

Chapter 2

Central Forces

2.1 Chapter Objectives

By the end of the chapter, students should be able to:

- Define a central force field and state its properties.
- State and mathematically prove the properties of a central force field.
- Show that energy is a constant of motion in central force fields.
- Show that central force fields are conservative.
- State and prove Kepler's laws of planetary motion.
- Solve numerical problems involving central force fields.

2.2 Introduction

Consider a particle of mass, m under the influence of a generic force \vec{F} . If the result of the force's action on the particle is such that:

- it is directed to move towards or away from a fixed point O ,
- the magnitude of the influence of \vec{F} on m depends only on the radial distance of the particle from the fixed point O ,

\vec{F} is called a central force and the region of influence of this force is correspondingly referred to as a central force field. The fixed point O is alternatively called the centre of force of \vec{F} . As such, Examples of central forces include and are not limited to the:

- Gravitational force between two masses m_1 and m_2 separated by a distance r ,

$$F_g(r) = \frac{Gm_1m_2}{r^2}, \quad (2.1)$$

- Electrostatic force between two charges q_1 and q_2 separated by a distance r ,

$$F_e(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1q_2}{r^2}. \quad (2.2)$$

2.3 Properties of Central Force Fields

Any particle that moves under the influence of a central force field describes motion with the following properties:

- The path of the orbit is in a plane.
 - Angular momentum of the particle is constant.
 - The radius vector drawn from the fixed point O to the particle, sweeps out equal areas in equal times.
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2.4 Motion of a Particle in a Central Force Field

2.4.1 Planar motion in a central force field

The property that a particle in a central force field moves in a plane with a path that is a curve has the mathematical implication that such motion can fully be described on a two dimensional surface, for example, the xy -plane. If we choose to transform the Cartesian xy -coordinates to polar coordinates (r, θ) as:

$$\begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta, \\ \rightarrow \theta &= \tan^{-1} \left(\frac{y}{x} \right). \end{aligned} \quad (2.3)$$

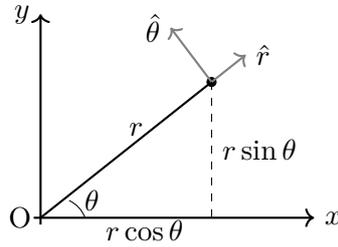


Figure 2.1: Transformation of Cartesian coordinates (x, y) to polar coordinates (r, θ) .

From the coordinates transformation in equation (2.3), the particle velocity is as expected obtained as time derivative of the particle displacement,

$$\begin{aligned} \dot{x} &= \dot{r} \cos \theta - r \dot{\theta} \sin \theta, \\ \dot{y} &= \dot{r} \sin \theta + r \dot{\theta} \cos \theta, \end{aligned} \quad (2.4)$$

Squaring the velocity components in equation (2.4) above yields:

$$\begin{aligned} \dot{x}^2 &= \dot{r}^2 \cos^2 \theta + r^2 \dot{\theta}^2 \sin^2 \theta - 2r \dot{r} \dot{\theta} \cos \theta \sin \theta, \\ \dot{y}^2 &= \dot{r}^2 \sin^2 \theta + r^2 \dot{\theta}^2 \cos^2 \theta + 2r \dot{r} \dot{\theta} \cos \theta \sin \theta, \\ v^2 &= \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2, \end{aligned} \quad (2.5)$$

Using our knowledge of calculus that, for a two dimensional vector $\vec{A} = A_x \hat{i} + A_y \hat{j}$, with \hat{i}, \hat{j} being the unit vectors along the x and y directions respectively. It follows that, $\vec{A}^2 = A^2 = A_x^2 + A_y^2$, we conclude that:

$$\boxed{\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}}, \quad (2.6)$$

where \hat{r} and $\hat{\theta}$ are the unit radial and angular directional vectors respectively. The expression of the particle acceleration is obtained by differentiating the velocity components in (2.4) as follows:

$$\begin{aligned} \ddot{x} &= \ddot{r} \cos \theta - 2\dot{r} \dot{\theta} \sin \theta - r \ddot{\theta} \sin \theta - r \dot{\theta}^2 \cos \theta, \\ \ddot{x}^2 &= \cos^2 \theta [\ddot{r} - r \dot{\theta}^2]^2 - \sin^2 \theta [2\dot{r} \dot{\theta} + r \ddot{\theta}]^2 \\ &\quad - 2 \sin \theta \cos \theta [\ddot{r} - r \dot{\theta}^2] [2\dot{r} \dot{\theta} + r \ddot{\theta}], \end{aligned} \quad (2.7)$$

$$\begin{aligned} \ddot{y} &= \ddot{r} \sin \theta + 2\dot{r} \dot{\theta} \cos \theta + r \ddot{\theta} \cos \theta - r \dot{\theta}^2 \sin \theta, \\ \ddot{y}^2 &= \cos^2 \theta [r \ddot{\theta} + 2\dot{r} \dot{\theta}]^2 + \sin^2 \theta [\ddot{r} - r \dot{\theta}^2]^2 \\ &\quad + 2 \sin \theta \cos \theta [r \ddot{\theta} + 2\dot{r} \dot{\theta}] [\ddot{r} - r \dot{\theta}^2], \end{aligned} \quad (2.8)$$

$$\begin{aligned} a^2 &= \ddot{x}^2 + \ddot{y}^2 = [r\ddot{\theta} + 2\dot{r}\dot{\theta}]^2 [\sin^2 \theta + \cos^2 \theta] + [\ddot{r} - r\dot{\theta}^2]^2 [\sin^2 \theta + \cos^2 \theta], \\ &= [r\ddot{\theta} + 2\dot{r}\dot{\theta}]^2 + [\ddot{r} - r\dot{\theta}^2]^2, \end{aligned}$$

Thus, a particle under the influence of a central force experiences an acceleration of the form:

$$\boxed{\vec{a} = [r\ddot{\theta} + 2\dot{r}\dot{\theta}] \hat{\theta} + [\ddot{r} - r\dot{\theta}^2] \hat{r}.} \quad (2.9)$$

