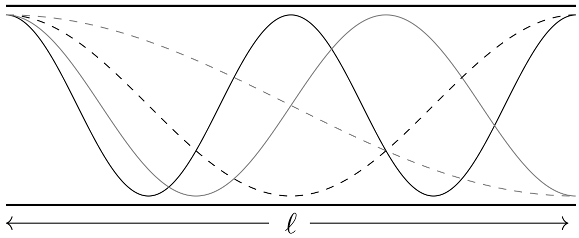

Figure 1.24: A pipe open at both ends showing the first four allowed modes. "A" represents the position of antinodes while "N" represents positions of nodes.

Table 1.1: Table showing the different modes and their relationship to the fundamental mode, f_1.

Mode	Pipe with one end closed	Pipe with both ends open
f_1	$\frac{v}{4\ell}$	$\frac{v}{2\ell}$
f_2	$3f_1$	$2f_1$
f_3	$5f_1$	$3f_1$
f_4	$7f_1$	$4f_1$
$\vdots$	$\vdots$	$\vdots$
f_n	$[2n-1]\,f_1$	nf_1

1.8 The Wave Equation

Consider a string stretched at both ends to a tension T, large enough so that the gravitational force on the string is negligible (Figure 1.25). We also make the following assumptions of the string:

1. The string is uniform with a constant linear density over the length of the string (that is, $\mu = \frac{m}{l} \equiv \text{constant}$).
2. The small displacement in the string is strictly vertical and that no particle of the string moves horizontally.

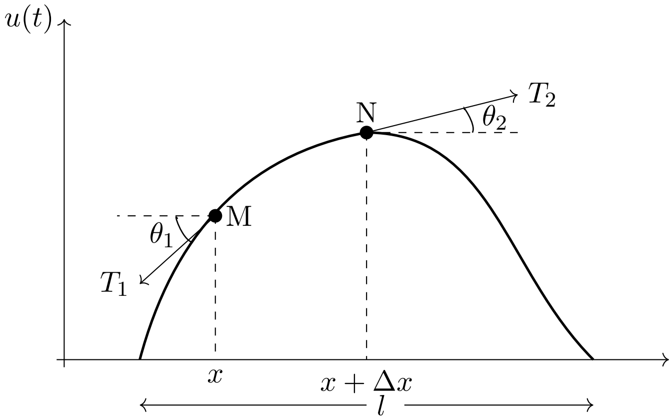

Figure 1.25: A string of total length l subjected to a tension T. The ends of the string, that is, $x = 0$ and $x = l$ are fixed in position.

We need to examine the forces on and the motion of the string segment MN. The horizontal forces are:

- At M: $T_1 \cos\theta_1$,

- AT N: $T_2 \cos\theta_2$

The assumption that all displacements be vertical, the horizontal forces must remain constant. Thus,

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T, \tag{1.96}$$

where T is the original tension of the string before it was slightly displaced. The vertical forces acting on the string segment MN are:

- At M: $-T_1 \sin\theta_1$,
- AT N: $T_2 \sin\theta_2$,

so that the resultant force in the vertical is, $F_\upsilon = T_2 \sin\theta_2 - T_1 \sin\theta_1$. Making use of Newton's second law of motion,

$$F_\upsilon = m\vec{a} = \mu \Delta x \frac{\partial^2 u}{\partial t^2}, \tag{1.97}$$

where $\mu\Delta x$ is the mass of the string segment MN. Therefore, the balance of vertical forces yields the equation of motion,

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = \mu \Delta x \frac{\partial^2 u}{\partial t^2}. \tag{1.98}$$

Dividing equation (1.98) by (1.96) yields,

$$\begin{aligned} \frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} &= \frac{\mu \Delta x}{T} \frac{\partial^2 u}{\partial t^2}, \\ \tan\theta_2 - \tan\theta_1 &= \frac{\mu \Delta x}{T} \frac{\partial^2 u}{\partial t^2}, \\ \frac{1}{\Delta x} [\tan\theta_2 - \tan\theta_1] &= \frac{\mu}{T} \frac{\partial^2 u}{\partial t^2}. \end{aligned} \tag{1.99}$$

