UGANDA ADVANCED LEVEL CERTIFICATE OF EDUCATION

PURE MATHEMATICS ITEMS WITH POSSIBLE RESPONSES:

"Revise, Reflect, Achieve"

PART I: 3D VECTORS:

ITEM ONE:

In Ntake village, O is the trading center. A and B are two homesteads.
The position vectors of two homesteads are such that; $\boldsymbol{OA} = \boldsymbol{a}$ and $\boldsymbol{OB} = \boldsymbol{b}$.
A community notice board G is placed on the line OA such that; $\overrightarrow{OG} = 2\overrightarrow{OA}$.
Point M is the midpoint of AB.
The village chief's palace is along the line OB such that a straight footpath from G to M connects to it at N.

Tasks:

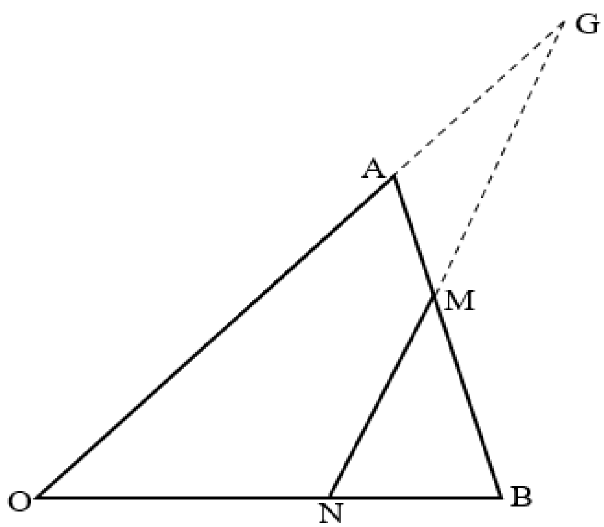
a). Find; $\overrightarrow{\mathrm{GM}}$, in terms of; $\boldsymbol{a}$ and $\boldsymbol{b}$.

b). Show that the position vector of the village chief's palace is given by;

$$\overrightarrow{ON} = \left(2 - \tfrac{3}{2}\beta\right)\boldsymbol{a} + \tfrac{1}{2}\beta\boldsymbol{b}$$

c). Hence, show that; $\boldsymbol{ON}:\boldsymbol{NB} = 2:1$

<u>Possible Responses:</u>



a). $\overrightarrow{\mathrm{AM}}:\overrightarrow{\mathrm{MB}} = 1:1 \;\Rightarrow \overrightarrow{\mathrm{AM}} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{\mathrm{OM}} = \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AM}} = \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$\Rightarrow \overrightarrow{\mathrm{OM}} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

Thus;

$\overrightarrow{\mathrm{GM}} = \overrightarrow{\mathrm{OM}} - \overrightarrow{\mathrm{OG}} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) - 2\boldsymbol{a}$

$\therefore \overrightarrow{\mathrm{GM}} = \frac{1}{2}(\boldsymbol{b} - 3\boldsymbol{a})$

b). Let; $\overrightarrow{\mathrm{GN}} = \beta\,\overrightarrow{\mathrm{GM}}$

$\overrightarrow{\mathrm{ON}} = \overrightarrow{\mathrm{OG}} + \overrightarrow{\mathrm{GN}} = \overrightarrow{\mathrm{OG}} + \beta\,\overrightarrow{\mathrm{GM}}$

$\overrightarrow{\mathrm{ON}} = 2\boldsymbol{a} + \beta\left[\frac{1}{2}(\boldsymbol{b} - 3\boldsymbol{a})\right]$

$\therefore \overrightarrow{\mathrm{ON}} = \left(2 - \frac{3}{2}\beta\right)\boldsymbol{a} + \frac{1}{2}\beta\boldsymbol{b}$

Hence proved.

c). Let; $\overrightarrow{ON} = \lambda\overrightarrow{OB}$

$\Rightarrow \overrightarrow{ON} = \lambda\boldsymbol{b}$ ....(*i*)

We thus equate the two expressions of $\overrightarrow{ON}$:

$\Rightarrow \left(2 - \frac{3}{2}\beta\right)\boldsymbol{a} + \frac{1}{2}\beta\boldsymbol{b} = \lambda\boldsymbol{b}$...........................(*ii*)

Equating corresponding co − efficients of; $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *in eqn* (*ii*)

For $\boldsymbol{a}$; $\left(2 - \frac{3}{2}\beta\right) = 0$

$\Rightarrow \beta = \frac{4}{3}$

For $\boldsymbol{b}$; $\frac{1}{2}\beta = \lambda$

$\frac{1}{2}\left(\frac{4}{3}\right) = \lambda$

$\Rightarrow \lambda = \frac{2}{3}$

Thus;

$\Rightarrow \overrightarrow{ON} = \frac{2}{3}\boldsymbol{b}$ [*From eqn* (*i*)]

Also;

$\overrightarrow{NB} = -\overrightarrow{ON} + \overrightarrow{OA} + \overrightarrow{AB} = -\frac{2}{3}\boldsymbol{b} + \boldsymbol{a} + (\boldsymbol{b} - \boldsymbol{a})$

$\Rightarrow \overrightarrow{NB} = \frac{1}{3}\boldsymbol{b}$

Hence; $\overrightarrow{ON}:\overrightarrow{NB} = \frac{\left(\frac{2}{3}\boldsymbol{b}\right)}{\left(\frac{1}{3}\boldsymbol{b}\right)} = 2:1$

$\therefore \overrightarrow{ON}:\overrightarrow{NB} = 2:1$ ***Hence, shown.***

ITEM TWO:

In a community, there are only two main roads; l_1 and l_2 which intersect at a trading center. Road l_1 starts from a point with a position vector; $\boldsymbol{8i - (k + j)}$ and is in the direction; $\boldsymbol{3i - j + k}$.