2.4.2 Relation between unit vectors

From figure 2.1, it is observed that the radial direction after the polar coordinate transformation is along the vector \vec{r} . Considering the horizontal and vertical components of the vector \vec{r} in equation (2.3) we obtain,

$$\vec{r} = x\hat{i} + y\hat{j} = r [\cos \theta \hat{i} + \sin \theta \hat{j}],$$

From the above argument,

$$\begin{aligned} \vec{r} &= r\hat{r} = r [\cos \theta \hat{i} + \sin \theta \hat{j}], \\ \boxed{\hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j}} \end{aligned} \quad (2.10)$$

The unit vector $\hat{\theta}$ in the angular direction is obtained by considering the velocity, that is, the time derivative of the position vector \vec{r} ,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r} [\cos \theta \hat{i} + \sin \theta \hat{j}] + r\dot{\theta} [-\sin \theta \hat{i} + \cos \theta \hat{j}], \quad (2.11)$$

Comparing equations (2.6) and (2.11) yields,

$$\begin{aligned} \hat{r} &= \cos \theta \hat{i} + \sin \theta \hat{j}, \\ \boxed{\hat{\theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}} \end{aligned} \quad (2.12)$$

Equations (2.10) and (2.12) give the relation between the unit vectors of polar coordinates and those of the Cartesian coordinates.

2.4.3 Conservation of angular momentum

With the form of the acceleration in (2.9) above and considering Newton's second law of motion, we have,

$$m\vec{a} = m [r\ddot{\theta} + 2\dot{r}\dot{\theta}] \hat{\theta} + m [\ddot{r} - r\dot{\theta}^2] \hat{r} = f(r)\hat{r}, \quad (2.13)$$

$$m [\ddot{r} - r\dot{\theta}^2] = f(r), \quad (2.14)$$

$$m [r\ddot{\theta} + 2\dot{r}\dot{\theta}] = 0, \quad (2.15)$$

$$\begin{aligned} r\ddot{\theta} &= -2\dot{r}\dot{\theta}, \\ \frac{\ddot{\theta}}{\dot{\theta}} &= -2\frac{\dot{r}}{r}, \end{aligned} \quad (2.16)$$

Integrating equation (2.16) yields,

$$\begin{aligned} \int \frac{\ddot{\theta}}{\dot{\theta}} &= -2 \int \frac{\dot{r}}{r}, \\ \ln \dot{\theta} &= -2 \ln r + c, \\ \ln(r^2 \dot{\theta}) &= c, \\ r^2 \dot{\theta} &= h, \end{aligned} \quad (2.17)$$

where h is a constant which proves the second property of central force fields. Considering the radial component of equation (2.13) i.e $m(\ddot{r} - r\dot{\theta}^2) = f(r)$,

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= \frac{f(r)}{m}, \\ \text{substituting for } \dot{\theta} &= \frac{h}{r^2}, \\ \ddot{r} - r\frac{h^2}{r^4} &= \frac{f(r)}{m}, \\ \ddot{r} - \frac{h^2}{r^3} &= \frac{f(r)}{m} \text{ As required.}\end{aligned}\tag{2.18}$$

Considering that we chose to express our dynamics in polar coordinates, it is more meaningful to express the dynamical equation (2.18) in terms of the (r, θ) coordinates, so that the time variability is kept implicit. To achieve these equivalent expressions, we proceed as follows:

$$\dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt} \frac{dr}{d\theta} = \dot{\theta} \frac{dr}{d\theta} = \frac{h}{r^2} \frac{dr}{d\theta},\tag{2.19}$$

where in the above expression, we have made use of equation (2.17).

$$\begin{aligned}\ddot{r} &= \frac{d\dot{r}}{dt} = \frac{d}{dt} \left[\frac{h}{r^2} \frac{dr}{d\theta} \right] = \frac{d\theta}{dt} \frac{d}{d\theta} \left[\frac{h}{r^2} \frac{dr}{d\theta} \right] = \dot{\theta} \frac{d}{d\theta} \left[\frac{h}{r^2} \frac{dr}{d\theta} \right], \\ &= \dot{\theta} \left[\frac{h}{r^2} \frac{d^2r}{d\theta^2} - \frac{2h}{r^3} \frac{dr}{d\theta} \frac{dr}{d\theta} \right] = \dot{\theta} \frac{h}{r^2} \left[\frac{d^2r}{d\theta^2} - \frac{2}{r} \frac{dr}{d\theta} \frac{dr}{d\theta} \right], \\ &= \frac{h^2}{r^4} \left[\frac{d^2r}{d\theta^2} - \frac{2}{r} \frac{dr}{d\theta} \frac{dr}{d\theta} \right] = \frac{h^2}{r^4} \left[\frac{d^2r}{d\theta^2} - \frac{2}{r} \left[\frac{dr}{d\theta} \right]^2 \right],\end{aligned}\tag{2.20}$$

Substituting the above form of \ddot{r} into equation (2.18) yields,

$$\frac{h^2}{r^4} \left[\frac{d^2r}{d\theta^2} - \frac{2}{r} \left[\frac{dr}{d\theta} \right]^2 \right] - \frac{h^2}{r^3} = \frac{f(r)}{m},$$

dividing the above equation by h^2/r^4 yields the dynamical equation of motion for a particle under the influence of a central force field,

$$\boxed{\frac{d^2r}{d\theta^2} - \frac{2}{r} \left[\frac{dr}{d\theta} \right]^2 - r = \frac{r^4}{h^2} \frac{f(r)}{m}}.\tag{2.21}$$

Alternatively, considering a new variable $u = 1/r$, the dynamical equation takes a closely related form obtained as follows:

$$\begin{aligned}\dot{r} &= \frac{dr}{dt} = \frac{d\theta}{dt} \frac{dr}{d\theta} = \dot{\theta} \frac{dr}{d\theta} = \frac{h}{r^2} \frac{dr}{d\theta}, \\ &= hu^2 \frac{d}{d\theta} \left[\frac{1}{u} \right] = hu^2 \cdot -\frac{1}{u^2} \frac{du}{d\theta} = -h \frac{du}{d\theta},\end{aligned}$$