Considering the terms in the parenthesis,

$$\begin{aligned} \tan\theta_2 &= \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} \text{ slope of the string at N,} \\ \tan\theta_1 &= \left.\frac{\partial u}{\partial x}\right|_{x} \text{ slope of the string at M.} \end{aligned} \tag{1.100}$$

Substituting equation (1.100) into (1.99) yields,

$$\begin{aligned} \frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] &= \frac{\mu}{T}\frac{\partial^2 u}{\partial t^2}, \\ \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} &= \frac{\mu}{T}\frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{1.101}$$

we therefore obtain the relation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 u}{\partial t^2}, \tag{1.102}$$

which can be re arranged to read as,

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{\upsilon^2}\frac{\partial^2 u}{\partial t^2}} \tag{1.103}$$

where the quantity,

$$\upsilon = \sqrt{\frac{T}{\mu}}, \tag{1.104}$$

is the velocity of the wave. Equation (1.104) is the one dimensional wave equation.

1.9 Solution of the Wave Equation

In this section, we proceed to solve the wave equation obtained in the previous section using the method of separation of variables. We start by outlining the wave equation again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\upsilon^2}\frac{\partial^2 u}{\partial t^2}. \tag{1.105}$$

A solution of the form below is proposed,

$$u(x,t) = X(x)T(t), \tag{1.106}$$

from which we obtain,

$$\frac{\partial^2 u}{\partial x^2} = T(t)\frac{\partial^2 X}{\partial x^2}, \quad \frac{\partial^2 u}{\partial t^2} = X(x)\frac{\partial^2 T}{\partial t^2}, \tag{1.107}$$

substituting equation (1.107) into (1.105) yields,

$$T(t)\frac{\partial^2 X}{\partial x^2} = \frac{1}{\upsilon^2}\cdot X(x)\frac{\partial^2 T}{\partial t^2}. \tag{1.108}$$

Dividing equation (1.108) by (1.106) yields,

$$\begin{aligned} \frac{T(t)}{X(x)T(t)}\frac{\partial^2 X}{\partial x^2} &= \frac{1}{\upsilon^2}\cdot\frac{X(x)}{X(x)T(t)}\frac{\partial^2 T}{\partial t^2}, \\ \frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} &= \frac{1}{\upsilon^2}\cdot\frac{1}{T(t)}\frac{\partial^2 T}{\partial t^2}. \end{aligned} \tag{1.109}$$

Since the left and right hand sides of equation (1.109) have different dependencies, their equality can only happen if the two terms are equal to a constant $(-\omega^2)$.

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} = \frac{1}{\upsilon^2}\cdot\frac{1}{T(t)}\frac{\partial^2 T}{\partial t^2} = -\omega^2. \tag{1.110}$$

The two equations that result are,

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} = -\omega^2, \quad \frac{\partial^2 X}{\partial x^2} + \omega^2 X(x) = 0, \tag{1.111}$$

$$\frac{1}{\upsilon^2}\cdot\frac{1}{T(t)}\frac{\partial^2 T}{\partial t^2} = -\omega^2, \quad \frac{\partial^2 T}{\partial t^2} + \omega^2\upsilon^2 T(t) = 0. \tag{1.112}$$

Equations (1.111) and (1.112) have solutions of the form:

$$X(x) = A\cos\omega x + B\sin\omega x, \tag{1.113}$$

$$T(t) = C\cos(\omega\upsilon t) + D\sin(\omega\upsilon t). \tag{1.114}$$

To obtain the constants A, B, C and D, we make use of the boundary and initial conditions. For boundary conditions, we consider that the displacement at the two fixed ends of the string, mathematically put as:

$$u(0,t) = 0, \qquad u(l,t) = 0. \tag{1.115}$$

Therefore, the first boundary condition yields,

$$\begin{aligned} X(0) &= A\cos 0 + B\sin 0 = 0, \qquad \to A = 0, \\ \therefore X(x) &= B\sin\omega x. \end{aligned} \tag{1.116}$$

