## Indices

There are five basic rules of indices

(a) $a^p \; x \; a^q = a^{p+q}$
(b) $\frac{a^p}{a^q} = a^{p-q}$
(c) $(a^p)^q = a^{pq}$
(d) $a^{\frac{1}{q}} = \sqrt[q]{a}$
(e) $a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p$

***Example 1***

Evaluate the following

(a) $2^2$ x $2^3$
(b) $\frac{4^3}{4^2}$
(c) $(3^2)^3$
(d) $2^{\frac{1}{2}} \; x \; 2^{\frac{1}{2}}$
(e) $\sqrt[3]{27}$
(f) $125^{\frac{2}{3}}$

Solution

(a) $2^2$ x $2^3$ = $2^{2+3} = 2^5 = 32$
(b) $\frac{4^3}{4^2}$ = $4^{3-2} = 4^1 = 4$
(c) $(3^2)^3 = 3^{2\,x\,3} = 3^6 = 729$
(d) $2^{\frac{1}{2}} \; x \; 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$
(e) $\sqrt[3]{27} = (3^3)^{\frac{1}{3}} = 3^{3\,x\frac{1}{3}} = 3^1 = 3$
(f) $125^{\frac{2}{3}}$ =$\left(\sqrt[3]{125}\right)^2 = 5^2 = 25$

***Example 2***

Evaluate the following

(a) $\left(\frac{125}{27}\right)^{\frac{4}{3}}$
(b) $81^{\frac{3}{4}}$

Solution

(a) $\left(\frac{125}{27}\right)^{\frac{4}{3}} = \left(\frac{125^{\frac{4}{3}}}{27^{\frac{4}{3}}}\right) = \left(\frac{\left(\sqrt[3]{125}\right)^4}{\left(\sqrt[3]{27}\right)^4}\right) = \frac{625}{81}$
(b) $81^{\frac{3}{4}}$ = $\left(\sqrt[4]{81}\right)^3$ = 27

### The zero index

From $\frac{a^p}{a^p} = a^{p-p} = a^0 = 1$

∴ Any number raised to power zero =1

i.e. $100^0 = 529^0 = 83^0 = 1$

### Negative indices

It can be shown that

$\frac{1}{a} = \frac{a^0}{a^1} = a^{0-1} = a^{-1}$

Also

$\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$

Hence a negative index is the inverse of a given number

***Example 3***

Evaluate the following

(a) $16^{\frac{-3}{2}}$
(b) $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

Solution

(a) $16^{\frac{-3}{2}} = \left(\frac{1}{16}\right)^{\frac{3}{2}} = \left(\frac{1}{\sqrt{16}}\right)^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$
(b) $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$=$\left(\frac{27}{62}\right)^{\frac{2}{3}} = \left(\frac{\sqrt[3]{27}}{\sqrt[3]{64}}\right)^2 = \frac{9}{16}$

### Solving equations with unknown indices

It involves making appropriate substation after expressing terms containing powers in simplified form

***Example 4***

Solve the equation

$2^{2x+1} - 7(2^x) + 6 = 0$

Solution

$2^{2x+1} - 7(2^x) + 6 = 0$

$2^1.2^{2x} - 7(2^x) + 6 = 0$

$2(2^x)^2 - 7(2^x) + 6 = 0$

Let p = $2^x$

⇨ $2p^2 - 7p + 6 = 0$
$(2p - 3)(p - 2) = 0$

Either 2p -3 = 0

p = $\frac{3}{2}$

or

p – 2 =0

p = 2

when p= $\frac{3}{2}$ => $2^x = \frac{3}{2}$

$\log 2^x = \log\frac{3}{2}$

$x \log 2 = \log\frac{3}{2}$

$x = \frac{\log\frac{3}{2}}{\log s} = 0.585$

When p = 2

$2^x = 2 = 2^1$

x = 1

Hence x = 1 and x = 0.585 (3d.p)