Road l_2 is represented by; $\frac{y-11}{2} = \frac{x+3}{-1} = z - 2$.

A community church is located along the road; l_1 such that it's $4\sqrt{11}$ metres from a point; $(8, -1, -1)$.

One unit represents one metre

Tasks:

a). Find the position vector of the community's trading center.

b). Find the coordinates of the community Church.

c). Hence, write the parametric equation that represents the community's flat surface.

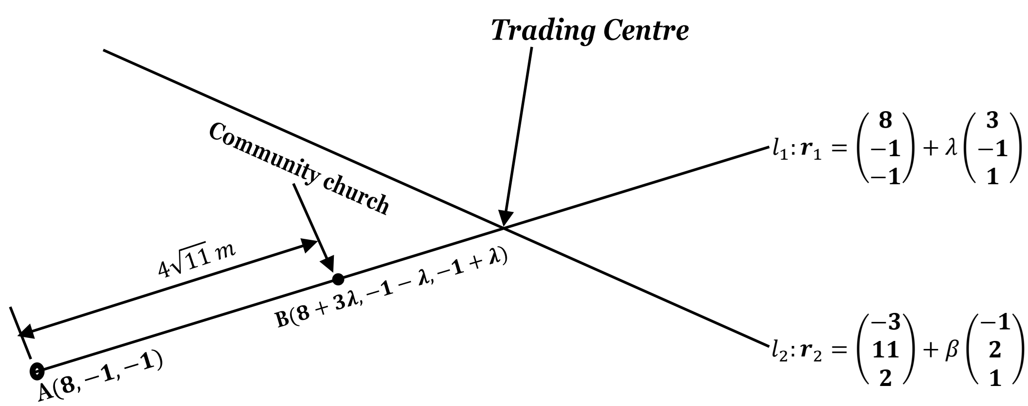
d). Obtain the cartesian of the community from c) above.

Figure 1: *An aerial picture of a community in Buziga village in Lyantonde district, Jan 2026*

Possible Responses:

$$\Rightarrow\ l_1: \boldsymbol{r}_1 = \begin{pmatrix} \mathbf{8} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$l_2: \boldsymbol{r}_2 = \begin{pmatrix} -3 \\ 11 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$



a). At the trading centre; $\boldsymbol{r}_1 = \boldsymbol{r}_2$

$$\Rightarrow \begin{pmatrix} \mathbf{8} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{3} \\ \mathbf{-1} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{-3} \\ \mathbf{11} \\ \mathbf{2} \end{pmatrix} + \beta \begin{pmatrix} \mathbf{-1} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix}$$

Equating co – efficients of corresponding unit vectors;

For $\boldsymbol{i}$; $8 + 3\lambda = -3 - \beta$

$\Rightarrow \beta + 3\lambda = -11$ ……………….. (i)

For $\boldsymbol{j}$; $-1 - \lambda = 11 + 2\beta$

$\Rightarrow \lambda + 2\beta = -12$ ……….……… (ii)

For $\boldsymbol{k}$; $-1 + \lambda = 2 + \beta$

$\Rightarrow \lambda = 3 + \beta$ ……………………. (iii)

Solving equations; (i) and (iii) simultaneously.

$\Rightarrow \beta + 3(3 + \beta) = -11$

$4\beta = -20$

$\therefore \beta = -5$

$\Rightarrow \lambda = 3 + (-5)$

$\therefore \lambda = -2$

Using the testing equation: $[eqn\ (ii)]$

$\lambda + 2\beta = -12$

$\Rightarrow -2 + 2(-5) = -12$: $\boldsymbol{which\ holds.}$

Thus, the two roads indeed intersect at the trading centre.

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix} + -2\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$\therefore$ The position vector of the trading center is; $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$.

b). We can easily see that; $|\overrightarrow{AB}| = 4\sqrt{11}$: $[From\ the\ diag\ above]$

We thus find; $\overrightarrow{AB}$ first.

$$\Rightarrow \overrightarrow{AB} = \begin{pmatrix} 8 + 3\lambda \\ -1 - \lambda \\ -1 + \lambda \end{pmatrix} - \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ -\lambda \\ \lambda \end{pmatrix}$$