The second boundary condition yields,

$$\begin{aligned} X(l) &= B\sin\omega l = 0, \\ \therefore \omega l &= n\pi; \quad \omega = \frac{n\pi}{l}, \quad n = 0,1,2,3,\ldots \end{aligned} \tag{1.117}$$

where the case $B = 0$ has been ignored to conserve the spatial dependence of the solution $u(x,t)$. Therefore,

$$X(x) = B\sin\left(\frac{n\pi}{l}x\right), \tag{1.118}$$

with a semi-intermediate general solution as:

$$\begin{aligned} u(x,t) &= X(x)T(t), \\ &= B\sin\left(\frac{n\pi}{l}x\right)\left[C\cos(\frac{n\pi}{l}\upsilon t) + D\sin(\frac{n\pi}{l}\upsilon t)\right] \end{aligned} \tag{1.119}$$

We now consider the initial conditions but we here try to remain as physical enough as possible. The initial conditions refer to the conditions of the system at time $t = 0$. Some of the examples of initial conditions are given in the figure below corresponding to the string being plucked in several positions. To keep our solution general, we consider two degenerate initial conditions:

$$u(x,0) = p(x), \quad \text{and} \tag{1.120}$$

$$\frac{\partial u}{\partial t}(x,0) = q(x). \tag{1.121}$$

Subjecting the above initial conditions to the solution in equation (1.121) yields:

$$u(x,0) = B\sin\left(\frac{n\pi}{l}x\right)[C\cos 0 + D\sin 0] = p(x), \tag{1.122}$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{n\pi}{l}\cdot B\sin\left(\frac{n\pi}{l}x\right)[-C\cdot 0 + D] = q(x). \tag{1.123}$$

Multiplying equation (1.122) by $\sin\frac{n\pi}{l}x$ and integrating with respect to x yields,

$$\begin{aligned} CB\sin\frac{n\pi}{l}x &= p(x), \\ \frac{1}{l}\int_{-l}^{l} CB\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}xdx &= \frac{1}{l}\int_{-l}^{l} p(x)\sin\frac{n\pi}{l}xdx, \\ BC &= \frac{1}{l}\int_{-l}^{l} p(x)\sin\frac{n\pi}{l}xdx \end{aligned} \tag{1.124}$$

For a choice of $B = 1$, we obtain:

$$C = \frac{1}{l}\int_{-l}^{l} p(x)\sin\frac{n\pi}{l}xdx \tag{1.125}$$

The supposed constant C is thus found to be the fourier transform of the initial condition function $p(x)$. Similarly, multiplying equation (1.123) by $\sin\frac{n\pi}{l}x$ and integrating with respect to x yields,

$$\begin{aligned} \frac{n\pi}{l}\upsilon B\sin\frac{n\pi}{l}x\cdot D &= q(x), \\ \frac{1}{l}\int_{-l}^{l}\left[\frac{n\pi}{l}\upsilon\right] DB\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}xdx &= \frac{1}{l}\int_{-l}^{l} q(x)\sin\frac{n\pi}{l}xdx, \\ \left[\frac{n\pi}{l}\upsilon\right] DB\cdot\frac{1}{l}\int_{-l}^{l}\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}xdx &= \frac{1}{l}\int_{-l}^{l} q(x)\sin\frac{n\pi}{l}xdx, \end{aligned} \tag{1.126}$$