Also,

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\theta}{dt} \frac{d\dot{r}}{d\theta} = \dot{\theta} \frac{d}{d\theta} \left[-h \frac{du}{d\theta} \right] = -h\dot{\theta} \frac{d^2u}{d\theta^2},$$

$$= -h^2 u^2 \frac{d^2 u}{d\theta^2}, \quad (2.22)$$

substituting equation (2.22) into equation (2.18) yields,

$$\left[-h^2 u^2 \frac{d^2 u}{d\theta^2} - h^2 u^3 \right] = \frac{f\left(\frac{1}{u}\right)}{m}, \quad (2.23)$$

dividing equation (2.23) by $-h^2/u^2$ yields,

$$\boxed{\frac{d^2 u}{d\theta^2} + u = -\frac{1}{h^2 u^2} \frac{f\left(\frac{1}{u}\right)}{m}}. \quad (2.24)$$

By solving the dynamical equations (2.21) and (2.24), it is possible to obtain the path (or orbit) of a particle under the influence of a central force f as $r(\theta)$ or $r(t)$. Since these equations express the relationship between the force and the resulting orbit, the central force can also be determined using the orbit of the particle. When equation (2.24) is used to obtain $u(\theta)$, the orbit is determined by inverting u as $r(\theta) = \frac{1}{u(\theta)}$.

2.4.4 Determining the Force from the Orbit

Obtaining the central force given the orbit of a particle is straight forward by substituting for $r(\theta)$ in equation (2.21) or alternatively $u(\theta) = 1/r(\theta)$ in equation (2.24). For an example, consider a particle of mass m acted on by a central force to describe a circular orbit $r(\theta) = 2a \cos \theta$, where a is the radius of the circular orbit. What is the central force?

- Approach I: Using the equation of motion (2.21):

$$\frac{d^2 r}{d\theta^2} - \frac{2}{r} \left[\frac{dr}{d\theta} \right]^2 - r = \frac{r^4}{h^2} \frac{f(r)}{m} \quad (2.25)$$

$$r = 2a \cos \theta, \quad (2.26)$$

$$\frac{dr}{d\theta} = -2a \sin \theta, \quad (2.27)$$

$$\frac{d^2 r}{d\theta^2} = -2a \cos \theta = -r, \quad (2.28)$$

substituting equations (2.26), (2.27) and (2.28) into (2.25) yields,

$$\begin{aligned} \frac{r^4}{h^2} \frac{f(r)}{m} &= -r - \frac{2}{r} [-2a \sin \theta]^2 - r, \\ &= -r - \frac{2}{r} \cdot 4a^2 \sin^2 \theta - r, \\ &= -r - \frac{2}{r} \cdot 4a^2 [1 - \cos^2 \theta] - r, \\ &= -r - \frac{8a^2}{r} + \frac{8a^2 \cos^2 \theta}{r} - r, \\ &= -2r - \frac{8a^2}{r} + \frac{2r^2}{r}, \\ &= -\frac{8a^2}{r}, \\ \rightarrow f(r) &= -\frac{8mh^2 a^2}{r^5} = \frac{\alpha}{r^5}, \end{aligned} \quad (2.29)$$

where $\alpha = -8mh^2 a^2 = \text{constant}$.

- Approach II: Using the equation of motion (2.24):

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{h^2u^2} \frac{f(\frac{1}{u})}{m}, f(\frac{1}{u}) = -mh^2u^2 \left[\frac{d^2u}{d\theta^2} + u \right]$$

$$u = \frac{1}{r} = \frac{1}{2a} \sec \theta, \quad (2.30)$$

$$\frac{du}{d\theta} = \frac{1}{2a} \sec \theta \tan \theta, \quad (2.31)$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{2a} [\sec^3 \theta + \tan^2 \theta \sec \theta], \quad (2.32)$$

substituting equations (2.30), (2.31), (2.32) into (2.30) yields,

$$\begin{aligned} f\left(\frac{1}{u}\right) &= -mh^2 \left[\frac{1}{4a^2} \sec^2 \theta \right] \left[\frac{1}{2a} [\sec^3 \theta + \tan^2 \theta \sec \theta] + \frac{1}{2a} \sec \theta \right], \\ &= -\frac{mh^2}{8a^3} \sec^2 \theta [\sec^3 \theta + [\sec^2 \theta - 1] \sec \theta + \sec \theta], \\ &= -\frac{mh^2}{8a^3} \sec^2 \theta \cdot 2 \sec^3 \theta = -\frac{mh^2}{4a^3} \sec^5 \theta, \\ &= -\frac{mh^2}{4a^3} \cdot \frac{32a^5}{[2a \cos \theta]^5} = -\frac{8mh^2a^2}{r^5}, \\ \rightarrow f(r) &= \frac{\alpha}{r^5}, \quad \text{where } \alpha = -8mh^2a^2 = \text{constant}. \end{aligned} \quad (2.33)$$

The resulting central force obtained from any of the two approaches is the same since the two equations of motions used in either one of the approaches are equivalent. As such, it is to the convenience of the user to choose the form of the equation to use in calculating the central force.

2.4.5 The law of areas

To prove the third and last property of central force fields, that the radius vector drawn from the fixed point O to the particle, sweeps out equal areas in equal time intervals, we compute the rate at which the radius vector covers a given distance. Consider an object under the influence of a central force that moves it from point P to Q through a small angle $\Delta\theta$ as shown in figure 2.2. The area

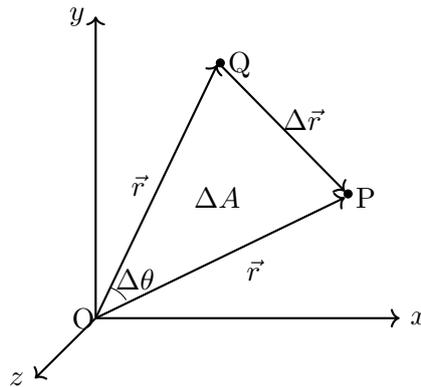


Figure 2.2: A particle under the influence of a central force which moves it from point P to Q with the origin O as the fixed point of the force.

element is given by the relation,

$$\Delta A = |\vec{r} \times \Delta\vec{r}|. \quad (2.34)$$

Supposing the particle took a time Δt in traversing from P to Q , we thus obtain the rate at which area is cut out by the radius vector drawn from the centre of force O to the particle, as the ratio

$\frac{\Delta A}{\Delta t}$ in the limit as $\Delta t \rightarrow 0$,

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \vec{v}|,$$

where $\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$.