Hence, $\sqrt{(3\lambda)^2 + (-\lambda)^2 + \lambda^2} = 4\sqrt{11}$

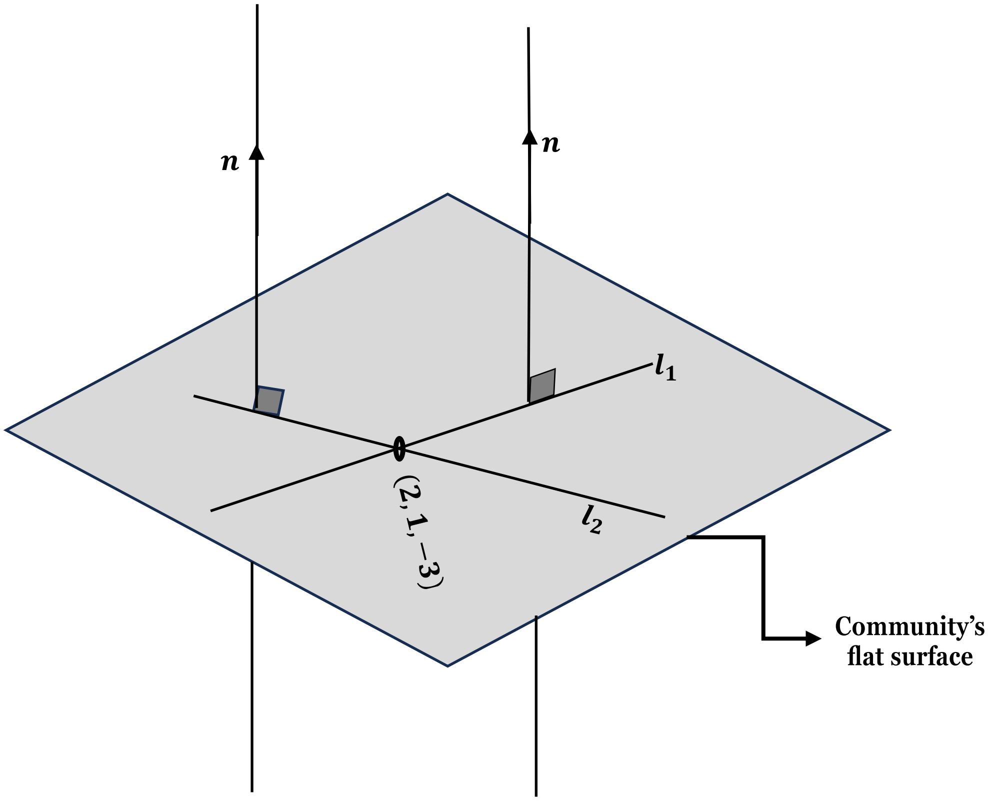
$\Rightarrow 11\lambda^2 = \left(4\sqrt{11}\right)^2$

$\lambda = 4$

$$\boldsymbol{r} = \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ -5 \\ 3 \end{pmatrix}$$

$\therefore$ The coordinates of the community church are; $(20, -5, 3)$.

c).



The parametric equation that represents the community's flat surface becomes;

$$\boldsymbol{r} = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{-3} \end{pmatrix} + t\begin{pmatrix} \mathbf{3} \\ \mathbf{-1} \\ \mathbf{1} \end{pmatrix} + \gamma\begin{pmatrix} \mathbf{-1} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix}$$

d). $\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = \boldsymbol{i}(-1-2) - \boldsymbol{j}(3+1) + \boldsymbol{k}(6-1)$

$\boldsymbol{n} = -3\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$

Then;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} . \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$$

Thus, the cartesian equation becomes; $3x + 4y - 5z = 25$.

ITEM THREE:
In a chemical plant, two chemical supply pipes; A and B run through the factory ceiling.
The Maintenance team wants to install the shortest possible straight support connecting the two pipes to maximize pipes' stability.
The pipe paths (**in metres**) are represented as;

- Pipe A: $\boldsymbol{r_A} = (8 + 3\lambda)\boldsymbol{i} + (10 + 7\lambda)\boldsymbol{k} - (16\lambda + 9)\boldsymbol{j}$
- Pipe B: $\boldsymbol{r_B} = 15\boldsymbol{i} + 29\boldsymbol{j} + 5\boldsymbol{k} + \mu(3\boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k})$

Figure 2: *An interior design (with pipes) of one of the containment buildings in Safic – Alcan Southern Chemical plant, Sandton, South Africa*

As the top Engineer, you are required to;

- confirm that the pipes do not intersect.
- find the specific coordinates on each of the chemical supply pipes onto weld the bar (support).
- calculate the exact length of the shortest possible straight support (bar).

Task(s):

Help the Maintenance team.

Possible Responses:

$\boldsymbol{r_A} = (8 + 3\lambda)\boldsymbol{i} + (10 + 7\lambda)\boldsymbol{k} - (16\lambda + 9)\boldsymbol{j}$
$\boldsymbol{r_B} = 15\boldsymbol{i} + 29\boldsymbol{j} + 5\boldsymbol{k} + \mu(3\boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k})$

At point of contact between Pipe A and Pipe B; $\boldsymbol{r}_A = \boldsymbol{r}_B$

$\Rightarrow \begin{pmatrix} 8 \\ -9 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$

Equating co – efficients of corresponding unit vectors.