The integral on the left hand side is equal to 1 and thus a good selection of the involved constants yields,

$$D = \frac{1}{l}\int_{-l}^{l} q(x)\sin\frac{n\pi}{l}x dx, \tag{1.127}$$

which is also a fourier sine series of the initial velocity of the string. It is also realized that the constants C and D are dependent of the integer n and hence, a subscript will be indicated on each one of them. To this end, the solution of the wave equation is:

$$u_n(x,t) = \sin\left(\frac{n\pi}{l}x\right)\left[C_n\cos(\frac{n\pi}{l}vt) + D_n\sin(\frac{n\pi}{l}vt)\right], \quad n = 1, 2, \ldots \tag{1.128}$$

The theory of differential equations has it that if an ordinary differential equation (ODE) has solutions u_n, the general solution is the sum of all solutions. Therefore the general solution to the one dimensional wave equation is,

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{l}x\right)\left[C_n\cos\left(\frac{n\pi}{l}vt\right) + D_n\sin\left(\frac{n\pi}{l}vt\right)\right] \tag{1.129}$$

From equation (1.129), there are infinitely many modes on the same string, with each n corresponding to one of the infinite modes. we have earlier one discussed that then when waves meet as obstacle, they are reflected to travel in the opposite direction. For the current, case, waves are surely reflected by one of the fixed points at the ends of the string. A sufficient solution like that in equation (1.129) must show these waves traveling in opposite directions as part of the general solution. To deduce that the obtained solution has all these waves incorporated, we consider the following simplification:

In the wave equation solution (1.129), if we open the brackets so that the equation reads as,

$$u(x,t) = \sum_{n=1}^{\infty}\left[C_n\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}vt\right)\right] + \left[D_n\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{n\pi}{l}vt\right)\right] \tag{1.130}$$

From the trigonometric identity,

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right), \tag{1.131}$$

comparing equations (1.130) and (1.131) leads to the relations:

$$\frac{A+B}{2} = \frac{n\pi}{l}x; \qquad A+B = \frac{2n\pi}{l}x, \tag{1.132}$$

$$\frac{A-B}{2} = \frac{n\pi}{l}vt; \qquad A-B = \frac{2n\pi}{l}vt. \tag{1.133}$$

Solving equations (1.132) and (1.133) yields;

$$A = \frac{n\pi}{l}[x+vt], \quad B = \frac{n\pi}{l}[x-vt],$$

$$\to \sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}vt\right) = \frac{1}{2}\left[\sin\frac{n\pi}{l}[x+vt] + \sin\frac{n\pi}{l}[x-vt]\right]. \tag{1.134}$$

Also, from the trigonometric identity,

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right), \tag{1.135}$$

we find that the second bracket in equation (1.130) can be expressed as:

$$\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{n\pi}{l}vt\right) = -\frac{1}{2}\left[\cos\frac{n\pi}{l}[x+vt] - \cos\frac{n\pi}{l}[x-vt]\right]. \tag{1.136}$$

Substituting equations (1.134) and (1.136)into (1.130) yields,

$$u(x,t)=\sum_{n=1}^{\infty}\frac{C_n}{2}\left[\sin\frac{n\pi}{l}\left[x+vt\right]+\sin\frac{n\pi}{l}\left[x-vt\right]\right]-\frac{D_n}{2}\left[\cos\frac{n\pi}{l}\left[x+vt\right]-\cos\frac{n\pi}{l}\left[x-vt\right]\right].$$

The solution of the wave equation can further be written as:

$$u(x,t) \;=\; \frac{1}{2}\left[f(x+vt)+h(x-vt)\right], \tag{1.137}$$

where the functions $f(x,t)$ are defined in equation (1.137) as:

$$f(x+vt) \;=\; C_n\sin\frac{n\pi}{l}\left[x+vt\right]-D_n\cos\frac{n\pi}{l}\left[x+vt\right], \quad \textit{and} \tag{1.138}$$

$$h(x-vt) \;=\; C_n\sin\frac{n\pi}{l}\left[x-vt\right]+D_n\cos\frac{n\pi}{l}\left[x-vt\right] \tag{1.139}$$

This form of the wave equation solution is quite informative in that it reveals that the general solution is a sum of waves travelling both in the positive $(f(x-vt))$ and in the negative $(f(x+vt))$ x-directions.