Using the expression of the velocity in polar coordinates (see equation (2.6)),

$$\begin{aligned} \vec{r} \times \vec{v} &= \vec{r} \times [\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}], \\ &= r\dot{r}[\hat{r} \times \hat{r}] + r^2\dot{\theta}[\hat{r} \times \hat{\theta}] = r^2\dot{\theta}\hat{k}. \end{aligned} \quad (2.35)$$

Substituting equation (2.35) into (2.34) yields,

$$\dot{A} = \frac{dA}{dt} = \frac{1}{2} |r^2\dot{\theta}\hat{k}| = \frac{1}{2} r^2\dot{\theta}, \quad (2.36)$$

The quantity $\dot{A} = \dot{A}\hat{k} = \frac{1}{2}r^2\dot{\theta}\hat{k}$ is known as the *areal* velocity. Further, making use of the expression in equation (2.17), it results that,

$$\dot{A} = \frac{1}{2}h, \text{ a constant.} \quad (2.37)$$

Hence, the radius vector drawn from the centre of force O to the particle cuts out equal areas in equal time intervals. This proves the third property of central force fields.

2.4.6 Confinement of Central Force fields to Planes

The first property of central force fields as stated in Section (2.3) is that the path of an orbit of a particle under the influence of such forces is in a plane. Here, we proceed to show that once a particle is acted on by a central force, it describes an orbit in a plane and remains in the same plane for all the time that the force remains active. Consider that at the time t_1 a body is contained in a plane Γ which thus contains its position, $\vec{r}(t_1)$ and velocity $\vec{v}(t_1)$ vectors. The angular momentum of the particle is given as,

$$\vec{L}(t_1) = \vec{r}(t_1) \times m\vec{v}(t_1). \quad (2.38)$$

At a later time t_2 , the angular momentum is,

$$\vec{L}(t_2) = \vec{r}(t_2) \times m\vec{v}(t_2). \quad (2.39)$$

However, we have obtained and proved in the previous sections that the angular momentum is a constant of motion for all bodies under the influence of central forces. Thus,

$$\vec{L}(t_1) = \vec{L}(t_2). \quad (2.40)$$

Since the angular momentum vector has remained the same, it has the implication that the position $\vec{r}(t_2)$ and velocity $\vec{v}(t_2)$ vectors are in the same plane Γ as at time t_1 .

2.5 Energy Conservation in Central Force Fields

Apart from the angular momentum which was shown in Section 2.4.3 as being conserved, we here show that energy as would be expected is also conserved in central force fields. To prove this, we start with equations of motion (2.14) and (2.15) as follows:

$$m [r\ddot{\theta} + 2\dot{r}\dot{\theta}] = 0, \quad (2.41)$$

$$m [\ddot{r} - r\dot{\theta}^2] = f(r), \quad (2.42)$$

multiplying equation (2.41) by $r\dot{\theta}$ and equation (2.42) by \dot{r} yielding,

$$m [r^2\dot{\theta}\ddot{\theta} + 2r\dot{r}\dot{\theta}^2] = 0, \quad (2.43)$$

$$m [\dot{r}\ddot{r} - r\dot{\theta}^2] = \dot{r}f(r). \quad (2.44)$$

Adding equations (2.43) and (2.44) yields

$$m [r^2\dot{\theta}\ddot{\theta} + r\dot{r}\dot{\theta}^2 + \dot{r}\ddot{r}] = \dot{r}f(r). \quad (2.45)$$

We now proceed to consider the derivative of the square of the velocity as follows:

$$\begin{aligned} \frac{d}{dt} [\dot{r}^2 + r^2\dot{\theta}^2] &= 2\dot{r}\ddot{r} + 2r\dot{r}\dot{\theta}^2 + 2r^2\dot{\theta}\ddot{\theta}, \\ &= 2 [\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta}], \\ \therefore \dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta} &= \frac{1}{2} \frac{d}{dt} [\dot{r}^2 + r^2\dot{\theta}^2]. \end{aligned} \quad (2.46)$$

Similarly,

$$\frac{d}{dt}f(r) = \frac{dr}{dt} \frac{df}{dr} = \dot{r} \frac{df}{dr}, \quad (2.47)$$

multiplying by dr and integrating yields,

$$\dot{r}f(r) = \frac{d}{dt} \int f(r)dr. \quad (2.48)$$

Substituting equation (2.46) and (2.47) into (2.45) yields,

$$\frac{1}{2}m \frac{d}{dt} [\dot{r}^2 + r^2\dot{\theta}^2] = \frac{d}{dt} \int f(r)dr \quad (2.49)$$

$$\frac{d}{dt} \left[\frac{1}{2}m [\dot{r}^2 + r^2\dot{\theta}^2] - \int f(r)dr \right] = 0 \quad (2.50)$$

The terms in the parenthesis are recognized as the kinetic energy, $KE = \frac{1}{2}mv^2$ and the potential energy $V = -\int f(r)dr$. It is also important to note that such a relationship between the force and potential, V is satisfied by a special class of force fields referred to as *conservative* and so we confirm that central force fields are conservative. The constant in the parenthesis is the total energy, E ,

$$\frac{1}{2}m [\dot{r}^2 + r^2\dot{\theta}^2] - \int f(r)dr = E, \text{ the total energy} \quad (2.51)$$

$$\therefore \frac{d}{dt}E = 0; \quad (2.52)$$

hence the total energy E is conserved and thus a constant of motion. The expression of the total energy, E in equation (2.51) can also be used to find the orbit of the particle in a central force field as follows:

$$\frac{1}{2}m [\dot{r}^2 + r^2\dot{\theta}^2] + V(\vec{r}) = E, \quad (2.53)$$

where we have used the fact that central forces are conservative such that $V = -\int f(r)dr$. If we further make use of the definition of angular momentum as,

$$mr^2\dot{\theta} = \ell. \quad (2.54)$$

This form of the angular momentum is closely related to the earlier definition in equation (2.16) for a unit mass. We therefore obtain,