For $\boldsymbol{i}$; $8 + 3\lambda = 15 + 3\mu$

$\Rightarrow 3\lambda - 3\mu = 7$ ……………………………….. (i)

For $\boldsymbol{j}$; $-9 - 16\lambda = 29 + 8\mu$

$\Rightarrow 16\lambda + 8\mu = -38$ …………………………… (ii)

For $\boldsymbol{k}$; $10 + 7\lambda = 5 - 5\mu$

$\Rightarrow 7\lambda + 5\mu = -5$ ……………………………….. (iii)

Solving equations; (i) and (ii) simultaneously gives;

$\therefore \lambda = -\frac{29}{36}$ and $\mu = -\frac{113}{36}$

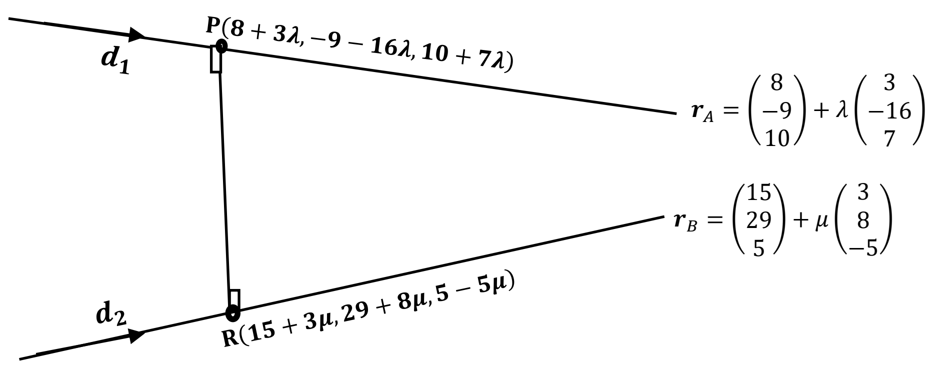
Using a testing equation: $[Eqn\ (iii)]$

$\Rightarrow 7\left(-\frac{29}{36}\right) + 5\left(-\frac{113}{36}\right) = -5$

$\frac{64}{3} \neq -5$: ***which doesn't hold.***

Thus, the two pipes do not intersect at all since; $LHS \neq RHS$.

$\therefore$ Two pipes are thus skew.



Displacement vector; $\overrightarrow{PR} = \begin{pmatrix} 15 + 3\mu \\ 29 + 8\mu \\ 5 - 5\mu \end{pmatrix} - \begin{pmatrix} 8 + 3\lambda \\ -9 - 16\lambda \\ 10 + 7\lambda \end{pmatrix} = \begin{pmatrix} 7 + 3\mu - 3\lambda \\ 38 + 8\mu + 16\lambda \\ -5 - 5\mu - 7\lambda \end{pmatrix}$

Since; $\overrightarrow{PR}$ is orthogonal to both pipes, then;

- $\overrightarrow{PR}.\boldsymbol{d_1} = 0$

$$\Rightarrow \begin{pmatrix} 7+3\mu-3\lambda \\ 38+8\mu+16\lambda \\ -5-5\mu-7\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} = 0$$

$157\lambda + 77\mu = -311$ .... (i)

- $\overrightarrow{\mathrm{PR}} . \boldsymbol{d_2} = 0$

$$\Rightarrow \begin{pmatrix} 7+3\mu-3\lambda \\ 38+8\mu+16\lambda \\ -5-5\mu-7\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} = 0$$

$11\lambda + 7\mu = -25$ .... (ii)

Solving the two equations; (i) and (ii) simultaneously gives;

$\lambda = -1$ and $\mu = -2$

Thus;

- $\boldsymbol{r}_A = \begin{pmatrix} 8 \\ -9 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \\ 10 \end{pmatrix} - 1 \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$
- $\boldsymbol{r}_B = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 15 \end{pmatrix}$

∴ The two closest points are; $(5, 7, 3)$ on pipe A, and; $(9, 13, 15)$ on pipe B.

Exact length of the support (bar) is the distance between the two closest points calculated above.

$\Rightarrow$ Exact length $= \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2}$
$= \sqrt{4^2 + 6^2 + 12^2}$
∴ Exact length $= 14m$

ITEM FOUR:

Joshua is using a drill bit to create small holes in building wall for easy passage of wire cables.

The drill bit (long and straight) is represented by a vector equation;

$$\boldsymbol{r} = (2+3t)\boldsymbol{i} - \boldsymbol{j}(1+t) - \boldsymbol{k}(t-4),$$

and it meets a rigid wall surface represented by a cartesian equation;

$$-3y + 4x + 5z + 34 = 0.$$

Figure 3: ***Joshua drilling the wall surface in one of the buildings in Mitooma Town Council, 2026.***

Task(s):

a). At what point does the drill tip penetrate the wall surface?
And calculate the acute angle between the drill bit and the wall surface.

b). Show that the two points with respective position vectors; $\boldsymbol{3i+2j+k}$ and $\boldsymbol{10j+i-3k}$**,** lie on opposite sides of the wall surface.

<u>Possible Responses:</u>

a). For the drill bit;

$$\boldsymbol{r}=\begin{pmatrix}2\\-1\\4\end{pmatrix}+t\begin{pmatrix}3\\-1\\-1\end{pmatrix}\;\Rightarrow\;\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-1\\4\end{pmatrix}+t\begin{pmatrix}3\\-1\\-1\end{pmatrix}$$

Thus; $\left.\begin{aligned}x&=2+3t\\y&=-1-t\\z&=4-t\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$

Substituting $eqn\ (i)$ in the wall surface equation;

$$-3(-1-t)+4(2+3t)+5(4-t)+34=0$$

$$\Rightarrow 10t+65=0$$

$$\therefore\ \ t=-6.5$$

Then; $\boldsymbol{r}=\begin{pmatrix}2\\-1\\4\end{pmatrix}+t\begin{pmatrix}3\\-1\\-1\end{pmatrix}=\begin{pmatrix}2\\-1\\4\end{pmatrix}-6.5\begin{pmatrix}3\\-1\\-1\end{pmatrix}=\begin{pmatrix}-17.5\\5.5\\10.5\end{pmatrix}$

Thus, the point of contact between the drill tip and the wall surface is; $(-17.5, 5.5, 10.5)$.