***Example 5***

Show that

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x} = 4$

Solution

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x} = 4$

Handling terms on the LHS

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x}$

$= \frac{3(2^x\, x2^1)-4(2^x.2^1)}{2^x.2^1-2^x}$

$= \frac{2^x(3\,.\,2^1-4\,.\,2^1)}{2^x(2^1-1)} = \frac{6-2}{1} = 4$

***Example 5***

Solve $x^{\frac{4}{3}} = 81$

$x^{\frac{4}{3}x\frac{3}{4}} = 81^{\frac{3}{4}}$

x = $\left(\sqrt[4]{81}\right)^2$= $3^3 = 27$

**Solving equations with squares**

***Example 6***

$\sqrt{2x+5} = x + 1$

Square both sides

$\left(\sqrt{2x+5}\right)^2 = (x+1)^2$

2x + 5 = $x^2$+ 2x + 1

$x^2$ = 4

x = $\pm 2$

Testing/checking using -2

$\sqrt{2x+5} = x + 1$

$\sqrt{2\,x-2+5} = -2 + 1$

1 ≠ -1

Hence -2 is **not** a solution to the equation

Testing/checking using 2

$\sqrt{2x+5} = x + 1$

$\sqrt{2\,x2+5} = 2 + 1$

3 = 3

Hence 2 is the solution to the equation

***Example 7***

Solve for x: $\sqrt{x+2} = 4$

Square both sides

$\left(\sqrt{x+2}\right)^2 = 4^2$

x + 2 = 16

x = 14

**Finding square roots of terms containing rational and irrational quantities**

When finding roots of terms expressed in the form a + $\sqrt{b}$, where a is a rational and b is an irrational quantity, we let the root to be in the form of $\pm(\sqrt{x_1} + \sqrt{x_2})$ where $x_1$ and $x_2$ are integers.

Example 8

Find the square root of 6 +2$\sqrt{5}$

Let $\pm(\sqrt{x_1} + \sqrt{x_2})$ be square root of 6 +2$\sqrt{5}$

⇨ $\pm(\sqrt{x_1} + \sqrt{x_2}) = \sqrt{6\ + 2\sqrt{5}}$

Squaring both sides

$(\sqrt{x_1} + \sqrt{x_2})^2 = \left(\sqrt{6+\sqrt{5}}\right)^2$

$x_1 + x_2 + 2\sqrt{x_1.x_2}$ =6 +2$\sqrt{5}$

Comparing terms on the two sides

$x_1 + x_2 = 6$

$x_1 = 6 - \ x_2$ ………………..(i)

$\mathrm{x}_1.\mathrm{x}_2 = 5$…………………….(ii)

Substituting eqn. (i) into eqn. (ii)

$(6 - \ x_2)x_2$ = 5

${x_1}^2 - 6x_2 + 5 = 0$

${x_1}^2 - x_2 - 5x_2 + 5 = 0$

$x_2(x_2 - 1) - 5(x_2 - 1) = 0$

$(x_2 - 1)((x_2 - 5) = 0$

Either: $x_2 - 1$ = 0 => $x_2 = 1$

Or $x_2 - 5$ = 0 => $x_2 = 5$

When$x_2 = 1$: $x_1 = 6 - 1$ = 5

When$x_2 = 5$: $x_1 = 6 - 5$ = 1

Hence the square root of 6 +2$\sqrt{5}$ is $\pm(1 + \sqrt{5})$

Example 9

Find the square root of 8 - 2$\sqrt{15}$

et $\pm(\sqrt{x_1} - \sqrt{x_2})$ be square root of 8 -2$\sqrt{15}$

$\pm(\sqrt{x_1} + \sqrt{x_2}) = \sqrt{8 - 2\sqrt{15}}$

Squaring both sides

$(\sqrt{x_1} - \sqrt{x_2})^2 = \left(\sqrt{8 - 2\sqrt{15}}\right)^2$

$x_1 + x_2 - 2\sqrt{x_1.x_2} = 8\ - 2\sqrt{15}$

Comparing terms on the two sides

$x_1 + x_2 = 8$

$x_1 = 8 - \ x_2$ ………………..(i)

$x_1.x_2 = 15$……………………(ii)

Substituting eqn. (i) into eqn. (ii)

$(8 - \ x_2)x_2$ = 15

${x_1}^2 - 8x_2 + 15 = 0$

${x_1}^2 - 3x_2 - 5x_2 + 15 = 0$

$x_2(x_2 - 3) - 5(x_2 - 3) = 0$

$(x_2 - 3)((x_2 - 5) = 0$

Either: $x_2 - 5$ = 0 => $x_2 = 5$

Or $x_2 - 3$ = 0 => $x_2 = 3$

When$x_2 = 5$: $x_1 = 8 - 5$ = 3

When$x_2 = 3$: $x_1 = 8 - 3$ = 5

Hence the square root of 8 -2$\sqrt{15}$ is $\pm(\sqrt{5} - \sqrt{3})$

## Revision exercise

1. Simplify
   (i) $9a^2 \div 27a^{-4}$ $\left[\frac{2}{3}a^6\right]$
   (ii) $(6a^{-3}) \div (9a^{-4})^2$ $\left[\frac{2}{27}a^5\right]$
   (iii) $\frac{2a^{-3}b^2}{7c^{-4}d^2}$ $\left[\frac{2b^2c^4}{7a^3d^2}\right]$

(iv) $(x^4yz^{-3})^2 \times \sqrt{x^{-5}y^2z} \div (xz)^{\frac{1}{2}}$ $[x^5yz^{-6}]$