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(\vec{r}), \\ \rightarrow \dot{r} &= \frac{dr}{dt} = \sqrt{\frac{2}{m} \left[E - V(\vec{r}) - \frac{\ell^2}{2mr^2} \right]^{\frac{1}{2}}}. \end{aligned} \quad (2.55)$$

The expression above is used to obtain the total time over which a particle is under the influence of a central force field as,

$$\begin{aligned} \tau &= \sqrt{\frac{m}{2}} \int \left[E - V(\vec{r}) - \frac{\ell^2}{2mr^2} \right]^{-\frac{1}{2}} dr, \\ &= \sqrt{\frac{m}{2}} \int \frac{dr}{\left[E - V(\vec{r}) - \frac{\ell^2}{2mr^2} \right]^{\frac{1}{2}}}. \end{aligned} \quad (2.56)$$

For completeness, the angular position of a particle in a central force field is obtained from the expression in equation (2.54) such that

$$\begin{aligned} mr^2\dot{\theta} &= \ell, \quad \frac{d\theta}{dt} = \frac{\ell}{mr^2}, \\ \rightarrow \theta &= \int \frac{\ell}{mr^2} dt. \end{aligned} \quad (2.57)$$

2.6 Kepler's Problem

Consider a two body system with masses m_1 and m_2 at the respective positions \vec{r}_1 and \vec{r}_2 as shown in figure 2.3, The relative position for the two masses m_1 and m_2 is

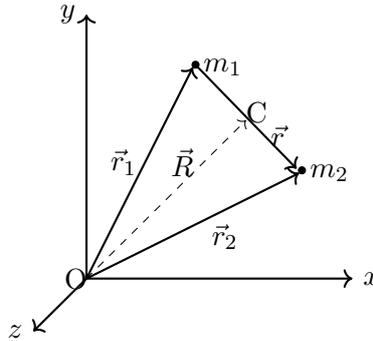


Figure 2.3: Masses m_1 and m_2 at position vectors \vec{r}_1 and \vec{r}_2 . C is the position of the centre of mass and O is the origin of the coordinate system.

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad (2.58)$$

upon which the corresponding potential for the 2 masses depends i.e $V = V(\vec{r})$. The point C represents the centre of mass for the system i.e the position of concentration of the joint mass of the system defined as:

$$\begin{aligned} \vec{R} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \\ \rightarrow M\vec{R} &= m_1\vec{r}_1 + m_2\vec{r}_2, \end{aligned} \quad (2.59)$$

where $M = m_1 + m_2$. Multiplying equation (2.58) by m_1 and m_2 in turns yields the expressions;

$$m_1 \vec{r} = m_1 \vec{r}_1 - m_1 \vec{r}_2, \quad (2.60)$$

$$m_2 \vec{r} = m_2 \vec{r}_1 - m_2 \vec{r}_2, \quad (2.61)$$

Equation (2.59) minus (2.60) yields,

$$\begin{aligned} M\vec{R} - m_1 \vec{r} &= m_2 \vec{r}_2 + m_1 \vec{r}_2 = M\vec{r}_2, \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{M} \vec{r}. \end{aligned} \quad (2.62)$$

Equation (2.59) plus (2.61) yields,

$$\begin{aligned} M\vec{R} + m_2 \vec{r} &= m_1 \vec{r}_1 + m_2 \vec{r}_1 = M\vec{r}_1, \\ \vec{r}_1 &= \vec{R} + \frac{m_2}{M} \vec{r}. \end{aligned} \quad (2.63)$$

From equations (2.62) and (2.63), we obtain,

$$\begin{aligned} \dot{\eta}_1 &= \dot{\vec{R}} + \frac{m_2}{M} \dot{\vec{r}}, \\ \dot{\eta}_2 &= \dot{\vec{R}} - \frac{m_1}{M} \dot{\vec{r}}. \end{aligned} \quad (2.64)$$

To understand the prevailing dynamics in this two-body system, we here use Lagrangian dynamics as discussed earlier. With the system kinetic energy T and potential energy V , the system Lagrangian is as such,

$$\begin{aligned} L &= T - V = \frac{1}{2} m_1 \dot{\eta}_1^2 + \frac{1}{2} m_2 \dot{\eta}_2^2 - V(\vec{r}), \\ &= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - V(\vec{r}), \end{aligned} \quad (2.65)$$

where the reduced mass μ is defined as,

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \quad (2.66)$$

The form of the system Lagrangian in equation (2.65) shows that the system's kinetic energy involves two contributions, namely;

- (i) The joint contribution of the two masses via the reduced mass i.e $(\frac{1}{2} \mu \dot{\vec{r}}^2)$,
- (ii) The appearance of the system to be a single mass concentrated at the centre of mass i.e $(\frac{1}{2} M \dot{\vec{R}}^2)$

It should also be noted that the potential $V(\vec{r})$ is a contribution from the two masses since it is a function of the relative positions of the masses.

With the Lagrangian in equation (2.65), Lagrange's equations of motion are obtained as:

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_\sigma} \right] - \frac{\partial L}{\partial q_\sigma} = 0, \quad (2.67)$$

where q_σ are the generalized coordinates. In the two body system, we have two generalized coordinates i.e $q_1 = \vec{R}$ and $q_2 = \vec{r}$. As such each generalized coordinate q_σ yields one equation of motion obtained as follows:

- (i) For $q_1 = \vec{R}$,

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{\vec{R}}} \right] - \frac{\partial L}{\partial \vec{R}} = 0, \quad (2.68)$$

$$\begin{aligned} \text{with } L &= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r}), \\ \frac{\partial L}{\partial \vec{R}} &= 0, \quad \frac{\partial L}{\partial \dot{\vec{R}}} = M\ddot{\vec{R}}. \end{aligned} \quad (2.69)$$

The equation of motion corresponding to the generalized coordinate \vec{R} is:

$$M\ddot{\vec{R}} = 0. \quad (2.70)$$

Generalized coordinates such as \vec{R} upon which the Lagrangian does not depend are referred to as *cyclic* coordinates.