Let θ be the acute angle between the drill bit and the wall surface.

$$\Rightarrow \theta = \sin^{-1}\left\{\frac{\left|\begin{pmatrix}4\\-3\\5\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\-1\end{pmatrix}\right|}{\sqrt{4^2+(-3)^2+5^2}\times\sqrt{3^2+(-1)^2+(-1)^2}}\right\}$$

$$\theta = \sin^{-1}\left\{\frac{10}{\sqrt{550}}\right\}$$

$$\therefore \ \theta \approx 25.2^0$$

Thus, the acute angle between the drill pit and the wall surface is; 25.2^0

b). Let the shortest ***displacement*** of the point; $(3, 2, 1)$ from the wall surface be; d_1

$$d_1 = \frac{4(3) - 3(2) + 5(1) + 34}{\sqrt{4^2 + (-3)^2 + 5^2}}$$

$$\Rightarrow d_1 = \frac{45}{\sqrt{30}} \ units$$

Also; let the shortest ***displacement*** of the point; $(1, 10, -3)$ from the wall surface be; d_2.

$$d_2 = \frac{4(1) - 3(10) + 5(-3) + 34}{\sqrt{4^2 + (-3)^2 + 5^2}}$$

$$\Rightarrow d_2 = \frac{-7}{\sqrt{30}} \ units$$

$\therefore$ Since d_1 and d_2 have opposite signs, then the two points with respective position vectors; $\boldsymbol{3i + 2j + k}$ and $\boldsymbol{10j + i - 3k}$, lie on opposite sides of the wall surface.

PART II: DIFFERENTIATION:

ITEM ONE:

A solid glass cylinder, which is used in an expensive laser amplifier, has a volume of $75\pi\ cm^3$.

The cost of polishing the surface area of this glass cyclinder is; £2 per cm^2 for the curved surface area, and £3 per cm^2 for the circular top and base areas.

The radius of the cylinder is $r\ cm$.

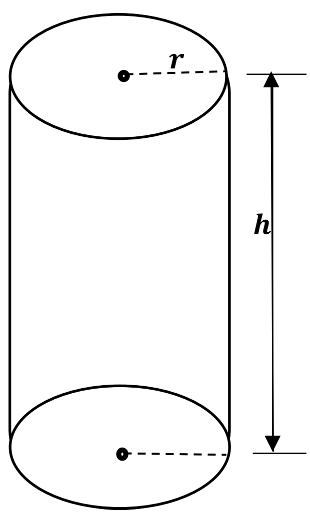
Tasks:

a). Show that the cost of polishing, £C, is given by; $\text{C} = 6\pi r^2 + \frac{300\pi}{r}$.

b). Use calculus to find the minimum cost of the polishing, giving your answers to the nearest pound.

c). Justify that the answer obtained in b) above is a minimum.

Possible Responses:



a). $V = \pi r^2 h$

$75\pi = \pi r^2 h$

$\Rightarrow h = \frac{75}{r^2}$ ....(i)

Total cost, $\text{C}(r)$ is calculated as;

$\text{C}(r, h) = 2\pi r h(2) + 2\pi r^2(3)$

$\Rightarrow \text{C}(r) = 4\pi r\left(\frac{75}{r^2}\right) + 6\pi r^2$

$\therefore\ \text{C}(r) = 6\pi r^2 + \frac{300\pi}{r}$: ***Hence, shown:***

b). $\frac{d}{dr}[\text{C}(r)] = \text{C}'(r) = 12\pi r - \frac{300\pi}{r^2}$

For minimum cost, $\text{C}'(r) = 0$

$\Rightarrow 12\pi r = \frac{300\pi}{r^2}$

$r^3 = 25$

$\therefore\ r \approx 2.924\ cm$

Thus, the minimum cost, $C_{min} = 6\pi(2.924)^2 + \frac{300\pi}{2.924}$

$$C_{min} \approx \textbf{£483}$$

c). $C''(r) = 12\pi + \frac{600\pi}{r^3}$

$C''(2.924) = 12\pi + \dfrac{600\pi}{(2.924)^3}$

$C''(2.924) \approx 113.0973$, ***which is positive:***

Thus, the cost calculated in b) above is minimum.

ITEM TWO:

Mr. Jotta is a wholesale trader in Lira City who offers free goods – packaging to every customer that orders in large quantities.

A customer has ordered for commodities from him and packaging requires a rectangular container (with a square base) of volume, 80 m^3.

The top and the base of the box are to be covered with lacquer that costs **UGX. 4000** per m^2, and the sides with lacquer that costs **UGX. 400** per m^2.