1.10 Phase and Group velocities

Consider two waves u_1 and u_2 of equal amplitudes but whose frequencies and wave numbers differ slightly. The mathematical representation of the two waves is:

$$\begin{aligned} u_1 &= u_0\cos(\omega t-kx), \quad \text{and} \\ u_2 &= u_0\cos([\omega+\Delta\omega]t-[k+\Delta k]x). \end{aligned}$$

When such disturbances simultaneously propagate through a medium, the net disturbance is the algebraic sum of the two individual disturbances. This has been referred to as the principle of superposition. Thus,

$$\begin{aligned} u &= u_1+u_2=u_0\left[\cos(\omega t-kx)+\cos([\omega+\Delta\omega]t-[k+\Delta k]x)\right], \\ &= 2u_0\cos\left(\frac{\omega t-kx+[\omega+\Delta\omega]t-[k+\Delta k]x}{2}\right)\cos\left(\frac{\omega t-kx-[\omega+\Delta\omega]t+[k+\Delta k]x}{2}\right), \\ &= 2u_0\cos\left(\left[\omega+\frac{\Delta\omega}{2}\right]t-\left[k+\frac{\Delta k}{2}\right]x\right)\cos\left(\frac{\Delta\omega t+\Delta kx}{2}\right). \end{aligned} \tag{1.140}$$

With the assumption that differences in the frequencies and wave numbers of the two waves being infinitesimal such that, $\Delta\omega << \omega$ and $\Delta k << k$, equation (1.140) simplifies to:

$$u=2u_0\cos(\omega t-kx)\cos\left(\frac{\Delta\omega}{2}t-\frac{\Delta k}{2}x\right). \tag{1.141}$$

The resultant disturbance in equation (1.141) is composed of two components, namely;

(i) the fast oscillating component with a frequency ω represented by the term $\cos(\omega t-kx)$ and,

(ii) the slowly oscillating component of frequency $\frac{\Delta\omega}{2}$ represented by the term $\cos\left(\frac{\Delta\omega}{2}t-\frac{\Delta k}{2}x\right)$.

A geometric representation of the disturbance in equation (1.141) is shown in figure 1.26. The wave velocity attributed to the fast moving wave is referred to as the *phase velocity* and is defined as:

$$v_{ph}=\frac{\omega}{k}. \tag{1.142}$$

This is the velocity of an individual particle in the wave.

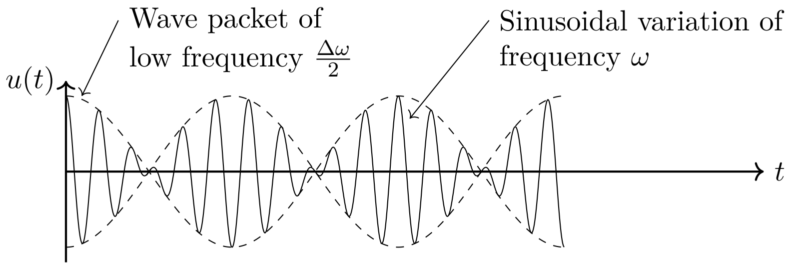

Figure 1.26: A schematic showing the superposition of two waves with equal amplitudes but whose frequencies and wave numbers differ infinitesimally. The pattern between any two consecutive nodes is referred to as a wave packet.

The wave velocity attributed to the slowly oscillating component is referred to as the *group velocity* and is defined as:

$$v_g = \frac{\Delta\omega}{\Delta k}, \tag{1.143}$$

and the limit of small Δk, the group velocity is obtained as the derivative of the angular frequency of the wave with respect to the wave number,

$$v_g = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \tag{1.144}$$

Further Reading

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