(ii) For $q_2 = \vec{r}$,

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{\vec{r}}} \right] - \frac{\partial L}{\partial \vec{r}} = 0, \quad (2.71)$$

$$\frac{\partial L}{\partial \vec{r}} = -\frac{\partial V(\vec{r})}{\partial \vec{r}}, \quad \frac{\partial L}{\partial \dot{\vec{r}}} = \mu\dot{\vec{r}}, \quad (2.72)$$

the equation of motion is obtained as:

$$\begin{aligned} \mu\ddot{\vec{r}} - \frac{\partial V}{\partial \vec{r}} &= 0, \\ \therefore \mu\ddot{\vec{r}} &= -\frac{\partial V}{\partial \vec{r}} = F(\vec{r}), \end{aligned} \quad (2.73)$$

where the latter equality asserts the conservative nature of central forces.

Equations (2.70) and (2.73) have the same form as the equations of motion discussed in the introduction. Equation (2.70) is a statement of the ensuing dynamics at an angle, essentially stating that angular momentum is conserved. Instead, equation (2.73) is a statement of Newton's second law of motion along the radial direction i.e ($F = m\vec{a}$).

Considering the two-body system in the centre of mass frame, in which case the origin is shifted to the centre of mass position such that $\vec{R} = 0$. The Lagrangian takes the form,

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r}), \quad (2.74)$$

which makes the problem much easier to solve compared to its fully fledged form in equation (2.65).

2.7 Particle trajectory in a Central Force Field

In this section, we want to obtain the orbit described by a particle under the influence of a central force field. As would be anticipated, the starting point is and should be any conservation statement arising from or directly related to the equations of motion. We therefore make use of equation (2.55) which was obtained as a statement of conservation of total energy in the system.

$$\dot{r} = \sqrt{\frac{2}{m} \left[E - V(\vec{r}) - \frac{\ell^2}{2mr^2} \right]^{\frac{1}{2}}}, \quad (2.75)$$

where E is the total energy, $V(\vec{r})$ is the potential energy, m is the mass of the particle and ℓ is the angular momentum, which we found to be conserved. Further, consider the two equations obtained earlier, the second of which is a restatement of the energy equation (2.75),

$$mr^2\dot{\theta} = \ell, \quad (2.76)$$

$$\frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) = E, \quad (2.77)$$

Manipulation of equation (2.77) yields the particle velocity as,

$$\dot{r} = \pm \sqrt{\frac{2}{m} \left[E - \frac{\ell^2}{2mr^2} - V(r) \right]^{\frac{1}{2}}}. \quad (2.78)$$

It is quickly observed that equation (2.78) can be solved by separation of variables and subsequent integration to yield,

$$\int \frac{dr}{\left[E - \frac{\ell^2}{2mr^2} - V(r) \right]^{\frac{1}{2}}} = \pm \sqrt{\frac{2}{m}} \int_{t_0}^t dt' = \pm [t - t_0] \sqrt{\frac{2}{m}} \quad (2.79)$$

By solving equation (2.79), we can obtain the trajectory of the particle as a function of time, that is, $r(t)$. The degree of ease or difficulty of calculating this integral depends on the form of the potential $V(r)$. As such, common considerations of equation (2.79) involve potentials of rather simple forms.

Alternatively, the particle trajectory could be obtained as a function of the second variable, that is, θ . In what follows, we discuss how the particle trajectory can be obtained for a general potential $V(r)$ before applying it to a specific form of potential which is of special interest to Physicists. From equation (2.77), we obtain,

$$\dot{r}^2 = \left[\frac{dr}{dt} \right]^2 = \frac{2}{m} \left[E - \frac{\ell^2}{2mr^2} - V(r) \right]. \quad (2.80)$$

Also from equation (2.76), we obtain

$$\frac{d\theta}{dt} = \frac{\ell}{mr^2}, \quad \left[\frac{d\theta}{dt} \right]^2 = \frac{\ell^2}{m^2 r^4}, \quad (2.81)$$

dividing equation (2.80) by (2.81) yields,

$$\begin{aligned} \left[\frac{dr}{d\theta} \right]^2 &= \left[\frac{2}{m} \right] \cdot \left[\frac{m^2 r^4}{\ell^2} \right] \cdot \left[E - \frac{\ell^2}{2mr^2} - V(r) \right], \\ \therefore \left[\frac{1}{r^2} \frac{dr}{d\theta} \right]^2 &= \frac{2m}{\ell^2} \left[E - \frac{\ell^2}{2mr^2} - V(r) \right] = \frac{2mE}{\ell^2} - \frac{1}{r^2} - \frac{2mV(r)}{\ell^2}. \end{aligned} \quad (2.82)$$

To this end, if the form of the potential $V(r)$ and the value of the energy E are given, equation (2.82) can be solved by separation of variables to obtain $r(\theta)$. In what follows, we solve equation (2.82) for an exemplary form of $V(r)$, that is, the gravitational potential. The potential arising from the gravitational force field takes the form,

$$V(r) = -\frac{K}{r}, \quad (2.83)$$

substituting equation (2.83) into (2.82) gives,

$$\left[\frac{1}{r^2} \frac{dr}{d\theta} \right]^2 = \frac{2m}{\ell^2} \left[E - \frac{\ell^2}{2mr^2} - V(r) \right] = \frac{2mE}{\ell^2} - \frac{1}{r^2} + \frac{2mK}{r\ell^2} \quad (2.84)$$

By inspection, the term $1/r^2$ is found to be recurrent in equation (2.84) prompting a consideration for a change of variables as follows:

$$\text{Let } u = \frac{1}{r}, \quad (2.85)$$

$$du = -\frac{1}{r^2} dr, \quad \rightarrow dr = -r^2 du, \quad (2.86)$$

substituting equations (2.85) and (2.86) into (2.84) yields,

$$\left[\frac{du}{d\theta}\right]^2 = \frac{2mE}{\ell^2} - u^2 + \frac{2mK}{\ell^2}u, \quad (2.87)$$

Taking square roots and separating variables is the quickest but rather lengthy approach to this problem at this point and is left to the reader as an assignment. Here, we instead stick to equation (2.87) re-arranging terms on the left hand side of it to read as,

$$\left[\frac{du}{d\theta}\right]^2 = -\left[u^2 - \frac{2mK}{\ell^2}u\right] + \frac{2mE}{\ell^2}. \quad (2.88)$$