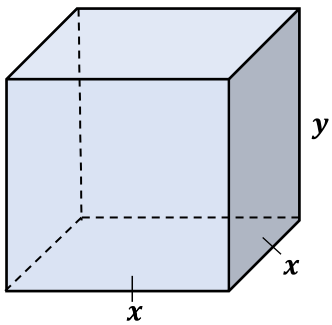
However, Mr. Jotta wants to incur less costs on packaging for his customer.

Figure 4: ***Mr. Jotta puzzled by the customer's order at his shop, Lira City, 2026***

Task(s):

Help Mr. Jotta come up with dimensions of the rectangular container that fit in his plan.

Possible Responses:



$Volume, V = x^2 y$

$80 = x^2 y$

$\Rightarrow y = \frac{80}{x^2}$ ....(i)

The total cost, $\text{P} = 2x^2(4000) + 4xy(400)$

$\Rightarrow \quad \text{P} = 8000x^2 + 1600x\left(\frac{80}{x^2}\right)$

$\text{P} = 8000x^2 + \frac{128000}{x}$

$\frac{d\text{P}}{dx} = 16000x - \frac{128000}{x^2}$

For minimum cost, $\frac{d\text{P}}{dx} = 0$

$\Rightarrow \quad 16000x = \frac{128000}{x^2}$

$x^3 = 8$

$\therefore \quad x = 2m$

From; $eqn\ (i)$

$y = \frac{80}{2^2} = 20m$

$\Rightarrow$ Length $=$ Width $= 2m$ and Height $= 20m$.

$\therefore$ Mr. Jotta has to use a $2m \times 2m \times 20m$ rectangular box to incur less costs on packaging for his customer.

ITEM THREE:

Towards a Mathematics competition in your school, your friend has been given the following set of questions in differential calculus by your Mathematics tutor to practice.

- Using small changes in differential calculus, show that; $(627.5)^{\frac{1}{4}} \approx 5\frac{1}{200}$
- If; $x(t) = (t-2)^{\frac{3}{2}}$ and $y(t) = t^2 - 1$, prove that; $\frac{d^2y}{dx^2} = \frac{4(t-4)}{9(t-2)^2}$
- If; $y(t) = tant$ and $x(t) = ln(1 + sint)$. Show that;
  - i). $\frac{dy}{dx} = sec^3\, t\,(1 + sint)$
  - ii). $\frac{d^2y}{dx^2} = sec^5\, t\,(1 + sint)^2(1 + 2sint)$
- A curve, $f(x)$ is given by parametric equations;
  $x(\emptyset) = 2cos\emptyset + sin2\emptyset$ and $y(\emptyset) = cos\emptyset - 2sin2\emptyset,\ 0 \le \emptyset \le 2\pi$
  The point, P lies on $f(x)$ where; $\emptyset = \frac{\pi}{4}$.
  - i). Find the coordinates of point, P.
  - ii). Show that the equation of the normal to $f(x)$ at point, P is given by; $4x + 2y = 5\sqrt{2}$.

Task(s):
Help your friend to prepare for the Mathematics competition.

Possible Responses:

- Let; $y = x^{\frac{1}{4}}$ ………………………………………… (i)
  Let; δx and δy be small increments in x and y respectively.
  $\Rightarrow\ y + \delta y = (x + \delta x)^{\frac{1}{4}}$ ………………………………… (ii)

  But; $(x + \delta x)^{\frac{1}{4}} \equiv (625 + 2.5)^{\frac{1}{4}}$; $\Rightarrow x = 625$ and $\delta x = 2.5$

  From $eqn\ (i)$; $y = (625)^{\frac{1}{4}}$
  $\therefore\ y = 5$

  Also; $\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$
  $\Rightarrow \frac{dy}{dx} = \frac{1}{4(625)^{\frac{3}{4}}} = \frac{1}{500}$

  For small increments, $\delta y \approx \left(\frac{dy}{dx}\right)\delta x$
  $\Rightarrow\ \delta y \approx \left(\frac{1}{500}\right) \times 2.5 = \frac{1}{200}$

From $eqn\ (ii)$; $\left(5+\frac{1}{200}\right) \approx (625+2.5)^{\frac{1}{4}}$

$$\therefore\ (627.5)^{\frac{1}{4}} \approx 5\frac{1}{200}$$: ***Hence, shown***

- $x = (t-2)^{\frac{3}{2}}$
  $\frac{dx}{dt} = \frac{3}{2}(t-2)^{\frac{1}{2}}$

  $y = t^2 - 1$
  $\frac{dy}{dt} = 2t$

  By chain rule; $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)$

  $$\Rightarrow\ \frac{dy}{dx} = (2t)\left(\frac{1}{\frac{3}{2}(t-2)^{\frac{1}{2}}}\right)$$

  $$\therefore\ \frac{dy}{dx} = \frac{4t}{3(t-2)^{\frac{1}{2}}}$$

  Again, by chain rule; $\frac{d^2y}{dx^2} = \left\{\frac{d}{dt}\left(\frac{dy}{dx}\right)\right\} \times \frac{dt}{dx}$

  $$\Rightarrow \frac{d^2y}{dx^2} = \frac{4}{3}\left\{\frac{d}{dt}\left(\frac{t}{(t-2)^{\frac{1}{2}}}\right)\right\} \times \frac{2}{3(t-2)^{\frac{1}{2}}}$$