(v) $\sqrt[4]{y^3}x\sqrt{y^{\frac{1}{2}}}$ $\left[y^{\frac{5}{4}}\right]$

2. Evaluate
   (a) $(64)^{-\frac{3}{2}}$ [16]
   (b) $\left(\frac{8}{27}\right)^{-\frac{1}{3}}\left[\frac{3}{2}\right]$
   (c) $\left(\frac{1}{25}\right)^{\frac{1}{2}}\left[\frac{1}{5}\right]$
   (d) $\left(\frac{8}{27}\right)^{\frac{2}{3}}\left[\frac{4}{9}\right]$
   (e) $\left(\frac{243}{512}\right)^{-\frac{2}{3}}$ [1.6445]
3. Solve the following equations
   (a) $98x^2 = 2$ [x= 0.1429]
   (b) $x^{-3} = 8$ $\left[x = \frac{1}{2}\right]$
   (c) $\frac{1}{32}x^3 = 8x^{-1}$[x = 4]
   (d) $\frac{9}{25}x = \frac{5}{3}x^{-2}$ $\left[x = \frac{5}{3}\right]$
   (e) $\frac{2}{14}x^{-2} + 14x = 0$ [x = -0.2169]
4. Solve for x

(a) $3^{2x+1} + 3 = 10(3^x)$ [x = 1 or x = -1]

(b) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$[x =0, x =1.585]

(c) $7^x = 3^{1-x}$ [x=0.3608]

(d) $7x^{\frac{1}{2}}+2 = 0\left[x = \frac{4}{49}\right]$

(e) $5x^{\frac{2}{3}} = x^{-\frac{1}{3}}$ $\left[x = \frac{1}{5}\right]$

(f) $4x^{-\frac{1}{3}} = 5x^{\frac{1}{6}}$ $\left[x = \frac{16}{25}\right]$

(g) $6x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{2}} = 0$ [x = 0.077]

(h) $8x^{-2} = 343x^{\frac{1}{2}}$ [x=0.003562]

5. Show that

(a) $\frac{(2^{2x}-3.2^{2x-2})(3^x-2.3^{x-2})}{3^{x-4}(4^{x+3}-2^{2x})} = \frac{1}{4}$

(b) $\frac{(1+a)^{\frac{1}{2}}-\frac{1}{3}a(1+a)^{-\frac{2}{3}}}{(1+a)^{\frac{2}{3}}} = \frac{3+2a}{3(1+a)^{\frac{4}{3}}}$

(c) $(a - a^{-1})\left(a^{\frac{4}{3}} - a^{\frac{2}{3}}\right) = \frac{a^2-a^{-2}}{a^{-\frac{1}{3}}}$

(d) $\frac{a^{\frac{1}{2}}+ab}{ab-b^2} - \frac{\sqrt{a}}{\sqrt{a-b}} = \sqrt{\frac{a}{b}}$

6. Solve
   (a) $x^{\frac{1}{3}} - 3 = 28x^{-\frac{1}{3}}$ [ x =-64, x = 343]
   (b) $2x^{\frac{1}{4}} = 9 - 4x^{-\frac{1}{4}}$[x =$\frac{1}{16}$, x = 256]
   (c) $x^3 + 8 = 9x^{\frac{3}{2}}$ [x = 1, x= 4]
   (d) $2x^{\frac{1}{3}} = \frac{81}{8}x^{-1}$ [x = 8.6967]
   (e) $49x^{-\frac{5}{6}} - \frac{8}{81}x^{\frac{7}{6}} = 0$ [x=22.2739]
   (f) $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 2 = 0$[x=-1]
   (g) $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0$[x=-1]
   (h) $6x^{\frac{1}{3}} + 5 + x^{-\frac{1}{3}} = 0$ $\left[x = \frac{1}{2}, x = \frac{1}{3}\right]$
7. Solve for x
   (a) $\sqrt{x+2} - x = 0$[x=2]
   (b) $\sqrt{1+x} = 1 + \sqrt{1-x}$ $\left[x = \frac{\sqrt{3}}{2}\right]$
   (c) $(3-x)^{\frac{1}{2}} = (1+x)^{\frac{1}{2}} + (2-x)^{\frac{1}{2}}$ [x = -0.92665]
   (d) $\sqrt{x+6} = \sqrt{1-3x} - \sqrt{4-x}$ [-5]
8. Without using mathematical tables or calculators, find the value of $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]
9. Find the square root of the following
   (a) $6 + 2\sqrt{5}$ $[\pm(1+\sqrt{5})]$
   (b) $18 - 2\sqrt{12}[\pm(\sqrt{0.695} - \sqrt{17.303})]$
   (c) $18 - 2\sqrt{2}[\pm(\sqrt{0.1118} - \sqrt{17.8882})]$

Thank you

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