The term in the parenthesis can be simplified further using completion of squares to the form,

$$u^2 - \frac{2mK}{\ell^2}u = \left[u - \frac{mK}{\ell^2}\right]^2 - \left[\frac{mK}{\ell^2}\right]^2, \quad (2.89)$$

and substituting equation (2.89) into (2.88) yields,

$$\left[\frac{du}{d\theta}\right]^2 = -\left[u - \frac{mK}{\ell^2}\right]^2 + \left[\frac{mK}{\ell^2}\right]^2. \quad (2.90)$$

The form of equation (2.90) prompts another change of variables which we carry out as follows;

$$\text{let } w = u - \frac{mK}{\ell^2}, \quad \rightarrow dw = du. \quad (2.91)$$

Substituting equation (2.91) into (2.90) yields,

$$\left[\frac{dw}{d\theta}\right]^2 = -w^2 + \left[\frac{mK}{\ell^2}\right]^2 \left[1 + \frac{2E\ell^2}{mK^2}\right], \quad (2.92)$$

$$\left[\frac{dw}{d\theta}\right]^2 = -w^2 + \xi, \quad \text{where } \xi = \left[\frac{mK}{\ell^2}\right]^2 \left[1 + \frac{2E\ell^2}{mK^2}\right]. \quad (2.93)$$

Next we consider taking square roots and separating variables to obtain upon taking the integral,

$$\int_{w_0}^w \frac{dw'}{\sqrt{\xi - w'^2}} = \int_{\theta_0}^{\theta} d\theta'. \quad (2.94)$$

Equation (2.94) is solved by the $\sin \theta$ or $\cos \theta$ substitutions which should be straight forward and is left for the reader as an exercise. Special attention must however be given to the change of limits when the substitution is made. When properly done, the integral in equation (2.94) yields,

$$w = \sqrt{\xi} \cos(\theta - \theta_0) = \sqrt{\xi} \cos \theta, \quad (2.95)$$

where the second equality is obtained by setting the problem such that $\theta_0 = 0$. Substituting for ξ and w in equation (2.95) gives,

$$\begin{aligned} u &= \frac{mK}{\ell^2} + \frac{mK}{\ell^2} \cdot \left[1 + \frac{2E\ell^2}{mK^2}\right]^{\frac{1}{2}} \cos \theta, \\ &= \left[\frac{mK}{\ell^2}\right] \left[1 + \left[1 + \frac{2E\ell^2}{mK^2}\right]^{\frac{1}{2}} \cos \theta\right]. \end{aligned} \quad (2.96)$$

Letting,

$$\epsilon = \left[1 + \frac{2E\ell^2}{mK^2} \right]^{\frac{1}{2}}, \quad (2.97)$$

$$u = \left[\frac{mK}{\ell^2} \right] [1 + \epsilon \cos \theta]. \quad (2.98)$$

The quantity ϵ is called the *eccentricity* of the motion described by a particle under the influence of a central force. By inverting u , we obtain the particle's orbit $r(\theta) = 1/u(\theta)$ as,

$$r(\theta) = \frac{\ell^2}{mK} \cdot \frac{1}{1 + \epsilon \cos \theta}, \quad (2.99)$$

and choosing,

$$p = \frac{\ell^2}{mK}, \quad (2.100)$$

we obtain,

$$\boxed{r(\theta) = \frac{p}{1 + \epsilon \cos \theta}}. \quad (2.101)$$

2.7.1 The Effective Potential

The position p corresponds to the location of the particle when the effective potential assumes its minimum value and the effective force is zero. The effective potential $U(r)$ is defined from the expression of the total energy, E in equation (2.77). Considering that the first term is the kinetic energy, the remaining terms are combined together to refer to the effective potential $U(r)$. Hence,

$$\boxed{U(r) = \frac{\ell^2}{2mr^2} + V(r)} \quad (2.102)$$

As can be observed outrightly, introduction of the effective potential into the energy conservation equation turns the problem into a one-dimensional one with the angular dimension disappearing. The effective potential is important because it helps to tell the form of the orbit of the particle without solving equation (2.101). With the introduction of the effective potential, the energy equation reads as,

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 + U(r), \\ \rightarrow \dot{r} &= \sqrt{\frac{2}{m} [E - U(r)]^{\frac{1}{2}}}, \end{aligned} \quad (2.103)$$

from which it is deduced that acceptable orbits occur only when $\dot{r} \geq 0$, which in turn implies that,

$$E \geq U(r). \quad (2.104)$$

This condition has the implication that admissible orbits are found at values of $U(r)$ lower than the selected value of the total energy E . We proceed to consider the form of the effective potential given a gravitational potential which takes the form $V(r) = -K/r$, such that,

$$U(r) = \frac{\ell^2}{2mr^2} - \frac{K}{r},$$

The minimum value U_{\min} of $U(r)$ is obtained as

$$U_{\min} = -\frac{mK^2}{2\ell^2}, \quad (2.105)$$

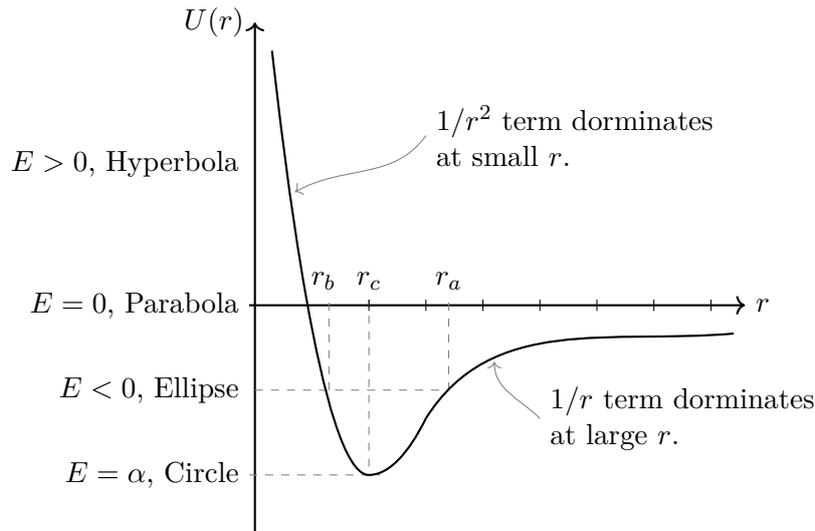


Figure 2.4: A sketch of the Effective potential $U(r)$ for a gravitational $V(r)$.

and it occurs at a constant radius,

$$r_c = \frac{\ell^2}{mK}, \quad (2.106)$$

obtaining the above expressions is straight forward and is left as an exercise for the reader. The resulting orbit in this case is a circle with a constant radius r_c . For $E < 0$, two values of radii are possible resulting into an elliptic orbit of semi-major axis r_a and semi-minor axis r_b .