  $$= \frac{8}{9}\left\{\frac{(t-2)^{\frac{1}{2}}(1) - t\left(\frac{1}{2}\right)(t-2)^{-\frac{1}{2}}}{(t-2)}\right\} \times \frac{1}{(t-2)^{\frac{1}{2}}}$$

  $$= \frac{8}{9}\left\{\frac{(t-2) - \frac{t}{2}}{(t-2)(t-2)^{\frac{1}{2}}}\right\} \times \frac{1}{(t-2)^{\frac{1}{2}}}$$

  $$= \frac{8}{9}\left\{\frac{\frac{1}{2}t - 2}{(t-2)^2}\right\}$$

  $$\therefore\ \frac{d^2y}{dx^2} = \frac{4(t-4)}{9(t-2)^2}$$: ***Hence, proved***

- $y = tant$
  $\frac{dy}{dt} = sec^2\, t$

  $x = ln(1 + sint)$
  $\frac{dx}{dt} = \frac{cost}{1+sint}$

  By chain rule, $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)$

$$\Rightarrow \frac{dy}{dx} = (sec^2 t)\left(\frac{1+sint}{cost}\right)$$

$\therefore \frac{dy}{dx} = \sec^3 t\,(1 + sint)$: ***Hence, shown***

Again, by chain rule; $\frac{d^2y}{dx^2} = \left\{\frac{d}{dt}\left(\frac{dy}{dx}\right)\right\} \times \frac{dt}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2} = \left\{\frac{d}{dt}\left(\frac{1+sint}{cos^3 t}\right)\right\} \times \left\{\frac{1+sint}{cost}\right\}$$

$$= \left\{\frac{cos^3 t(cost) - (1+sint)(-3\,cos^2 t sint)}{cos^6 t}\right\} \times \left\{\frac{1+sint}{cost}\right\}$$

$$= \left\{\frac{cos^2 t + 3sint(1+sint)}{cos^5 t}\right\} \times (1 + sint)$$

$$= \left\{\frac{2\sin^2 t + 3sint + 1}{\cos^5 t}\right\} \times (1 + sint)$$

But; $2\,sin^2 t + 3sint + 1 = (1 + sint)(1 + 2sint)$

$$\Rightarrow \frac{d^2y}{dx^2} = \left\{\frac{(1+sint)(1+2sint)}{cos^5 t}\right\} \times (1 + sint)$$

$\therefore \frac{d^2y}{dx^2} = \sec^5 t\,(1 + sint)^2(1 + 2sint)$: ***Hence, shown***

- At point P, $\emptyset = \frac{\pi}{4}$

$$\Rightarrow \quad x = 2\cos\left(\frac{\pi}{4}\right) + sin2\left(\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) + 1$$

$$x = \left(1 + \sqrt{2}\right)$$

Also, $\Rightarrow y = cos\left(\frac{\pi}{4}\right) - 2sin2\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - 2$

$$y = \frac{1}{2}\left(\sqrt{2} - 4\right)$$

$$\therefore \quad P\left[\left(1 + \sqrt{2}\right), \frac{1}{2}\left(\sqrt{2} - 4\right)\right]$$

$x = 2cos\emptyset + sin2\emptyset$

$\frac{dx}{d\emptyset} = -2sin\emptyset + 2cos2\emptyset$

$y = cos\emptyset - 2sin2\emptyset$

$\frac{dy}{d\emptyset} = -sin\emptyset - 4cos2\emptyset$

By chain rule, $\frac{dy}{dx} = \left(\frac{dy}{d\emptyset}\right)\left(\frac{d\emptyset}{dx}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{4cos2\emptyset + sin\emptyset}{2sin\emptyset - 2cos2\emptyset}$

At point P, $\emptyset = \frac{\pi}{4}$

$$\Rightarrow \frac{dy}{dx} = \frac{4cos2\left(\frac{\pi}{4}\right) + sin\left(\frac{\pi}{4}\right)}{2\,sin\left(\frac{\pi}{4}\right) - 2cos2\left(\frac{\pi}{4}\right)} = \frac{\left(\frac{\sqrt{2}}{2}\right)}{2\left(\frac{\sqrt{2}}{2}\right)}$$

$\therefore\ \frac{dy}{dx} = \frac{1}{2} = \mathrm{M_T}$

But, $\mathrm{M_N} = -\frac{1}{\mathrm{M_T}} = -\frac{1}{\frac{1}{2}}$

$\therefore\ \mathrm{M_N} = -2$

Thus, the equation of the normal to $f(x)$ at P can be obtained as;

$$\frac{y + \frac{1}{2}(4 - \sqrt{2})}{x - (1 + \sqrt{2})} = -2$$

$\Rightarrow -2(x - 1 - \sqrt{2}) = y + \frac{1}{2}(4 - \sqrt{2})$

$-4x + 4 + 4\sqrt{2} = 2y + 4 - \sqrt{2}$

$\Rightarrow 4x + 2y = 4\sqrt{2} + \sqrt{2}$

$\therefore\ 4x + 2y = 5\sqrt{2}$: ***Hence, shown***

ITEM FOUR:

A construction site in Busiro – East has a spherical water tank. The site manager measures the radius of the same tank as $210cm$.

However, after a hot afternoon, the plastic expands slightly increasing the radius to $210.5cm$.

Figure 5: ***A construction site in Budunga village, Busiro - East, Central Uganda, 2026***

Task(s):

Using differential calculus, help the site manager approximate the change in volume of water the tank can hold. (Use: $\pi = \frac{22}{7}$).

Possible Responses:

Volume of a spherical tank, $V = \frac{4}{3}\pi r^3$

$\Rightarrow \dfrac{dV}{dr} = 4\pi r^2$

Original radius, $r = 210cm$,

Increase in tank's radius, $\delta r = (210.5 - 210)cm$

$\therefore \ \delta r = 0.5cm$

For small increments, $\delta V \approx \left(\frac{dV}{dr}\right)\delta r$

$\Rightarrow \ \delta V \approx (4\pi r^2) \times 0.5 = \left(4 \times \frac{22}{7} \times 210^2\right) \times 0.5 \ cm^3$

$\delta V \approx 277200 \ cm^3$

$\therefore$ The approximate change in volume of water the tank can hold is; $277200 \ cm^3$

ITEM FIVE:

Ssemwanga is a student at Kooki High School.

He is in a school laboratory performing a titration experiment that produces a gas.

The volume of the gas, $V\ cm^3$, collected in a syringe over time, $t\ minutes$, is modelled by;

$$V = 9t^2 - t^3 + 21t$$

The experiment is being monitored for **only** ten minutes.

Figure 6: *Ssemwanga in a school laboratory titrating, Kooki High School, Tooro region, Oct 2025*

Task(s):

a). Find the exact volume of the gas collected when the reaction is most vigorous.

And, at what time will it be?

b). Sketch the curve of V against t.

Possible Responses:

a). When the reaction is most vigorous, then the volume of the gas collected is maximum.

$V = 9t^2 - t^3 + 21t$

$\Rightarrow \dfrac{dV}{dt} = 18t - 3t^2 + 21$

Reaction is vigorous when; $\frac{dV}{dt} = 0$

$\Rightarrow \ 18t - 3t^2 + 21 = 0$

$-3(t^2 - 6t - 7) = 0$

$(t - 7)(t + 1) = 0$

$\Rightarrow \ t = 7$ and $t = -1$

Since t denotes time, then; $t \neq -1$

$\therefore t = 7\ minutes$

Determining the nature of the coordinate point obtained above.

$\dfrac{d^2V}{dt^2} = 18 - 6t$

At $t = 7$, $\frac{d^2V}{dt^2} = 18 - 6(7)$

$\frac{d^2V}{dt^2} = -24 < 0$; ***thus, the point is a maxima***.

$\Rightarrow V_{max} = 9(7)^2 - (7)^3 + 21(7)$

$V_{max} = 245\ cm^3$

$\therefore$ The exact volume of the gas collected when the reaction is most vigorous is; $245\ cm^3$
And, it will be after $7\ minutes$.

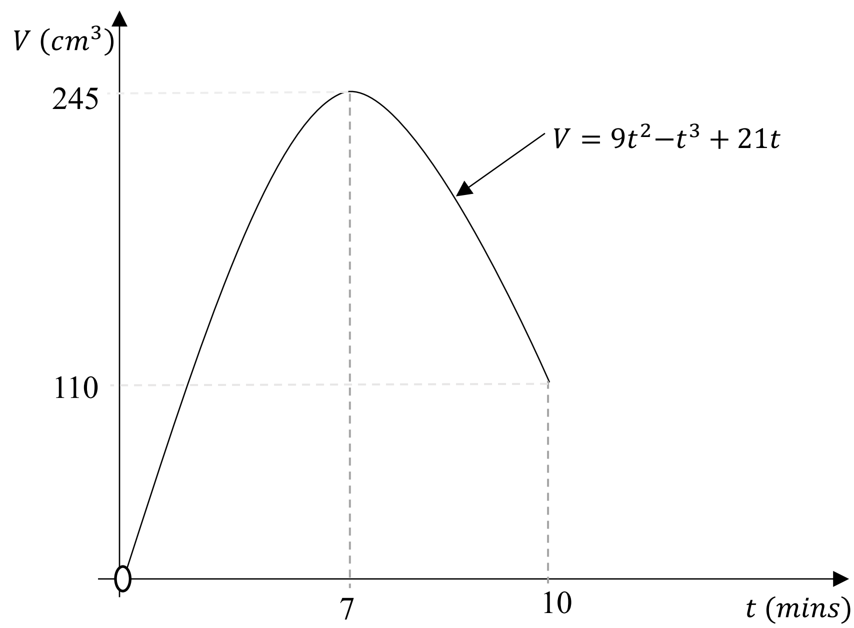
b). To sketch the curve, we have to consider the extreme point obtained in part a) above.

The maxima point is; $(7, 245)$.

- $V -$ intercept, $t = 0$
  $V = 9(0)^2 - (0)^3 + 21(0)$
  $V = 0$
  $\therefore$ The V −intercept is; $(0,0)$.

- At $t = 10$,
  $V = 9(10)^2 - (10)^3 + 21(10)$
  $V = 110$
  $\therefore$ The point is; $(10, 110)$

Sketch:



END:

"Wishing you countless success to Everyone and to all S.6 candidates of 2026"