2.7.2 Conic Sections

In equation (2.101), we have obtained the equation of the trajectory of a particle under the influence of a central force. Although it is out of the scope of this monograph, the same equation which we shall state again for emphasis represents orbits of particles in a plane, whose emergence may or may not be caused by the action of central forces. In polar coordinates (r, θ) , the orbit takes the form,

$$r(\theta) = \frac{p}{1 + \epsilon \cos \theta}, \quad (2.107)$$

which we have obtained using the equations of motion in the previous section. The same expression can be obtained from a geometrical consideration with the use of figure 2.5 but this is left as an exercise to the reader. The shape of the orbit depends on the constant ϵ known as the eccentricity, defined as,

$$\epsilon = \frac{d_2}{d_1}, \quad (2.108)$$

the ratio of the distance of the particle from the fixed point O to the fixed axis AB. The point O is called the focus and the fixed axis AB is called the directrix. The particle trajectory could take any of the following geometries depending on the value of ϵ as follows;

- Circle when $\epsilon = 0$,
- Ellipse when $0 < \epsilon < 1$,
- Parabola when $\epsilon = 1$,
- Hyperbola when $\epsilon > 1$.

2.7.3 Kepler's Laws of Planetary Motion

Kepler - a German astrophysicist of the 16th Century put forward statements, which according to the time, were deemed to govern the motion of planets around the sun. In relation to central forces

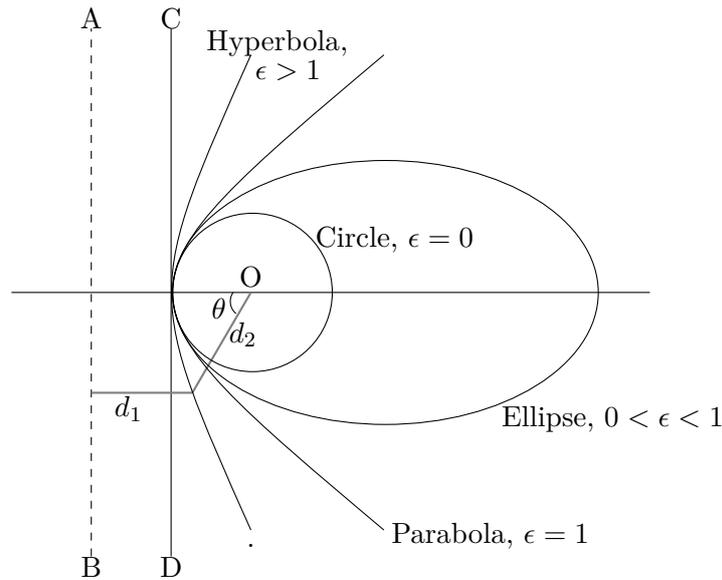


Figure 2.5: Probable orbits of a particle under the influence of a central force.

discussed this far, the position of the sun acting as the focus O . The statements (now taken as law) given by Kepler are:

1. Planets move in elliptical orbits with the sun as the focus. This has since been referred to as the law of ellipses.
2. The radius vector drawn from the planet to the sun cuts out equal areas in equal time intervals. This is called the law of equal areas.
3. The squares of the periods of revolution of the planets are proportional to the cubes of the elliptical semi-major axis.

The first and second of Kepler's laws need not to be proved again as they have already been covered in the context of central forces. Instead, we proceed to prove Kepler's third law of planetary motion as follows: The law of areas proved in Section 2.4.5 which in fact directly corresponds to Kepler's second law, showed that the areal velocity,

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} \hat{k} = \frac{1}{2} h \hat{k},$$

The time a planet takes to complete a single revolution can thus be obtained as the ratio, $T = \text{area}/\text{areal velocity}$. Consider an ellipse with semi-major and semi-minor axes a and b respectively (see figure 2.6). In this case,

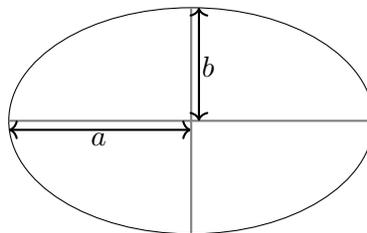


Figure 2.6: An ellipse with semi-major axis a and semi-minor axis b .

$$\begin{aligned} T &= \frac{A}{\dot{A}} = \frac{\pi ab}{\frac{1}{2} \ell}, \\ &= \frac{2\pi ab}{\ell}. \end{aligned}$$

It can however be shown that for an ellipse, $b = a[1 - \epsilon^2]$, such that,

$$\begin{aligned}\tau &= \frac{2\pi a^2[1 - \epsilon^2]}{\ell} = \frac{2\pi a^2[1 - \epsilon^2]}{\sqrt{pmK}}, \\ &= \frac{2\pi a^2[1 - \epsilon^2]}{\sqrt{mK} \cdot [a[1 - \epsilon^2]]^{\frac{1}{2}}} = \frac{2\pi a^{\frac{3}{2}}[1 - \epsilon^2]^{\frac{1}{2}}}{\sqrt{mK}},\end{aligned}\tag{2.109}$$

where we have used equation (2.100) to obtain $\ell = \sqrt{pmK}$ and that $p = a[1 - \epsilon^2]$ for an ellipse. Squaring both sides of equation (2.109) yields,

$$\begin{aligned}\tau^2 &= \frac{4\pi^2 a^3[1 - \epsilon^2]}{mK} = \beta a^3, \\ \text{where } \beta &= \frac{4\pi^2[1 - \epsilon^2]}{mK} = \text{constant}, \\ \therefore \boxed{\tau^2 \propto a^3}\end{aligned}\tag{2.110}$$

hence Kepler's third law.

Further Reading

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